

Unsupervised Learning

Graph clustering: spectral clustering and the Stochastic Bloc Model

Polytechnique MAP 573, 2019 – Julien Chiquet

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<https://github.com/jchiquet/CourseUnsupervisedLearningX>



Outline

① Basic notions on graphs and networks

- Definitions

- Representations

② Graph Partitioning

- Hierarchical clustering

- Spectral Clustering

③ The Stochastic Block Model (SBM)

- Some Graphs Models and their limitations

- Mixture of Erdős-Rényi and the SBM

- Inference in SBM with variational EM

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① Basic notions on graphs and networks

Definitions

Representations

② Graph Partitioning

③ The Stochastic Block Model (SBM)

References



Statistical Analysis of Network Data: Methods and Models,
Eric Kolaczyk
Chapter 2, Section 1



Analyse statistique de graphes,
Catherine Matias
Chapitre 1

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① Basic notions on graphs and networks

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Graphs, Networks: some definitions

Definition (Network versus Graph)

- A **Network** is a collection of interacting entities
- A **Graph** is the mathematical representation of a network

Definition (Graph)

A graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is a mathematical structure consisting of

- a set $\mathcal{V} = \{1, \dots, n\}$ of **vertices** or **nodes**
- a set $\mathcal{E} = \{e_1, \dots, e_p : e_k = (i_k, j_k) \in (\mathcal{V} \times \mathcal{V})\}$ of **edges** or **links**
- The number of vertices $N_v = |\mathcal{V}|$ is called the **order**
- The number of edges $N_e = |\mathcal{E}|$ is called the **size**

Definition (Vocabulary)

subgraph, induced subgraph, (un)directed graph, weighted graph, bipartite graph, tree, DAG, etc.

Paths, Cycles, Connected Components

Definition (Path)

In a undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ a path between $i, j \in \mathcal{V}^2$ is a series of edges e_1, \dots, e_k such that

- $\forall 1 \leq \ell < k$, all edges $(e_\ell, e_{\ell+1})$ share a vertex in \mathcal{V}
- e_1 starts from i , e_k ends to j .

Vocabulary

- A **cycle** is a path from i to itself.
- A **connected component** is a subset $\mathcal{V}' \subset \mathcal{V}$ such that there exists an path between any $i, j \in \mathcal{V}'$.
- A graph is **connected** when there is a path between every node pairs.

Proposition (Decomposition)

Any graph can be decomposed in a unique set of maximal connected components. The number of connected component is a least $n - |\mathcal{E}|$

Neighborhood, Degree

Definition (Neighborhood)

The neighbors of a vertex are the nodes directly connected to this vertex:

$$\mathcal{N}(i) = \{j \in \mathcal{V} : (i, j) \in \mathcal{E}\}.$$

Definition (Degree)

The degree d_i of a node i is given by its number of neighbors, i.e. $|\mathcal{N}(i)|$.

Remark

In digraphs, vertex degree is replaced by **in-degree** and **out-degree**.

Proposition

*In a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ the sum of the degree is given by $2|\mathcal{E}|$. Hence **this is always an even quantity**.*

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Adjacency matrix and list of edges

Definition (Adjacency matrix)

The connectivity of $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is captured by the $|\mathcal{V}| \times |\mathcal{V}|$ matrix \mathbf{A} :

$$(\mathbf{A})_{ij} = \begin{cases} 1 & \text{if } i \sim j, \\ 0 & \text{otherwise.} \end{cases}$$

Proposition

The degree of \mathcal{G} are then simply obtained as the row-wise and/or column-wise sums of \mathbf{A} .

Remark

If the list of vertices is known, the only information which needs to be stored is the list of edges. In terms of storage, this is equivalent to a sparse matrix representation.

Incidence matrix

Definition (Incidence matrix)

The connectivity of $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is captured by the $|\mathcal{V}| \times |\mathcal{E}|$ matrix \mathbf{B} :

$$(\mathbf{B})_{ij} = \begin{cases} 1 & \text{if } i \text{ is incident to edge } j, \\ 0 & \text{otherwise.} \end{cases}$$

Proposition (Relationship)

Let $\tilde{\mathbf{B}}$ be a modified *signed* version of \mathbf{B} where $\tilde{B}_{ij} = 1 / -1$ if i is incident to j as tail/head. Then

$$\tilde{\mathbf{B}}\tilde{\mathbf{B}}^T = \mathbf{D} - \mathbf{A},$$

where $\mathbf{D} = \text{diag}(\{d_i, i \in \mathcal{V}\})$ is the diagonal matrix of degrees.

$\rightsquigarrow \tilde{\mathbf{B}}\tilde{\mathbf{B}}^T$ is called the Laplacian matrix and will be studied latter.

