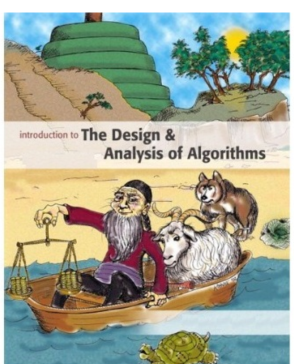
Introduction to

Algorithm Design and Analysis

[14] Minimum Spanning Tree



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In the last class…

- **Undirected and Symmetric Digraph**
  - DFS skeleton
- **Biconnected Components**
  - Articulation point
  - Bridge
- **Other undirected graph problems**
  - Orientation for undirected graphs
  - MST based on graph traversal

当年我们学贪心算法

定义 7 **一个图论算法的计算量**$f(\nu,\varepsilon)=O(P(\nu,\varepsilon))$**时，则称此算法为有效算法或好算法，其中**$P(\nu,\varepsilon)$**是某个多项式，**$\nu$**与**$\varepsilon$ **分别是图的顶数与边数。**

Dijkstra **算法**（u,v 不相邻时，$w(uv)=\infty$）

（1）令 $l(u_0)=0$；$l(v)=\infty$，$v\neq u_0$；$S_0=\{u_0\}$，$i=0$。

（2）对每一个 $v\in\bar{S}_i$（$\bar{S}_i$ 指 S_i 以外的顶所成之集合），用 $\min\{l(v),l(u_i)+w(u_iv)\}$ 代替 $l(v)$；设 u_{i+1} 是使 $l(v)$ 取最小值的 $\bar{S}_i$ 中的顶，令 $S_{i+1}=S_i\cup\{u_{i+1}\}$；

（3）若 $i=\nu-1$，止；若 $i<\nu-1$，用 $i+1$ 代替 i，转（2）。

由上述算法知：

（1）S_i 中各顶标 $l(u)$ 即为 u_0 到 u 的距离。又因 $\nu<\infty$，故有限步之后，$V(G)$ 中每一顶都标志了与 u_0 的距离，从而可以找到各顶到 u_0 的最短轨。

（2）Dijkstra 算法的时间复杂度 $f(\nu,\varepsilon)=O(\nu^2)$，所以是有效算法。

Greedy Strategy

- **Optimization Problem**
- **Greedy Strategy**

- **MST Problem**
  - Prim's Algorithm
  - Kruskal's Algorithm
- **Single-Source Shortest Path Problem**
  - Dijkstra's Algorithm

Greedy Strategy for Optimization Problems

- **Coin change Problem**
  - [candidates] A finite set of coins, of 1, 5, 10 and 25 units, with enough number for each value
  - [constraints] Pay an exact amount by a selected set of coins
  - [optimization] a smallest possible number of coins in the selected set
- **Solution by greedy strategy**
  - For each selection, choose the highest-valued coin as possible.

Greedy Fails Sometimes

We have to pay 15 in total

- **If the available types of coins are {1,5,12}**
  - The greedy choice is {12,1,1,1}
  - But the smallest set of coins is {5,5,5}

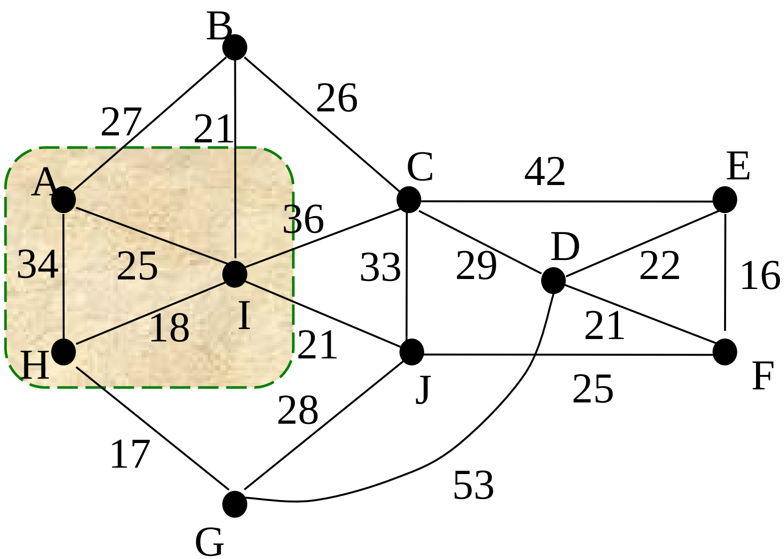
- **If the available types of coins are {1,5,10,25}**
  - The greedy choice is always correct

