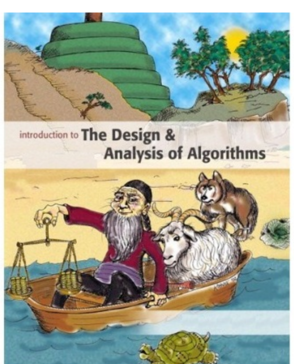
Introduction to

# *Algorithm Design and Analysis*

## [11] Graph Traversal

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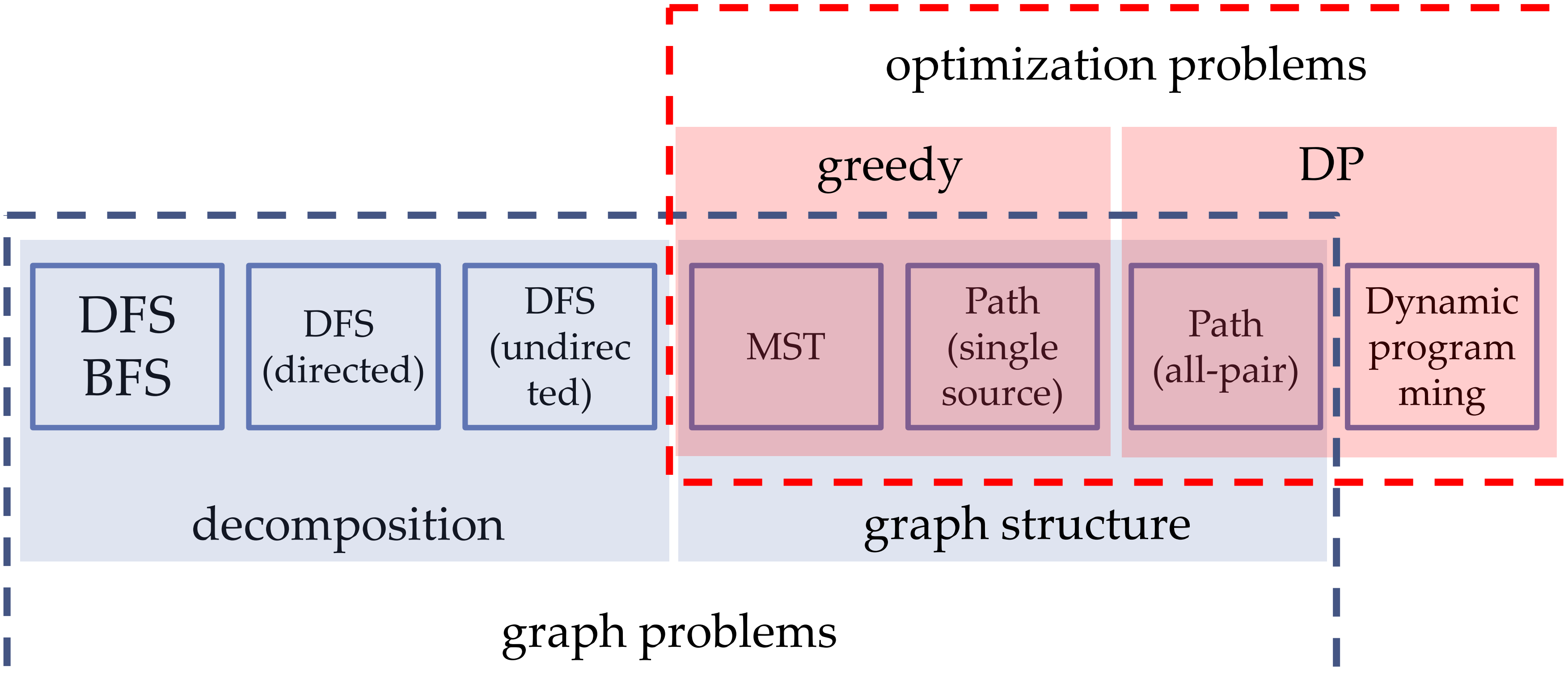
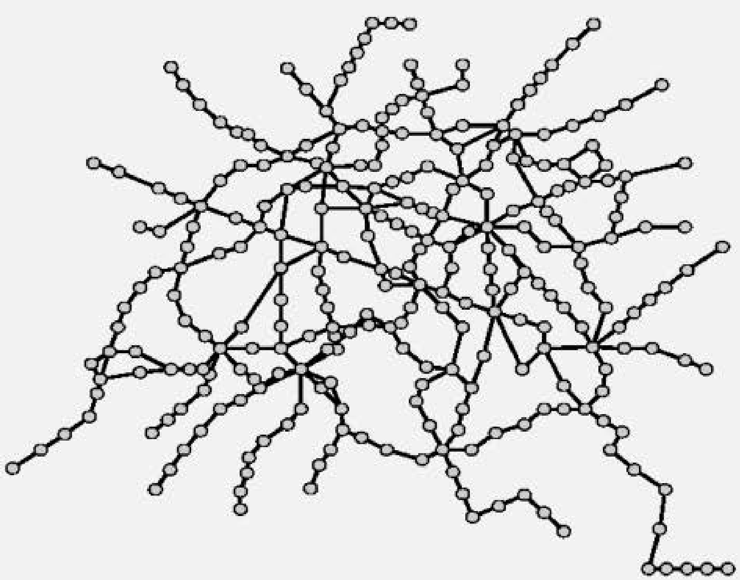
# Course Contents

