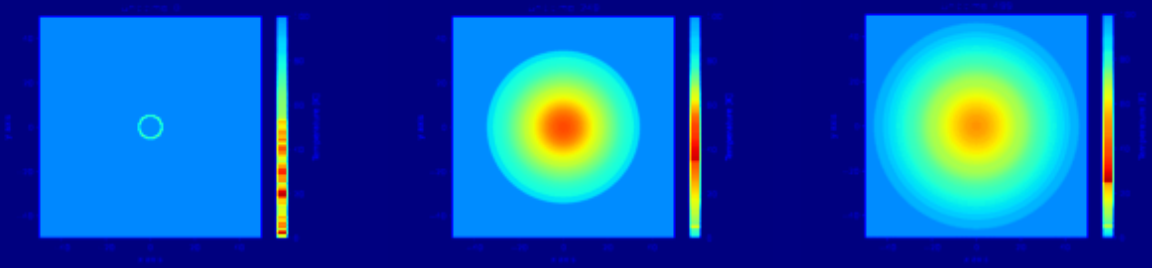
Partial Differential Equations I



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Last change: October 28, 2024 at 12:15am.

Useful links

Click on each vignette for updated resources or the book cover on next slides.

- Pad + general information
- Meeting link `Mon, Fri 09:00:00 PM -05`
- Beamer slides + Report lecture
- Analytical methods for solve the wave equation (1D, 2D and 3D) course + books
- Live recordings + Jason Bramburger's lectures
- Repository
- Animations with matplotlib
- Shared folder + exercises

Remark

We'll try to follow this outline `https://math.dartmouth.edu/~m53f22`.

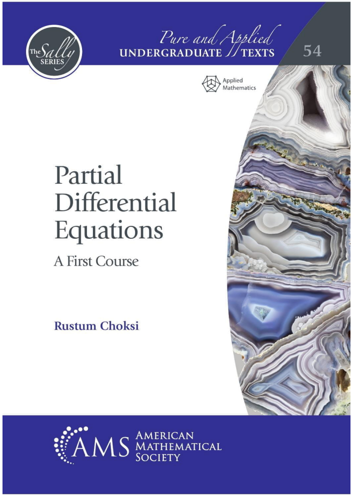
VisualPDE

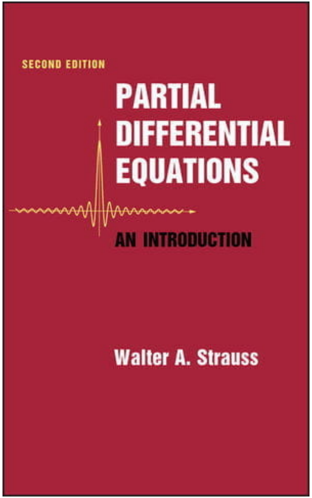
Every time we explore a new PDE we are likelihood to visualize the animation on `https://visualpde.com`.

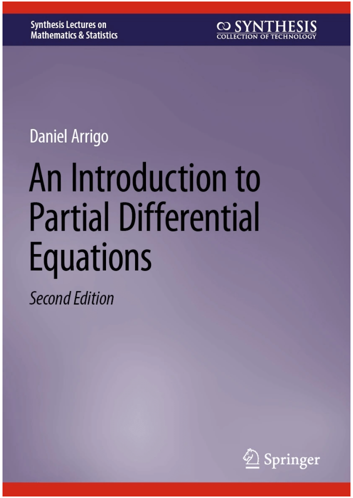
Universal document viewer

Okular is a PDF viewer that allows interaction with forms, e.g., display animations of time dependent PDE solutions.

