COMPUTER LAB 6 SUPPORTING CHAPTER 8

Numerical experiments with the hyperbolic model PDE-problem

Given the model problem for a hyperbolic PDE

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0, \quad a > 0, \quad 0 < x < 2, \quad t > 0$$

with initial condition $u(x,0) = 0, \quad 0 < x \leq 2$. As for the boundary condition at $x = 0$ assume a square wave is entering from the left, i.e.

$$u(0,t) = \begin{cases} 1, & -(n+1)\frac{T}{2} < t \leq -n\frac{T}{2}, \quad n = 0, 2, 4, .. \\ -1, & -(n+1)\frac{T}{2} < t \leq -n\frac{T}{2}, \quad n = 1, 3, 5, .. \end{cases}$$

where T is the period time.

Make a numerical experiment showing how the 1) upwind, 2) Lax-Friedrich and 3) Lax-Wendroff methods behave on this PDE-problem. Discretize the x-interval into N equidistant subintervals and define gridpoints $x_i = ih_x, i = 0, 1, 2, \ldots, N$, where $h_x = 2/N$. The Courant number σ is $\sigma = ah_t/h_x$.

A) Write programs and run the three methods on the time interval $(0, 2T)$. Let $N = 100$, $T = 1$ and $a = 1$. Present the results in graphs with $u(x, 2T)$ as a function of x. Show graphs for the σ-values $\sigma = 0.8$, $\sigma = 1$ and $\sigma = 1.1$. In all there are nine graphs to be presented. Are your experimental results in agreement with the theoretical stability results? Which methods will smooth the solution and which will introduce spurious oscillations?

B) In this part the exchanger in Example 8.1 is studied. A fluid of temperatute $T(x,t)$ is flowing with constant speed v in a pipe. Outside the pipe there is a cooling medium that keeps a constant low temperature T_{cool}. The temperature of the fluid in the pipe is initially cool, i.e. $T = T_{cool}$ but within a short time period hot fluid enters the pipe. The task is to study how the temperature $T(x,t)$ of the fluid in the pipe depends on x and t.

The following PDE-model is given:

$$\frac{\partial T}{\partial t} + v\frac{\partial T}{\partial x} + a(T - T_{cool}) = 0, \quad 0 < x < L, \quad t > 0$$

with the initial condition

$$T(x,0) = T_{cool}$$

and the boundary condition

$$T(0,t) = \begin{cases} \mathrm{T}_{cool} + (T_{hot} - T_{cool})sin(\pi t) & 0 \leq t \leq 0.5 \\ \mathrm{T}_{hot} & 0.5 \leq t \leq 4 \\ \mathrm{T}_{hot} + T_{cool}sin(\pi(t-4)) & t > 4 \end{cases}$$

The length of the heat exchanger is $L = 3$. The heat exchange parameter $a = 0.1$, the velocity of the fluid is $v = 1$, the cooling temperature $T_{cool} = 5$ and the hot temperature is $T_{hot} = 100$.

a) Use the upwind method to simulate the temperature in the pipe. Choose suitable stepsizes h_x and h_t and present the result in a 3D graph.

b) Work out a difference formula of Lax-Wendroff type that solves the PDE-problem in B). Present the result as a difference formula of the type

$$u_{i,k+1} = c_1 u_{i-1,k} + c_2 u_{i,k} + c_3 u_{i+1,k} + c_4$$

where the coefficients c_1, c_2, c_3 and c_4 are to be determined by you.

c) Program the Lax-Wendroff formula in b), run it for a suitable choice of h_x and h_t and present the result in a 3D plot. Make a comparison with the graph from a).