



Introduction to

Algorithm Design and Analysis

[6] MergeSort



Yu Huang

<http://cs.nju.edu.cn/yuhuang>
Institute of Computer Software
Nanjing University



In the Last Class...

- **Heap: Structure + PO**
 - FixHeap
 - ConstructHeap
 - Array-based implementation
- **HeapSort**
 - Accelerated HeapSort
- **Heap: more than sort**
 - Generalizations
 - Other applications



MergeSort

- **MergeSort**
 - Worst-case analysis of MergeSort
 - More than sort: the MergeSort D&C
- **Lower Bounds for *comparison-based sorting***
 - Worst-case
 - Average-case

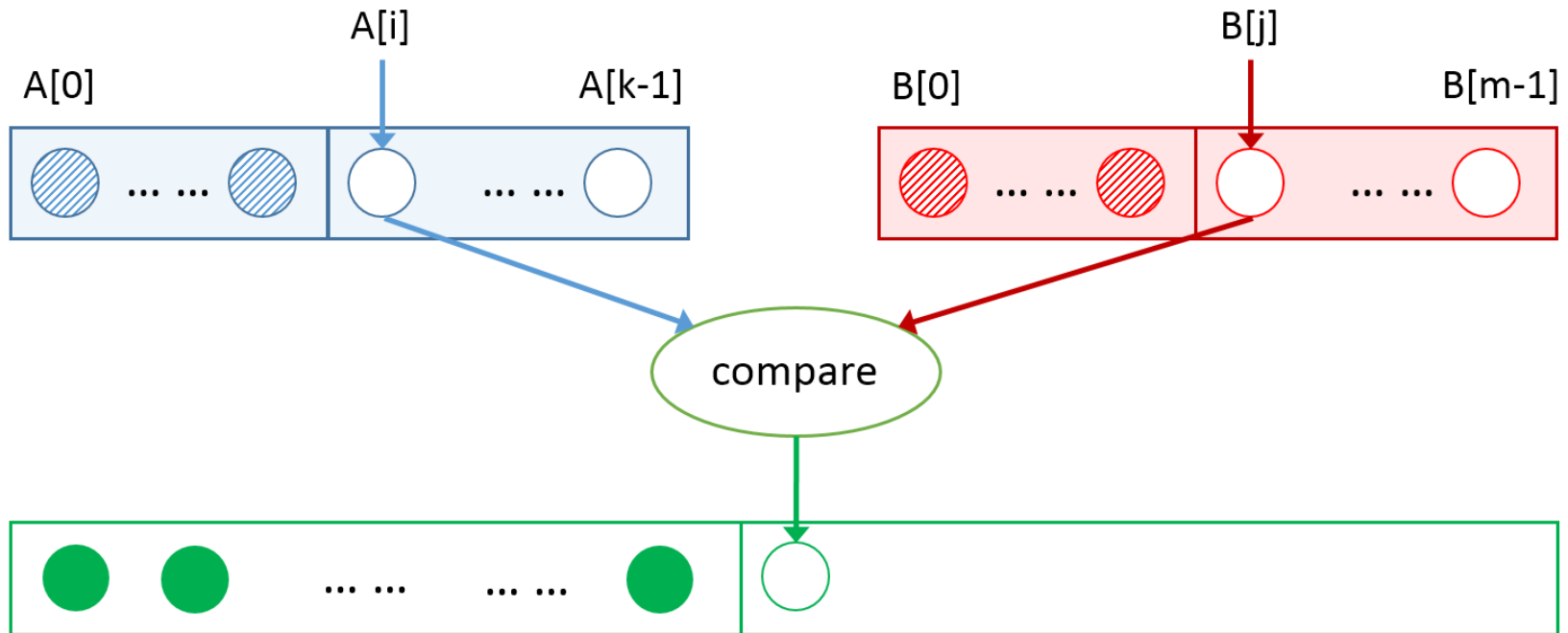
$n \log n$



MergeSort: the Strategy

- **Easy division**
 - No comparison is conducted during the division
 - Minimizing the size difference between the divided subproblems
- **Merging two sorted subranges**
 - Using *Merge*

Merging Sorted Arrays



Merge: the Specification

- **Input**

- Array A with k elements and B with m elements, whose keys are in non-decreasing order

- **Output**

- Array C containing $n = k + m$ elements from A and B in non-decreasing order
- C is passed in and the algorithm fills it

