Project Report

Polytech Nice

Simple Road Traffic Modeling

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1 Presentation of the Subject

1.1 Useful Definition

Ordinary Differential Equation (ODE):

An ODE is a mathematical equation that relates a function to its derivatives with respect to one or more independent variables. ODEs are commonly represented given a function F of x, y, and derivatives of y. Then, an equation of the form

$$F\left(x, y, y', \ldots, y^{(n-1)}\right) = y^{(n)}$$

SMRT

SMRT means Simple road traffic simulation

1.2 Simple Road Traffic Modeling

Road traffic modeling involves studying how vehicles behave on road networks, with the aim of simulating and analyzing aspects such as traffic flow, congestion, and driver behavior. This field employs mathematical and computer models to understand and predict traffic patterns, playing a crucial role in urban planning, traffic management, and the development of intelligent transportation systems (as shown in Figure 1).

Explore the integration of a driving simulator with a traffic simulator in Jeihani et al.'s study [1], revealing enhanced traffic density and Variable Message Sign (VMS) reliability. The findings suggest that integration positively influences compliance behavior and factors affecting route diversion.

Figure 1: **Road Traffic:** An illustrative example of a road traffic phenomenon, which is the focus of our study.

1.2.1 The Objective of SMRT

The primary goal of simple road simulation is to analyze and understand vehicle behavior in a controlled and reproducible environment. Researchers and engineers use these simulations to study the impact of various factors on traffic flow, safety, and efficiency. This contributes to the development and testing of traffic management strategies and vehicle control systems, providing insights into the dynamic interactions between vehicles and the road environment.

1.2.2 Use Case to Explain the Interest of These Simulations

Simple road simulations find applications in various fields, including urban planning, transportation engineering, and autonomous vehicle development. For instance, they can assess the effectiveness of new traffic signal timings, study the implications of road design changes, or test the performance of autonomous vehicles in different traffic scenarios. These simulations offer a cost-effective and risk-free way to evaluate real-world interventions and innovations, enhancing decision-making processes and supporting the development of sustainable and efficient transportation systems.

1.2.3 Field of Application

Simple road simulation is applied across a wide range of fields, including transportation research, traffic engineering, and urban planning. It plays a crucial role in the development and validation of autonomous vehicle technologies. The ability to model and simulate various road conditions helps researchers and practitioners make informed decisions, improving the design and management of transportation systems. Simple road simulation serves as a powerful tool for enhancing our understanding of traffic dynamics and contributes to the ongoing evolution of smart and adaptive transportation systems.

1.2.4 Using GitHub for Project Management

In this project, we selected GitHub as the primary management tool to facilitate our collaboration, track changes, and manage the source code. The following representation provides an overview of our collaborative use of GitHub as a pair throughout the project.

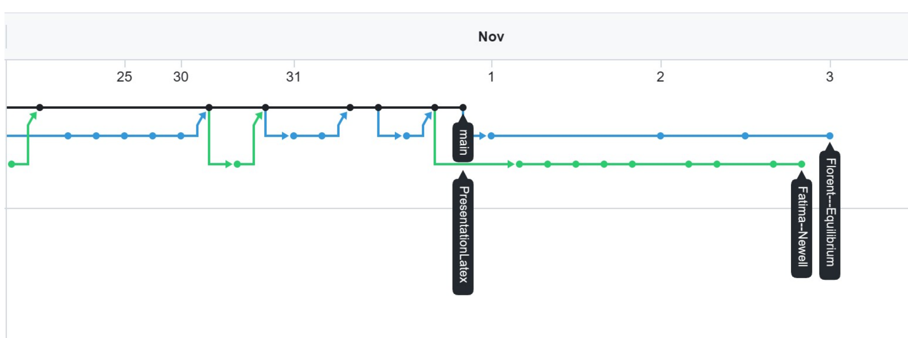

Figure 2: **Road traffic :** In this picture, you can see an example of a road traffic phenomenon that we could study

2 Project Objectives

In this modeling, we will focus on the position and velocity of multiple vehicles following each other in a single lane. We will assume that vehicles accelerate when they have open space, do nothing when they are close to the vehicle in front, and brake more strongly as they get very close. We will arrange two and then three vehicles, one behind the other, with the first vehicle maintaining a constant velocity, and observe the behavior of the following vehicles.
So, the main objectives of the project are:

- Modeling the behavior using coupled ordinary differential equations.
- Implementing this simplistic modeling.
- Studying the outcomes based on different parameters.

2.1 Problems Encountered

3 The equation for SMRT

3.1 Differential Ordinary Equation

In this part, the idea is to resolve two types of systems of Ordinary Differential Equations (ODEs) that allow us to simulate traffic flow. To achieve this, we will use the Euler Explicit method to numerically solve the solutions. The Euler Explicit method is given by the folowing eqution :

- EDO to solve: $\boxed{y'(t) = f\big(t, y(t)\big)}$.
- First step of the resolution: $\boxed{y_0 = y(t_0)}$.
- Recursive process to find the n-th solution of the EDO: $\boxed{y_{n+1} = y_n + hf(t_n, y_n)}$

3.1.1 Linear Model

Mathematical Theory:

In this section, we're going to explain the math behind our model for understanding how cars behave in traffic. To make things simple, we use a discrete model, which means we look at cars one at a time and how they interact on the road.

