Chapter 10. Applied Projects on Differential Equations

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The following projects have been used for a long time in a first advanced course on Numerical Solution of Differential Equations at KTH, Stockholm. Most of the projects have been developped and modified during several years by my colleague Gerd Eriksson, and I have her permission to present them in updated form in this book. Projects 4 and 5 are taken from one of the early manuals of COMSOL MULTIPHYSICS, see also Appendix B.2. They have kindly given me permission to publish them in updated form.

Project 1. Signal propagation in a long electrical conductor

Given a ten kilometer long conductor with resistance R, inductance L and capacitance C. From $x = 0$ signals of amplitude 1 V are sent frequently during time intervals of different lengths. Denote this time-dependent voltage by $u_0(t)$. The voltage in the conductor is a function $u(x, t)$ of the position x on the conductor and time t and is modeled by the following hyperbolic PDE

$$\frac{\partial^2 u}{\partial t^2} + \frac{R}{L}\frac{\partial u}{\partial t} = \frac{1}{LC}\frac{\partial^2 u}{\partial x^2}, \quad 0 \le x \le X, \quad t > 0$$

which is the wave equation with damping. At $t = 0$ the initial conditions are

$$u(x, 0) = 0, \qquad \frac{\partial u}{\partial t}(x, 0) = 0, \quad 0 < x \le X$$

where $X = 10^4$.

The boundary condition at $x = 0$ is the given signal $u_0(t)$, i.e.

$$u(0, t) = u_0(t)$$

At $x = X$ the conductor is open, i.e. the signal is not reflected but disappears out. The BC fulfilling this condition is the advection equation, i.e.,

$$\frac{\partial u}{\partial t}(X, t) + \frac{1}{\sqrt{LC}}\frac{\partial u}{\partial x}(X, t) = 0$$

Discretize the conductor into 100 subintervals and use central difference approximations for the space and time derivatives in the damped wave equation. As for the approximation of the BC at $x = X$, the upwind method (FTBS) is appropriate.

For the conductor the following parameter values are given: $R = 0.004$ ohm, $L = 10^{-6}$ H and $C = 0.25 \cdot 10^{-8}$ F. Simulate the signal propagation during a sufficiently long time, e.g. during three milliseconds. First use the maximum allowed time step fulfilling the stability condition, then use a time step being 80 percent of the maximum time step. What is your comments on the result of the two time steps that have been used? Is the smoothing of the solution the effect of damping or of the method used?

Start your simulations by testing $u_0(t)$ with the MATLAB function given below. It corresponds to three short signals repeatedly sent with a period of 0.0004 seconds.

```
function usignal=uzero(t)
tau=rem(t,0.0004);
T0=0; T1=0.00005; T2= 0.00010; T3= 0.00013; T4= 0.00018; T5= 0.00025;
u0=1;
m=length(t);u1=zeros(m,1);
ind=find(t0<=tau & tau<=T1 | T2<=tau & tau <=T3 | T4<=tau & tau<=T5);
u1(ind)=u0*ones(size(ind));
usignal=u1;
```

You can of course try your own signal sequences.

Project 2. Flow in a cylindrical pipe

In a long straight pipe with circular cross section with radius R a fluid is streaming and we want to find out how the flow velocity varies in the pipe. Let the velocity vector be $(u, v)^T$ where u is the velocity in the length direction of the pipe and v the velocity in the radial direction. The flow is assumed to be circular symmetric, i.e. in cylindrical coordinates $u(r, z)$ and $v(r, z)$ depends only on r and z, not φ.

At the inlet $z = 0$ the velocity of the fluid is $u_0 = 0.1$ $[m/s]$ in the z-direction, i.e. $u(r, 0) = u_0$, $v(r, 0) = 0$, $0 \le r \le R$. The radius $R = 0.05$ $[m]$. The fluid has density $\rho = 1000$ $[kg/m^3]$ and the viscosity is $\nu = 10^{-5}$ $[m^2/s]$. In fluid problems the Reynolds number $Re = u_0 R/\nu$ is an important constant. When $Re >> 1$, which is the case here, it is possible to simplify the complicated Navier-Stokes PDEs which are used to model $u(r, z)$ and $v(r, z)$.

