Multiplying the Fourier series by $\sin(m\theta)$, $m \in \mathbb{N}$ and integrating term by term:

$$\int_0^{2\pi} \sin(m\theta) f(\theta)\, d\theta = \int_0^{2\pi} \sin(m\theta) \frac{a_0}{2}\, d\theta + \int_0^{2\pi} \sin(m\theta) \left(\sum_{k=1}^{\infty} a_k \cos(k\theta) + b_k \sin(k\theta) \right) d\theta.$$

$$\int_0^{2\pi} f(\theta) \sin(m\theta)\, d\theta = 0 + \sum_{k=1}^{\infty} \left(a_k \int_0^{2\pi} \cos(k\theta) \sin(m\theta)\, d\theta + b_k \int_0^{2\pi} \sin(k\theta) \sin(m\theta)\, d\theta \right).$$

$$\int_0^{2\pi} f(\theta) \sin(m\theta)\, d\theta = \sum_{k=1}^{\infty} \left(\frac{a_k}{2} \int_0^{2\pi} \sin((m+k)\theta) + \sin((m-k)\theta)\, d\theta + \frac{b_k}{2} \int_0^{2\pi} \cos((m-k)\theta) - \cos((m+k)\theta)\, d\theta \right).$$

When $m \neq k$ both integrals vanish, thus the infinite sum reduces to m-th addend.

$$\int_0^{2\pi} f(\theta) \sin(m\theta)\, d\theta = a_m \int_0^{2\pi} \cos(m\theta) \sin(m\theta)\, d\theta + b_m \int_0^{2\pi} \sin^2(m\theta)\, d\theta.$$

$$\int_0^{2\pi} f(\theta) \sin(m\theta)\, d\theta = a_m \cdot 0 + \frac{b_m}{2} \int_0^{2\pi} 1 - \cos(2m\theta)\, d\theta.$$

$$\int_0^{2\pi} f(\theta) \sin(m\theta)\, d\theta = b_m \pi. \implies \boxed{b_m = \frac{1}{\pi} \int_0^{2\pi} f(\theta) \sin(m\theta)\, d\theta.}$$

Fourier transform

Fourier transform

$$\frac{\mathrm{d}^n f}{\mathrm{d}x^n} = i^n k^n$$

$$\widehat{v}(\xi) = \mathcal{F}[v](\xi) = \int\limits_{\mathbb{R}^d} \exp(-ix\xi)\, v(x)\, \mathrm{d}x$$

$$\left(\partial_t u \widehat{- \alpha^2 \triangle u}\right)(k,t) = \widehat{\left(\partial_t u\right)}(k,t) - \widehat{\alpha^2 \triangle u}(k,t) = \widehat{0} = 0$$

$$\widehat{\left(\partial_t u\right)}(k,t) = \int\limits_{-\infty}^{\infty} \partial_t u(x,t) \exp(-ikx)\, \mathrm{d}x = \partial_t \left(\int\limits_{-\infty}^{\infty} u(x,t)\right) \exp(-ikx)\, \mathrm{d}x = \partial_t \widehat{u}(k,t) = \partial_t \widehat{u} =$$

$$\widehat{\partial_x^2 u}\,(k,t) = (ik)^2\,\widehat{u}\,(k,t) = -k^2\widehat{u}\,(k,t) =$$

Para cada k, $f\,(t) := \widehat{u}\,(k,t)$

$$F\,(x) = \frac{1}{\sqrt{4\pi t}}\exp\left(-\frac{x^2}{4t}\right)$$

$$u\,(x,t) = \frac{1}{\sqrt{4\pi\alpha t}}\int_{-\infty}^{\infty}\exp\left(-\frac{(x-y)^2}{4\alpha t}\right)g\,(y)\,\mathrm{d}y$$

Proof.

$$u\,(x,t) - u_0\,(x_0) = \frac{1}{(4\pi t)^{d/2}}\int_{\mathbb{R}^d}\exp\left(-\frac{\|x-y\|_2^2}{4t}\right)u_0\,(y)\,\mathrm{d}y - u_0\,(x_0)$$

□

Find all maps $T\colon \mathbb{R}_+ \to \mathbb{C}$ satisfying the functional equation

$$\begin{cases} T(t+s) &= T(t)\,T(s)\,, \quad \forall t,s \geq 0, \\ T(0) &= 1. \end{cases}$$

$$\forall a \in \mathbb{C} : t \mapsto \exp(ta)\,.$$

Definition (Semigroup)

A function $U\colon \mathbb{R}_{>0} \to B(E)$ is called semigroup iff

- $\forall t, t' \geq 0 : U_{t+t'} = U_t U_{t'}$.
- $U_0 = 1$.

