Note on Quantum Simulation of High Energy Physics

Yifei Wang

School of Physics, Peking University, Beijing 100871, China

`wang_yifei@pku.edu.cn`

Abstract

This note is about the quantum simulation of high energy physics.

Throughout this note we take $c = \hbar = k_B = 1$. Einstein summation convention is taken unless otherwise specified. The Lorentz metric for flat spacetime is assumed as $\eta_{\mu\nu} = \mathrm{diag}(-1, +1, +1, +1)$.

Created date: August 13, 2021

Version: September 1, 2021

Contents

1 Lattice gauge theory

This section is mainly based on the thesis [1].

1.1 Yang-Mills theory in the continuum

The well known Lagrangian density of a free Dirac fermion is given by

$$\mathcal{L} = \overline{\psi}(x)\left(\mathrm{i}\gamma^{\mu}\partial_{\mu} - m\right)\psi(x). \tag{1.1}$$

Here ψ is a spinor of some group G, γ^{μ}'s are the Dirac matrices fulfilling the anti-commutation relation $\{\gamma^{\mu}, \gamma^{\nu}\} = -2\eta^{\mu\nu}$ with η being the metric tensor, and $\overline{\psi} = \psi^{\dagger}\gamma^{0}$. This Lagrangian density is invariant under global symmetry transformation defined by the same representation of G as the field ψ,

$$\psi(x) \to V\psi(x), \quad \overline{\psi}(x) \to \overline{\psi}(x)V^{\dagger}. \tag{1.2}$$

We now restrict ourselves to the compact Lie groups $\mathrm{SU}(N)$ and $\mathrm{U}(1)$ with generators t^a fulfilling

$$\left[t^a, t^b\right] = \mathrm{i}f^{abc}t^c, \tag{1.3}$$

where f^{abc} are the antisymmetric structure constants of the group. We further impose the normalization condition

$$\mathrm{tr}\left(t^a t^b\right) = \frac{1}{2}\delta^{ab}. \tag{1.4}$$

We want to construct a theory that is invariant under local symmetry transformations

$$\psi(x) \to V(x)\psi(x), \quad \overline{\psi}(x) \to \overline{\psi}(x)V^{\dagger}(x). \tag{1.5}$$

Clearly the derivative along the direction n^{μ}

$$n^{\mu}\partial_{\mu}\psi(x) = \lim_{h\to 0}\frac{\psi(x+hn) - \psi(x)}{h} \tag{1.6}$$

does not have a simple transformation behaviour. To compensate for the different transformations at different spacetime points, we introduce the comparator $U(x', x)$ which is a unitary matrix transformed as

$$U(x', x) \to V(x')U(x', x)V^\dagger(x) \tag{1.7}$$

and $U(x, x) = \mathbb{I}$. Hence, the covariant derivative

$$n^\mu D_\mu \psi(x) := \lim_{h\to 0} \frac{\psi(x + hn) - U(x + hn, x)\psi(x)}{h} \tag{1.8}$$

has the simple transformation. Now expand the comparator and introduce the connection $A^a_\mu(x)$ and the coupling constant g

$$U(x + hn, x) = \mathbb{I} + \mathrm{i}ghn^\mu A^a_\mu(x)t^a + \mathcal{O}(h^2). \tag{1.9}$$

The comparator along a lath $\mathcal{C}$ between two points y and z is given by

$$U(z, y) = \mathcal{P} \exp\left(\mathrm{i}g \int_{\mathcal{C}} \mathrm{d}x^\mu\, A^a_\mu\,(x)\, t^a\right), \tag{1.10}$$

where P indicates path ordering. It follows from the transformation of the comparator Eq. (1.7) that the gauge field $A_\mu = A^a_\mu t^a$ transforms as

$$A_\mu(x) \to V(x)\left(A_\mu(x) + \frac{\mathrm{i}}{g}\partial_\mu\right)V^\dagger(x). \tag{1.11}$$

And we can use the gauge field to write the covariant derivative explicitly

$$D_\mu = \partial_\mu - \mathrm{i}gA_\mu(x). \tag{1.12}$$

Define the field strength

$$\begin{aligned} F_{\mu\nu}(x) &:= \frac{\mathrm{i}}{g}\left[D_\mu, D_\nu\right] \\ &= \partial_\mu A_\nu - \partial_\nu A_\mu - \mathrm{i}g\left[A_\mu, A_\nu\right]. \end{aligned} \tag{1.13}$$

It is clear that $F_{\mu\nu}$ transforms similar to D_μ, that is,

$$F_{\mu\nu}(x) \to V(x)F_{\mu\nu}(x)V^\dagger(x). \tag{1.14}$$

Therefore we can construct the Yang-Mills Lagrangian density which is invariant under local symmetry transformation

$$\mathcal{L}_{\mathrm{YM}} = -\frac{1}{2}\,\mathrm{tr}\left(F^{\mu\nu}F_{\mu\nu}\right). \tag{1.15}$$

To express it more explicitly, we can expand the field strength in terms of the generator matrices,

$$F_{\mu\nu} = F^a_{\mu\nu} t^a, \quad F^a_{\mu\nu} = 2\,\mathrm{tr}\left(F_{\mu\nu} t^a\right). \tag{1.16}$$