Greedy Strategy

- **Expanding the partial solution step by step.**
- **In each step, a selection is made from a set of candidates. The choice made must be:**
  - [Feasible] it has to satisfy the problem's constraints
  - [Locally optimal] it has to be the best local choice among all feasible choices on the step
  - [Irrevocable] the choice cannot be revoked in subsequent steps

```
set greedy(set candidate)
  set S=Ø;
  while not solution(S) and   candidate≠Ø
    select locally optimizing x from candidate;
    candidate=candidate-{x};
    if feasible(x) then S=S∪{x};
  if solution(S) then return S
       else return ("no solution")
```

Weighted Graph and MST

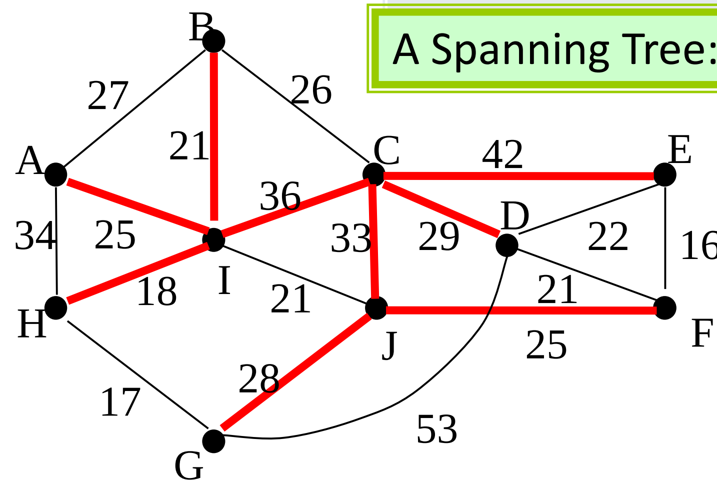


A weighted graph

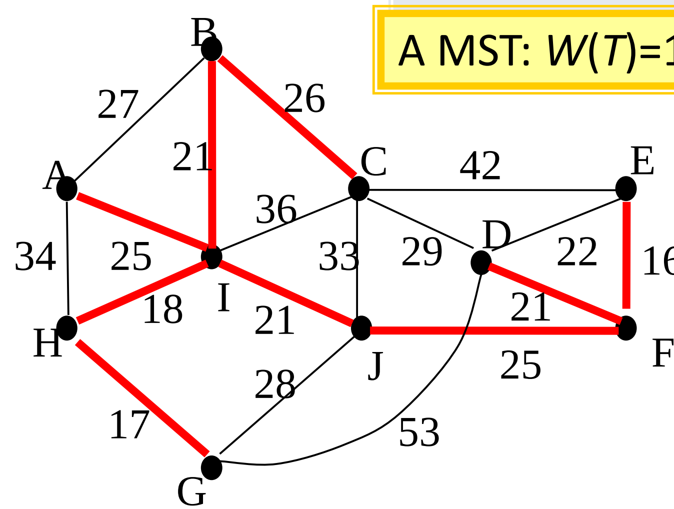
The nearest neighbor of vertex ***I*** is ***H***