Distribution

Distribution

Definition (Distribution)

A distribution F is a rule, assigning to each test function $\phi \in C_c^\infty(\mathbb{R})$ a real number, which is linear and continuous.

$$\begin{aligned} F\colon C_c^\infty(\mathbb{R}) &\longrightarrow \mathbb{R} \\ \phi &\longmapsto \langle F, \phi \rangle . \end{aligned}$$

Remark

Let $f(x)$ be a locally integrable function on $\mathbb{R}$. Then $f(x)$ can be interpreted as the distribution F_f where

$$\forall \phi \in C_c^\infty(\mathbb{R}) : \left\langle F_f, \phi \right\rangle := \int\limits_{-\infty}^{\infty} f(x)\,\phi(x)\,\mathrm{d}x.$$

Think of F_f as the distribution generated by the function $f(x)$.

Definition (Derivative of a Distribution)

Let F be any distribution. Then F' is also a distribution defined by $\forall \phi \in C_c^\infty(\mathbb{R}) : \langle F', \phi \rangle = -\langle F, \phi' \rangle$.

The distributional derivative of the distribution generated by a smooth function is simply the distribution generated by the classical derivative.

Delta "function"

Definition (Delta function)

$$\forall \phi \in C_c^\infty (\mathbb{R}) : \langle \delta_0, \phi \rangle = \phi (0) .$$

Definition (Derivative of the Delta Function)

$$\forall \phi \in C_c^\infty (\mathbb{R}) : \left\langle \delta_0', \phi \right\rangle = - \left\langle \delta_0, \phi' \right\rangle = -\phi' (0) .$$

Definition (Convergence of a Sequence of Distributions)

A sequence of distributions $\{F_n\}_{n\in\mathbb{N}}$ converges to a distribution F iff

$$\forall \phi \in C_c^\infty (\mathbb{R}) : \langle F_n, \phi \rangle \xrightarrow{n\to\infty} \langle F, \phi \rangle .$$

Definition (Sobolev Space $W^{k,p}\left(\mathbb{R}^d\right)$)

Let $k \in \mathbb{N}$, $p \geq 1$, and let $u \colon \mathbb{R}^d \to \mathbb{R}$ be a locally integrable function. Suppose the following:

- For any multi-index α of order less than or equal to k, $\partial^\alpha u$ in the sense of distributions is generated by a locally integrable function v_α; that is,

$$\forall \phi \in C_c^\infty\left(\mathbb{R}^d\right) : \langle \partial^\alpha u, \phi \rangle = (-1)^{|\alpha|} \langle u, \partial^\alpha \phi \rangle = (-1)^{|\alpha|} \int \underset{\mathbb{R}^d}{\cdots} \int u\left(x\right) \partial^\alpha \phi\left(x\right) \mathrm{d}x = \int \underset{\mathbb{R}^d}{\cdots} \int v_\alpha\left(x\right) \phi\left(x\right) \mathrm{d}x.$$

- The function v_α satisfies

$$\int \underset{\mathbb{R}^d}{\cdots} \int \left|v_\alpha\left(x\right)\right|^p \mathrm{d}x < \infty.$$

Then we say $u \in W^{k,p}\left(\mathbb{R}^d\right)$.

Wave operator

Let the d'Alembertian operator $\Box := \partial_t^2 - c^2\triangle = \partial_t^2 - c^2\sum_1^n \partial_j^2$ on $\mathbb{R}^n \times \mathbb{R}$.

The wave equation

$$\Box u = 0 \tag{10}$$

is fulfilled by waves with propagation speed c in a homogeneous isotropic medium.

Jean-Baptiste le Rond D'Alembert (1717–1783).

Invariance

The operator $\Box$ is invariant under time-reversal $(x,t) \to (x,-t)$.