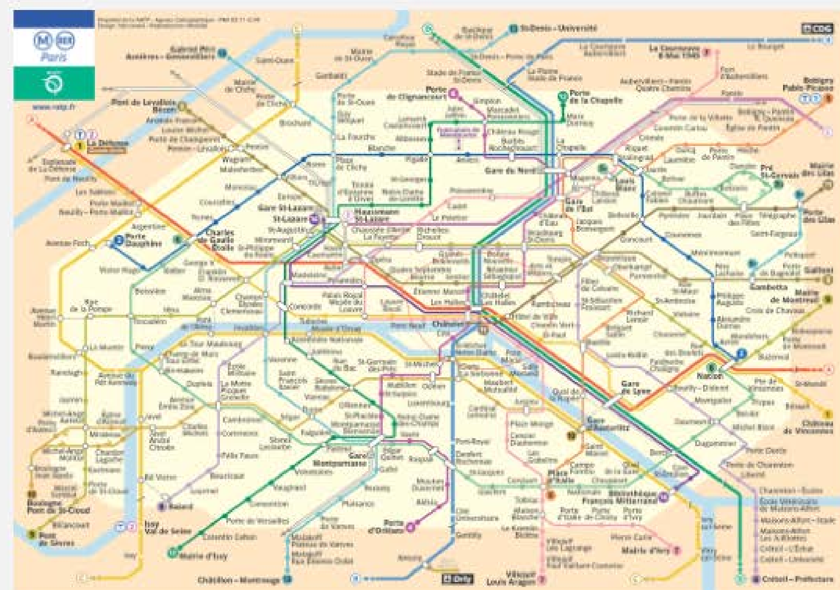


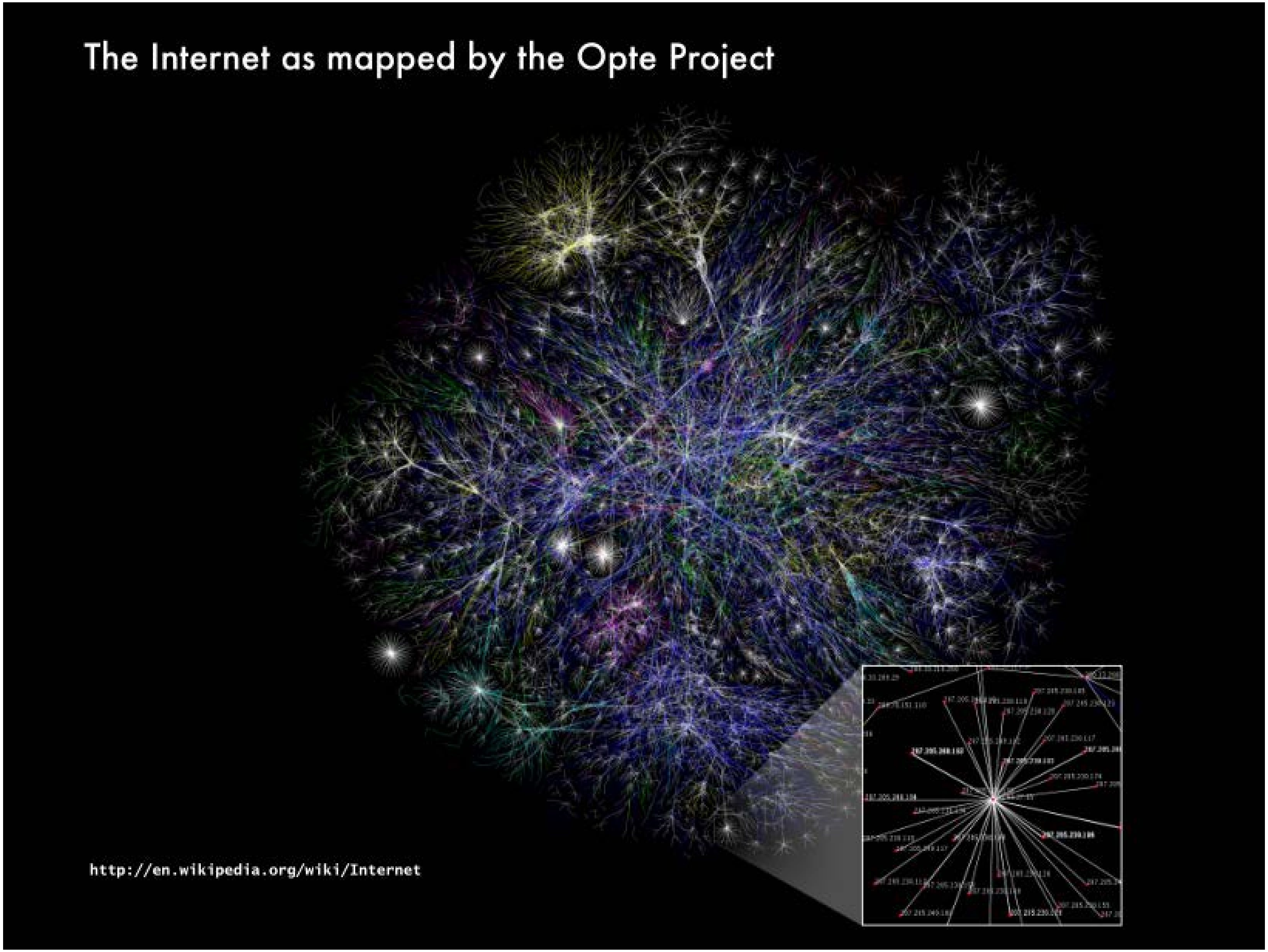
# In the Last Class…

- **Dynamic Equivalence Relation**
- **Implementing *disjoint set* by Union-Find**
  - Straight Union-Find
  - Making Shorter Tree by Weighted Union
  - Compressing Path by Compressing Find
    - Amortized analysis of *wUnion-cFind*

