



Introduction to

Algorithm Design and Analysis

[2] Asymptotics



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In the Last Class...

- **Algorithm – the spirit of computing**
 - Model of computation
- **Algorithm design and analysis**
 - Design
 - Correctness proof by induction
 - Analysis
 - Worst-case / average-case complexity



Asymptotic Behavior

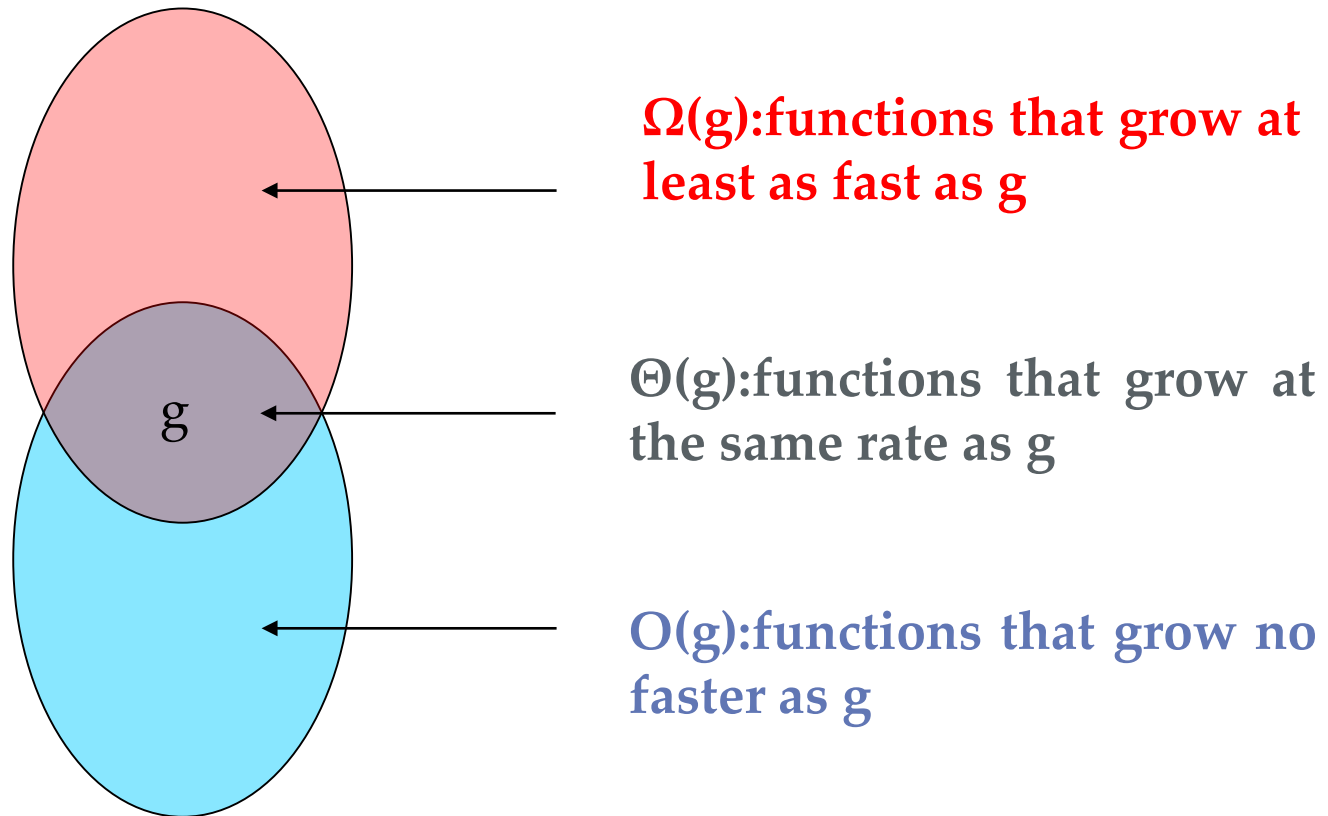
- **Asymptotic growth rate of functions**
 - Basic idea
- **Key notations**
 - O, Ω, Θ
 - o, ω
- **Brute force enumeration**
 - By iteration
 - By recursion