The explicit form of $F^a_{\mu\nu}$ is

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu. \tag{1.17}$$

As a result,

$$\mathcal{L}_{\mathrm{YM}} = -\frac{1}{4} F^{a,\mu\nu} F^a_{\mu\nu}. \tag{1.18}$$

The classical Lagrangian density including the massive fermions and the massless gauge bosons as the mediator for interaction hence reads

$$\mathcal{L} = \overline{\psi}(x) \left(\mathrm{i}\gamma^\mu D_\mu - m\right) \psi(x) - \frac{1}{4} F^{a,\mu\nu} F^a_{\mu\nu}. \tag{1.19}$$

The Euler-Lagrange equation of the Lagrangian (1.19) are given by

$$\begin{gathered} \left(\mathrm{i}\gamma^\mu D_\mu - m\right) \psi(x) = 0, \\ \partial^\mu F^a_{\mu\nu}(x) + g f^{abc} A^{b,\mu}(x) F^c_{\mu\nu}(x) = -g\overline{\psi}(x)\gamma_\nu t^a \psi(x). \end{gathered} \tag{1.20}$$

It is worth mentioning that if the symmetry group is U(1), the first eqution becomes the equation of motion of a fermion in the external electromagnetic field and the second equation reduces to the Maxwell equation with the source flow given by $J_\mu = -g\overline{\psi}\gamma_\mu\psi$. Note that the sourceless Maxwell equations are just equivalent to the antisymmetric property of the field $F_{\mu\nu}$.

Path integral formalism can be used to quantize the theory above. Ground state expectation values of an observable O are given by the path integral

$$\langle O \rangle = \frac{1}{Z} \int \mathcal{D}A\, \mathcal{D}\psi\, \mathcal{D}\overline{\psi}\, O \exp\left(\mathrm{i}S\right), \tag{1.21}$$

where $S = \int \mathrm{d}^4 x\, \mathcal{L}$. Applying the Wick rotation $t \to \mathrm{i}\tau$ makes the spacetime a Euclidean space, in which the expectation (1.21) is analogous to that obtained in statistical mechanics. This analogy allows Monte Carlo methods to be applied to the lattice gas theory.

Another quantization procedure is the canonical quantization in which we divide the spacetime into slices with equal time. We have the momenta for fermion field and gauge field respectively,

$$\begin{aligned} \pi^l_F &= \frac{\partial \mathcal{L}}{\partial\left(\partial_0 \psi^l(x)\right)} = \mathrm{i}\psi^{\dagger l}(x), \\ \pi^{a,0} &= \frac{\partial \mathcal{L}}{\partial\left(\partial_0 A^a_0(x)\right)} = 0, \\ \pi^{a,k} &= \frac{\partial \mathcal{L}}{\partial\left(\partial_0 A^a_k(x)\right)} = F^{a,k0}(x), \end{aligned} \tag{1.22}$$

where $k = 1, 2, 3$ refers to the spatial indices of the gauge field. The Hamiltonian is therefore

$$\begin{aligned} H &= \int \mathrm{d}^3x \, \left(\pi_F^l(\boldsymbol{x})\partial_0\psi^l(\boldsymbol{x}) + \pi^{a,k}(\boldsymbol{x})\partial_0 A_k^a(\boldsymbol{x}) - \mathcal{L}\right) \\ &= \int \mathrm{d}^3x \, \psi^\dagger(\boldsymbol{x}) \left(-\mathrm{i}\boldsymbol{\alpha}\cdot\nabla - g\boldsymbol{\alpha}\cdot\boldsymbol{A}(\boldsymbol{x}) + \gamma^0 m\right)\psi(\boldsymbol{x}) \\ &\quad + \int \mathrm{d}^3x \, \left(-\frac{1}{2}\pi^{a,k}(\boldsymbol{x})\pi_k^a(\boldsymbol{x}) + \frac{1}{4}F^{a,jk}(\boldsymbol{x})F_{jk}^a(\boldsymbol{x})\right) \\ &\quad - \int \mathrm{d}^3x \, A_0^a(\boldsymbol{x}) \left(\partial^k\pi_k^a(\boldsymbol{x}) + gf^{abc}A^{b,k}(\boldsymbol{x})\pi_k^c(\boldsymbol{x}) + g\psi^\dagger(\boldsymbol{x})t^a\psi(\boldsymbol{x})\right), \end{aligned} \tag{1.23}$$

where $\boldsymbol{\alpha}^k = \gamma^0\gamma^k$ and $\boldsymbol{A}^k = A^{a,k}t^a$. Since $\pi^{a,0} = 0$, we choose the temperal or Weyl gauge that the temporal component of the gauge field is zero, $A_0^a = 0$. We can now impose the canonical quantization condition

$$\left\{\psi^a(\boldsymbol{x}), \psi^{\dagger b}(\boldsymbol{y})\right\} = \delta(\boldsymbol{x}-\boldsymbol{y})\delta_{ab}, \quad \left\{\psi^a(\boldsymbol{x}), \psi^b(\boldsymbol{y})\right\} = 0 \tag{1.24}$$