Each car's movement is governed by a basic equation:

$$\dot{x_i}(t) = V_i = \alpha_i(x_{i-1} - x_i)$$

In this equation, $x_i(t)$ represents where the i-th car is at a given time, V_i is how fast the i-th car is going, and α_i is a number that describes how that car behaves. The right side of the equation, $\alpha_i(x_{i-1} - x_i)$, tells us how the car's speed changes based on how close it is to the car in front.

When we put this equation to work for all the cars, we end up with a bunch of equations (one for each car), which helps us understand how they all move together in traffic. These equations give us a dynamic view of how cars influence each other as they drive.

The system of equations is written like this for each car, where i can be 1, 2, and so on, up to the number of cars:

$$\begin{cases} \dot{x}_1 & = V_1 \\ \dot{x}_2(t) & = \alpha_2(x_1 - x_2) \\ & \vdots \\ \dot{x}_n(t) & = \alpha_n(x_{n-1} - x_n) \end{cases}$$

These equations help us understand how traffic flows and how individual cars influence one another on the road.

In the first part of the study, we consider α_i as a constant. However, in the next part of the simulation, we define some functions $\alpha_i(t)$.

In fact, we have three types of simulations:

1. **Constant value:**

$$\alpha_i(t) = C_i$$

2. **Sinusoidal with noise:**

$$\alpha_i(t) = |W \cdot \sin(\omega t + \phi) + \mathcal{N}(0, 0.1)|$$

3. **Stochastic Driver Model:**

Consider a random driver model with an output labeled as $\alpha_i(t)$, which relies on various factors like the average, spread, and time t. This model produces a random noise part from a usual distribution with an average and spread of $\sqrt{\text{spread}}$. If a limit is given, the generated noise is confined within the interval $[-\text{limit}, \text{limit}]$.

Mathematically, we can express this as:

$$\alpha_i(t) = \begin{cases} \left|\mathcal{N}(\text{average}, \sqrt{\text{spread}})\right|, & \text{if } -\text{limit} \le \alpha_i(t) \le \text{limit}, \\ -\text{limit}, & \text{if } \alpha_i(t) < -\text{limit}, \\ \text{limit}, & \text{if } \alpha_i(t) > \text{limit}. \end{cases}$$

Here, average signifies the mean of the acceleration, and spread stands for the acceleration's spread. The limit is defined to prevent or create specific situations, like accidents, depending on the desired result.

Implementation :

Algorithm 1 $\dot{x_1}$

Output:
- $\dot{x_1} = V_1$

Algorithm:
- $\dot{x_1} = 130 \times \frac{1000}{3600}$
- **return** $\dot{x_1}$

Algorithm 2 $\dot{x_i}$

Input:
- t := "time step"
- x_i := Table of position for the i-th Car
- x_{i-1} := Table of position for the (i-1)-th Car

Output:
- $\dot{x_i} = V_i$

Algorithm:
- $\dot{x_i}(t) = V_i = \alpha_i(x_{i-1}[t] - x_i[t])$
- **return** $\dot{x_i}$

This algorithm allows to define an orthonormed local reference frame and we delete the lines to eliminate for reducing matrix

Algorithm 3 sinusoidal_model

Input:
- W := "Amplitude of the perturbation"
- ω := "Angular frequency"
- t := "time"
- ϕ := "Phase (in radians)"

Output:
- This function returns a value of acceleration ($\alpha_i(t)$) following a sinusoidal model with random noise.

Algorithm:
- $\alpha_i(t) = |W \cdot \sin(\omega \cdot t + \phi) + N|$
- N:="random noise such as N $\sim \mathcal{N}(0, 0.01)$"
- **return** $\alpha_i(t)$

Algorithm 4 Update Positions and Velocities

Input:
- $t :=$ time step
- $x_1Pos :=$ Table of position for the 1st Car
- $x_2Pos :=$ Table of position for the 2nd Car
- $v_1 :=$ Table of velocity for the 1st Car
- $v_2 :=$ Table of velocity for the 2nd Car
- $time :=$ Table of time values
- $h :=$ time step size

Output:
- x_1Pos
-
- x_2Pos
- $accident :=$ Flag indicating whether an accident occurred

Algorithm:
- **for** t in range(1, n_steps) **do**
-     $x_1Pos[t] = x_1Pos[t-1] + \dot{x_1}() \times h$
-     $x_2Pos[t] = x_2Pos[t-1] + \dot{x_2}(t-1) \times h$
-     $v_1[t] = \dot{x_1}()$
-     $v_2[t] = \dot{x_2}(t)$
-     $time[t] = time[t-1] + h$
-     **if** isAccident(t) **then**
-         $accident =$ True
-         **break**
-     **end if**
- **end for**

Return x_1Pos, x_2Pos

3.1.2 Newell's Model

Mathematical Theory: In this section, we model the velocity using an exponential model, so the movement of each car is given by:

$$\dot{x_i}(t) = V_i(1 - e^{-\frac{\lambda_i}{V_i}(x_{i-1}(t) - x_i(t) - d_i})$$

V_i, λ_i, d_i parameters associated with the ith car:

- V_i : the maximum velocity of the ith car.
- λ_i : the capacity of acceleration/decelaration.

- d_i : the minimum headway(safe following distance).