Merge: Recursive Version

merge(*A*,*B*,*C*)

if (*A* is empty)
 rest of *C* = rest of *B*
else if (*B* is empty)
 rest of *C* = rest of *A*

Base cases

else

if (first of *A* ≤ first of *B*)
 first of *C* = first of *A*
 merge(rest of *A*, *B*, rest of *C*)

else

 first of *C* = first of *B*
 merge(*A*, rest of *B*, rest of *C*)

return



Worst Case Complexity of Merge

- Observations

- Worst case is that the last comparison is conducted between $A[k-1]$ and $B[m-1]$
 - After each comparison, one element is inserted into Array C, *at least*.
 - After entering Array C, an element will never be compared again
 - After the last comparison, at least two elements (the two just compared) have not yet been moved to Array C. *So at most $n-1$ comparisons are done.*

- In worst case, *$n-1$* comparisons are done, where $n=k+m$



Optimality of Merge

- Any algorithm to merge two sorted arrays, each containing $k=m=n/2$ entries, by comparison of keys, does at least $n-1$ comparisons in the worst case.

- Choose keys so that:

$$b_0 < a_0 < b_1 < a_1 < \dots < b_i < a_i < b_{i+1}, \dots, < b_{m-1} < a_{k-1}$$

- Then the algorithm must compare a_i with b_i for every i in $[0, m-1]$, and must compare a_i with b_{i+1} for every i in $[0, m-2]$, so, there are $n-1$ comparisons.

Valid for $|k-m| \leq 1$, as well.

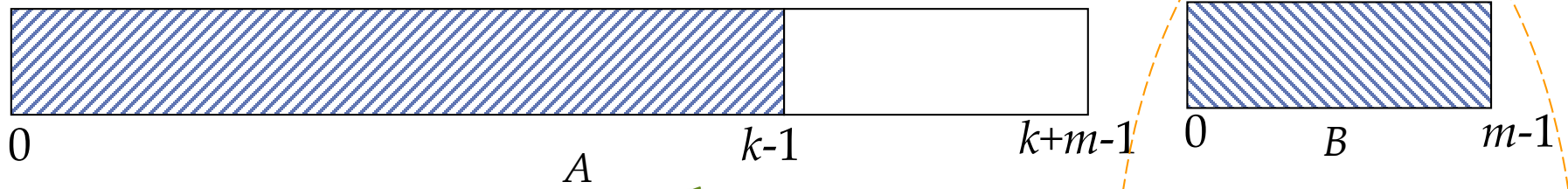
Space Complexity of Merge

- A algorithm is “in space”
 - If the extra space it has to use is in $\Theta(1)$.
- Merge is **not** an in place algorithm
 - Since it needs $O(n)$ extra space to store the merged sequence during the merging process.

