Integral form of the Conservation Law

In general, a conservation law in one dimension takes the form

$$\partial_t u + \partial_x f\left(u\right) = 0.$$

Let $\Omega_x \subset \mathbb{R}$ be a domain and $u \in C\left(\Omega_x \times \Omega_t\right)$. Suppose that $\partial_t u \in C\left(\Omega_x \times \Omega_t\right)$ and exists $g\left(x\right)$ and $h\left(x\right)$ such that

$$\forall t \in \Omega_t : \left|u\left(x,t\right)\right| \leq g\left(x\right)$$

and

$$\left|\partial_t u\right| \leq h\left(x\right)$$

$$\frac{\partial u(x,t)}{\partial t} + \frac{\partial f(u(x,t))}{\partial x} = 0.$$

Integrate with respect to x

$$\int_a^b \left(\frac{\partial u(x,t)}{\partial t} + \frac{\partial f(u(x,t))}{\partial x} \right) \mathrm{d}x = \int_a^b 0 \, \mathrm{d}x.$$

By linearity of integral

$$\int_a^b \frac{\partial u(x,t)}{\partial t} \mathrm{d}x + \int_a^b \frac{\partial f(u(x,t))}{\partial x} \mathrm{d}x = 0.$$

Deriving under the integral sign

$$\frac{\partial}{\partial t} \int_a^b u(x,t) \, \mathrm{d}x + \int_a^b \frac{\partial f(u(x,t))}{\partial x} \mathrm{d}x = 0.$$

Applying the Fundamental theorem of calculus

$$\int_a^b u(x,t) \, \mathrm{d}x - \int_a^b u(x,0) \, \mathrm{d}x + \int_0^t f(a,\tau) \, \mathrm{d}\tau - \int_0^t f(b,\tau) \, \mathrm{d}\tau = 0.$$

$$\boxed{\int_a^b u(x,0) \, \mathrm{d}x + \int_0^t f(a,\tau) \, \mathrm{d}\tau - \int_0^t f(b,\tau) \, \mathrm{d}\tau = \int_a^b u(x,t) \, .}$$

Basic definitions

Basic definitions

Definition (Partial differential equation (PDE))

It is an equation that involves an unknown function u and its partial derivatives together with the independent variables. It is written as

$$\mathcal{L}\,(\text{independent variables}, u, \text{partial derivatives of } u) = 0. \tag{7}$$

Definition (Domain)

A domain Ω is an open and connected subset of $\mathbb{R}^d$ having a piecewise linear boundary of class C^1.

Remark

From now on Ω will always be a domain.

Definition (Classical PDE solution)

It is a sufficiently smooth function $u\colon \Omega \to \mathbb{R}$ that satisfies (7) for any $x \in \Omega$.

Definition (Auxiliary condition)

An *auxiliary condition* in a general solution is an equality that specifies the value of the unknown function on a subset of Ω.

Definition (Initial Value Problem (IVP))

Let $u\colon \Omega \times [0,T] \to \mathbb{R}$ be the solution to (7). An initial value problem is a differential equation together with a set of auxiliar conditions that specify the solution and/or its derivatives at $t = 0$.

Example (IVP for the Korteweg-de Vries equation)

$$
\begin{cases} \partial_t u + \partial_x^3 u + 6u\partial_x u = 0 & \text{for } (x,t) \in \Omega \times [0,T]. \\ u(x,0) = u_0(x) & \text{for } x \in \Omega. \end{cases} \tag{8}
$$

Definition (Boundary Value Problem (BVP))

Let $u\colon \Omega \to \mathbb{R}$ be a solution of (7). A boundary value problem is a differential equation together with a set of auxiliary conditions that specify the solution and/or its derivatives in $\partial\Omega$.

Example (Laplace's equation with Inhomogeneous Dirichlet Boundary Conditions)

$$
\begin{cases} \triangle u = 0 & \text{for } (x,y) \in \Omega. \\ u = f & \text{for } \partial\Omega. \end{cases}
$$

Example (Three widely used boundary conditions for a $u(x)$ on $[a,b]$)

- Dirichlet boundary conditions are $u(a) = u(b) = 0$.
- Neumann boundary conditions are $u'(a) = u'(b) = 0$.
- Periodic boundary conditions are $u(a) = u(b)$ and $u'(a) = u'(b)$.

Definition (Well-posed problem)

It is an EDP with auxiliary conditions that the conditions holds

- **Existence** for a given choice of auxiliary conditions, there exists a solution to the PDE that satisfies it.
- **Uniqueness** there is only one solution.
- **Stability** if we perturb slightly the auxiliary condition, then the new solution does not change much with respect to the original solution.

Theorem (Malgrange-Ehrenpreis (1950))

Every constant-coefficients linear PDE on $\mathbb{R}^d$ can be solved.

Example (Lewy operator (1957))

Not every linear PDE with polynomial coefficients has a solution.

$$\partial_x u + i\partial_y u - 2i\left(x + iy\right)\partial_t u = f\left(t\right).$$

Proof.

See https://people.maths.ox.ac.uk/trefethen/pdectb/lewy2.pdf. □

Example (Mizohata operator (1962))

$$\partial_x u + ix\partial_y u = g\left(x, y\right).$$

Definition (Bulk integrals)

Integrate a function $f(x)$ on some domain $\Omega \subset \mathbb{R}^d$.

$$\int \underset{\Omega}{\cdots} \int f(x)\,\mathrm{d}x$$

where $\mathrm{d}x = \mathrm{d}x_1 \ldots \mathrm{d}x_d$.

Definition (Flux integrals)

Let S be an orientable surface with its well-defined unit normal vector field n varying continuously across S.

$$\iint_S F \cdot n \,\mathrm{d}S.$$

For a closed surfaces S which are the boundary of a domain $\Omega \subset \mathbb{R}^3$, we will denote the surface by $\partial\Omega$.

$$\iint_{\partial\Omega} F \cdot n \,\mathrm{d}S.$$

Definition (Integrable function)

A function f defined on a domain $\Omega \subset \mathbb{R}^d$ is integrable on Ω iff

$$\int \underset{\Omega}{\cdots} \int |f(x)| \, \mathrm{d}x$$

exists and is a finite number.