How to Compare Two Algorithms

- **Algorithm analysis, with *simplifications***
 - Measuring the cost by the number of critical operations
 - Large input size only
 - Essential part only
 - Only the leading term in $f(n)$ is considered
 - Constant coefficients are ignored
- **Capturing the essential part in the cost in a mathematical way**
 - Asymptotic growth rate of $f(n)$

Relative Growth Rate



“Big Oh”

- **Basic idea** $f(n) \in O(g(n))$
 - For sufficiently large input size, $g(n)$ is an upper bound for $f(n)$
- **Definition – “ $\varepsilon - N$ ”**
 - Giving $g: \mathbb{N} \rightarrow \mathbb{R}^+$, then $O(g)$ is the set of $f: \mathbb{N} \rightarrow \mathbb{R}^+$, such that for some $c \in \mathbb{R}^+$ and some $n_0 \in \mathbb{N}$, $f(n) \leq cg(n)$ for all $n \geq n_0$
- **Definition – “ $\lim_{n \rightarrow \infty}$ ”**
 - $f \in O(g)$ if $\lim_{n \rightarrow \infty} \frac{f(n)}{g(n)} = c < \infty$

The limit may not exist, though it usually does.

Example


- Let $f(n)=n^2$, $g(n)=n\log n$, then:

- $f \notin O(g)$, since

$$\lim_{n \rightarrow \infty} \frac{n^2}{n \log n} = \lim_{n \rightarrow \infty} \frac{n}{\log n} = \lim_{n \rightarrow \infty} \frac{1}{\frac{1}{n \ln 2}} = +\infty$$

