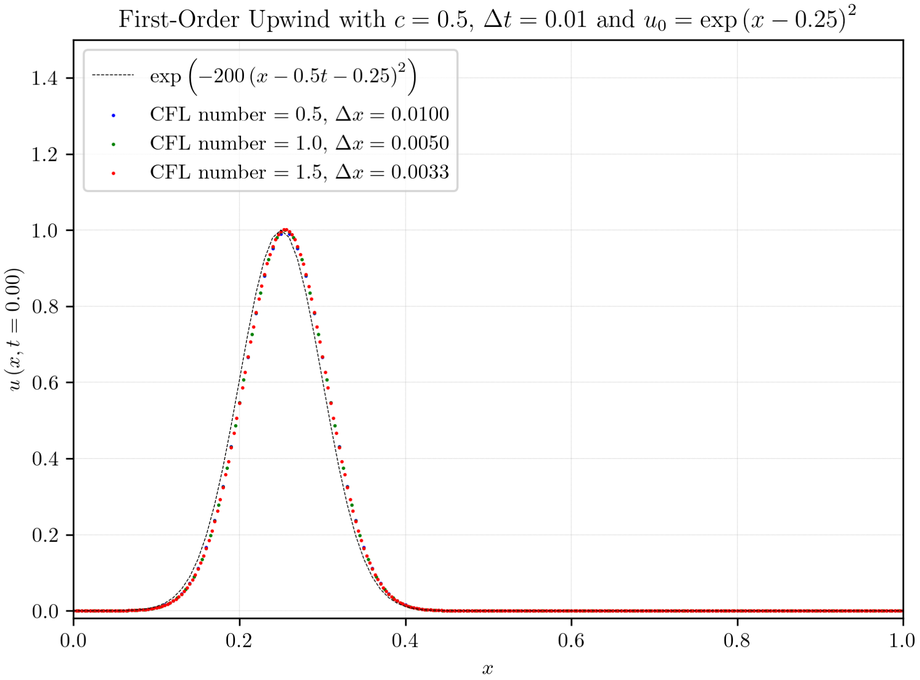
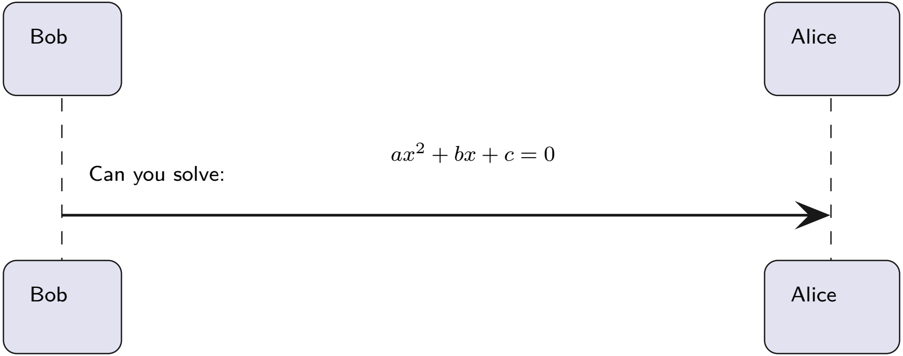

First-Order Upwind with $c = 0.5$, $\Delta t = 0.01$ and $u_0 = \exp{(x - 0.25)}^2$
$\exp\left(-200\,(x - 0.5t - 0.25)^2\right)$
CFL number $= 0.5$, $\Delta x = 0.0100$
CFL number $= 1.0$, $\Delta x = 0.0050$
CFL number $= 1.5$, $\Delta x = 0.0033$
$u\,(x, t = 0.00)$
x
0.0
0.2
0.4
0.6
0.8
1.0
1.2
1.4

Flow chart

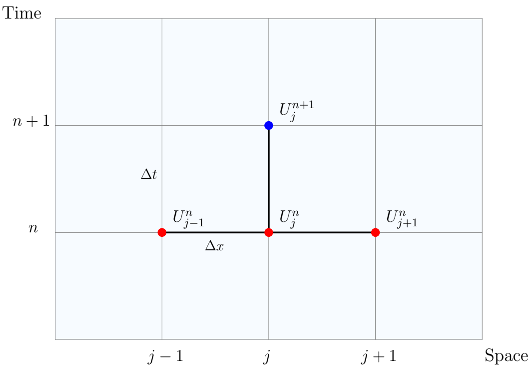


Fourier stability analysis

Example (Forward-Time Central-Space (FTCS))

$$0 = \frac{U_j^{n+1} - U_j^n}{\Delta t} + c\frac{U_{j+1}^n - U_{j-1}^n}{2\Delta x}.$$

$$U_j^{n+1} = U_j^n - \frac{r}{2}\left(U_{j+1}^n - U_{j-1}^n\right), \quad r = c\frac{\Delta t}{\Delta x}.$$