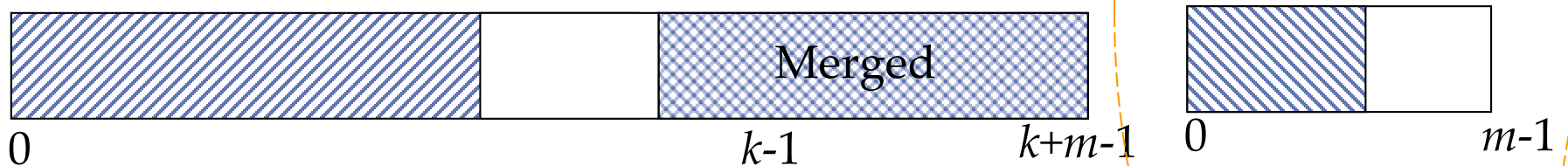


Overlapping Arrays for Merge

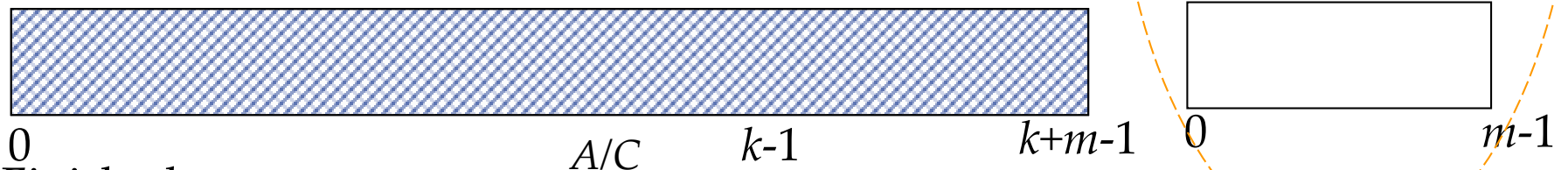
Before the merge



Partly finished



Finished



MergeSort

- Input: Array E and indexes $first$, and $last$, such that the elements of $E[i]$ are defined for $first \leq i \leq last$.
- Output: $E[first], \dots, E[last]$ is a sorted rearrangement of the same elements.
- Procedure

```
void mergeSort(Element[] E, int first, int last)
    if (first < last)
        int mid = (first + last) / 2;
        mergeSort(E, first, mid);
        mergeSort(E, mid + 1, last);
        merge(E, first, mid, last)
    return
```



Analysis of MergeSort

- The recurrence equation for Mergesort
 - $W(n) = W(\lfloor n/2 \rfloor) + W(\lceil n/2 \rceil) + n - 1$
 - $W(1) = 0$

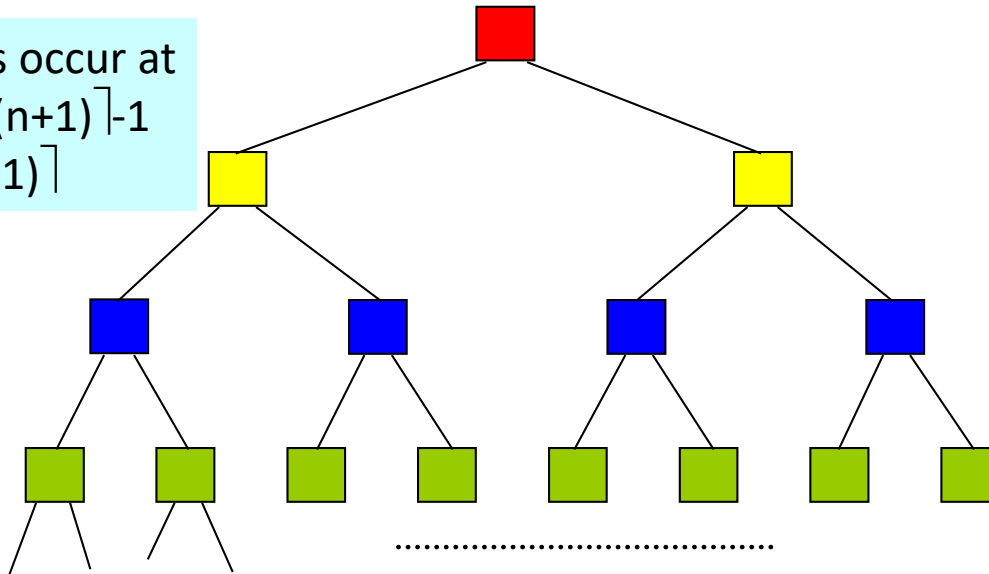
Where $n = \text{last} - \text{first} + 1$, the size of range to be sorted

- The *Master Theorem* applies for the equation, so:

$$W(n) \in \Theta(n \log n)$$

Recursion Tree for Mergesort

Base cases occur at depth $\lceil \lg(n+1) \rceil - 1$ and $\lceil \lg(n+1) \rceil$