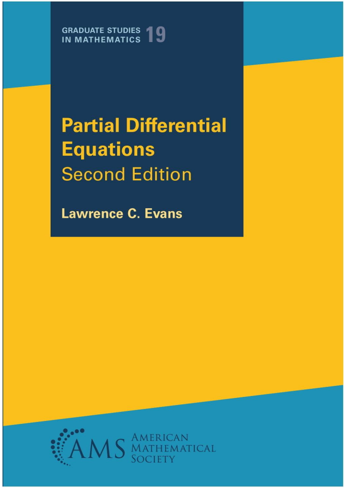
References with foundations on ODE

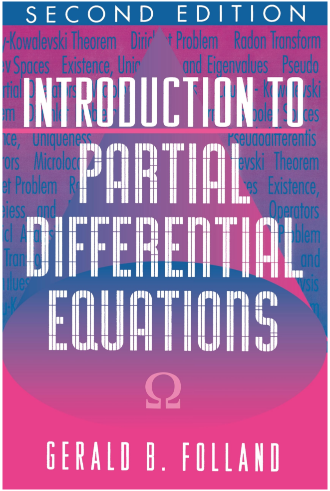


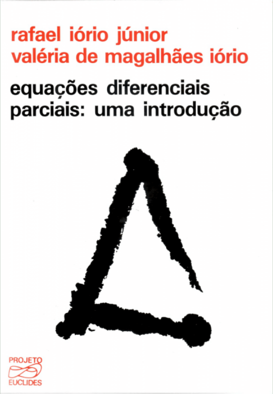




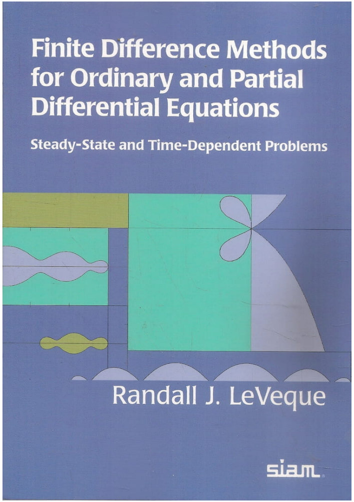
References with foundations on Functional Analysis

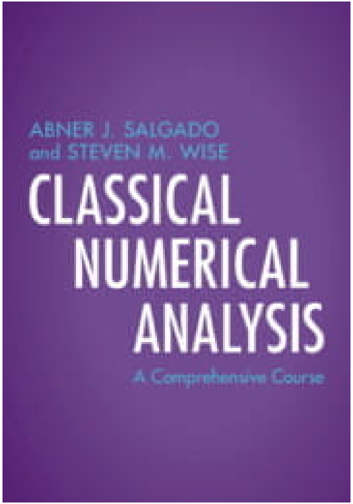


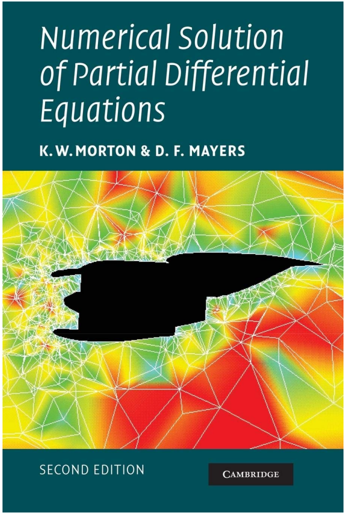




References with foundations on Numerical Analysis







Contents

Review of ODEs

An ordinary differential equation (ODE) is a *functional equation* that relates some function with its derivatives.

Example (Classification of ODEs)

- Heterogeneous first-order linear constant coefficient.

$$\frac{\mathrm{d}u}{\mathrm{d}x} = \pi u + \cos(x).$$

- Homogeneous second-order linear.

$$\frac{\mathrm{d}^2u}{\mathrm{d}x^2} - x\frac{\mathrm{d}u}{\mathrm{d}x} + u = 0.$$

- Homogeneous second-order linear constant coefficient.

$$\frac{\mathrm{d}^2u}{\mathrm{d}x^2} + \alpha^2 u = 0.$$

- Heterogeneous first-order nonlinear.

$$\frac{\mathrm{d}u}{\mathrm{d}x} = u^5 + 1.$$

For functions of several variables, an ODE becomes in a PDE.

Example (PDE models)

- Models the concentration of a substance flowing in a fluid at a constant rate $c \in \mathbb{R} \setminus \{0\}$.

  (Advection) $$\partial_t u + c\partial_x u = 0.$$

  Its general solution is $u\left(x, t\right) = \phi\left(x - ct\right)$ where ϕ is an arbitrary function.
- Type of propagating disturbance that moves faster than the speed of sound in a medium.