# Graph Everywhere



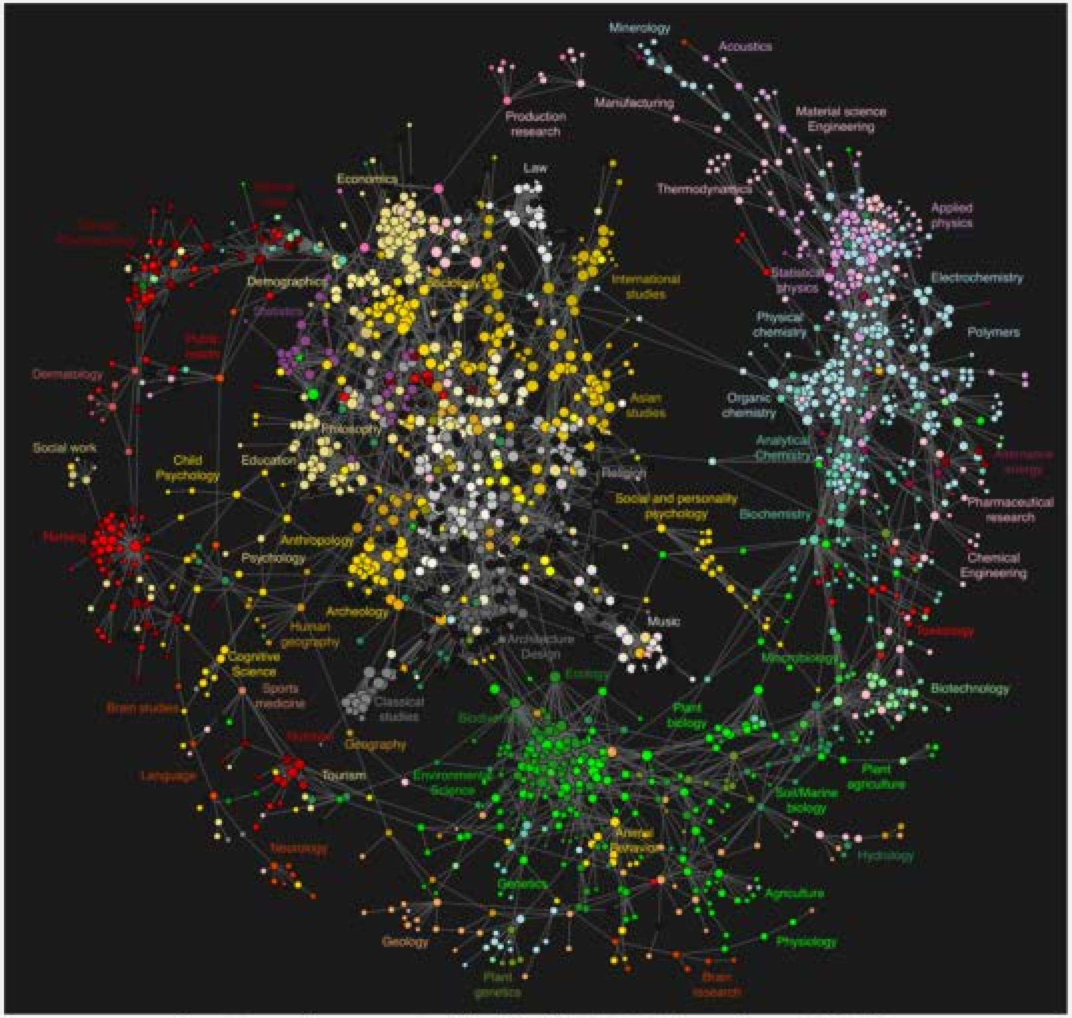



The Internet as mapped by the Opte Project
http://en.wikipedia.org/wiki/Internet

## 10 million Facebook friends


"Visualizing Friendships" by Paul Butler

# Map of science clickstreams




http://www.plosone.org/article/info:doi/10.1371/journal.pone.0004803

# Graph Basics

- **Node**
  - Entities of interest
  - $V(G)$
- **Edge**
  - Relations of interest
  - $E(G) \subseteq V \times V$

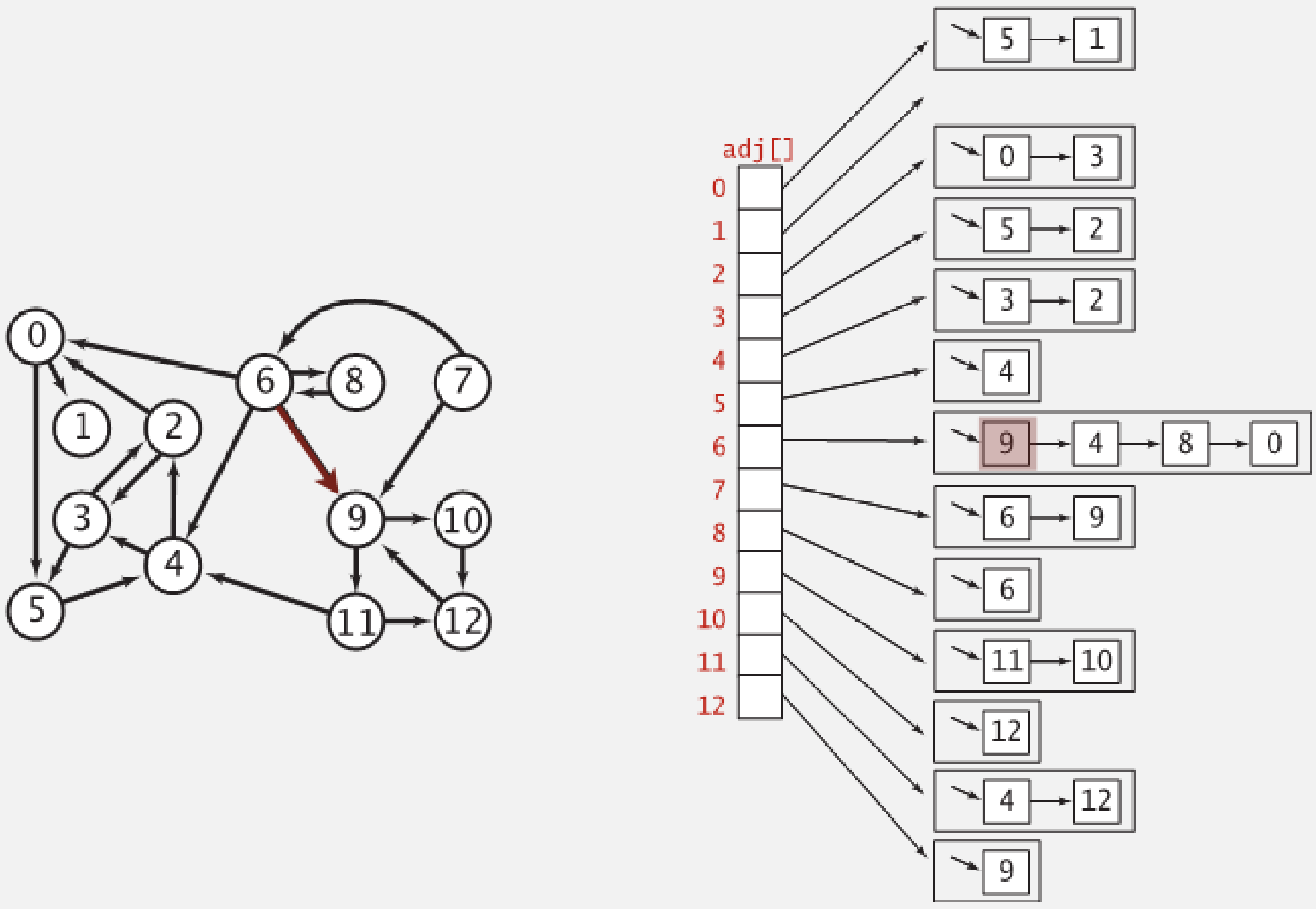
# Graph Traversals

- Depth-First and Breadth-First Search
- Finding Connected Components

- **General DFS/BFS Skeleton**
- **Depth-First Search Trace**

# Adjacency-lists digraph representation

Maintain vertex-indexed array of lists.