$n-1$ Level 0

$n-2$ Level 1

$n-4$ Level 2

$n-8$ Level 3

$k/2$ may be $\lceil k/2 \rceil$ or $\lfloor k/2 \rfloor$



$T(n)$	$n-1$
--------	-------



$T(n/2)$	$n/2-1$
----------	---------



$T(n/4)$	$n/4-1$
----------	---------



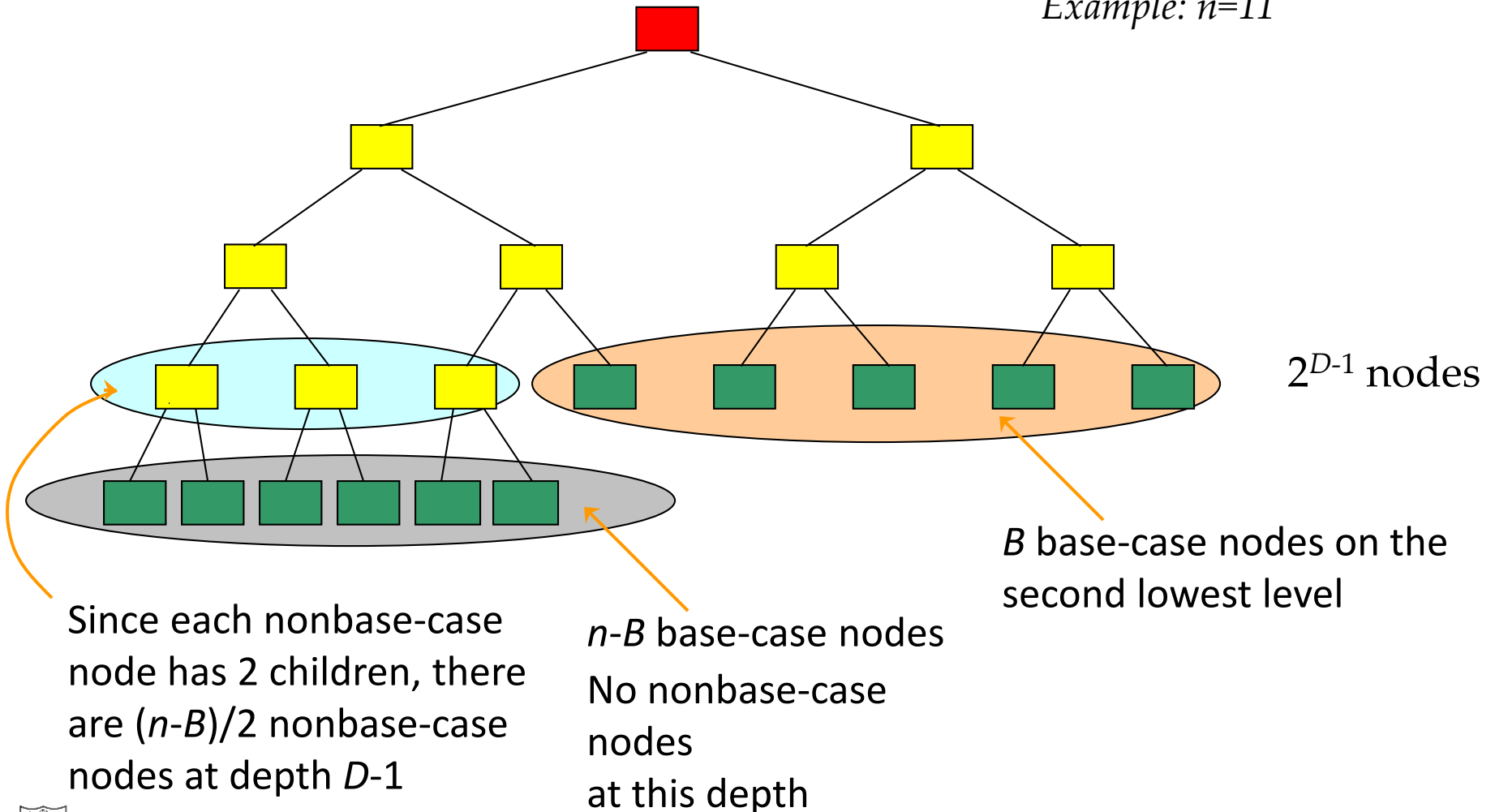
$T(n/8)$	$n/8-1$
----------	---------

Note:

nonrecursive costs on level k is $n-2^k$ for all level without basecase node

Non-complete Recursion Tree

Example: $n=11$



Number of Comparison of MergeSort

- The maximum depth D of the recursive tree is $\lceil \log(n+1) \rceil$.
- Let B base case nodes on depth $D-1$, and $n-B$ on depth D , (Note: base case node has nonrecursive cost 0).
- $(n-B)/2$ nonbase case nodes at depth $D-1$, each has nonrecursive cost 1.
- So:

$$W(n) = \sum_{d=0}^{D-2} (n - 2^d) + \frac{n-B}{2} = n(D-1) - (2^{D-1} - 1) + \frac{n-B}{2}$$

$$\text{Since } (2^D - 2B) + B = n, \text{ that is } B = 2^D - n$$

$$\text{So, } W(n) = nD - 2^D + 1$$

$$\text{Let } \frac{2^D}{n} = 1 + \frac{B}{n} = \alpha, \text{ then } 1 \leq \alpha < 2, \quad D = \log n + \log \alpha$$