  (Shock waves) $$\partial_x u + u\partial_y u = 0.$$

  Like a common wave, a shock wave carries energy and can propagate through a medium, but is characterized by an abrupt, almost discontinuous change in the pressure, temperature, and density of the medium.
- Models the constant heat flow in a region where the temperature is fixed at the boundary.

  (Laplace) $$\triangle u = 0.$$

More classifications of differential equations

- An integro-differential equation involving both the derivatives and its anti-derivatives of a solution.

(RLC circuit)

$$L\frac{\mathrm{d}I(t)}{\mathrm{d}t} + RI(t) + \frac{1}{C}\int_0^t I(\tau)\,\mathrm{d}\tau = E(t).$$

- A functional differential equation with deviating argument and more applicable than ODEs.

(Population growth)

$$\frac{\mathrm{d}u(t)}{\mathrm{d}t} = \rho u(t)\left(1 - \frac{u(t-\tau)}{k}\right).$$

- A stochastic differential equation is composed in terms of stochastic process.

(Arithmetic Brownian motion)

$$\mathrm{d}X_t = \mu\,\mathrm{d}t + \sigma\,\mathrm{d}B_t.$$

- A differential algebraic equation involves differential and algebraic terms.
- Stiff PDE, Delay PDE, Controlled PDE, Fractional PDE, Neural PDE and so on.

Review of ODEs

Let the IVP

$$\begin{cases} \dfrac{\mathrm{d}u}{\mathrm{d}t} & = -\dfrac{u}{2}, \quad t \in [0, 10]. \\ u(0) & = a_i, \end{cases}$$

where $a_1 = 2$, $a_2 = 4$, $a_3 = 6$ and $a_4 = 8$.

```python
import numpy as np
from jaxtyping import Array, Float
from scipy.integrate import solve_ivp


def exponential_decay(
    t: Float[Array, "dim"], u: Float[Array, "dim"]
) -> Float[Array, "2"]:
    return -0.5 * u


sol = solve_ivp(
    fun=exponential_decay,
    t_span=(0, 10),
    y0=(2, 4, 6, 8),
    t_eval=np.linspace(start=0, stop=10),
    dense_output=True,
)
```

Program : Recovered from https://docs.scipy.org/doc/scipy-1.14.1/reference/generated/scipy.integrate.solve_ivp.html.

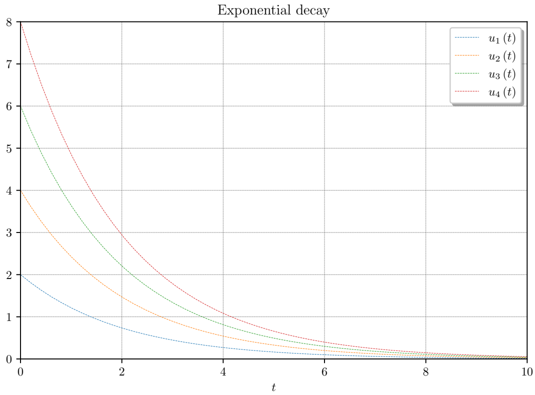


Figure: Numerical solution.

The BVP

$$\frac{\mathrm{d}u}{\mathrm{d}x} + \exp\left(u\right) = 0, \quad u\left(0\right) = u\left(1\right) = 0.$$

```python
import numpy as np
from jaxtyping import Array, Float
from scipy.integrate import solve_bvp


def fun(x: Float[Array, "dim"], u: Float[Array, "2"]) → Float[Array, "2"]:
    return np.vstack((u[1], -np.exp(u[0])))


def bc(ua: float, ub: float) → Float[Array, "2"]:
    return np.array([ua[0], ub[0]])


x = np.linspace(start=0, stop=1, num=5)
u_a = np.zeros(shape=(2, x.size))
u_b = np.copy(a=u_a)
u_b[0] = 3

sol_a = solve_bvp(fun=fun, bc=bc, x=x, y=u_a)
sol_b = solve_bvp(fun=fun, bc=bc, x=x, y=u_b)
```

Program : Recovered from https://docs.scipy.org/doc/scipy-1.14.1/reference/generated/scipy.integrate.solve_bvp.html.

