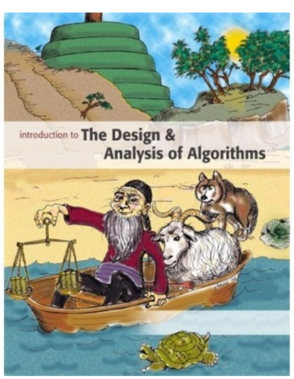
Introduction to

Algorithm Design and Analysis

[3] Recursion



Yu Huang

http://cs.nju.edu.cn/yuhuang
Institute of Computer Software
Nanjing University

In the Last Class …

- **Asymptotic growth rate**
  - o O, Ω, Θ
  - o o, ω
- **Brute force algorithms**
  - o By iteration
  - o By recursion

Recursion

- **Recursion in algorithm design**
  - The divide and conquer strategy
  - Proving the correctness of recursive procedures
- **Solving recurrence equations**
  - Some elementary techniques
  - Master theorem

Recursion in Algorithm Design

- **Computing n! with Fac(n)**
  - if n=1 then return 1 else return Fac(n-1)*n

M(1)=0 and M(n)=M(n-1)+1 for n>0
(critical operation: multiplication)

- **Hanoi Tower**
  - if n=1 then move d(1) to peg3 else
    Hanoi(n-1, peg1, peg2); move d(n) to peg3; Hanoi(n-1, peg2, peg3)

M(1)=1 and M(n)=2M(n-1)+1 for n>1
(critical operation: move)

Recursion in Algorithm Design

- **Counting the Number of Bits**
  - Input: a positive decimal integer n
  - Output: the number of binary digits in n's binary representation

Int BitCounting (int n)
1. If(n==1) return 1;
2. Else
3. return BitCounting(n div 2) +1;

$$T(n) = \begin{cases} 0 & n = 1 \\ T(\lfloor n/2 \rfloor) + 1 & n > 1 \end{cases}$$

Divide and Conquer

- **Divide**
  - Divide the "big" problem to smaller ones
- **Conquer**
  - Solve the "small" problems by recursion
- **Combine**
  - Combine results of small problems, and solve the original problem

Divide and Conquer

The general pattern

solve(I)

  n=size(I);

  if ($n \le$smallSize)

    solution=directlySolve(I);

  else

    divide I into I_1,… I_k;

    for each $i \in \{1,\dots,k\}$

      S_i=solve(I_i);

    solution=combine(S_1 ,… ,S_k);

  return solution

$T(n)=B(n)$ for $n \le smallSize$

$$T(n)=D(n)+\sum_{i=1}^{k} T(size(I_i))+C(n)$$

for $n > smallSize$

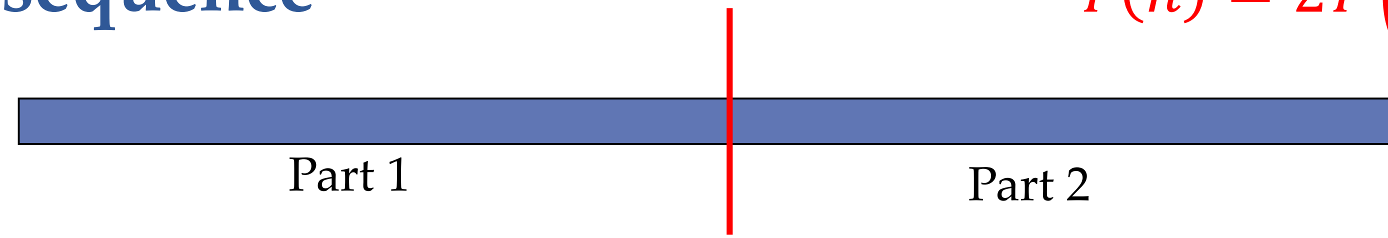
Divide Conquer

- **The BF recursion**
  - o Problem size: often decreases linearly
    - "n, n-1, n-2, …"
- **The D&C recursion**
  - o Problem size: often decrease exponentially
    - "n, n/2, n/4, n/8, …"