Definition (Locally integrable function)

A function $f \colon \mathbb{R}^d \to \mathbb{R}$ is locally integrable iff for every $K \subset \mathbb{R}^d$ which is closed and bounded, we have

$$\int \underset{K}{\cdots} \int |f(x)| \, \mathrm{d}x < \infty.$$

Theorem (Divergence)

If F be a smooth vector field on a bounded domain with outer normal, then

$$\iiint\limits_{\Omega} \operatorname{div} F \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z = \iint\limits_{\partial\Omega} F \cdot n \, \mathrm{d}S.$$

Theorem (Green)

If $P(x,y)$ and $Q(x,y)$ be smooth functions on a bounded $2D$ domain Ω and $C = \partial\Omega$, then

$$\iint\limits_{\Omega} \left(\frac{\partial Q(x,y)}{\partial x} - \frac{\partial P(x,y)}{\partial y} \right) \mathrm{d}x\,\mathrm{d}y = \int\limits_{C} \left(P(x,y)\,\mathrm{d}x + Q(x,y)\,\mathrm{d}y \right).$$

Theorem (Componentwise Divergence)

If $f = f(x)$ is smooth function on a bounded domain $\Omega \subset \mathbb{R}^3$, then

$$\forall i = 1,2,3 : \iiint\limits_{\Omega} f_{x_i}(x)\,\mathrm{d}x = \iint\limits_{\partial\Omega} f n_i\,\mathrm{d}S$$

where n_i denotes the i-th component of the outer unit normal n.

Theorem (Integration by Parts)

If f and g are smooth functions of one variable x, then

$$\int\limits_a^b f(x)\,g'(x)\,\mathrm{d}x = \left[f(x)\,g(x)\right]\Big|_a^b - \int\limits_a^b f'(x)\,g(x)\,\mathrm{d}x.$$

Theorem (Vector Field Integration by Parts Formula)

$$\iiint\limits_{\Omega} u \cdot \nabla u \,\mathrm{d}x = -\iiint\limits_{\Omega} (\operatorname{div} u)\, v \,\mathrm{d}x + \iint\limits_{\partial\Omega} (vu) \cdot n \,\mathrm{d}S.$$

Here n denotes the outer unit normal to $\partial\Omega$.

Theorem (Green's First Identity)

$$\iiint\limits_{\Omega} v \triangle u \,\mathrm{d}x = \iint\limits_{\partial\Omega} u \frac{\partial u}{\partial n} \,\mathrm{d}S - \iiint\limits_{\Omega} \nabla u \cdot \nabla v \,\mathrm{d}x.$$

Theorem (Green's Second Identity)

$$\iiint\limits_{\Omega} \left(v \triangle u - u \triangle v\right) \,\mathrm{d}x = \iint\limits_{\partial\Omega} \left(v \frac{\partial u}{\partial n} - u \frac{\partial v}{\partial n}\right) \mathrm{d}S.$$

Theorem (One-Dimensional Integral to Pointwise Theorem I)

$$f \in C(\mathbb{R}) \text{ and } \forall I \subsetneq \mathbb{R} \text{ bounded interval}: \int\limits_{I} f(x)\, \mathrm{d}x = 0 \implies f \equiv 0.$$

Theorem (One-Dimensional Integral to Pointwise Theorem II)

$$f \in C(\mathbb{R}) \text{ and } \forall g \in C_c(\mathbb{R}) \text{ continuous function with compact support}: \int\limits_{\mathbb{R}} f(x)\, g(x)\, \mathrm{d}x = 0 \implies f \equiv 0.$$

Theorem (General Integral to Pointwise Theorem I)

$$f \in C(\Omega) \text{ and } \forall W \subset \Omega \text{ bounded subdomain}: \int \underset{W}{\cdots} \int f(x)\, \mathrm{d}x = 0 \implies f \equiv 0.$$

Theorem (General Integral to Pointwise Theorem II)

$$f \in C(\Omega) \text{ and } \forall g \in C_c(\Omega) \text{ continuous function with compact support}: \int \underset{\Omega}{\cdots} \int f(x)\, g(x)\, \mathrm{d}x = 0 \implies f \equiv 0.$$

Integration

Theorem (Lebesgue Dominated Convergence)

Let $\{f_n\}_{n\in\mathbb{N}}$ *be a sequence of integrable functions on* $I \subset \mathbb{R}$ *and let* f *be an integrable function on* I *such that* $\forall x \in I : \lim\limits_{n\to\infty} f_n(x) = f(x)$. *Suppose that there exists an integrable function* $g(x) \geq 0$ *on* I *such that*

$$\forall n \in \mathbb{N} : \forall x \in I : |f_n(x)| \leq g(x).$$

Then

$$\lim_{n\to\infty} \int\limits_I f_n(x)\, \mathrm{d}x = \int\limits_I f(x)\, \mathrm{d}x.$$

Theorem (Monotone Convergence)

Let $\{f_n\}_{n\in\mathbb{N}}$ *be a sequence of nonnegative integrable functions on* $I \subset \mathbb{R}$ *and let* f *be an integrable function on* I *such that* $\forall x \in I : \lim\limits_{n\to\infty} f_n(x) = f(x)$. *Suppose that* $\forall n \in \mathbb{N} : \forall x \in I : 0 \leq f_n(x) \leq f_{n+1}(x)$. *Then*

$$\lim_{n\to\infty} \int\limits_I f_n(x)\, \mathrm{d}x = \int\limits_I f(x)\, \mathrm{d}x.$$

Theorem (Differentiation under the Integral Sign)

Let $f \in C(\Omega \times (a,b))$ *for some* $-\infty \leq a < b \leq \infty$. *Suppose the following:*

- $\partial_t f(x,t) \in C(\Omega \times (a,b))$.
- *There exist integrable functions* $g(x)$ *and* $h(x)$ *on* Ω *such that*

$$|f(x,t)| \leq g(x) \text{ and } \forall t \in (a,b) : \left|\partial_t f(x,t)\right| \leq h(x).$$

Then the function

$$t \longmapsto \int \underset{\Omega}{\cdots} \int f(x,t)\, \mathrm{d}x$$

is differentiable (continuous) on $t \in (a,b)$ *and*

$$\partial_t \left(\int \underset{\Omega}{\cdots} \int f(x,t)\, \mathrm{d}x \right) = \int \underset{\Omega}{\cdots} \int \partial_t f(x,t)\, \mathrm{d}x.$$

Proof.

