Key Terminology (from [3], p. 17)

- **DEFINITION** → describes the objects and notions we use
  - objects: numbers, functions, etc
  - notions: integration, summation, etc.
- **STATEMENT** → expresses that an object has a certain property
  - ↳ can be true or false, but cannot be ambiguous or imprecise
  - ↳ must end in a period
- **PROOF** → a convincing logical argument that a statement is true
- **THEOREM** → a mathematical statement proved true
  - ↳ used to build proofs

Reference Books

1.) The Calculus 7 of a Single Variable (1996) by Leithold, L.

2.) The Calculus with Analytic Geometry 2nd ed. (1972) by Leithold, L

3.) Intro to Theory of Computing 3rd ed. (2012) by Sipser, M.

Anti-differentiation

ANTIDERIVATIVE (definition from [1] p. 314 & [2] p. 260)

→ A function (F) is called an antiderivative of the function (f) on an interval (I) if $F'(x) = f(x)$ for every value of x in the interval (I).

↳ the interval (I) is the domain of F and f

↳ e.g.

$F(x)$		$f(x)$
$4x^3 + 2x + 5$	← antidifferentiate — — differentiate →	$12x^2 + 2$

if f is the derivative of F, then F is the antiderivative of f

THEOREM 1 ([1], p. 315)

→ if F is the antiderivative of f on the interval (I), then every antiderivative of f is given by

$$F(x) + C$$

↳ where C is any arbitrary constant

↳ Nota Bene (By Extension): there is a whole set of possible antiderivatives for a function. That is because you have to account for all the constants lost in differentiation

↳ Theorem 1 e.g.:

$$4x^3 + x^2 + 5 \xrightarrow{\text{differentiate}} \overbrace{12x^2 + 2x + 0}^{f(x)}$$

$$\underbrace{4x^3 + x^2}_{F(x)} + \underset{\text{any constant}}{\uparrow\;?} \xleftarrow{\text{antidifferentiate}}$$

ANTIDIFFERENTIATION therefore, is the process of finding the set of all antiderivatives of a function. (definition; [1], p. 316)

$$\int d(F(x)) = F(x) + C$$

integral symbol (inverses derivative operation)

"the antiderivative of the derivative of $F(x)$ is $F(x) + C$"

↳ e.g. of antidifferentiation

$$f(x) = 3x^2 + 4x + 3$$

$$f'(x) = 6x + 4 \; dx$$

dx tells us that we differentiated respect to x

$$\int f'(x) = \int 6x + 4 \, dx = 3x^2 + 4x + C.$$

THEOREMS/FORMULAS ON ANTIDIFFERENTIATION

([1], p. 317 or [2], p. 263)

THEOREM 2

$$\int dx = x + C$$

THEOREM 3 - Constant Coefficient

$$\int a\, f(x)\, dx = a \int f(x)\, dx$$

→ where a is a coefficient of $f(x)$

↳ e.g. Theorem 3

when expression is binomial and above, add parentheses

$$\int (4x^3 + 8x + 4)\, dx = \int 4(x^3 + 2x + 1)\, dx$$

(factored out 4)

$$= 4 \int (x^3 + 2x + 1)\, dx.$$

THEOREM 4 - SUM & DIFFERENCE

$$\int \left(a_1 f_1(x) + a_2 f_2(x) + \cdots + a_n f_n(x)\right) dx$$

$$= a_1 \int f_1(x)\, dx + a_2 \int f_2(x)\, dx + \cdots + a_n \int f_n\, dx$$

→ Sum & difference rule allows you to split an antiderivative expression

↳ all functions must have the same domain

$$\{f_1, f_2, \cdots f_n\}$$

↳ e.g. Theorem 4

$$\int (4x^3 + 5x)\,dx$$

$$= 4\int x^3\,dx + 5\int x\,dx.$$

THEOREM 5 – POWER RULE

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C, \quad n \neq (-1)$$

→ n is a rational number and is not (-1)

↳ e.g. Theorem 5

$$\int x^5\,dx = \frac{x^{5+1}}{5+1} + C = \frac{x^6}{6} + C.$$

↳ THEOREM 5 PROOF

$$\frac{d}{dx}\left[\frac{x^{n+1}}{n+1} + C\right] = \frac{1}{n+1}(n+1)\,x^{n+1-1} + 0 = x^n.$$

THEOREM 6 - CHAIN RULE FOR ANTIDERIVATIVES

$$\int f(g(x)) \cdot g'(x)\,dx = F(g(x)) + C$$

→ this is the basis for <u>u-substitution</u>

↳ if we let $g(x)$ be u, then the formula becomes:

$$\int f(u)\,du = F(u) + C$$

↳ THEOREM 6 PROOF

antiderivative of f by definition

$$\frac{d}{dx}\left[F(g(x)) + C\right] = F'(g(x)) \cdot g'(x) \qquad F'(x) = f(x)$$

$$= f(g(x)) \cdot g'(x)$$

↳ e.g Theorem 6

$$\int (5+2x^3)^8 x^2\,dx$$

$$= \int f(g(x)) \left(\tfrac{1}{6}\right) g'(x)$$

Theorem 3 & 6 $= \frac{1}{6}\left(\int x^8\,dx\right)$

Theorem 4 & 5 $= \frac{1}{6}\left(\frac{x^{8+1}}{8+1} + C\right)$

$$= \frac{(5+2x^3)^9}{54} + C.$$

Let $g(x)$ be $5+2x^3$

Let $f(x)$ be x^8

$$f(g(x)) = (5+2x^3)^8$$

$$g'(x) = 6x^2\,dx$$

$$\frac{1}{6} g'(x) = x^2\,dx$$

...

EXAMPLES

1.) $\int (2\sqrt{x} - 4x^3)\,dx$

$= 2\int x^{1/2} - 4\int x^3\,dx$ by Theorem 3 nd 4

$= 2\left(\frac{x^{1/2+1}}{1/2+1}\right) + C_1 - \left(4\left(\frac{x^{3+1}}{3+1}\right) + C_2\right)$ by Theorem 5

$= \frac{4}{3}(x^{3/2}) + 1 - x^4 - C_2$

$= \frac{4}{3}\sqrt{x^3} - x^1 + C$, where $C = C_1 \ C_2$.

(since C is any constant, we can merge them)

2.) $\int x^2\sqrt{1+x}\,dx$

Let u be $1+x$:

$x = (1+x) - 1 = u - x$

$x^2 = (u-1)^2$

$du = 0 + 1\,dx = dx$

We can now define the whole expression in terms of u:

$\int x^2\sqrt{1+x}\,dx = \int (u-1)^2\sqrt{u}\,du$

$= \int (u^2 - 2u + 1)\sqrt{u}\,du$

$= \int u^{5/2} - 2u^{3/2} + u^{1/2}\,du$

$= \int u^{5/2}du - 2\int u^{3/2}du + \int u^{1/2}du$

↳ by Theorem 4

$= \frac{u^{5/2+1}}{5/2+1} + C_1 - \left(2\left(\frac{u^{3/2+1}}{3/2+1}\right) + C_2\right)$

$+ \left(\frac{u^{1/2+1}}{1/2+1} + C_3\right)$

↳ by Theorem 5

$= \frac{2}{7}(x^{7/2}) - \frac{4}{5}(x^{5/2}) + \frac{2}{3}(u^{3/2})$

$+ C$,

where $C = C_1 - C_2 + C_3$.