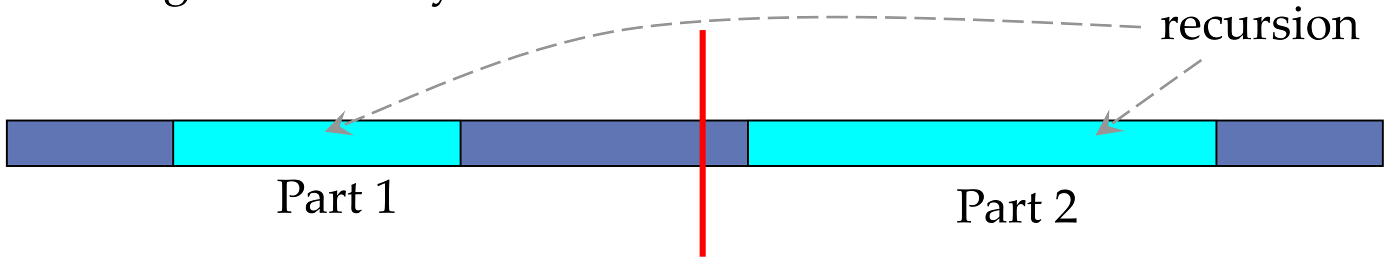
Examples

Max sum subsequence

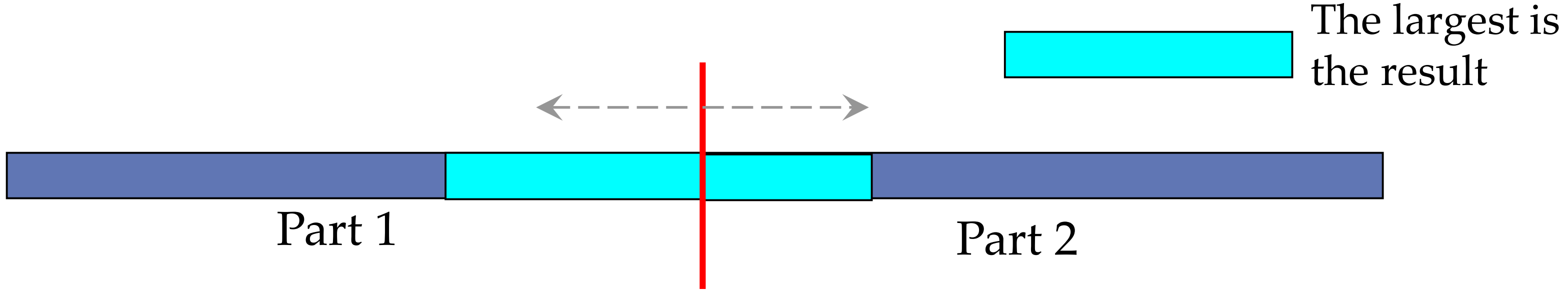
$$T(n) = 2T\left(\frac{n}{2}\right) + n$$



the sub with largest sum may be in:



or:



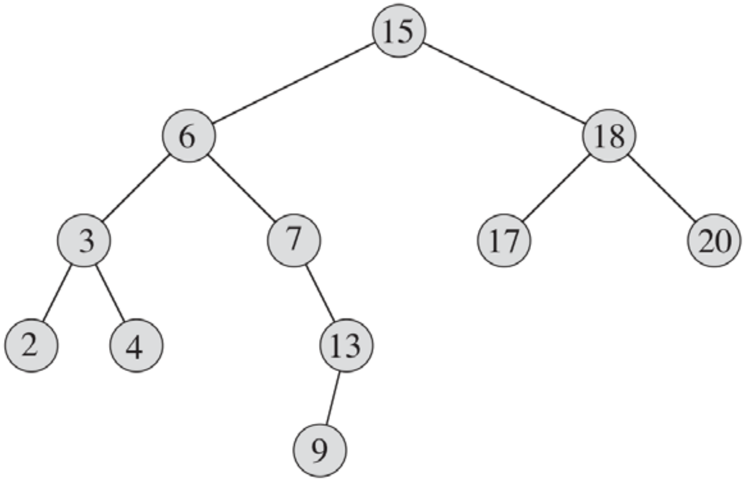
Examples

- **Maxima**
- **Frequent element**
- **Multiplication**
    - o Integer
    - o Matrix
- **Nearest point pair**

Examples

- **Arrays**

- **Trees**



Workhorse

- "Hard division, easy combination"
- "Easy division, hard combination"