Remark

- If the solution of (10) is $u \in C^2(\mathbb{R} \times \mathbb{R})$, then ∂_x and ∂_t commute and the wave operator is
$$\Box = \partial_t^2 - c^2\partial_x^2 = \left(\partial_t - c\partial_x\right)\left(\partial_t + c\partial_x\right).$$
Thus, the one-dimensional wave equation becomes $\left(\partial_t - c\partial_x\right)\left[\left(\partial_t + c\partial_x\right)u\right] = 0$. These are the first order operator of the transport equation, whose solution is of the form $u(x,t) = g(x-ct)$.
- The wave equation solution is the sum of two solutions of transport equations: one moves to the left and the other moves to the right both with propagation speed c.

Theorem (General solution of the one-dimensional wave equation)

The solution of (10) *is* $u(x,t) = f(x+ct) + g(x-ct)$, *where* $f, g \in C^2(\mathbb{R})$.

Remark

- If $\phi \in C^2(\mathbb{R})$, then $u_\pm(x,t) = \phi(x \pm ct)$ fulfills (10) since $\partial_t^2 u_\pm = c^2 \partial_x^2 u_\pm = \frac{\mathrm{d}^2\phi}{\mathrm{d}x^2}(x \pm ct)$.
- If ϕ is locally integrable, then u is a distributional solution.

Proof.

Let $f, g \in C^2(\mathbb{R})$ and the changes of variable $\xi = x + ct$, $\eta = x - ct$. The chain rule gives us

$$\partial_x u = \left(\partial_\xi + \partial_\eta\right) u, \qquad \partial_t u = c\left(\partial_\xi - \partial_\eta\right) u.$$

Combining these equations, we find

$$\left(\partial_t - c\partial_x\right) u = -2c\partial_\eta u, \qquad \left(\partial_t + c\partial_x\right) u = 2c\partial_\xi u.$$

Thus, (10) results

$$\Box u = \left(\partial_t - c\partial_x\right)\left(\partial_t + c\partial_x\right) u = \left(-2c\partial_\eta u\right)\left(2c\partial_\xi u\right) = -4c^2\partial_{\xi\eta} u = 0.$$

His solution is $u(\xi,\eta) = f(\xi) + g(\eta)$. That is, $u(x,t) = f(x+ct) + g(x-ct)$ with the original variables. $\square$

Global homogeneous Cauchy problem for the one-dimensional wave equation

The boundary value problem for (10) is a Cauchy problem.

(11)
$$\begin{cases} \Box u = 0 & \text{for } x \in \mathbb{R},\ t > 0. \\ u(x,0) = \phi(x), \quad \partial_t u(x,0) = \psi(x) & \text{for } x \in \mathbb{R}, \end{cases}$$

where $\phi \in C^2(\mathbb{R})$ and $\psi \in C^1(\mathbb{R})$ are the initial displacement and the initial velocity, respectively.

Theorem (d'Alembert's formula)

If $u(x,t)$ is solution of (11), then

(12)
$$u(x,t) = \frac{1}{2}\left[\phi(x+ct) + \phi(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\,\mathrm{d}s.$$

Proof.

By theorem 93, we have that the general solution of (10) is $u(x,t) = f(x+ct) + g(x-ct)$ with $f, g \in C^2(\mathbb{R})$.
Let us look for a relation between the functions f y g con ϕ y ψ. Note that

$$u(x,0) = f(x) + g(x) = \phi(x), \quad \partial_t u(x,0) = f'(x) + g'(x) = \psi(x).$$

Replace x by α

$$\phi(\alpha) = f(\alpha) + g(\alpha), \quad \psi(\alpha) = f'(\alpha) + g'(\alpha).$$

Proof (Cont.)

Solving the system of equations for f' and g' in terms of ϕ' and ψ'.

$$f'(\alpha) = \frac{1}{2}\left[\phi'(\alpha) + \psi(\alpha)\right], \qquad g'(\alpha) = \frac{1}{2}\left[\phi'(\alpha) - \psi(\alpha)\right].$$

Integrating with respect to α gives us

$$f(\alpha) = \frac{1}{2}\phi(\alpha) + \frac{1}{2}\int_0^\alpha \psi(s)\,\mathrm{d}s + C_1, \qquad g(\alpha) = \frac{1}{2}\phi(\alpha) - \frac{1}{2}\int_0^\alpha \psi(s)\,\mathrm{d}s + C_2.$$

$$\therefore \phi(\alpha) = f(\alpha) + g(\alpha) = \frac{1}{2}\phi(\alpha) + \frac{1}{2}\int_0^\alpha \psi(s)\,\mathrm{d}s + C_1 + \frac{1}{2}\phi(\alpha) - \frac{1}{2}\int_0^\alpha \psi(s)\,\mathrm{d}s + C_2 = \phi(\alpha) + C_1 + C_2.$$

