NOTES FOR PHYSICS
CREATED: AUGUST 13, 2021
LAST MODIFIED: DECEMBER 23, 2021

Note on Quantum Simulation of Lattice Gauge Theory

Yifei Wang

School of Physics, Peking University,
No.5 Yiheyuan Road, Beijing 100871, P. R. China

E-mail: wang_yifei@pku.edu.cn

ABSTRACT: This note is about the quantum simulation of lattice gauge theory. We first introduce the general formalism of continuous gauge field theory and lattice gauge theory. To perform the quantum simulation, we may need to map the theory to some finite dimensional model. We introduce some mostly used mapping paradigms after the discussion of fundamental theory. Based on these finite dimensional models, several experiments were proposed and some of them were realised, important examples of which were also included in this note.

Throughout this note we take $c = \hbar = k_B = 1$. Einstein summation convention is taken unless otherwise specified. The Lorentz metric for flat spacetime is assumed as $\eta_{\mu\nu} = \mathrm{diag}(-1, +1, +1, +1)$.

Contents

1 Yang-Mills theory

This section is based on the review in [1, 2]. The original text by Yang and Mills is [3].

The well known Lagrangian density of a free Dirac fermion is given by

$$\mathcal{L} = \overline{\psi}(x)\left(\mathrm{i}\gamma^{\mu}\partial_{\mu} - m\right)\psi(x). \tag{1.1}$$

Here ψ is a spinor of some group G, γ^{μ}'s are the Dirac matrices fulfilling the anti-commutation relation $\{\gamma^{\mu}, \gamma^{\nu}\} = -2\eta^{\mu\nu}$ with η being the metric tensor, and $\overline{\psi} = \psi^{\dagger}\gamma^{0}$. This Lagrangian density is invariant under global symmetry transformation defined by the same representation of G as the field ψ,

$$\psi(x) \to V\psi(x), \quad \overline{\psi}(x) \to \overline{\psi}(x)V^{\dagger}. \tag{1.2}$$

We now restrict ourselves to the compact Lie groups SU(N) and U(1) with generators t^a fulfilling

$$\left[t^a, t^b\right] = \mathrm{i}f^{abc}t^c, \tag{1.3}$$

where f^{abc} are the antisymmetric structure constants of the group. We further impose the normalization condition

$$\mathrm{tr}\left(t^a t^b\right) = \frac{1}{2}\delta^{ab}. \tag{1.4}$$

We want to construct a theory that is invariant under local symmetry transformations

$$\psi(x) \to V(x)\psi(x), \quad \overline{\psi}(x) \to \overline{\psi}(x)V^\dagger(x). \tag{1.5}$$

Clearly the derivative along the direction n^μ

$$n^\mu \partial_\mu \psi(x) = \lim_{h\to 0} \frac{\psi(x+hn) - \psi(x)}{h} \tag{1.6}$$

does not have a simple transformation behaviour. To compensate for the different transformations at different spacetime points, we introduce the comparator $U(x', x)$ which is a unitary matrix transformed as

$$U(x', x) \to V(x')U(x', x)V^\dagger(x) \tag{1.7}$$

and $U(x, x) = \mathbb{I}$. Hence, the covariant derivative

$$n^\mu D_\mu \psi(x) := \lim_{h\to 0} \frac{\psi(x+hn) - U(x+hn, x)\psi(x)}{h} \tag{1.8}$$

has the simple transformation. Now expand the comparator and introduce the connection $A^a_\mu(x)$ and the coupling constant g

$$U(x+hn, x) = \mathbb{I} + \mathrm{i}ghn^\mu A^a_\mu(x)t^a + \mathcal{O}(h^2). \tag{1.9}$$

The comparator along a lath $\mathcal{C}$ between two points y and z is given by

$$U(z, y) = \mathcal{P}\exp\left(\mathrm{i}g \int_{\mathcal{C}} \mathrm{d}x^\mu \, A^a_\mu\,(x)\, t^a\right), \tag{1.10}$$

where P indicates path ordering. It follows from the transformation of the comparator Eq. (1.7) that the gauge field $A_\mu = A^a_\mu t^a$ transforms as

$$A_\mu(x) \to V(x)\left(A_\mu(x) + \frac{\mathrm{i}}{g}\partial_\mu\right)V^\dagger(x). \tag{1.11}$$