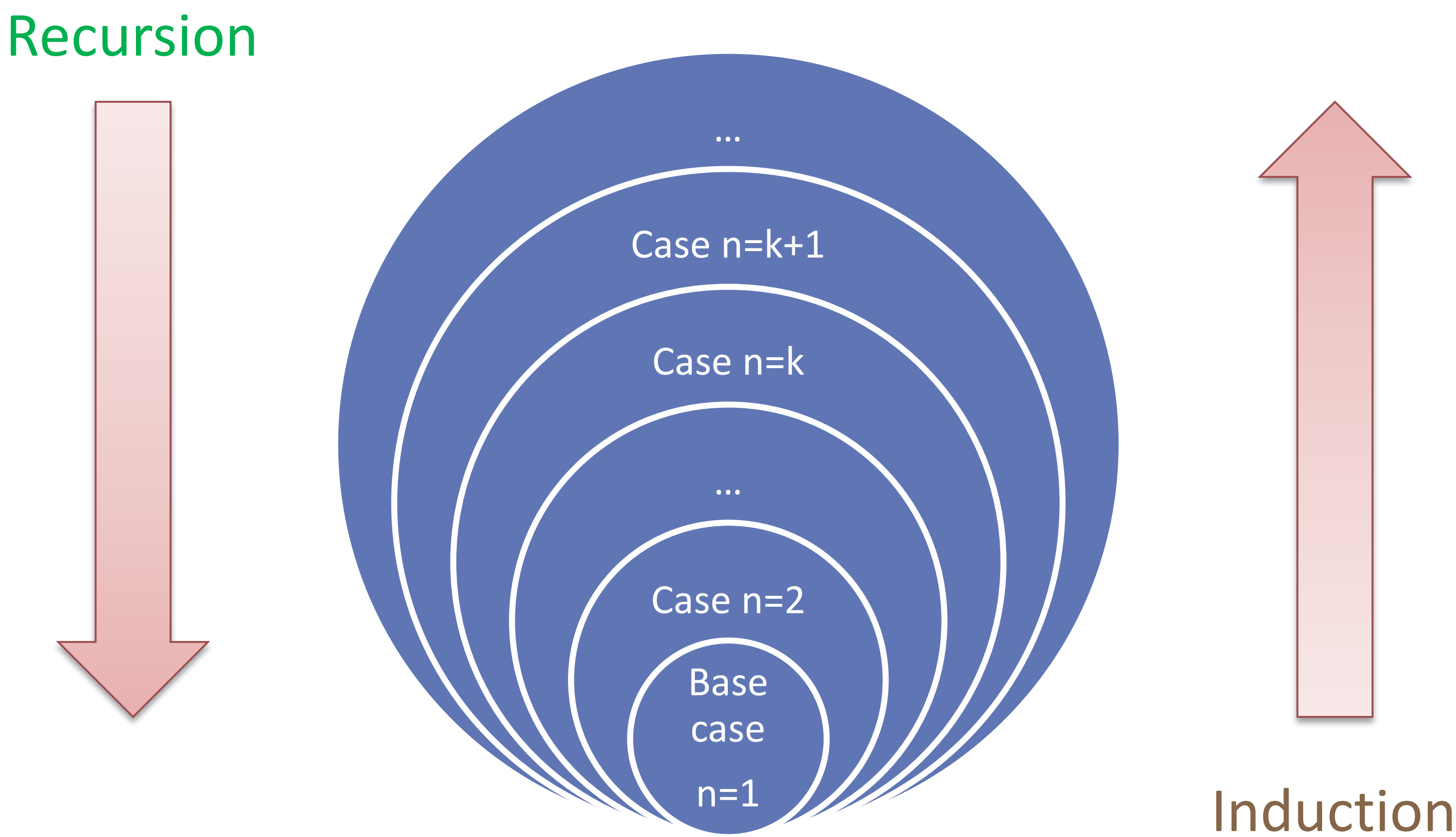
Usually, the "real work" is in one part.

More Applications of D&C

The simplest, most important tool for reducing complexity is the divide-and-conquer technique: analyze or design the system as a collection of interacting subsystems, called *modules*. The power of this technique lies primarily in being able to consider interactions among the components within a module without simultaneously thinking about the components that are inside other modules.

J. Saltzer et al., Principles of Computer System Design, Chap. 1.3.1.

Correctness of Recursion



Analysis of Recursion

- **Solving recurrence equations**

- **E.g., Bit counting**
  - Critical operation: add
  - The recurrence relation

$$T(n) = \begin{cases} 0 & n = 1 \\ T(\lfloor n/2 \rfloor) + 1 & n > 1 \end{cases}$$

Analysis of Recursion

- **Backward substitutions**

By the recursion equation : $T(n) = T\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + 1$

For simplicity , let $n = 2^k$ (k is a nonnegativ e integer),

that is, $k = \log n$

$$T(n) = T\left(\frac{n}{2}\right) + 1 = T\left(\frac{n}{4}\right) + 1 + 1 = T\left(\frac{n}{8}\right) + 1 + 1 + 1 = \ldots\ldots$$

$$T(n) = T\left(\frac{n}{2^k}\right) + \log n = \log n \qquad (\ T(1)=0\ )$$

Smooth Functions

- $f(n)$
  - Nonnegative eventually non-decreasing function defined on the set of natural numbers
- **$f(n)$ is called smooth**
  - If $f(2n) \in \Theta(f(n))$.