And we have that $C_1 + C_2 = 0$. Finally,

$$u(x,t) = f(x+ct) + g(x-ct) = \frac{1}{2}\left[\phi(x+ct) + \phi(x-ct)\right] + \frac{1}{2}\int_0^{x+ct} \psi(s)\,\mathrm{d}s - \frac{1}{2}\int_0^{x-ct} \psi(s)\,\mathrm{d}s.$$

$$u(x,t) = \frac{1}{2}\left[\phi(x+ct) + \phi(x-ct)\right] + \frac{1}{2}\int_0^{x+ct} \psi(s)\,\mathrm{d}s - \left(-\frac{1}{2}\int_{x-ct}^{0} \psi(s)\,\mathrm{d}s\right).$$

$$u(x,t) = \frac{1}{2}\left[\phi(x+ct) + \phi(x-ct)\right] + \frac{1}{2}\int_{x-ct}^{x+ct} \psi(s)\,\mathrm{d}s.$$

$\square$

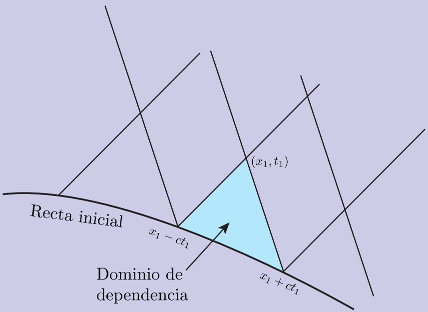
Example

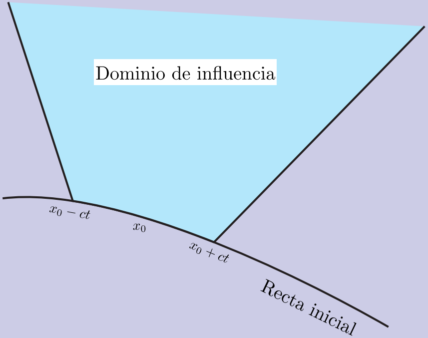
Let $l > 0$ and suppose that $\phi(x) = \sin\left(\frac{x}{l}\right)$ and $\psi(x) = 0$. Then, the d'Alembert's formula gives

$$u(x,t) = \frac{1}{2}\left[\sin\left(\frac{x+t}{l}\right) + \sin\left(\frac{x-t}{l}\right)\right] = \sin\left(\frac{x}{l}\right)\cos\left(\frac{ct}{l}\right).$$

Definition (Domain of dependence and influence)

- Fix a point (x_1, t_1) in space-time and obtain an initial data $u(x_1, t_1)$ and look to the past.
- Fix a point x_0 at $t = 0$ and ask yourself after a time $t_1 > 0$, ¿which points on the string are influenced by the displacement/velocity at x_0 for $t = 0$?





The following principle is related to the method of variation of Parameters in ODEs.

Global inhomogeneous Cauchy problem for the one-dimensional wave equation

Now suppose that the infinite string is under to a vertical external force at position x and time t.

(13)
$$\begin{cases} \square u = f(x,t) & \text{for } x \in \mathbb{R}, t > 0. \\ u(x,0) = \phi(x), \quad \partial_t u(x,0) = \psi(x) & \text{for } x \in \mathbb{R}. \end{cases}$$

Theorem

If $u(x,t)$ is a solution of (13), then

(14)
$$u(x,t) = \frac{1}{2}\left[\phi(x+ct) + \phi(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\,\mathrm{d}s + \frac{1}{2c}\iint_D f(y,\tau)\,\mathrm{d}y\,\mathrm{d}\tau,$$

where D is the dependency domain associated with (x,t), es decir, the triangle in the plane x versus t and the base points $(x-ct,0)$ and $(x+ct,0)$.

Remark

If $f \equiv 0$ in (13), then we recover d'Alembert's formula in (14).

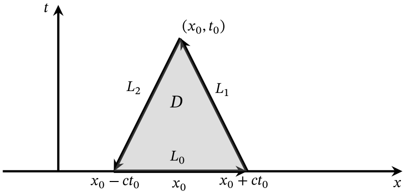
Proof.