. $\square$

Theorem (Leibnitz Rule)

If $f(t,s)$ be a smooth function in both variables, then

$$\partial_t \left(\int_0^t f(t,s)\,\mathrm{d}s \right) = f(t,t) + \int_0^t \partial_t f(t,s)\,\mathrm{d}s.$$

Proof.

By the change of variable $\tau = \frac{s}{t}$, we have $\int_0^t f(t,s)\,\mathrm{d}s = \int_0^1 t f(t,t\tau)\,\mathrm{d}\tau$. Hence by differentiation under the integral sign

$$\partial_t \left(\int_0^1 (t f(t,t\tau))\,\mathrm{d}\tau \right) = \int_0^1 \partial_t (t f(t,t\tau))\,\mathrm{d}\tau = \int_0^1 \Big(f(t,t\tau) + t\partial_t f(t,t\tau) + t\partial_s f(t,t\tau)\,\tau \Big)\,\mathrm{d}\tau.$$

Changing variables back to s, we find $\int_0^1 t\partial_t f(t,t\tau)\,\mathrm{d}\tau = \int_0^t \partial_t f(t,s)\,\mathrm{d}s$. Finally, by the Fundamental Theorem of Calculus,

$$\int_0^1 f(t,t\tau) + t f_s(t,t\tau)\,\tau\,\mathrm{d}\tau = \int_0^1 \frac{d}{d\tau}(\tau f(t,t\tau))\,\mathrm{d}\tau = f(t,t)\,.$$

□

Theorem (Fubini-Tonelli)

Let $f(x, y)$ be a (possibly) complex-valued function on $\mathbb{R}^2$. The

$$\int_{\mathbb{R}} \left(\int_{\mathbb{R}} |f(x, y)| \, \mathrm{d}x \right) \mathrm{d}y = \int_{\mathbb{R}} \left(\int_{\mathbb{R}} |f(x, y)| \, \mathrm{d}y \right) \mathrm{d}x.$$

Furthermore, if either of the above iterated integrals in (A.22) is finite (yielding that f is)

$$\int_{\mathbb{R}} \left(\int_{\mathbb{R}} f(x, y) \, \mathrm{d}x \right) \mathrm{d}y = \int_{\mathbb{R}} \left(\int_{\mathbb{R}} f(x, y) \, \mathrm{d}y \right) \mathrm{d}x.$$

Definition (Order)

We define the order of a PDE to be the order of the highest derivative which appears in the equation.

Let's write the PDE in the following form:

All terms containing u and its derivatives $=$ All terms involving only the independent variables.

Thus, any PDE for $u(x)$ can be written as

$$\mathcal{L}(u) = f(x),$$

for some operator $\mathcal{L}$ and some function f.

Definition (Linear)

We say the PDE is linear if $\mathcal{L}$ is linear in u. That is,

$$\forall \alpha \in \mathbb{R} : \forall u_1, u_2 : \mathcal{L}(\alpha u_1 + u_2) = \alpha \mathcal{L}(u_1) + \mathcal{L}(u_2).$$

Otherwise, we say the PDE is nonlinear.

Example (Advection equation, $\mathcal{L}(u) = \partial_t u + c\partial_x u$)

$$\begin{aligned}
\mathcal{L}(\alpha u_1 + u_2) &= \partial_t(\alpha u_1 + u_2) + c\partial_x(\alpha u_1 + u_2). \\
&= \alpha\partial_t u_1 + \alpha c\partial_x u_1 + \partial_t u_2 + c\partial_x u_2. \\
&= \alpha\mathcal{L}(u_1) + \mathcal{L}(u_2).
\end{aligned}$$

Definition (Semilinear, Quasilinear and Fully Nonlinear PDE)

A PDE of order k is called:

semilinear iff all ocurrences of derivatives of order k appear with a coefficient which only depends on the independent variables,

quasilinear iff all ocurrences of derivatives of order k appear with a coefficient which only depends on the independent variables, u, and its derivatives of order strictly less than k,

fully nonlinear iff it is not quasilinear.

Example

Linear

$$(xy)\,\partial_x u + e^y \partial_y u + (\sin x)\,u = x^3 y^4.$$

Semilinear

$$(xy)\,\partial_x u + e^y \partial_y u + (\sin x)\,u = u^2.$$

Quasilinear

$$u\partial_x u + \partial_y u = 0.$$

Fully Nonlinear

$$\left(\partial_x u\right)^2 + \left(\partial_y u\right)^2 = 1.$$

Remark

By definition, we have the strict inclusions

$$\{\text{linear PDEs}\} \subsetneq \{\text{semilinear PDEs}\} \subsetneq \{\text{quasilinear PDEs}\}.$$

Explicit form for first-order PDEs in two independent variables x and y

Linear means the PDE can be written in the form

$$\boxed{a\,(x,y)\,\partial_x u\,(x,y)+b\,(x,y)\,\partial_y u\,(x,y)=c_1\,(x,y)\,u+c_2\,(x,y)}$$

for some given real-valued functions a, b, c_1 and c_2 of x and y.

Semilinear means the PDE can be written in the form

$$\boxed{a\,(x,y)\,\partial_x u\,(x,y)+b\,(x,y)\,\partial_y u\,(x,y)=c\,(x,y,u)}$$

for some given real-valued functions a and b of x and y, and a real-valued function c of x, y, and u.