- **Examples of smooth functions**
  - $\log n$, n, $n\log n$ and n^{α} ($\alpha \geq 0$)
  - E.g., $2n\log 2n = 2n(\log n + \log 2) \in \Theta(n\log n)$

Even Smoother

- **Let $f(n)$ be a smooth function, then for any fixed integer $b \geq 2$, $f(bn) \in \Theta(f(n))$.**
  - That is, there exist positive constants c_b and d_b and a nonnegative integer n_0 such that

$$d_b f(n) \leq f(bn) \leq c_b f(n) \quad \textbf{for } n \geq n_0.$$

It is easy to prove that the result holds for $b = 2^k$,

for the second inequality :

$$f(2^k n) \leqslant c_2^k f(n) \quad \text{for } k = 1,2,3\ldots \text{ and } n \geqslant n_0.$$

For an arbitrary integer $b \geqslant 2, 2^{k-1} \leqslant b \leqslant 2^k$

Then, $f(bn) \leqslant f(2^k n) \leqslant c_2^k f(n)$, *we can use* c_2^k *as* c_b.

Smoothness Rule

- **If $T(n) \in \Theta(f(n))$ for values of n that are powers of $b (b \geq 2)$, then $T(n) \in \Theta(f(n))$.**
  - Let $T(n)$ be an eventually non-decreasing function and $f(n)$ be a smooth function.

$$\begin{aligned}
&\textit{Just proving the big-Oh part:}\\
&\textit{By the hypothsis: } T(b^k) \leqslant cf(b^k) \textit{ for } b^k \geqslant n_0.\\
&\textit{By the prior result: } f(bn) \leqslant c_b f(n) \textit{ for } n \geqslant n_0.\\
&\textit{Let } n_0 \leqslant b^k \leqslant n \leqslant b^{k+1},\\
&\underbrace{T(n) \leqslant T(b^{k+1})}_{\text{Non-decreasing}} \leqslant cf(b^{k+1}) = cf(bb^k) \leqslant cc_b f(b^k) \leqslant cc_b f(n)
\end{aligned}$$

Non-decreasing hypothesis Prior result Non-decreasing

Smoothness Rule

- **If $T(n) \in \Theta(f(n))$ for values of n that are powers of b ($b \geq 2$), then $T(n) \in \Theta(f(n))$.**
  - Let $T(n)$ be an eventually non-decreasing function and $f(n)$ be a smooth function.

By the recursion equation: $T(n) = T\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + 1$

For simplicity, let $n = 2^k$ (k is a nonnegative integer), that is, $k = \log n$

$$T(n) = T\left(\frac{n}{2}\right) + 1 = T\left(\frac{n}{4}\right) + 1 + 1 = T\left(\frac{n}{8}\right) + 1 + 1 + 1 = \ldots\ldots$$

$$T(n) = T\left(\frac{n}{2^k}\right) + \log n = \log n \qquad (T(1)=0)$$

Guess and Prove

- **Example: $T(n)=2T(\lfloor n/2 \rfloor) + n$**
- **Guess**
  - o $T(n) \in O(n)$?
    - $T(n) \le cn$, to be pro
  - o $T(n) \in O(n^2)$?
    - $T(n) \le cn^2$, to be prove
  - o **Or maybe**, $T(n) \in O(n\log$
    - $T(n) \le cn\log n$, to be prove
- **Prove**
  - o by substitution

Try to prove $T(n) \le cn$:

However:

$$
\begin{aligned}
T(n) &= 2T(\lfloor n/2 \rfloor) + n \\
&\le 2(c\lfloor n/2 \rfloor \log(\lfloor n/2 \rfloor)) + n \\
&\le cn\log(n/2) + n \\
&= cn\log n - cn\log 2 + n \\
&= cn\log n - cn + n \\
&\le cn\log n \quad \text{for } c \ge 1
\end{aligned}
$$

Divide and Conquer Recursions

- **Divide and conquer**
  - Divide the "big" problem to smaller ones
  - Conquer the "small" problems by recursion
  - Combine results of small problems, and solve the original problem
- **Divide and conquer recursion**