In this flow problem Navier-Stokes equations are approximated by

$$u\frac{\partial u}{\partial z} = \nu(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}) - v\frac{\partial u}{\partial r} - \frac{1}{\rho}\frac{dp}{dz} \qquad (1)$$

$$\frac{\partial u}{\partial z} + \frac{1}{r}\frac{\partial (vr)}{\partial r} = 0 \tag{2}$$

The pressure p is only z-dependent. At $z = 0$, $p = p_0$, where $p_0 = 10^4$ $[Pa]$. At $r = 0$ we have the following BC

$$\frac{\partial u}{\partial r}(0, z) = 0, \quad v(0, z) = 0$$

At the wall of the pipe the two velocities are zero, i.e.

$$u(R, z) = 0, \quad v(R, z) = 0$$

Far away in the pipe, several meters from the inlet, the velocity will have a stationary distribution with parabolic shape

$$u = 2u_0(1 - (\frac{r}{R})^2), \quad v = 0 \tag{3}$$

The task is to compute how the velocities vary with r and z. It is also included to compute how far away from the inlet the stationary distribution (3) is attained with an acceptable tolerance.

The differential equations (1) and (2) must be treated specially at $r = 0$ where both are singular. Show that with l'Hospital's rule they turn into

$$u\frac{\partial u}{\partial z} = 2\nu\frac{\partial^2 u}{\partial r^2} - \frac{1}{\rho}\frac{dp}{dz}, \quad \frac{\partial u}{\partial z} + 2\frac{\partial v}{\partial r} = 0, \quad at \quad r = 0$$

For numerical treatment the MoL with $n = 50$ subintervals in the r-direction is used. Then there will be $2n$ unknown variables being functions of z. These are $u_1(z), u_2(z), \ldots, u_n(z), \sigma(z), v_2(z), v_3(z), \ldots, v_n(z)$. $u_1(z)$ and $v_1(z)$ are velocity functions at $r = 0$. We know that $v_1(z) = 0$, hence this component is not among the unknowns. The variable σ is defined as

$$\sigma(z) = \frac{1}{\rho}\frac{dp}{dz}$$

With difference approximations of all derivatives in the r-direction we will have n ODEs each of (1) and (2), hence $2n$ in total. To solve these Euler's implicit method shall be used with stepsize h_z. The equations (1) and (2) will then be approximated by

$$u_i\frac{u_i - u_i^{old}}{h_z} - F_i(u_{i-1}, u_i, u_{i+1}, \sigma, v_i) = 0, \quad \frac{u_i - u_i^{old}}{h_z} + G_i(v_{i-1}, v_i, v_{i+1}) = 0$$

where u_i and v_i denote the unknown velocity components at $z+h_z$ and where u_i^{old} denotes the already computed velocity component at the previous value z. Find the expressions for F_i and G_i.

At the end of the day there will be a nonlinear system of $2n$ algebraic equations to be solved at each step in the z-direction. This is done with Newton's method. If the unknowns are written in the order $u_1, u_2, \ldots, u_n, \sigma, v_2, v_3, \ldots, v_n$ the jacobian J will have a block structure consisting of four $n \times n$-matrices

$$J = \begin{pmatrix} J_1 & J_2 \\ J_3 & J_4 \end{pmatrix}$$

where J_1 is tridiagonal, J_2 has nonzero elements only in the first column and in the diagonal, J_3 is diagonal and J_4 is tridiagonal (perhaps with exception of the first row, depending on the difference approximation of $2(\partial v/\partial r)_{r=0}$.

Close to the inlet there will be large variations in the velocities. The first steps should be $h_z = 0.001$ up to $z = 0.005$. Continue with $h_z = 0.005$ to $z = 0.04$. Further up in the pipe the step h_z can be successively larger. Continue the calculations of u and v and the pressure p up to the value of z where the velocity has approximately has attained its stationary distribution. Present the result graphically!

Project 3. Soliton waves

From Wikipedia we get the following information about soliton waves:

"In mathematics and physics, a soliton is a self-reinforcing solitary wave (a wave packet or pulse) that maintains its shape while it travels at constant speed. Solitons are caused by a cancellation of nonlinear and dispersive effects in the medium. (The term dispersive effectsrefers to a property of certain systems where the speed of the waves varies according to frequency.) Solitons arise as the solutions of a widespread class of weakly nonlinear dispersive partial differential equations describing physical systems."