The nearest neighbor of shaded subset of vertex is ***G***

A Spanning Tree: $W(T)$=257



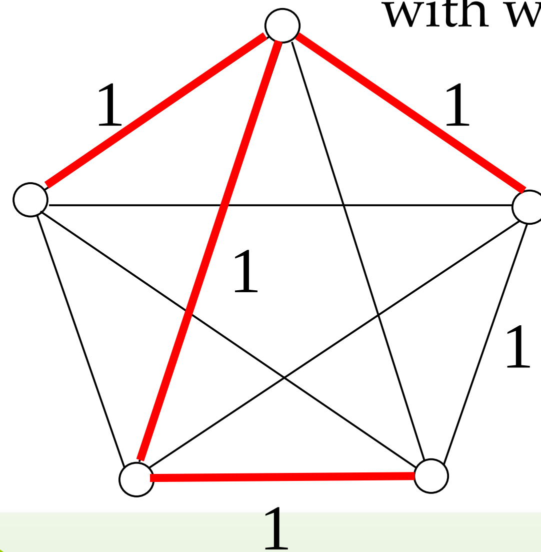
A MST: $W(T)$=190



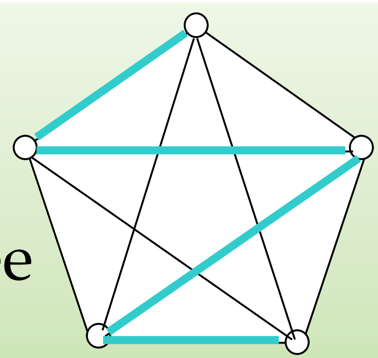
Graph Traversal and MST

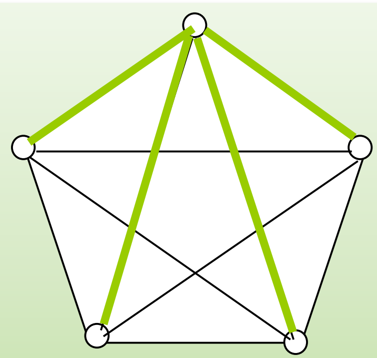
There are cases that graph traversal tree **cannot** be minimum spanning tree, with the vertices explored in any order.

All other edges with weight 5



DFS tree





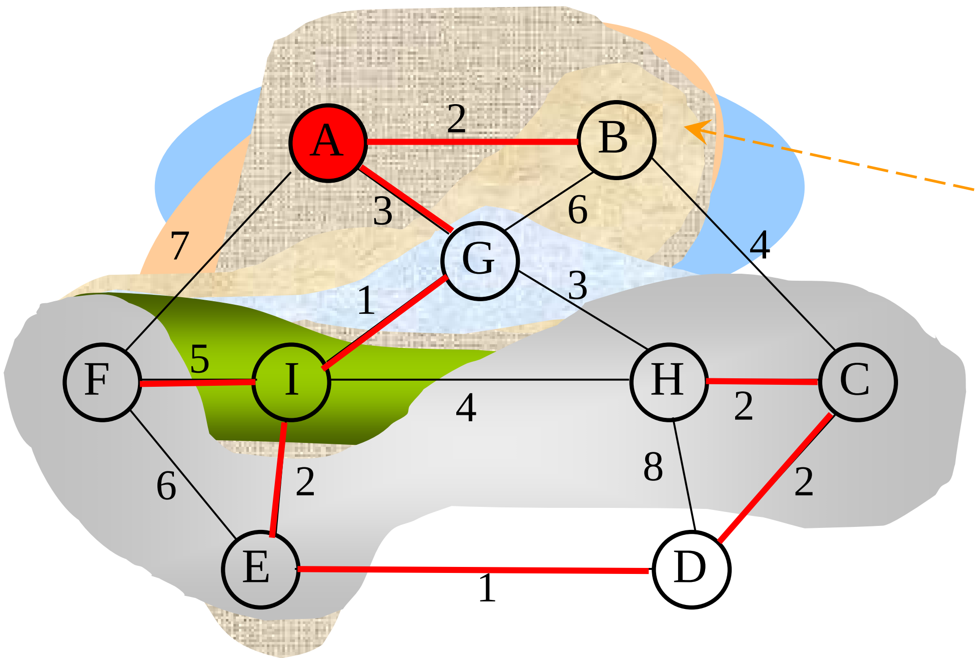
BFS tree

in any ordering of vertex

Greedy Algorithms for MST

- **Prim's algorithm:**
  - Difficult selecting: "best local optimization means **no cycle and small weight under limitation**.
  - Easy checking: doing nothing
- **Kruskal's algorithm:**
  - Easy selecting: smallest in primitive meaning
  - Difficult checking: **no cycle**

Prim's Algorithm



edges included in the MST

Greedy strategy:
For each set of fringe vertex, select the edge with the minimal weight, that is, local optimal.

