

Project 8: Rotating Fluid in a cylinder

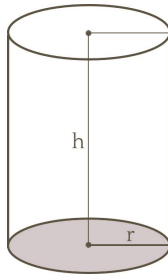
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Problem description

- Rotating cylinder filled with water is stopped at $t = 0$.
Navier-Stokes equation in cylindrical coordinates
- Velocity only has angular component
 $\mathbf{u}(r, \phi, z, t) = u_\phi(r, z, t)\mathbf{e}_{\phi}$
- Symmetry of the domain \rightarrow we consider a rectangular slice



Problem description

Equation for $t > 0$:

$$\frac{\partial u}{\partial t} = \nu \left(\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u}{\partial r} \right) - \frac{u}{r^2} + \frac{\partial^2 u}{\partial z^2} \right) \text{ for } 0 < r < R \text{ and } 0 < z < H.$$

Initial condition :

$$u = \omega r \text{ for } 0 \leq r \leq R$$

Boundary conditions : speed is zero on the boundary

$$u(r = 0) = u(r = R) = 0$$

$$u(z = 0) = u(z = H) = 0$$

High cylinder approximation

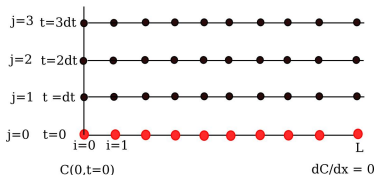
- u is assumed independent of z
- Equation becomes :

$$\frac{\partial u}{\partial t} = f(u) = \nu \left(\frac{1}{r} \frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial r^2} - \frac{u}{r^2} \right)$$

- Parabolic equation

High cylinder approximation

- Discretization of the domain with one spatial dimension



- Discrete equation with 2nd order central finite difference

$$\begin{aligned} f(U_i) &= \frac{\nu}{r_i} \frac{(U_{i+1} - U_{i-1}))}{2h} + \nu \frac{U_{i+1} - 2U_i + U_{i-1}}{h^2} - \frac{\nu}{r_i^2} U_i \\ &= \frac{\nu}{h} \left(\frac{1}{h} - \frac{1}{2r_i} \right) U_{i-1} - \nu \left(\frac{2}{h^2} + \frac{1}{r_i^2} \right) U_i + \frac{\nu}{h} \left(\frac{1}{h} + \frac{1}{2r_i} \right) U_{i+1} \end{aligned}$$

High cylinder approximation

- Tridiagonal matrix $f(U) = AU + b(t)$ with b^t for boundary conditions
- $b(t) = 0$ for $t > 0$ and $b(0) = [0 \cdots 0 *]^T$ accounts for non zero initial condition at $r = R$.
- ODE system $\frac{dU}{dt} = AU + b(t)$ is stiff
($h = R/10 \implies \lambda_{\max} = -0.0091, \lambda_{\min} = -0.2571$), requires implicit method CRANK-NICHOLSON

- CRANK-NICHOLSON method is of 2nd order

$$\begin{aligned}U^{t+1} &= U^t + 0.5 * h_t \left(f(U^t) + f(U^{t+1}) \right) \\&= U^t + 0.5 * h_t \left(A * U^t + b^t + A * U^{t+1} + b^{t+1} \right)\end{aligned}$$

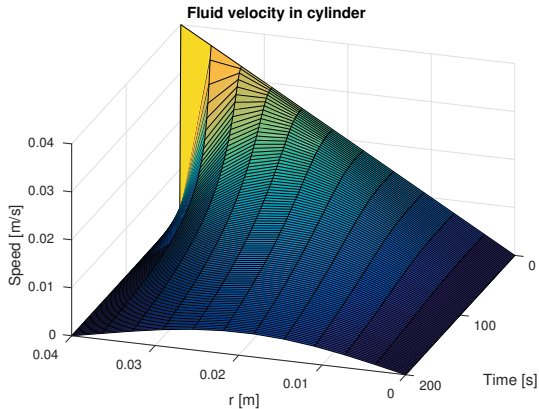
Therefore,

$$(Id - 0.5 * h_t * A) U^{t+1} = (Id + 0.5 * h_t * A) U^t + 0.5 * h_t * b^t$$

- Linear system that can easily be solved for U^{t+1} with sparse solver in Matlab

