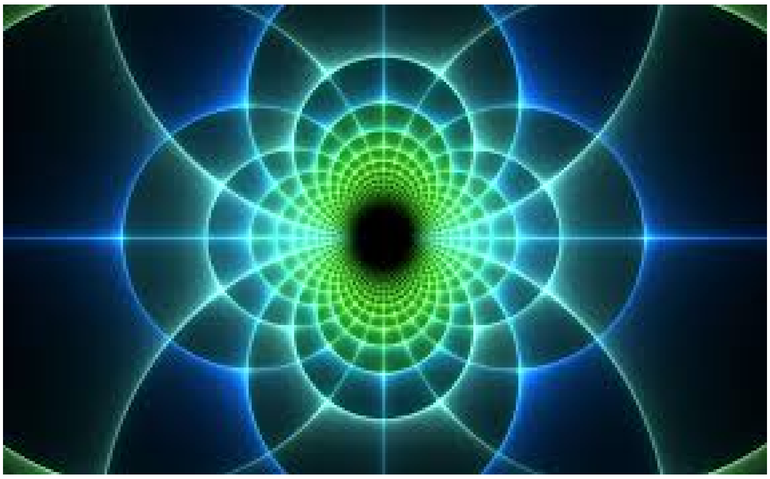
Liège University

Brain Inspired Computing

The Symmetry Perspective



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1 Introduction

In the last years, lots of research have been conducted about symmetry, especially in the context of dynamical systems. Indeed, in many cases, it appeared evident that symmetry of ordinary differential equations allowed to analyze pattern formation, especially in the biology domain. Actually some behaviours can be deduced only from the symmetry of the system. In a sense, all models with a given symmetry explore the same range of pattern-forming behaviors. The details of the system are still relevant to know which model the system will pick from this set of pattern behaviours. That range of behaviors can be studied in its own right without reference to many details of the model. However, this two-stage approach is not always the most relevant as for lower complexity system, simple calculations might be much more efficient.

The framework of nonlinear dynamical systems predominantly engages with trajectories within an abstract phase space : a domain of variables dictating the system's state. However, the genesis of this theory lies in qualitative inquiries, such as the existence of periodic or chaotic solutions, rather than quantitative inquiries. Consequently, theoretical endeavors often entail unspecified transformations of coordinates within phase space, potentially leading to a disconnect between the theory's variables and those observed empirically.

This disparity becomes particularly conspicuous when dealing with systems described by partial differential equations (PDEs). Consider, for instance, a complex fluid flow pattern in physical space, which may be condensed into a single point within phase space. The elucidation of this point's significance, such as its representation by Fourier coefficients, becomes imperative in such contexts.

In this context, symmetric dynamical systems offer a notable advantage. In many cases (though not universally), the system's symmetry possesses well-defined physical and phase space interpretations. For instance, if an experimental setup exhibits circular symmetry, a group of rotations operates both in physical space and phase space. While these actions may not be identical, the elements undergoing transformation remain consistent.

Consequently, a symmetry property of a solution in phase space typically corresponds to a symmetry property in physical space, and vice versa. For instance, discerning that a periodic solution manifests as a rotating or standing wave stems from its spatial and temporal symmetries. Subsequently, model-specific details elucidate the precise characteristics of these waves. Thus, symmetries serve as a crucial conduit bridging the abstract realm of theory with empirical observations.

In this project, we will investigate the effect of symmetry on dynamical systems rather than static ones.

Here is a brief summary of the different aspects this project will cover :

- This project commences with a brief reminder on group theory. This theoretical framework provides the necessary groundwork for comprehending the symmetries inherent in ODE systems and their implications for equilibrium states.
- Following the elucidation of group theory, the concept of symmetry breaking is introduced. This entails defining the criteria for an ODE system to exhibit symmetry and exploring how solutions within such systems can exhibit symmetrical properties.
- The theoretical discussions are augmented with practical examples, facilitating a deeper understanding of the concepts elucidated this far. Real-world applications and scenarios are used to illustrate the theoretical blocks presented earlier
- As the project progresses, attention shifts towards introducing two indispensable tools for symmetry breaking within ODE systems.
  1. Liapunov-Schmidt Reduction : This tool serves to streamline bifurcation problems

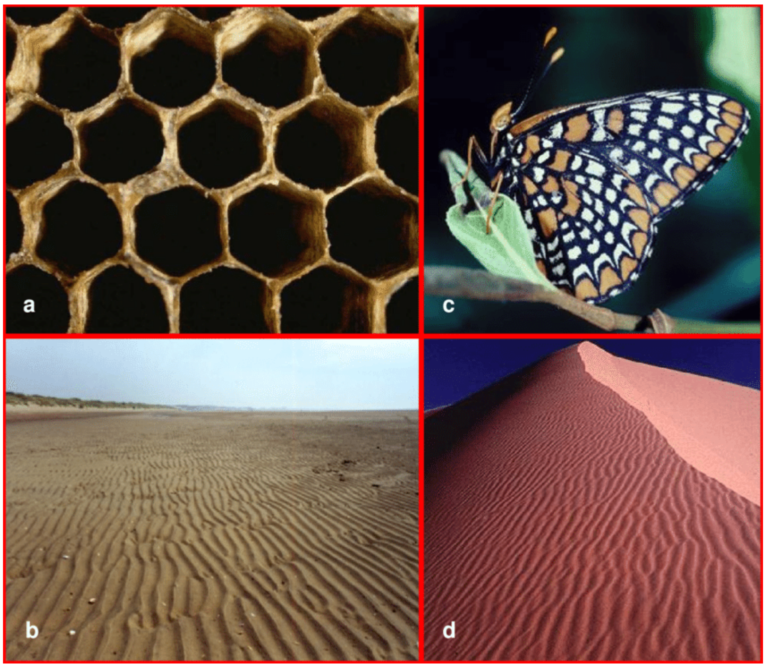

Figure 1 – Some examples of symmetry in the wildness

by reducing them to manageable dimensions, transitioning from infinite-dimensional problems to finite ones.

2. Equivariant Branching Lemma : This lemma furnishes the necessary conditions for the existence of bifurcation branches within specific types of symmetry groups, thereby facilitating a deeper understanding of symmetry-breaking phenomena.

2 Group Theory

2.1 Definition of a group

Group Definition

A **group** G is a set equipped with a law of composition $(\cdot)$ satisfying the following properties :

1. **Closure :** For all $a, b \in G$, $a \cdot b \in G$.
2. **Associativity :** For all $a, b, c \in G$, $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.
3. **Identity Element :** There exists an element $e \in G$ such that for all $a \in G$, $a \cdot e = e \cdot a = a$.
4. **Inverse Element :** For each $a \in G$, there exists an element $a^{-1} \in G$ such that $a \cdot a^{-1} = a^{-1} \cdot a = e$.

1. Let $G = \mathbb{Z}$, the set of integers, and define the operation $(\cdot)$ as integer addition. Then $(\mathbb{Z}, +)$ forms a group.
   (a) **Closure :** For any two integers $a, b \in \mathbb{Z}$, $a + b \in \mathbb{Z}$.
   (b) **Associativity :** For any integers $a, b, c \in \mathbb{Z}$, $(a + b) + c = a + (b + c)$.
   (c) **Identity Element :** The identity element is 0, as for any integer a, $a+0 = 0+a = a$.
   (d) **Inverse Element :** For any integer a, the inverse element is $-a$, as $a + (-a) = (-a) + a = 0$.
2. The set of invertible n x n matrices with real elements, called the general linear group, whose law of composition is matrix multiplication
   (a) **Closure :** For any two invertible matrices $A, B \in GL(n, \mathbb{R})$, their product AB is also invertible and belongs to $GL(n, \mathbb{R})$.
   (b) **Associativity :** For any matrices $A, B, C \in GL(n, \mathbb{R})$, $(AB)C = A(BC)$, as matrix multiplication is associative.
   (c) **Identity Element :** The identity matrix I_n serves as the identity element, as for any matrix $A \in GL(n, \mathbb{R})$, we have $AI_n = I_nA = A$.
   (d) **Inverse Element :** For any matrix $A \in GL(n, \mathbb{R})$, its inverse A^{-1} exists in $GL(n, \mathbb{R})$ such that $AA^{-1} = A^{-1}A = I_n$.

   In other words, the general linear group is the subgroup of matrices whose determinant is not null.

By a law of composition, we mean a binary operation for combining pairs a and b of G to get another element $a.b$ of G.

More formally, a law of composition is a function of two variables in G with values in G. In other words, it is a map :

$$G \times G \longrightarrow G$$

where $G \times G$ denotes all the pairs of elements (a,b) in G.

One can show a practical example of the implications of the law of composition on the general linear group. Let us have $a, b \in GL(2, \mathbb{R})$ such that

$$\text{a} = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}$$

$$\text{b} = \begin{pmatrix} 2 & 1 \\ 3 & 0 \end{pmatrix}$$

These matrices indeed belong to the general linear group since their determinant is not null. Closure is verified since

c=a.b $= \begin{pmatrix} 8 & 1 \\ 9 & 0 \end{pmatrix}$

Closure is respected since $c \in GL(2, \mathbb{R})$ because its determinant is not 0. Note that the element a, b can not be recovered once the composition law has been applied. One source of confusion might arise since the law of composition is associative but not commutative since $a.b \neq b * a$. This why when talking of group operation, we usually make use of the multiplication since it doesn't imply commutativity. The groups whose law of composition is commutative form a subgroup of G called the abelian groups.

Let us show with another example that composition of function is not commutative. Consider D_3, the dihedral group containing all the possible rotations and reflections of a set of 3 elements (or preserving a triangle in plane). Consider the 3 vertices of an triangle denoted as $T = a, b, c$. The 6 maps are defined as :

— The rotations :
  1. α_1 A clockwise rotation of 120° such that $T' = \alpha_1(T) = c, a, b$
  2. α_2 A clockwise rotation of 240°, such that $T' = \alpha_2(T) = b, c, a$
  3. α_3 A clockwise rotation of 360°, this is equivalent to the identity element of the group, such that $T' = \alpha_3(T) = a, b, c$

— The reflections :
  1. β_1 that permutes all the vertices but not the first one, such that $T' = \beta_1(T) = a, c, b$
  2. β_2, that permutes all the vertices but not the second one, such that $T' = \beta_2(T) = c, b, a$
  3. β_3, that permutes all the vertices but not the third one, such that $T' = \beta_3(T) = b, a, c$

One can easily see the law of composition is not commutative since $\alpha_1(\beta_1(T)) = \alpha_1(a, c, b) = b, a, c$ and $\beta_1(\alpha_1(T)) = \beta_1(c, a, b) = c, b, a$

2.2 The Symmetric Group

Bijectivity

A map

$$f : T \to T$$

is bijective if it satisfies the following conditions :

1. **Injective (One-to-One) :** For every pair of distinct elements x, y in the domain T, if $f(x) = f(y)$, then $x = y$.
2. **Surjective (Onto) :** For every element y in the codomain T, there exists an element x in the domain T such that $f(x) = y$.