Layout and Visualization

- Visualization of large networks is a field of research in its own
- Be carefull with graphical interpretation of (large) networks

```
library(igraph)
library(sand)
GLattice <- graph.lattice(c(5,5,5))
GBlog <- aidsblog
```

Layout and Visualization

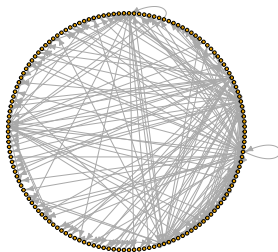
Example with circle plot

```
par(mfrow=c(1,2))  
plot(GLattice, layout=layout.circle); title("5x5x5 lattice")  
plot(GBlog , layout=layout.circle); title("blog network")
```

5x5x5 lattice



blog network

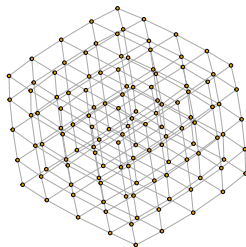


Layout and Vizualization

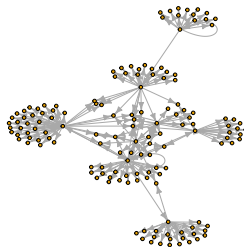
Example with Fruchterman and Reingold

```
par(mfrow=c(1,2))  
plot(GLattice, layout=layout.fruchterman.reingold); title("5x5x5 lattice")  
plot(GBlog , layout=layout.fruchterman.reingold); title("blog network")
```

5x5x5 lattice



blog network



Layout and Visualization: **ggraph** way I

```
library(ggraph)
library(gridExtra)
g1 <- ggraph(GBlog, layout = "fr") +
  geom_edge_link(color = "lightgray") + geom_node_point() + theme_void()

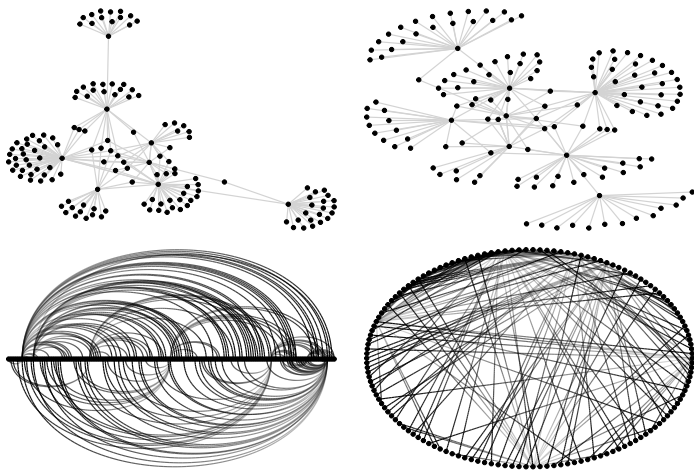
g2 <- ggraph(GBlog, layout = "kk") +
  geom_edge_link(color = "lightgray") + geom_node_point() + theme_void()

g3 <- ggraph(GBlog, layout = "linear") +
  geom_edge_arc(aes(alpha=..index..), show.legend = FALSE) +
  geom_node_point() + theme_void()

g4 <- ggraph(GBlog, layout = "linear", circular = TRUE) +
  geom_edge_link(aes(alpha=..index..), show.legend = FALSE) +
  geom_node_point() + theme_void()

grid.arrange(g1, g2, g3, g4, nrow = 2, ncol = 2)
```





Layout and Visualization: **ggraph** way II



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 - Hierarchical clustering
 - Spectral Clustering
- 3 The Stochastic Block Model (SBM)

References

-  Statistical Analysis of Network Data: Methods and Models,
Eric Kolaczyk
Chapter 4, Section 4
-  Statistical Analysis of Network Data with R,
Eric Kolaczyk, Gábor Csárdi
Chapter 4, Section 4
-  Analyse statistique de graphes,
Catherine Matias
Chapitre 3
-  A Tutorial on Spectral Clustering,
Ulrike von Luxburg

Principle of graph partitionning

Definition (Partition)

A decomposition $\mathcal{C} = \{C_1, \dots, C_K\}$ of the vertices \mathcal{V} such that

- $C_k \cap C_{k'} = \emptyset$ for any $k \neq k'$
- $\bigcup_k C_k = \mathcal{V}$

Goal of graph partitionning

Form a partition of the vertices with unsupervised approach where the \mathcal{C} is composed by "cohesive" sets of vertices, for instance,

- ① vertices well connected among themselves
- ② well separated from the remaining vertices

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Principle

Input: n individuals with p attributes)

1. Compute the dissimilarity between groups
2. Regroup the two most similar elements

Iterate until all element are in a single group

Output: n nested partitions from $\{\{1\}, \dots, \{n\}\}$ to $\{\{1, \dots, n\}\}$

Algorithm 1: Agglomerative hierarchical clustering

Ingredients

- ① a dissimilarity measure between singleton
- ② a distance measure between sets

Dissimilarity measures

Standards

Use standard distances on adjacency matrix:

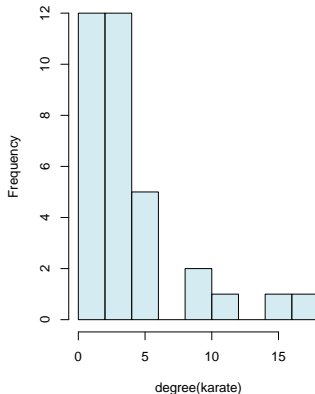
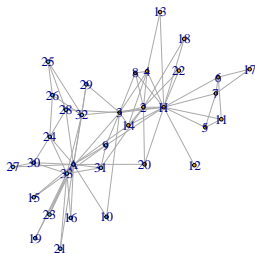
- Euclidean distance: $x_{ij} = \sqrt{\sum_{k} (A_{ik} - A_{jk})^2}$
- Manhattan distance: $x_{ij} = \sum_{k} |A_{ik} - A_{jk}|$
- etc. . .

