The Symmetry Perspective

Brain Inspired Computing

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Presentation Overview

1. Introduction
2. Group Theory
3. Equivariance
4. Symmetry Breaking
5. Conclusion

What is this project about?

- Introduce Group Theory as simply as possible
- Present Steady-States Bifurcations
- Bring mathematical tools to explore them

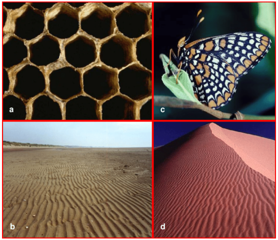
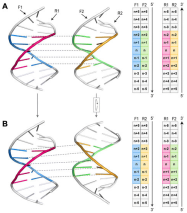


Figure: Symmetry patterns observed in the environment

For which applications?

A Two-step Approach

- Physics: In crystallography, the arrangement of atoms in a crystal can be described using symmetry groups.
- Molecular Structure: Symmetry helps determine the shapes of molecules, predict their vibrations, and understand their spectra.
- Biology : Genetics: Symmetry is used to study patterns in genetic sequences and understand evolutionary relationships.



What is a Group?

Definition

A **group** G is a set equipped with a binary operation $(\cdot)$ satisfying the following properties:

1. **Closure:** For all $a, b \in G$, $a \cdot b \in G$.
2. **Associativity:** For all $a, b, c \in G$, $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.
3. **Identity Element:** There exists an element $e \in G$ such that for all $a \in G$, $a \cdot e = e \cdot a = a$.
4. **Inverse Element:** For each $a \in G$, there exists an element $a^{-1} \in G$ such that $a \cdot a^{-1} = a^{-1} \cdot a = e$.

A Simple Group Example : General Linear Group

The set of invertible n x n matrices with real elements, called the general linear group, whose law of composition is matrix multiplication

1. **Closure:** For any two invertible matrices $A, B \in GL(n, \mathbb{R})$, their product AB is also invertible and belongs to $GL(n, \mathbb{R})$.
2. **Associativity:** For any matrices $A, B, C \in GL(n, \mathbb{R})$, $(AB)C = A(BC)$, as matrix multiplication is associative.
3. **Identity Element:** The identity matrix I_n serves as the identity element, as for any matrix $A \in GL(n, \mathbb{R})$, we have $AI_n = I_nA = A$.
4. **Inverse Element:** For any matrix $A \in GL(n, \mathbb{R})$, its inverse A^{-1} exists in $GL(n, \mathbb{R})$ such that $AA^{-1} = A^{-1}A = I_n$.

The Symmetric Group

- **Definition:** The **symmetric group**, denoted by $\mathbf{S_n}$, is the group of all permutations of n distinct elements.
- **Order:** The order of S_n is $n!$, where $n!$ represents the factorial of n.
- **Example:** One has $\mathbf{S_3} = \mathbf{D_3}$ where $\mathbf{D_n}$ is the dihedral group consisting in all rotations and reflections in plane preserving a n-gone (Check on equilateral triangle for $n = 3$).
- **Properties:**
  - It is a finite group with $n!$ elements.
  - Symmetric groups play a fundamental role in permutation group theory and combinatorics.

Subgroups

Definition

Let G be a group. A subset H of G is called a **subgroup** of G if it satisfies the following conditions:

1. **Closure:** For all a, b in H, $a \cdot b$ is also in H.
2. **Identity Element:** The identity element of G is in H.
3. **Inverse Element:** For every a in H, its inverse a^{-1} is also in H.

As example, think of the upper triangular matrix group of dimension n $T(n)$ as a subgroup of the general linear group $GL(n)$.

Group Isomorphism

Definition

Let G and H be groups. A function $f : G \to H$ is called an **isomorphism** if it satisfies the following conditions:

1. **Injective (One-to-One):** For every pair of distinct elements x, y in G, if $f(x) = f(y)$, then $x = y$.
2. **Surjective (Onto):** For every element y in H, there exists an element x in G such that $f(x) = y$.
3. **Preserves Group Operation:** For all x, y in G, $f(x \cdot y) = f(x) \cdot f(y)$.