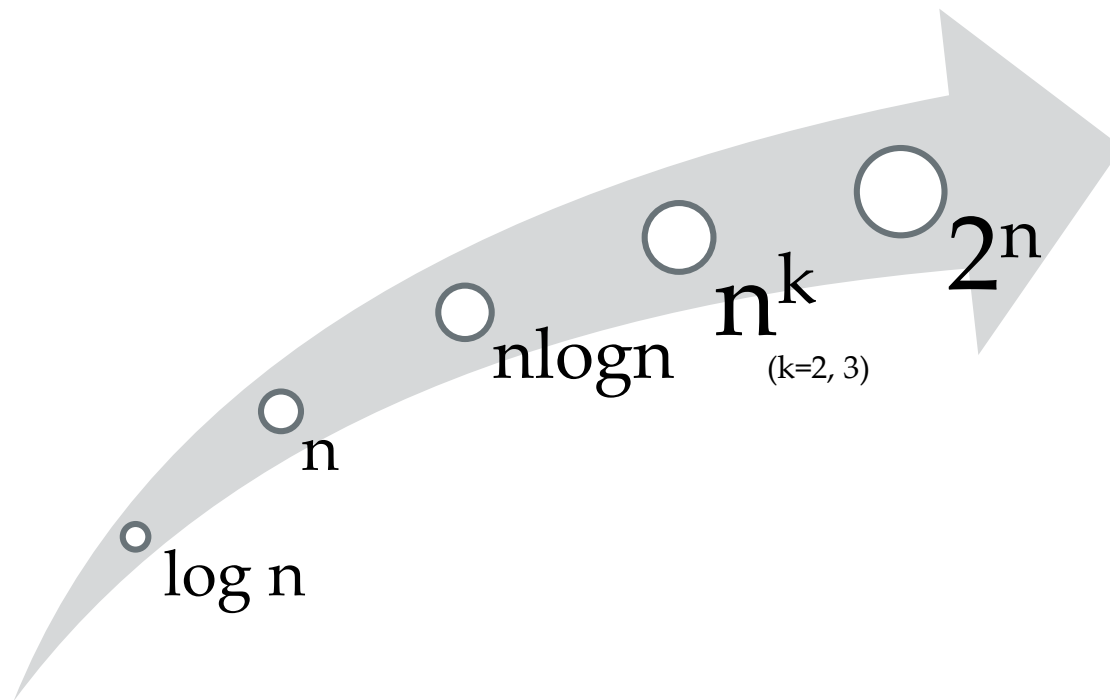
- $g \in O(f)$, since

$$\lim_{n \rightarrow \infty} \frac{n \log n}{n^2} = \lim_{n \rightarrow \infty} \frac{\log n}{n} = \lim_{n \rightarrow \infty} \frac{1}{n \ln 2} = 0$$



L'Hospital's
rule

Asymptotic Growth Rate



Asymptotic Order

- Logarithm $\log n$

$$\log n \in O(n^\alpha) \text{ for any } \alpha > 0$$

- Power n^k

$$n^k \in O(c^n) \text{ for any } c > 1$$

- Factorial $n!$

$$n! \approx \sqrt{2\pi n} \left(\frac{n}{e}\right)^n \quad (\text{Stirling's formula})$$

“Big Ω ”

- **Basic idea of $f(n) \in \Omega(g(n))$**
 - Dual of “O”
- **Definition – “ $\varepsilon - N$ ”**
 - Giving $g: \mathbb{N} \rightarrow \mathbb{R}^+$, then $\Omega(g)$ is the set of $f: \mathbb{N} \rightarrow \mathbb{R}^+$, such that for some $c \in \mathbb{R}^+$ and some $n_0 \in \mathbb{N}$, $f(n) \geq cg(n)$ for all $n \geq n_0$
- **Definition – “ $\lim_{n \rightarrow \infty}$ ”**
 - $f \in \Omega(g)$ if $\lim_{n \rightarrow \infty} \frac{f(n)}{g(n)} = c > 0$ (the limit may be ∞)

The Set Θ

- **Basic idea of $f(n) \in \Theta(g(n))$**
 - Roughly the same
 - $\Theta(g) = O(g) \cap \Omega(g)$
- **Definition – “ $\varepsilon - N$ ”**
 - Giving $g:\mathbb{N} \rightarrow \mathbb{R}^+$, then $\Theta(g)$ is the set of $f:\mathbb{N} \rightarrow \mathbb{R}^+$, such that for some $c_1, c_2 \in \mathbb{R}^+$ and some $n_0 \in \mathbb{N}$,
$$0 \leq c_1 g(n) \leq f(n) \leq c_2 g(n), \text{ for all } n \geq n_0$$
- **Definition – “ $\lim_{n \rightarrow \infty}$ ”**
 - $f \in \Theta(g)$ if $\lim_{n \rightarrow \infty} \frac{f(n)}{g(n)} = c$ ($0 < c < \infty$)



Some Empirical Data

algorithm		1	2	3	4
Run time in <i>ns</i>		$1.3n^3$	$10n^2$	$47n\log n$	$48n$
time for size	10^3	1.3s	10ms	0.4ms	0.05ms
	10^4	22m	1s	6ms	0.5ms
	10^5	15d	1.7m	78ms	5ms
	10^6	41yrs	2.8hrs	0.94s	48ms
	10^7	41mill	1.7wks	11s	0.48s
max Size in time	sec	920	10,000	1.0×10^6	2.1×10^7
	min	3,600	77,000	4.9×10^7	1.3×10^9
	hr	14,000	6.0×10^5	2.4×10^9	7.6×10^{10}
	day	41,000	2.9×10^6	5.0×10^{10}	1.8×10^{12}
time for 10 times size		$\times 1000$	$\times 100$	$\times 10+$	$\times 10$

on 400Mhz Pentium II, in C

from: Jon Bentley: *Programming Pearls*



Properties of O , Ω and Θ

- **Transitive property:**
 - If $f \in O(g)$ and $g \in O(h)$, then $f \in O(h)$
- **Symmetric properties**
 - $f \in O(g)$ if and only if $g \in \Omega(f)$
 - $f \in \Theta(g)$ if and only if $g \in \Theta(f)$
- **Order of sum function**
 - $O(f+g) = O(\max(f, g))$