Fourier stability analysis

Theorem (Stability analysis for FTCS scheme)

$$\forall r > 0 \iff |\lambda(k)| > 1.$$

Proof.

$$U_{j+1}^{n} = U_j^n - \frac{r}{2}\left(U_{j+1}^n - U_{j-1}^n\right).$$

$$\lambda(k)^{n+1} e^{ik(j\Delta x)} = \lambda(k)^n e^{ikj\Delta x} - \frac{r}{2}\lambda(k)^n \left(e^{ik(j+1)\Delta x} - e^{ik(j-1)\Delta x}\right).$$

$$\lambda(k) = 1 - \frac{r}{2}\left(e^{ik\Delta x} - e^{-ik\Delta x}\right).$$

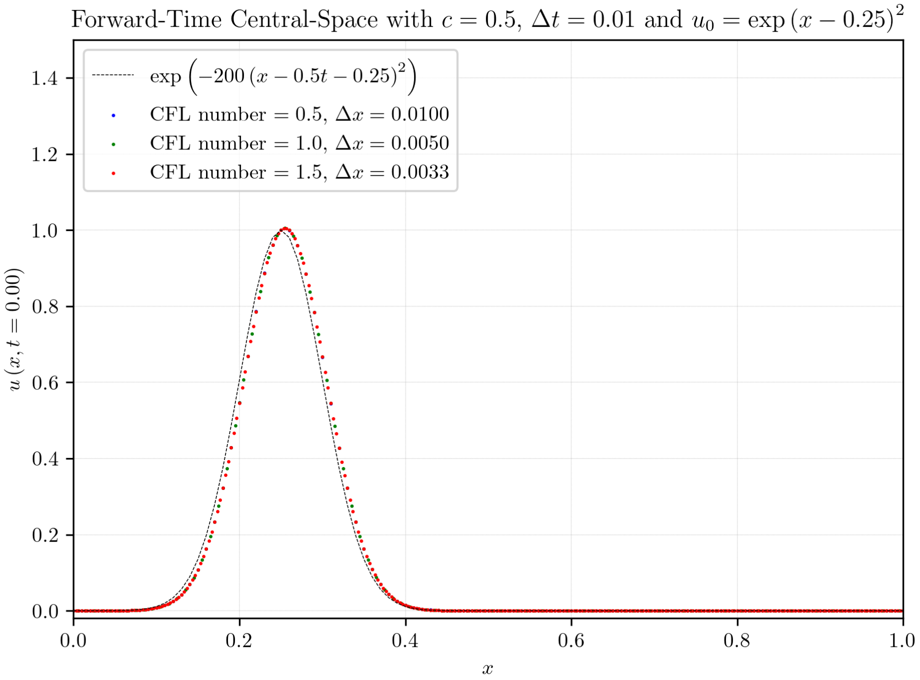
$$\lambda(k) = 1 - ir\sin(k\Delta x).$$

$$\begin{aligned} |\lambda(k)|^2 &= 1 + r^2\sin^2(k\Delta x) \\ &> 1 \iff \forall r > 0. \end{aligned}$$

□

Remark

We say that the FTCS scheme is unconditionally unstable.

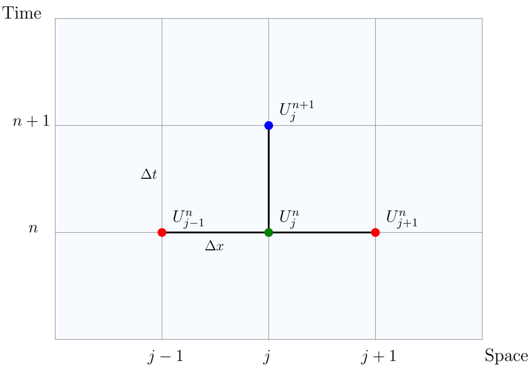


Forward-Time Central-Space with $c = 0.5$, $\Delta t = 0.01$ and $u_0 = \exp(x - 0.25)^2$
$\exp\left(-200\,(x - 0.5t - 0.25)^2\right)$
CFL number $= 0.5$, $\Delta x = 0.0100$
CFL number $= 1.0$, $\Delta x = 0.0050$
CFL number $= 1.5$, $\Delta x = 0.0033$
$u\,(x, t = 0.00)$
x
0.0
0.2
0.4
0.6
0.8
1.0
1.2
1.4