for fermions and

$$\left[A_k^a(\boldsymbol{x}), \pi_j^b(\boldsymbol{y})\right] = \mathrm{i}\delta_{ab}\delta_{kj}\delta(\boldsymbol{x}-\boldsymbol{y}), \quad \left[A_k^a\left(\boldsymbol{x}\right), A_j^b\left(\boldsymbol{y}\right)\right] = \left[\pi_k^a\left(\boldsymbol{x}\right), \pi_j^b\left(\boldsymbol{y}\right)\right] = 0 \tag{1.25}$$

for gauge bosons. This gauge choice and canonical quantization condition yields the quantized Hamiltonian

$$\begin{aligned} H = \int \mathrm{d}^3x \, \psi^\dagger(\boldsymbol{x}) \left(-\mathrm{i}\boldsymbol{\alpha}\cdot\nabla - g\boldsymbol{\alpha}\cdot\boldsymbol{A}(\boldsymbol{x}) + \gamma^0 m\right)\psi(\boldsymbol{x}) \\ + \int \mathrm{d}^3x \, \left(-\frac{1}{2}\pi^{a,k}(\boldsymbol{x})\pi_k^a(\boldsymbol{x}) + \frac{1}{4}F^{a,jk}(\boldsymbol{x})F_{jk}^a(\boldsymbol{x})\right). \end{aligned} \tag{1.26}$$

Introduce the vector potential $\boldsymbol{A}^a\left(\boldsymbol{x}\right)$ whose k-component is $A^{a,k}\left(\boldsymbol{x}\right)$, we can define the colour-electric and colour-magnetic field as

$$\begin{aligned} \boldsymbol{E}^a\left(\boldsymbol{x}\right) &:= -\partial_t\boldsymbol{A}^a\left(\boldsymbol{x}\right), \\ \boldsymbol{B}^a\left(\boldsymbol{x}\right) &:= \nabla\times\boldsymbol{A}^a\left(\boldsymbol{x}\right) + \frac{1}{2}gf^{abc}\boldsymbol{A}^b\left(\boldsymbol{x}\right)\times\boldsymbol{A}^c\left(\boldsymbol{x}\right), \end{aligned} \tag{1.27}$$

whose components are just

$$E^{a,k} = -F^{a,k0}, \quad B^{a,k} = \frac{1}{2}\epsilon_{ijk}F^{a,ij}. \tag{1.28}$$

Therefore we can rewrite second line of the Hamiltonian (1.26) as $H_{\mathrm{el}} + H_{\mathrm{mag}}$, where

$$\begin{aligned} H_{\mathrm{el}} &:= \frac{1}{2}\int \mathrm{d}^3x \, \boldsymbol{E}^a\left(\boldsymbol{x}\right)\cdot\boldsymbol{E}^a\left(\boldsymbol{x}\right) \\ H_{\mathrm{mag}} &:= \frac{1}{2}\int \mathrm{d}^3x \, \boldsymbol{B}^a\left(\boldsymbol{x}\right)\cdot\boldsymbol{B}^a\left(\boldsymbol{x}\right). \end{aligned} \tag{1.29}$$

The generator for local gauge transformation is the Gauss law components

$$G^a(\boldsymbol{x}) = \partial^k \pi_k^a(\boldsymbol{x}) + g f^{abc} A^{b,k}(\boldsymbol{x}) \pi_k^c(\boldsymbol{x}) + \rho^a(\boldsymbol{x}), \tag{1.30}$$

where $\rho^a(\boldsymbol{x}) = g\psi^\dagger(\boldsymbol{x}) t^a \psi(\boldsymbol{x})$ are the charge density components. The operators satisfy the commutation relation

$$\left[G^a(\boldsymbol{x}), G^b(\boldsymbol{y})\right] = \mathrm{i} g f^{abc} G^a(\boldsymbol{x}) \delta(\boldsymbol{x} - \boldsymbol{y}). \tag{1.31}$$

Since we have chosen that the temperal component of the gauge field be zero, the Gauss's law is absent from our Hamiltonian formulation. We should add an additional constraint on the physical states that the local gauge transformation yield a global gauge transformation and thus has no physical effect. This condition is equivalent to

$$G^a(\boldsymbol{x}) |\Psi\rangle = q^a(\boldsymbol{x}) |\Psi\rangle. \tag{1.32}$$

This is nothing but the Gauss's law. Since G^a commutes with the Hamiltonian, the external charge distribution $q^a(\boldsymbol{x})$ is conserved.

1.2 Lattice formulation

There are two approaches to the lattice formulation of the theory. The first is proposed by Wilson [2] from the path integral formulation. The second is proposed by Kogut and Susskind [3] from the Hamiltonian formulation. In this section we introduce the second one for the convenience of applying the quantum simulation and tensor network (TN).

1.2.1 The doubling problem

The trivial discretization of Hamiltonian (1.26) causes problems. Let us illustrate this problem in the simplest $1+1$ dimensional free massless fermion case. The Hamiltonian of such fermions is

$$H = \int \mathrm{d}x\, \psi^\dagger(x) \left(-\mathrm{i}\gamma^0\gamma^1\partial_x\right) \psi(x). \tag{1.33}$$

The equation of motion $\mathrm{i}\partial_t\psi(x) = [\psi(x), H]$ is simple the Dirac equation

$$\partial_t\psi(x) = -\gamma^0\gamma^1\partial_x\psi(x). \tag{1.34}$$

Using the Weyl representation for Dirac matrices

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^5 = \gamma^0\gamma^1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{1.35}$$

and the plain wave ansatz $\psi(x) = \exp(\mathrm{i}(Et - px))\, u(p, E)$, we have the eigenvalue problem

$$\begin{pmatrix} p & 0 \\ 0 & -p \end{pmatrix} u(p, E) = E u(p, E). \tag{1.36}$$

The solution is the usual dispersion relation $E = \pm p$ with corresponding eigenvectors $u_{\pm}(p, E)$, which are the eigenvectors of γ^5 with well defined chirality ± 1.

