Note on Quantum Simulation of High Energy Physics

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Abstract

This note is about the quantum simulation of high energy physics.

Throughout this note we take $c = \hbar = k_B = 1$. Einstein summation convention is taken unless otherwise specified. The Lorentz metric for flat spacetime is assumed as $\eta_{\mu\nu} = \mathrm{diag}(-1, +1, +1, +1)$.

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Contents

1 Lattice gauge theory

This section is mainly based on the thesis [1].

1.1 Yang-Mills theory in the continuum

The well known Lagrangian density of a free Dirac fermion is given by

$$\mathcal{L} = \overline{\psi}(x)\left(\mathrm{i}\gamma^\mu \partial_\mu - m\right)\psi(x). \tag{1.1}$$

Here ψ is a spinor of some group G, γ^μ's are the Dirac matrices fulfilling the anti-commutation relation $\{\gamma^\mu, \gamma^\nu\} = -2\eta^{\mu\nu}$ with η being the metric tensor, and $\overline{\psi} = \psi^\dagger \gamma^0$. This Lagrangian density is invariant under global symmetry transformation defined by the same representation of G as the field ψ,

$$\psi(x) \to V\psi(x), \quad \overline{\psi}(x) \to \overline{\psi}(x)V^\dagger. \tag{1.2}$$

We now restrict ourselves to the compact Lie groups $\mathrm{SU}(N)$ and $\mathrm{U}(1)$ with generators t^a fulfilling

$$\left[t^a, t^b\right] = \mathrm{i}f^{abc}t^c, \tag{1.3}$$

where f^{abc} are the antisymmetric structure constants of the group. We further impose the normalization condition

$$\mathrm{tr}\left(t^a t^b\right) = \frac{1}{2}\delta^{ab}. \tag{1.4}$$

We want to construct a theory that is invariant under local symmetry transformations

$$\psi(x) \to V(x)\psi(x), \quad \overline{\psi}(x) \to \overline{\psi}(x)V^\dagger(x). \tag{1.5}$$

Clearly the derivative along the direction n^μ

$$n^\mu \partial_\mu \psi(x) = \lim_{h\to 0} \frac{\psi(x+hn) - \psi(x)}{h} \tag{1.6}$$

does not have a simple transformation behaviour. To compensate for the different transformations at different spacetime points, we introduce the comparator $U(x', x)$ which is a unitary matrix transformed as

$$U(x', x) \to V(x')U(x', x)V^\dagger(x) \tag{1.7}$$

and $U(x,x)=\mathbb{I}$. Hence, the covariant derivative

$$n^\mu D_\mu \psi(x) := \lim_{h\to 0} \frac{\psi(x+hn) - U(x+hn,x)\psi(x)}{h} \tag{1.8}$$

has the simple transformation. Now expand the comparator and introduce the connection $A^a_\mu(x)$ and the coupling constant g

$$U(x+hn,x) = \mathbb{I} + \mathrm{i} g h n^\mu A^a_\mu(x) t^a + \mathcal{O}(h^2). \tag{1.9}$$

It follows from the transformation of the comparator Eq. (1.7) that the gauge field $A_\mu = A^a_\mu t^a$ transforms as

$$A_\mu(x) \to V(x) \left(A_\mu(x) + \frac{\mathrm{i}}{g}\partial_\mu \right) V^\dagger(x). \tag{1.10}$$

And we can use the gauge field to write the covariant derivative explicitly

$$D_\mu = \partial_\mu - \mathrm{i} g A_\mu(x). \tag{1.11}$$

Define the field strength

$$\begin{aligned} F_{\mu\nu}(x) &:= \frac{\mathrm{i}}{g}\left[D_\mu, D_\nu\right] \\ &= \partial_\mu A_\nu - \partial_\nu A_\mu - \mathrm{i}g\left[A_\mu, A_\nu\right]. \end{aligned} \tag{1.12}$$

It is clearly that $F_{\mu\nu}$ transforms similar to D_μ, that is,

$$F_{\mu\nu}(x) \to U(x) F_{\mu\nu}(x) U^\dagger(x). \tag{1.13}$$

Therefore we can construct the Yang-Mills Lagrangian density which is invariant under local symmetry transformation

$$\mathcal{L}_{\mathrm{YM}} = -\frac{1}{2}\operatorname{tr}\left(F^{\mu\nu}F_{\mu\nu}\right). \tag{1.14}$$