“Little Oh”

- **Basic idea of $f(n) \in o(g(n))$**
 - **Non-ignorable** gap between f and its upper bound g
- **Definition – “ $\epsilon - N$ ”**
 - Giving $g: \mathbb{N} \rightarrow \mathbb{R}^+$, then $o(g)$ is the set of $f: \mathbb{N} \rightarrow \mathbb{R}^+$, such that for **any** $c \in \mathbb{R}^+$, there **exists some** $n_0 \in \mathbb{N}$,
$$0 \leq f(n) < cg(n), \text{ for all } n \geq n_0$$
- **Definition – “ $\lim_{n \rightarrow \infty}$ ”**
 - $f \in o(g)$ if $\lim_{n \rightarrow \infty} \frac{f(n)}{g(n)} = 0$

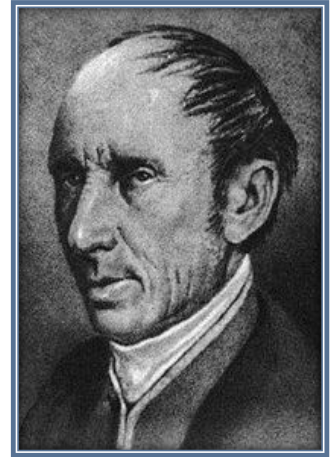
“Little ω ”

- **Basic idea of $f(n) \in \omega(g(n))$**
 - Dual of “o”
- **Definition – “ $\varepsilon - N$ ”**
 - Giving $g:N \rightarrow \mathbb{R}^+$, then $\omega(g)$ is the set of $f:N \rightarrow \mathbb{R}^+$, such that for **any** $c \in \mathbb{R}^+$, there **exists some** $n_0 \in \mathbb{N}$,
$$0 \leq cg(n) < f(n), \text{ for all } n \geq n_0$$
- **Definition – “ $\lim_{n \rightarrow \infty}$ ”**
 - $f \in \omega(g)$ if $\lim_{n \rightarrow \infty} \frac{f(n)}{g(n)} = \infty$

Do You Know Infinity

- **Mathematical analysis**
(differentiation / integration)
 - Firm foundation

Cauchy



- **How to talk about *infinity*?**
 - $(\varepsilon - N)$ -definition
 - $(\varepsilon - \delta)$ -definition

Weierstrass



Brute Force Enumeration by Iteration

- **Swapping array elements**
 - $\langle \text{time, space} \rangle$
 - From $\langle O(n^2), O(1) \rangle$
 - To $\langle O(n), O(n) \rangle$
 - To $\langle O(n), O(1) \rangle$
- **Maximum subsequence sum**
 - Time
 - From $O(n^3)$
 - To $O(n^2)$
 - To $O(n \log n)$
 - To $O(n)$



Swapping Array Elements

- E.g., 1,2,3,4 | 5,6,7 \Rightarrow 5,6,7,1,2,3,4
- Brute force

	Time	Space
BF1	$O(n^2)$	$O(1)$
BF2	$O(n)$	$O(n)$
Your Task	$O(n)$	$O(1)$

Space sensitive

Time sensitive

- Your task
 - Both time and space efficient

Max-sum Subsequence

- The problem: Given a sequence S of integer, find the **largest sum** of a **consecutive subsequence** of S . (0, if all negative items)

○ An example: -2, 11, -4, 13, -5, -2; the result 20: (11, -4, 13)

A brute-force algorithm:

MaxSum = 0;

```
for (i = 0; i < N; i++)
```

```
  for (j = i; j < N; j++)
```

```
  {
```

```
    ThisSum = 0;
```

```
    for (k = i; k <= j; k++)
```

```
      ThisSum += A[k];
```