Correctness: How to Prove

of nodes selected

$T^{(1)} \quad T^{(2)} \quad \ldots \quad T^{(k\text{-}1)} \quad T^{(k)} \quad T^{(k+1)} \quad \ldots \quad T^{(n)}$

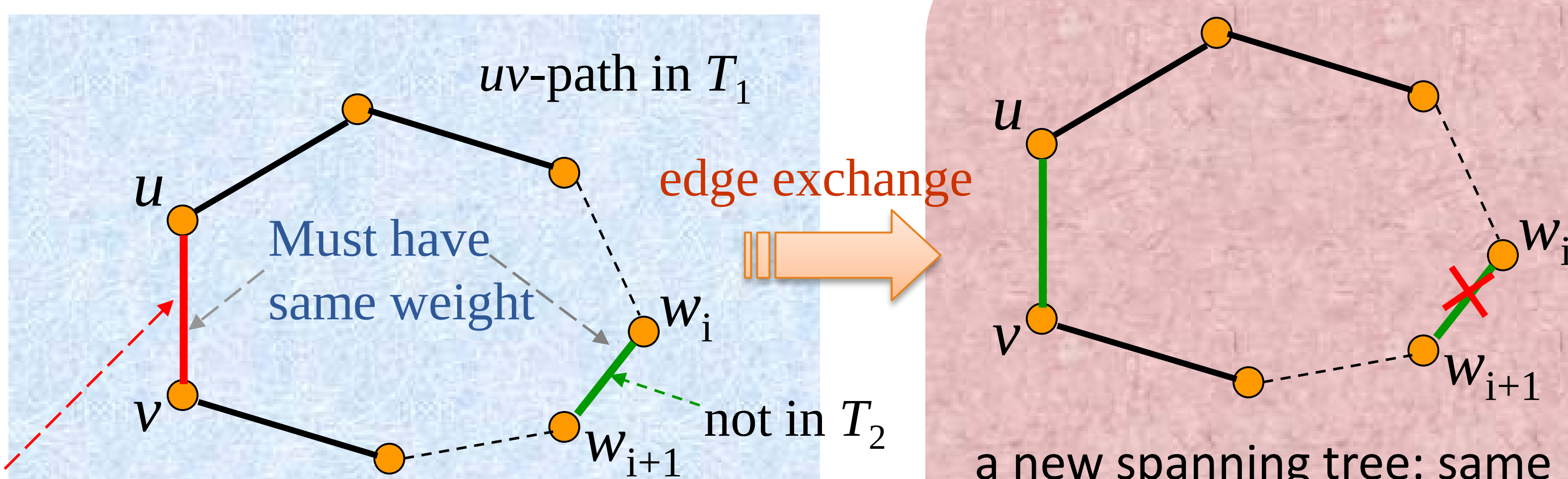
Invariance: MST

- Spanning tree
- Min weight

Computational thinking

Minimum Spanning Tree Property

- **A spanning tree T of a connected, weighted graph has MST property if and only if for any non-tree edge uv, $T \cup \{uv\}$ contain a cycle in which uv is one of the maximum-weight edge.**
- **All the spanning trees having MST property have the same weight.**