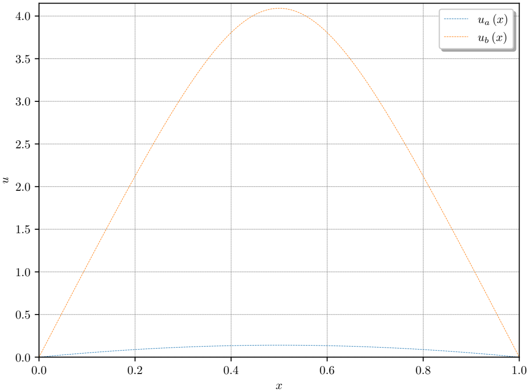


Figure: Numerical solution.

Theorem (Existence and Uniqueness of solutions - Picard-Lindelöf)

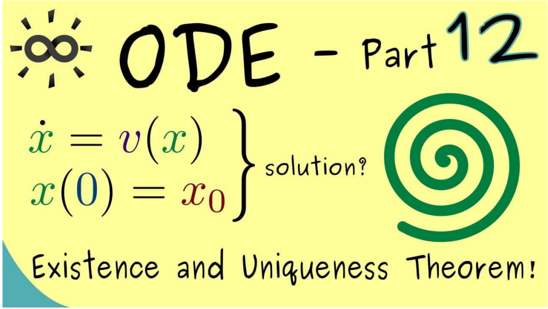
Consider the initial value problem

(1)
$$\begin{cases} \frac{\mathrm{d}u}{\mathrm{d}x} = f(x,u), \\ u(\xi) = \eta. \end{cases}$$

Here it is assumed that $f(\cdot,\cdot)$ is continuous on $[\xi, \xi+a] \times \mathbb{R}$ where $a > 0$, and furthermore satisfies

(Lipschitz condition)
$$|f(x,u) - f(x,\overline{u})| \le L\,|u - \overline{u}|$$

for some $L \in \mathbb{R}_{\ge 0}$; here all $x \in [\xi, \xi+a]$, u, $\overline{u} \in \mathbb{R}$ are allowed. Then (1) admits precisely one C^1-solution $u(x)$ on $[\xi, \xi+a]$.



Idea of proof.

1. Formulation as a fixed point problem.

$$u(x) = \eta + \int_{\xi}^{x} f(t, u(t)) \, \mathrm{d}t.$$

2. Introduction of a Banach space, verifying contraction property.

$$\begin{aligned} T \colon C^0(I_b) &\longrightarrow C^0(I_b) \\ u &\longmapsto \eta + \int_{\xi}^{x} f(t, u(t)) \, \mathrm{d}t. \end{aligned}$$

3. Application of Contraction Principle, construction of local solution.

$\square$

Theorem (Peano)

For $I = [\xi, \xi + a]$, $J = [\eta - b, \eta + b]$, *we have* $f \in C^0 (I \times J), |f|_{C^0(I \times J)} \leq M$ *for some* $M, a, b > 0$, *there exists a solution* $u(x) \in C^1 \left(\left[\xi, \xi + \min \left\{ a, \frac{b}{M+1} \right\} \right] \right)$.

Idea of proof.

1. The idea is to reduce to the situation in Picard's theorem.
2. The mollification of f is now given by the family of functions.

$$f_\varepsilon (x, u) := f *_u \chi_\varepsilon (x, u) = \int_{\mathbb{R}} f (x, u - z) \chi_\varepsilon (z) \, \mathrm{d}z.$$

3. In order to be able to invoke the version of Picard's theorem, we need to extend $f_\varepsilon (x, u)$ to all $\mathbb{R}$.

$$|f_\varepsilon (x, u) - f_\varepsilon (x, \overline{u})| \leq \frac{C}{\varepsilon} M |u - \overline{u}| .$$

4. Use the Arzelà-Ascoli theorem.

□

Separable equation

If the right hand side of the equation

$$\frac{\mathrm{d}u}{\mathrm{d}x} = g\left(x\right)p\left(u\right)$$

can be expressed as function $g\left(x\right)$ that depends only of x times a function $p\left(u\right)$ that depends only on u, the differential equation is called separable.