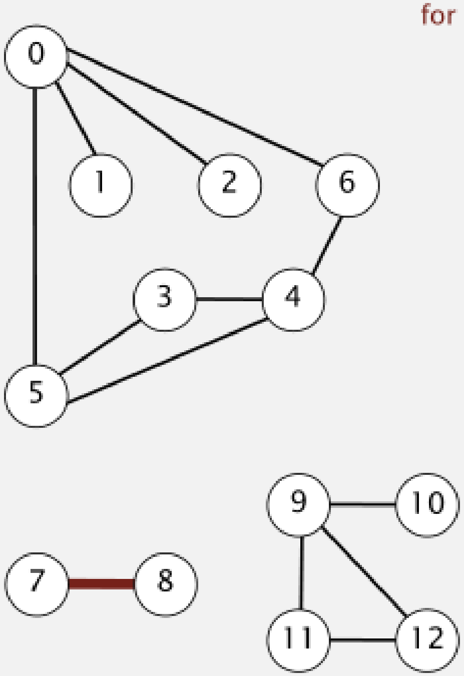


## Adjacency-matrix graph representation

Maintain a two-dimensional $V$-by-$V$ boolean array;

for each edge $v–w$ in graph: `adj[v][w] = adj[w][v] = true`.

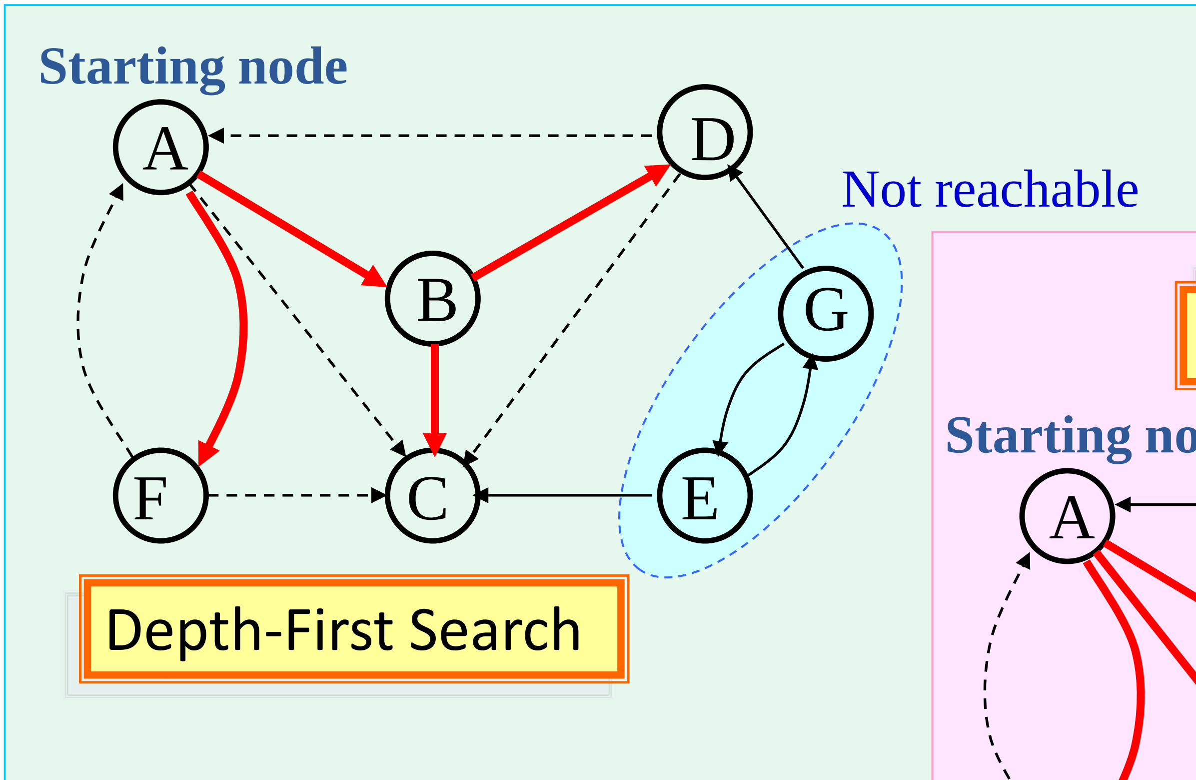
**Directed** vs. **Undirected** graphs

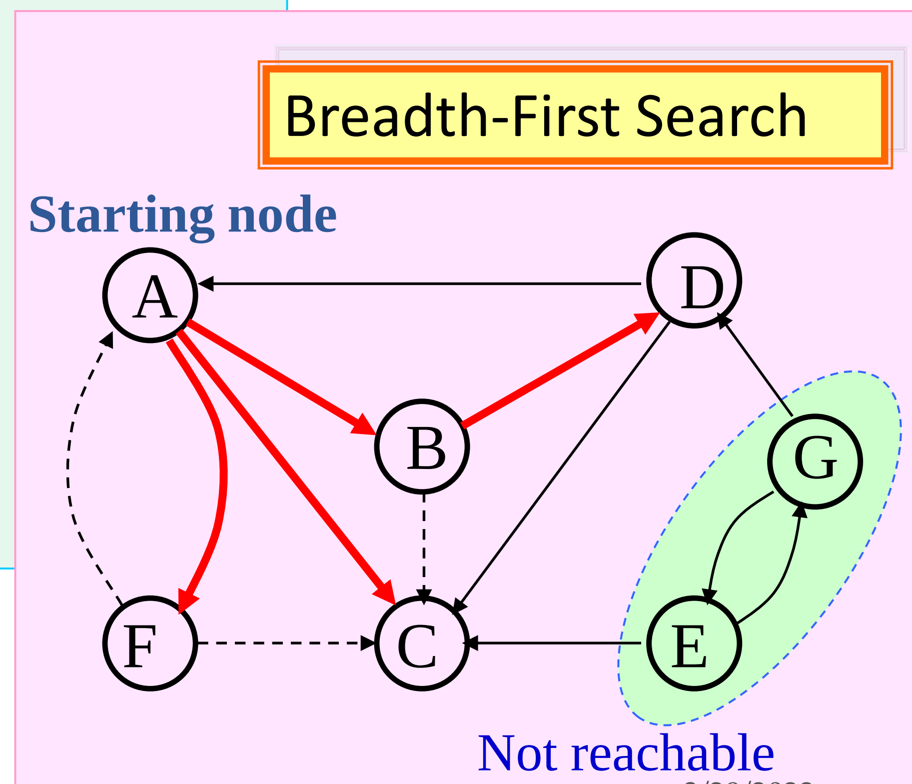



two entries
for each edge

| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 1 | 1 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 2 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 3 | 0 | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 4 | 0 | 0 | 0 | 1 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 |
| 5 | 1 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 6 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 7 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 |
| 8 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 |
| 9 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 1 |
| 10 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 |
| 11 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 1 |
| 12 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 1 | 0 |