Graph-specific

For instance, Modularity (studied during tutorial)

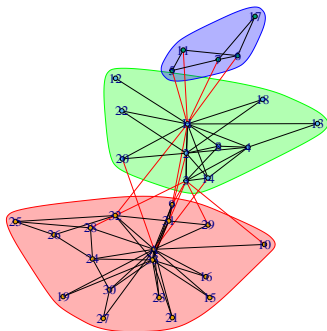
Example: karaté club

```
library(sand) data(karate)
par(mfrow=c(1,2)) plot(karate)
hist(degree(karate), col=adjustcolor("lightblue", alpha.f = 0.5), main="")
```



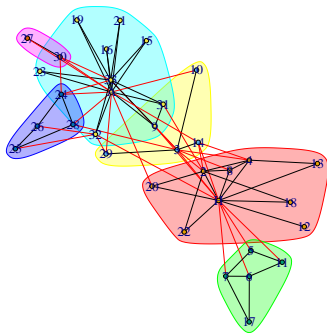
Examples of graph clustering I

```
hc <- cluster_fast_greedy(karate)  
plot(hc, karate)
```



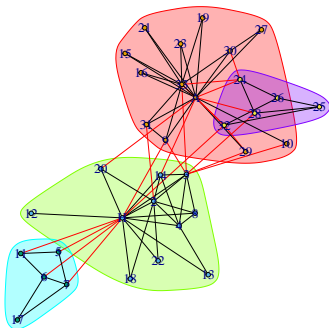
Examples of graph clustering II

```
hc <- cluster_edge_betweenness(karate)  
plot(hc, karate)
```



Examples of graph clustering III

```
hc <- cluster_walktrap(karate)  
plot(hc, karate)
```



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Graph Laplacian

Definition ((Un-normalized) Laplacian)

The Laplacian matrix \mathbf{L} , resulting from the modified incidence matrix $\tilde{\mathbf{B}}$ $\tilde{B}_{ij} = 1 / -1$ if i is incident to j as tail/head, is defined by

$$\mathbf{L} = \tilde{\mathbf{B}}\tilde{\mathbf{B}}^T = \mathbf{D} - \mathbf{A},$$

where $\mathbf{D} = \text{diag}(d_i, i \in \mathcal{V})$ is the diagonal matrix of degrees.

Remark

- \mathbf{L} is called Laplacian by analogy to the second order derivative (see below).
- Spectrum of \mathbf{L} has much to say about the structure of the graph \mathcal{G} .

Graph Laplacian: spectrum

Proposition (Spectrum of \mathbf{L})

The $n \times n$ matrix \mathbf{L} has the following properties:

$$\mathbf{x}^\top \mathbf{L} \mathbf{x} = \frac{1}{2} \sum_{i,j} A_{ij} (x_i - x_j)^2, \quad \forall \mathbf{x} \in \mathbb{R}^n.$$

- \mathbf{L} is a symmetric, positive semi-definite matrix,
- the smallest eigenvalue is 0 with associated eigenvector $\mathbf{1}$.
- \mathbf{L} has n positive eigenvalues $0 = \lambda_1 < \dots < \lambda_n$.

Corollary (Spectrum and Graph)

- The multiplicity of the first eigen value (0) of \mathbf{L} determines the number of connected components in the graph.
- The larger the second non trivial eigenvalue, the higher the connectivity of \mathcal{G} .

Normalized Graph Laplacian

Definition ((Normalized) Laplacian)

The normalized Laplacian matrix \mathbf{L} is defined by

$$\mathbf{L}_N = \mathbf{D}^{-1/2} \mathbf{L} \mathbf{D}^{-1/2} = \mathbf{I} - \mathbf{D}^{-1/2} \mathbf{A} \mathbf{D}^{-1/2}.$$

Proposition

The $n \times n$ matrix \mathbf{L}_N has the following properties:

$$\mathbf{x}^\top \mathbf{L}_N \mathbf{x} = \frac{1}{2} \sum_{i,j} A_{ij} \left(\frac{x_i}{\sqrt{d_i}} - \frac{x_j}{\sqrt{d_j}} \right)^2, \text{quad} \forall \mathbf{x} \in \mathbb{R}^n.$$

- \mathbf{L} is a symmetric, positive semi-definite matrix, with n nonnegative eigenvalues $0 = \lambda_1 < \dots < \lambda_n$
- the smallest eigenvalue is 0 with associated eigenvector $\mathbf{D}^{-1/2} \mathbf{1}$.

Absolute Graph Laplacian

Definition ((Absolute) Laplacian)

The absolute Laplacian matrix \mathbf{L}_{abs} is defined by

$$\mathbf{L}_{abs} = \mathbf{D}^{-1/2} \mathbf{A} \mathbf{D}^{-1/2} = \mathbf{I} - \mathbf{L}_N,$$

with eigenvalues $1 - \lambda_n \leq \dots \leq 1 - \lambda_2 \leq 1 - \lambda_1 = 1$, where $0 = \lambda_1 \leq \dots \leq \lambda_n$ are the eigenvalues of \mathbf{L}_N .