By using this equation for each car, we get a bunch of equations—one for each car. This helps us grasp how they all move together in traffic. These equations give us a dynamic picture of how cars affect each other as they drive.

The set of equations is written like this for each car, where i can be 1, 2, and so on, up to the total number of cars:

$$\begin{cases} \dot{x}_1 &= V_1 \\ \dot{x}_2(t) &= V_2(1 - e^{-\frac{\lambda_2}{V_2}(x_1(t) - x_2(t) - d_2}) \\ &\vdots \\ \dot{x}_n(t) &= V_n(1 - e^{-\frac{\lambda_n}{V_n}(x_{n-1}(t) - x_n(t) - d_n}) \end{cases}$$

And so, the position of each vehicle is:

$$\begin{cases} x_1(t+\Delta t) &= x_1(t) + \Delta t V_1 \\ x_2(t+\Delta t) &= x_2(t) + \Delta t V_2(1 - e^{-\frac{\lambda_2}{V_2}(x_1(t) - x_2(t) - d_2}) \\ &\vdots \\ x_n(t+\Delta t) &= x_n(t) + \Delta t V_n(1 - e^{-\frac{\lambda_n}{V_n}(x_{n-1}(t) - x_n(t) - d_n}) \end{cases}$$

Implementation:

for $\dot{x_1}$, we use the same algorithm than for the Linear model., However, to calculate the $\dot{x_i}$ values we use the following algorithm.

Algorithm 5 $\dot{x_i}$

Input:
- t := "time step"
- x_i := Table of position for the (i)th Car
- x_{i-1} := Table of position for the (i-1)th Car
- d_i := Safe following distance the (i)th Car

Output:
- $\dot{x_i} := V_i$

Algorithm:
- $\dot{x_i}(t) := V_i(1 - e^{-\frac{\lambda_i}{V_i}(x_{i-1}(t) - x_i(t) - d_i})$
- **return** $\dot{x_i}$

This algorithm determines the derivative of x_i with respect to time, $\dot{x_i}$, based on the positions of the ith and $(i-1)$th cars, the safe following distance d_i, and the time step t. The result represents the velocity V_i of the ith car.

3.2 Partial Differential Equations

4 Types of Simulations Performed

4.1 Simulation with Drunk Drivers

4.2 Simulation with Unpredictable Drivers

4.3 Simulation with Drivers Reacting Similarly

4.3.1 Accordion Phenomenon

For this parts, we make the following assumptions:

- The first car maintains a constant speed over time.
- The second car reaction is based on the behavior of the first one.
- The acceleration of the second car is governed by a constant function, denoted as alpha, which remains fixed. In the algorithm 5 we took $\alpha_2 = 2.0$

Simulation with the Linear Model For Two cars

The initial simulation we aimed to conduct was intended to demonstrate and illustrate the behavior of drivers in real-life scenarios. Figure 3 accurately depicts this phenomenon. Specifically, we

Similar to real-life situations, we observe that if the second car is too far behind the first one, it decelerates. Conversely, when it is adequately distant from the first car, it increases its speed.

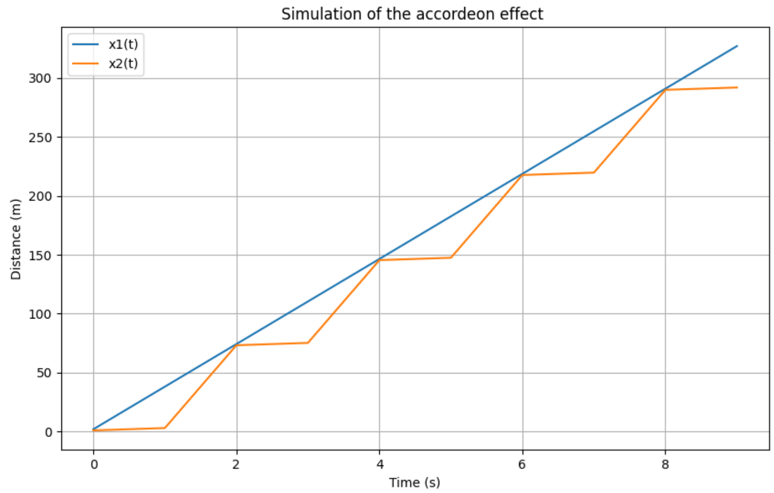

Figure 3: **Simulation of accordion phenomenon :** In this picture, you can observe the position evolutions of two cars, the first one (blue) maintaining a constant speed, and the second one (orange) following behind the first one. We could see that orange graph behaves like a periodic accordion with respect to time.

Simulation with the Newell's Model For Two cars

Similar to the previous case, applying Newell's method yields identical results. As depicted in Figure 4, the graph illustrates the accordion phenomenon, featuring distinct periods of acceleration followed by deceleration.

