Project 8: Rotating Fluid in a cylinder

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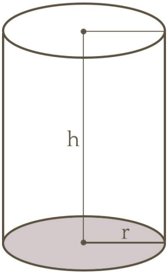
Problem description

- Rotating cylinder filled with water is stopped at $t = 0$. Navier-Stokes equation in cylindrical coordinates
- Velocity only has angular component $\mathbf{u}(r, \varphi, z, t) = u_\varphi(r, z, t)\mathbf{e}_\varphi$
- Symmetry of the domain $\rightarrow$ we consider a rectangular slice



Problem description

Equation for $t > 0$:

$$\frac{\partial u}{\partial t} = \nu\Big(\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial u}{\partial r}) - \frac{u}{r^2} + \frac{\partial^2 u}{\partial z^2}\Big) \text{ for } 0 < r < R \text{ and } 0 < z < H.$$

Initial condition :

$$u = \omega r \text{ for } 0 \leq r \leq R$$

Boundary conditions : speed is zero on the boundary

$$u(r=0) = u(r=R) = 0$$

$$u(z=0) = u(z=H) = 0$$

High cylinder approximation

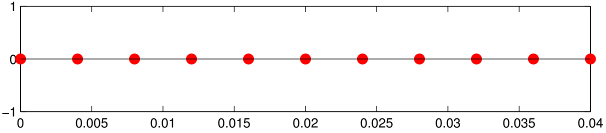
- u is assumed independent of z
- Equation becomes :

$$\frac{\partial u}{\partial t} = f(u) = \nu\Big(\frac{1}{r}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial r^2} - \frac{u}{r^2}\Big)$$

- Parabolic equation

High cylinder approximation

- Discretization of the domain with one spatial dimension



- Discrete equation with 2nd order central finite difference

$$\begin{aligned}\frac{dU_i}{dt} &= \frac{\nu}{r_i}\frac{(U_{i+1}-U_{i-1})}{2h} + \nu\frac{U_{i+1}-2U_i+U_{i-1}}{h^2} - \frac{\nu}{r_i^2}U_i \\ &= \frac{\nu}{h}\left(\frac{1}{h}-\frac{1}{2r_i}\right)U_{i-1} - \nu\left(\frac{2}{h^2}+\frac{1}{r_i^2}\right)U_i + \frac{\nu}{h}\left(\frac{1}{h}+\frac{1}{2r_i}\right)U_{i+1}\end{aligned}$$

High cylinder approximation

- Tridiagonal matrix $\dfrac{dU}{dt} = AU + b(t)$ with $b(t)$ for boundary conditions
- $b(t) = 0$ for $t > 0$ and $b(0) = [0 \cdots 0 \ *]^T$ accounts for non zero initial condition at $r = R$.
- ODE system $\dfrac{dU}{dt} = AU + b(t)$ is stiff ($h = R/10 \Longrightarrow \lambda_{max} = -0.0091, \lambda_{min} = -0.2571$), requires implicit method Crank-Nicholson

High cylinder approximation

- Crank-Nicholson method is of 2nd order

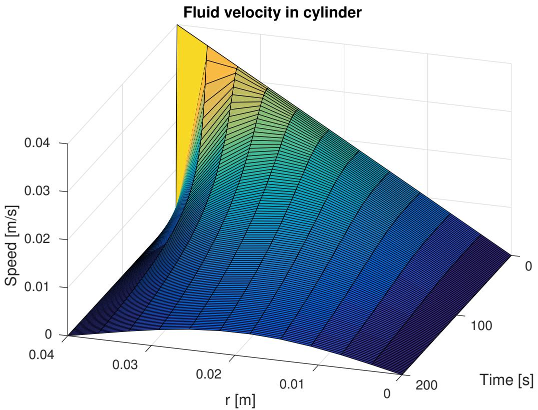
$$
\begin{aligned}
U^{t+1} &= U^t + 0.5 * h_t\Big(f(U^t) + f(U^{t+1})\Big) \\
&= U^t + 0.5 * h_t\Big(A * U^t + b^t + A * U^{t+1} + b^{t+1}\Big)
\end{aligned}
$$

Therefore,

$$
\Big(Id - 0.5 * h_t * A\Big)U^{t+1} = (Id + 0.5 * h_t * A)U^t + 0.5 * h_t * b^t
$$

- Linear system that can easily be solved for U^{t+1} with sparse solver in Matlab

- Solution in Matlab



High cylinder approximation : stability analysis

Consider any perturbation of the form $U_i^t = U^t e^{jkr_i}$. Define the constants $a_i = \frac{\nu}{h^2}\left(1 - \frac{1}{2i}\right)$, $b_i = \frac{-\nu}{h^2}\left(2 + \frac{1}{i^2}\right)$ and $c_i = \frac{\nu}{h^2}\left(1 + \frac{1}{2i}\right)$. The scheme is

$$\frac{U_i^{t+1} - U_i^t}{\Delta t} = \frac{1}{2}\left(a_i U_{i-1}^t + b_i U_i^t + c_i U_{i+1}^t + a_i U_{i-1}^{t+1} + b_i U_i^{t+1} + c_i U_{i+1}^{t+1}\right).$$