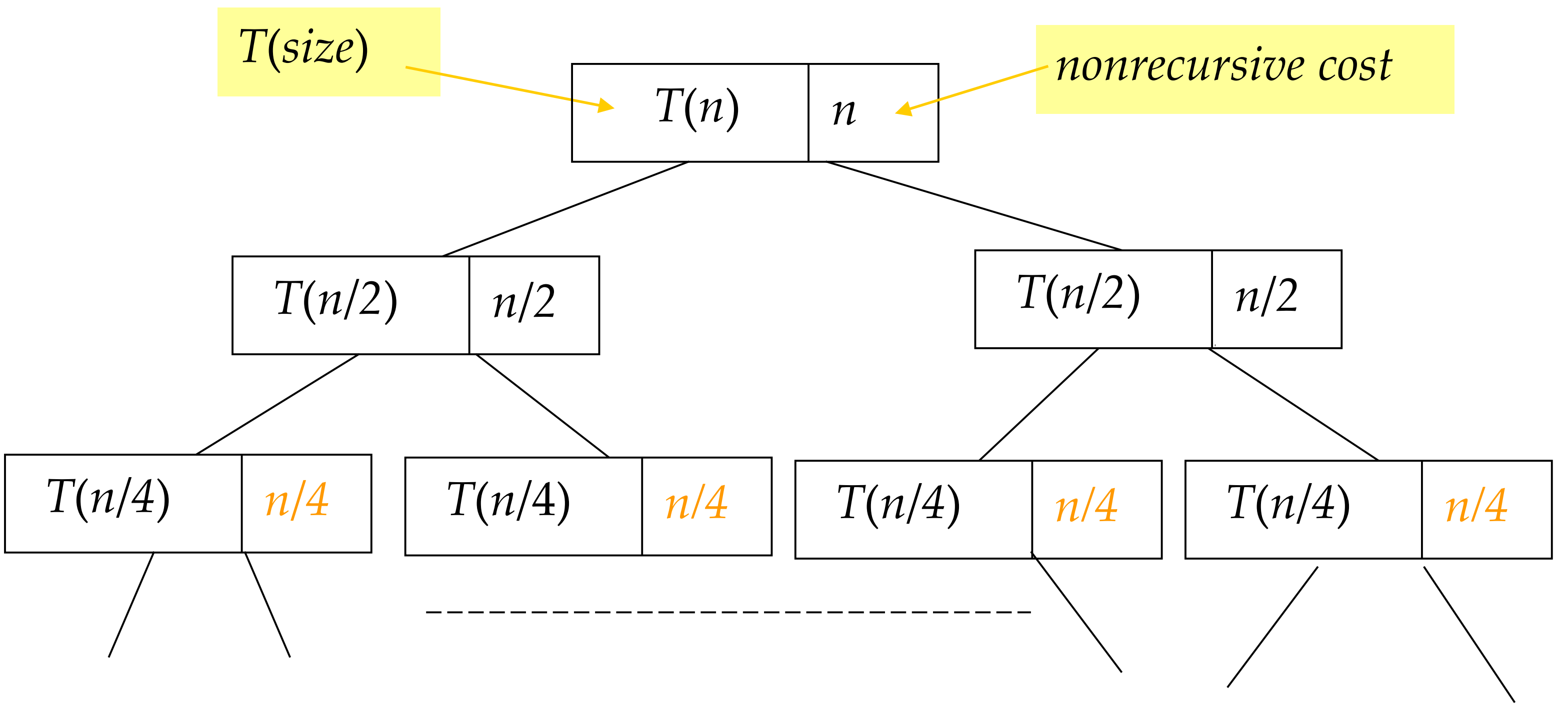
$$T(n) = b\,T(n/c) + f(n)$$

divide

Combine (and divide)

conquer

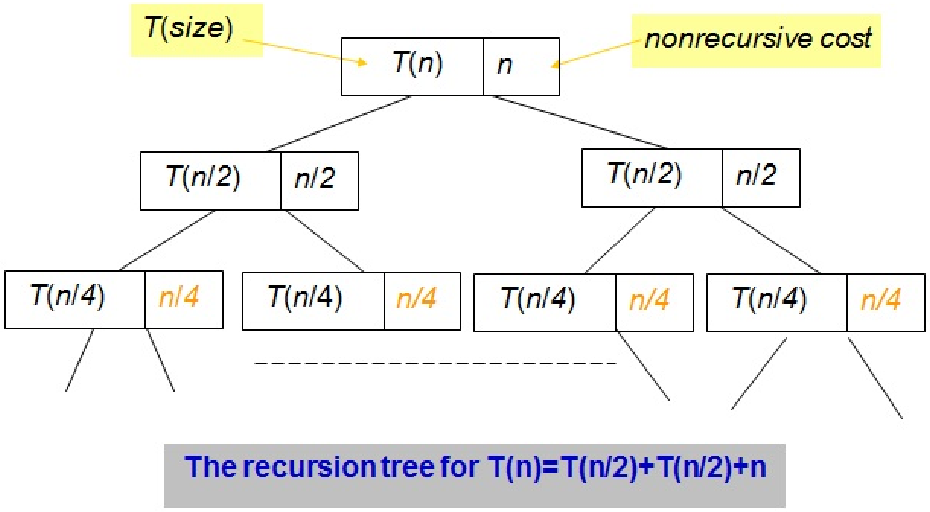
Recursion Tree



The recursion tree for T(n) = 2T(n/2) + n

Recursion Tree

- **Node**
  - Non-leaf
    - Non-recursive cost
    - Recursive cost
  - Leaf
    - Base case
- **Edge**
  - Recursion