Spectral Clustering

Principle

- ① Use the spectral property of \mathbf{L} to perform clustering in the eigen space
- ② If the network have K connected components, the first K eigenvectors are $\mathbf{1}$ span the eigenspace associated with eigenvalue 0
- ③ Applying a simple clustering algorithm to the rows of the K first eigenvectors separate the components

↪ This principle generalizes to a graph with a single component: spectral clustering tends to separates groups of nodes which are highly connected together

Normalized Spectral Clustering

Input: Adjacency matrix and number of classes Q

Compute the normalized graph Laplacian \mathbf{L}

Compute the eigen vectors of \mathbf{L} associated with the Q **smallest eigenvalues**

Define \mathbf{U} , the $p \times Q$ matrix that encompasses these Q vectors

Define $\tilde{\mathbf{U}}$, the row-wise normalized version of \mathbf{U} : $\tilde{u}_{ij} = \frac{u_{ij}}{\|\mathbf{U}_i\|_2}$

Apply k-means to $(\tilde{\mathbf{U}}_i)_{i=1,\dots,p}$

Output: vector of classes $\mathbf{C} \in \mathcal{Q}^p$, such as $C_i = q$ if $i \in q$

Algorithm 2: Spectral Clustering by Ng, Jordan and Weiss (2002)

Absolute Spectral Clustering

Input: Adjacency matrix and number of classes Q

Compute the graph Laplacian \mathbf{L}_{abs}

Compute the eigen vectors of \mathbf{L}_{abs} associated with the Q **largest** absolute eigenvalues

Define \mathbf{U} , the $p \times Q$ matrix that encompasses these Q vectors

Apply k-means to $(\mathbf{U}_i)_{i=1,\dots,p}$

Output: vector of classes $\mathbf{C} \in \mathcal{Q}^p$, such as $C_i = q$ if $i \in q$

Algorithm 3: Spectral Clustering by Rohe et al. (2011)

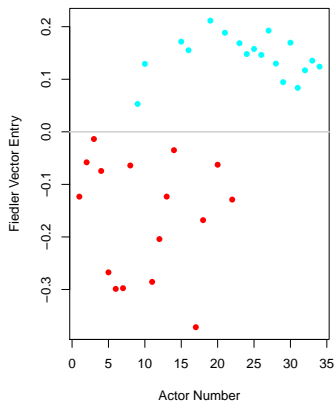
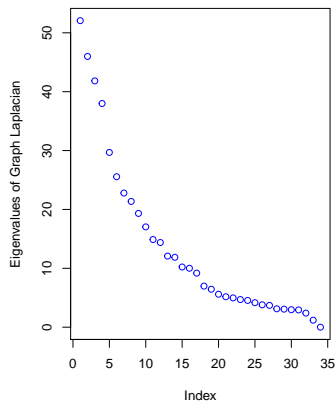
Example: Karate club and Fiedler vector and eigenvalue I

```
k.lap <- graph.laplacian(karate)
eig.anal <- eigen(k.lap)

f.vec <- eig.anal$vectors[, 33]
faction <- igraph::get.vertex.attribute(karate, "Faction")
f.colors <- as.character(length(faction))
f.colors[faction == 1] <- "red"
f.colors[faction == 2] <- "cyan"

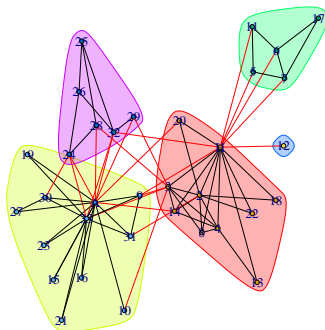
par(mfrow=c(1,2))
plot(eig.anal$values, col="blue",
     ylab="Eigenvalues of Graph Laplacian")
plot(f.vec, pch=16, xlab="Actor Number",
     ylab="Fiedler Vector Entry", col=f.colors)
abline(0, 0, lwd=2, col="lightgray")
```

Example: Karate club and Fiedler vector and eigenvalue II



Clustering based on the first non null eigenvalue

```
hc <- cluster_leading_eigen(karate)  
plot(hc, karate)
```



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Inference in SBM with variational EM

References



Statistical Analysis of Network Data: Methods and Models

Eric Kolaczyk

Chapters 5 and 6



Mixture model for random graphs, Statistics and Computing

Daudin, Robin, Picard

pbil.univ-lyon1.fr/members/fpicard/franckpicard_fichiers/pdf/DPR08.pdf



Analyse statistique de graphes,

Catherine Matias

Chapitre 4, Section 4

Motivations

Last section: find an underlying organization in a observed network

Spectral or hierachical clustering for network data

↪ Not model-based, thus no statistical inference possible

now: clustering of network based on a probabilistic model of the graph

Become familiar with

- the stochastic block model, a random graph model tailored for clustering vertices,
- the variational EM algorithm used to infer SBM from network data.

hierarchical/kmeans clustering \leftrightarrow Gaussian mixture models



hierarchical/spectral clustering for network \leftrightarrow Stochastic block model

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A mathematical model: Erdős-Rényi graph

Definition

Let $\mathcal{V} = 1, \dots, n$ be a set of fixed vertices. The (simple) Erdős-Rényi model $\mathcal{G}(n, \pi)$ assumes random edges between pairs of nodes with probability π . In other word, the (random) adjacency matrix \mathbf{X} is such that

$$X_{ij} \sim \mathcal{B}(\pi)$$

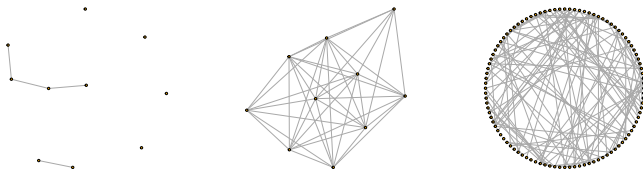
Proposition (degree distribution)