In simpler terms, a bijective map from T to itself establishes a one-to-one correspondence between the elements of the set T. Then such a map is a permutation of T. The group of permutations of the n elements is called symmetric group, denoted $\mathbf{S_n}$.

There are $n!$ permutations possible for n elements, so the order of that group is $n!$. The group law is the composition of functions.

The trivial group $\mathbf{S_2} = i, \tau$ is made of two permutations. i is the identity and τ is the transpose. This group is then of little interest. Notice that this group is abelian since the composition of function this specific group is commutative.

Let us interest ourselves to $\mathbf{S_3}$. Indeed, this group has the particularity to be the symmetric group with the smallest order whose law of composition is not commutative. The group elements are the same than the one we used to illustrate the fact that the composition is not always symmetric, indeed we prove that if $a, b \in \mathbf{S_3}$ then $a.b(T) \neq b.a(T)$. We also showed that $\mathbf{S_3} = \mathbf{D_3}$ which is not the case for larger n. Indeed, one can easily show that $\mathbf{S_4} \neq \mathbf{D_4}$ because the order of $\mathbf{S_4}$ is 24, but those 24 permutations don't maintain the geometry of a square in the plane.

To describe this group, we can use permutation matrix, indeed consider :

$$\alpha_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\alpha_2 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

One has $\alpha_2 = \alpha_1 * \alpha_1$, so the law of composition is the product of the permutation matrice, which is commutative in this particular case, but for example this is not true for $\alpha_1 * \beta_1 = \beta_1 * \alpha_1$.

2.3 Subgroups

Subgroup Definition

Let G be a group. A subset H of G is called a **subgroup** of G if it satisfies the following conditions :

1. **Closure :** For all a, b in H, $a \cdot b$ is also in H.
2. **Identity Element :** The identity element of G is in H.
3. **Inverse Element :** For every a in H, its inverse a^{-1} is also in H.

Notice that the first law just points out that the law of composition of the group G is also a law of composition in the sens of H. Associativity therefore automatically transfer from G to H. Please note that every group G has at least two subgroups : The whole group and the group consisting of the identity element of G.

As an example of subgroup, one can mention the group $T \in GL(n, \mathbf{R})$ which is the group of the upper (the lower is also valid) triangular matrices, which is indeed a subgroup of the general linear subgroup G.

Let us focus once more on the symmetric group S_3, we can consider the subgroup that leaves the first element unchanged. We note this subgroup $H \in S_3$ where $H =< \alpha_3, \beta_1 >$ (see previous sections to see the corresponding actions of the group elements).

2.4 Isomorphism

Isomorphism Definition

Let G and H be groups. A function $f : G \to H$ is called an **isomorphism** if it satisfies the following conditions :

1. **Injective (One-to-One) :** For every pair of distinct elements x, y in G, if $f(x) = f(y)$, then $x = y$.
2. **Surjective (Onto) :** For every element y in H, there exists an element x in G such that $f(x) = y$.
3. **Preserves Group Operation :** For all x, y in G, $f(x \cdot y) = f(x) \cdot f(y)$.

If there exists an isomorphism between two groups G and H, we say that G is **isomorphic** to H, denoted as $G \cong H$. Note that this definition is equivalent to stating that f is bijective and that f is compatible with the laws of composition.

More intuitively, if $a, b \in G$ and $a', b' \in G'$ than ab in G corresponds to $a'b'$ in G'. So the properties of G and G' must be preserved.

For example, the subgroup P of permutation matrices is such that $P \in GL(n, \mathbf{R})$. P is isomorphic with S_n. Let us note the element of S_n σ and $f(\sigma)$ the corresponding permutation matrix. Indeed, let's check from the definition

1. Injective : If $f(\sigma_1) = f(\sigma_2)$ then $\sigma_1 = \sigma_2$
2. Surjective : For any permutation matrix M in the group of permutation matrix, there exists a permutation σ such that $f(\sigma) = M$.
3. Preserving Group operation : For any $\sigma_1, \sigma_2 \in S_n$, $f(\sigma_1 * \sigma_2) = f(\sigma_1).f(\sigma_2)$ where * denotes composition of permutation while . denotes permutation matrices multiplications.

Since these groups have the same properties, it is convenient to blur the distinction between them.

Let us now introduce the notion of automorphism. Automorphism is an isomorphism from a group to itself.

Consider the permutation group S_3, which consists of all permutations of three elements : $\{1, 2, 3\}$.

Let σ be a permutation in S_3. We can define an automorphism f of S_3 as follows :

1. Define $f(\sigma)$ as the permutation that maps i to $\sigma(i)$ for $i = 1, 2, 3$.

   For example, if $\sigma = (1\ 2\ 3)$, then $f(\sigma) = (1\ 2\ 3)$, because $f(\sigma)$ is just the permutation σ itself(it is the identity transformation).
2. Extend f to all of S_3 by defining f as a bijection from S_3 to itself.

This function f is an automorphism of S_3 because it satisfies the properties of a bijective homomorphism :

— **Bijective :** f is bijective because it is a bijection from S_3 to itself.
— **Homomorphism :** f preserves the group operation of permutations, meaning that for any two permutations σ_1 and σ_2, $f(\sigma_1 \circ \sigma_2) = f(\sigma_1) \circ f(\sigma_2)$, where $\circ$ represents composition of permutations.

In this case, f is the identity automorphism of S_3, which maps each permutation to itself.

This example illustrates the concept of an automorphism of the permutation group S_3. It is worth noting that there are other automorphisms of S_3, but the identity automorphism is the simplest example. The identity automorphism exists in any group, but there can sometimes be less trivial ones.

The point of introducing automorphism is to introduce conjugation. Let $a \in G$, the conjugation by b is the map defined as :

$$\phi(x) = bxb^{-1}$$

So if $a \in G$, the conjugate of a by b is $a' = bab^{-1}$

2.5 Homomorphisms

Homomorphism Definition

Let G and H be groups. A function $f : G \to H$ is called a **homomorphism** if it satisfies the following property :
For all x, y in G, $f(x \cdot y) = f(x) \cdot f(y)$, where $\cdot$ represents the group operation in G and $\cdot$ represents the group operation in H.

In simpler terms, a homomorphism is a function that preserves the group operation between two groups. Its definition is really close from isomorphisms however, it is not bijective. Let us illustrate homomorphisms with simple examples.

The determinant of an element of $GL(n, \mathbf{R})$ is an homomorphism using the mapping determinant : $GL_n(\mathbf{R}) \longrightarrow \mathbf{R}$. Indeed, it maps a matrix from the general linear group to a real number. Notice that it is indeed an isomorphism because the law of composition of $GL(n, \mathbf{R})$ is multiplication as well as in the real group. So one has if $A, B \in GL(n, \mathbf{R})$ than $det(A.B) = det(A) * det(B)$ where "." denotes matrix multiplication. However, the relation is not bijective since the matrix can be retrieved solely based on its determinant, so it's not an isomorphism.

Now we can focus on two homomorphisms we will need later.

1. The image : let us consider an homomorphism h :$G \longrightarrow G'$:

$$im_h = \{b \in G' | b = h(a), a \in G\}$$

   Thus this is a subgroup of G' because the relation is not bijective, otherwise it would be an isomorphism.
2. The kernel of h, it is the set of elements of G that are mapped to identity in G' :

$$kerh = \{a \in G | h(a) = 1\}$$

   This is also a subgroup of G. This definition is consistent with the concept of the kernel in linear algebra. In the context of group homomorphisms, the kernel represents the "dead space" of the homomorphism, the elements in the domain group G that get mapped to the identity element in the codomain group G'.

We can now introduce the special linear group, a subgroup of the general linear group which forms the kernel of the determinant homomorphism.

$$SL_n(\mathbf{R} = \{M(\mathbf{R}_{n*n} | det(M) = 1\}$$

2.6 Group Product

The notion of group product will be evoked many times in this project, for the sake of clarity, let us clearly define what it is.

Let us have two groups G and G'. The product $G \times G'$ forms another group.

Let G_1 and G_2 be two groups. The product of groups G_1 and G_2, denoted $G_1 \times G_2$, is defined as the Cartesian product of the underlying sets of G_1 and G_2 (S_1 and S_2), equipped with a group operation defined component-wise.

Formally, let $G_1 = (S_1, \cdot_1, e_1)$ and $G_2 = (S_2, \cdot_2, e_2)$ be two groups. Then the product of groups G_1 and G_2 is defined as :

$[G_1 \times G_2 = (S_1 \times S_2, \cdot, (e_1, e_2))]$

where (e_1, e_2) are the identity element of both groups. $\cdot$ is the component-wise group operation defined as :

$[(a_1, a_2) \cdot (b_1, b_2) = (a_1 \cdot_1 b_1, a_2 \cdot_2 b_2)]$

for all $(a_1, a_2), (b_1, b_2) \in S_1 \times S_2$, where $a_1, b_1 \in S_1$ and $a_2, b_2 \in S_2$

As a reminder, the Cartesian product of two sets A and B, denoted $A \times B$, is the set of all possible ordered pairs (a, b) where $a \in A$ and $b \in B$.

For example, if $A = \{1, 2\}$ and $B = \{a, b, c\}$, then their Cartesian product $A \times B$ is :

$[A \times B = \{(1, a), (1, b), (1, c), (2, a), (2, b), (2, c)\}]$

This means that every element of set A is paired with every element of set B.

The order of the group $G \times G'$ is the product of both groups'order.

As an example, the group $S_3 \times S_4$ represents the direct product of two symmetric groups :

1. S_3 : The symmetric group of degree 3, consisting of all possible permutations of 3 elements, with $3! = 6$ elements.

2. S_4 : The symmetric group of degree 4, consisting of all possible permutations of 4 elements, with $4! = 24$ elements.

The direct product $S_3 \times S_4$ creates a new group whose elements are ordered pairs, where the first component comes from S_3 and the second component comes from S_4. The group operation in $S_3 \times S_4$ is defined component-wise, meaning that the group operation is performed separately on each component.