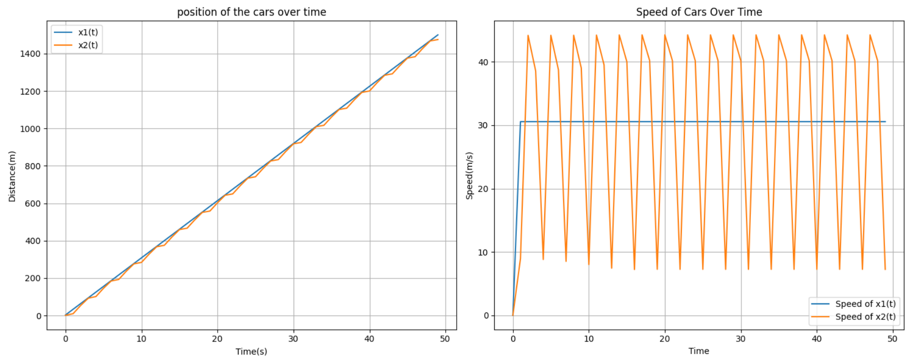

Figure 4: **Simulation of accordion phenomenon :** In this picture, you can observe the positional evolutions of two cars. The first one (blue) maintains a constant speed, while the second one (orange) follows behind the first. It is evident that the orange graph behaves like a periodic accordion with respect to time. In the right figure, we can also see the evolutions of the car speeds, respectively represented by the same colors.

4.3.2 Accident Phenomenon For The Linear Model

Simulation with the Linear Model For Two cars

For this case, the objectif is to show that if one driver have a bigger time reaction capacity, there is an accident. Onto this case, we took the same assumptions than before, but with a capacity of accelleration a little bit lower and a highter time reaction. In the algorithm 5 we took $\alpha_2 = 1.75$ and in the algorithm 4 we took $h = 1.5$

On the figure 5 accident is confirmed because there is a curve intersection (the position of the car number two (orange) is greater than the first car (blue)) So it means that the second car is crashed into the first car.

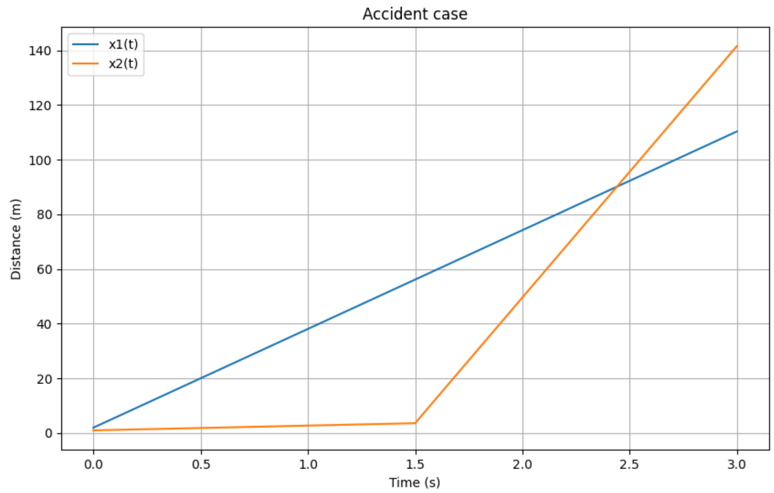

Figure 5: **Simulation of Accident case between two cars :** In this picture, you can observe the position evolutions of two cars, the first one (blue) maintaining a constant speed, and the second one (orange) following behind the first one. We could see on The graph shows that the acceleration of the orange car is too large. This implies an accident between two cars.

Simulation with the Newell's Model For Two cars

In the three subsequent figures 6, 7, and 8, we can observe a similar graph pattern to that of the linear model. However, in this scenario, we have the flexibility to vary additional parameters, such as the maximum speed of the cars (for instance, introducing a new car with a more powerful motor) and the safety distance, which is an inherent aspect of the Newell model. This variation allows us to witness accidents occurring at each curve intersection on the graph.

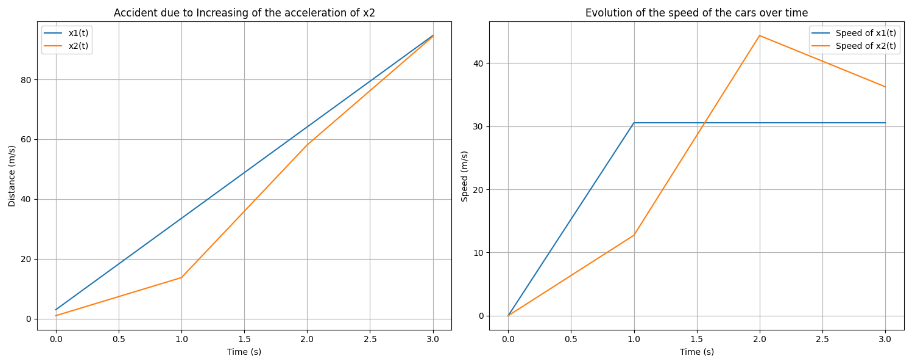

Figure 6: **Simulation of Accident case between two cars :** In this picture, you can observe the position evolutions of two cars, the first one (blue) maintaining a constant speed, and the second one (orange) following behind the first one. We could see on The graph shows that the acceleration of the orange car is too large. This implies an accident between two cars.