A trivial discretization is defined by the maps

$$\psi(x) \to \frac{\widehat{\psi}_n}{\sqrt{a}}, \quad \partial_x \psi(x) \to \frac{1}{2a^{3/2}} \left(\widehat{\psi}_{n+1} - \widehat{\psi}_{n-1} \right), \quad \int \mathrm{d}x \to a \sum_n . \tag{1.37}$$

The resulting lattice Hamiltonian is

$$\widehat{H} = \sum_n \widehat{\psi}_n \left(-\mathrm{i}\gamma^0\gamma^1 \frac{1}{2a} \left(\delta_{k,n+1} - \delta_{k,n-1} \right) \right) \widehat{\psi}_k, \tag{1.38}$$

and the equation of motion

$$\partial_t \widehat{\psi}_n = -\gamma^0\gamma^1 \frac{1}{2a} \left(\widehat{\psi}_{n+1} - \widehat{\psi}_{n-1} \right). \tag{1.39}$$

Still using the plain wave ansatz $\widehat{\psi}_n = \exp\left(\mathrm{i}\left(Et - pna\right)\right) \widehat{u}(p, E)$, we have the eigenvalue equation

$$\begin{pmatrix} \sin(pa)/a & 0 \\ 0 & -\sin(pa)/a \end{pmatrix} \widehat{u}(p, E) = E\widehat{u}(p, E), \tag{1.40}$$

with the momentum restricted to the first Brillouin zone $-\pi/a \leqslant p \leqslant \pi/a$. With the same eigenstates as the continuous case, the corresponding dispersion relation becomes $E = \pm\sin(pa)/a$. Comparing to the continuous case, additional low energy states appear at $p = \pi/a$, and for each chirality there exist a left moving mode and a right moving mode. These new modes do not vanish at the limit $a \to 0$. In general, for a Dirac theory with d spatial dimensions, there occur 2^d zeros in the first Brillion zone, that is, for each dimension the zeros are doubled. This is the so called doubling problem.

Furthermore, it is shown by Nielsen and Ninomiya [4–6] that any local, translationally invariant, hermitian lattice formulation of a chiral fermion gives rise to a equal number of left and right movers. There are several solutions to this problem from which we introduce the staggered fermion approach for the convenience of quantum simulaton and TN.

1.2.2 The Kogut-Susskind Hamiltonian formulation

Contrary to the naive Hamiltonian (1.38), consider a Hamiltonian with a single component fermionic field $\widehat{\phi}_n$ on each lattice site n,

$$\widehat{H} = -\frac{\mathrm{i}}{2a} \sum_n \left(\widehat{\phi}_n^\dagger \widehat{\phi}_{n+1} - \widehat{\phi}_{n+1}^\dagger \widehat{\phi}_n \right). \tag{1.41}$$

The fields still fulfill the anticommutation relation $\left\{ \widehat{\phi}_n, \widehat{\phi}_m \right\} = 0$, $\left\{ \widehat{\phi}_n^\dagger, \widehat{\phi}_m \right\} = \delta_{mn}$. The equation of motion is therefore

$$\partial_t \widehat{\phi}_n = \frac{1}{2a} \left(\widehat{\phi}_{n+1} - \widehat{\phi}_{n-1} \right), \tag{1.42}$$

which is the same as lower component of the spinor in Eq. (1.39) and as a result yields a single left and a single right moving solution. Label field $\widehat{\phi}_N$ as $\widehat{\psi}_{\mathrm{u},n}$ for even $N = 2n$ and $\widehat{\psi}_{\mathrm{l},n}$ for odd $N = 2n + 1$, we have

$$\partial_t \widehat{\psi}_{\mathrm{u},n} = \frac{1}{2a}\left(\widehat{\psi}_{\mathrm{l},n} - \widehat{\psi}_{\mathrm{l},n-1}\right), \quad \partial_t \widehat{\psi}_{\mathrm{l},n} = \frac{1}{2a}\left(\widehat{\psi}_{\mathrm{u},n+1} - \widehat{\psi}_{\mathrm{u},n}\right). \tag{1.43}$$

Taking the continuum limit $a \to 0$, and taking the representation

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \gamma^5 = \gamma^0\gamma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{1.44}$$

we recover the dirac equation (1.34) with the map

$$\lim_{a\to 0} \begin{pmatrix} \widehat{\psi}_{\mathrm{u},n}/\sqrt{a} \\ \widehat{\psi}_{\mathrm{l},n}/\sqrt{a} \end{pmatrix} \to \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \end{pmatrix} = \psi(x). \tag{1.45}$$

The recovery of continuum model at the limit $a \to 0$ can also be checked by the Hamiltonian directly.

