Applied Numerical Methods - Computer Lab 1

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Part 1. Solution of ODE-systems with constant coefficients

Starting from the described electrical circuit, we get the differential equation

$$L\ddot{q} + R\dot{q} + \frac{1}{C}q = E, \quad q(0) = 0, \quad \dot{q}(0) = 0,$$

after defining the variable $\dot{q} = i$. We easily rewrite this as a first order system of linear equations. Setting

$$\mathbf{y} = \begin{pmatrix} y(1) \\ y(2) \end{pmatrix} = \begin{pmatrix} q \\ \dot{q} \end{pmatrix},$$

this gives

$$\begin{pmatrix} \dot{y}(1) \\ \dot{y}(2) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{1}{LC} & -\frac{R}{L} \end{pmatrix} \begin{pmatrix} y(1) \\ y(2) \end{pmatrix} + \begin{pmatrix} 0 \\ E \end{pmatrix}.$$

Lets define the matrix

$$A = \begin{pmatrix} 0 & 1 \\ -\frac{1}{LC} & -\frac{R}{L} \end{pmatrix}$$

for some parameters R, L, C.

This problem is linear and the stability analysis of the problem is therefore quite straightforward. The problem will be stable if the real part of all the eigenvalues of A are negatives. Assume any combination of R, L, C that are all positives, we have $\det(A) = \lambda_1\lambda_2 = \dfrac{1}{LC} > 0$ and $\mathrm{trace}(A) = \lambda_1 + \lambda_2 = -\frac{R}{L} < 0$. This clearly ensures that all eigenvalues are negative and guaranties stability.

Below is our Matlab code for this question. The chosen value of E was 1 because we can see in the analytical solution that it is just a scaling factor.

The solution of the equation is shown on figure 1 for al the values of R. As we saw, the system is stable for all positive values of R, L, C. For small values of R, we can see that the system is oscillating with decreasing amplitude. As R increases, the system reaches the equilibrium faster. We can note that for $R = 10$, the system behaves like an RC-system. This happens because the inductance becomes neglectable due to the small current.

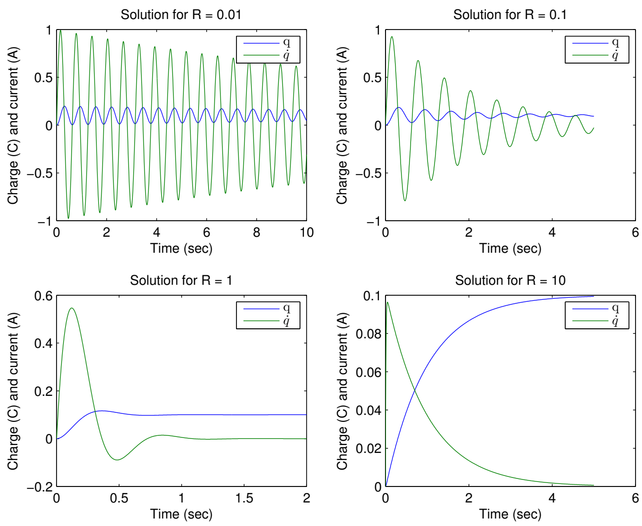

Figure 1: Solutions of the system in part 1 for various values of R

```
% LAB1_1.m - Script for question 1
% Authors: Florentin Goyens and David Weicker

close all;
L = 0.1;
C = 0.1;
R = [0.01 0.1 1 10];
E = 1;
g = [0 E/L]';
N = 1000;
tfinal = [10 5 2 5];
y = zeros(2,N+1); % Preallocation and initial condition

tit = {'Solution for R = 0.01','Solution for R = 0.1','Solution for R = 1','Solution ↩
    for R = 10'};

for i=1:4 % For each value of R
    h = tfinal(i)/N;
    t = 0:h:tfinal(i);
    A = [0 1;-1/(L*C) -R(i)/L];
    for j=2:N+1
        y(:,j) = A\((expm(A*t(j))-eye(2))*g); % Using (2.26) in notes and zero initial↩
            condition
    end
    subplot(2,2,i);
    plot(t,y);title(tit{i}); xlabel('Time (sec)');ylabel('Charge (C) and current (A)')↩
        ; hl=legend('q','$\dot{q}$');
    set(hl, 'Interpreter', 'latex');
end
```

Part 2. Stability of ODE-systems and equilibrium points

a. Stability of the solutions of an ODE-system of LCC-type

The purpose of this section is to investigate the stability of an ODE while a parameter changes continuously.