The soliton phenomenon was first described in 1834 by John Scott Russell (1808-1882) who observed a solitary wave in the Union Canal in Scotland. He reproduced the phenomenon in a wave tank and named it the Åave of Translation".

Solitons are modeled by the following nonlinear PDE formulated by Korteweg and de Vries in 1895

$$\frac{\partial u}{\partial t} = -6u\frac{\partial u}{\partial x} - \frac{\partial^3 u}{\partial x^3}$$

We study the x-interval $-12 \leq x \leq 12$ with periodic BCs

$$u(12,t) = u(-12,t), \quad \frac{\partial^k u}{\partial x^k}(12,t) = \frac{\partial^k u}{\partial x^k}(-12,t), k = 1, 2, ..$$

For a soliton to appear the IC must have a special form and amplitude. The following IC can be used

$$u(x,0) = \frac{27}{cosh(x) + cosh(3x)}$$

Use the Method of Lines with 240 subintervals for the discretization of the x-interval. The problem then turns into a system of ODEs

$$\frac{d\mathbf{u}}{dt} = \mathbf{F}(\mathbf{u})$$

Perform computer simulations from $t = 0$ to $t_{end} = 1.2$ with the Runge-Kutta classical method RK4 with the time step $h_t = 0.001$.

Show and explain what happens if you take a larger step $h_t = 0.002$.

Test how sensitive the soliton wave is when the amplitude of the IC is changed. How much enlargement/reduction in the amplitude is allowed for keeping the soliton effect?

An implicit method like the trapezoidal method is an alternative to RK4. What are the advantages and disadvantages when using this method?

Project 4. Wave scattering in a waveguide

A plane wave is sent into a waveguide at the left end. Due to reflexes from the walls the plane wave is scattered. We want to investigate how the bend of the waveguide affects the wave for different frequences at the end of the waveguide.

The waves are modeled by the 2D wave equation

$$\frac{\partial^2 U}{\partial t^2} = \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2}$$

The time dependence is eliminated if the wave is monocromatic with a given wave length λ. Let $\omega = 2\pi/\lambda$ and make the ansatz

$$U(x,y,t) = u(x,y)e^{i\omega t}$$

The wave equation is then reduced to Helmholtz' equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -\omega^2 u$$

with appropriate BCs. On the reflective walls of metal there are no waves, hence $u = 0$ at the walls. At the exit of the wave guide we have an absorbing boundary condition, which means that the wave disappears out from the guide. This is modeled by the advection equation

$$\frac{\partial U}{\partial t} + \frac{\partial U}{\partial y} = 0$$

if the y-axis is directed downwards. If the time-dependence is eliminated the BC at $y = y_{out}$ is

$$\frac{\partial u}{\partial y} + i\omega u = 0$$

At the inlet of the guide where $x = 0$ the wave $U(0, y, t)$ consists of two parts, one incoming plane wave $e^{i\omega(t-x)}$ and one representing reflections from the interior, $V = U(0, y, t) - e^{i\omega(t-x)}$. For the wave part V there is the same kind of absorbing BC as at the exit, i.e.

$$\frac{\partial V}{\partial t} - \frac{\partial V}{\partial x} = 0$$

Inserting the expression for V gives after some formula manipulation the BC at $x = 0$

$$\frac{\partial u}{\partial x} - i\omega u + 2i\omega = 0$$

We now have a 2D elliptic PDE with mixed Dirichlet and Neumann conditions.

The geometrical form of the wave guide is inserted into a square with the side 0.20 m. The width of the wave guide is 0.04 m . The contour of the wave guide is obtained from the following (x, y)-coordinates, where the y-axis is directed downwards

```
x=[0,0.16,0.20,0.20,0.16,0.16,0.14,0]
y=[0,0,0.04,0.20,0.20,0.06,0.04,0.04]
```

It is interesting to investigate cases where the wave length of the incoming wave has about the same size as the width of the wave guide. Experiment with wave lengths λ in the region 30 mm to 80 mm.

Use first the stepsize $h_x = h_y = 0.005$ then $h_x = h_y = 0.0025$. What advantages and disadvantages is there with another halving of the stepsize?