$$\text{So, } W(n) = n \log n - (\alpha - \log \alpha)n + 1$$

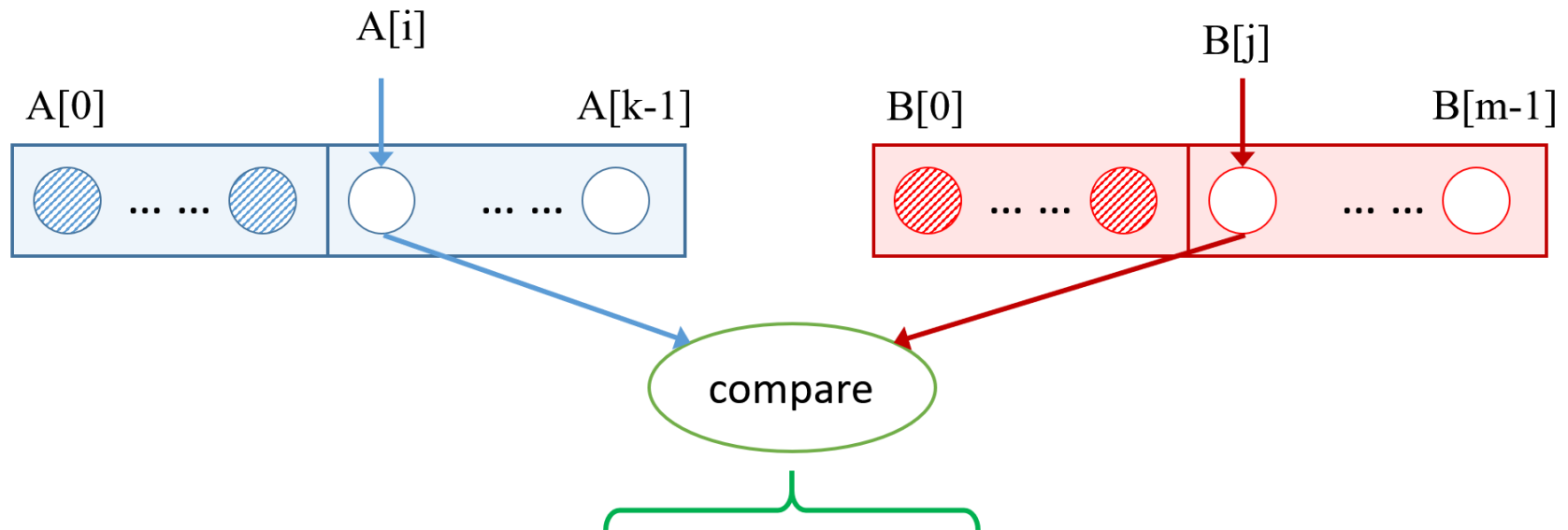
- $\lceil n \log(n) - n + 1 \rceil \leq \text{number of comparison} \leq \lceil n \log(n) - 0.914n \rceil$



The MergeSort D&C

- **Counting the number of inversions**
 - Brute force: $O(n^2)$
 - Can we use divide & conquer
 - In $O(n \log n) \Rightarrow$ combination in $O(n)$
- **MergeSort as the **carrier****
 - Sorted subarrays
 - $A[0..k-1]$ and $B[0..m-1]$
 - Compare the *left* and the *right* elements
 - $A[i]$ vs. $B[j]$