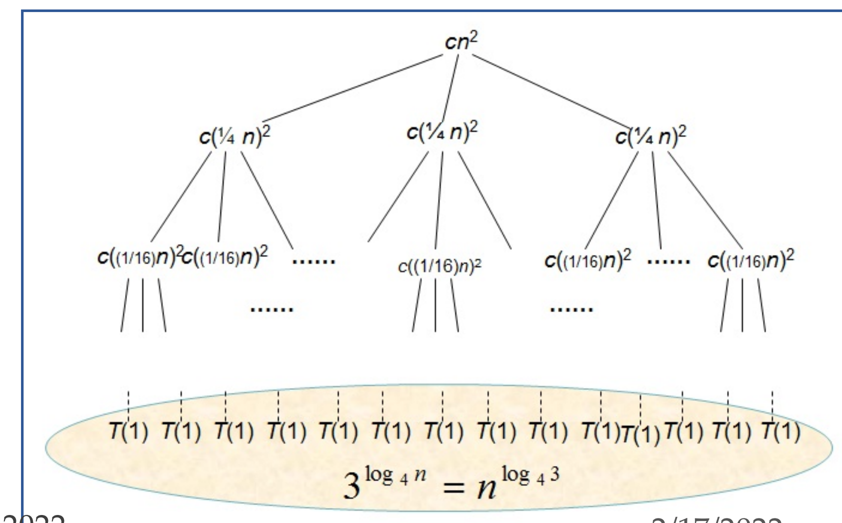
Recursion Tree

- $$T(n) = \overbrace{\underbrace{3}_{\text{\# of sub-problems}}T(n/\underbrace{4}_{\text{size of sub-problems}})}^{\text{Recursive cost}} + \overbrace{\Theta(n^2)}^{\text{Non-recursive cost}}$$