The problem involves complex-valued quantities which implies that the numerical solution components $u_{i,j}$ have complex values. In the graphical presentation plot the real part of u which can be interpreted as the wave propagation in the wave guide at a frozen time-point.

Project 5. Metal block with heat source and thermometer.

Given a homogeneous metal block with rectangular cross-section 0.30×0.20 $[m^2]$. In the block there is a stationary heat distribution modelled by Poisson's equation in 2D with appropriate BCs. Inside the block there is a heat source with cross-section 0.04×0.04 $[m^2]$ with its left side situated 0.04 $[m]$ from the left side of the rectangle and symmetrically positioned in the y-direction. The metal block has heat exchange with the environment through its left wall ($x = 0$) through a thin layer of glass. The remaining three walls are heat insulated.

The task here is to compute the temperature distribution in the metal block at stationary conditions. The temperature $u(x, y)$ is modeled by Poisson's equation

$$\beta\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = -q(x, y)$$

where $\beta = 45$ $[W/(m \cdot K)]$ is the thermal conductivity of the metal. This value is also valid in the quadratic region with the heat sourse. We assume that the heat source gives an evenly distributed heat flow of $q(x, y) = 20000/0.04^2$ W/m^3 inside the quadratic region. Outside of this region $q(x, y) = 0$. In grid points on the boundary of the quadratic region half the q-value is used.

The temperature of the environment is $u_{out} = 20$ $[^oC]$. The heat transfer coefficient of the glass layer is $K_c = 900$. Hence the BC at $x = 0$ is

$$\beta\frac{\partial u}{\partial x}(0, y) = -K_c(u_{out} - u(0, y))$$

The isolated walls of the metal block gives the BCs $\partial u/\partial y = 0$ at the upper ($y = 0.20$) and the lower ($y = 0$) wall and $\partial u/\partial x = 0$ at the right wall ($x = 0.30$).

Discretize the problem with three different stepsizes, $h_x = h_y = 0.02, 0.01$ and 0.005. Plot contour curves of the temperature distribution and note specially the maximum temperature of the metal block (which is found inside the quadratic region). Also note the temperature at the position of the thermometer which is the point $(0.16, 0.10)$.

Project 6. Deformation of a circular metal plate.

A circular metal plate with thickness $t = 5$ $[mm]$ and the radius $R = 50$ $[mm]$ is simply supported on a circular frame with the same radius. The plate is loaded transversally with an equally distributed pressure q $[Pa]$. In this problem there is no φ-dependence in the deformation $u(r, \varphi)$, when described in polar coordinates. Hence the dependent variable in the ODE modeling the deformation is r.

$$\frac{d^4u}{dr^4} + \frac{2}{r}\frac{d^3u}{dr^3} - \frac{\gamma^2}{r^2}\frac{d^2u}{dr^2} + \frac{\gamma^2}{r^3}\frac{du}{dr} = -\frac{q}{D_r}, \quad D_r = \frac{t^3}{12}\frac{E_r}{(1-\nu_{\varphi r}^2/\gamma^2)}$$

The material constants are $E_\varphi = 40000$ $[MPa]$, $E_r = 10000$ $[MPa]$, $\gamma^2 = E_\varphi/E_r$, $\nu_{\varphi r} = 0.24$. The load pressure is constant $q = 0.15$.

The BCs of this fourth order ODE is at $r = 0$ $u'(0) = 0$, $u'''(0) = 0$. At $r = R$ the deformation and the moment M_r are both zero. Since $M_r = -D_r(u'' + \nu_{\varphi r}u'/r)$ we have the BCs $u(R) = 0$ and $u''(R) + \nu_{\varphi r}u'(R)/R = 0$.

Discretize the problem in N subintervals and approximate all derivatives in the problem with difference quotients of second order. This will lead to a linear system of equations with a bandmatrix of band width five. Use the symmetry of the problem, i.e. that $u_{-1} = u_1$ and $u_{-2} = u_2$. Start with $N = 50$ giving the stepsize $h = R/N$ and continue with the stepsizes $h/2$ and $h/4$. Plot the result in a graph showing the three stepsize approximations of $u(r)$ in the same figure. To solve the linear system of equations use sparse technique available in e.g. MATLAB(R).

The ODE is singular at $r = 0$. Use l'Hopital's rule to find the form of the ODE at $r = 0$.