The MergeSort D&C



- If $A[i] > B[j]$

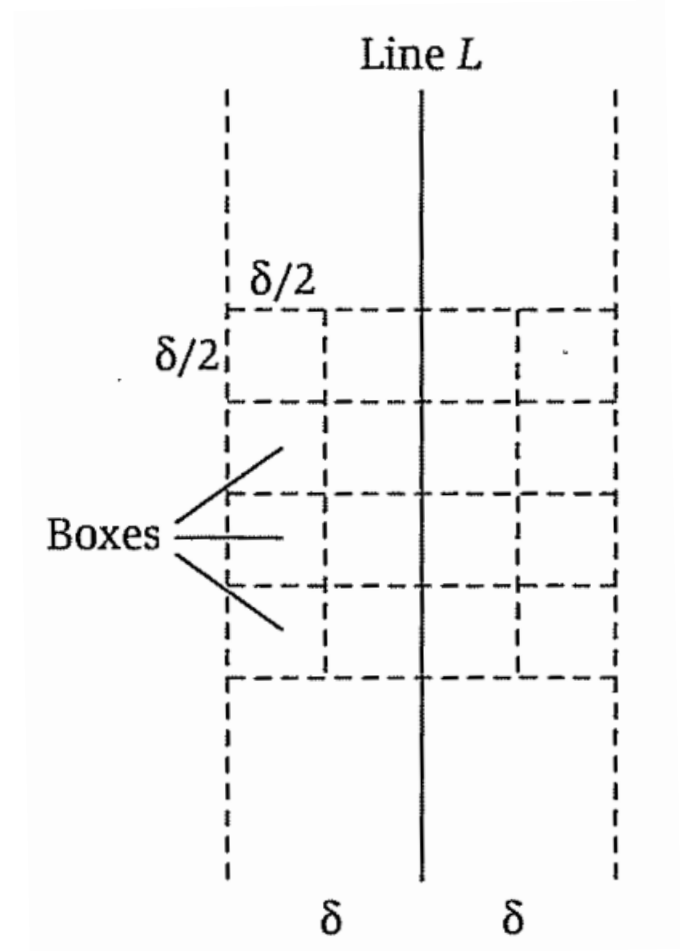
- (i, j) is an inversion
- For all $i' > j$,
 (i', j) are inversions ($i' > i$)
- $B[j]$ is selected

- If $A[i] < B[j]$

- No inversion found
- $A[i]$ is selected

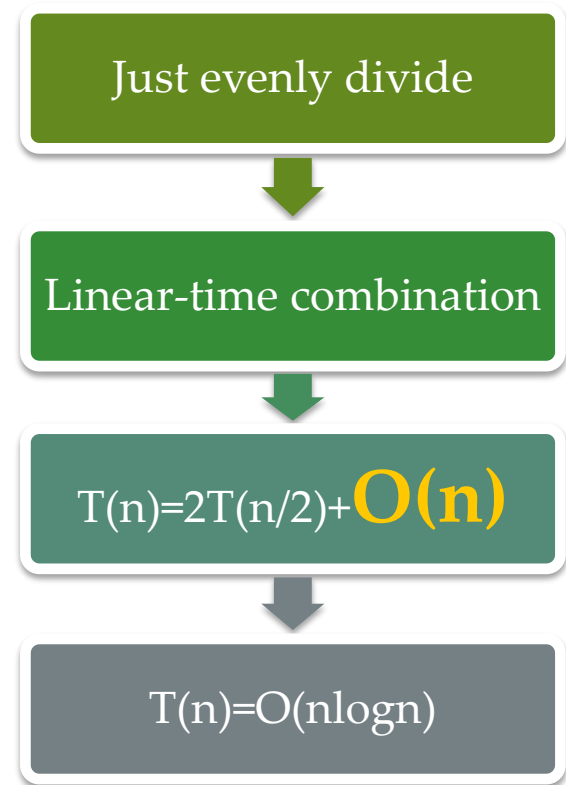
The MergeSort DC

- **The nearest pair**
 - n nodes on a plane
 - The pair with the minimum distance
- **The MergeSort DC**
 - $T(n) = 2T(n/2) + f(n)$
 - $f(n)$ must be $O(n)$
 - How?



The MergeSort D&C

- Max-sum subsequence
- Maxima on a plane
- Finding the *frequent* element
- Integer/matrix multiplication
- ...