Example (Lax-Friedrichs scheme)

$$
0 = \frac{U_j^{n+1} - \frac{U_{j+1}^n + U_{j-1}^n}{2}}{\Delta t} + c\frac{U_{j+1}^n - U_{j-1}^n}{2\Delta x}.
$$

$$
U_j^{n+1} = \frac{U_{j+1}^n + U_{j-1}^n}{2} - \frac{r}{2}\left(U_{j+1}^n - U_{j-1}^n\right), \quad r = c\frac{\Delta t}{\Delta x}.
$$



Theorem (Stability analysis for Lax-Friedrichs)

$$r \in (0,1] \iff |\lambda(k)| \leq 1.$$

Proof.

$$U_j^{n+1} = \frac{U_{j+1}^n + U_{j-1}^n}{2} - \frac{r}{2}\left(U_{j+1}^n - U_{j-1}^n\right).$$

$$\lambda(k)^{n+1} e^{ik(j\Delta x)} = \frac{\lambda(k)^n}{2}\left(e^{ik(j+1)\Delta x} + e^{ik(j-1)\Delta x}\right) - \frac{r\lambda(k)^n}{2}\left(e^{ik(j+1)\Delta x} - e^{ik(j-1)\Delta x}\right).$$

$$\lambda(k) = \frac{e^{ik\Delta x} + e^{-ik\Delta x}}{2} - r\frac{e^{ik\Delta x} - e^{-ik\Delta x}}{2}.$$

$$\lambda(k) = \cos(k\Delta x) - ir\sin(k\Delta x).$$

$$|\lambda(k)|^2 = \cos^2(k\Delta x) + r^2\sin^2(k\Delta x).$$

$$|\lambda(k)|^2 = 1 - \sin^2(k\Delta x) + r^2\sin^2(k\Delta x).$$

$$|\lambda(k)|^2 = 1 - \left(1 - r^2\right)\sin^2(k\Delta x).$$

$$\leq 1 \iff r \in (0,1].$$

$\square$

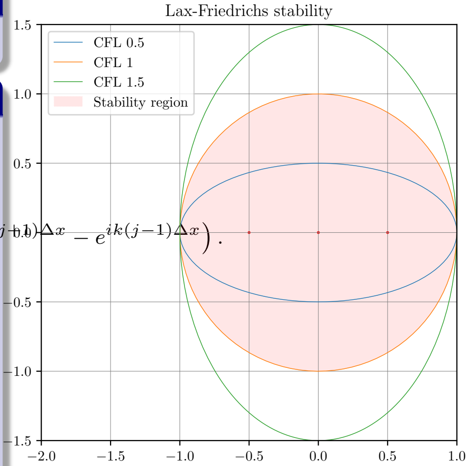
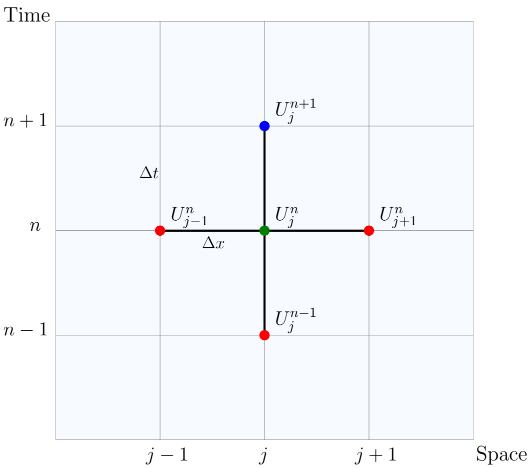


Figure: The Lax-Friedrichs scheme is oscillates.



Example (Leapfrog scheme)

$$0 = \frac{U_j^{n+1} - U_j^{n-1}}{2\Delta t} + c\frac{U_{j+1}^n - U_{j-1}^n}{2\Delta x}.$$

$$U_j^{n+1} = U_j^{n-1} - \frac{c\Delta t}{\Delta x}\frac{U_{j+1}^n - U_{j-1}^n}{2}.$$

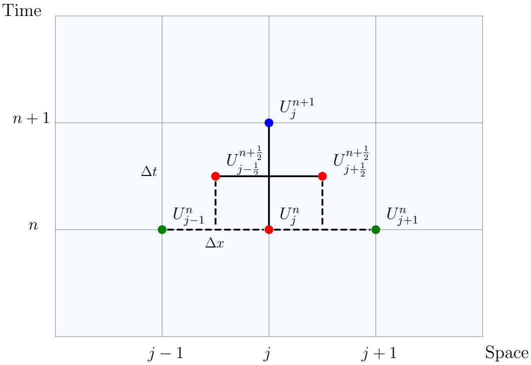
$$\left|U_j^{n+1}\right| = \left|U_j^{n-1} - r\frac{U_{j+1}^n - U_{j-1}^n}{2}\right| = \left|0 - r\frac{e^{ikj\Delta x}e^{ik\Delta x} - e^{ikj\Delta x}e^{-ik\Delta x}}{2}\right|.$$