Let $(x_0, t_0) \in D$ fixed and arbitrary. Suppose that (13) has a solution.

$$f(x,t) = \Box u = \partial_t^2 u - c^2 \partial_x^2 u.$$

$$\begin{aligned}
\iint\limits_D f(x,t)\,\mathrm{d}x\,\mathrm{d}t &= \iint\limits_D \left(\partial_t^2 u - c^2\partial_x^2 u\right)\,\mathrm{d}x\,\mathrm{d}t. \\
&= \iint\limits_D \left(-c^2\partial_x u\right)_x - \left(-\partial_t u\right)_t \mathrm{d}x. \\
&\overset{G}{=} \int\limits_{\partial D} -c^2\partial_x u - \partial_t u\,\mathrm{d}x. \\
&= \int\limits_{L_0\cup L_1\cup L_2} -c^2\partial_x u - \partial_t u\,\mathrm{d}x. \\
&= \sum_{i=0}^{2}\int\limits_{L_i} -c^2\partial_x u - \partial_t u\,\mathrm{d}x.
\end{aligned}$$

With the help of Green's theorem with $P = -\partial_t u$ y $Q = -c^2\partial_x u$.



Proof (Cont.)

We calculate each line integral

$$\int_{L_0} -c^2\partial_x u\,\mathrm{d}t - \partial_t u\,\mathrm{d}x = -\int_{x_0-ct_0}^{x_0+ct_0} \psi\left(x\right)\,\mathrm{d}x.$$

$$\int_{L_1} -c^2\partial_x u\,\mathrm{d}t - \partial_t u\,\mathrm{d}x = \int_0^{t_0} \left(-c^2\partial_x u + c\partial_t u\right)\,\mathrm{d}t = cu\left(x_0, t_0\right) - c\phi\left(x_0 + t_0\right).$$

$$\int_{L_2} -c^2\partial_x u - \partial_t u\,\mathrm{d}x = -c\phi\left(x_0 - ct_0\right) + cu\left(x_0, t_0\right).$$

Then,

$$\iint_D f\left(x,t\right)\,\mathrm{d}x\,\mathrm{d}t = -\int_{x_0-ct_0}^{x_0+ct_0} \psi\left(x\right)\,\mathrm{d}x + cu\left(x_0,t_0\right) - c\phi\left(x_0+t_0\right) - c\phi\left(x_0-ct_0\right) + cu\left(x_0,t_0\right).$$

Therefore,

$$u\left(x_0,t_0\right) = \frac{1}{2}\left[\phi\left(x_0+ct_0\right) + \phi\left(x_0-ct_0\right)\right] + \frac{1}{2c}\int_{x_0-ct_0}^{x_0+ct_0} + \frac{1}{2c}\iint_D f\left(x,t\right)\,\mathrm{d}x\,\mathrm{d}t.$$

□

Example

$$\begin{cases} \Box u = xt & \text{for } x \in \mathbb{R}, t > 0. \\ u(x,0) = \sin(x), \quad \partial_t u(x,0) = \cos(x) & \text{for } x \in \mathbb{R}. \end{cases}$$

Solution

Then, (14) gives

$$u(x,t) = \frac{1}{2}\left[\sin(x+ct) + \sin(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} \cos(s)\, \mathrm{d}s + \frac{1}{2c}\int_0^t \int_{x-c(t-\tau)}^{x+c(t-\tau)} y\tau \, \mathrm{d}y \, \mathrm{d}\tau.$$

$$u(x,t) = \frac{1}{2}\left[2\sin(x)\cos(ct)\right] + \frac{1}{2c}\left[\sin(x+ct) - \sin(x-ct)\right] + \frac{1}{2}\int_0^t \int_{x-ct+c\tau}^{x+ct-c\tau} y\tau \, \mathrm{d}y \, \mathrm{d}\tau.$$

$$u(x,t) = \sin(x)\cos(ct) + \frac{1}{2}\left[2\cos(x)\sin(ct)\right] + \frac{1}{2}\int_0^t \left[\frac{(x+ct-c\tau)^2}{2} - \frac{(x-ct+c\tau)^2}{2}\right]\tau \, \mathrm{d}\tau.$$

$$u(x,t) = \sin(x)\cos(ct) + \cos(x)\sin(ct) + \frac{1}{4}\int_0^t 4cx(t-\tau)\tau \, \mathrm{d}\tau = \sin(x+ct) + cx\int_0^t \left(t\tau - \tau^2\right) \mathrm{d}\tau.$$

$$u(x,t) = \sin(x+ct) + \frac{cxt^3}{6}.$$

Mechanical wave

Recall the physical concepts of the law of conservation of energy and the equation of motion of the mechanical wave, and we will deduce its formulas for kinetic and potential energy.
The period T and the frequency f of the wave are given by

$$\frac{\omega}{2\pi} = f = \frac{1}{T}.$$

$$v = \frac{\omega}{k} = \frac{\lambda}{T} = \lambda f.$$

The speed of a wave on a stretched string with tension t and linear density μ is

$$v = \sqrt{\frac{\tau}{\mu}},$$

that is, it depends only on the properties of the string and the medium.

