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Applied Numerical Methods, SF2520, Numerical Algebra part

Lab 8, SVD-factorization and Integral Equation

A. SVD-factorization

Singular Value Decomposition, SVD, of an $m \times n$ matrix A is defined as $A = USV^T$, where U is an $m \times m$ orthogonal matrix, v is an $n \times n$ orthogonal matrix and S is an $m \times n$ matrix built up by a diagonal matrix and a matrix with zeroes.

The diagonal elements of the diagonal matrix are called the singular values s_i, $i = 1, 2 \ldots, k$ where $k = min(m, n)$. They fulfill $s_1 \geq s_2 \geq ... \geq s_k \geq 0$.

If $m > n$ the zero matrix is situated under the $n \times n$ diagonal matrix denoted by S_n (the last element is the singular value s_n). If $m < n$ the zero matrix is situated to the right of the diagonal matrix S_m. If $m = n$ all matrices are $n \times n$.

For an underdetermined system $A\mathbf{x} = \mathbf{y}$ where A is $m \times n$ with $m < n$ there is no unique solution $\mathbf{x}$. If we add the constraint that $\|x\|_2$ should be as small as possible the solution $\hat{\mathbf{x}}$ will be unique.

To see how $\hat{\mathbf{x}}$ is computed look at the norm of the residual

$$\|\mathbf{y} - A\mathbf{x}\|_2^2 = \|\mathbf{d} - S\mathbf{z}\|_2^2$$

where $\mathbf{d} = U^T\mathbf{y}$ is a known vector with m components and $\mathbf{z} = V^T\mathbf{x}$ has n components of unknowns. The residual is zero for $z_i = d_i/s_i, \quad i = 1, 2, \ldots, m$. But $\mathbf{z}$ has n components, how should the remaining $z_{m+1}, \ldots, z_n$ be chosen? Since

$$\|\mathbf{x}\|_2^2 = \|V\mathbf{z}\|_2^2 = \|\mathbf{z}\|_2^2 = (\frac{d_1}{s_1})^2 + (\frac{d_2}{s_2})^2 + \ldots + (\frac{d_m}{s_m})^2 + z_{m+1}^2 + \ldots + z_n^2$$

we get the smallest norm solution by putting $z_{m-1}, \ldots, z_n$ to zero and then let $\mathbf{x} = V\mathbf{z}$.

When the SVD for an underdetermined system is truncated to a numerical rank r, where $r < m$, all singular values $s_{r+1}, \ldots s_m$ are set to zero. To get the solution we proceed in the same way as above, i.e. the solution $z_i = d_i/s_i, \quad i = 1, 2, \ldots, r$, the remaining $z_{r+1}, \ldots, z_n$ are set to zero. The minimum norm solution $\hat{\mathbf{x}}$ will be smaller since

$$\|\mathbf{x}\|_2^2 = \|V\mathbf{z}\|_2^2 = (\frac{d_1}{s_1})^2 + (\frac{d_2}{s_2})^2 + \ldots + (\frac{d_r}{s_r})^2$$

is smaller the smaller the rank r is.

In MATLAB the singular value decomposition of A is obtained from `[U,S,V]=svd(A)`.

Numerical solution of Fredholm's integral equation

This computer lab is borrowed from a kompendium by Gerd Eriksson, teacher and collegue of mine for many years at Numerical Analysis.

For many measuring instruments the behaviour is modeled by an integral equation named *Fredholm's integral equation of the first kind*:

$$\int_{-\infty}^{\infty} K(x, y)p(x)dx = f(y)$$

Here $p(x)$ is the solution function we want to compute. We know in advance that $p(x)$ is zero outside a known interval $a < x < b$, hence we solve

$$\int_a^b K(x, y)p(x)dx = f(y)$$

$K(x,y)$ is a known *apparatus function* (K as in kernel). The right hand side consists of observed values of $f(y)$, $(y_i, f(y_i))$, $i = 1, 2, .., m$.

In the ideal case $K(x,y)$ is the deltafunction, in which case the wanted parameterfunction p is identical with the observed function f. Here we assume that the function $K(x,y)$ is given by the following expression

$$K(x,y) = \frac{1}{2\beta}(1 + \cos\frac{\pi(y-x)}{\beta}), \quad |y-x| < \beta \quad and \quad K(x,y) = 0 \quad otherwise$$

If the β-value is large the measuring instrument is not very good, the measured values are smeared out giving bad information about p.

Since measured data consists of m observed values $f_i = f(y_i)$ we can write the integral equation on the form

$$\int_a^b K(x, y_i)p(x)dx = f_i, \quad i = 1, 2, \ldots, m$$

We now have m equations to determine the unknown function $p(x)$. The integral is approximated by the trapezoidal rule, where the interval $[a, b]$ is divided into N subinterval of size h, i.e. $Nh = b - a$.

The discretized integral equation is written as a linear system of equations

$$A\mathbf{p} = \mathbf{f}$$

where A is an $m \times n$ matrix, The system can be over- or underdetermined. The more intervals n in the trapezoidal rule, the better is the integral approximation. However, the system of equations will be large and more underdetermined. In many applications the system of equations is ill-conditioned.

Solve the problem by making an SVD-factorization of A. With a truncated SVD we try to find a good solution by inspecting the graphs of p. If the graph contains many oscillations, the corresponding truncation takes too many singular values into account and should be rejected. On the other hand the choice of the "best" graph is a subjective choice.

In this lab we use 36 measured points in the interval $[0, 6]$, obtained with the kernel $K(x,y)$ with $\beta = 1.5$ applied to the function $p(x)$. We know that $p(x) \geq 0$ in the interval. In the trapezoidal rule use 60 intervals giving a 36×59 matrix. Hence the number of singular values are 36 of which many are small.

Compute the solution candidates of $p(x)$ for different degree of truncated SVD. Plot the given measured values $f(y_i)$, $\quad i = 1, 2, \ldots, m$, and a representative number of the solution function $p(x)$. Give a comment on the candidate you choose and the numerical rank this corresponds to.

Simulate the measured values f by generating yourself the 36 values from

$$p(x) = 0.8\cos(\pi x/6) - 0.4\cos(\pi x/2) + 1, \quad 0 \leq x \leq 6$$

This is in fact cheating but on the other hand you have full control of the solution p computed from the integral equation and the true solution. Plot the true solution with the candidate you have chosen. To make the problem a little more realistic add noise to your measured values by adding to $f(y_i)$ small random values `f=f+0.01*randn(1,length(y))`.