Quasilinear means that the PDE can be written in the form

$$\boxed{a\,(x,y,u)\,\partial_x u\,(x,y)+b\,(x,y,u)\,\partial_y u\,(x,y)=c\,(x,y,u)}$$

for some given real-valued functions a, b, and c of x, y, and u.

Remark

Note that in all cases, the coefficient real-valued functions a, b, and c need not be linear in their arguments.

Theorem (Cauchy-Kowalewska)

Let the functions f_i of the system

$$\forall i = 1, \ldots, n : \frac{\partial u_i}{\partial t} = f_i \left(x, t, u, \partial_x u \right),$$

where $u(x,t) = (u_1(x,t), \ldots, u_n(x,t))$ with an initial condition

$$u_0(x,0) = (\phi_1(x), \ldots, \phi_n(x))$$

be analytical in some neighborhood of the point

$$t = 0, x = 0, u = 0, \partial_x u = 0.$$

Furthermore, let the initial data (6.1.2) be analytical at x = 0. It follows that the Cauchy problem (6.1.1), (6.1.2) admits a unique analytical solution in some neighborhood of the point $x = t = 0$.

Classification of Linear Second Order Partial Differential Equations

Definition (Classification of Linear Second Order PDE)

Let the second order linear EDP with constant coefficients $\mathcal{L}u(x) = f(x)$ and $x \in \Omega$, where the differential operador $\mathcal{L}$ is given by

$$\begin{aligned} \mathcal{L}u(x) &:= \sum_{j=1}^{d}\sum_{i=1}^{d} a_{ij}\partial_i\partial_j u(x) + \sum_{i=1}^{d} b_i\partial_i u(x) + cu(x). \\ &= \left\langle A, \nabla^2 u(x)\right\rangle_F + \langle b, \nabla u(x)\rangle + cu(x), \end{aligned}$$

where $A = [a_{ij}] \in \mathbb{R}^{d\times d}$, $b = [b_i]^T \in \mathbb{R}^d$, $\nabla^2 u(x)$ is the Hessian matrix of u at x, $\nabla u(x)$ is the gradient of u at the same point and the Frobenius inner product is definided as

$$\forall A, B \in \mathbb{R}^{d\times d} : \langle A, B\rangle_F := \operatorname{tr}\left(A^T B\right).$$

We say that the second-order partial differential operator with constant coefficients $\mathcal{D}$ is

- **Elliptic** iff A has d eigenvalues with the same sign, that is, either $\sigma(A) \subset \mathbb{R}_+$ or $\sigma(A) \subset \mathbb{R}_-$.
- **Parabolic** iff A has exactly $d-1$ eigenvalues, whether positive or negative, and zero is an eigenvalue of multiplicity one.
- **Hyperbolic** iff A has $d-1$ positive or negative eigenvalues, and the remaining one is non-zero and of opposite sign.
- **Ultra-parabolic** iff zero is a multiple eigenvalue and all the remaining eigenvalues have the same sign.
- **Ultra-hyperbolic** iff zero is not an eigenvalue and there is more than one positive eigenvalue and more than one negative eigenvalue.

Assume that A is a symmetric matrix. By performing a linear coordinate transformation $\xi = F(x)$. Note that

$$\frac{\partial \xi_i}{\partial x_j} = F_{ij}.$$

$$\frac{\partial u}{\partial x_i} = \sum_{j=1}^{d} \frac{\partial u}{\partial \xi_j}\frac{\partial \xi_j}{\partial x_i} = \sum_{j=1}^{d} \frac{\partial u}{\partial \xi_j} F_{ji}.$$

$$\frac{\partial^2 u}{\partial x_i \partial x_j} = \sum_{l=1}^{d}\sum_{k=1}^{d} \frac{\partial \xi_k}{\partial x_i}\frac{\partial^2 u}{\partial \xi_k \partial \xi_l}\frac{\partial \xi_l}{\partial x_j} = \sum_{l=1}^{d}\sum_{k=1}^{d} F_{ki}\frac{\partial^2 u}{\partial \xi_k \partial \xi_l} F_{lj}.$$

After the coordinate transformation, the differential equation takes the form

$$0 = \sum_{j=1}^{d}\sum_{i=1}^{d} a_{ij}\left[\sum_{l=1}^{d}\sum_{k=1}^{d} F_{ki}\frac{\partial^2 u}{\partial \xi_k \partial \xi_l} F_{lj}\right] + \sum_{i=1}^{d} b_i \left[\sum_{j=1}^{d} \frac{\partial u}{\partial \xi_j} F_{ji}\right] + cu.$$

$$0 = \sum_{l=1}^{d}\sum_{k=1}^{d}\left[\sum_{j=1}^{d}\sum_{i=1}^{d} F_{ki} a_{ij} F_{lj}\right]\frac{\partial^2 u}{\partial \xi_k \partial \xi_l} + \sum_{j=1}^{d}\left[\sum_{i=1}^{d} F_{ji} b_i\right]\frac{\partial u}{\partial \xi_j} + cu.$$

We would like to choose the matrix F so that $D = FAF^T$ is diagonal. Recall that we can diagonalize a symmetric matrix by means of an orthogonal change of variables. In other words, we can choose F to be an orthogonal matrix.

Remark

- If D has nonzero diagonal entries all of the same sign, the differential equation is elliptic.
- If D has nonzero diagonal entries with one entry of different sign from the others, then the differential equation is hyperbolic.
- If D has one zero diagonal entry, the equation may be parabolic.

Example (Khokhlov-Zabolotskaya-Kuznetsov (KZK) equation)

$$\partial_{x_1}^2 u + \partial_{x_2}^2 u - \partial_{x_3}\partial_{x_4} u = 0.$$

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2} \\ 0 & 0 & -\frac{1}{2} & 0 \end{bmatrix}, \quad F = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & -\frac{1}{2} \end{bmatrix}.$$

The KZK equation is hyperbolic.