The principle of conservation of mechanical energy

Energy is a scalar quantity associated with the state of one or more bodies.

Definition (Energía cinética)

Es la energía asociada con el estado de movimiento de un cuerpo.

$$K = \frac{1}{2} m v^2.$$

$$K = \frac{1}{2} \int_0^l \rho \partial_t^2 u(x,t)\, \mathrm{d}x.$$

$$U = \frac{1}{2} \int_0^l T \partial_x^2 u(x,t)\, \mathrm{d}x.$$

$$E = K + U = \frac{1}{2} \int_0^l \left(\rho \partial_t^2 u(x,t) + T \partial_x^2 u(x,t) \right) \mathrm{d}x.$$

$$\partial_t E = 0.$$

Energy of vibrating string

Wave equation with two spatial dimensions

Diffusion operator

Theorem

Let $u_0 \in C_b\left(\mathbb{R}^d\right) \cap L^2\left(\mathbb{R}^d\right)$.

$$\forall x_0 \in \mathbb{R}^d : \lim_{(x,t)\to(x_0,0)} u\left(x,t\right) = u_0\left(x_0\right).$$

Example

Let

$$u_0\left(x\right) = \exp\left(-\frac{x^2}{2}\right).$$

Theorem (Nonexpansiveness)

For any $t > 0$.

$$
\begin{aligned}
E(t) : C_b\left(\mathbb{R}^d\right) &\longrightarrow C_b\left(\mathbb{R}^d\right) \\
x &\longmapsto \frac{1}{(4\pi t)^{\frac{d}{2}}} \int\limits_{\mathbb{R}^d} \exp\left(-\frac{\|x-y\|_2^2}{4t}\right) u(y)\,\mathrm{d}y.
\end{aligned}
$$

$$
\forall t > 0 : \left\|u_1(\cdot,t) - u_2(\cdot,t)\right\|_{L^\infty\left(\mathbb{R}^d\right)} \leq \left\|u_0^1 - u_0^2\right\|_{L^\infty\left(\mathbb{R}^d\right)}
$$

Definition (Weak solution)

Let $T \in (0, \infty]$.

$$u \in L^2\left(0, T; H_0^1(\Omega)\right), \quad \partial_t u \in L^2\left(0, T; H_0^1(\Omega)'\right)$$

Theorem (Maximum principle)

Let $u\colon \overline{\mathcal{C}} \to \mathbb{R}$

$$\partial_t u\,(x,t) - \triangle u\,(x,t) \leq 0$$

Example (IVP for the diffusion equation)

$$
\begin{cases} \partial_t u - \alpha^2 \triangle u = 0 & \text{for } (x,t) \in \Omega \times [0,T]. \\ u(x,0) = \phi(x), \quad \partial_t u(x,0) = \psi(x) & \text{for } x \in \Omega. \end{cases} \tag{15}
$$

Hopf-Cole transformation for the one-dimensional quasi-linear advection-diffusion

Is a fundamental PDE that occurs in fluid mechanics, nonlinear acoustics, gas dynamics and traffic flow. We consider the

(Viscous Burger's equation)
$$\begin{cases} \partial_t u + u\partial_x u = \nu \partial_x^2 u & \text{for } a < x < b,\ t > 0. \\ u(x,0) = \phi(x) & \text{for } a < x < b. \\ u(a,t) = f(t) \text{ and } u(b,t) = g(t) & \text{for } t > 0. \end{cases}$$

From where

- u is the fluid velocity.
- ν is the kinetic viscosity coefficient.

The Hopf-Cole transformation provides an interesting method for solving the above PDE and other higher order PDEs, this describes $u(x,t)$ by a function $w(x,t)$ given as

(16)
$$\boxed{u(x,t) = -2\nu \frac{\partial_x w}{w}.}$$

We compute the following derivatives from (16).