edge uv in T_2 but not in T_1 , with minimum weight among all different edges

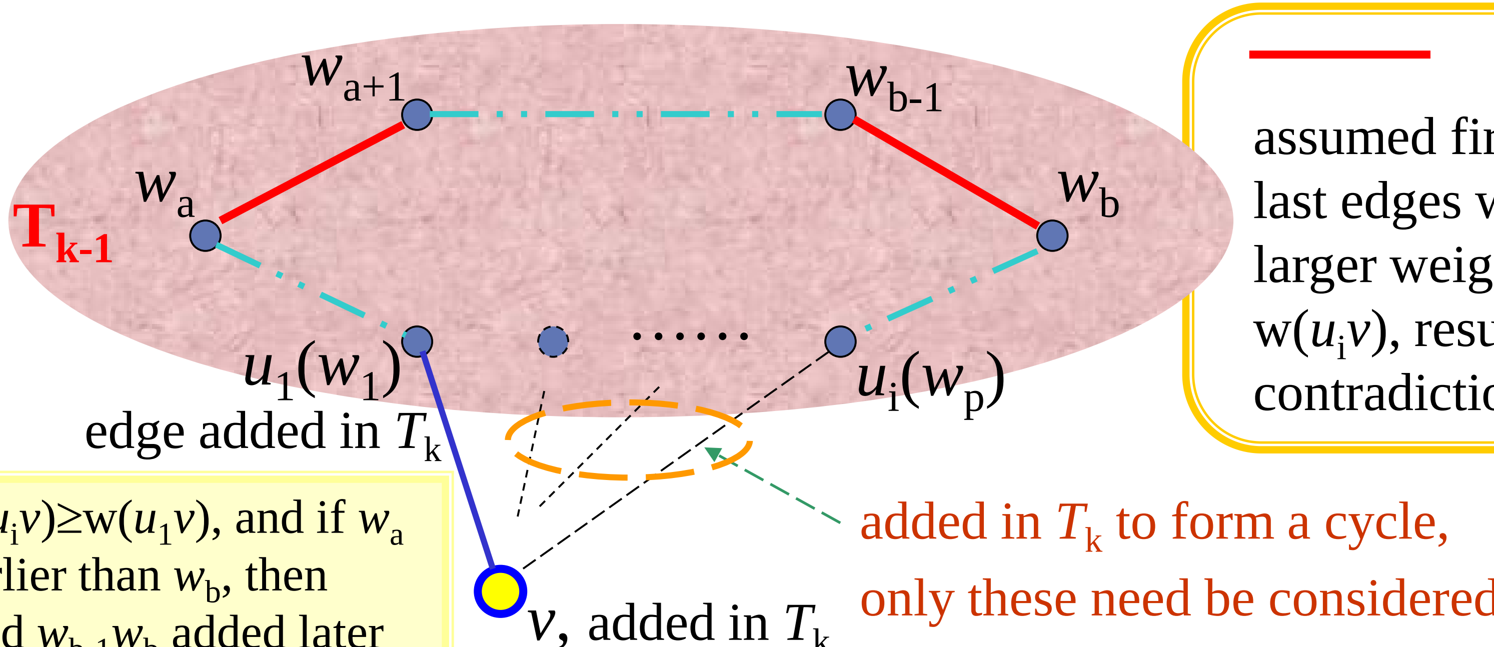
a new spanning tree: same weight as T_1, less different edges from that of T_2

MST Property and Minimum Spanning Tree

- **In a connected, weighted graph G=(V,E,W), a tree T is a minimum spanning tree if and only if T has the MST property.**
- **Proof**
  - $\Rightarrow$ For a minimum spanning tree T, if it does not have MST property. So, there is a non-tree edge uv, and **$T \cup \{uv\}$ contain an edge xy with weight larger than that of uv**. Substituting uv for xy results a spanning tree with less weight than T. Contradiction.
  - $\Leftarrow$ As claimed above, any minimum spanning tree has the MST property. Since T has MST property, it has the same weight as any minimum spanning tree, i.e. T is a minimum spanning tree as well.

Correctness of Prim's Algorithm

- Let T_k be the tree constructed after the k^{th} step of Prim's algorithm is executed. Then T_k has the MST property in G_k, the subgraph of G induced by vertices of T_k.



assumed first and last edges with larger weight than $w(u_i v)$, resulting contradictions.

Note: $w(u_i v) \geq w(u_1 v)$, and if w_a added earlier than w_b, then $w_a w_{a+1}$ and $w_{b-1} w_b$ added later than any edges in $u_1 w_a$-path, and v as well.

Key Issue in Implementation

- **Maintaining the set of fringe vertices**
  - Create the set and update it after each vertex is "selected" (***deleting*** the vertex having been selected and ***inserting*** new fringe vertices)
  - Easy to decide the vertex with "highest priority"
  - Changing the priority of the vertices (***decreasing key***).
- **The choice: priority queue**

Implementation

Main Procedure

```
primMST(G,n)
  Initialize the priority queue pq as empty;
  Select vertex s to start the tree;
  Set its candidate edge to (-1,s,0);
  insert(pq,s,0);
  while (pq is not empty)
    v=getMin(pq); deleteMin(pq);
    add the candidate edge of v to the tree;
    updateFringe(pq,G,v);
  return
```

getMin(*pq*) always be the vertex with the smallest key in the fringe set.