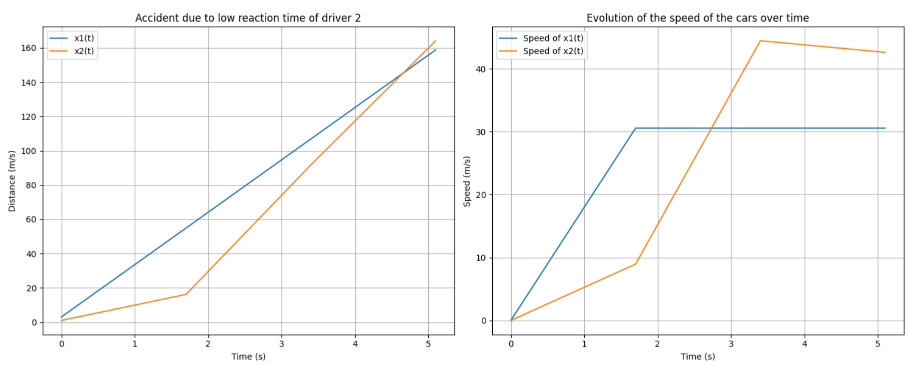

Figure 7: **Simulation of Accident case between two cars :** In this picture, you can observe the position evolutions of two cars, the first one (blue) maintaining a constant speed, and the second one (orange) following behind the first one. However the low reaction time of driver 2 implies an accident between two cars.

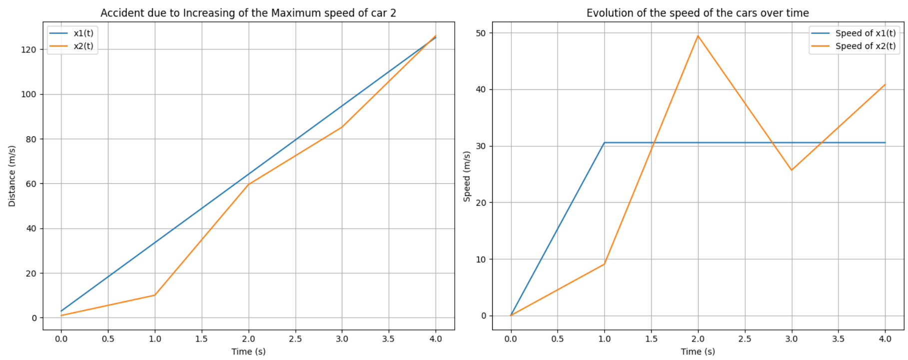

Figure 8: **Simulation of Accident case between two cars :** In this picture, you can observe the position evolutions of two cars, the first one (blue) maintaining a constant speed, and the second one (orange) following behind the first one. However the increasing of the maximum speed of car 2 implies an accident between two cars.

4.4 Study of Equilibrium, Stability, and Instability of the Solution

4.4.1 System Stability and Equilibrium for the Linear Model

In this part, the idea is to study the distance difference between two cars in order to determine in which case the solutions is stable and if there is an equilibrium for the solution. For doing that, we studied the solutions of the following system of equation :

$$\begin{cases} \dot{d_1} &= V_1 - \alpha_2 \cdot d_1 \\ & \vdots \\ \dot{d_n} &= \alpha_n \cdot d_{n-1} - \alpha_{n+1} \cdot d_n \end{cases}$$

So, In our context, for the two cars, we have a 1D system. With 3 cars, we have a 2D system of equations.

Now talk about the system stability and equilibrium :

With 2 cars :

So, the equation to study is the following one :

$$\dot{d_1} = V_1 - \alpha_2 \cdot d_1$$

After the resolution (6.1.1) we obtain the next equation for $d_1(t)$:

$$\boxed{d_1(t) = \frac{V_1 - V_1 \cdot e^{-\alpha_2(t-t_0)} + \alpha_2 \cdot d_1(t_0) \cdot e^{-\alpha_2(t-t_0)}}{\alpha_2}}$$

The first intuition before seeing the equation was to think that when the $\alpha_2 < 0$, the solution is divergent because one car is go back and the other go forward. Now, if we study the Equation,

the terms V_1 and $\alpha_2 \cdot d_1(t_0)$ are constant so they are not interesting.

However, it's important to note that both terms, $e^{-\alpha_2(t-t_0)}$, are exponential functions of time. If the value of α_2 is negative, these exponentials will increase exponentially over time.

So we could easily see that the equation is stable and converge onto the equilibrium $d_1(t) = \frac{V_1}{\alpha_2}$.

graphically we can see it by :