For example, if σ is a permutation in S_3 and τ is a permutation in S_4, then the product $(\sigma, \tau) \cdot (\sigma', \tau')$ is defined as $(\sigma \cdot \sigma', \tau \cdot \tau')$, where $\sigma \cdot \sigma'$ and $\tau \cdot \tau'$ are the products in S_3 and S_4 respectively.

The resulting group $S_3 \times S_4$ contains all possible combinations of permutations of 3 and 4 elements, with a total of $6 \times 24 = 144$ elements.

3 Steady-States Bifurcation

We will start our trip through symmetry equations with the simplest dynamical systems of equations : The steady-states ones, where there are no dynamics. Indeed, the simplest states of a dynamical system are steady-states where $x(t) = x_0 \forall t$

3.1 Symmetries of differential equations

The symmetries of a system of ODEs makes use of the notion of group transormation on the set of variables that preserves the structure of the equation.

3.1.1 Group Actions

We want to define how a group *acts* on a set, ie consider the group as a set of transformations in space.

Group Action Definition

Let G be a group and X be a set. A *group action* of G on X is a map $\cdot : G \times X \to X$ such that for all $g, h \in G$ and $x \in X$, the following properties hold :

1. **Identity Preservation :** The identity element e of G leaves every element in X unchanged : $e \cdot x = x$ for all $x \in X$.
2. **Compatibility :** The action is compatible with the group operation $*$ in G : $(g * h) \cdot x = g \cdot (h \cdot x)$ for all $g, h \in G$ and $x \in X$.

A group action can also be denoted by $(g, x) \mapsto gx$ or simply gx. Intuitively, it's a way for a group to act on a set. It can be seen as an homomorphism.

As an example, let us consider the permutation group S_3 and the set of 3 elements X. As a reminder, S_3 posesses 6 elements : 3 rotations including identity, as well as 3 reflections.

The identity preservation is fullfilled because applying the identity of S_3 to X leaves X unchanged. The compatibility property is respected since the group operation for S_3 is composition of function, so if for example α_1 is a 1 element rotation of all the elements in clockwise direction and α_2 of 2 elements. One has

$$(\alpha_1 * \alpha_1).x = \alpha_2.x = \alpha_1.(\alpha_1.x)$$

$\forall x \in X$. Indeed, making two times the 1 element rotation is equivalent to a single two element rotation. It can be more intuitive to think of S_3 has the group of permutation matrices with $n = 3$ and realize that this group acts on X through matrix multiplication.

3.1.2 Equivarient Mappings

We now want to define what does it mean for a system of ODEs to be symmetric under a group action. Let us consider the system :

$$\dot{x} = f(x, \lambda)$$

with $f : \mathbf{R^n} \times \mathbf{R^r} : \mathbf{R^n}$ At the moment λ is of little interest since steady-state is assumed.

The group element $g \in G$ is a symmetry of the system if for any solution $x(t)$ of the system, the action of the group element g on $x(t)$, ie. gx is also a solution of the system. Let us proove a sufficiant condition for g to be a symmetry.

$$\dot{y}(t) = f(y(t)) = f(gx(t))$$

and

$$\dot{y}(t) = g\dot{x}(t) = gf(x(t))$$

so one has $f(gx(t)) = gf(x(t))$ so f is $g - equivariant$. If this is valid for $\forall g \in G$, then one can say that f is G-equivarient.

Let us have the system of 3 equations :

$$\begin{cases} \dot{x} &= (\lambda x + x^2) - y - z \\ \dot{y} &= (\lambda y + y^2) - x - z \\ \dot{z} &= (\lambda z + z^2) - x - y \end{cases}$$

Let us check that this system is $\mathbf{S_3}$ equivarient. if $\gamma \in \mathbf{S_3}$, one should have $\gamma f(x) = f(\gamma x)$. For $f(\gamma x)$, the system becomes :

$$\begin{cases} \dot{\gamma x} &= (\lambda \gamma x + \gamma x^2) - \gamma y - \gamma z \\ \dot{\gamma y} &= (\lambda \gamma y + \gamma y^2) - \gamma x - \gamma z \\ \dot{\gamma z} &= (\lambda \gamma z + \gamma z^2) - \gamma x - y \end{cases}$$

γ acts by permuting the coordinates, so γx_i represents the rearranged variables under the permutation. Now if we permute the system using $\gamma f(x)$, the system becomes :

$$\begin{cases} \dot{\gamma x} &= \gamma((\lambda x + x^2) - y - z) \\ \dot{\gamma y} &= \gamma((\lambda y + y^2) - x - z) \\ \dot{\gamma z} &= \gamma((\lambda z + z^2) - x - y) \end{cases}$$

which is the same using the compatibility rule of a group action. To make it more intuitive, let us say that $\gamma : \mathbf{R} \longrightarrow \mathbf{R}$ is a clockwise permutation so that its action on $X = \{x, y, z\}$ is

— $x \longrightarrow y$
— $y \longrightarrow z$
— $z \longrightarrow x$

$f(\gamma x)$ swaps all the coordinates so the system becomes :

$$\begin{cases} \dot{y} &= (\lambda y + y^2) - z - x \\ \dot{z} &= (\lambda z + z^2) - y - x \\ \dot{x} &= (\lambda x + x^2) - y - z \end{cases}$$

and $\gamma f(x)$ swaps the equation so the system becomes :

$$\begin{cases} \dot{y} &= (\lambda y + y^2) - z - x \\ \dot{z} &= (\lambda z + z^2) - y - x \\ \dot{x} &= (\lambda x + x^2) - y - z \end{cases}$$

which prooves that the system is $\gamma - equivariant$, and it can be shown the same way for any element of the set in $\mathbf{S_3}$. Now let us show the difference with a system which is not symmetric under the group action of $\mathbf{S_3}$ to make sure you understand the difference. We perform one change with respect to the previous system, we replace the first x in the first equation by a y :

$$\begin{cases} \dot{x} &= (\lambda y + x^2) - y - z \\ \dot{y} &= (\lambda y + y^2) - x - z \\ \dot{z} &= (\lambda z + z^2) - x - y \end{cases}$$

$f(\gamma x)$ becomes

$$\begin{cases} \dot{y} &= (\lambda z + y^2) - z - x \\ \dot{z} &= (\lambda z + z^2) - y - x \\ \dot{x} &= (\lambda x + x^2) - y - z \end{cases}$$

and $\gamma f(x)$:

$$\begin{cases} \dot{y} &= (\lambda y + y^2) - z - x \\ \dot{z} &= (\lambda z + z^2) - y - x \\ \dot{x} &= (\lambda y + x^2) - y - z \end{cases}$$

As you can see, those two systems are now different and this is a sufficiant condition to prove that this system is no more S_3 equivariant.

Finally, we state that the system of N equations is symmetric if it's $\mathbf{S_N}$ equivariant.

We now want to show in a more algebraic manner what does it mean for a system to be $\Gamma - \textit{equivariant}$.

Consider the two following situations :

— A neuron network consisting of 5 neurons inhibiting their respective neighbors.
— A neuron network consisting of the same 5 neurons, but where the boundaries neuron don't inhibit each other anymore.