ADT operation executions:

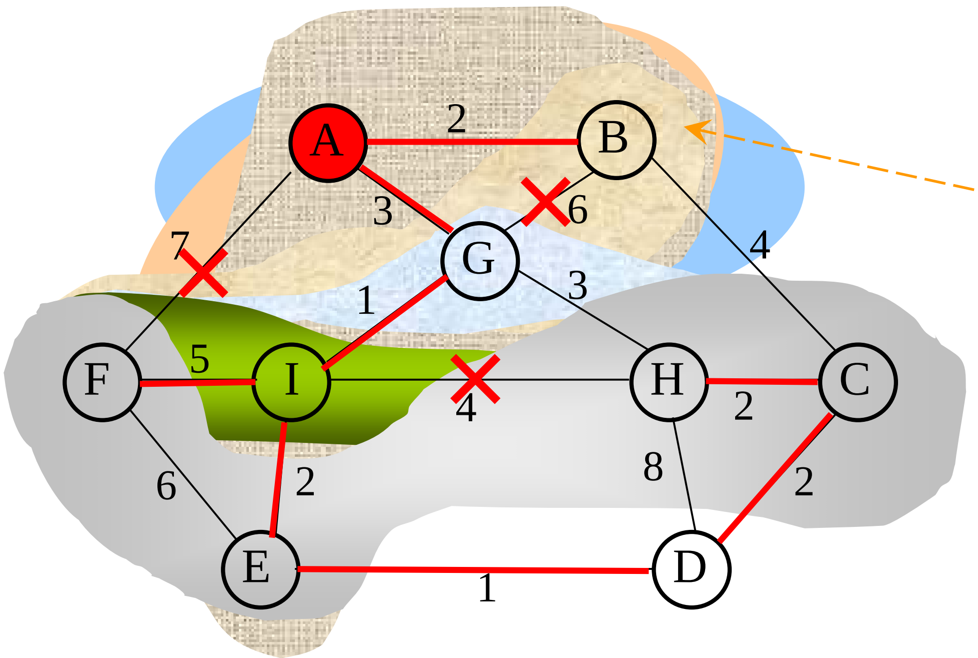
insert, getMin, deleteMin: *n* times

decreaseKey: *m* times

Updating the Queue

```
updateFringe(pq,G,v)
  For all vertices w adjcent to v //2m loops
    newWgt=w(v,w);
    if w.status is unseen then
      Set its candidate edge to (v,w,newWgt);
      insert(pq,w,newWgt)
    else
      if newWgt<getPriorty(pq,w)
        Revise its candidate edge to (v,w,newWgt);
        decreaseKey(pq,w,newWgt)
  return
```

Prim's Algorithm



Greedy strategy:
For each set of fringe vertex, select the edge with the minimal weight, that is, local optimal.

edges included in the MST

Complexity

- **Operations on ADT priority queue:** **(for a graph with *n* vertices and *m* edges)**
  - insert: n ; getMin: n ; deleteMin: n ;
  - decreasKey: m (appears in $2m$ loops, but execute at most m)
- **So,**

$$T(n,m) = O(nT(\text{getMin})+nT(\text{deleteMin+insert})+mT(\text{decreaseKey}))$$

- **Implementing priority queue using array, we can get** $\Theta(n^2+m)$

Some History

Robert C. Prim

From Wikipedia, the free encyclopedia

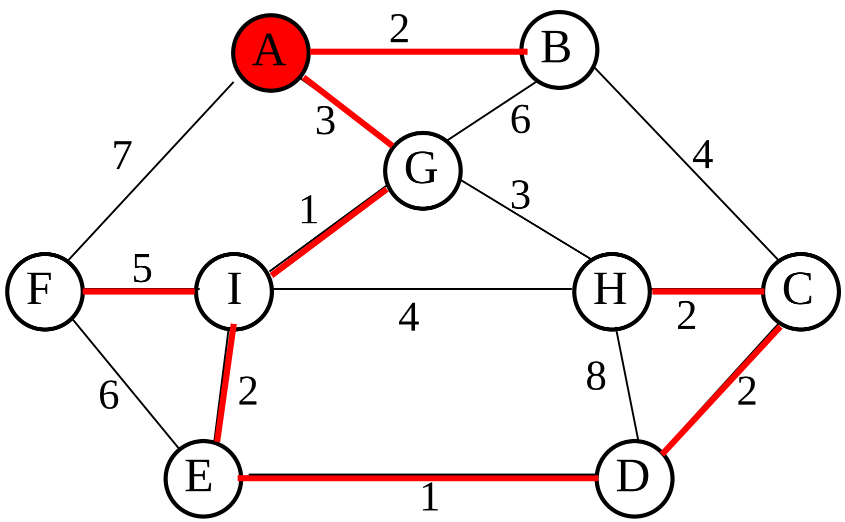
Robert Clay Prim (born September 25, 1921[1] in Sweetwater, Texas) is an American mathematician and computer scientist.