The mass term can be added as follows. The mass term

$$\int \mathrm{d}x\, m\overline{\psi}\,(x)\,\psi\,(x) = \int \mathrm{d}x\, m\left(\psi_1^\dagger\,(x)\,\psi_1\,(x) - \psi_2^\dagger\,(x)\,\psi_2\,(x)\right) \tag{1.46}$$

can be written in the discretized version

$$m\sum_n \left(\widehat{\psi}^\dagger_{\mathrm{u},n}\widehat{\psi}_{\mathrm{u},n} - \widehat{\psi}^\dagger_{\mathrm{l},n}\widehat{\psi}_{\mathrm{l},n}\right) = m\sum_n (-1)^n \widehat{\phi}^\dagger_n \widehat{\phi}_n. \tag{1.47}$$

Finally, the Hamiltonian for d spatial dimensional lattice reads

$$\widehat{H} = \epsilon_d \sum_{\boldsymbol{n}} \sum_{k=1}^{d} \left(\widehat{\phi}^\dagger_{\boldsymbol{n}+e_k}\widehat{\phi}_{\boldsymbol{n}} \mathrm{e}^{\mathrm{i}\theta_k} + \mathrm{h.c.}\right) + m\sum_{\boldsymbol{n}} (-1)^{\sum_{k=1}^{d} n_k}\, \widehat{\phi}^\dagger_{\boldsymbol{n}}\widehat{\phi}_{\boldsymbol{n}}, \tag{1.48}$$

where $\boldsymbol{n} \in \mathbb{Z}^d$ is a lattice vector with components n_k and unit vector e_k, ϵ_d is a dimension dependent prefactor and $\exp\left(\mathrm{i}\theta_k\right)$ are direction dependent phase factors. This approach cannot remove the doubling solutions completely, for example, in three dimensions two flavours of Dirac fermions in the continuum limit are recovered instead of 8. To consider the colour degrees of freedom one just need to replace the field with a colou spinor.

From now on, we use $\psi_{\boldsymbol{n}}$ instead of $\widehat{\phi}_{\boldsymbol{n}}$ for convenience.

However, the hopping term $\widehat{\phi}^\dagger_{\boldsymbol{n}}\widehat{\phi}_{\boldsymbol{n}+e_k}$ is not gauge invariant under a local gauge transformation. To recover the gauge invariance, we need to include the gauge field into the Hamiltonian.

First define the $U_{\boldsymbol{n},k}$ linking the sites $\boldsymbol{n}$ and $\boldsymbol{n} + e_k$ which under gauge transformations transforms as

$$U_{\boldsymbol{n},k} \to V_{\boldsymbol{n}+e_k} U_{\boldsymbol{n},k} V^\dagger_{\boldsymbol{n}}. \tag{1.49}$$

Also we require that $U_{\boldsymbol{n}+e_k,-k} = U^\dagger_{\boldsymbol{n},k}$. The gauge invariant hopping term is now clearly

$$\psi^\dagger_{\boldsymbol{n}+e_k} U_{\boldsymbol{n},k} \psi_{\boldsymbol{n}}. \tag{1.50}$$

For compact Lie groups U(1) and SU(N), we can write $U_{\boldsymbol{n},k}$ in the exponential form

$$U_{\boldsymbol{n},k} = \exp\left(\mathrm{i}\Lambda^a_{\boldsymbol{n},k} t^a\right), \tag{1.51}$$

where $\Lambda^a_{\boldsymbol{n},k}$ are parameters. The infinitesimal form is

$$U_{\boldsymbol{n},k} = \mathbb{I} + \mathrm{i}\Lambda^a_{\boldsymbol{n},k} t^a + \mathcal{O}\left(\Lambda^2\right). \tag{1.52}$$

Comparing to the continuous case, we have

$$\frac{1}{ag}\Lambda^a_{\boldsymbol{n},k} \to A^a_k\left(\boldsymbol{x}\right). \tag{1.53}$$

Now we consider the colour magnetic field Hamiltonian. Another gauge invariant term is the trace of the product of $U_{\boldsymbol{n},k}$ around a closed loop. Locality leads us to make the smallest loop possible, which are plaquettes on the lattice, denoted by $\square_p$. We have

$$U_p \equiv U_{\boldsymbol{n}+e_j,-j} U_{\boldsymbol{n}+e_k+e_j,-k} U_{\boldsymbol{n}+e_k,j} U_{\boldsymbol{n},k} = U^\dagger_{\boldsymbol{n},j} U^\dagger_{\boldsymbol{n}+e_j,k} U_{\boldsymbol{n}+e_k,j} U_{\boldsymbol{n},k}. \tag{1.54}$$

To see the continuous limit, expand

$$\begin{aligned} \Lambda^a_{\boldsymbol{n}+e_k,j} &= \Lambda^a_{\boldsymbol{n},j} + a\partial_k \Lambda^a_{\boldsymbol{n},j} + \mathcal{O}\left(a^2\right) \\ \Lambda^a_{\boldsymbol{n}+e_j,k} &= \Lambda^a_{\boldsymbol{n},k} + a\partial_j \Lambda^a_{\boldsymbol{n},k} + \mathcal{O}\left(a^2\right) \end{aligned} \tag{1.55}$$