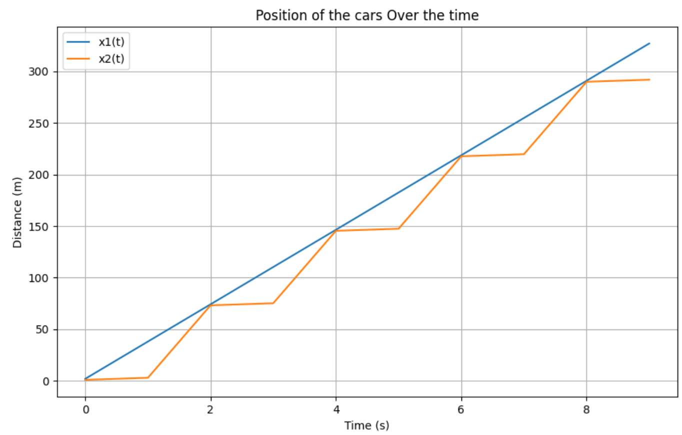

Figure 9: View of the old window of temrec3D

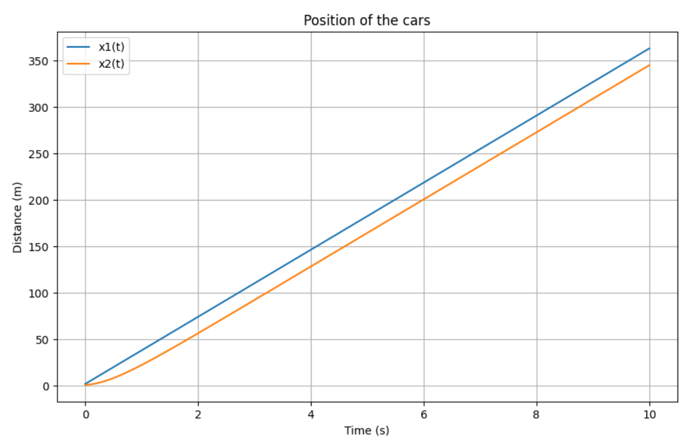

Figure 10: View of the new window of temrec3D

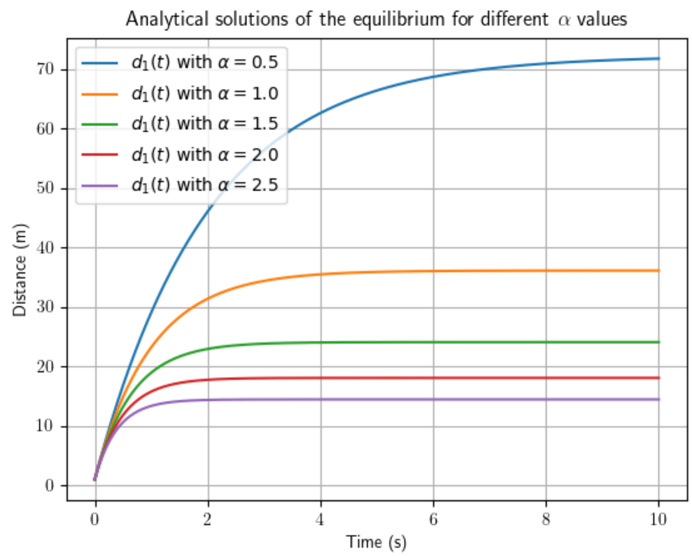

Figure 11: View of the old window of temrec3D

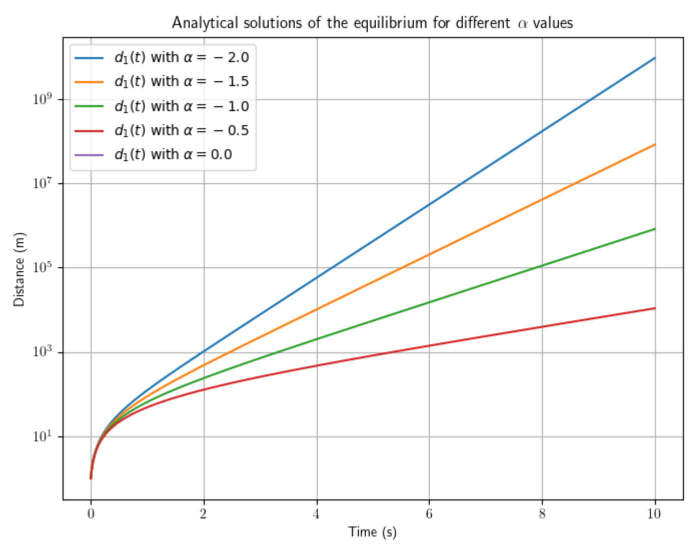

Figure 12: View of the new window of temrec3D

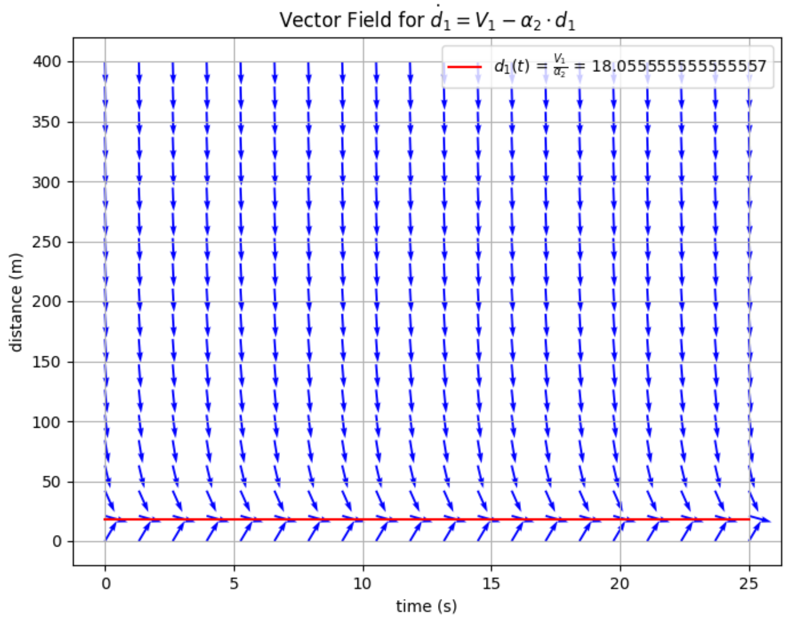

Figure 13: **Road traffic :** In this picture, you can see an example of a road traffic phenomenon that we could study