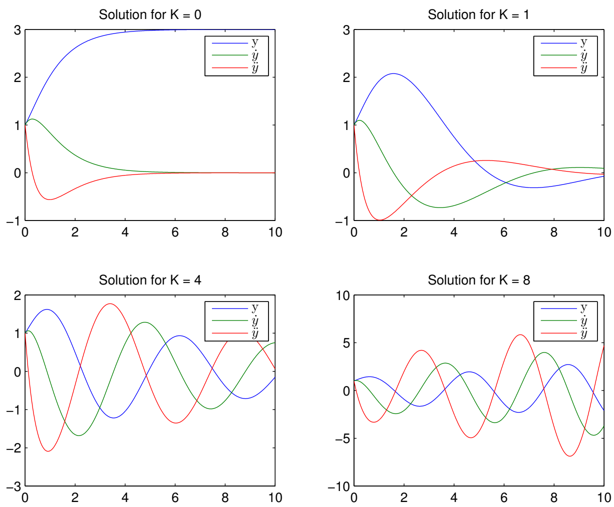

Figure 2: Solutions of the system in part 2a for various values of K

The given third-order equation is the following :

$$\begin{aligned} y''' + 3y'' + 2y' + Ky = &\ 0 \\ y(0) = &\ 1 \\ y'(0) = &\ 1 \\ y''(0) = &\ 1 \end{aligned}$$

It is possible to rewrite this as a system of ODE's, introducing new variables.

$$\mathbf{u'} = \begin{pmatrix} y' \\ y'' \\ y''' \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -K & -2 & -3 \end{pmatrix} \begin{pmatrix} y \\ y' \\ y'' \end{pmatrix} = \mathbf{Au}$$

The initial conditions are rewritten by :

$$\mathbf{u(0)} = \begin{pmatrix} y(0) \\ y'(0) \\ y''(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

The solutions, computed analytically for different values of K, are given in figure 2. The Matlab code is given at the end of this section. It is possbile to see that for the first three values of K, the maximal amplitude of the blue curve (that is the function y) tends to decrease. This being an LCC system, we know that a perturbation will follow the same ODE so we can conclude that the system is stable for those values of K. On the other hand, for the last value ($K = 8$), this amplitude increases. Hence, the system is not stable for $K = 8$. By continuity, that means that the system becomes unstable for a value of K between 4 and 8. Looking at the graphs, we can say that this value is probably arround 6.

We then drew a root locus to see how the system would evolve if K varies continuously. Again, the code is given at the end of the section. We used the built-in function *rlocus*. In order to do so, we first have to compute the transfert function of the process that is equivalent to our differential equation with an unit feedback with gain K. One can show that this transfert function $H(s)$ is :

$$H(s) = \frac{1}{s^3 + 3s^2 + 2s}$$

The figure 3 shows the rootlocus. The discrete values of K were chosen so that the curve would be smooth. We can see that as K increases, one eigenvalue is getting more and more negative while the two others first are getting closer then leave the real axis. We can also see that at some point, those two eigenvalues cross the imaginary axis thus making the system unstable.

The next step was to compute the first value of K making the system unstable. It is possible to check each time we increase K if the eigenvalue have a positive real part but the result is only an approximation. It is also possible to compute this value analytically and that is what we did. By continuity, we know that for this value, two eigenvalues are located on the imaginary axis so :

$$\lambda = xj$$

With x a real number. We then plug that into the caracteristic equation.

$$\begin{aligned} \lambda^3 + 3\lambda^2 + 2\lambda + K &= 0 \\ -x^3 j - 3x^2 + 2xj + K &= 0 \end{aligned}$$

Two complex numbers are equal if and only if their real part and their imaginary part are equal. This yields :

$$\begin{pmatrix} K - 3x^2 \\ 2x - x^3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

We have the trivial solution ($x = K = 0$) but we are not interested in that one. The two others are :

$$\begin{aligned} x &= \pm\sqrt{2} \\ K &= 6 \end{aligned}$$

So we can conclude that :