Up to the first order in a,

$$\begin{aligned} \operatorname{tr} U_p &= \operatorname{tr}\exp\left(\mathrm{i}a\left(\partial_k \Lambda^a_{\boldsymbol{n},j} - \partial_j \Lambda^a_{\boldsymbol{n},k}\right) t^a - \mathrm{i} f^{abc} \Lambda^b_{\boldsymbol{n},j} \Lambda^c_{\boldsymbol{n},k} t^a\right) \\ &= \operatorname{tr}\exp\left(\mathrm{i}a^2 g F^a_{kj} t^a\right) \quad \text{at } a \to 0. \end{aligned} \tag{1.56}$$

Then at the continuous limit

$$\operatorname{tr} U_p = 1 - \frac{1}{4} a^4 g^2 \left(F^a_{kj}\right)^2. \tag{1.57}$$

Therefore we can construct the magnetic Hamiltonian as

$$H_{\text{mag}} = \frac{a^{d-4}}{g^2} \sum_{\square_p} \left(1 - \frac{1}{2}\left(\operatorname{tr} U_p + \text{h.c.}\right)\right) \tag{1.58}$$

If we are going to construct the Lagrangian form of a four-dimensional Euclidean lattice gauge theory with the time dimension also discrete, what we have done is enough. However, since we are going to construct the Hamiltonian form with a continuous time, we need to

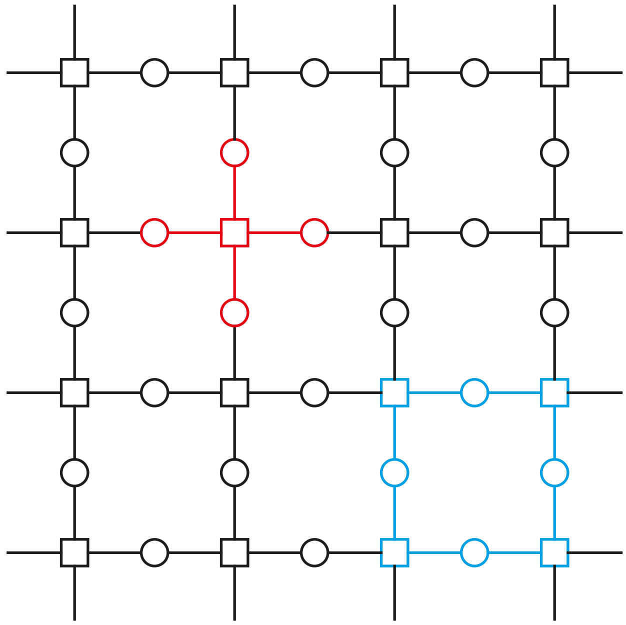

Figure 1: A 2+1 D lattice model. The squares are sites carrying fermionic Hilbert space, the circles are links carrying the gauge field Hilbert space. The blue square is a plaquette. The red star is a vertex.

restrict the magnetic field to the spatial dimensions, and construct another electric field by canonical quantization. We also need to point out the Hilbert space the operators acting on [7].

There are sites, links and vertices in a lattice system, see Figure. 1. Fermions are located on the sites. The natural choice of the Hilbert spaces for comparators to act on are the Hilbert spaces located on the links. The states in these Hilbert spaces are labelled by representation of the gauge group $|jmn\rangle$ corresponding to $(U^j)_{mn}$, where U^j is the j'th representation. For these Hilbert spaces on links we can define the generators for gauge transformations whose effects are left and right actions for the comparator. That is, for generator $L_{\boldsymbol{n},k}$, we have

$$\exp\left(-\mathrm{i}\lambda^a L^a_{\boldsymbol{n},k}\right)\left(U^j_{\boldsymbol{n},k}\right)_{\mu\nu}\exp\left(\mathrm{i}\lambda^a L^a_{\boldsymbol{n},k}\right) = \left(\exp\left(\mathrm{i}\lambda^a t^a\right)\right)_{\mu\rho}\left(U^j_{\boldsymbol{n},k}\right)_{\rho\nu}, \tag{1.59}$$

and for generator $R^a_{\boldsymbol{n},k}$, we have

$$\exp\left(-\mathrm{i}\lambda^a R^a_{\boldsymbol{n},k}\right)\left(U^j_{\boldsymbol{n},k}\right)_{\mu\nu}\exp\left(\mathrm{i}\lambda^a R^a_{\boldsymbol{n},k}\right) = \left(U^j_{\boldsymbol{n},k}\right)_{\mu\rho}\left(\exp\left(-\mathrm{i}\lambda^a t^a\right)\right)_{\rho\nu}. \tag{1.60}$$