With 3 cars :

After the resolution (6.1.2) we obtain the next equation for $d_1(t)$ and $d_2(t)$:

4.4.2 System Stability and Equilibrium for the Newell's Model

The Study of the equilibrium for two cars :

The objective here, as before, is to study the stability of the model. The 1D equation is as follows:

$$\dot{d_1}(t) = \dot{x_1}(t) - \dot{x_2}(t) = V_1 - V_2 + V_2 \cdot e^{-\frac{\alpha_2}{V_2}(d_1 - d_2^{sec})} = f(d_1, t),$$
$$\begin{cases} V_1 := \text{Maximum speed for car number 1,} \\ V_2 := \text{Maximum speed for car number 2,} \\ \alpha_2 := \text{capacity of acceleration for car number 2,} \\ d_2^{sec} := \text{security distance that car number 2 maintains.} \end{cases}$$

Firstly, let's determine the equilibrium:

$$
\begin{aligned}
V_1 - V_2 + V_2 \cdot e^{-\frac{\alpha_2}{V_2}(d_1 - d_2^{sec})} &= 0 \\
e^{-\frac{\alpha_2}{V_2}(d_1 - d_2^{sec})} &= \frac{V_2 - V_1}{V_2} \\
e^{-\frac{\alpha_2}{V_2}d_1} &= \frac{V_2 - V_1}{V_2} e^{-\frac{\alpha_2}{V_2}d_2} \\
-\frac{\lambda_2}{V_2} d_1 &= \ln\left(\frac{V_2 - V_1}{V_2} e^{-\frac{\alpha_2}{V_2}d_2}\right) \\
d_1^* &= -\frac{V_2}{\lambda_2} \ln\left(\frac{V_2 - V_1}{V_2} e^{-\frac{\alpha_2}{V_2}d_2}\right)
\end{aligned}
$$

The challenge with this equation is the inability to find an analytical solution. Consequently, we apply Lyapunov's Indirect Theorem to investigate the equilibrium.

In the context of 1D analysis, we begin by calculating $f'(d_1, t)$, which results in:

$$f'(d_1, t) = -\alpha_2 e^{-\frac{\alpha_2}{V_2}(d_1 - d_2^{sec})}$$

It is evident that $f'(d_1, t)$ is negative for all values of α_2 greater than zero. Therefore, in our specific scenario, the equilibrium remains stable at all times, as negative acceleration is not considered.

From the equilibrium, we also observe that the equilibrium exists if and only if $V_2 > V_1$.

Clearly, the condition for a stable equilibrium is $\alpha_2 > 0$ and $V_2 > V_1$.

The Study of the equilibrium for three cars :

$$
\begin{cases}
\dot{d_1}(t) = \dot{x_1}(t) - \dot{x_2}(t) = V_1 - V_2 + V_2 \cdot e^{-\frac{\alpha_2}{V_2}(d_1 - d_2^{sec})} = f(d_1, t), \\
\dot{d_2}(t) = \dot{x_2}(t) - \dot{x_3}(t) = V_2 - V_2 \cdot e^{-\frac{\alpha_2}{V_2}(d_1 - d_2^{sec})} - V_3 + V_3 \cdot e^{-\frac{\alpha_3}{V_3}(d_2 - d_3^{sec})}
\end{cases}
$$

$$
\begin{cases}
V_1 := \text{Maximum speed for car number 1}, \\
V_2 := \text{Maximum speed for car number 2}, \\
\alpha_2 := \text{capacity of acceleration for car number 2}, \\
d_2^{sec} := \text{security distance that car number 2 maintains.}
\end{cases}
$$

again, we apply Lyapunov's Indirect Theorem to investigate the equilibrium.

We are going to calculkate the Jacobienne matrix :

$$
J_{\bar{x}} = \begin{pmatrix}
-\alpha_2 e^{-\frac{\alpha_2}{V_2}(d_1 - d_2^{sec})} & 0 \\
\alpha_2 e^{-\frac{\alpha_2}{V_2}(d_1 - d_2^{sec})} & -\alpha_3 e^{-\frac{\alpha_3}{V_3}(d_2 - d_3^{sec})}
\end{pmatrix}
$$

However, in the 2D case, Lyapunov tells us that the equilibrium is stable if:

$$
\boxed{\begin{array}{c} Tr(J_{\bar{x}}) < 0 \\ \det(J_{\bar{x}}) > 0 \end{array}}
$$

The both condition are respected iff $\lambda_2 > 0, \lambda_3 > 0$

5 Summary

6 Annexe

6.1 Calculation of the Analytical solutions

6.1.1 Linear Model for Two Cars

$$\dot{d}_1(t) = V_1 - \alpha_2 \cdot d_1$$

We use the variable separation method to Solve this EDO, so we obtain :

$$\begin{aligned}
&\int_{x(t_0)}^{x(t_f)} \frac{1}{V_1 - \alpha_2 \cdot d_1(t)} d\ d_1(t) = \int_{t_0}^{t_f} dt \\
&-\frac{1}{\lambda_2} \left[\ln |V_1 - \alpha_2 \cdot d_1(t)|\right]_{x(t_0)}^{x(t_f)} = t_f - t_0 \\
&-\frac{1}{\lambda_2} \ln \left| \frac{V_1 - \alpha_2 \cdot d_1(t_f)}{V_1 - \alpha_2 \cdot d_1(t_0)} \right| = t_f - t_0
\end{aligned}$$