For example, the subgroup P of permutation matrices is such that $P \in GL(n, \mathbf{R})$. P is isomorphic with S_n.

Group Homomorphism

Definition

Let G and H be groups. A function $f : G \to H$ is called a **homomorphism** if it satisfies the following property:
For all x, y in G, $f(x \cdot y) = f(x) \cdot f(y)$, where $\cdot$ represents the group operation in G and $\cdot$ represents the group operation in H.

So it's a function that preserves the group operations within two groups. The determinant of an element of $GL(n, \mathbf{R})$ is an homomorphism using the mapping determinant: $GL_n(\mathbf{R}) \longrightarrow \mathbf{R}$.

Group Product

Definition

Let $G_1 = (S_1, \cdot_1, e_1)$ and $G_2 = (S_2, \cdot_2, e_2)$ be two groups. Then the product of groups G_1 and G_2 is defined as:

$$G_1 \times G_2 = (S_1 \times S_2, \cdot, (e_1, e_2))$$

where (e_1, e_2) are the identity element of both groups. $\cdot$ is the component-wise group operation defined as:

$$(a_1, a_2) \cdot (b_1, b_2) = (a_1 \cdot_1 b_1, a_2 \cdot_2 b_2)$$

for all $(a_1, a_2), (b_1, b_2) \in S_1 \times S_2$(cartesian product), where $a_1, b_1 \in S_1$ and $a_2, b_2 \in S_2$

Group Action

Definition

Let G be a group and X be a set. A *group action* of G on X is a map $\cdot : G \times X \to X$ such that for all $g, h \in G$ and $x \in X$, the following properties hold:

1. **Identity Preservation:** The identity element e of G leaves every element in X unchanged: $e \cdot x = x$ for all $x \in X$.
2. **Compatibility:** The action is compatible with the group operation * in G: $(g * h) \cdot x = g \cdot (h \cdot x)$ for all $g, h \in G$ and $x \in X$.

Intuitively, it is how a group **acts** on a set.

Equivariant Mappings

Goal

The goal here is to explain what it means for a system of Ordinary Differential Equations (ODEs) to be symmetric under a group of action.

System of ODEs

Let us consider a system of ODEs:

$$\dot{x} = f(x, \lambda)$$

where λ is the bifurcation parameter, with $f : \mathbf{R}^n \times \mathbf{R}^r \to \mathbf{R}^n$.

Equivariant Mapping

Symmetry under a Group Action

Statement

The element $\gamma \in \mathbf{O}(n)$ is a symmetry of $f(x, \lambda)$ if for any solution $x(t)$, $\gamma x(t)$ is also a solution.

Equation Transformation

Let $y(t) = \gamma x(t)$, then we have:

$$\dot{y}(t) = f(y(t)) = f(\gamma x(t))$$

and

$$\dot{y}(t) = \gamma \dot{x}(t) = \gamma f(x(t))$$

Equivariant Mapping

Symmetry under a Group Action

Condition

The condition $f(\gamma x) = \gamma f(x)$ indicates that f is γ-equivariant.

Generalization

However, we are interested in group equivariance rather than element equivariance. This principle can be generalized:
Let Γ act on $\mathbf{R}^n$. f is γ-equivariant if $f(\gamma x, \lambda) = \gamma f(x, \lambda)$ for all $\gamma \in \Gamma$ and $x \in \mathbf{R}^n$.

Equivariance : First Example

Consider the following example $\begin{cases} \dot{x} &= (\lambda x + x^2) - y - z \\ \dot{y} &= (\lambda y + y^2) - x - z \\ \dot{z} &= (\lambda z + z^2) - x - y \end{cases}$

If $\gamma \in \mathbf{S_3}$, γx_i means : $\begin{cases} x \longrightarrow y \\ y \longrightarrow z \\ z \longrightarrow x \end{cases}$ while $\gamma f(x)$ means to swap the equations. So it is perfectly equivalent.