Example (Linear Second Order PDE)

Let $d \in \mathbb{N}$ be the spatial dimension.

Elliptic

$$-\triangle u(x) = \left\langle -I_d, \nabla^2 u(x) \right\rangle_F = -\sum_{i=1}^{d} \partial_i^2 u(x).$$

Parabolic

$$H(x,t) = \partial_t u(x,t) + \left\langle \begin{bmatrix} -I_d & \mathbf{0} \\ \mathbf{0}^T & 0 \end{bmatrix}, \nabla^2 u(x,t) \right\rangle_F = \partial_t u(x,t) - \triangle u(x,t).$$

Hyperbolic

$$\Box u(x,t) = \left\langle \begin{bmatrix} -I_d & \mathbf{0} \\ \mathbf{0}^T & 1 \end{bmatrix}, \nabla^2 u(x,t) \right\rangle_F = \partial_t^2 u(x,t) - \triangle u(x,t).$$

Example (Euler-Tricomi)

Let us consider an equation that changes type.

$$\partial_x^2 u + x \partial_y^2 u = 0. \tag{9}$$

This equation is elliptic in the right half-plane $\left\{(x,y) \in \mathbb{R}^2 \mid x > 0\right\}$, is parabolic in the plane $\left\{(x,y) \in \mathbb{R}^2 \mid x = 0\right\}$ and is hyperbolic in the left half-plane $\left\{(x,y) \in \mathbb{R}^2 \mid x < 0\right\}$.

Remark

(9) is useful in the study of transonic flow (flight at or near the speed of sound).

Classification of Linear Second Order Partial Differential Equations

Theorem (Canonical form of a linear second order PDE)

For any second order linear PDE, there exists a linear change of variables $\xi(x,y)$, $\eta(x,y)$ such that in the new coordinates (ξ,η), the PDE is transformed as follows

- *If PDE is elliptic, then*

$$\partial_\xi^2 u + \partial_\eta^2 u + F\left(\partial_\xi u, \partial_\eta u, u\right) = 0.$$

- *If PDE is parabolic, then*

$$\partial_\eta^2 u + F\left(\partial_\xi u, \partial_\eta u, u\right) = 0.$$

- *If PDE is hyperbolic, then*

$$\partial_\xi \partial_\eta u + F\left(\partial_\xi u, \partial_\eta u, u\right) = 0.$$

In each case, F is some linear function of three variables.

Proof.

Use the Sylvester's law of inertia. □

Definition

A system of quasi-linear partial differential equations will be called of

hyperbolic iff its homogeneous part admits wave-like solutions. This implies that a hyperbolic set of equations will be associated to propagating waves and that the behavior and properties of the physical system described by these equations will be dominated by wave-like phenomena.
In other words, a hyperbolic system describes convection phenomena and inversely, convection phenomena are described by hyperbolic equations.

parabolic iff the equations admit solutions corresponding to damped waves.

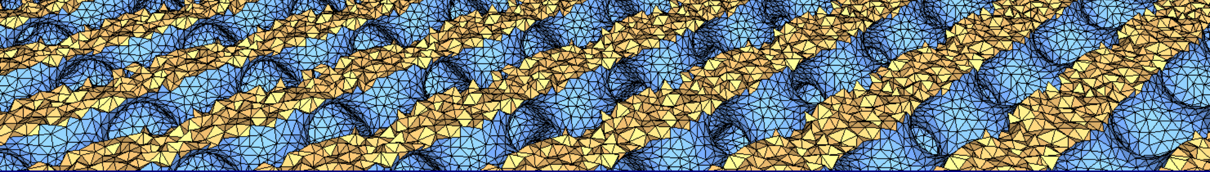
elliptic if it does not admit wave-like solutions.

Example

$$
\begin{aligned}
a\partial_x u + c\partial_y v &= f_1.\\
b\partial_x v + c\partial_y u &= f_2.
\end{aligned}
$$

Example (Stationary Euler equations)

$$
\begin{bmatrix} u & \rho & 0 \\ \frac{c^2}{\rho} & u & 0 \\ 0 & 0 & u \end{bmatrix} \partial_x \begin{bmatrix} \rho \\ u \\ v \end{bmatrix} + \begin{bmatrix} v & 0 & \rho \\ 0 & v & 0 \\ \frac{c^2}{\rho} & 0 & v \end{bmatrix} \partial_y \begin{bmatrix} \rho \\ u \\ v \end{bmatrix} = 0.
$$



Method of characteristics

Consider the problem for the explicit form of linear first-oder PDEs in two independent variables

$$\begin{cases} a\left(x,y\right)\partial_x u + b\left(x,y\right)\partial_y u = c_1\left(x,y\right)u + c_2\left(x,y\right), \\ u\left(x,y\right) \text{ given for } \left(x,y\right) \in \Gamma. \end{cases}$$

to be solved in some domain $\Omega \subset \mathbb{R}^2$ with data given on some curve $\Gamma \subset \overline{\Omega}$.

Remark

Often the $\Gamma \subset \partial\Omega \subset \mathbb{R}^2$ it will just be one of the coordinate axes.