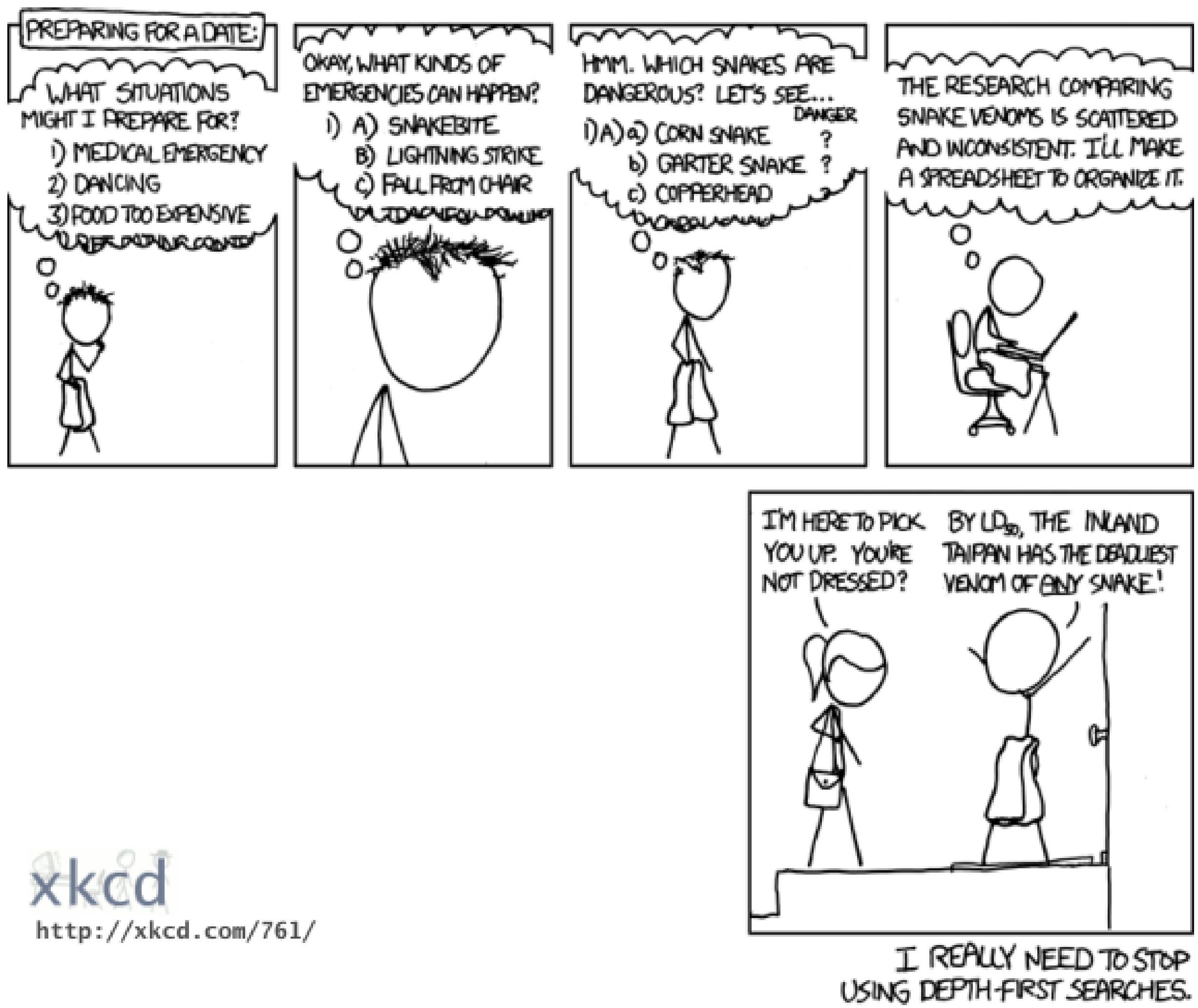
# Graph Traversal




- - - - - - - - - *Edges only "checked"*

# Depth-first search application: preparing for a date



# Outline of Depth-First Search

- dfs(G,v)
- Mark v as "discovered".
- For each vertex w that edge vw is in G:
- If w is undiscovered:
- dfs(G,w)
- Otherwise:
- "Check" vw without visiting w.
- Mark v as "finished".

A vertex must be exact one of three different status:
- undiscovered
- discovered but not finished
- finished

That is: exploring vw, visiting w, exploring from there as much as possible, and backtrack from w to v.

# Outline of Breadth-First Search

- Bfs(G,*s*)
- Mark *s* as "discovered";
- **enqueue**(pending,*s*);
- while (pending is nonempty)
- **dequeue**(pending, *v*);
- For each vertex *w* that edge *vw* is in G:
- If *w* is "undiscovered"
- Mark *w* as "discovered" and **enqueue**(pending, *w*)
- Mark *v* as "finished";

# Finding Connected Components

- **Input: a symmetric digraph G, with *n* nodes and 2*m* edges(interpreted as an undirected graph), implemented as a array *adjVertices*[1,…*n*] of adjacency lists.**
- **Output: an array *cc*[1..*n*] of component number for each node $v_i$**

```
• void connectedComponents(Intlist[ ] adjVertices, int n,
        int[ ] cc) // This is a wrapper procedure
•     int[ ] color=new int[n+1];
•     int v;
•     <Initialize color array to white for all vertices>
•     for (v=1; v≤n; v++)
•         if (color[v]==white)
•             ccDFS(adjVertices, color, v, v, cc);
•     return
```

Depth-first search

# ccDFS: the procedure

- void ccDFS(IntList[ ] *adjVertices*, int[ ] *color*, int *v*, int *ccNum*, int [ ] *cc*)//*v* as the code of current connected component
- int *w*;
- IntList remAdj;
- color[v]=gray;
- cc[v]=ccNum;
- remAdj=adjVertices[v];
- while (remAdj≠nil)
- w=first(remAdj);
- if (color[w]==white)
- ccDFS(adjVertices, color, w, ccNum, cc);
- remAdj=rest(remAdj);
- color[v]=black;
- return

The elements of *remAdj* are neighbors of *v*

Processing the next neighbor, if existing, another depth-first search to be incurred

*v* finished

# Analysis of CC Algorithm

- **connectedComponents, the wrapper**
  - Linear in $n$ (color array initialization+for loop on *adjVertices* )
- **ccDFS, the depth-first searcher**
  - In one execution of ccDFS on $v$, the number of instructions(rest(*remAdj*)) executed is proportional to the size of *adjVertices*[$v$].
  - Note: Σ(size of *adjVertices*[$v$]) is $2m$, and the adjacency lists are traveresed **only once**.