- **Total cost**

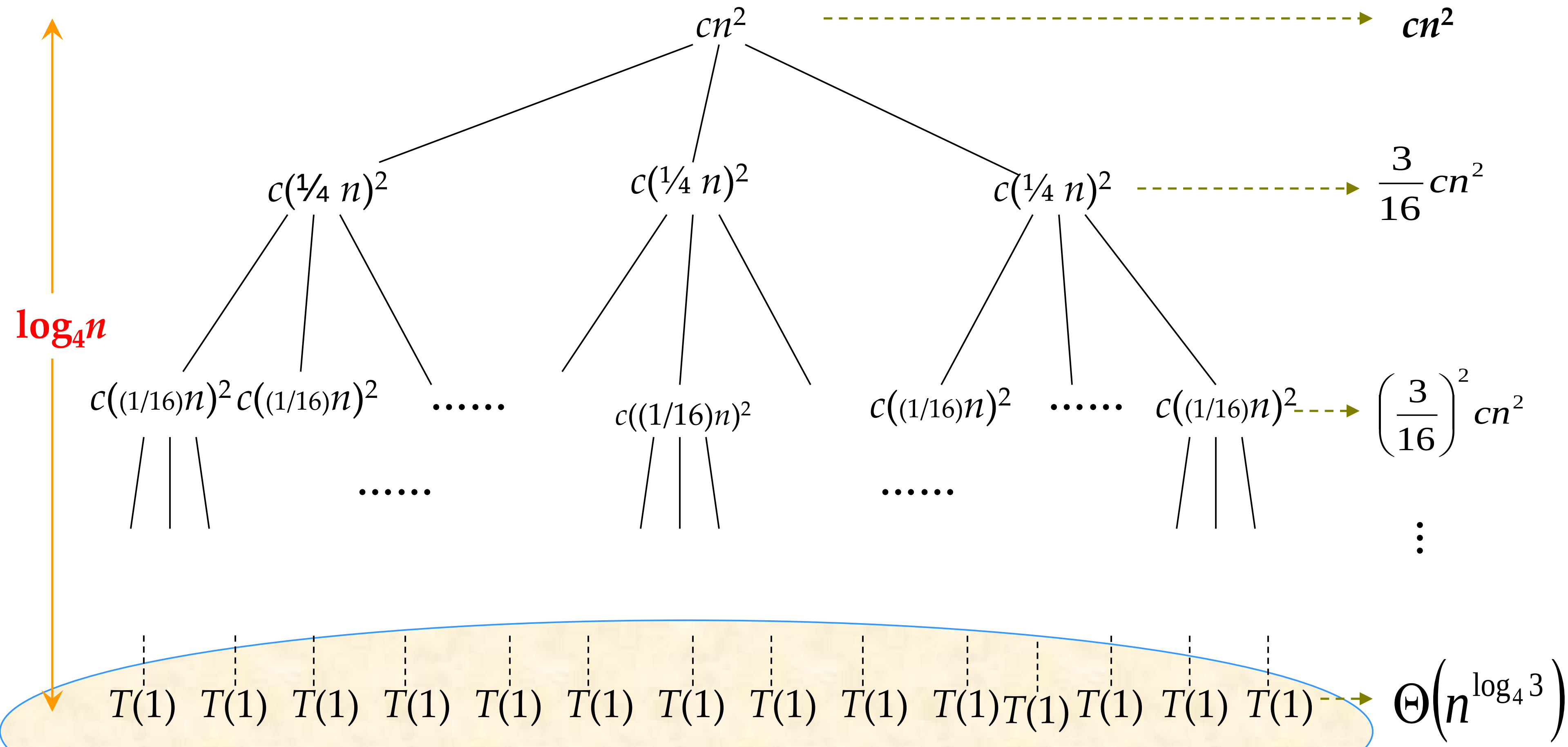
Σ

Sum of row sums



Sum of Row-sums

$T(n) = 3T(\lfloor n/4 \rfloor) + \Theta(n^2)$



Note: $3^{\log_4 n} = n^{\log_4 3}$

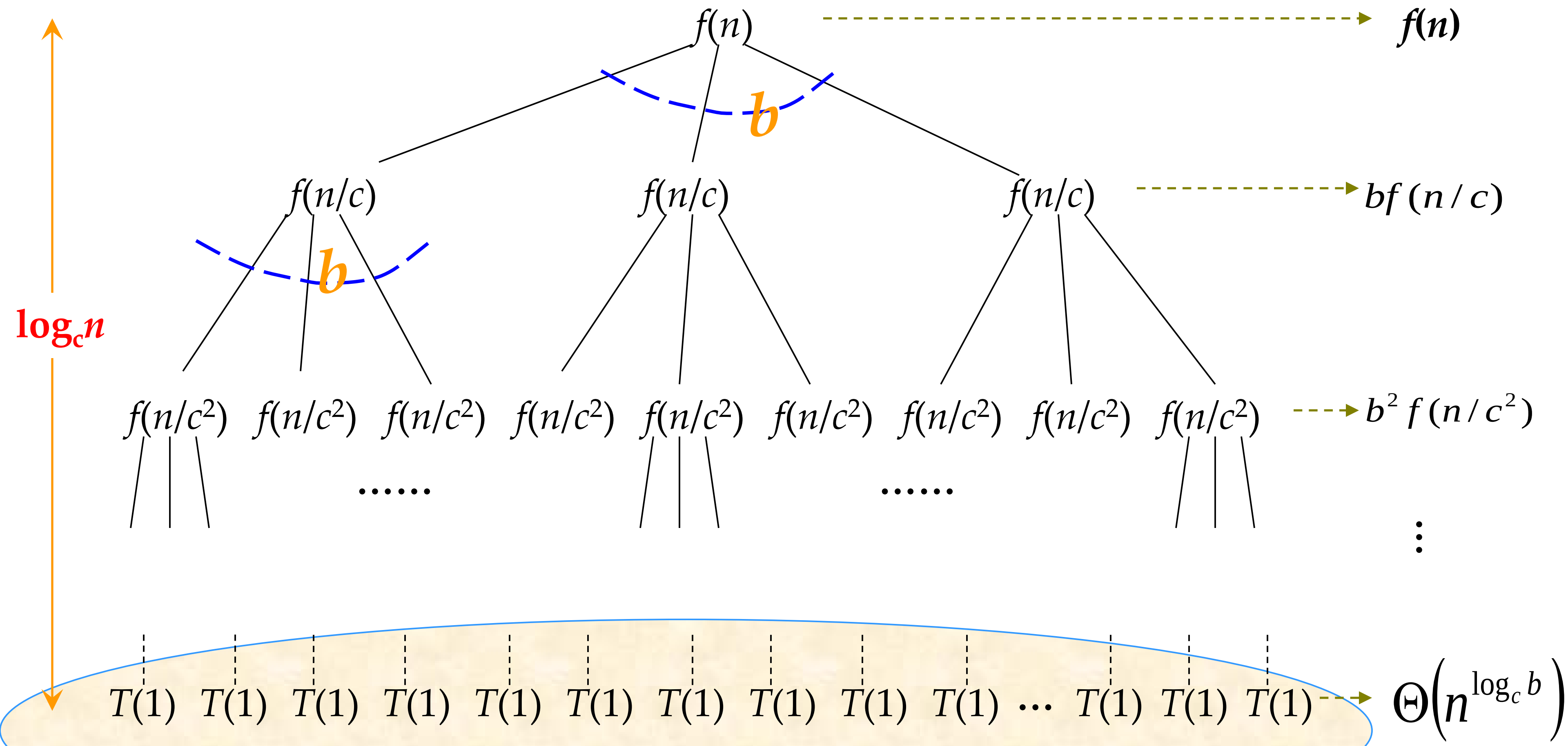
Total: $\Theta(n^2)$

Solving the Divide-and-Conquer Recurrence

- **The recursion equation for divide-and-conquer, the general case: $T(n)=bT(n/c)+f(n)$**
- **Observations:**
  - Let base-cases occur at depth D(leaf), then $n/c^D=1$, that is $D=\log(n)/\log(c)$
  - Let the number of leaves of the tree be L, then $L=b^D$, that is $L=b^{(\log(n)/\log(c))}$.
  - By a little algebra: $L=n^E$, where $E=\log(b)/\log(c)$, called *critical exponent*.

Recursion Tree for

$T(n)=bT(n/c)+f(n)$



Note: $b^{\log_c n} = n^{\log_c b}$

Total ?

Divide-and-Conquer - the Solution

- **The solution of divide-and-conquer equation is the non-recursive costs of all nodes in the tree, which is the sum of the row-sums**
  - The recursion tree has depth $D=\log(n)/\log(c)$, so there are about that many row-sums.

- **The 0^{th} row-sum**
  - is $f(n)$, the nonrecursive cost of the root.
- **The D^{th} row-sum**
  - is n^E, assuming base cases cost 1, or $\Theta(n^E)$ in any event.

Solution by Row-sums

- **[Little Master Theorem] Row-sums decide the solution of the equation for divide-and-conquer:**
  - Increasing geometric series: $T(n) \in \Theta(n^E)$
  - Constant: $T(n) \in \Theta(f(n) \log n)$
  - Decreasing geometric series: $T(n) \in \Theta(f(n))$