We could remove the $|.|$ because the sign is always the same. And we get :

$$\begin{aligned}
&\ln \left(\frac{V_1 - \alpha_2 \cdot d_1(t_f)}{V_1 - \alpha_2 \cdot d_1(t_0)} \right) = -\lambda_2(t_f - t_0) \\
&\frac{V_1 - \alpha_2 \cdot d_1(t_f)}{V_1 - \alpha_2 \cdot d_1(t_0)} = e^{-\lambda_2(t_f - t_0)} \\
&V_1 - \alpha_2 \cdot d_1(t_f) = (V_1 - \alpha_2 \cdot d_1(t_0))e^{-\lambda_2(t_f - t_0)} \\
&\boxed{d_1(t) = \frac{V_1 - [V_1 - \alpha_2 \cdot d_1(t_0)]e^{-\lambda_2(t_f - t_0)}}{\alpha_2}}
\end{aligned}$$

6.1.2 Linear Model for Three Cars

The system under matricial form coul be write like that :

$$\dot{D} = (t) \begin{pmatrix} -\alpha_2 & 0 \\ \alpha_2 & -\alpha_3 \end{pmatrix} \begin{pmatrix} d_1 \\ d_2 \end{pmatrix} + \begin{pmatrix} V_1 \\ 0 \end{pmatrix}$$

The first step is to perfom the eigenvectors of the matrix:

$$\begin{vmatrix} X + \alpha_2 & 0 \\ \alpha_2 & X + \alpha_3 \end{vmatrix} = (X + \alpha_2)(X + \alpha_3 = X^2 + X(\alpha_3 + \alpha_2) + \alpha_2\alpha_3$$

by simple resolution of the second order polynomial, we could fin two eigenvalues: $\lambda_1 = -\alpha_2, \lambda_2 = -\alpha_3$ we looking for the two eigen vectors :

$$\begin{cases} \left[\begin{array}{cc|c} 0 & 0 & 0 \\ \alpha_2 & -\alpha_3+\alpha_2 & 0 \end{array}\right] \\ \\ \left[\begin{array}{cc|c} -\alpha_2+\alpha_3 & 0 & 0 \\ \alpha_2 & 0 & 0 \end{array}\right] \end{cases}$$

It is easy to see that the two following vector are vector that verify the condition :

$$\begin{cases} u_1 = \begin{pmatrix} 1 \\ \frac{\alpha_3-\alpha_2}{\alpha_2} \end{pmatrix} \\ \\ u_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{cases}$$

Then, we know that a general solution of the equation is :

$\bar{D}(t) = \sum_{i=1}^{2} X_i$
where $X_i = C_i e^{\lambda_i t} u_i$,
with C_i being constants to be determined, and u_i as the eigenvectors, **and**λ_i**the eigenvalues**.

So :

$$X_1 = \begin{pmatrix} \frac{C_1(\alpha_3-\alpha_2)}{\alpha_2 e^{\alpha_2 t}} \\ \\ \frac{C_1}{e^{\alpha_2 t}} \end{pmatrix}$$

$$X_2 = \begin{pmatrix} 0 \\ \frac{C_2}{e^{\alpha_3 \cdot t}} \end{pmatrix}$$

Finally the general solution is given by :

$$\bar{X} = \begin{pmatrix} \frac{C_1(\alpha_3-\alpha_2)}{\alpha_2 e^{\alpha_2 t}} \\ \\ \frac{C_1}{e^{\alpha_2 t}} + \frac{C_2}{e^{\alpha_3 \cdot t}} \end{pmatrix}$$

Then we are going to find for particular solution :

$$\begin{cases} \dot{d}_1 &= 0 \\ \dot{d}_2 &= 0 \end{cases} \iff \begin{cases} d_1^* &= \frac{V_1}{\alpha_2} \\ d_2^* &= \frac{V_1}{\alpha_3} \end{cases}$$

In clear, we have :

$$\begin{cases} \dot{d}_1(t) & = \frac{C_1(\alpha_3-\alpha_2)}{\alpha_2 e^{\alpha_2 t}} + \frac{V_1}{\alpha_2} \\ \dot{d}_2(t) & = \frac{C_1}{e^{\alpha_2 t}} + \frac{C_2}{e^{\alpha_3 \cdot t}} + \frac{V_1}{\alpha_3} \end{cases}$$

we set the initial condition to
$d_1(0) = m, d_2(0) = n$ So we find easily $C_1 and C_2$:

$$C_1 = \frac{m\alpha_2 - V_1}{\alpha_3 - \alpha_2}$$

$$C_2 = n - \frac{V_1}{\alpha_3} - C_1$$

References

[1] Mansoureh Jeihani, Shiva NarooieNezhad and Kaveh Bakhsh Kelarestaghi. "Integration of a driving simulator and a traffic simulator case study: Exploring drivers' behavior in response to variable message signs". In: *IATSS Research* 41.4 (2017), pp. 164–171. ISSN: 0386-1112. DOI: 10.1016/j.iatssr.2017.03.001. URL: https://www.sciencedirect.com/science/article/pii/S0386111217300304.