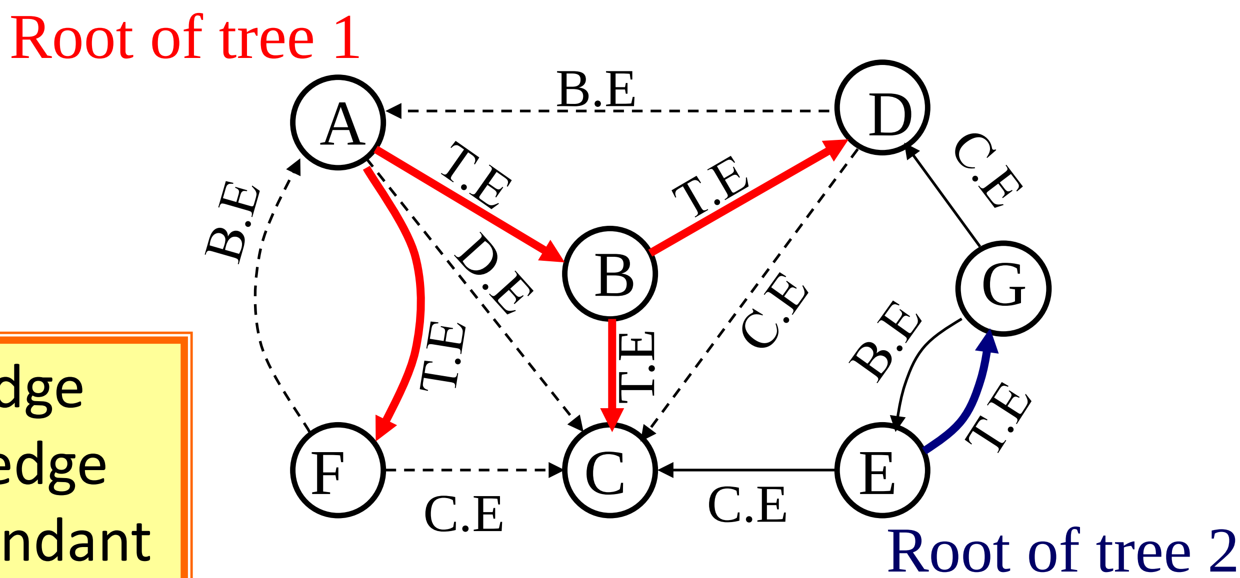
- **So, the time complexity is in Θ(m+n)**
  - Extra space requirements:
    - Color array
    - Activation frame stack for recursion

# Visits On a Vertex

- **Classification for the visits on a vertex**
  - First visit(exploring): status: white→gray
  - (Possibly) multi-visits by backtracking to: status keeps gray
  - Last visit(no more branch-finished): status: gray→black
- **Different operations can be done, during the different visits on a specific vertex**
  - On the vertex
  - On (selected) incident edges

# Depth-first Search Trees

DFS forest={(DFS tree1), (DFS tree2)}




T.E: tree edge
B.E: back edge
D.E: descendant edge
C.E: cross edge

A finished vertex is never revisited, such as C

# Depth-First Search – Generalized Skeleton

- Input: Array *adjVertices* for graph G
- Output: Return value depends on application.
- int dfsSweep(IntList[] *adjVertices*,int n, …)
- int ans;
- <Allocate color array and initialize to white>
- For each vertex *v* of G, in some order
- if (color[v]==white)
- int vAns=dfs(*adjVertices*, color, v, …);
- <Process vAns>
- // Continue loop
- return ans;

# Depth-First Search – Generalized Skeleton

- int dfs(IntList[] *adjVertices*, int[] color, int v, …)
-     int w;
-     IntList remAdj;
-     int ans;
-     color[v]=gray;
-     <Preorder processing of vertex v>
-     remAdj=*adjVertices*[v];
-     while (remAdj≠nil)
-         w=first(remAdj);
-         if (color[w]==white)
-             <Exploratory processing for tree edge vw>
-             int wAns=dfs(*adjVertices*, color, w, …);
-             < Backtrack processing for tree edge vw , using wAns>
-         else
-             <Checking for nontree edge vw>
-         remAdj=rest(remAdj);
-     <Postorder processing of vertex v, including final computation of ans>
-     color[v]=black;
-     return ans;

If partial search is used for a application, tests for termination may be inserted here.

**Specialized for connected components:**
- parameter added
- preorder processing inserted – cc[v]=ccNum

# Breadth-First Search - Skeleton

- Input: Array *adjVertices* for graph G
- Output: Return value depends on application.

```
void bfsSweep(IntList[] adjVertices,int n, …)
    int ans;
    &lt;Allocate color array and initialize to white&gt;
    For each vertex v of G, in some order
        if  (color[v]==white)
            void bfs(adjVertices, color, v, …);
    // Continue loop
    return;
```

# Breadth-First Search - Skeleton

- void bfs(IntList[] *adjVertices*, int[] color, int v, …)
-     int w; IntList remAdj; Queue pending;
-     color[v]=gray;   enqueue(pending, *v*);
-     while (pending is nonempty)
-         *w*=dequeue(pending); remAdj=*adjVertices*[*w*];
-         while (remAdj≠nil)
-             *x*=first(remAdj);
-             if (color[*x*]==white)
-                 color[*x*]=gray; enqueue(pending, *x*);
-             remAdj=rest(remAdj);
-         <processing of vertex w>
-         color[*w*]=black;
-     return ;

***can be further generalized***

# DFS vs. BFS Search

- **Processing opportunities for a node**
  - Depth-first: 2
    - At discovering
    - At finishing
  - Breadth-first: only 1, when de-queued
  - At the second processing opportunity for the DFS, the algorithm can make use of information about the descendants of the current node.