The physical meaning is that the effect of the right generator is the same as generating a gauge transformation on site $\boldsymbol{n}$, and the effect of the left generator is the same as generating a transformation on site $\boldsymbol{n}+e_k$. Using Baker-Hausdorff formula we have the commutation relations

$$\begin{aligned}\left[L^a_{\boldsymbol{n},k},\left(U^j_{\boldsymbol{n}',k'}\right)_{\mu\nu}\right]&=-\delta_{\boldsymbol{n}\boldsymbol{n}'}\delta_{kk'}\left(T^a\right)_{\mu\rho}\left(U^j_{\boldsymbol{n},k}\right)_{\rho\nu},\\ \left[R^a_{\boldsymbol{n},k},\left(U^j_{\boldsymbol{n}',k'}\right)_{\mu\nu}\right]&=\delta_{\boldsymbol{n}\boldsymbol{n}'}\delta_{kk'}\left(U^j_{\boldsymbol{n},k}\right)_{\mu\rho}\left(T^a\right)_{\rho\nu},\end{aligned}\tag{1.61}$$

where T^a is the representation of the Lie algebra and we use indices i, j, l to make clear that it is a matrix product. The commutation relations between the generators are *set* as

$$\begin{aligned}\left[L^a_{\boldsymbol{n},k},L^b_{\boldsymbol{n}',k'}\right]&=\mathrm{i}\delta_{\boldsymbol{n}\boldsymbol{n}'}\delta_{kk'}f^{abc}L^c_{\boldsymbol{n},k},\\ \left[R^a_{\boldsymbol{n},k},R^b_{\boldsymbol{n}',k'}\right]&=\mathrm{i}\delta_{\boldsymbol{n}\boldsymbol{n}'}\delta_{kk'}f^{abc}R^c_{\boldsymbol{n},k},\\ \left[L^a_{\boldsymbol{n},k},R^b_{\boldsymbol{n}',k'}\right]&=0.\end{aligned}\tag{1.62}$$

The quadratic Casimir operator $\boldsymbol{J}^2_{\boldsymbol{n},k}\equiv\sum_a L^a_{\boldsymbol{n},k}L^a_{\boldsymbol{n},k}=\sum_a R^a_{\boldsymbol{n},k}R^a_{\boldsymbol{n},k}$. In the continuous model, the electric field $E^{a,k}=F^{a,0k}=-\pi^{a,k}$ fulfills the commutation relation

$$\left[E^{a,k}(\boldsymbol{x}),U(y+hn,y)\right]=-\delta\left(\boldsymbol{y}-\boldsymbol{x}\right)ghn^k t^a U(y+nh,y).\tag{1.63}$$

By comparing this equation with the commutation relation between $L_{\boldsymbol{n},k}$ and $U_{\boldsymbol{n},k}$, we have the that in the continuous limit

$$a^{1-d}gL_{\boldsymbol{n},k}\to E^{a,k}\left(\boldsymbol{x}\right).\tag{1.64}$$

Therefore, the colour-electric energy is given by

$$H_{\mathrm{el}}=\frac{a^{2-d}g^2}{2}\sum_{\boldsymbol{n},k}\boldsymbol{J}^2_{\boldsymbol{n},k}.\tag{1.65}$$

To summarize, the gauge invariant lattice Hamiltonian reads

$$\begin{aligned}H=\epsilon_d\sum_{\boldsymbol{n},k}\left(\psi^\dagger_{\boldsymbol{n}+e_k}U_{\boldsymbol{n},k}\psi_{\boldsymbol{n}}\mathrm{e}^{\mathrm{i}\theta_k}+\mathrm{h.c.}\right)+m\sum_{\boldsymbol{n}}\left(-1\right)^{\sum_k n_k}\psi^\dagger_{\boldsymbol{n}}\psi_{\boldsymbol{n}}\\ +\frac{a^{d-4}}{g^2}\sum_{\Box_p}\left(1-\frac{1}{2}\left(\mathrm{tr}\,U_p+\mathrm{h.c.}\right)\right)+\frac{a^{2-d}g^2}{2}\sum_{\mathrm{links}}\boldsymbol{J}^2.\end{aligned}\tag{1.66}$$

Finally we demonstrate that the effect of local gauge transformation is performed on vertices, which includes the site, d ingoing links and d outgoing links, see Figure. 1. The transformation is

$$V_{\boldsymbol{n}}\left(\lambda^a\right)=\sum_i\exp\left(\mathrm{i}\lambda^a L^a_i\right)\sum_o\exp\left(\mathrm{i}\lambda^a R^a_o\right)\exp\left(\mathrm{i}\lambda^a Q^a_{\boldsymbol{n}}\right),\tag{1.67}$$

where i indicates 'incoming', o indicates 'outcoming' and Q^a is the generator for gauge transformation of fermion fields, fulfilling

$$\left(\exp\left(-\mathrm{i}\lambda^a Q^a\right)\right)^j\psi_\mu\left(\exp\left(\mathrm{i}\lambda^a Q^a\right)\right)^j=D^j_{\mu\nu}\left(\lambda^a\right)\psi_\nu.\tag{1.68}$$

Here $D^j(\lambda^a)$ is the j'th representation for transforming by λ^a. It can be proved that [7]

$$(\exp(-\mathrm{i}\lambda^a Q^a))^j = \exp\left(\mathrm{i}\psi_\mu^\dagger q_{\mu\nu}\psi_\nu\right)\det\left(D^j(-\lambda^a)\right)^N, \tag{1.69}$$

where $q_{\mu\nu}$ is the component of the matrix q^j such that $D^j(\lambda^a) = \exp(\mathrm{i}q^j)$ and that $N = 0$ for a vertex in the even sublattice and $N = 1$ for the odd one. Then we have immediately that for SU(N) group the generator for gauge transformation of fermion fields is

$$Q^j_{\mathrm{SU}(N)} = \psi_\mu^\dagger T^j_{\mu\nu}\psi_\nu, \tag{1.70}$$

and for U(1) group the generator is

$$Q_{\mathrm{U}(1)} = \psi^\dagger\psi - \frac{1}{2}\left(1-(-1)^N\right). \tag{1.71}$$