The (random) degree D_i of vertex i follows a binomial distribution:

$$D_i \sim b(n - 1, \pi).$$

Erdős-Rényi - example

```
G1 <- igraph::sample_gnp(10, 0.1)
G2 <- igraph::sample_gnp(10, 0.9)
G3 <- igraph::sample_gnp(100, .02)
par(mfrow=c(1,3))
plot(G1, vertex.label=NA) ; plot(G2, vertex.label=NA)
plot(G3, vertex.label=NA, layout=layout.circle)
```



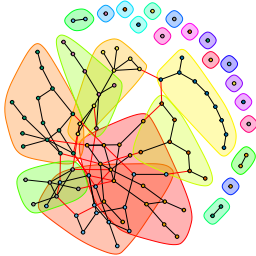
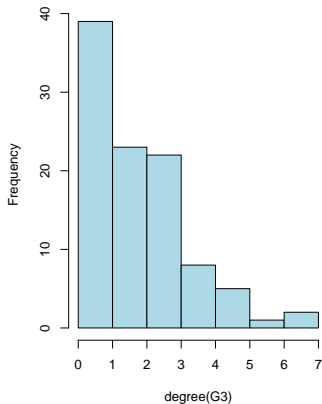
Erdős-Rényy - limitations: very homogeneous

```
average.path.length(G3); diameter(G3)
```

```
## [1] 5.41448
```

```
## [1] 16
```

Histogram of degree(G3)



Mechanism-based model: preferential attachment

The graph is defined dynamically as follows

Definition

Start from a initial graph $\mathcal{G}_0 = (\mathcal{V}_0, \mathcal{E}_0)$, then for each time step,

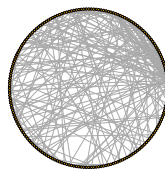
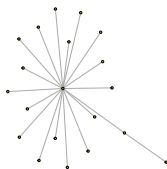
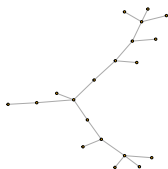
- ① At t a new node V_t is added
- ② V_t is connected to $i \in V_{t-1}$ with probability

$$D_i^\alpha + \text{cst.}$$

\rightsquigarrow Nodes with high degree get more connections thus **richers get richers**

Preferential attachment - example

```
G1 <- igraph::sample_pa(20, 1, directed=FALSE)
G2 <- igraph::sample_pa(20, 5, directed=FALSE)
G3 <- igraph::sample_pa(200, directed=FALSE)
par(mfrow=c(1,3))
plot(G1, vertex.label=NA) ; plot(G2, vertex.label=NA)
plot(G3, vertex.label=NA, layout=layout.circle)
```



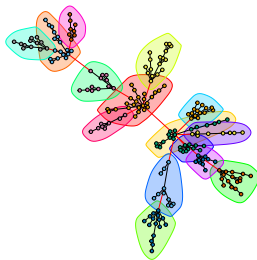
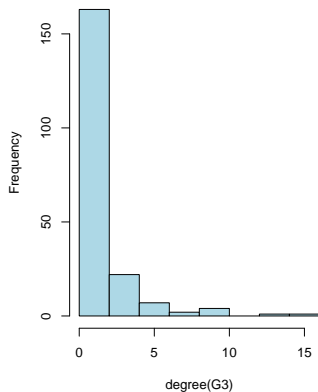
Preferential attachment - limitations

```
average.path.length(G3); diameter(G3)
```

```
## [1] 6.26809
```

```
## [1] 15
```

Histogram of degree(G3)



Limitations

- Erdős-Rényi

The ER model does not fit well real world network

- As can be seen from its degree distribution
- ER is generally too homogeneous

- Preferential attachment

- Is defined through an algorithm so performing statistics is complicated
- Is stucked to the power-law distribution of degrees

The Stochastic Block Model

The SBM¹ generalizes ER in a mixture framework. It provides

- a statistical framework to adjust and interpret the parameters
- a flexible yet simple specification that fits many existing network data

¹Other models exist (e.g. exponential model for random graphs) but less popular.

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Stochastic Block Model: definition

Mixture model point of view: mixture of Erdős-Rényi

Latent structure

Let $\mathcal{V} = \{1, \dots, n\}$ be a fixed set of vertices. We give each $i \in \mathcal{V}$ a **latent label** among a set $\mathcal{Q} = \{1, \dots, Q\}$ such that

- $\alpha_q = \mathbb{P}(i \in q), \quad \sum_q \alpha_q = 1;$
- $Z_{iq} = \mathbf{1}_{\{i \in q\}}$ are independent hidden variables.

The conditional distribution of the edges

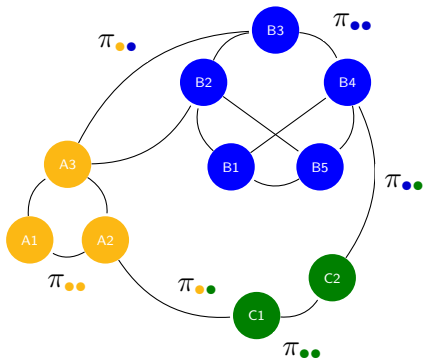
Connexion probabilities depend on the node class belonging:

$$X_{ij} | \{i \in q, j \in \ell\} \sim \mathcal{B}(\pi_{q\ell}) \quad \left(\Leftrightarrow X_{ij} | \{Z_{iq}Z_{j\ell} = 1\} \sim \mathcal{B}(\pi_{q\ell}). \right)$$

