Applied Numerical Methods - Lab 3

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Stationary heat conduction in 1-D

In a one dimensional pipe we are interested in the temperature evolution along the z-axis. We will study the behaviour of the numerical solution based on finite differences.

1) Distretization to a linear system of equations

The z-axis is first discretized in $N+1$ points spreading from 0 to L. The differential equation for the temperature is the following

$$-\kappa \frac{d^2T}{dz^2} + v\rho C \frac{dT}{dz} = Q(z)$$

with

$$Q(z) = \begin{cases} 0 & if \quad 0 \leq z < a \\ Q_0 \sin\left(\frac{z-a}{b-a}\pi\right) & if \quad a \leq z \leq b \\ 0 & if \quad b \leq z \leq L. \end{cases}$$

We assumed that κ as a constant value through the pipe.

The boundary conditions are

$$T(0) = T_0$$

and

$$-\kappa \frac{dT}{dz}(L) = k_v(T(L) - T_{out}).$$

For the discretization, let $T_0, T_1, \ldots, T_N$ be the unknown temperature we will solve for. The step is $h = L/N$. Define $z_i = i * h$ and $T_i \approx T(z_i)$. It is clear that $z_0 = 0$ and $z_N = L$. We also define the ghost point $z_{N+1} = L + h$ that will be used implicitly to write the system. The real unknowns we will include in the system are $T_1, \ldots, T_N$. Obviously T_0 is known from the start. We therefore need N linearly independent equations.

The figure 1 illustrates the discretization and the usage of ghost points depending on the boundary conditions.

Figure 1: Discretization and ghost points.

With finite difference approximation we rewrite the problem as

$$-\kappa\frac{T_{i+1}-2T_i+T_{i-1}}{h^2}+v\rho C\frac{T_{i+1}-T_{i-1}}{2h}=Q(z_i),$$

for all the points, $i=1,\dots N$. This is equivalent to

$$\underbrace{\Big(-\frac{v\rho Ch}{2}-\kappa\Big)}_{\alpha}T_{i-1}+\underbrace{\Big(2k\Big)}_{\beta}T_{i-1}+\underbrace{\Big(\frac{v\rho Ch}{2}-\kappa\Big)}_{\gamma}T_{i+1}=h^2Q(z_i),$$

The previous are N linear equations that feature the $N+2$ variables $T_0, T_1, \dots, T_{N+1}$. The first boundary condition gives the straightforward $T_0=400$ which removes T_0 from the problem.

We also have

$$-\kappa\frac{T_{N+1}-T_{N-1}}{h}=k_v(T_N-T_{out}).$$

This allows to express T_{N+1} with other variables and remove it from the system of equations. We will then have N unknowns in our system of N difference equations.

$$T_{N+1}=T_{N-1}-\underbrace{\frac{2hk_v}{k}}_{\delta}T_N+\underbrace{\frac{2hk_v}{k}}_{\delta}T_{out}. \tag{1}$$

The final difference equation is the only one where T_{N+1} appears. That is

$$\alpha T_{N-1}+\beta T_N+\gamma T_{N-1}=h^2Q(z_N).$$

We remove T_{N+1} using equation 1 and this yields

$$(\alpha+\gamma)T_{N-1}+(\beta-\gamma\delta)T_N+\gamma\delta T_{out}=h^2Q(z_N).$$

We now have a linear system of N equations. This system is tridiagonal and the Matlab *band solver* will be very efficient is the matrix is defined as sparse.

2) Convergence of solution without convection

We now set $v=0$ and solve the equation for increasing values of N. Results on figure 2.

We see that increasing N has a great effect on the quality of the solution. Furthermore the solution corresponds visually well to the impulse $Q(z)$.