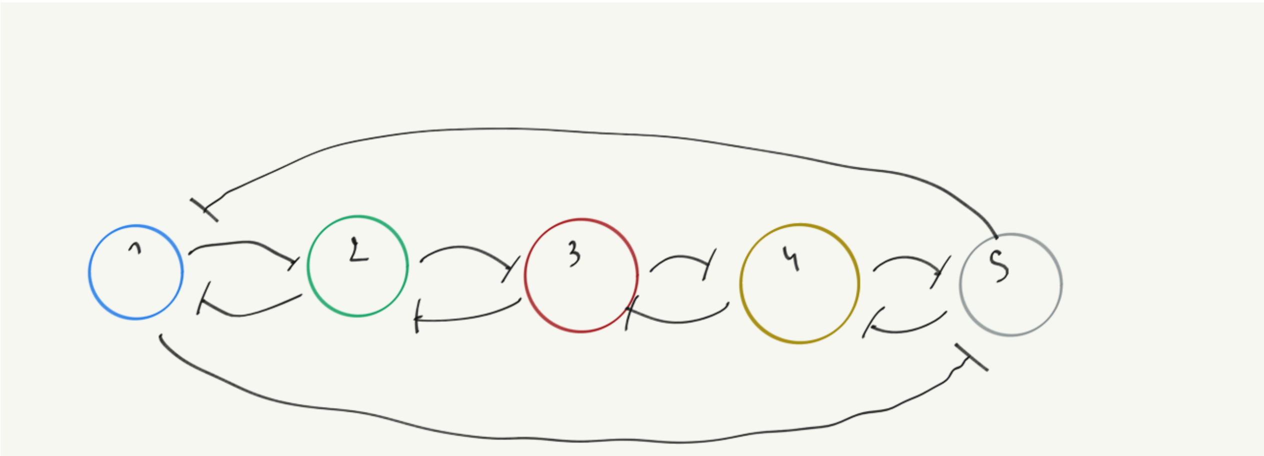

Figure 2 – Complete Model

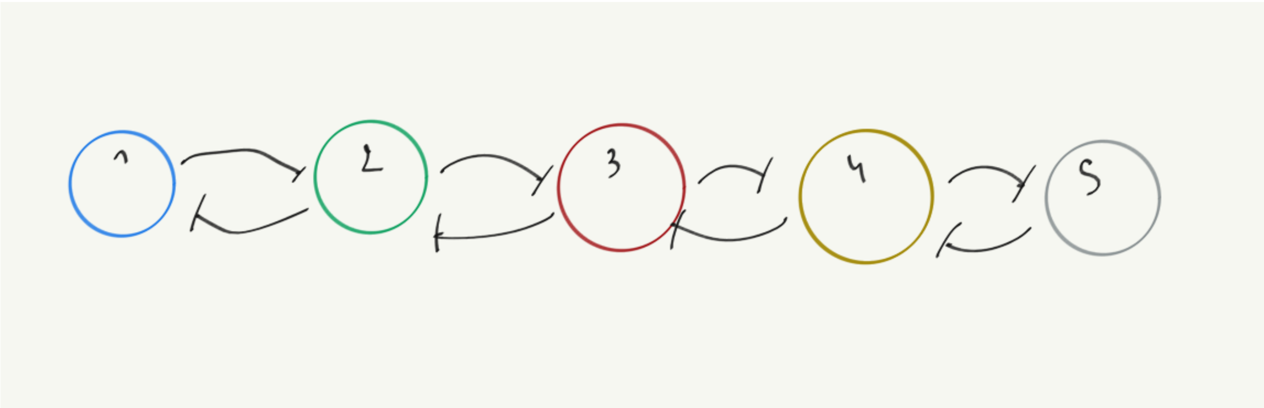

Figure 3 – Incomplete Model

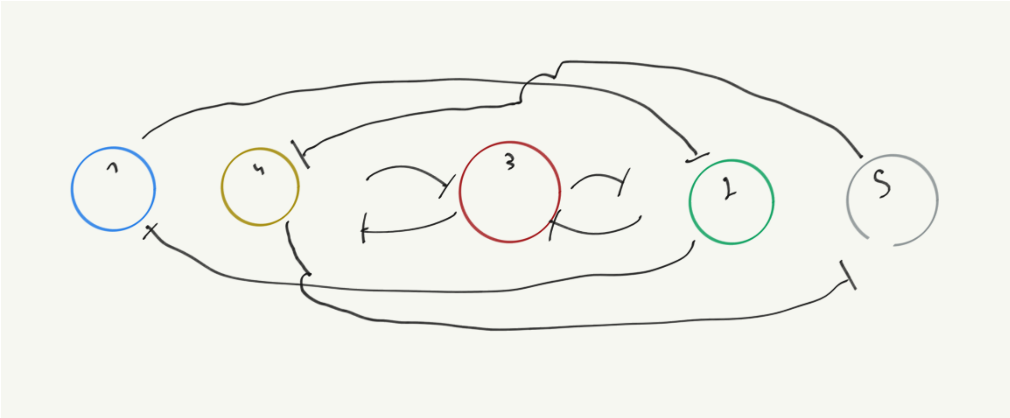

FIGURE 4 – Model after applying the permutation

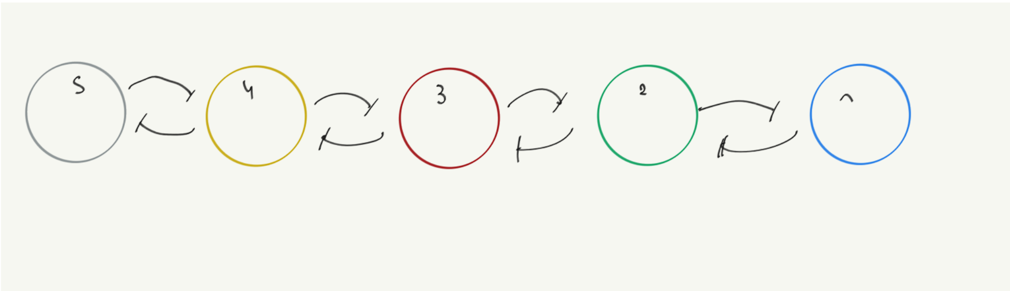

FIGURE 5 – Model after applying the second permutation

Even though these networks might look very similar, let us show that their equivariant group are radically disimilar.

We begin our analyzis with the incomplete model.

Let us note the group element (xy) the element that permutes element x and y. Therefore, we note the system $f(\underline{x}) = \{1,2,3,4,5\}$ the system modeling our network. Let us consider $\sigma \in \mathbf{S_5}$ so that $\sigma = (24)$. Taking its action on the system gives

$$\sigma.f(\underline{x}) = \{1,4,3,2,5\}$$

. The figure 4 shows that the network is therefore absolutely different than before applying the simulation (because the two neurons on the left don't inhibit each other and the same for the two last ones).

Now one can consider applying the group element $\theta =< (15),(24) >$ using composition of function as a law of composition to the network and observe that this time the network is equivalent to the initial one (figure 5). Using this information, one has that $\theta * \theta = i$ where i is the identity element of $\mathbf{S_5}$, $\theta^{-1} = \theta$ so it is straightforward that the system if $\mathbf{C_2} - equivariant$. This finite group has 2 elements $< e, g >$ where e is the identity and g the generator such that $g.g = e$, chararcterized by the Cayley Table :

$$\left[\begin{array}{c|cc} \cdot & e & g \\ \hline e & e & g \\ g & g & e \end{array}\right]$$

Let us now try to understand why θ is the generator of g while σ is not.

We will model the system using a state matrix where $a_{i,j}$ is
— 0 if there is no connection from neuron i to neuron -1

— -1 if there is an inhibitory connection from neuron i to neuron j
— +1 if there is an excitatory connection from neuron i to neuron j.
Each neuron x_i can therefore gather all the connection that starts from it, with x_{i_j} denoting connection from neuron i to j.

Therefore we have the following :
— $(x_1)^T = [0, -1, 0, 0, 0]$
— $(x_2)^T = [-1, 0, -1, 0, 0]$
— $(x_3)^T = [0, -1, 0, -1, 0]$
— $(x_4)^T = [0, 0, -1, 0, -1]$
— $(x_5)^T = [0, 0, 0, -1, 0]$

So the state matrix is given by $A(\underline{x}) = [x_1, x_2, x_3, x_4, x_5]$. So $A(\underline{x}) = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 \end{pmatrix}$ which is indeed symmetrical of course.

Now consider the permutation matrix Σ which is the isomorphism of σ, ie. the permutation matrix swapping element 2 and 4, it is $\Sigma = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$ Remember the definition of equivariance with respect to a group element γ : $\gamma.f(x) = f(\gamma.x)$

So now we have :
— $(\Sigma * x_1)^T = [0, 0, 0, -1, 0]$
— $(\Sigma * x_2)^T = [-1, 0, -1, 0, 0]$
— $(\Sigma * x_3)^T = [0, -1, 0, -1, 0]$
— $(\Sigma * x_4)^T = [0, 0, -1, 0, -1]$
— $(\Sigma * x_5)^T = [0, -1, 0, 0, 0]$

so $A(\Sigma * x) = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & -1 \\ 0 & -1 & 0 & -1 & 0 \\ -1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \end{pmatrix}$

while $\Sigma * A(x) = \begin{pmatrix} 0 & 0 & 0 & -1 & 0 \\ -1 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & -1 \\ 0 & -1 & 0 & 0 & 0 \end{pmatrix}$

So one can observe $\Sigma * A(x) \neq A(\Sigma * x)$.

Now let us have Θ the isomorphism of θ so $\Theta = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$ which swaps columns 1 and 5 as well as columns 2 and 4. We now have :
— $(\Theta * x_1)^T = [0, 0, 0, -1, 0]$
— $(\Theta * x_2)^T = [0, 0, -1, 0, -1]$

— $(\Theta * x_3)^T = [0, -1, 0, -1, 0]$
— $(\Theta * x_4)^T = [-1, 0, -1, 0, 0]$
— $(\Theta * x_5)^T = [0, -1, 0, 0, 0]$

so $A(\Theta * x) = \begin{pmatrix} 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & -1 \\ 0 & -1 & 0 & -1 & 0 \\ -1 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \end{pmatrix}$

and

$$\Theta * A(x) =$$

$$\begin{pmatrix} 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & -1 \\ 0 & -1 & 0 & -1 & 0 \\ -1 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \end{pmatrix}$$

So the system is indeed $\Theta - \textit{equivariant}$ which is logic since it's the isomorphism of θ and the system is $\theta - \textit{equivariant}$. Remember Θ is the isomorphism of θ which is the generator of the cyclic group. Note that adding for example one neuron left and one right without linking the two boundaries would lead the group to still be $\mathbf{C_2} - \textit{equivariant}$ the difference would be that i would be the $7 * 7$ identity matrix and $g =< (17), (26), (35) >$, g's isomorphism as permutation matrix would become : $\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

Now let's come back to the case where we link the two boundaries neurons with one another.

The state matrix becomes $A'(x) = \begin{pmatrix} 0 & -1 & 0 & 0 & -1 \\ -1 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & -1 \\ -1 & 0 & 0 & -1 & 0 \end{pmatrix}$

A' is $\Gamma - \textit{equivariant}$ where Γ contains the identity element and all the reflections of the pentagons with respect to each element (subgroup of $\mathbf{S_5}$). It is an isomorphism with the group containing the identity 5*5 matrix and the following ones :

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

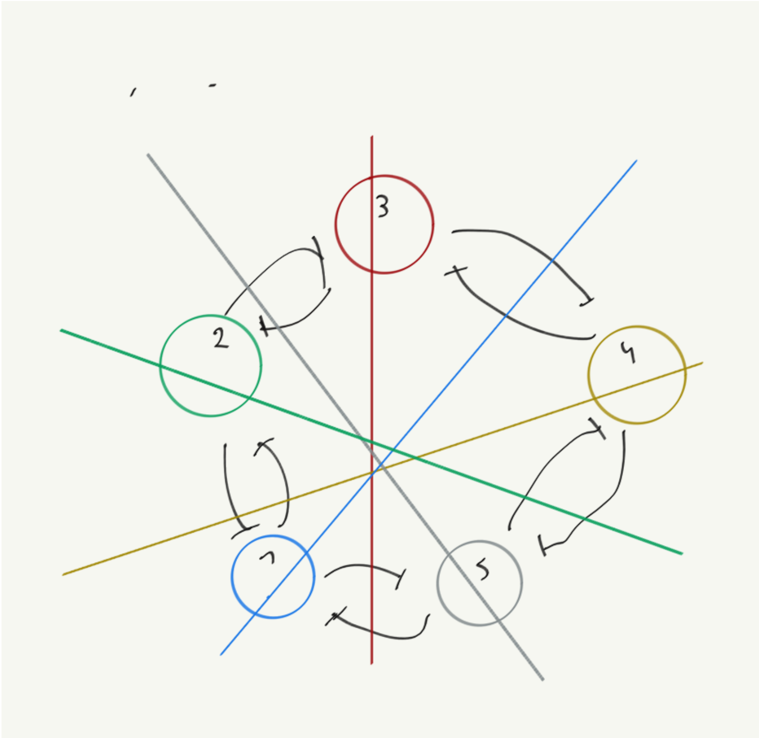

FIGURE 6 – Possible Symmetries on complete model

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$
$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$
$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

One can easily observe that when we removed the two connections, the equivariant group became a subgroup of Γ. These 5 permutations matrices correspond to the 5 permutations with respect to each neuron as can be see on figure 6.

3.2 Symmetry of a Solution of a symmetric ODE

3.2.1 Isotropy Subgroups

We need to define what isotropy subgroups consist in, in order to explain the notion of symmetry of a solution of an equivariant ODE. Remember we consider only steady states at the moment, so the solution writes $f(x) = 0$

A symmetry g is an element of G that leaves x invariant. The set of all g forms a subgroup of G known as the isotropy subgroup of x. Note that this notion strongly depends on the set x.