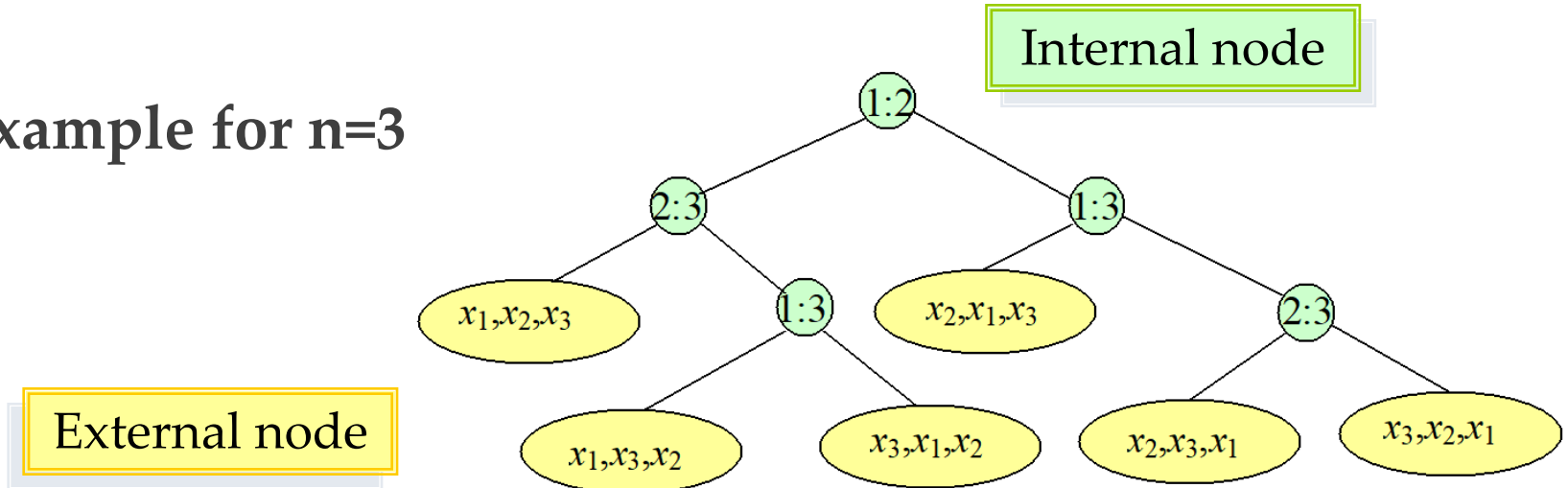
Lower Bounds for Comparison-based Sorting

- Upper bound, e.g., worst-case cost
 - For **any** possible input, the cost of the **specific** algorithm A is no more than the *upper bound*
 - $\text{Max}\{\text{Cost}(i) \mid i \text{ is an input}\}$
- Lower bound, e.g., comparison-based sorting
 - For **any** possible (comparison-based) sorting algorithm A, the worst-case cost is no less than the *lower bound*
 - $\text{Min}\{\text{Worst-case}(a) \mid a \text{ is an algorithm}\}$



Decision Tree for Sorting

A example for $n=3$



- Decision tree is a 2-tree (assuming no same keys)
- The action of Sort on a particular input corresponds to following on path in its decision tree from the root to a leaf associated to the specific output

Characterizing the Decision Tree

- For a sequence of **n** distinct elements, there are **$n!$** different permutation
 - So, the decision tree has at least **$n!$** leaves, and exactly $n!$ leaves can be reached from the root.
 - So, for the purpose of lower bounds evaluation, we use trees with exactly $n!$ leaves.
- The number of comparison done in the *worst case* is the **height** of the tree.
- The *average* number of comparison done is the **average** of the **lengths** of all paths from the root to a leaf.



Lower Bound for Worst Case

- **Theorem:** Any algorithm to sort n items by comparisons of keys must do at least $\lceil \log n! \rceil$, or approximately $\lceil n \log n - 1.443n \rceil$, key comparisons in the worst case.
 - Note: Let $L=n!$, which is the number of leaves, then $L \leq 2^h$, where h is the height of the tree, that is $h \geq \lceil \log L \rceil = \lceil \log n! \rceil$
 - Lemma: let L be the number of leaves in a binary tree and h be its height. Then $L \leq 2^h$
 - For the asymptotic behavior:

$$\log(n!) \geq \log(n(n-1) \cdots (\lceil \frac{n}{2} \rceil)) \geq \log(\frac{n}{2})^{\frac{n}{2}} = \frac{n}{2} \log(\frac{n}{2}) \in \Theta(n \log n)$$

External Path Length(EPL)