And we can use the gauge field to write the covariant derivative explicitly

$$D_\mu = \partial_\mu - \mathrm{i}gA_\mu(x). \tag{1.12}$$

Define the field strength

$$\begin{aligned} F_{\mu\nu}(x) &:= \frac{\mathrm{i}}{g}\left[D_\mu, D_\nu\right] \\ &= \partial_\mu A_\nu - \partial_\nu A_\mu - \mathrm{i}g\left[A_\mu, A_\nu\right]. \end{aligned} \tag{1.13}$$

It is clear that $F_{\mu\nu}$ transforms similar to D_μ, that is,

$$F_{\mu\nu}(x) \to V(x)F_{\mu\nu}(x)V^\dagger(x). \tag{1.14}$$

Therefore we can construct the Yang-Mills Lagrangian density which is invariant under local symmetry transformation

$$\mathcal{L}_{\mathrm{YM}} = -\frac{1}{2} \operatorname{tr} \left(F^{\mu\nu} F_{\mu\nu} \right). \tag{1.15}$$

To express it more explicitly, we can expand the field strength in terms of the generator matrices,

$$F_{\mu\nu} = F^a_{\mu\nu} t^a, \quad F^a_{\mu\nu} = 2 \operatorname{tr} \left(F_{\mu\nu} t^a \right). \tag{1.16}$$

The explicit form of $F^a_{\mu\nu}$ is

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu. \tag{1.17}$$

As a result,

$$\mathcal{L}_{\mathrm{YM}} = -\frac{1}{4} F^{a,\mu\nu} F^a_{\mu\nu}. \tag{1.18}$$

The classical Lagrangian density including the massive fermions and the massless gauge bosons as the mediator for interaction hence reads

$$\mathcal{L} = \overline{\psi}(x) \left(\mathrm{i}\gamma^\mu D_\mu - m \right) \psi(x) - \frac{1}{4} F^{a,\mu\nu} F^a_{\mu\nu}. \tag{1.19}$$

The Euler-Lagrange equation of the Lagrangian (1.19) are given by

$$\begin{gathered} \left(\mathrm{i}\gamma^\mu D_\mu - m \right) \psi(x) = 0, \\ \partial^\mu F^a_{\mu\nu}(x) + g f^{abc} A^{b,\mu}(x) F^c_{\mu\nu}(x) = -g\overline{\psi}(x)\gamma_\nu t^a \psi(x). \end{gathered} \tag{1.20}$$

It is worth mentioning that if the symmetry group is U(1), the first eqution becomes the equation of motion of a fermion in the external electromagnetic field and the second equation reduces to the Maxwell equation with the source flow given by $J_\mu = -g\overline{\psi}\gamma_\mu\psi$. Note that the sourceless Maxwell equations are just equivalent to the antisymmetric property of the field $F_{\mu\nu}$.

Path integral formalism can be used to quantize the theory above. Ground state expectation values of an observable O are given by the path integral

$$\langle O \rangle = \frac{1}{Z} \int \mathcal{D}A \, \mathcal{D}\psi \, \mathcal{D}\overline{\psi} \, O \exp \left(\mathrm{i}S \right), \tag{1.21}$$

where $S = \int \mathrm{d}^4 x \, \mathcal{L}$. Applying the Wick rotation $t \to \mathrm{i}\tau$ makes the spacetime a Euclidean space, in which the expectation (1.21) is analogous to that obtained in statistical mechanics. This analogy allows Monte Carlo methods to be applied to the lattice gas theory.