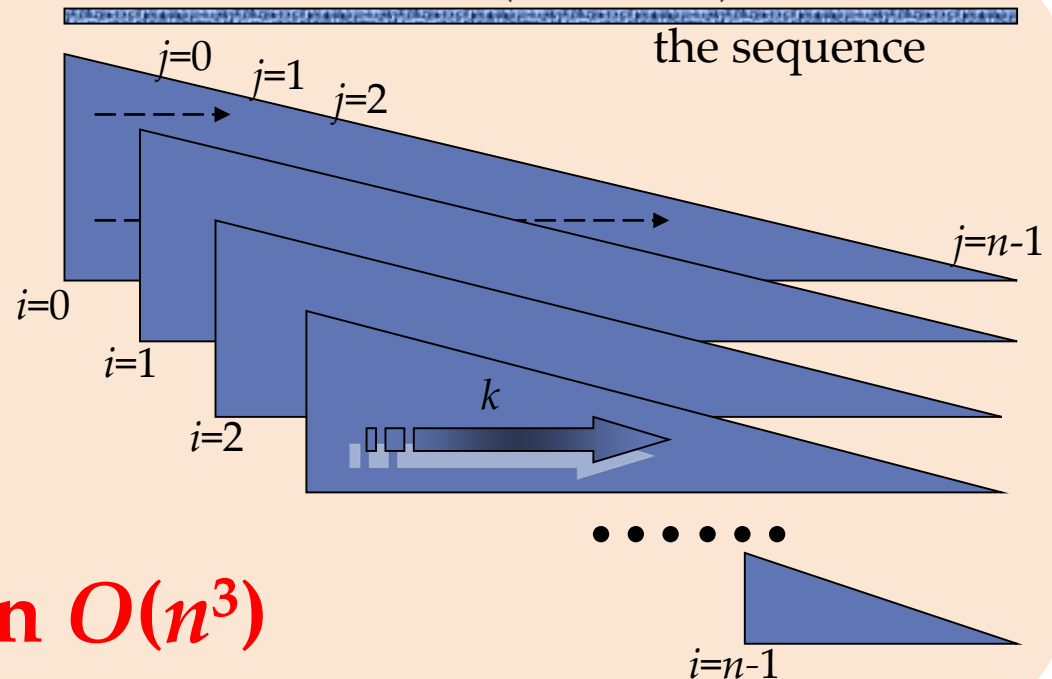
```
      if (ThisSum > MaxSum)
```

```
        MaxSum = ThisSum;
```

```
  }
```

```
return MaxSum;
```

in $O(n^3)$



More Precise Complexity

The total cost is : $\sum_{i=0}^{n-1} \sum_{j=i}^{n-1} \sum_{k=i}^j 1$

$$\sum_{k=i}^j 1 = j - i + 1$$

$$\sum_{j=i}^{n-1} (j - i + 1) = 1 + 2 + \dots + (n - i) = \frac{(n - i + 1)(n - i)}{2}$$

$$\sum_{i=0}^{n-1} \frac{(n - i + 1)(n - i)}{2} = \sum_{i=1}^n \frac{(n - i + 2)(n - i + 1)}{2}$$

$$= \frac{1}{2} \sum_{i=1}^n i^2 - \left(n + \frac{3}{2}\right) \sum_{i=1}^n i + \frac{1}{2} (n^2 + 3n + 2) \sum_{i=1}^n 1$$

$$= \frac{n^3 + 3n^2 + 2n}{6}$$



Decreasing the Number of Loops

An improved algorithm

```
MaxSum = 0;
```

```
for (i = 0; i < N; i++)
```

```
{
```

```
    ThisSum = 0;
```

```
    for (j = i; j < N; j++)
```

```
    {
```

```
        ThisSum += A[j];
```