# Time Relation on Changing Color

- **Keeping the order in which vertices are encountered for the first or last time**
  - A global integer time: 0 as the initial value, incremented with each color changing for *any* vertex, and the final value is $2n$
  - Array *discoverTime*: the $i$ th element records the time vertex $v_i$ turns into gray
  - Array *finishTime*: the $i$ th element records the time vertex $v_i$ turns into black
  - The active interval for vertex $v$, denoted as $active(v)$, is the duration while $v$ is gray, that is:

$$discoverTime[v], \ldots, finishTime[v]$$

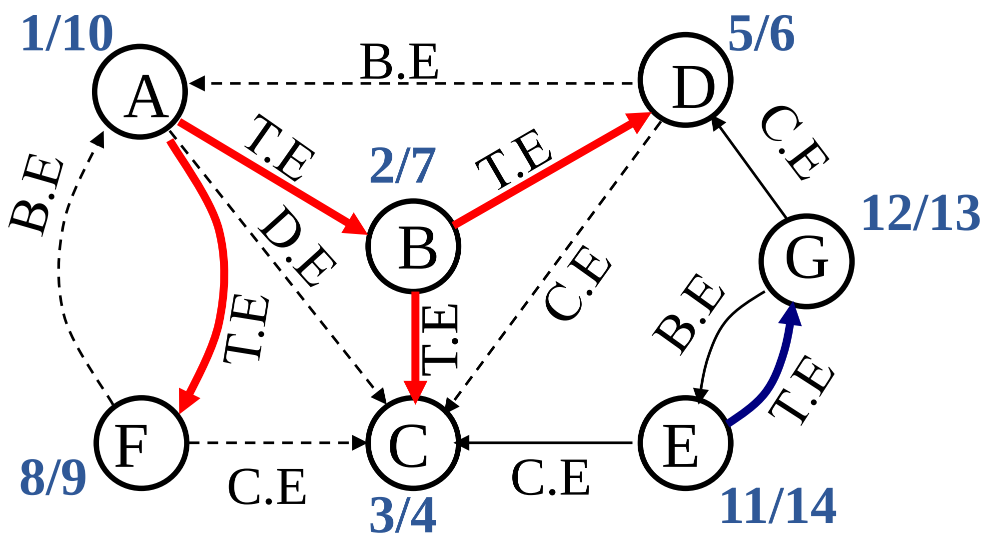
# Depth-First Search Trace

- **General DFS skeleton modified to compute discovery and finishing times and "construct" the depth-first search forest.**
- **int dfsTraceSweep(IntList[ ] *adjVertices*,int n, int[ ] *discoverTime*, int[ ] *finishTime*, int[ ] *parent*)**
- **int ans; int *time*=0**
- **<Allocate color array and initialize to white>**
- **For each vertex *v* of G, in some order**
- **if (color[v]==white)**
- **parent[v]=-1**
- **int vAns=dfsTrace(*adnVertices*, color, v, *discoverTime*, *finishTime*, *parent*, *time* );**
- **// Continue loop**
- **return ans;**

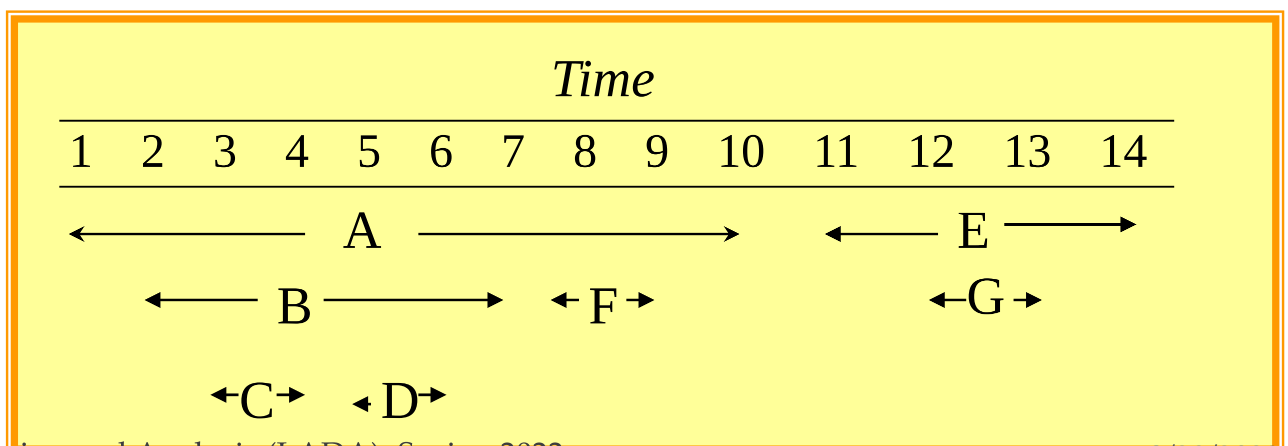
# Depth-First Search Trace

- int dfsTrace(intList[ ] *adjVertices*, int[ ] color, int v,  int[ ] *discoverTime*,
- int[ ] *finishTime*, int[ ] *parent*  int *time*)
- int w; IntList remAdj; int ans;
- color[v]=gray; time++; *discoverTime*[v]=time;
- remAdj=*adjVertices*[v];
- while (remAdj≠nil)
- w=first(remAdj);
- if (color[w]==white)
- parent[w]=v;
- int wAns=dfsTrace(*adjVertices*, color, w, *discoverTime, finishTime, parent, time*);
- else  <Checking for nontree edge vw>
- remAdj=rest(remAdj);
- time++; *finishTime*[v]=time; color[v]=black;
- return ans;

# Active Interval




The relations are summarized in the next frame



# Properties of Active Intervals(1)

- **If $w$ is a descendant of $v$ in the DFS forest, then $active(w) \subseteq active(v)$, and the inclusion is proper if $w \neq v$.**
- **Proof:**
  - Define a partial order <: w<v iff. w is a proper descendants of v in its DFS tree. The proof is by induction on <.
  - If v is minimal. The only descendant of v is itself. Trivial.
  - Assume that for all x<v, if w is a descendant of x, then $active(w) \subseteq active(x)$.
  - Let w be any proper descendant of v in the DFS tree, there must be some x such that vx is a tree edge on the tree path to w, so w is a descendant of x. According to dfsTrace, we have $active(x) \subset active(v)$, by inductive hypothesis, $active(w) \subset active(v)$.

# Properties of Active Intervals(2)

- **If $active(w) \subseteq active(v)$, then $w$ is a descendant of $v$. And if $active(w) \subset active(v)$, then $w$ is a proper descendant of $v$.**

**That is: *w* is discovered while *v* is active.**

- **Proof:**
  - If w is **not** a descendant of v, there are two cases:
    - v is a proper descendant of w, then $active(v) \subset active(w)$, so, it is impossible that $active(w) \subseteq active(v)$, contradiction.
    - There is no ancestor/descendant relationship between v and w, then $active(w)$ and $active(v)$ are disjoint, contradiction.