The isotropy group of x writes

$$\Sigma_x = \{g \in G : gv = v\}$$

As a simple example, consider S_3 and a set X. If $x_a \in X = \{x_1, x_2, x_3\}$, the isotropy subgroup of S_3 for x consists in the identity element of S_3, so $\Sigma_{x_a} = \{i\}$. However, if $x_b = \{x_1, x_1, x_3\}$, then the isotropy subgroup is $\Sigma_{x_b} = \{i, \beta_1\}$ where $\beta_1 \in S_3$ is the permutation of the two first coordinates and finally if $x_c = \{x_1, x_1, x_1\}$, then $\Sigma_{x_c} = S_3$. We can thus distinguish these solutions by their isotropy subgroups.

3.3 Symmetry Breaking

This happens when solutions of an equation have less symmetries than equations themselves.

3.3.1 Introduction with a simple example

To illustrate such a problem with a simple example, I implemented a simple winner take all model :

$$\dot{x_i} = \frac{1}{\tau}(-x_i + S(K - \sum_{j=1}^{5} x_j))$$

with $i = 1, ..., 5$ and $j = 1, ..., 5 \neq i$ with

$$S(x) = \frac{100 * x^2}{(40^2 + x^2)}$$

if $x > 0$, 0 otherwise.

As you can observe, the system of equation is $\mathbf{S_5} - equivariant$, because $\gamma f(x) = f(\gamma x) \forall \gamma \in \mathbf{S_5}$, so the system of equation is symmetric.

Now let us look at the simulation of this winner take all model. I searched for the equilibrium of such system for many input K (parameter ramping). Figure 7 shows the corresponding results for initial conditions that are $[0.1, 0.1, 0., 0., 0.]$. You can observe that the solution are first the same whatever the neuron, then **symmetry breaking arise** : the solutions are not the same. Let us consider x, the vector modeling the state of the 5 neurons.

— Before symmetry breaking : $\Sigma_x = \mathbf{S_5}$

— After symmetry breaking : $\Sigma_x = \mathbf{S_2} \times \mathbf{S_3}$

In practice, it means that before the bifurcation, any permutation element of $\mathbf{S_5}$ leaves the solution (the state of the five neurons) unchanged. After the bifurcation, any element of the group $\mathbf{S_2} \times \mathbf{S_3}$, leaves the state of the five neurons unchanged. Remember that the group $\mathbf{S_2} \times \mathbf{S_3}$ has element that are composed of one component from $\mathbf{S_2}$ and one from $\mathbf{S_3}$. The group operation is performed separately on each component.

As a conclusion to this example, one can observe that the isotropy subgroup of a solution of an equivariant system provides usefull information about the shape of this solution.

Before explaining how one can find the solution for given symmetries, a few more concepts might be usefull to master in order to understand the rest of the project.

We now show the symmetry breaking when we use random initial conditions and see what the behaviour looks like.

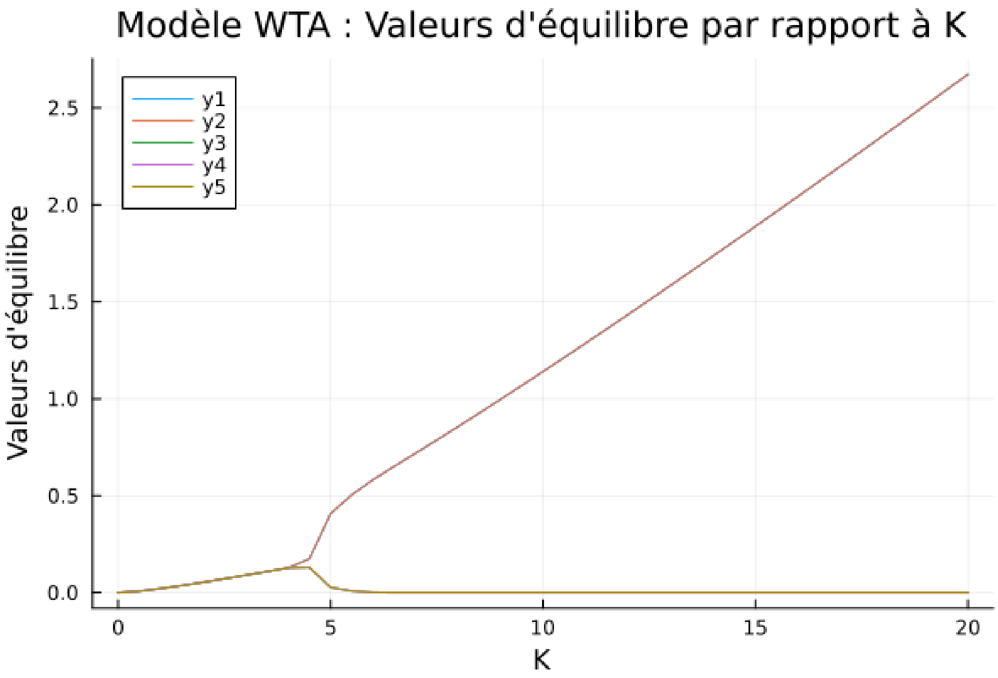

FIGURE 7 – Winner take all parameter ramping

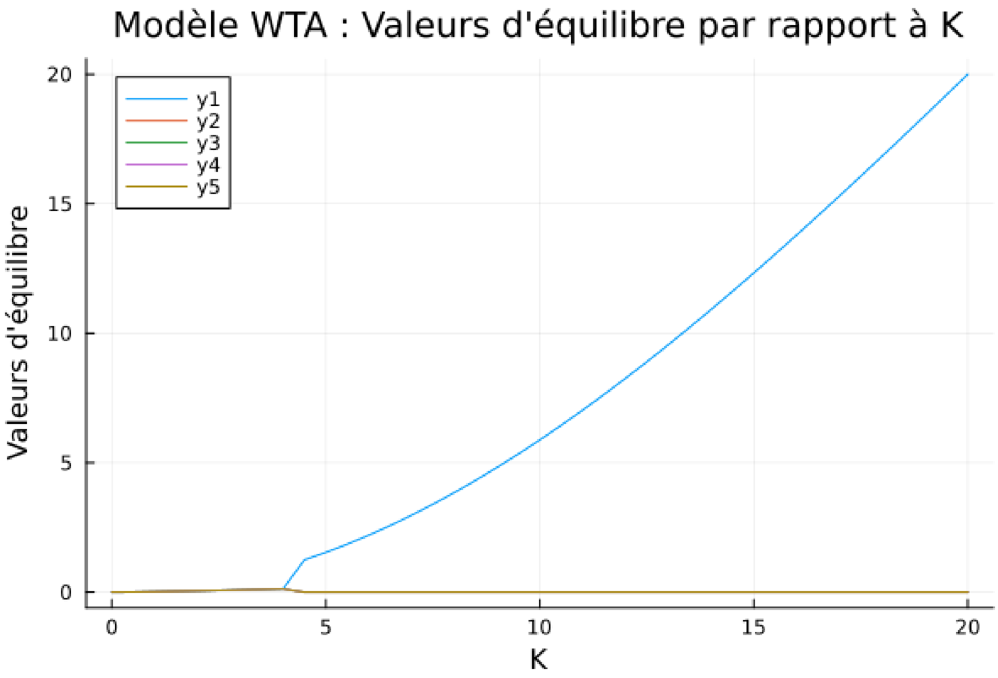

FIGURE 8 – Winner take all parameter ramping with random IC

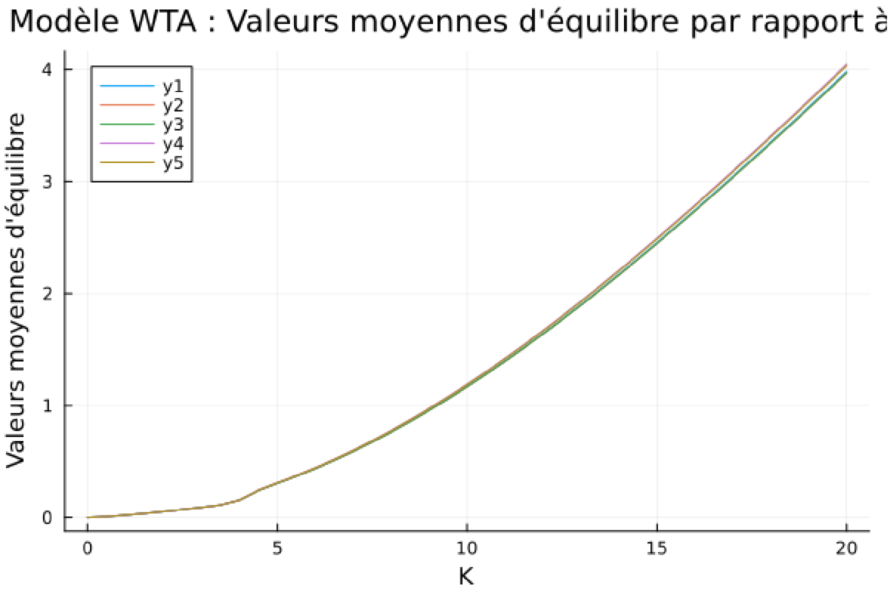

FIGURE 9 – Winner take all parameter ramping average results for 100000 simulations

You can see on figure 8 that there is again symmetry breaking, and there is only one possible winner here so $\Sigma_x = \mathbf{S_1} \times \mathbf{S_4}$. This winner changes at each simulation due to the randomness of the initial conditions.

There is a more fascinating aspect about group symmetries that can be investigated. If we do the same simulation many times for random initial condition so $x_i \in [0, 1]$ and that we average the results, we will see that symmetry is retrieved. Indeed, as you can see on figure 8, the state of all neurons is appareantly equal(there is a slight difference on the picture that can be explained by the finite number of simulations that can be achieved in a reasonnable time). If one could run an infinite number of simulations than the curves would be perfectly superimposed.

Before explaining how one can find the solution for given symmetries, a few more concepts might be usefull to master in order to understand the rest of the project.

3.3.2 Isotropy Lattice

Let us have a $\Gamma - \textit{equivariant}$ system f(x). If $x \in \mathbf{R^n}$ is an equilibrium of the system, then if $\gamma \in \Gamma$, γx is also an equilibrium because $f(\gamma x) = \gamma f(x) = \gamma 0 = 0$.