To express it more explicitly, we can expand the field strength in terms of the generator matrices,

$$F_{\mu\nu} = F^a_{\mu\nu} t^a, \quad F^a_{\mu\nu} = 2\operatorname{tr}\left(F_{\mu\nu} t^a\right). \tag{1.15}$$

The explicit form of $F^a_{\mu\nu}$ is

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu. \tag{1.16}$$

As a result,

$$\mathcal{L}_{\mathrm{YM}} = -\frac{1}{4} F^{a,\mu\nu} F^a_{\mu\nu}. \tag{1.17}$$

The classical Lagrangian density including the massive fermions and the massless gauge bosons as the mediator for interaction hence reads

$$\mathcal{L} = \overline{\psi}(x)\left(\mathrm{i}\gamma^{\mu} D_{\mu} - m\right)\psi(x) - \frac{1}{4}F^{a,\mu\nu}F^{a}_{\mu\nu}. \tag{1.18}$$

The Euler-Lagrange equation of the Lagrangian (1.18) are given by

$$\begin{gathered}\left(\mathrm{i}\gamma^{\mu} D_{\mu} - m\right)\psi(x) = 0,\\ \partial^{\mu}F^{a}_{\mu\nu}(x) + gf^{abc}A^{b,\mu}(x)F^{c}_{\mu\nu}(x) = -g\overline{\psi}(x)\gamma_{\nu}t^{a}\psi(x).\end{gathered} \tag{1.19}$$

It is worth mentioning that if the symmetry group is U(1), the first eqution becomes the equation of motion of a fermion in the external electromagnetic field and the second equation reduces to the Maxwell equation with the source flow given by $J_{\mu} = -g\overline{\psi}\gamma_{\mu}\psi$. Note that the sourceless Maxwell equations are just equivalent to the antisymmetric property of the field $F_{\mu\nu}$.

Path integral formalism can be used to quantize the theory above. Ground state expectation values of an observable O are given by the path integral

$$\langle O\rangle = \frac{1}{Z}\int \mathcal{D}A\,\mathcal{D}\psi\,\mathcal{D}\overline{\psi}\,O\exp\left(\mathrm{i}S\right), \tag{1.20}$$

where $S = \int \mathrm{d}^4x\,\mathcal{L}$. Applying the Wick rotation $t \to \mathrm{i}\tau$ makes the spacetime a Euclidean space, in which the expectation (1.20) is analogous to that obtained in statistical mechanics. This analogy allows Monte Carlo methods to be applied to the lattice gas theory.

Another quantization procedure is the canonical quantization in which we divide the spacetime into slices with equal time. We have the momenta for fermion field and gauge field respectively,

$$\begin{aligned}\pi^{l}_{F} &= \frac{\partial\mathcal{L}}{\partial\left(\partial_0\psi^{l}(x)\right)} = \mathrm{i}\psi^{\dagger l}(x),\\ \pi^{a,0} &= \frac{\partial\mathcal{L}}{\partial\left(\partial_0 A^{a}_{0}(x)\right)} = 0,\\ \pi^{a,k} &= \frac{\partial\mathcal{L}}{\partial\left(\partial_0 A^{a}_{k}(x)\right)} = F^{a,k0}(x),\end{aligned} \tag{1.21}$$

where $k = 1, 2, 3$ refers to the spatial indices of the gauge field. The Hamiltonian is therefore

$$\begin{aligned}H &= \int \mathrm{d}^3x\,\left(\pi^{l}_{F}(\boldsymbol{x})\partial_0\psi^{l}(\boldsymbol{x}) + \pi^{a,k}(\boldsymbol{x})\partial_0 A^{a}_{k}(\boldsymbol{x}) - \mathcal{L}\right)\\ &= \int \mathrm{d}^3x\,\psi^{\dagger}(\boldsymbol{x})\left(-\mathrm{i}\boldsymbol{\alpha}\cdot\nabla - g\boldsymbol{\alpha}\cdot\boldsymbol{A}(\boldsymbol{x}) + \gamma^{0}m\right)\psi(\boldsymbol{x})\\ &\quad + \int \mathrm{d}^3x\,\left(-\frac{1}{2}\pi^{a,k}(\boldsymbol{x})\pi^{a}_{k}(\boldsymbol{x}) + \frac{1}{4}F^{a,jk}(\boldsymbol{x})F^{a}_{jk}(\boldsymbol{x})\right)\\ &\quad - \int \mathrm{d}^3x\,A^{a}_{0}(\boldsymbol{x})\left(\partial^{k}\pi^{a}_{k}(\boldsymbol{x}) + gf^{abc}A^{b,k}(\boldsymbol{x})\pi^{c}_{k}(\boldsymbol{x}) + g\psi^{\dagger}(\boldsymbol{x})t^{a}\psi(\boldsymbol{x})\right),\end{aligned} \tag{1.22}$$