Plugging the ansatz gives

$$\left(\frac{1}{\Delta t} - \frac{a_i}{2} e^{-jkh} - \frac{b_i}{2} - \frac{c_i}{2} e^{jkh}\right) U^{t+1} = \left(\frac{1}{\Delta t} + \frac{a_i}{2} e^{-jkh} + \frac{b_i}{2} + \frac{c_i}{2} e^{jkh}\right) U^t.$$

We get a stable scheme when the factor that amplifies perturbations has a complex module smaller than 1

$$\frac{|U^{t+1}|}{|U^t|} \leq 1$$

Let $\beta = \dfrac{a_i e^{-jkh} + b_i + c_i e^{jkh}}{2}$,

$$\frac{|1+\Delta t\beta|}{|1-\Delta t\beta|} \leq 1$$

$$\frac{Re\{1+\Delta t\beta\}^2 + Im\{1+\Delta t\beta\}^2}{Re\{1-\Delta t\beta\}^2 + Im\{1-\Delta t\beta\}^2} \leq 1$$

Developments show that this will be satisfied when

$$a_i \cos(kh) + b_i + c_i \cos(kh) \leq 0.$$

This is indeed true because $a_i, c_i > 0$ and

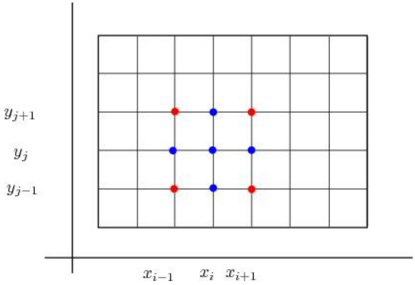
$$a_i \cos(kh) + b_i + c_i \cos(kh) \leq a_i + b_i + c_i = \frac{-\nu}{(ih)^2} < 0.$$

2 dimensional problem

- Solution u now depends on z, this gives 2 dimensions for the space grid

2 dimensional problem

Straightforward generalization of 1D case for domain and equation



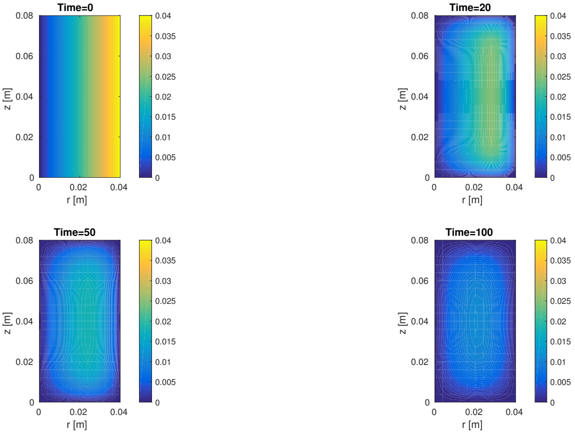
$$\frac{dU_{i,j}}{dt} = \frac{\nu}{r_i}\frac{(U_{i+1,j} - U_{i-1,j})}{2h} + \nu\frac{U_{i+1,j} - 2U_{i,j} + U_{i-1,j}}{h^2}$$
$$- \frac{\nu}{r_i^2}U_{i,j} + \frac{U_{i,j+1} - 2U_{i,j} + U_{i,j-1}}{h^2}$$

2 dimensional problem

- Renumbering : $k = i + Nr * (j-1)$ allows minimal band width given space of domain

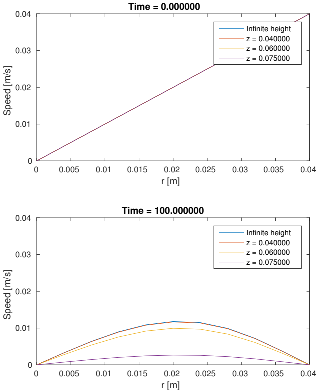
$$\frac{dU_k}{dt} = \nu \left(\cdot\cdot \frac{1}{h^2} \cdot\cdot \quad (\frac{1}{h^2} - \frac{1}{2hr_i}) \quad (\frac{-4}{h^2} - \frac{1}{r_i^2}) \quad (\frac{1}{h^2} + \frac{1}{2hr_i}) \quad \cdot\cdot \frac{1}{h^2} \cdot\cdot \right) \begin{pmatrix} \vdots \\ U_{k-N_r} \\ \vdots \\ U_{k-1} \\ U_k \\ U_{k+1} \\ \vdots \\ U_{k+N_r} \\ \vdots \end{pmatrix}$$

2 dimensional problem : solution



Comparison of the 2 solutions

- We want to discuss the decision to consider z-dependency



Thank you for your attention