In 1941, Prim received his B.S. in Electrical Engineering from The University of Texas at Austin,[2] where he also met his wife Alice (Hutter) Prim (1921–2009), whom he married in 1942. Later in 1949, he received his Ph.D. in Mathematics from Princeton University, where he also worked as a research associate from 1948 until 1949.

During the climax of World War II (1941–1944), Prim worked as an engineer for General Electric. From 1944 until 1949, he was hired by the United States Naval Ordnance Lab as an engineer and later a mathematician. At Bell Laboratories, he served as director of mathematics research from 1958 to 1961. There, Prim developed Prim's algorithm. Also during his tenure at Bell Labs, Robert Prim assisted the Weapons Reliability Committee at Sandia National Laboratory chaired by Walter McNair in 1951.[3] After Bell Laboratories, Prim became vice president of research at Sandia National Laboratories.

During his career at Bell Laboratories, Robert Prim along with coworker Joseph Kruskal developed two different algorithms (see greedy algorithm) for finding a minimum spanning tree in a weighted graph, a basic stumbling block in computer network design. His self-named algorithm, Prim's algorithm, was originally discovered in 1930 by mathematician Vojtěch Jarník and later independently by Prim in 1957. It was later rediscovered by Edsger Dijkstra in 1959. It is sometimes referred to as the *DJP algorithm* or the *Jarník algorithm*.

Kruskal's Algorithm



edges included in the MST

Also Greedy strategy:
From the set of edges not yet included in the partially built MST, select the edge with the minimal weight, that is, local optimal, in another sense.

Key Issue in Implementation

- **How to know an insertion of edge will result in a cycle *efficiently*?**
- **For correctness: the two endpoints of the selected edge *can not* be in the same connected components.**
- **For the efficiency: connected components are implemented as dynamic equivalence classes using union-find.**

Kruskal's Algorithm: the Procedure

- kruskalMST(G,n,F) //outline
-     int count;
-     Build a minimizing priority queue, pq, of edges of G, prioritized by weight.
-     Initialize a Union-Find structure, sets, in which each vertex of G is in its own set.
-
- F=ϕ;
-     while (isEmpty(pq) == false)
-         vwEdge = getMin(pq);
-         deleteMin(pq);
-         int vSet = find(sets, vwEdge.from);
-         int wSet = find(sets, vwEdge.to);
-         if (vSet ≠ wSet)
-             Add vwEdge to F;
-             union(sets, vSet, wSet)
-     return

Simply sorting, the cost will be $\Theta(m\log m)$

Prim vs. Kruskal

- **Lower bound for MST**
  - For a correct MST, each edge in the graph should be examined at least once.
  - So, the lower bound is $\Omega(m)$
- **$\Theta(n^2+m)$ and $\Theta(m\log m)$, which is better?**
  - Generally speaking, depends on the density of edge of the graph.