(17)
$$\begin{cases} \partial_t u &= -2\nu \partial_t \left(\frac{\partial_x w}{w}\right) = -2\nu \frac{w\partial_t \partial_x w - \partial_x w \partial_t w}{w^2} = 2\nu \frac{\partial_x w \partial_t w - w \partial_t \partial_x w}{w^2}. \\ \partial_x u &= -2\nu \partial_x \left(\frac{\partial_x w}{w}\right) = -2\nu \frac{w\partial_x^2 w - \left(\partial_x w\right)^2}{w^2} = 2\nu \frac{\left(\partial_x w\right)^2 - w\partial_x^2 w}{w^2}. \\ \partial_x^2 u &= -2\nu \partial_x \partial_x \left(\frac{\partial_x w}{w}\right) = 2\nu \frac{-w^2 \partial_x^3 w + 3w\partial_x w \partial_x^2 w - 2w_x^3}{w^3}. \end{cases}$$

Now, replace the equations (16) and (17) into (Viscous Burger's equation).

Left hand side

$$u_t = 2\nu \frac{w_x w_t - w w_{xt}}{w^2}.$$

Right hand side

$$\begin{aligned} \nu u_{xx} - u u_x &= 2\nu^2 \frac{-w^2 w_{xxx} + 3 w w_x w_{xx} - 2 w_x^3}{w^3} - \left(-2\nu \frac{w_x}{w}\right) 2\nu \frac{w_x^2 - w w_{xx}}{w^2} \\ &= 2\nu^2 \frac{-w^2 w_{xxx} + 3 w w_x w_{xx} - 2 w_x^3}{w^3} + 4\nu^2 \frac{w_x^3 - w w_x w_{xx}}{w^3} \\ &= -2\nu^2 \frac{w w_{xxx} - w_x w_{xx}}{w^2}. \end{aligned}$$

Finally, we have

$$\begin{aligned} \text{Left hand side} &= \text{Right hand side} \\ 2\nu \frac{w_x w_t - w w_{xt}}{w^2} &= -2\nu^2 \frac{w w_{xxx} - w_x w_{xx}}{w^2}. \\ \frac{w_x w_t - w w_{xt}}{w^2} &= \nu \frac{w_x w_{xx} - w w_{xxx}}{w^2}. \end{aligned}$$

We get

$$\left(\frac{w_t}{w}\right)_x = \nu \left(\frac{w_{xx}}{w}\right)_x.$$

Integrating with respect to x

$$\frac{w_t}{w} = \nu \frac{w_{xx}}{w}.$$

$$w_t = \nu w_{xx}. \tag{4}$$

We see that (Viscous Burger's equation) had been transformed into the heat equation.

Laplace operator

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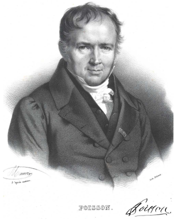

Siméon Denis Poisson
(1781–1840).

Theorem (Maximum principle)

Let $u \in C^2(\Omega) \cap C\left(\overline{\Omega}\right)$.

$$\forall x \in \Omega : \mathcal{L}u(x) \leq 0.$$

- *If* $c \equiv 0$, *then the function* u *attains its maximum at the boundary*

$$\max_{x \in \overline{\Omega}} \leq \max_{x \in \partial\Omega}$$

- $$\forall x \in \Omega : c(x) \geq 0 \implies \max_{x \in \overline{\Omega}} u(x) \leq \max_{x \in \partial\Omega} \left\{0, \max_{x \in \partial\Omega} u(x)\right\}.$$

Our goal is to find a equilibrium (steady-state) heat distribution in a rectangle $R = [0, a] \times [0, b]$ where we specify the temperature (heat) on the boundary. This means that we are given four functions (of one variable) f_1, f_2, f_3 and f_4 and we wish to solve

$$\begin{cases} u_{xx} + u_{yy} = 0, & 0 < x < a, 0 < y < b, \\ u\left(x, 0\right) = f_1\left(x\right), & u\left(x, b\right) = f_2\left(x\right), 0 \leq x \leq a, \\ u\left(x, 0\right) = f_1\left(y\right), & u\left(x, b\right) = f_2\left(y\right), 0 \leq x \leq b, \\ u\left(x, 0\right) = f_2\left(y\right), & u\left(x, 0\right) = f_2\left(y\right), 0 \leq x \leq b, \end{cases}$$