High cylinder approximation

- Solution in Matlab



High cylinder approximation : stability analysis

Consider any perturbation of the form $U_i^t = U^t e^{jkr_i}$. Define the constants $a_i = \frac{\nu}{h^2} \left(1 - \frac{1}{2i}\right)$, $b_i = \frac{-\nu}{h^2} \left(2 + \frac{1}{i^2}\right)$ and $c_i = \frac{\nu}{h^2} \left(1 + \frac{1}{2i}\right)$. The scheme is

$$\frac{U_i^{t+1} - U_i^t}{\Delta t} = \frac{1}{2} \left(a_i U_{i-1}^t + b_i U_i^t + c_i U_{i+1}^t + a_i U_{i-1}^{t+1} + b_i U_i^{t+1} + c_i U_{i+1}^{t+1} \right).$$

Plugging the ansatz gives

$$\left(\frac{1}{\Delta t} - \frac{a_i}{2} e^{-jkh} - \frac{b_i}{2} - \frac{c_i}{2} e^{jkh} \right) U^{t+1} = \left(\frac{1}{\Delta t} + \frac{a_i}{2} e^{-jkh} + \frac{b_i}{2} + \frac{c_i}{2} e^{jkh} \right) U^t.$$

We get a stable scheme when the factor that amplifies perturbations has a complex module smaller than 1

$$\frac{|U^{t+1}|}{|U^t|} \leq 1$$

High cylinder approximation : stability analysis

$$\text{Let } \beta = \frac{a_i e^{-jkh} + b_i + c_i e^{jkh}}{2},$$

$$\frac{|1 + \Delta t \beta|}{|1 - \Delta t \beta|} \leq 1$$

$$\frac{\operatorname{Re}\{1 + \Delta t \beta\}^2 + \operatorname{Im}\{1 + \Delta t \beta\}^2}{\operatorname{Re}\{1 - \Delta t \beta\}^2 + \operatorname{Im}\{1 - \Delta t \beta\}^2} \leq 1$$

Developments show that this will be satisfied when

$$a_i \cos(kh) + b_i + c_i \cos(kh) \leq 0.$$

This is indeed true because $a_i, c_i > 0$ and

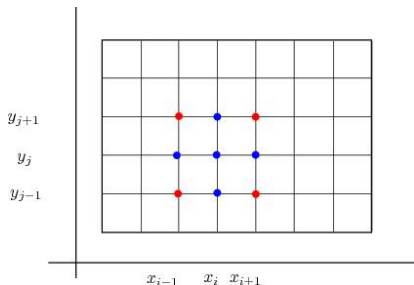
$$a_i \cos(kh) + b_i + c_i \cos(kh) \leq a_i + b_i + c_i = \frac{-\nu}{(ih)^2} < 0.$$

2 dimensional problem

- Solution u now depends on z , this gives 2 dimensions for the space grid

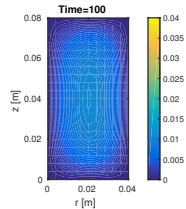
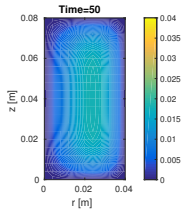
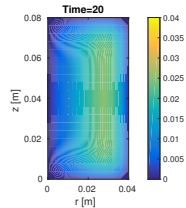
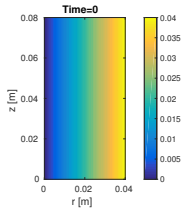
2 dimensional problem

Straightforward generalization of 1D case for domain and equation



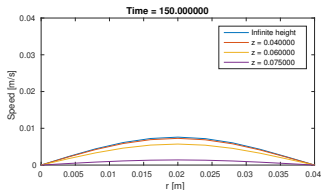
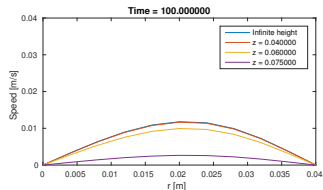
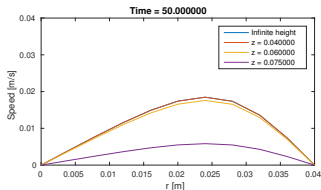
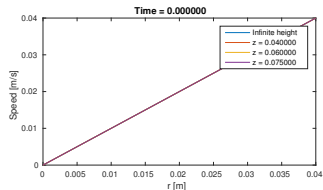
$$\begin{aligned} \frac{dU_{i,j}}{dt} = & \frac{\nu}{r_i} \frac{(U_{i+1,j} - U_{i-1,j})}{2h} + \nu \frac{U_{i+1,j} - 2U_{i,j} + U_{i-1,j}}{h^2} \\ & - \frac{\nu}{r_i^2} U_{i,j} + \frac{U_{i,j+1} - 2U_{i,j} + U_{i,j-1}}{h^2} \end{aligned}$$

2 dimensional problem : solution



Comparison of the 2 solutions

- We want to discuss the decision to consider z -dependency



Thank you for your attention