In the problem discretizations of $u'''(0)$ and $u^{IV}(r_i)$ are needed. Use the ansatz

$$u'''(0) = \frac{au(2h) + bu(h) + cu(0) + du(-h) + eu(2h)}{h^3}$$

and determine the coefficients a, b, c, d, e so that the difference quotient is of second order. Make a similar ansatz to approximate the fourth derivative.

Project 7. Cooling of a crystal ball.

The cooling of crystal glass needs attention in the production process. Quick cooling is wanted to keep the cost lower, but may cause the glass to break. A crystal ball with radius R shall be cooled down from 980 $[^oC]$ to room temperature. The temperature lowering in the oven takes place in a controlled way and depends on the adjusted temperature parameter T that starts at $t = 0$ until $t = T$ according to

$$f(t) = 980e^{-3.9t/T}$$

For $t > T$ we have $f(t) = 980e^{-3.9} = 19.8$, hence normal room temperature.

The heat equation for a homogeneous sphere follows the PDE

$$\frac{\partial u}{\partial t} = \frac{D}{r}\frac{\partial^2(ru)}{\partial r^2}, \quad 0 < r < R, \quad t > 0$$

The thermal diffusivity for crystal glass is $D = 4.0 \cdot 10^{-7}$ $[m^2/s]$. At $r = 0$ we have the BC $\frac{\partial u}{\partial r} = 0$. Hence at this boundary the heat equation takes the form

$$\frac{\partial u}{\partial t} = 3D\frac{\partial^2 u}{\partial r^2}$$

Prove this with the help of l'Hopital's rule. The BC at $r = R$ is $u(R, t) = f(t)$. When the cooling starts the temperature of the ball is 980 $[^oC]$.

Of special interest is the temperature gradient in the r-direction - the glass may break if $|\frac{\partial u}{\partial r}|$ is too large. In our case we assume that the glass will break if the the gradient on any occasion exceeds 6000 $[^oC/m]$, which will happen if the cooling is too quick. Make numerical experiments with the parameter T and find the smallest value of T in the cases $R = 6$ $[cm]$, $R = 12$ $[cm]$ and $R = 18$ $[cm]$.

Compute the temperature $u(r, t)$ in the glass ball from $t = 0$ until its surface has cooled down to room temperature.

Try discretization with 60 intervals in the r-direction. Use the MoL and an appropriate ODE-solver or use the implicit Euler method with small time steps in the beginning and then increase the time step.

Visualize the result graphically.

Project 8. Rotating fluid in a cylinder.

In a cylinder there is a viscous fluid. The radius of the cylinder is $R = 40$ $[mm]$. The cylinder with its content is rotating with an angular velocity ω, so that $u_\varphi = \omega r, \quad 0 \le r \le R$. The fluid has no velocity in the r- or z-direction, only the velocity component $u = u_\varphi$.

Suddenly at $t = 0$ the cylinder stops. The movement of the fluid for $t > 0$ is modeled by the PDE

$$\frac{\partial u}{\partial t} = \nu(\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial u}{\partial r}) - \frac{u}{r^2} + \frac{\partial^2 u}{\partial z^2}), \quad 0 < r < R, \quad 0 < z < H$$

where H is the height of the cylinder and ν is the viscosity coefficient. The BCs are: $u = 0$ at $r = 0$ and $r = R$. Also $u = 0$ at $z = 0$ and $z = H$. The paramter ν has the value 10^{-6} $[m^2 s]$ (the value for water). The angular velocity has the value $\omega = 1$.

We want to compute and plot curves to see how the velocity $u(r, z, t)$ of the fluid changes with time.

a) Assume that the cylinder is very high. Then the velocity is independent of z. This model simplification makes the PDE depend on only t and r and hence 1D in space. Use e.g. the Crank-Nicolson method to compute the velocity distribution $u(r, t)$ in the fluid. Plot curves with the velocity of the fluid as function of the radius at different time points.

b) Now assume the cylinder has the height $H = 8$ $[cm]$. The fact that the velocity is zero at $z = 0$ and $z = H$ will influence the velocity distribution in the cylinder. Use e.g. the Crank-Nicolson method again for the numerical solution of this 2D problem. Plot the velocity distribution at different time points with the `contourf`-command in MATLAB (R). Discuss the difference of the two results.