We find the characteristics, i.e., the curves which follow these directions, by solving

$$\frac{\mathrm{d}x}{\mathrm{d}s} = a\left(x\left(s\right), y\left(s\right)\right), \qquad \frac{\mathrm{d}y}{\mathrm{d}s} = b\left(x\left(s\right), y\left(s\right)\right).$$

Now suppose u is a solution to the PDE. Let $z\left(s\right)$ denote the values of the solution u along a characteristic; i.e.,

$$z\left(s\right) \coloneqq u\left(x\left(s\right), y\left(s\right)\right).$$

Then by the chain rule, we have

$$\begin{aligned}
\frac{\mathrm{d}z}{\mathrm{d}s} &= \partial_x u\left(x\left(s\right), y\left(s\right)\right)\frac{\mathrm{d}x}{\mathrm{d}s}\left(x\left(s\right), y\left(s\right)\right) + \partial_y u\left(x\left(s\right), y\left(s\right)\right)\frac{\mathrm{d}y}{\mathrm{d}s}\left(x\left(s\right), y\left(s\right)\right). \\
\frac{\mathrm{d}z}{\mathrm{d}s} &= \partial_x u\left(x\left(s\right), y\left(s\right)\right)a\left(x\left(s\right), y\left(s\right)\right) + \partial_y u\left(x\left(s\right), y\left(s\right)\right)b\left(x\left(s\right), y\left(s\right)\right). \\
\frac{\mathrm{d}z}{\mathrm{d}s} &= c_1\left(x\left(s\right), y\left(s\right)\right)z\left(s\right) + c_2\left(x\left(s\right), y\left(s\right)\right).
\end{aligned}$$

Definition (Characteristics equations)

There are three dependent variables x, y and z and one independent variable s.

$$\begin{cases} \frac{\mathrm{d}x}{\mathrm{d}s}(s) &= a\left(x(s), y(s)\right). \\ \frac{\mathrm{d}y}{\mathrm{d}s}(s) &= b\left(x(s), y(s)\right). \\ \frac{\mathrm{d}z}{\mathrm{d}s}(s) &= c_1\left(x(s), y(s)\right) z(s) + c_2\left(x(s), y(s)\right). \end{cases}$$

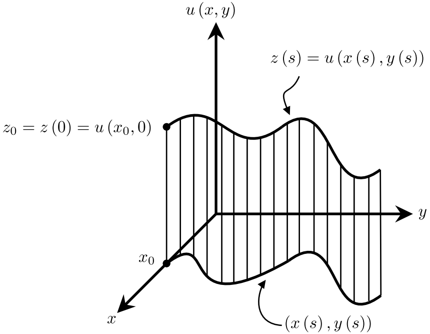


Figure: The solution u is described by the surface defined by $z = u(x, y)$. From any point x_0 on the x-axis, there is a curve $(x(s), y(s))$ in the xy-plane, upon which wa can calculate the solution $z = u(x(s), y(s))$. Knowing only the structure of the PDE, x_0 and z_0 we can solve ODEs to find the part of the solution surface which lies above the curve.

Example (Advection)

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Linear advection equation

$$\partial_t u + c\partial_x u = 0$$

with an initial condition $u(x,0) = u_0(x)$ and $c \in \mathbb{R} \setminus \{0\}$.

We can see that $u(x,t) = u_0(x - ct)$ satisfies the PDE. Let $z(x,t) = x - ct$, then from the chain rule we have

$$\begin{aligned}
\partial_t u_0(x - ct) + c\partial_t u_0(x - ct) &= \partial_t u_0(z(x,t)) + c\partial_x u_0(z(x,t)).\\
&= u'(z)\,\partial_t z + cu_0'(z).\\
&= -cu_0'(z) + cu_0'(z).\\
&= 0.
\end{aligned}$$

This tell us that the solution transports (or advects) the initial condition with speed c.
The characteristics are paths in the xt-plane, denoted by $(X(t), t)$ on which the solution is constant.

For $\partial_t u + c\partial_x u = 0$ we have $X(t) = X_0 + ct$, since

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} u(X(t), t) &= \partial_t u(X(t), t) + \partial_x u(X(t), t) \frac{\mathrm{d}X(t)}{\mathrm{d}t}. \\
&= \partial_t u(X(t), t) + \partial_x u(X(t), t) c. \\
&= 0.
\end{aligned}$$

Hence $u(X(t), t) = u(X(0), 0) = u_0(X_0)$, i.e. the initial condition is transported along characteristics.
Characteristics have important implications for the direction of flow of information, and for boundary conditions.
More generally, if we have a non-zero right hand side in the PDE, then the situation is a bit more complicated on each characteristic.
Consider $\partial_t u + c\partial_x u = f(x, t, u(x, t))$, and $X(t) = X_0 + ct$.

$$
\begin{aligned}
\frac{\mathrm{d}u\left(X\left(t\right),t\right)}{\mathrm{d}t} &= \partial_t u\left(X\left(t\right),t\right) + \partial_x u\left(X\left(t\right),t\right)\frac{\mathrm{d}X\left(t\right)}{\mathrm{d}t} \\
&= \partial_t u\left(X\left(t\right),t\right) + \partial_x u\left(X\left(t\right),t\right)\frac{\mathrm{d}X\left(t\right)}{\mathrm{d}t} \\
&= f\left(X\left(t\right),t,u\left(X\left(t\right),t\right)\right).
\end{aligned}
$$

In this case, the solution is no longer constant on $\left(X\left(t\right),t\right)$, but we have reduced a PDE to a set of ODEs, so that

$$
u\left(X\left(t\right),t\right) = u_0\left(X_0\right) + \int\limits_0^t f\left(X\left(t\right)t, u\left(X\left(t\right),t\right)\right)\,\mathrm{d}t.
$$

The domain of dependence of the exact solution $u\left(t_{n+1}, x_j\right)$ is determined by the characteristics curve passing through $\left(t_{n+1}, x_j\right)$.