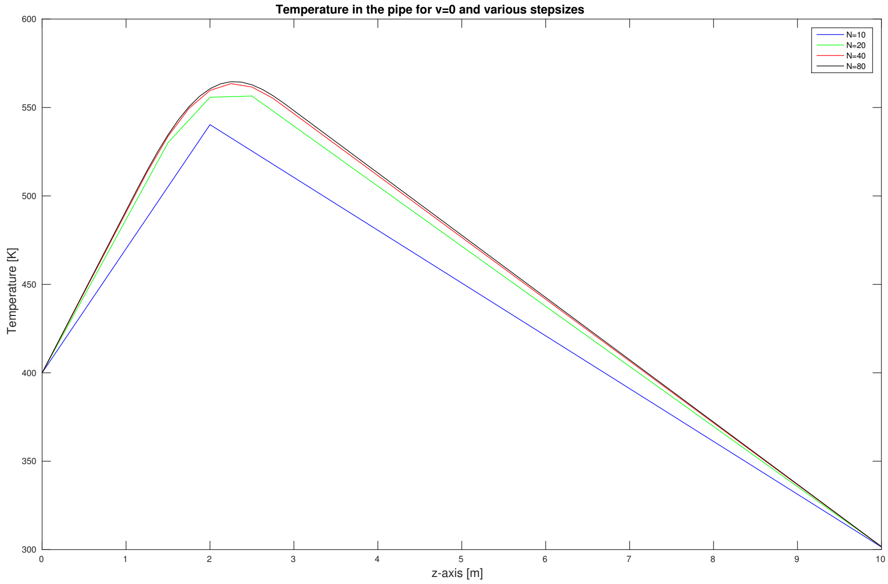

Figure 2: Convergence of solution without convection.

3) Increasing speed for $N = 40$

Let us set $N = 40$. We will now have v vary and take the values $0.1, 0.5, 1, 10$. Results on figure 3.

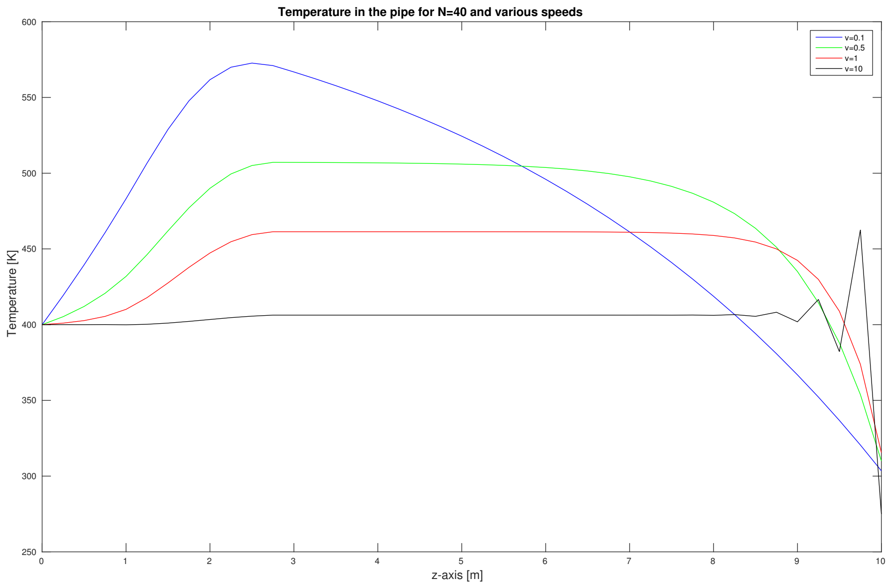

Figure 3: Solution with increasing speed v.

We clearly notice that oscillations occur when $v = 10$.

As we want to see better how the oscillation appears, we increase the precision with N for v fixed at 10. Results on figure 4.

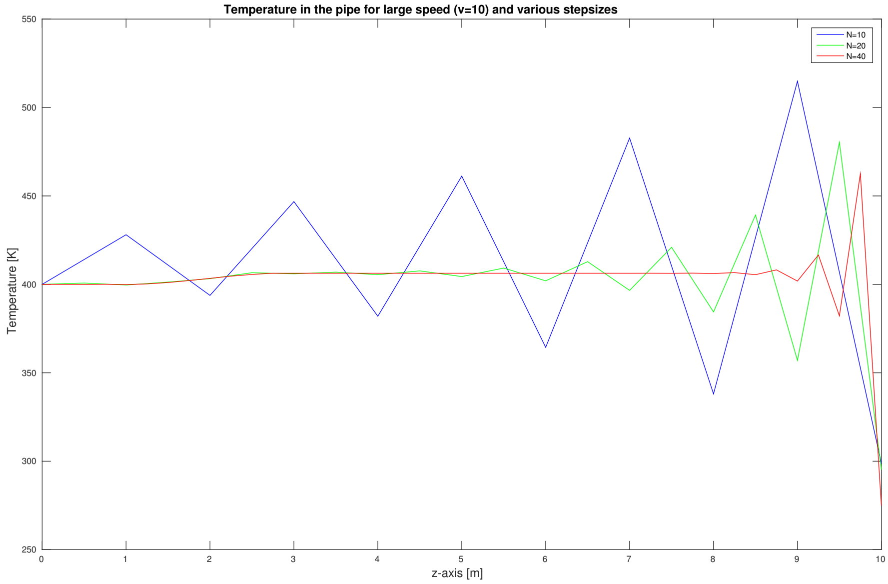

Figure 4: Capture the oscillation with increasing N.

As the level of the discretization increases with N, we see more and more oscillation.