The $Q \times Q$ matrix π gives for all couple of labels

$$\pi_{q\ell} = \mathbb{P}(X_{ij} = 1 | i \in q, j \in \ell).$$

Stochastic Block Model: the big picture



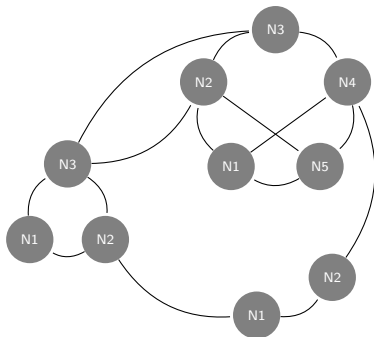
Stochastic Block Model

Let n nodes divided into

- $\mathcal{Q} = \{\bullet, \bullet, \bullet\}$ classes
- $\alpha_{\bullet} = \mathbb{P}(i \in \bullet), \bullet \in \mathcal{Q}, i = 1, \dots, n$
- $\pi_{\bullet\bullet} = \mathbb{P}(i \leftrightarrow j | i \in \bullet, j \in \bullet)$

$$Z_i = \mathbf{1}_{\{i \in \bullet\}} \sim^{\text{iid}} \mathcal{M}(1, \alpha), \quad \forall \bullet \in \mathcal{Q},$$
$$X_{ij} | \{i \in \bullet, j \in \bullet\} \sim^{\text{ind}} \mathcal{B}(\pi_{\bullet\bullet})$$

Stochastic Block Model: unknown parameters



Stochastic Block Model

Let n nodes divided into

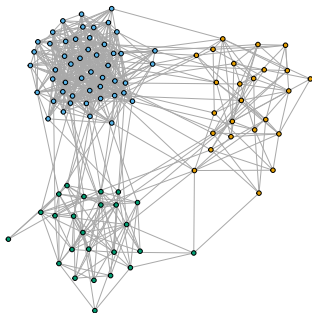
- $\mathcal{Q} = \{\bullet, \bullet, \bullet\}$, $\text{card}(\mathcal{Q})$ known
- $\alpha_{\bullet} = ?$,
- $\pi_{\bullet\bullet} = ?$

$$Z_i = \mathbf{1}_{\{i \in \bullet\}} \sim^{\text{iid}} \mathcal{M}(1, \alpha), \quad \forall \bullet \in \mathcal{Q},$$
$$X_{ij} \mid \{i \in \bullet, j \in \bullet\} \sim^{\text{ind}} \mathcal{B}(\pi_{\bullet\bullet})$$

Stochastic block models – examples of topology

Community network

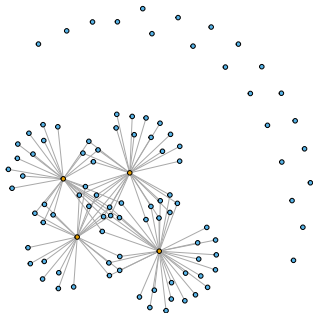
```
pi <- matrix(c(0.3,0.02,0.02,0.02,0.3,0.02,0.02,0.02,0.3),3,3)
communities <- igraph::sample_sbm(100, pi, c(25, 50, 25))
plot(communities, vertex.label=NA, vertex.color = rep(1:3,c(25, 50, 25)))
```



Stochastic block models – examples of topology

Star network

```
pi <- matrix(c(0.05,0.3,0.3,0),2,2)
star <- igraph::sample_sbm(100, pi, c(4, 96))
plot(star, vertex.label=NA, vertex.color = rep(1:2,c(4,96)))
```



Degree distributions

Conditional degree distribution

The conditional degree distribution of a node $i \in q$ is

$$D_i | i \in q \sim \text{b}(n-1, \bar{\pi}) \approx \mathcal{P}(\lambda_q), \quad \bar{\pi}_q = \sum_{\ell=1}^Q \alpha_\ell \pi_{q\ell} \quad \lambda_q = (n-1) \bar{\pi}_q$$

Conditional degree distribution

The degree distribution of a node i can be approximated by a mixture of Poisson distributions:

$$\mathbb{P}(D_i = k) = \sum_{q=1}^Q \alpha_q \exp\{-\lambda_q\} \frac{\lambda_q^k}{k!}$$

Likelihoods

Complete-data loglikelihood

$$\log L(\mathbf{X}, \mathbf{Z}) = \sum_{i,q} Z_{iq} \log \alpha_q + \sum_{i < j, q, \ell} Z_{iq} Z_{j\ell} \log \pi_{q\ell}^{X_{ij}} (1 - \pi_{q\ell})^{1-X_{ij}}.$$

Conditional expectation of the complete-data loglikelihood

$$\mathbb{E}_{\mathbf{Z}|\mathbf{X}}[\log L(\boldsymbol{\theta}; \mathbf{X}, \mathbf{Z})] = \sum_{i,q} \tau_{iq} \log \alpha_q + \sum_{i < j, q, \ell} \eta_{ijq\ell} \log \pi_{q\ell}^{X_{ij}} (1 - \pi_{q\ell})^{1-X_{ij}}$$

where $\tau_{iq}, \eta_{ijq\ell}$ are the posterior probabilities:

- $\tau_{iq} = \mathbb{P}(Z_{iq} = 1 | \mathbf{X}) = \mathbb{E}[Z_{iq} | \mathbf{X}]$.
- $\eta_{ijq\ell} = \mathbb{P}(Z_{iq} Z_{j\ell} = 1 | \mathbf{X}) = \mathbb{E}[Z_{iq} Z_{j\ell} | \mathbf{X}]$.