$$\textit{The system is unstable} \iff K > 6$$

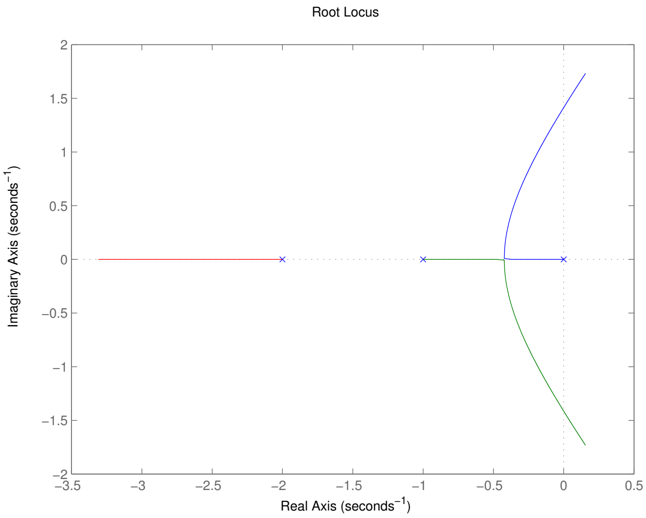

Figure 3: Root locus of $y''' + 3y'' + 2y' + Ky = 0$

```
function [] = LAB1_21()
%LAB1_21
%Matlab code for part 2a
%Author : David Weicker and Florentin Goyens
close all;
K = [0 1 4 6];
u0 = [1 1 1]';
N = 100;
u = [u0 zeros(3,N)];
tfinal = 10;
h = tfinal/N;
t=0:h:tfinal;

for i=1:4
    A=[0 1 0;0 0 1;-K(i) -2 -3];
    for j=2:N+1
        u(:,j) = expm(A*t(j))*u0;
    end
    subplot(2,2,i);
    string = sprintf('Solution for K = %d',K(i));
    plot(t,u);hl=legend('y','$\dot{y}$','$\ddot{y}$');title(string);
    set(hl, 'Interpreter', 'latex');
end

%root locus
sys = tf(1,[1 3 2 0]);
figure;
k = 0:0.005:10;
rlocus(sys,k);
end
```

b. Stability of the critical points of a nonlinear ODE-system

In this section, we will look at the stability. As this sytem is nonlinear, we have to analyse the stability of the critical points with the help of the jacobian. For this system, the jacobian is analytically given by :

$$D_f(\mathbf{u}) = \begin{pmatrix} 5-u_3 & 4 & -u_1 \\ 1 & 4-u_3 & -u_2 \\ 2u_1 & 2u_2 & 0 \end{pmatrix}$$

This jacobian is a function of $\mathbf{u}$ but we are only interested in the stability of the critical points. So we only have to check the eigenvalues of $D_f(\mathbf{u}^*)$ where $\mathbf{u}^*$ is a solution of the equation $f(\mathbf{u}) = 0$.

So we first have to compute all the critical points. There are four of them. The Matlab code shown at the end of this section is used to do that. We have used the built-in function $fsolve$. This give the following critical points, numeroted $\mathbf{u}^i$:

$$\mathbf{u}^1 = \begin{pmatrix} 7.9446 \\ -5.0876 \\ 2.4384 \end{pmatrix}$$

$$\mathbf{u}^2 = \begin{pmatrix} -7.9446 \\ 5.0876 \\ 2.4384 \end{pmatrix}$$

$$\mathbf{u}^3 = \begin{pmatrix} 8.7881 \\ 3.4308 \\ 6.5616 \end{pmatrix}$$

$$\mathbf{u}^4 = \begin{pmatrix} -8.7881 \\ -3.4308 \\ 6.5616 \end{pmatrix}$$

The jacobian of the preceding critical points have the following eigenvalues :

	$\Lambda_D(\mathbf{u}^1)$	$\Lambda_D(\mathbf{u}^2)$	$\Lambda_D(\mathbf{u}^3)$	$\Lambda_D(\mathbf{u}^4)$
λ_1	$13.3417\jmath$	$13.3417\jmath$	$13.3417\jmath$	$13.3417\jmath$
λ_2	$-13.3417\jmath$	$-13.3417\jmath$	$-13.3417\jmath$	$-13.3417\jmath$
λ_3	4.1231	4.1231	-4.1231	-4.1231

We can see that the two first critical points ($\mathbf{u}^1$,$\mathbf{u}^2$) are unstable because there are each time one eigenvalue with a stricly positive real part.

The last two ($\mathbf{u}^3$,$\mathbf{u}^4$) however are stable because there is no eigenvalue with a stricly positive real part and the ones with a real part equal to zero are simple.

```
function [sol,lambda] = LAB1_22()
%LAB1_22
%Matlab code for part 2b
%Author : David Weicker and Florentin Goyens
close all;

approx = [8 -5 2; -8 5 2; 9 3 7; -9 -3 7];
sol = zeros(3,4);
lambda = zeros(3,4);
for i=1:4
    sol(:,i) = fsolve(@twoB,approx(i,:))';
    jacobian = [5-sol(3,i) 4 -sol(1,i);1 4-sol(3,i) -sol(2,i);2*sol(1,i) 2*sol(2,i) ↩
        0]; %analytical jacobian
    lambda(:,i) = eig(jacobian);
end
```

```
end

function dudx = twoB(u)
%Computes derivative of u in part 2b
dudx = u;
dudx(1) = 5*u(1) + 4*u(2) - u(1)*u(3);
dudx(2) = u(1) + 4*u(2) - u(2)*u(3);
dudx(3) = u(1)*u(1) + u(2)*u(2) - 89;
end
```