Equivariance : Second Example

Consider a neuron network consisting of 5 neurons inhibiting their respective neighbors.

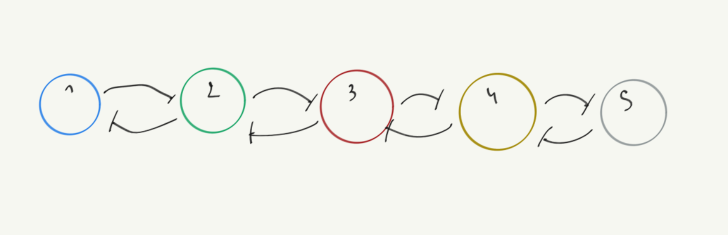


Figure: Model to consider

Equivariance : Second Example

For $\sigma =< (24) >$, $f(\sigma x) \neq \sigma f(x)$.
For $\theta =< (15),(24) >$,one has that $\theta * \theta = i$ where i is the identity element of $\mathbf{S_5}$, $\theta^{-1} = \theta$ so it is straightforward that the system if $\mathbf{C_2} - equivariant$.
How to show it with the isotropy group of permutation matrices?
The state matrix is given by A=

$$\begin{pmatrix} 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 \end{pmatrix}$$

Equivariance : Second Example

So we have

- $(x_1)^T = [0, -1, 0, 0, 0]$
- $(x_2)^T = [-1, 0, -1, 0, 0]$
- $(x_3)^T = [0, -1, 0, -1, 0]$
- $(x_4)^T = [0, 0, -1, 0, -1]$
- $(x_5)^T = [0, 0, 0, -1, 0]$

and

- $(\Sigma * x_1)^T = [0, 0, 0, -1, 0]$
- $(\Sigma * x_2)^T = [-1, 0, -1, 0, 0]$
- $(\Sigma * x_3)^T = [0, -1, 0, -1, 0]$
- $(\Sigma * x_4)^T = [0, 0, -1, 0, -1]$
- $(\Sigma * x_5)^T = [0, -1, 0, 0, 0]$

Equivariance : Second Example

So $A(\Sigma * x)$: $\begin{pmatrix} 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & -1 \\ 0 & -1 & 0 & -1 & 0 \\ -1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \end{pmatrix}$

while $\Sigma * A(x) = \begin{pmatrix} 0 & 0 & 0 & -1 & 0 \\ -1 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & -1 \\ 0 & -1 & 0 & 0 & 0 \end{pmatrix}$

prooving that $A(\Sigma x) \neq \Sigma A(x)$. One can perform the same test for Θ and will that this time it is equal.

Isotropy Subgroups

Generalization

The isotropy group of x writes

$$\Sigma_x = \{g \in G : gx = x\}$$

Less formally, it's the subset of G that leaves x unchanged.

Symmetry Breaking

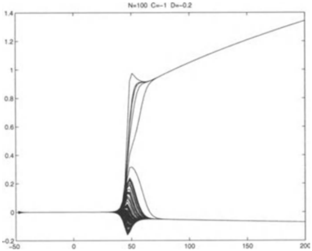


Figure: Simulation for N=100

Symmetry breaking happens when solutions have less symmetry than equations. In other words, if f is $\Gamma -$ *equivariant*, it happens when the isotropy subgroup is a subgroup of Γ (not the whole group).