Another quantization procedure is the canonical quantization in which we divide the spacetime into slices with equal time. We have the momenta for fermion field and gauge field respectively,

$$\begin{aligned} \pi^l_F &= \frac{\partial \mathcal{L}}{\partial \left(\partial_0 \psi^l(x) \right)} = \mathrm{i}\psi^{\dagger l}(x), \\ \pi^{a,0} &= \frac{\partial \mathcal{L}}{\partial \left(\partial_0 A^a_0(x) \right)} = 0, \\ \pi^{a,k} &= \frac{\partial \mathcal{L}}{\partial \left(\partial_0 A^a_k(x) \right)} = F^{a,k0}(x), \end{aligned} \tag{1.22}$$

where $k = 1, 2, 3$ refers to the spatial indices of the gauge field. The Hamiltonian is therefore

$$
\begin{aligned}
H &= \int \mathrm{d}^3x \, \left(\pi_F^l(\boldsymbol{x}) \partial_0 \psi^l(\boldsymbol{x}) + \pi^{a,k}(\boldsymbol{x}) \partial_0 A_k^a(\boldsymbol{x}) - \mathcal{L} \right) \\
&= \int \mathrm{d}^3x \, \psi^\dagger(\boldsymbol{x}) \left(-\mathrm{i}\boldsymbol{\alpha} \cdot \nabla - g\boldsymbol{\alpha} \cdot \boldsymbol{A}(\boldsymbol{x}) + \gamma^0 m \right) \psi(\boldsymbol{x}) \\
&\quad + \int \mathrm{d}^3x \, \left(-\frac{1}{2} \pi^{a,k}(\boldsymbol{x}) \pi_k^a(\boldsymbol{x}) + \frac{1}{4} F^{a,jk}(\boldsymbol{x}) F_{jk}^a(\boldsymbol{x}) \right) \\
&\quad - \int \mathrm{d}^3x \, A_0^a(\boldsymbol{x}) \left(\partial^k \pi_k^a(\boldsymbol{x}) + g f^{abc} A^{b,k}(\boldsymbol{x}) \pi_k^c(\boldsymbol{x}) + g \psi^\dagger(\boldsymbol{x}) t^a \psi(\boldsymbol{x}) \right),
\end{aligned}
\tag{1.23}
$$

where $\boldsymbol{\alpha}^k = \gamma^0 \gamma^k$ and $\boldsymbol{A}^k = A^{a,k} t^a$. Since $\pi^{a,0} = 0$, we choose the temperal or Weyl gauge that the temporal component of the gauge field is zero, $A_0^a = 0$. We can now impose the canonical quantization condition

$$
\left\{ \psi^a(\boldsymbol{x}), \psi^{\dagger b}(\boldsymbol{y}) \right\} = \delta(\boldsymbol{x} - \boldsymbol{y}) \delta_{ab}, \quad \left\{ \psi^a(\boldsymbol{x}), \psi^b(\boldsymbol{y}) \right\} = 0 \tag{1.24}
$$

for fermions and

$$
\left[A_k^a(\boldsymbol{x}), \pi_j^b(\boldsymbol{y}) \right] = \mathrm{i} \delta_{ab} \delta_{kj} \delta(\boldsymbol{x} - \boldsymbol{y}), \quad \left[A_k^a\left(\boldsymbol{x}\right), A_j^b\left(\boldsymbol{y}\right) \right] = \left[\pi_k^a\left(\boldsymbol{x}\right), \pi_j^b\left(\boldsymbol{y}\right) \right] = 0 \tag{1.25}
$$

for gauge bosons. This gauge choice and canonical quantization condition yields the quantized Hamiltonian

$$
\begin{aligned}
H = \int \mathrm{d}^3x \, \psi^\dagger(\boldsymbol{x}) \left(-\mathrm{i}\boldsymbol{\alpha} \cdot \nabla - g\boldsymbol{\alpha} \cdot \boldsymbol{A}(\boldsymbol{x}) + \gamma^0 m \right) \psi(\boldsymbol{x}) \\
+ \int \mathrm{d}^3x \, \left(-\frac{1}{2} \pi^{a,k}(\boldsymbol{x}) \pi_k^a(\boldsymbol{x}) + \frac{1}{4} F^{a,jk}(\boldsymbol{x}) F_{jk}^a(\boldsymbol{x}) \right).
\end{aligned}
\tag{1.26}
$$