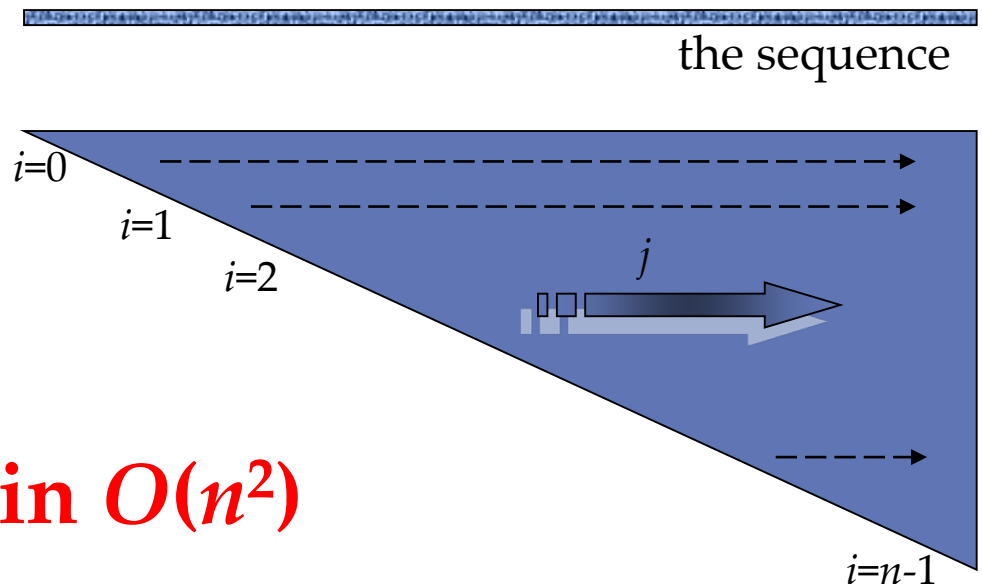
```
        if (ThisSum > MaxSum)
```

```
            MaxSum = ThisSum;
```

```
    }
```