Outline

① Basic notions on graphs and networks

② Graph Partitioning

③ The Stochastic Block Model (SBM)

Some Graphs Models and their limitations

Mixture of Erdős-Rényi and the SBM

Inference in SBM with variational EM

The EM strategy does not apply directly for SBM

Ouch: another intractability problem

- the Z_{iq} are **not independent** in the SBM framework. . .
- we cannot compute $\eta_{ijql} = \mathbb{P}(Z_{iq}Z_{jl} = 1|\mathbf{X}) = \mathbb{E}[Z_{iq}Z_{jl}|\mathbf{X}]$,
- the conditional expectation $Q(\boldsymbol{\theta})$, i.e. the main EM ingredient, is **intractable**.

Solution: mean field approximation

Approximate η_{ijql} by $\tau_{iq}\tau_{jl}$, i.e., **assume independence between Z_{iq}**

\rightsquigarrow This can be formalized in the variational framework

Revisiting the EM algorithm I

Proposition

Consider a distribution \mathbb{Q} for the $\{Z_{iq}\}$. We have

$$\log L(\boldsymbol{\theta}; \mathbf{X}) = \mathbb{E}_{\mathbb{Q}}[\log L(\boldsymbol{\theta}, \mathbf{X}, \mathbf{Z})] + \mathcal{H}(\mathbb{Q}) + \text{KL}(\mathbb{Q} \mid \mathbb{P}(\mathbf{Z}|\mathbf{X}; \boldsymbol{\theta})),$$

where \mathcal{H} is the entropy and $\text{KL}(\cdot|\cdot)$ is the Kullback-Leibler divergence:

$$\mathcal{H}(\mathbb{Q}) = - \sum_z \mathbb{Q}(z) \log \mathbb{Q}(z) = -\mathbb{E}_{\mathbb{Q}}[\log \mathbb{Q}(Z)]$$

$$\text{KL}(\mathbb{Q} \mid \mathbb{P}(\mathbf{Z}|\mathbf{X}; \boldsymbol{\theta})) = \sum_z \mathbb{Q}(z) \log \frac{\mathbb{Q}(z)}{\mathbb{P}(\mathbf{Z}|\mathbf{X}; \boldsymbol{\theta})} = \mathbb{E}_{\mathbb{Q}} \left[\log \frac{\mathbb{Q}(z)}{\mathbb{P}(\mathbf{Z}|\mathbf{X}; \boldsymbol{\theta})} \right]$$

Revisiting the EM algorithm II

Let

$$J(\mathbb{Q}, \boldsymbol{\theta}) \triangleq \mathbb{E}_{\mathbb{Q}} (\log L(\boldsymbol{\theta}; \mathbf{X}, \mathbf{Z})) + \mathcal{H}(\mathbb{Q})$$

The steps in the EM algorithm may be viewed as:

Expectation step : choose \mathbb{Q} to maximize $J(\mathbb{Q}; \boldsymbol{\theta}^{(t)})$

The solution is $\mathbb{P}(\mathbf{Z}|\mathbf{X}; \boldsymbol{\theta}^{(t)})$

Maximization step : choose $\boldsymbol{\theta}$ to maximize $J(\mathbb{Q}^{(t)}; \boldsymbol{\theta})$

The solution maximizes $\mathbb{E}_{\mathbf{Z}|\mathbf{X}; \boldsymbol{\theta}^{(t)}} (\log L(\boldsymbol{\theta}; \mathbf{X}, \mathbf{Z}))$

Variational approximation for SBM

Problem for SBM

$\mathbb{P}(\mathbf{Z}|\mathbf{X}; \boldsymbol{\theta}^{(t)})$ cannot be computed thus the E-step cannot be solved.

Idea

Choose \mathbb{Q} in a class of function so that the E-step can be solved.

Family of distribution that factorizes

We chose \mathbb{Q} so as the Z_{iq} are marginally independents:

$$\mathbb{Q}(\mathbf{Z}) = \prod_{i=1}^n \mathbb{Q}_i(Z_i) = \prod_{i=1}^n \prod_{q=1}^Q \tau_{iq}^{Z_{iq}},$$

where $\tau_{iq} = \mathbb{Q}_i(Z_i = q) = \mathbb{E}Q(Z_{iq})$, with $\sum_q \tau_{iq} = 1$ for all $i = 1, \dots, n$.

Variational EM for SBM: the criterion

Lower bound of the loglikelihood

Since \mathbb{Q} is an approximation of $\mathbb{P}(\mathbf{Z}|\mathbf{X})$, the Kullback-Leibler divergence is non-negative and

$$\log L(\boldsymbol{\theta}; \mathbf{X}) \geq \mathbb{E}_{\mathbb{Q}}[\log L(\boldsymbol{\theta}, \mathbf{X}, \mathbf{Z})] + \mathcal{H}(\mathbb{Q}) = J(\mathbb{Q}, \boldsymbol{\theta}).$$

For the SBM,

$$J(\mathbb{Q}, \boldsymbol{\theta}) = \sum_{i,q} \tau_{iq} \log \alpha_q + \sum_{i < j, q, \ell} \tau_{iq} \tau_{j\ell} \log b(X_{ij}; \pi_{q\ell}) - \sum_{i,q} \tau_{iq} \log(\tau_{iq}),$$

\rightsquigarrow we optimize the loglikelihood lower bound $J(\mathbb{Q}, \boldsymbol{\theta}) = J(\boldsymbol{\tau}, \boldsymbol{\theta})$ in $(\boldsymbol{\tau}, \boldsymbol{\theta})$.