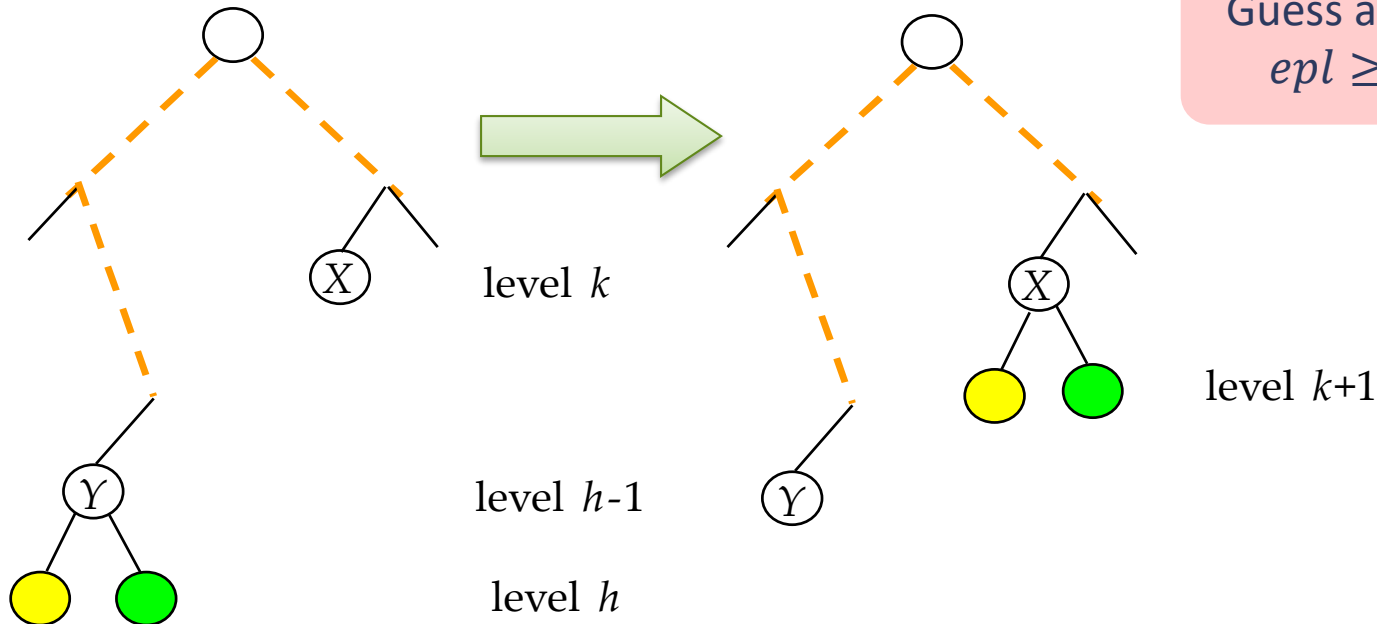
- **EPL – sum of path length to every leaf**
 - The EPL t is recursively defined as follows:
 - [Base case] 0 for a single external node
 - [Recursion] t is non-leaf with sub-trees L and R , then the sum of:
 - the external path length of L ;
 - the number of external node of L ;
 - the external path length of R ;
 - the number of external node of R ;



Lower Bound for Average Behavior

- Since a decision tree with L leaves is a 2-tree, the average path length from the root to a leaf is $\frac{epl}{L}$.
 - Recall that $epl \geq L \log(L)$.
- **Theorem:** The average number of comparison done by an algorithm to sort n items by comparison of keys is at least $\log(n!)$, which is about $n \log n - 1.443n$.

More Balanced 2-tree, Less EPL

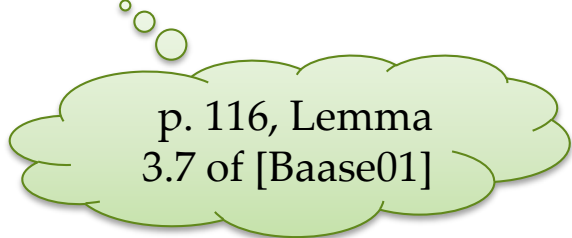


Guess and prove:
 $epl \geq L \log L$

Assuming that $h-k>1$, when calculating epl , $h+h+k$ is replaced by $(h-1)+2(k+1)$. The net change in epl is $k-h+1<0$, that is, the epl decreases.

Properties of EPL

- Let t be a 2-tree, then the epl of t is the sum of the paths from the root to each external node.
- $epl \geq m \log(m)$, where m is the number of external nodes in t
 - $epl = epl_L + epl_R + m \geq m_L \log(m_L) + m_R \log(m_R) + m$,
 - note $f(x) + f(y) \geq 2f((x+y)/2)$ for $f(x) = x \log x$
 - so,
$$epl \geq 2((m_L + m_R)/2) \log((m_L + m_R)/2) + m$$
$$= m(\log(m) - 1) + m = m \log m.$$



p. 116, Lemma
3.7 of [Baase01]

MergeSort Has Optimal Average Performance

- The **average** number of comparisons done by an algorithm to sort n items by comparison of keys is at least about $n\log n - 1.443n$
- The **worst** complexity of MergeSort is in $\Theta(n\log n)$
- So, MergeSort is optimal as for its average performance

Thank you!

Q & A

Yu Huang

<http://cs.nju.edu.cn/yuhuang>