WTA example

Let us create a WTA model consisting of 5 neurons, that will bifurcate for some value of the bifurcation parameter

$$\dot{x}_i = \frac{1}{\tau}(-x_i + S(K - \sum_{j=1}^{5} x_j))$$

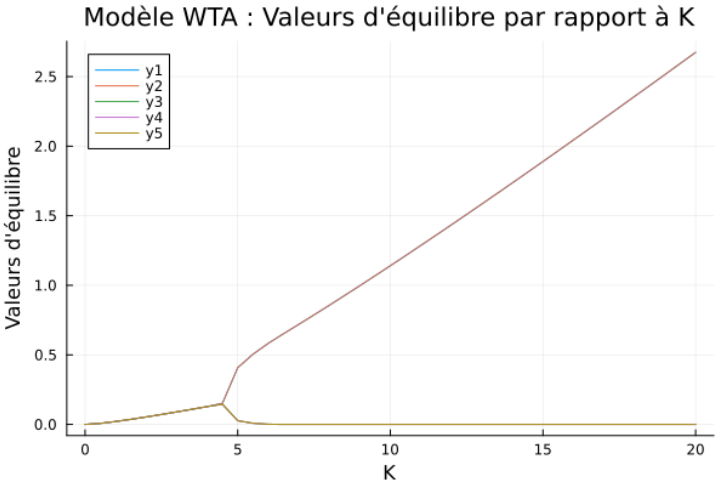
with $i = 1, ..., 5$ and $j = 1, ..., 5 \neq i$ with

$$S(x) = \frac{100 * x^2}{(40^2 + x^2)}$$

if $x > 0$, 0 otherwise.
This system is $\mathbf{S_5}$ equivariant.

WTA example : Parameter ramping

We simulate the model by increasing at each step the bifurcation.



WTA simu

- Before symmetry breaking: $\Sigma_x = \mathbf{S_5}$
- After symmetry breaking: $\Sigma_x = \mathbf{S_1} \times \mathbf{S_4}$

However, symmetry is kept "on average":

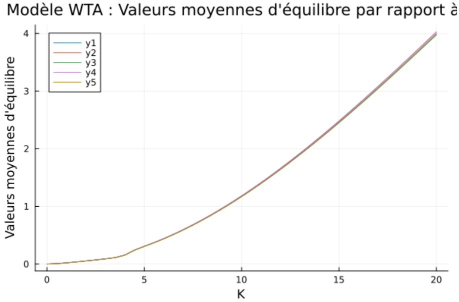


Figure: Winner take all parameter ramping average for 100000 simulations

Invariant Subspaces

Definition

An invariant subspace of a dynamical system is a subspace that remains unchanged under the action of that transformation or system.

If $\mathbf{S_N}$ acts by permutating coordinates on $\mathbf{R_n}$:

$$V : \{(x, x, ..., x) : x \in \mathbf{R}\}$$

is $\mathbf{S_N}$ $-$ *invariant* because permutating the coordinates won't affect the space properties of V.

Steady-States Bifurcations : Linearization

Denoting by dF the derivative of F and M the linear map at the bifurcation point:

$$A_0 = (dF)_{y(\lambda_0),0}$$

If A_0 is singular, then G acts irreducibly on the state space
Therefore:

- 0 is the only eigenvalue on the imaginary axis
- The generalized eigenspace corresponding to 0 is $ker(A_0)$
- Γ acts irreducibly on $ker(A_0)$

Steady-States Bifurcations

Let us consider the system:

$$\dot{x}_i = \lambda x_i - (x_1 + \ldots + x_n) + x_i^2 - x_i^3$$

for dimension 10, let us simulate to get the eigenvalues at the bifurcation

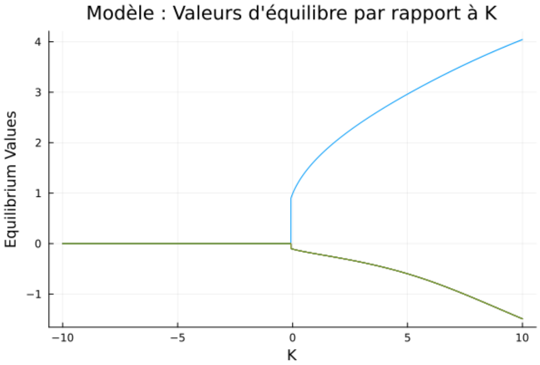


Figure: Equilibrium Values of the system

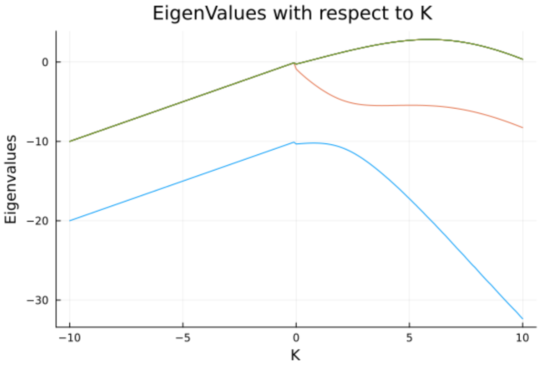


Figure: Eigenvalues Values of the system

Now we prooved that the only eigenvalue on the imaginary axis at the bifurcating point is 0, one must show that S_N acts irreducibly on $ker(A_0)$. Note that there is a single dominant eigenvalue, indicating a pitchfork bifurcation at this point.

Liapunov- Schmidt reduction

We apply it to the system of dimension 3 for simplicity.

1. Linearize the system around the equilibrium (here **0**)
2. Evaluate the Jacobian matrix at the equilibrium:
$$\begin{pmatrix} \lambda - 1 & -1 & -1 \\ -1 & \lambda - 1 & -1 \\ -1 & -1 & \lambda - 1 \end{pmatrix}$$
3. Evaluate the range and the kernel, for $\lambda \longrightarrow 3$, one can find a non trivial solution.
4. For that case, the kernel is spanned by $(1, 1, 1)$ (1 dimensionnal case). We now decompose the space in $\mathbf{x} = c\mathbf{k} + \mathbf{o}$.

LPS

1. Project: Close to bifurcation ,the high dimensionnal dynamics is dominated by the 1-dimensionnal dynamics. The critical component (projected on the kernel) is:

$$\dot{c} = (\lambda c) - 3c + c^2 - c^3$$

The non critical is given by the projection of the jacobian on the range plus higher order terms:

$$\dot{o} = J\mathbf{o} + H$$

2. Analyze the results

Equivariant Branching Lemma

Assuming it is a G-equivariant system, that the jacobian evaluated at (0,0) has at least one zero eigenvalue and that the others are real, consider a subgroup Σ such that the dimension of the invariable subspace is 1.
The lemma states that there exists a unique equilibrium solution branch bifurcating from $x = 0$ such that the isotropy subgroup of solutions on the branch is Σ.
Remember in the last example, the kernel was spanned by $(1, 1, 1)$, which is invariant under the action of $\mathbf{S_3}$, so there must be a pitchfork bifurcation for this value of parameter.

EBL: Bifurcation Diagram

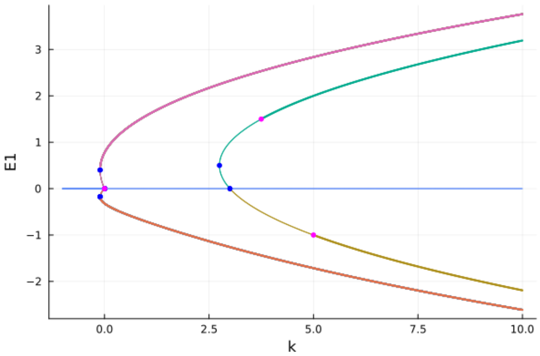


Figure: Bifurcation Diagram of the system, works because unique dominant eigenvalue

Conclusion

Brivanlou:*The more we're looking at this, the more I appreciate that life is made of continuous symmetry breaking*

Thanks for your attention