Note that γx and x have conjugate isotropy subgroups :

$$\Sigma_{\gamma x} = \gamma \Sigma_x \gamma^{-1}$$

A simple example can illustrate that, if we denote $c_{i,j}$ the permutation of element i and j and p_i the clockwise permutation of i elements :

If $x = (1, 2, 2)$, $\gamma \in \mathbf{S_3} = p_1$, meaning γ^{-1} must then be p_2 which is equivalent to anticlockwise rotation of one element in this case. Suppose $x = (1, 2, 2)$ and so $\gamma x = (2, 1, 2)$

One has $\Sigma_x = \{i, c_{2,3}\}$. $\Sigma_{\gamma x} = \{i, c_{1,3}\}$. $\gamma \Sigma_x = \{p_1, p_1 * c_{2,3}\}$ is, by the composition of functions (denoted *) which is the law of composition in $\mathbf{S_3}$. Finally, $\gamma \Sigma_x \gamma^{-1} = \{i, p_1 * c_{2,3} * p_2\}$, which is stricly equivalent to $\Sigma_{\gamma x} = \{i, p1 * c_{1,3}\}$

The isotropy lattice is a hierarchical structure that organizes these isotropy subgroups according to their inclusion relationships.

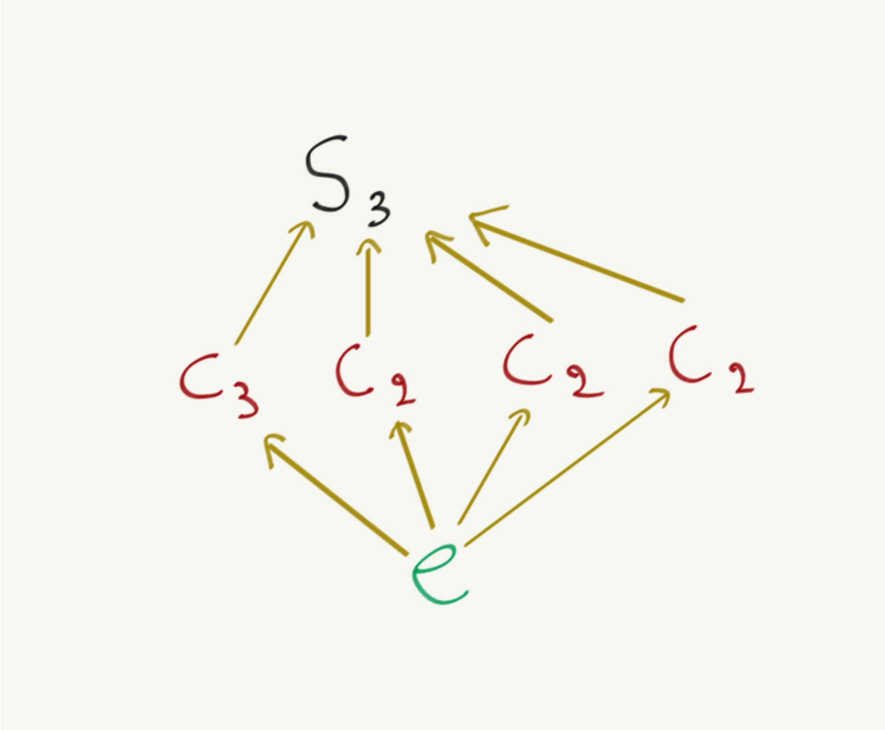

FIGURE 10 – Isotropy Lattice of S_3

It is composed of :
- Nodes : Each node in the lattice represents an isotropy subgroup of the main group G on its action on $\mathbf{R^n}$
- Edges : An edge between two nodes indicates that one isotropy subgroup is a subgroup of the other, reflecting the inclusion relationship.

In other words, it classifies the possible ways for equilibria to break symmetry where smaller isotropy subgroups correspond to breaking more symmetries.

Let us take back an example with our favorite group $\mathbf{S_3}$. This group has many subgroups :
- The group $\mathbf{S_3}$ itself.
- The cyclic group $\mathbf{C_3}$ corresponding to all the rotations (It contains 3 elements)
- The three subgroups $\mathbf{C_2}$ corresponding to permuting two elements
- The trivial subgroup containing only the identity element

Let us try to build the isotropy lattice of $\mathbf{S_3}$ after gathering these informations. The result is shown on 10 (where e denotes the identity subgroup containing only the identity element). The point of building the isotropy lattice is that it can help us to carry a systematic search for solutions. We can define the partial ordering of the isotropy lattice. In this example, one has

$$\mathbf{S_3} < \mathbf{C_3} < e$$

Indeed, e corresponds to breaking more symmetries than $\mathbf{C_3}$ for example because the number of elements in the identity isotropy subgroup is 1, and 3 in the one of C_3.

3.3.3 Fixed Point Subspace

In the study of dynamical systems with symmetry, the fixed point subspace is a subset of the state space that remains invariant under the action of a symmetry group or a subgroup.

The formal definition could be written as

$$Fix(G) = \{x \in \mathbf{R^n} : gx = x, \forall g \in G\}$$

Do not confuse with the isotropy definition, which as a reminder, it the subset of G that leaves any element of the solution unchanged.

For example, the fixed point subspace of $\mathbf{C_2} \in \mathbf{S_3}$ consisting of the element 2 and 3 is $\{(x, y, y) : x, y \in \mathbf{R}$.

This notion is fondamental in symmetry breaking since when a system undergoes symmetry breaking, it usually moves from higher to lower symmetry state, corresponding to moving from a larger fixed point subspace to a smaller one.

This leads us to the following theorem :

If $f : \mathbf{R^n} \longrightarrow \mathbf{R^n}$ is $G - equivariant$ and if one has H a subgroup of G, then

$$f(Fix(H)) \in Fix(H)$$

This theorem is very usefull since it tells us that we can find an equilibrium solution with isotropy subgroup H by restricting the state space to Fix(H). This makes the problem of finding solution easier since the space of Fix(H) is usually of lower dimensions. A systematic search of solution can than consist in starting with the largest isotropy subgroups and then go down in the lattice.

Consider the symmetric group S_3, which acts on the state space $\mathbb{R}^3$, where each coordinate (x_1, x_2, x_3) represents the state of one of three coupled oscillators. The fixed point subspaces corresponding to different subgroups of S_3 are given below :

— **Trivial Subgroup** $\{e\}$
  — **Subgroup** : $\{e\}$ (identity only)
  — **Fixed Point Subspace** : The entire space $\mathbb{R}^3$
  — **Dimension** : 3

— **Cyclic Subgroup** C_3
  — **Subgroup** : $C_3 = \langle(123)\rangle$ (cyclic permutations)
  — **Fixed Point Subspace** : Points where $x_1 = x_2 = x_3$
  — **Dimension** : 1

[Fix(C_3) $= \{(x, x, x) \mid x \in \mathbb{R}\}$]

— **Transpositions** C_2
  — **Subgroup** : $C_2 = \langle(12)\rangle$
  — **Fixed Point Subspace** : Points where $x_1 = x_2$, with x_3 free
  — **Dimension** : 2

[Fix(C_2) $= \{(x, x, y) \mid x, y \in \mathbb{R}\}$]

We can then generalize this principle : consider $\mathbf{S_N}$ acting on $\mathbf{R^n}$, consider the isotropy subgroup $\Gamma \in \mathbf{S^n}$ acting by non trivial partitions P=p,q where $p + q = N$, then one can easily observe that

$$Fix(\Gamma) = (u, ..., u, v, ..., v)$$

where there are p u and q v elements, so $dim(Fix(\Gamma) = 2$; More generally, if Γ is a partition in k blocks, $dim(Fix(\Gamma) = k$

3.3.4 Invariant Subspaces

An invariant subspace of a dynamical system is a subspace that remains unchanged under the action of that transformation or system.

As example, if $\mathbf{S_N}$ act on $\mathbf{R^n}$ by permuting coordinates and we consider

— $V : \{(x, x, ..., x) : x \in \mathbf{R}\}$
— $W : \{(x_1, x_2, ..., x_n) : x_1 * x_2 * ... * x_n = 0\}$

are both $\mathbf{S^n}$ invariant because permutating the coordinates won't affect the space properties of V and W.

Note that a subspace $V \in \mathbf{R^n}$is $G - irreducible$ if the only $G - invariant$ subspaces of V is the origin. It is absolutely irreducible if the $G - invariant$ subsets are both the origin and the multiple of the identity matrix.

If you take back V and W definitions, they are both S_N irreducible. Considering the action of $\mathbf{S_n}$ on W, and considering again the subgroup Γ partitionning into k blocks, now $dim(Fix(\Gamma)) = k - 1$ rather than k since the relation $x_1 * ... * x_n = 0$ reduces the dimension of the problem.

3.3.5 Linearization

In order to understand the symmetric bifurcation problem, one should linearize the system.

Consider $M : V \longrightarrow V$ as a linear map. M is $-equivariant$ if it commutes with G : $Mg = gM \longrightarrow A$ is $G - equivariant$.

The point of this statement is to notice that if G acts irreducibly, $kerA = \{0\}$ or $kerA = \mathbf{R^n}$, meaning that A is either invertible (its determinant is not zero) or A=0 (because multiplying it by any vector would give 0).

A representation of a group G on a vector space V is said to be absolutely irreducible if V cannot be further decomposed into nontrivial invariant subspaces under the action of G.

The permutation representation of $\mathbf{S_3}$ on $\mathbf{R^3}$ is absolutely irreducible. This means that there are no nontrivial invariant subspaces under the action of $\mathbf{S_3}$ on $\mathbf{R^3}$. . In other words, $\mathbf{S_3}$ cannot be decomposed into smaller, invariant subspaces.

3.3.6 Steady States Bifurcartion

A steady state bifurcation occurs at an equilibrium when the linearized equation has a zero eigenvalue and no other eigenvalues on the imaginary axis. We can make use of the absolutely irreducible representations because they form the generic representations.

Let us consider a group-inavriant (to G) equilibria $x(\lambda)$ to the ODE $\dot{x} = F(x, \lambda)$ and state that the bifurcation occurs at $\lambda = \lambda_0$. Denoting by dF the derivative of F and M the linear map :

$$A_0 = (dF)_{y(\lambda_0),0}$$

If A_0 is singular, then G acts irreducibly on the state space

Therefore :

— 0 is the only eigenvalue on the imaginary axis
— The generalized eigenspace corresponding to 0 is $ker(A_0)$
— Γ acts irreducibly on $ker(A_0)$

So finding the equilibria in bifurcations problem can be reduced to find the zeros of mappings on $ker(A_0)$ where A_0 is the jacobian matrix.