Consider the problem for the explicit form of semilinear first-oder PDEs in two independent variables

$$\begin{cases} a\left(x,y\right)\partial_x u + b\left(x,y\right)\partial_y u = c\left(x,y,u\right), \\ u\left(x,y\right) \text{ given for } \left(x,y\right) \in \Gamma. \end{cases}$$

Consider the problem for the explicit form of quasilinear first-oder PDEs in two independent variables

$$\begin{cases} \dot{x}(s) &= a(x(s), y(s), z(s)), \\ \dot{y}(s) &= b(x(s), y(s), z(s)), \\ \dot{z}(s) &= c(x(s), y(s), z(s)). \end{cases}$$

Example

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$$\begin{cases} \dot{x}_i(s) = \partial_{p_i} F(p(s), z(s), x(s)), \\ \dot{z}(s) = \sum_{j=1}^{N} p_j(s) \partial_{p_j} F(p(s), z(s), x(s)), \\ \dot{p}_i(s) = -\partial_{x_i} F(p(s), z(s), x(s)) - \partial_z F(p(s), z(s), x(s)) p_i(s) \end{cases}$$

$$\begin{cases} \dot{x}(s) = \nabla_p F(p(s), z(s), x(s)), \\ \dot{z}(s) = \nabla_p F(p(s), z(s), x(s)) \cdot p(s), \\ \dot{p}(s) = -\nabla_x F(p(s), z(s), x(s)) - \dfrac{\partial}{\partial} \end{cases}$$

Example

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Thus we consider the problem

$$\begin{cases} \frac{\mathrm{d}^2\psi(t)}{\mathrm{d}t^2} - \lambda\psi\left(t\right) & = 0 \\ \frac{\mathrm{d}^2\varphi(x)}{\mathrm{d}x^2} - \lambda\varphi\left(x\right) & = 0. \end{cases}$$

Trigonometric Fourier Series

The subject was founded by Jean-Baptiste Joseph Fourier, who discovered what we would recognize as the basics of Fourier Analysis in his studies of heat flow in the 1820s.

Jean-Baptiste Joseph Fourier (1768–1830).

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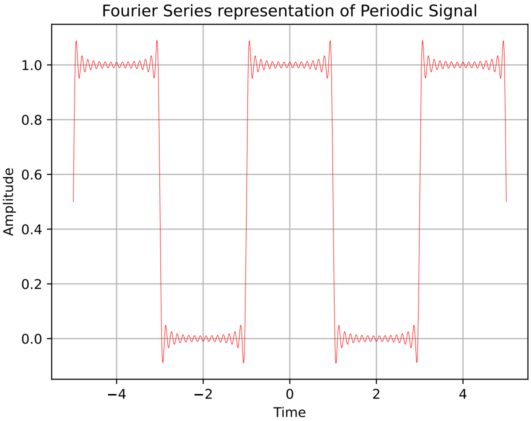
Fourier Series representation of Periodic Signal
Amplitude
Time
1.0
0.8
0.6
0.4
0.2
0.0
−4
−2
0
2
4

Definition (Fourier series of f relative)

Let $f \in L^2(I)$ and $\{\varphi_k\}_{k\in\mathbb{N}}$ an orthonormal sequence on $I \subset \mathbb{R}$. The Fourier series of f relative of $\{\varphi_k\}_{k\in\mathbb{N}}$ is $\sum\limits_{k\in\mathbb{N}} c_k\varphi_k(\theta)$, where $\forall k \in \mathbb{N} : c_k := \langle f, \varphi_k\rangle = \int\limits_I f(\theta)\overline{\varphi_k(\theta)}\,\mathrm{d}\theta$ are the Fourier coefficients of f relative of $\{\varphi_k\}_{k\in\mathbb{N}}$.

Example

If $I = [0, 2\pi]$ and two orthonormal sequences of trigonometric functions $\{\varphi_k\}_{k\in\mathbb{N}}, \{\phi_k\}_{k\in\mathbb{Z}}$:

$$\text{real}\quad \varphi_0(\theta) = \frac{1}{\sqrt{2\pi}},\quad \varphi_{2k-1}(\theta) = \frac{\cos(k\theta)}{\sqrt{\pi}},\quad \varphi_{2k}(\theta) = \frac{\sin(k\theta)}{\sqrt{\pi}}.$$

$$\text{complex}\quad \phi_k(\theta) = \frac{e^{ik\theta}}{\sqrt{2\pi}} = \frac{\cos(k\theta) + i\sin(k\theta)}{\sqrt{2\pi}}.$$

Then, the Fourier series of f relative of $\{\varphi_k\}_{k\in\mathbb{N}}$ and $\{\phi_k\}_{k\in\mathbb{N}}$ are

$$\text{real}\quad \frac{a_0}{2} + \sum_{k\in\mathbb{N}} a_k\cos(k\theta) + b_k\sin(k\theta). \qquad \text{complex}\quad \sum_{k\in\mathbb{Z}} \alpha_k e^{ik\theta},\quad \alpha_k = \frac{1}{2\pi}\int\limits_0^{2\pi} f(\theta)\,e^{-ik\theta}\,\mathrm{d}\theta.$$

Remark

The set of functions $\left\{ \frac{1}{\sqrt{2\pi}}, \frac{\cos(m\theta)}{\sqrt{\pi}}, \frac{\sin(n\theta)}{\sqrt{\pi}} \right\}_{m,n\in\mathbb{N}} \subset L^2([0,2\pi])$ is orthonormal.

Indeed, $\forall n, m \in \mathbb{N}$:

- $\displaystyle\int_0^{2\pi} \left(\frac{1}{\sqrt{2\pi}}\right)^2 \mathrm{d}\theta = \int_0^{2\pi} \frac{1}{2\pi} \mathrm{d}\theta = \frac{1}{2\pi}\theta\Big|_0^{2\pi} = 1.$
- $\displaystyle\int_0^{2\pi} \left(\frac{\cos(m\theta)}{\sqrt{\pi}}\right)^2 \mathrm{d}\theta = \int_0^{2\pi} \frac{\cos^2(m\theta)}{\pi} \mathrm{d}\theta = \frac{1}{2\pi}\int_0^{2\pi} 1 + \cos(2m\theta)\ \mathrm{d}\theta = \frac{1}{2\pi}\left(\theta + \frac{\sin(4m\theta)}{4m}\right)\Big|_0^{2\pi} = 1.$
- $\displaystyle\int_0^{2\pi} \left(\frac{\sin(n\theta)}{\sqrt{\pi}}\right)^2 \mathrm{d}\theta = \int_0^{2\pi} \frac{\sin^2(n\theta)}{\pi} \mathrm{d}\theta = \frac{1}{2\pi}\int_0^{2\pi} 1 - \cos(2m\theta)\ \mathrm{d}\theta = \frac{1}{2\pi}\left(\theta - \frac{\sin(4m\theta)}{4m}\right)\Big|_0^{2\pi} = 1.$
- $\displaystyle\int_0^{2\pi} \frac{1}{\sqrt{2\pi}} \frac{\cos(m\theta)}{\sqrt{\pi}} \mathrm{d}\theta = \frac{1}{\sqrt{2\pi}}\int_0^{2\pi} \cos(m\theta)\ \mathrm{d}\theta = 0.$
- $\displaystyle\int_0^{2\pi} \frac{1}{\sqrt{2\pi}} \frac{\sin(n\theta)}{\sqrt{\pi}} \mathrm{d}\theta = \frac{1}{\sqrt{2\pi}}\int_0^{2\pi} \sin(n\theta)\ \mathrm{d}\theta = 0.$
- $\displaystyle\int_0^{2\pi} \frac{\cos(m\theta)}{\sqrt{\pi}} \frac{\sin(n\theta)}{\sqrt{\pi}} \mathrm{d}\theta = \frac{1}{\pi}\int_0^{2\pi} \sin(n\theta)\cos(m\theta)\ \mathrm{d}\theta = \frac{1}{\pi}\int_0^{2\pi} \frac{\sin((n+m)\theta) - \sin((n-m)\theta)}{2} \mathrm{d}\theta = 0.$

Definition (Fourier series generated by f)

Let $f \in L^2([0, 2\pi])$. The Fourier coefficients of f are given by

$$a_0 = \frac{1}{\pi} \int_0^{2\pi} f(\theta)\, \mathrm{d}\theta, \quad a_k = \frac{1}{\pi} \int_0^{2\pi} f(\theta) \cos(k\theta)\, \mathrm{d}\theta, \quad b_k = \frac{1}{\pi} \int_0^{2\pi} f(\theta) \sin(k\theta)\, \mathrm{d}\theta.$$

and the n-th partial Fourier sum is

$$\boxed{s_n f(\theta) = \frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos(k\theta) + b_k \sin(k\theta).}$$

Indeed, from the equalities $\forall k \in \mathbb{N}$:

- $\displaystyle\int_0^{2\pi} \frac{a_0}{2}\, \mathrm{d}\theta = \frac{a_0}{2}\theta \Big|_0^{2\pi} = \pi a_0.$
- $\displaystyle\int_0^{2\pi} \cos(k\theta)\, \mathrm{d}\theta = \frac{\sin(k\theta)}{k}\Big|_0^{2\pi} = 0.$
- $\displaystyle\int_0^{2\pi} \sin(k\theta)\, \mathrm{d}\theta = \frac{-\cos(k\theta)}{k}\Big|_0^{2\pi} = 0.$

If we integrate the Fourier series term by term

$$\int_0^{2\pi} f(\theta)\,\mathrm{d}\theta = \int_0^{2\pi} \frac{a_0}{2}\,\mathrm{d}\theta + \int_0^{2\pi} \left(\sum_{k=1}^{\infty} a_k \cos(k\theta) + b_k \sin(k\theta) \right) \mathrm{d}\theta.$$

Then,

$$\int_0^{2\pi} f(\theta)\,\mathrm{d}\theta = \frac{a_0}{2} \int_0^{2\pi} \mathrm{d}\theta + \sum_{k=1}^{\infty} \left(a_k \int_0^{2\pi} \cos(k\theta)\,\mathrm{d}\theta + b_k \int_0^{2\pi} \sin(k\theta)\,\mathrm{d}\theta \right).$$

$$\int_0^{2\pi} f(\theta)\,\mathrm{d}\theta = \pi a_0 + \sum_{k=1}^{\infty} (a_k \cdot 0 + b_k \cdot 0). \quad \Longrightarrow \quad \boxed{a_0 = \frac{1}{\pi} \int_0^{2\pi} f(\theta)\,\mathrm{d}\theta.}$$

Multiplying the Fourier series by $\cos(m\theta)$, $m \in \mathbb{N}$ and integrating term by term:

$$\int_0^{2\pi} \cos(m\theta) f(\theta)\, d\theta = \int_0^{2\pi} \cos(m\theta) \frac{a_0}{2}\, d\theta + \int_0^{2\pi} \cos(m\theta) \left(\sum_{k=1}^{\infty} a_k \cos(k\theta) + b_k \sin(k\theta) \right) d\theta.$$

$$\int_0^{2\pi} f(\theta) \cos(m\theta)\, d\theta = 0 + \sum_{k=1}^{\infty} \left(a_k \int_0^{2\pi} \cos(k\theta)\cos(m\theta)\, d\theta + b_k \int_0^{2\pi} \sin(k\theta)\cos(m\theta)\, d\theta \right).$$

$$\int_0^{2\pi} f(\theta) \cos(m\theta)\, d\theta = \sum_{k=1}^{\infty} \left(\frac{a_k}{2} \int_0^{2\pi} \cos((m+k)\theta) + \cos((m-k)\theta)\, d\theta + \frac{b_k}{2} \int_0^{2\pi} \sin((m+k)\theta) + \sin((m-k)\theta)\, d\theta \right).$$

When $m \neq k$ both integrals vanish, thus the infinite sum reduces to m-th addend.

$$\int_0^{2\pi} f(\theta) \cos(m\theta)\, d\theta = a_m \int_0^{2\pi} \cos^2(m\theta)\, d\theta + b_m \int_0^{2\pi} \sin(m\theta)\cos(m\theta)\, d\theta.$$

$$\int_0^{2\pi} f(\theta) \cos(m\theta)\, d\theta = \frac{a_m}{2} \int_0^{2\pi} 1 + \cos(2m\theta)\, d\theta + b_m \cdot 0.$$

$$\int_0^{2\pi} f(\theta) \cos(m\theta)\, d\theta = a_m \pi. \qquad \implies \boxed{a_m = \frac{1}{\pi} \int_0^{2\pi} f(\theta) \cos(m\theta)\, d\theta.}$$