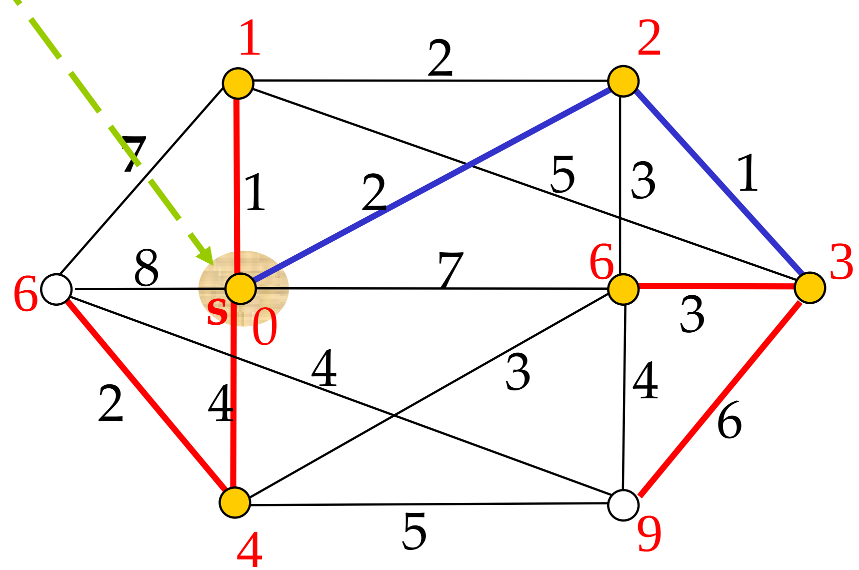
Single Source Shortest Paths

The single source

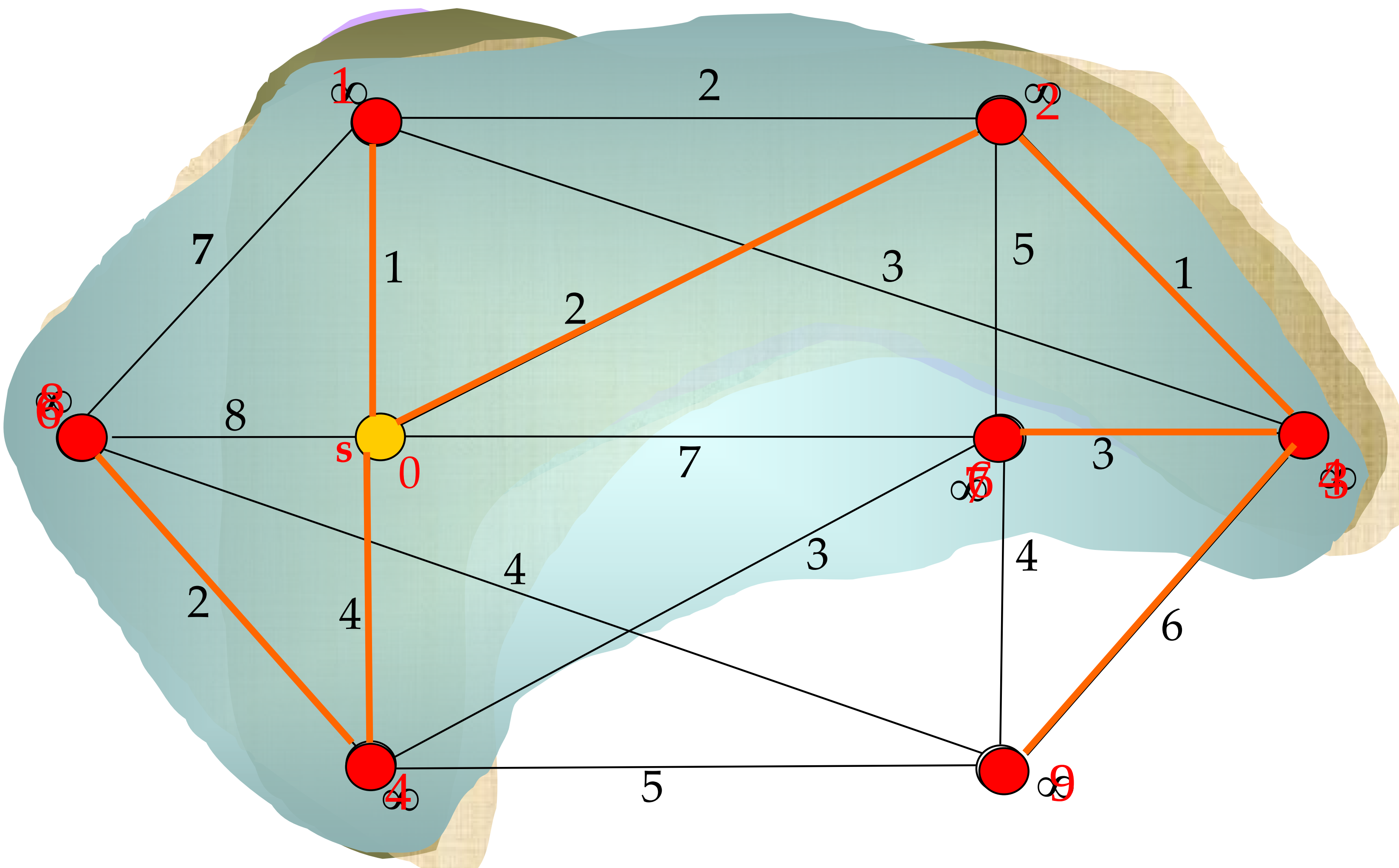
Red labels on each vertex is the length of the shortest path from s to the vertex.

Note:

The shortest [0, 3]-path doesn't contain the shortest edge leaving s, the edge [0,1]



Dijkstra's Algorithm



Thank you!

Q & A

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