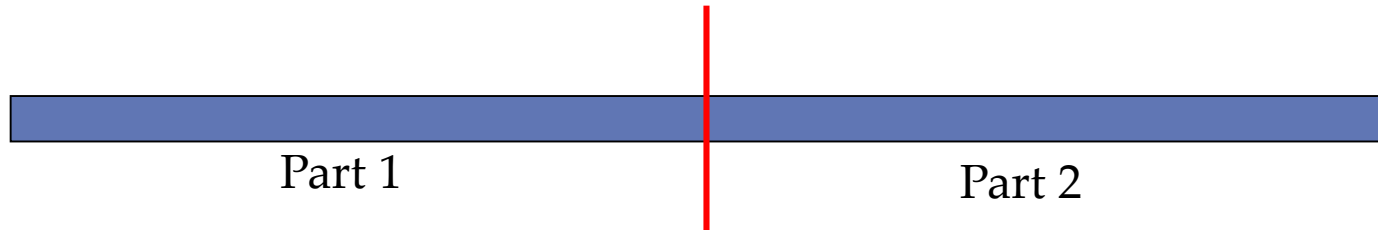
```
}
```

```
return MaxSum;
```

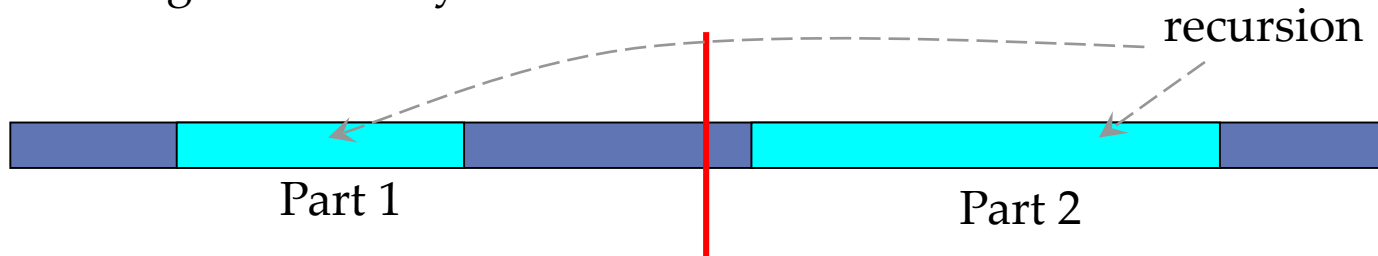


in $O(n^2)$

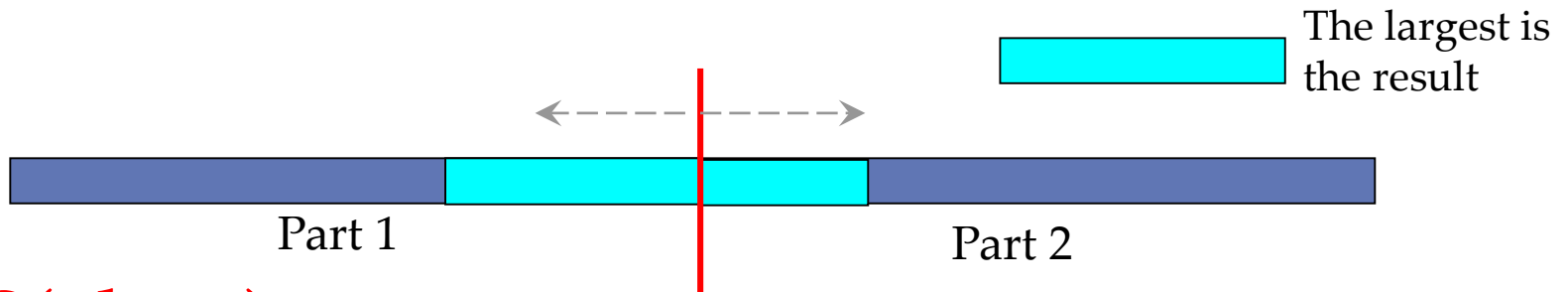
Power of Divide and Conquer



the sub with largest sum may be in:



or:



in $O(n \log n)$

Power of Divide and Conquer

```
Center = (Left + Right) / 2;  
MaxLeftSum = MaxSubSum(A, Left, Center); MaxRightSum = MaxSubSum(A, Center + 1,  
Right);
```

```
MaxLeftBorderSum = 0; LeftBorderSum = 0;  
for (i = Center; i >= Left; i--)  
{  
    LeftBorderSum += A[i];  
    if (LeftBorderSum > MaxLeftBorderSum)    MaxLeftBorderSum = LeftBorderSum;  
}
```

```
MaxRightBorderSum = 0; RightBorderSum = 0;  
for (i = Center + 1; i <= Right; i++)  
{  
    RightBorderSum += A[i];  
    if (RightBorderSum > MaxRightBorderSum)    MaxRightBorderSum = RightBorderSum;  
}
```

```
return Max3(MaxLeftSum, MaxRightSum,  
            MaxLeftBorderSum + MaxRightBorderSum);
```

Note: this is the core part of the procedure, with base case and wrap omitted.



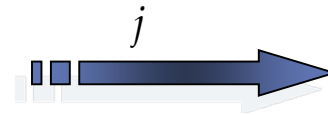
A Linear Algorithm

First scan the array to eliminate the case of “all negative integers”

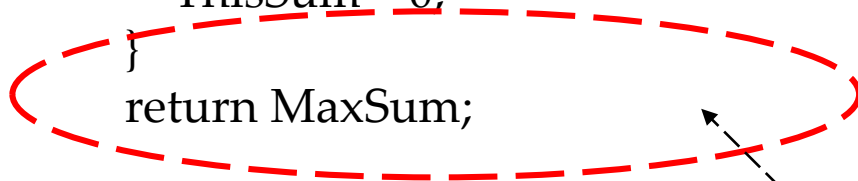
```
ThisSum = MaxSum = 0;
for (j = 0; j < N; j++)
{
    ThisSum += A[j];
    if (ThisSum > MaxSum)
        MaxSum = ThisSum;
    else if (ThisSum < 0)
        ThisSum = 0;
}
return MaxSum;
```



the sequence



This is an example of
“online algorithm”



Negative item or subsequence
cannot be a prefix of the
subsequence we want.