Example (Separable equation)

$$\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{x-5}{u^2}.$$

Solution

$$u^2\,\mathrm{d}u = \left(x-5\right)\,\mathrm{d}x.$$

$$\int u^2\,\mathrm{d}u = \int \left(x-5\right)\,\mathrm{d}x.$$

$$\frac{u^3}{3} = \frac{x^2}{2} - 5x + C \implies u\left(x\right) = \left(\frac{3x^2}{2} - 15x + K\right)^{\frac{1}{3}}.$$

Linear equation

In order to solve the ODE in the standard form

(2)
$$\frac{\mathrm{d}u}{\mathrm{d}x} + P(x)\, u(x) = Q(x).$$

Calculate the integrating factor $\mu(x)$ by

(3)
$$\mu(x) = \exp\left[\int P(x)\, \mathrm{d}x\right].$$

And multiply (2) by (3)

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\mu(x)\, u(x)\right] = \mu(x)\, Q(x).$$

And obtain the solution

$$u(x) = \frac{1}{\mu(x)}\left[\int \mu(x)\, Q(x)\, \mathrm{d}x + C\right].$$

Example (Linear equation)

$$\frac{\mathrm{d}u}{\mathrm{d}x} + 2u(x) = 50\exp(-10x).$$

Homogeneous linear second order ode

Let be $a \in \mathbb{R} \setminus \{0\}$.

$$a\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} + b\frac{\mathrm{d}u}{\mathrm{d}x} + cu = 0.$$

Find a solution of the form $u(x) = e^{rx}$.

$$ar^2 e^{rx} + bre^{rx} + ce^{rx} = 0.$$
$$e^{rx}\left(ar^2 + br + c\right) = 0.$$

Since $e^{rx} > 0$

$$ar^2 + br + c = 0.$$

Example (Homogeneous linear second order)

$$\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} + 5\frac{\mathrm{d}u}{\mathrm{d}x} - 6u = 0.$$

Solution

$$r^2 + 5r - 6 = (r-1)(r+6) = 0.$$

e^{x} *and* e^{-6x} *are solutions.*

Example ( )

$$\frac{\mathrm{d}^3 u}{\mathrm{d}x^3} + 3\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} - \frac{\mathrm{d}u}{\mathrm{d}x} - 3u = 0.$$

Nonhomogeneous

$$a\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} + b\frac{\mathrm{d}u}{\mathrm{d}x} + cu = f\left(x\right).$$

Example (Nonhomogeneous)

$$\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} + 3\frac{\mathrm{d}u}{\mathrm{d}x} + 2u = 3x.$$

Method of Variation of Parameters

$$a\frac{\mathrm{d}^2u}{\mathrm{d}x^2}+b\frac{\mathrm{d}u}{\mathrm{d}x}+cu=f\left(x\right).$$

$$u_h\left(x\right)=c_1u_1\left(x\right)+c_2u_2\left(x\right).$$
$$u_p\left(x\right)=v_1\left(x\right)y_1\left(x\right)+v_2\left(x\right)y_2\left(x\right).$$

Example ( )

$$\frac{\mathrm{d}^2u}{\mathrm{d}x^2}+u=\tan x.$$

Solution

The homogeneous equation $\frac{\mathrm{d}^2u}{\mathrm{d}x^2}+u=0$ *are* $\cos x$ *and* $\sin x$.

$$\begin{aligned}
u_p\left(x\right)&=v_1\left(x\right)\cos\left(x\right)+v_2\left(x\right)\sin\left(x\right).\\
v_1\left(x\right)&=\sin\left(x\right)-\ln\left|\sec x+\tan x\right|+C_1.\\
v_2\left(x\right)&=-\cos x+C_2.\\
u\left(x\right)&=c_1\cos x+c_2\sin x-\left(\cos x\right)\ln\left(\sec x+\tan x\right).
\end{aligned}$$

Using Python to Solve PDEs