# Properties of Active Intervals(3)

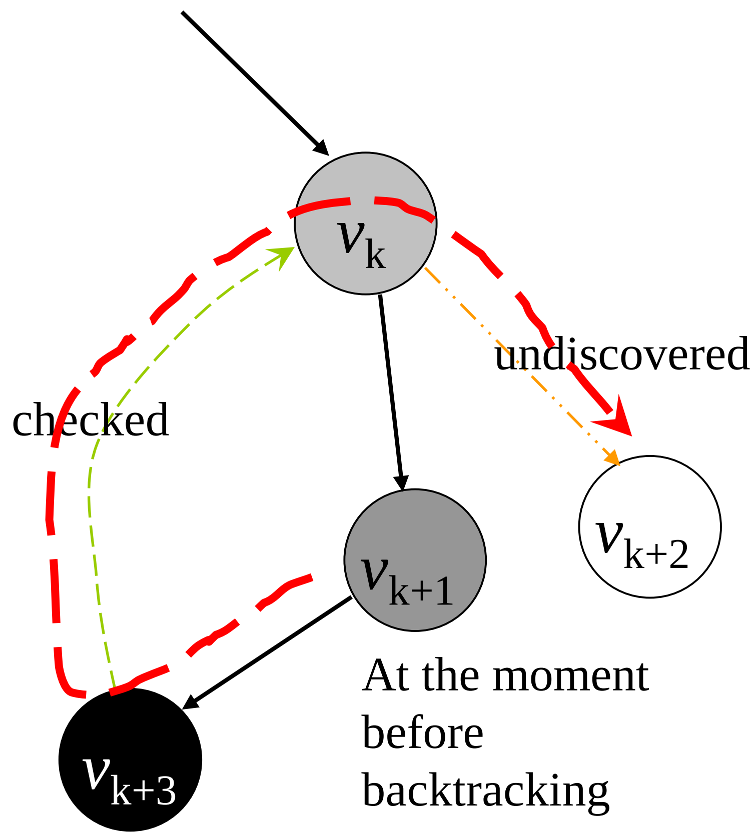
- **If v and w have no ancestor/descendant relationship in the DFS forest, then their active intervals are disjoint.**
- **Proof:**
  - If v and w are in different DFS tree, it is trivially true, since the trees are processed one by one.
  - Otherwise, there must be a vertex c, satisfying that there are tree paths c to v, and c to w, without edges in common. Let the leading edges of the two tree path are cy, cz, respectively. According to dfsTrace, *active*($y$) and *active*($z$) are disjoint.
  - We have *active*($v$)$\subseteq$*active*($y$), *active*($w$)$\subseteq$*active*($z$). So, *active*($v$) and *active*(w) are disjoint.

# Properties of Active Intervals(4)

- **If edge vw$\in E_G$, then**
  - vw is a **cross edge** iff. *active*(w) entirely precedes *active*(v).
  - vw is a **descendant edge** iff. there is some third vertex $x$, such that $active(w) \subset active(x) \subset active(v)$,
  - vw is a **tree edge** iff. $active(w) \subset active(v)$, and there is no third vertex $x$, such that $active(w) \subset active(x) \subset active(v)$,
  - vw is a **back edge** iff. $active(v) \subset active(w)$,

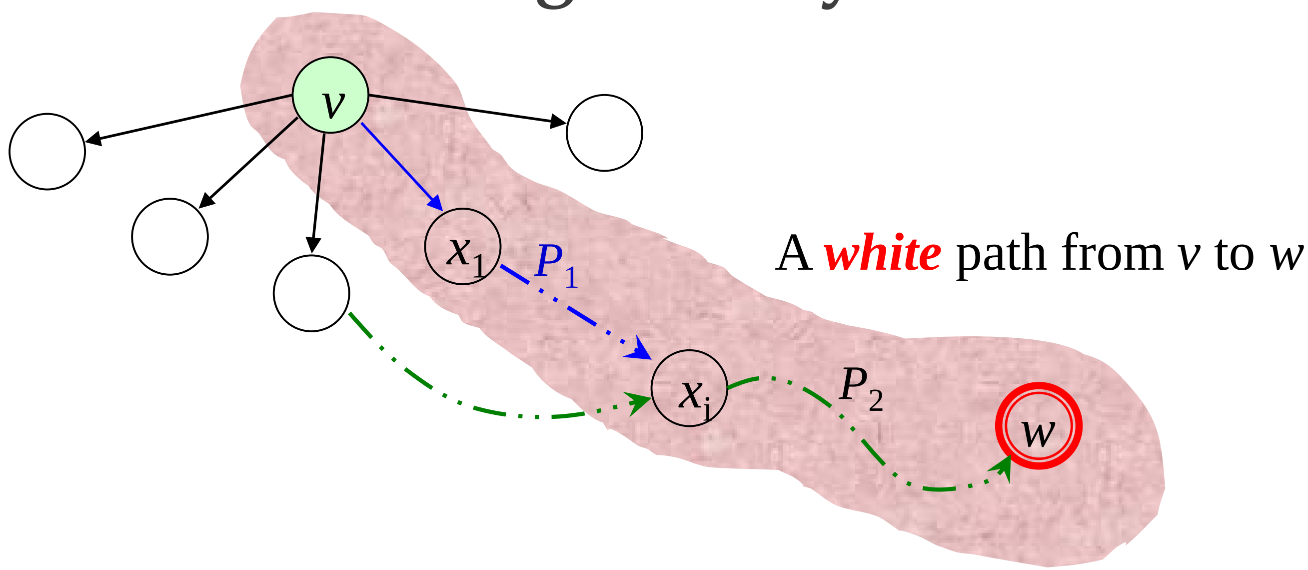
# Ancestor and Descendant

- That $w$ is a descendant of $v$ in the DFS forest means that there is a direct path from $v$ to $w$ in some DFS tree.
- The path is also a path in $G$.
- **However, if there is a direct path from $v$ to $w$ in G, is $w$ necessarily a descendant of $v$ in *the* DFS forest?**



# DFS Tree Path

- [**White Path Theorem**] **w is a descendant of v in a DFS tree iff. at the time v is discovered(just to be changing color into gray), there is a path in G from v to w consisting entirely of white vertices.**



# Proof of White Path Theorem

- **Proof**
  - o $\Rightarrow$ All the vertices in the path are descendants of v.
  - o $\Leftarrow$ by induction on the length $k$ of a white path from v to w.
    - When $k$=0, v=w.
    - For $k$>0, let $P$=(v, $x_1$,$x_2$,…$x_k$=w). There must be some vertex on $P$ which is discovered during the active interval of v, e.g. $x_1$, Let $x_i$ is earliest discovered among them. Divide $P$ into $P_1$ from v to $x_i$, and $P_2$ from $x_i$ to w. $P_2$ is a white path with length less than $k$, so, by inductive hypothesis, w is a descendant of $x_i$. Note: $\mathit{active}(x_i) \subseteq \mathit{active}(v)$, so $x_i$ is a descendant of v. By transitivity, w is a descendant of v.

# *Thank you!*

# *Q & A*

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