The first thing we note is that it

$$\begin{cases} u\left(x, 0\right) = 0, & 0 \leq x < a, 0 < y < b, \\ u\left(0, y\right) = 0, & 0 \leq x \leq a, 0 < y \leq b \end{cases}$$

Laplace's Equation in a Disk

$$\begin{cases} \Delta u = 0, & \mathbf{x} \in D = \{(x,y) : x^2 + y^2 < a^2\}, \\ u = h, & \mathbf{x} \in \partial D. \end{cases}$$

Let us solve this problem via separation of variables. Switching to polar coordinates,
Rewriting the Laplacian operator in terms of r and θ yields

$$\Delta u = \partial_r^2 u + \frac{1}{r}\partial_r u + \frac{1}{r^2}\partial_\theta^2 u.$$

Therefore, we are looking to solve

$$\begin{cases} \partial_r^2 u + \frac{1}{r}\partial_r u + \frac{1}{r^2}\partial_\theta^2 u = 0, & r < a, \\ u = h(\theta), & r = a. \end{cases}$$

look for separated solutions of the form $u(r,\theta) = R(r)\Theta(\theta)$ for which the PDE requires that

$$R''\Theta + \frac{1}{r}R'\Theta + \frac{1}{r^2}R\Theta'' = 0.$$

Dividing by $R\Theta$ and multiplying by r^2, we obtain

$$r^2\frac{R''}{R} + r\frac{R'}{R} = -\frac{\Theta''}{\Theta} = \lambda.$$

The Separation of Variables Algorithm for Boundary Value Problems

The Separation of Variables Algorithm for Boundary Value Problems

Given a linear PDE with boundary conditions and/or initial conditions, the Separation of Variables Algorithm is based upon the following steps:

1. We look for separated solutions to the PDE and the boundary conditions of the form
$$u(x,t) = X(x)\,T(t).$$
2. The boundary conditions carry over to the eigenvalue problem involving $X(x)$. We solve this boundary value / eigenvalue problem to find countably many eigenvalues λ_n for which there exist nontrivial solutions $X_n(x)$.
3. We solve the eigenvalue problem of $T(t)$ for each eigenvalue λ_n found in the previous step. We thus arrive at countably many separated solutions
$$u_n(x,t) = X_n(x)\,T_n(t).$$
to the PDE and the boundary conditions.
4. We note that any finite linear combination u_n of these separated solutions will also be a solution to the PDE and the boundary conditions. We boldly consider an infinite linear combination of the form
$$\sum_{n=1}^{\infty} a_n X_n(x)\,T_n(t).$$
with coefficients a_n.
5. We note that achieving the initial conditions amounts to choosing coefficients appropriately. When the eigenvalue problems are for $-\frac{\mathrm{d}^2}{\mathrm{d}x^2}$ on some interval with a symmetric boundary condition, we arrive at what we previously called a general Fourier series for the data. We find these coefficients by exploiting orthogonality and spanning properties of the eigenfunctions. This effectively means we find each coefficient via projection onto the respective eigenfunction.

Remark

We note that finding these separated solutions reduces to solving eigenvalue problems for each of the components X and T with the same eigenvalue.

Definition (Bessel's functions)

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Green's function

Green's function

Definition (Green's function)

Is a function defined in $\overline{\Omega} \setminus \{x_0\}$ the following hold:

- $$\forall x_0 \in \Omega : \exists H_{x_0}(x) \in C\left(\overline{\Omega}\right) \text{ such that } \forall x \neq x_0 : G(x, x_0) = \Phi(x - x_0) + H_{x_0}(x).$$
- $$\forall x \in \partial\Omega : G(x, x_0) = 0.$$

Theorem (Symmetry of the Green's Function)

$$\forall x \neq x_0 \in \Omega : G(x, x_0) = G(x_0, x).$$

Remark

Fundamental solutions and Green's functions (with different boundary conditions) are actually common to all linear PDEs.

$$0.$$

$$\Phi\left(x,t\right) \coloneqq \frac{1}{\sqrt{4\pi\alpha t}} \exp\left(-\frac{x^2}{4\alpha t}\right).$$

Green's Function of the 1D Wave Equation

$$\Phi(x,t) \coloneqq \frac{1}{2c} H(ct - |x|) = \begin{cases} \dfrac{1}{2c} & |x| < ct, t > 0 \\ 0 & |x| \geq ct, t > 0 \end{cases}$$

where H is the Heaviside function.

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