As an example, let us build the system

$$\dot{x_i} = x_i - (x_1 + ... + x_n) + x_i^2 - x_i^3$$

$\forall i \in \{1, ..., N\}$ where the higher order terms stand there for stability. For the sake of simplicity, let us take $N = 10$.

One can easily observe this system is $\mathbf{S_{10}} - equivariant$. Indeed,denoting the system by F, $\forall\gamma \in \mathbf{S_{10}}$, one has $\gamma F(x) = F(\gamma x)$.

In this system, the plot showing the equilibrium values with respect to λ is shown on 11. As you can easily observe, there seems to be a bifurcation of the system for $\lambda = 0$.

Let us now analyze the evolution of the eigenvalues as λ varies. One can see on figure 12 that there is indeed a 0 eigenvalue at the bifurcation plot.

The numerical methods I used (jacobian evaluation using ForwardDiff) lead me to the conclusion the bifurcation takes place in $\lambda_0 \approx -0.07$, indeed, it is difficult to find with infinite precision the point where bifurcation takes place using numerical methods. In this case, A_0 is the jacobian of the system (otherwise stated it's the linearized system) evaluated in λ_0. Its eigenvalues are

$[-10.074272537271733, -0.08647429006033942, -0.08647429006033862, -0.08647429006033862,$

-0.0864742900603386, -0.08647429006033856, -0.08647429006033856,

-0.08647429006033848, -0.08647429006033792, 0.024697981612968906]

These values are very close to 0, but not exactly because I solved the system numerically rather than anatycally, I could try to get smaller one by keeping decreasing the step value for λ but I was satisfied with that result which was precise enough regarding the scale of the other eigenvalues.

- The only eigenvalue of A_0 on the imaginary axis is indeed 0.
- The general eigenspace (the set of all generalized eigenvectors associated with λ_0) corresponding to 0 is the kernel of A_0
- $\mathbf{S_{10}}$ acts irreducibly on A_0 : there are no nontrivial invariant subspaces under the action of $\mathbf{S_{10}}$ on $\mathbf{R^{10}}$

3.4 Liapunov Schmidt Reduction

The Liapunov-Schmidt reduction is a mathematical technique used in the analysis of nonlinear equations, particularly in bifurcation theory and the study of stability. It reduces a high-dimensional problem to a lower-dimensional one, making it more tractable to analyze.

3.4.1 Introducing the problem

Consider a problem consisting of a system of equation written

$$f(x, \lambda)$$

with

- $x \in \mathbf{R^n}$ is the state vector .
- $\lambda \in \mathbf{R}$ is the bifurcation parameter.
- f, a non linear mapping, such that $f : \mathbf{R^n} \times \mathbf{R} \longrightarrow \mathbf{R^n}$.

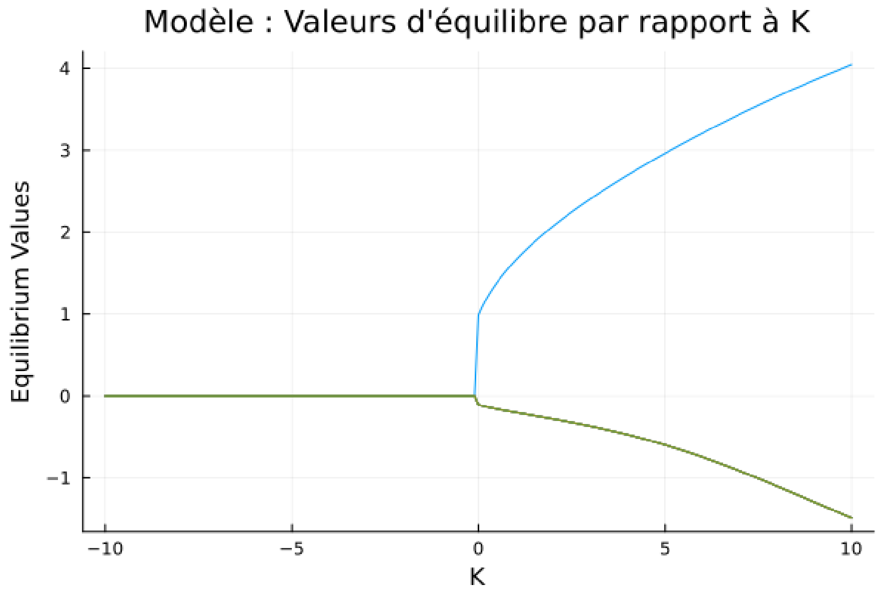

FIGURE 11 – Equilibrium Values of the system

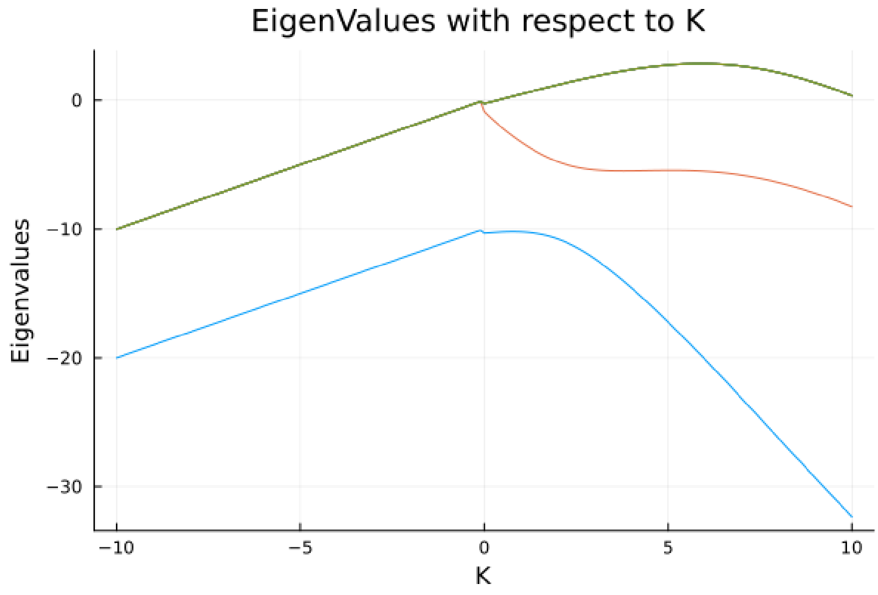

FIGURE 12 – Eigenvalues of the system

3.4.2 Step 1 : Linearization around a solution

Let us consider a state x_0 such that $f(x_0, \lambda_0) = 0$

We derive the system with respect to x, ie. we evaluate the jacobian of the solution in x_0, λ_0 :

$$A_0 = (df)_{x_0,\lambda_0}$$

Note that A_0's size must be $n \times n$

3.4.3 Step 2 : Splitting the Space

One divides $\mathbf{R^n}$ in two distincts parts : the range of A_0, ie the set of all possible output vectors that can be obtained by multiplying the matrix by a vector from its domain ;

$$Range(A_0) = \{y \in \mathbf{R^n} | y = A_0 x, x \in \mathbf{R^n}\}$$

The other part is the complement of the range subgroup of $\mathbf{R^n}$ with respect to the range of A_0, that we denote $N(A_0)$

Finally, one has

$$\mathbf{R^n} = (A_0) \oplus N(A_0)$$

3.4.4 Step 3 : Projecting in the subspaces

We define Q, the projection operator that maps any element of $\mathbf{R^n}$ to $R(A_0)$ and the projection operator $I - Q$ that maps any element of $\mathbf{R^n}$ in $N(A_0)$.

3.4.5 Step 4 :Decomposing the non linear Problem

Express the system using projections :

$$f(x, \lambda) = Qf(x, \lambda) + (I - Q)f(x, \lambda)$$

3.4.6 Step 5 : Solve the problems Separately

Solving $Qf(x, \lambda)$ and $(I - Q)f(x, \lambda)$ are lower dimensionnal problems that might be easier to solve.

3.4.7 Step 6 : Constructing Solutions to the Original Problem

Once solutions of the reduced equations are found, they can be used to construct solutions to the original problem.

3.4.8 Example of LPS Reduction

Let us try to apply this procedure to the system we just used to illustrate symmetry breaking :

$$\dot{x_i} = \lambda x_i - (x_1 + ... + x_n) + x_i^2 - x_i^3$$

with i ranging from 1 to 3.

1. Linearize the system around an equilibrium : The equilibrium is found by solving $\lambda x_i - (x_1 + ... + x_n) + x_i^2 - x_i^3 \forall i$. One has for example the solution $x_i = 0 \forall i$
2. Evaluate the Jacobian matrix at the equilibrium : $\begin{pmatrix} \lambda - 1 & -1 & -1 \\ -1 & \lambda - 1 & -1 \\ -1 & -1 & \lambda - 1 \end{pmatrix}$
3. Evaluate the range of the Jacobian, and the nullspace (which is actually the kernel of the Jacobian since it is all the vector in $\mathbf{R^3}$ that are mapped to 0).

   To find the nullspace of the jacobian, one solves

   $$J_{x0}\mathbf{v} = 0$$

   where J_{x0} is the jacobian evaluated at the equilibrium.

   $$(\lambda - 1)v_i - v_j - v_k = 0, \quad \text{for } i \neq j \neq k.$$

   Summing the 3 equations, the system has to respect th equality :

   $$\lambda * (x_1 + x_2 + x_3) - 3x_1 - 3x_2 - 3 * x_3 = 0$$

   It is trivial that this system has a non-trivial solution (different than **v**=**0**) for $= 3$.There are other non trivial solutions of higher dimensiosn but applying the following procedure would work but be less straightforward.
4. The kernel of J_{x0} for $\lambda \longrightarrow 3$ is $k = (\frac{1}{\lambda - 1}(x + y), x, y) \forall x, y \in \mathbb{R}$

   The orthogonal subspace o of J_{x0} is its range in $\mathbb{R}^3$. Let us take $\lambda = 3$ to simplify the maths. In that case, the kernel (which in this case is one dimensionnal) is spanned by the vector $k = (1, 1, 1)$. Let us keep on the calculations using this particular case for λ as it is the most simple case as possible (one dimensionnal). Using higher dimensionnal case would imply to follow the same procedure but with much more complicated projections.
5. We project the system onto the critical and non-critical subspaces. Let $\mathbf{x} = c\mathbf{k} + \mathbf{o}$, where $\mathbf{o}$ is in the range of J_{X0}.

   The critical component : This critical subspace represents the directions in which the system undergoes bifurcations. The projection yields a reduced equation for the critical variable c, which captures the behavior near the bifurcation point.