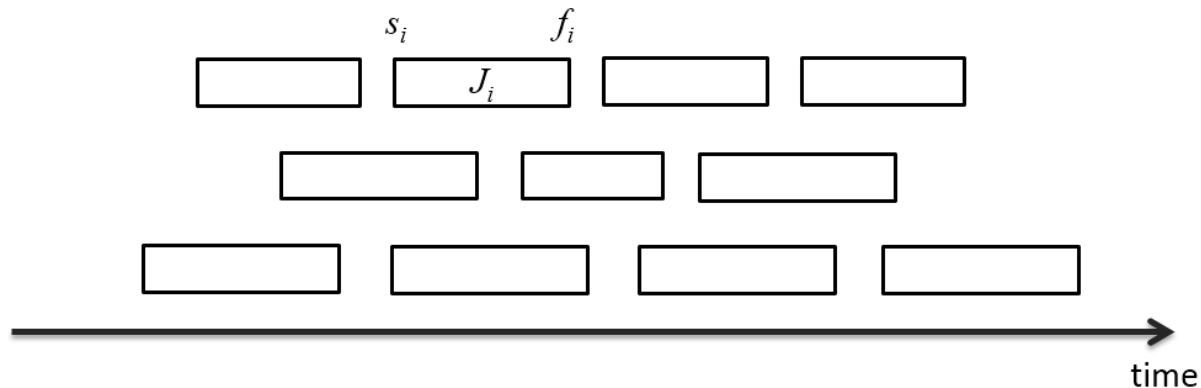
in $O(n)$

Brute Force Enumeration by Recursion

- **Job scheduling**
 - Problem definition
 - Brute force recursion
 - Further improvements
- **Matrix chain multiplication**
 - Problem definition
 - Brute force recursion(s)
 - Further improvements

Job Scheduling

- Jobs: $J_i = [s_i, f_i)$
- Max number of **compatible** jobs



Job Scheduling

- **Brute force recursion**
 - Select job 'a'
 - Case 1: the result does not contain 'a'
 - Recursion on $J \setminus \{a\}$
 - Case 2: the result contains 'a'
 - Recursion on $J \setminus \{a\} \setminus \{\text{tasks overlapping with 'a'}\}$
- **Further improvements**
 - Dynamic programming (L16)
 - Greedy algorithms (L14)



Matrix Chain Multiplication

- The task:

Find the product: $A_1 \times A_2 \times \dots \times A_{n-1} \times A_n$

A_i is 2-dimensional array of different legal size

- The Challenge:

- Matrix multiplication is associative
- Different computing order results in great difference in the number of operations

- The problem:

- Which is the best computing order



Cost of Matrix Multiplication

An example: $A_1 \times A_2 \times A_3 \times A_4$
 $30 \times 1 \quad 1 \times 40 \quad 40 \times 10 \quad 10 \times 25$

Let $C = A_{p \times q} \times A_{q \times r}$

$((A_1 \times A_2) \times A_3) \times A_4$: 20700 multiplications

$A_1 \times (A_2 \times (A_3 \times A_4))$: 11750

$(A_1 \times A_2) \times (A_3 \times A_4)$: 41200

$A_1 \times ((A_2 \times A_3) \times A_4)$: 1400

$$c_{i,j} = \sum_{k=1}^q a_{ik} b_{kj}$$

There are q multiplication

C has $p \times r$ elements as $c_{i,j}$

So, pqr multiplications altogether



Solutions

- **Brute force recursion (L16)**
 - BF1
 - BF2
- **Dynamic programming (L16)**
 - Based on brute force recursion 2

Thank you!

Q & A

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