Fourier stability analysis

Example (Lax-Wendroff scheme)

$$0 = \frac{U_i^{n+1} - U_i^{n-1}}{2\Delta t} + c\frac{U_{i+1}^n - U_{i-1}^n}{2\Delta x}.$$

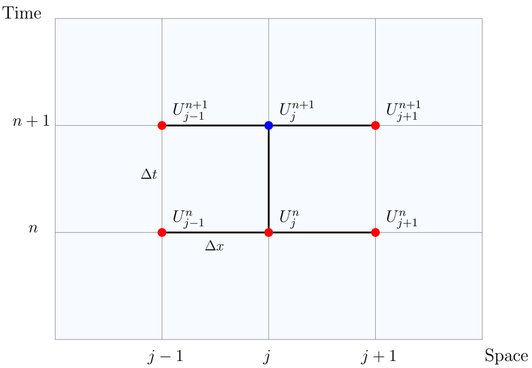
$$U_j^{n+1} = U_j^n - \frac{c\Delta t}{\Delta x}\left(U_{j+\frac{1}{2}}^{n+\frac{1}{2}} - U_{j-\frac{1}{2}}^{n+\frac{1}{2}}\right).$$



Example (Crank-Nicolson scheme)

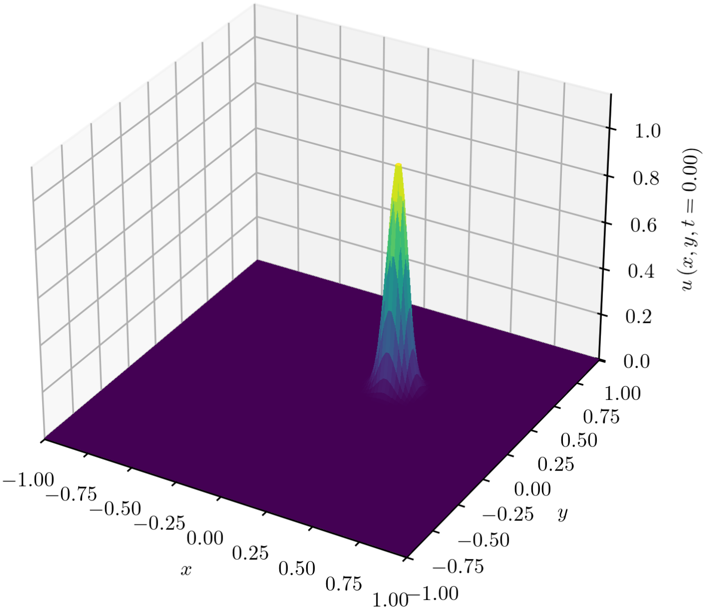
$$0 = \frac{U_i^{n+1} - U_i^{n-1}}{2\Delta t} + c\frac{U_{i+1}^n - U_{i-1}^n}{2\Delta x}.$$

$$U_j^{n+1} = U_j^n - \frac{c\Delta t}{\Delta x}\frac{U_{j+1}^n - U_{j-1}^n + U_{j+1}^{n+1} - U_{j-1}^{n+1}}{4}.$$



Theorem (Lax-Richtmyer equivalence)

A consistent finite-difference scheme for a PDE for which the initial-value problem is well posed is convergent iff it is stable.



1.0
0.8
0.6
0.4
0.2
0.0
$u(x, y, t = 0.00)$
1.00
0.75
0.50
0.25
0.00
−0.25
−0.50
−0.75
−1.00
y
−1.00
−0.75
−0.50
−0.25
0.00
0.25
0.50
0.75
1.00
x

Fourier stability analysis

Since it underlies the mass conservation law, the advection equation

$$\partial_t u + \nabla \cdot (cu) = 0$$

is said to be in conservative form (also called conservation or flux form).
From the identity

$$\nabla \cdot (cu) = u\nabla \cdot c + c \cdot \nabla u,$$

$$\partial_t u + a \cdot \nabla u = 0$$

if the vector field c is divergence-free, that is if

$$\nabla \cdot c = 0.$$

If we define the characteristics $(\xi(t), t)$ in the (x, t) space of solutions of the ODE

$$\frac{\mathrm{d}\xi(t)}{\mathrm{d}t} = c(\xi(t), t),$$

then it follows that

$$\frac{\mathrm{d}u(\xi(t), t)}{\mathrm{d}t} = 0,$$

and hence the solution $u(x, t)$ is constant along the characteristics.