Since our model is invariant by the local gauge transformations, the physical states $|\mathrm{phy}\rangle$ satisfy

$$V_{\boldsymbol{n}}(\lambda^a)|\mathrm{phy}\rangle = \mathrm{e}^{\mathrm{i}\lambda^a q^a}|\mathrm{phy}\rangle, \tag{1.72}$$

where q_a is just a real number and the transformation comes to a global phase shift. The generator for local gauge transformation reads

$$G^a_{\boldsymbol{n}} = \sum_i L^a_i + \sum_o R^a_o + Q^a_{\boldsymbol{n}}. \tag{1.73}$$

Since the Hamiltonian is gauge invariant, it commutes with $G^a_{\boldsymbol{n}}$. The Gauss's law is therefore recovered

$$G^a_{\boldsymbol{n}}|q^a\rangle = q^a|q^a\rangle. \tag{1.74}$$

Here q^a can be viewed as some static, external charge distribution.

2 Lattice Schwinger model and its simulation

2.1 The model

Schwinger model is the lattice model for QED in 1+1 dimensions. The gauge group is U(1) and the only generator is 1 with no structure constants. The magnetic field is also absent. The Hamiltonian is then

$$\begin{aligned}H = &-\frac{\mathrm{i}}{2a}\sum_k\sum_f\left(\psi^\dagger_{k+1,f}\mathrm{e}^{\mathrm{i}\Lambda_k}\psi_{k,f} - \mathrm{h.c.}\right)\\ &+\sum_k\sum_f\left((-1)^k m_f + \kappa_f\right)\psi^\dagger_{k,f}\psi_{k,f} + \frac{ag^2}{2}\sum_k R_k^2,\end{aligned} \tag{2.1}$$

where f is the flavour index and κ is the chemical potential. Note that the continuous Schwinger model is simply the 1+1 dimensional QED model. In our lattice model, for the ground state $|0\rangle$ that $R_k|0\rangle = 0$, the commutation relation $[R_k, U_j] = \delta_{kj} U_j$ gives that

$$R_k (U_j)^{r_k} |0\rangle = [R_k, (U_j)^{r_k}] |0\rangle = \delta_{kj} r_k (U_k)^{r_k} |0\rangle , \tag{2.2}$$

where $U_k = \mathrm{e}^{\mathrm{i}\Lambda_k}$. So we can define $|r_k\rangle = (U_k)^{r_k}|0\rangle$. It is clear that $R_k|r_k\rangle = r_k|r_k\rangle$ and $L_k|r_k\rangle = -r_k|r_k\rangle$ for $L_k = -R_k$. Substitute $L = -R$ in U(1) case into the Gauss's law component (1.73), the divergence $\nabla \cdot \boldsymbol{E}$ is recovered in the continuous limit.

For simplicity, consider the single flavour model without chemical potential. The dimensionless Hamiltonian now is

$$W = \frac{2H}{ag^2} = -\mathrm{i}x \sum_{n=1}^{N-1} \left(\psi_{n+1}^\dagger U_n \psi_n - \text{h.c.} \right) + \mu \sum_{n=1}^{N} (-1)^n \psi_n^\dagger \psi_n + \sum_{n=1}^{N-1} R_n^2, \tag{2.3}$$

where $x = 1/(ag)^2$ and $\mu = m/ag^2$. The link operator $U_n = \exp(\mathrm{i}\Lambda_n)$, with the commutation relation $[\Lambda_n, R_m] = \mathrm{i}\delta_{nm}$, $\Lambda_n \in [0, 2\pi]$. The physical states fulfill the Gauss's law $G_n|q_n\rangle = q_n|q_n\rangle$, with $G_n = L_{n-1} + R_n + \psi_n^\dagger \psi_n - (1 - (-1)^n)/2$.

References

[1] Stefan Kühn. *Quantum and classical simulation of High Energy Physics.* PhD thesis, Munich, Tech. U., 10 2017.

[2] Kenneth G. Wilson. Confinement of quarks. *Phys. Rev. D*, 10:2445–2459, Oct 1974.

[3] John Kogut and Leonard Susskind. Hamiltonian formulation of wilson's lattice gauge theories. *Phys. Rev. D*, 11:395–408, Jan 1975.

[4] H.B. Nielsen and M. Ninomiya. A no-go theorem for regularizing chiral fermions. *Physics Letters B*, 105(2):219–223, 1981.

[5] H.B. Nielsen and M. Ninomiya. Absence of neutrinos on a lattice: (i). proof by homotopy theory. *Nuclear Physics B*, 185(1):20–40, 1981.

[6] H.B. Nielsen and M. Ninomiya. Absence of neutrinos on a lattice: (ii). intuitive topological proof. *Nuclear Physics B*, 193(1):173–194, 1981.

[7] Erez Zohar and Michele Burrello. Formulation of lattice gauge theories for quantum simulations. *Phys. Rev. D*, 91:054506, Mar 2015.