   Using the expression of $\mathbf{x}$ projected in the two subspaces, the equation

   $$\dot{c} = (\lambda c) - 3c + c^2 - c^3$$

   describes the dynamics of the critical component c. You can observe the linear terms cancel each other at $\lambda = 3$, indicating bifurcation.

   The equation $\dot{o} = J\mathbf{o} + H$ where $\mathbf{o}$ is in the range of the jacobian. It describes the dynamics of the non-critical component, ie. the components that do not undergo bifurcation. H accounts for high order terms, responsible for non-linearities.
6. We analyze the reduced system equation obtained for the critical component :

   $$\dot{c} = (\lambda c) - 3c + c^2 - c^3$$

   — For $\lambda < 3$, $(\lambda - 3) < 0$, meaning that we have stable fixed point at $c = 0$ and order stable fixed point given by the high order terms

   — For $\lambda > 3$, $(\lambda - 3) > 0$ so we have unstability for $c = 0$ and new stable or unstable fixed points (depending on the high order terms) appearing.

This LPS reduction allowed us to analyze the behaviour of the system and to understand how the system's dynamics change as the parameter λ varies. The same analyzis could be done for $\lambda = 0$, therefore the kernel is spanned by for example $(0, -1, 1)$.

3.5 The Equivariant Branching Lemma

3.5.1 Overview

The goal of this lemma is to proove that after symmetry breaking, the solution subgroup is a maximum isotropy subgroup. Otherwise stated, in the LPS reduction, it has a one dimension fixed space.

3.5.2 Definition

A subgroup is *axial* providing the dimension of its fixed point space is 1.

Without any loss of generality, let us assume that the bifurcation occurs at $\lambda = 0$ in $x = 0$

Consider a finite group G acting irreducibly on $\mathbf{R^n}$, let us have the usual system :

$$\frac{dx}{dt} = f(x, \lambda)$$

Assuming it is a G-equivariant system, that the jacobian evaluated at (0,0) has at least one zero eigenvalue and that the others are real, consider a subgroup Σ such that :

$$dim(Fix(\Sigma)) = 1$$

The lemma states that there exist a unique equilibrium solution branch bifurcating from $x = 0$ such that the isotropy subgroup of solutions on the branch is Σ

Before giving an example, one notice the following :
— The equivariant branching lemma does not tell us if the equilibrium solution branches bifurcate into positive or negative values of λ
— The equivariant branching lemma does not address the stability properties of the equilibria.
— The statement that the branch of equilibria is unique assumes that the symmetry-related equilibria created in a pitchfork bifurcation are identified.
— The lemma states that one branch will bifurcate from this point, but it doesn't restrict the problem to this branch, other branches could bifurcate from $x = 0$
— When $\Sigma < G$, the bifurcation points in Σ have lower symmetry than in G (the basic solution isotropy subgroup is G than Σ), this effect is called spontaneous symmetry breaking.

3.5.3 Example

Let us consider once again the previous example we worked on to show an example of the LPS reduction.

Consider $\Gamma = \mathbf{S_3}$, let us have $\mathbf{x} = (x_1, x_2, x_3)$

Remember the system has the form

$$\dot{x_i} = x_i - (x_1 + ... + x_n) + x_i^2 - x_i^3$$

with $i = 1, 2, 3$ and $n = 3$.

1. The system is $S_3 - equivariant$ as shown before
2. The trivial solution $\mathbf{x} = (0, 0, 0)$ is valid $\forall \lambda$

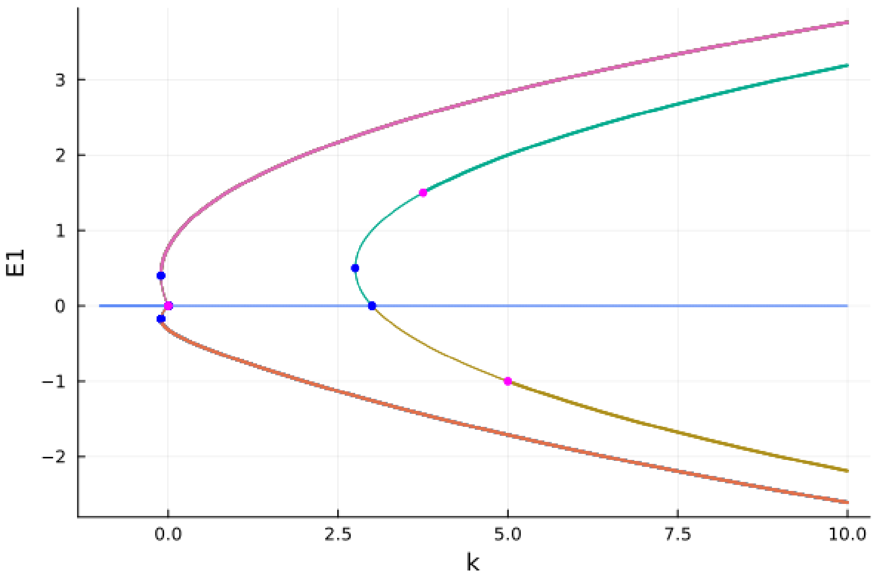

FIGURE 13 – Pitchfork Bifurcation for 1 branch at the spot predicted by the equivariant branching lemma using Julia bifurcation kit ($k = \lambda$)

3. The Jacobian evaluated at $\mathbf{x} = (0, 0, 0)$ is : $\begin{pmatrix} \lambda - 1 & -1 & -1 \\ -1 & \lambda - 1 & -1 \\ -1 & -1 & \lambda - 1 \end{pmatrix}$

   At $\lambda = 3$, J has a zero eigenvalue corresponding to the eigenvector $\mathbf{v}_0 = (1, 1, 1)$

   The isotropy subgroup Σ is the entire $\mathbf{S_3}$ since it leaves $\mathbf{v_0}$ unchanged.

   So $dim(Fix(\Sigma)) = 1$ and the equivariant branching lemma states that there will be a branch of non-trivial solution bifurcating in $x = 0$ for $\lambda = 3$. Remember the kernel was spanned by $(1, 1, 1)$, which is invariant under the action of $\mathbf{S_3}$, so there must be a pitchfork bifurcation for this value of parameter (see on plot 13). Another usefull information can be extracted from it, indeed, we can see the other bifurcation takes place at $\lambda = 0$ (for any N), which is the case, we analyzed at the previous point. Note that one can increase the dimension of the problem, for example 5, and the corressponding bifurcation point will move to 5 as well, since it is for $\lambda = 5$ that the kernel is spanned by $(1, 1, 1, 1, 1)$.

4 Conclusion

In this project, we delved into the stunning world of symmetry and its implications in the study of steady state bifurcations, drawing from the basic principles of group theory. We began by introducing basic concepts of group theory, which provided the necessary mathematical framework to explore symmetries in various systems. Understanding group actions and their representations was essential as it set the stage for analyzing how symmetries can influence the behavior of solutions to differential equations.

We then examined steady state bifurcations, focusing on the role of symmetry in these phenomena. By defining equivariance, we highlighted how symmetrical properties of a system can be preserved under certain transformations, leading to invariant solutions. This exploration included a detailed look at symmetry breaking, where the introduction of perturbations can cause a system to transition from a symmetric state to one where the symmetry is no longer apparent.

A portion of our discussion centered on the Liapunov-Schmidt reduction method, a powerful tool used to reduce the complexity of bifurcation problems. This reduction simplifies the analysis by transforming the original problem into a lower-dimensional one, making it easier to study the bifurcations and their stability.

Finally, we explored the Equivariant Branching Lemma, which provides conditions under which symmetric solutions branch from trivial solutions in equivariant bifurcation problems. This lemma is instrumental in predicting the existence and structure of bifurcating solutions, offering deep insights into the behavior of systems with symmetries.

In conclusion, this project has highlighted the critical role that symmetry plays in understanding and analyzing steady state bifurcations. The tools and concepts from group theory, coupled with advanced methods like the Liapunov-Schmidt reduction and the Equivariant Branching Lemma, offer a robust framework for studying these complex phenomena. By appreciating the interplay between symmetry and bifurcations, we gain a deeper understanding of the underlying structures that govern the behavior of various physical, biological, and engineering systems.

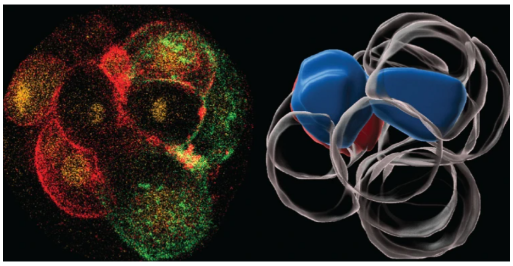

FIGURE 14 – Illustration related to cell differentiation

For example, in the field of biology, it can explain how a single cell develops into a more complex organism made of multiple kinds of cells. It's not yet clear what causes the asymmetry that leads to this differentiation. However, knowing which cells are more likely to form the fetus could allow IVF clinics to better screen embryos to find those that are most likely to lead to successful pregnancies. This theory still has a long path to go before we can make use of it, but its application might be game-changing.

Finally, I will end a statement of Brivanlou from [3] that I discovered thanks to professor A.Franci that means a lot for me since I dived into this project : "The more we're looking at this, the more I appreciate that life is made of continuous symmetry breaking".

Références

[1] Golubitsky Stewart , The Symmetry Perspective, From Equilibrium to Chaos in Phase Space and Physical Space.pdf

[2] ALGEBRA,Michael Artin,Massachusetts Institute of Technology, Chapter 2 : Groups

[3] Human embryos embrace asymmetry to form the body,Sara Reardon,https://www.nature.com/articles/d41586-024-01403-y?WT.ec_id=NATURE-202405&sap-outbound-id=36DAA593269BB9D36D97C2383A1600F884C1A4F6

[4] Group Theory, Wikipedia, 24/05/2024,https://en.wikipedia.org/wiki/Group_theory

[5] Group Theory, Jaeyi Song and Sophia Hou,MIT

[6] Equivariant bifurcation theory,Dr. Jeff Moehlis, University of California, Santa Barbara, California,Dr. Edgar Knobloch, Department of Physics, University of California, Berkeley, CA, USA. http://www.scholarpedia.org/article/Equivariant_bifurcation_theory

[7] AARMS SUMMER SCHOOL–LECTURE V : AN INTRODUCTION TO THE FINITE AND INFINITE DIMENSIONAL REDUCTION METHOD,MANUEL DEL PINO AND JUNCHENG WEI, AARMSSUMMERSCHOOLâĂŞLECTUREV:ANINTRODUCTIONTOTHEFINITEANDINFINITEDIMENSIONALREDUCTIONMETHOD