where $\boldsymbol{\alpha}^k = \gamma^0\gamma^k$ and $\boldsymbol{A}^k = A^{a,k}t^a$. Since $\pi^{a,0} = 0$, we choose the temperal or Weyl gauge that the temporal component of the gauge field is zero, $A_0^a = 0$. We can now impose the canonical quantization condition

$$\left\{\psi^a(\boldsymbol{x}), \psi^{\dagger b}(\boldsymbol{y})\right\} = \delta(\boldsymbol{x}-\boldsymbol{y})\delta_{ab}, \quad \left\{\psi^a(\boldsymbol{x}), \psi^b(\boldsymbol{y})\right\} = 0 \tag{1.23}$$

for fermions and

$$\left[A_k^a(\boldsymbol{x}), \pi_j^b(\boldsymbol{y})\right] = \mathrm{i}\delta_{ab}\delta_{kj}\delta(\boldsymbol{x}-\boldsymbol{y}), \quad \left[A_k^a\left(\boldsymbol{x}\right), A_j^b\left(\boldsymbol{y}\right)\right] = \left[\pi_k^a\left(\boldsymbol{x}\right), \pi_j^b\left(\boldsymbol{y}\right)\right] = 0 \tag{1.24}$$

for gauge bosons. This gauge choice and canonical quantization condition yields the quantized Hamiltonian

$$\begin{aligned} H = &\int \mathrm{d}^3x\, \psi^\dagger(\boldsymbol{x}) \left(-\mathrm{i}\boldsymbol{\alpha}\cdot\nabla - g\boldsymbol{\alpha}\cdot\boldsymbol{A}(\boldsymbol{x}) + \gamma^0 m\right)\psi(\boldsymbol{x}) \\ &+ \int \mathrm{d}^3x \left(-\frac{1}{2}\pi^{a,k}(\boldsymbol{x})\pi_k^a(\boldsymbol{x}) + \frac{1}{4}F^{a,jk}(\boldsymbol{x})F_{jk}^a(\boldsymbol{x})\right). \end{aligned} \tag{1.25}$$

Introduce the vector potential $\boldsymbol{A}^a\left(\boldsymbol{x}\right)$ whose k-component is $A^{a,k}\left(\boldsymbol{x}\right)$, we can define the colour-electric and colour-magnetic field as

$$\begin{aligned} \boldsymbol{E}^a\left(\boldsymbol{x}\right) &:= -\partial_t \boldsymbol{A}^a\left(\boldsymbol{x}\right), \\ \boldsymbol{B}^a\left(\boldsymbol{x}\right) &:= \nabla\times\boldsymbol{A}^a\left(\boldsymbol{x}\right) + \frac{1}{2}gf^{abc}\boldsymbol{A}^b\left(\boldsymbol{x}\right)\times\boldsymbol{A}^c\left(\boldsymbol{x}\right), \end{aligned} \tag{1.26}$$

whose components are just

$$E^{a,k} = -F^{a,k0}, \quad B^{a,k} = \frac{1}{2}\epsilon_{ijk}F^{a,ij}. \tag{1.27}$$

Therefore we can rewrite second line of the Hamiltonian (1.25) as $H_{\mathrm{el}} + H_{\mathrm{mag}}$, where

$$\begin{aligned} H_{\mathrm{el}} &:= \frac{1}{2}\int \mathrm{d}^3x\, \boldsymbol{E}^a\left(\boldsymbol{x}\right)\cdot\boldsymbol{E}^a\left(\boldsymbol{x}\right) \\ H_{\mathrm{mag}} &:= \frac{1}{2}\int \mathrm{d}^3x\, \boldsymbol{B}^a\left(\boldsymbol{x}\right)\cdot\boldsymbol{B}^a\left(\boldsymbol{x}\right). \end{aligned} \tag{1.28}$$

References

[1] Stefan Kühn. *Quantum and classical simulation of High Energy Physics*. PhD thesis, Munich, Tech. U., 10 2017.