This can be generalized to get a result not using explicitly row-sums.

Master Theorem

- **Loosening the restrictions on $f(n)$**
  - Case 1: $f(n) \in O(n^{E-\varepsilon})$, ($\varepsilon>0$), then:

    $$T(n) \in \Theta(n^E)$$

  - Case 2: $f(n) \in \Theta(n^E)$, as all node depth contribute about equally:

    $$T(n) \in \Theta(f(n)\log(n))$$

  - case 3: $f(n) \in \Omega(n^{E+\varepsilon})$, ($\varepsilon>0$), and if $bf(n/c) \leq \theta f(n)$ for some constant $\theta < 1$ and all sufficiently large n, then:

    $$T(n) \in \Theta(f(n))$$

The positive ε is critical, resulting gaps between cases as well

Using Master Theorem

- Example 1: $T(n) = 9T(\frac{n}{3}) + n$

  $b = 9, c = 3, E = 2, f(n) = n = O(n^{E-1})$

  Case 1 applies: $T(n) = \Theta(n^2)$

- Example 2: $T(n) = T(\frac{2}{3}n) + 1$

  $b = 1, c = \frac{3}{2}, E = 0, f(n) = 1 = \Theta(n^E)$

  Case 2 applies: $T(n) = \Theta(\log n)$

- Example 3: $T(n) = 3T(\frac{n}{4}) + n \log n$

  $b = 3, c = 4, E = \log_4 3, f(n) = \Omega(n^{E+\epsilon})$

  $bf(\frac{n}{4}) = \frac{3}{4} n \log n - \frac{3}{2} n$

  Case 3 applies: $T(n) = \Theta(n \log n)$

Using Master Theorem

- **T(n) = 2T(n/2) + nlogn**
  - Does Case 3 apply? Why?

- **T(n)=$\sqrt{n}$ T($\sqrt{n}$) + n**

- **The gap between the 3 cases**
  - Often, none of the 3 cases apply
  - Your task: design more non-solvable recursions

Thank you!

Q & A

Yu Huang
yuhuang@nju.edu.cn
http://cs.nju.edu.cn/yuhuang