E and M steps for SBM

Variational E-step

Maximizing $J(\boldsymbol{\tau})$ for fixed $\boldsymbol{\theta}$, we find a fixed-point relationship:

$$\hat{\tau}_{iq} \propto \alpha_q \prod_j \prod_\ell b(X_{ij}, \pi_{q\ell})^{\hat{\tau}_{j\ell}} \quad (1)$$

M-step

Maximizing $J(\boldsymbol{\theta})$ for fixed $\boldsymbol{\tau}$, we find,

$$\hat{\alpha}_q = \frac{1}{n} \sum_i \hat{\tau}_{iq}, \quad \hat{\pi}_{q\ell} = \frac{\sum_{i \neq j} \hat{\tau}_{iq} \hat{\tau}_{j\ell} X_{ij}}{\sum_{i \neq j} \hat{\tau}_{iq} \hat{\tau}_{j\ell}}. \quad (2)$$

Model selection

We use our lower bound of the loglikelihood to compute an approximation of the ICL

$$\begin{aligned} \text{vICL}(Q) = \mathbb{E}_{\hat{\mathbb{Q}}}[\log L(\hat{\boldsymbol{\theta}}; \mathbf{X}, \mathbf{Z})] \\ - \frac{1}{2} \left(\frac{Q(Q+1)}{2} \log \frac{n(n-1)}{2} + (Q-1) \log(n) \right), \end{aligned}$$

where

$$\mathbb{E}_{\hat{\mathbb{Q}}}[\log L(\hat{\boldsymbol{\theta}}; \mathbf{X}, \mathbf{Z})] = J(\hat{\boldsymbol{\tau}}, \hat{\boldsymbol{\theta}}) - \mathcal{H}(\hat{\mathbb{Q}}).$$

The variational BIC is just

$$\text{vBIC}(Q) = J(\hat{\boldsymbol{\tau}}, \hat{\boldsymbol{\theta}}) - \frac{1}{2} \left(\frac{Q(Q+1)}{2} \log \frac{n(n-1)}{2} + (Q-1) \log(n) \right).$$

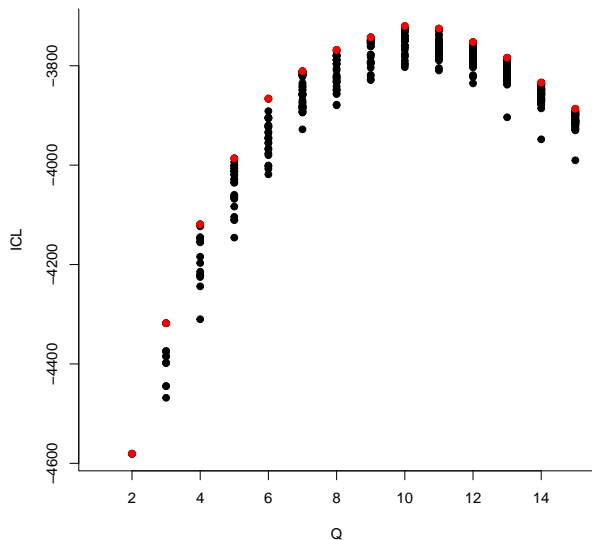
Example on the French blogosphere (I)

```
library(blockmodels)
library(sand)

adj_blog <- upgrade_graph(fbblog) %>%
  as_adjacency_matrix() %>%
  as.matrix()

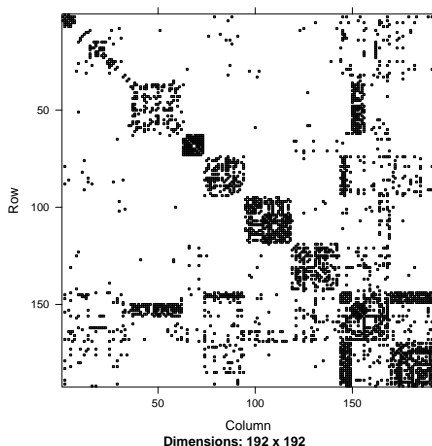
mySBM_collection <- BM_bernoulli(
  "SBM_sym",
  adj_blog, verbosity = 0,
  plotting = "figures/ICL_fbblog.pdf"
)
mySBM_collection$estimate()
```

Example on the French blogosphere (II)



Example on the French blogosphere (III)

```
library(Matrix)
clusters <-
  apply(mySBM_collection$memberships[[10]]$Z, 1, which.max)
image(Matrix(adj_blog[order(clusters), order(clusters)]))
```



Example on the French blogosphere (IV) I

```
library(RColorBrewer); pal <- brewer.pal(10, "Set3")

g <- graph_from_adjacency_matrix(
  adj_blog,
  mode = "undirected",
  weighted = TRUE,
  diag = FALSE
)
V(g)$class <- clusters
V(g)$size <- 5
V(g)$frame.color <- "white"
V(g)$color <- pal[V(g)$class]
V(g)$label <- ""
E(g)$arrow.mode <- 0

par(mar = c(0,0,0,0))
plot(g, edge.width=E(g)$weight)
```

Example on the French blogosphere (IV) II