Introduce the vector potential $\boldsymbol{A}^a\left(\boldsymbol{x}\right)$ whose k-component is $A^{a,k}\left(\boldsymbol{x}\right)$, we can define the colour-electric and colour-magnetic field as

$$
\begin{aligned}
\boldsymbol{E}^a\left(\boldsymbol{x}\right) &:= -\partial_t \boldsymbol{A}^a\left(\boldsymbol{x}\right), \\
\boldsymbol{B}^a\left(\boldsymbol{x}\right) &:= \nabla \times \boldsymbol{A}^a\left(\boldsymbol{x}\right) + \frac{1}{2} g f^{abc} \boldsymbol{A}^b\left(\boldsymbol{x}\right) \times \boldsymbol{A}^c\left(\boldsymbol{x}\right),
\end{aligned}
\tag{1.27}
$$

whose components are just

$$
E^{a,k} = -F^{a,k0}, \quad B^{a,k} = \frac{1}{2} \epsilon_{ijk} F^{a,ij}. \tag{1.28}
$$

Therefore we can rewrite second line of the Hamiltonian (1.26) as $H_{\mathrm{el}} + H_{\mathrm{mag}}$, where

$$
\begin{aligned}
H_{\mathrm{el}} &:= \frac{1}{2} \int \mathrm{d}^3x \, \boldsymbol{E}^a\left(\boldsymbol{x}\right) \cdot \boldsymbol{E}^a\left(\boldsymbol{x}\right) \\
H_{\mathrm{mag}} &:= \frac{1}{2} \int \mathrm{d}^3x \, \boldsymbol{B}^a\left(\boldsymbol{x}\right) \cdot \boldsymbol{B}^a\left(\boldsymbol{x}\right).
\end{aligned}
\tag{1.29}
$$

The generator for local gauge transformation is the Gauss law components

$$
G^a\left(\boldsymbol{x}\right) = \partial^k \pi_k^a(\boldsymbol{x}) + g f^{abc} A^{b,k}(\boldsymbol{x}) \pi_k^c(\boldsymbol{x}) + \rho^a\left(\boldsymbol{x}\right), \tag{1.30}
$$

where $\rho^a(\boldsymbol{x}) = g\psi^\dagger(\boldsymbol{x})t^a\psi(\boldsymbol{x})$ are the charge density components. The operators satisfy the commutation relation

$$\left[G^a(\boldsymbol{x}), G^b(\boldsymbol{y})\right] = \mathrm{i}gf^{abc}G^a(\boldsymbol{x})\,\delta(\boldsymbol{x}-\boldsymbol{y}). \tag{1.31}$$

Since we have chosen that the temperal component of the gauge field be zero, the Gauss's law is absent from our Hamiltonian formulation. We should add an additional constraint on the physical states that the local gauge transformation yield a global gauge transformation and thus has no physical effect. This condition is equivalent to

$$G^a(\boldsymbol{x})\,|\Psi\rangle = q^a(\boldsymbol{x})\,|\Psi\rangle\,. \tag{1.32}$$

This is nothing but the Gauss's law. Since G^a commutes with the Hamiltonian, the external charge distribution $q^a(\boldsymbol{x})$ is conserved.

2 Lattice gauge theory

2.1 Path integral formulation of lattice gauge theory

Wilson [4] proposed the path integral formulation of the lattice gauge theory to study the confinement of quarks. We introduce this theory here following the book [5].

2.1.1 General formalism

First, the d dimensional Euclidean spacetime is discretised into a d dimensional lattice with a lattice space a. Sites on this lattice is labelled by x taking discrete values $(n_1a, \cdots, n_da)$. Suppose, though we have not pointed out the true meaning yet, that there is a matter field $\psi(x)$ on site x, transforming under local gauge transformation as

$$\psi(x) \to V(x)\psi(x). \tag{2.1}$$

Like the comparator in continuous case, define the $U_\mu(x)$ linking the sites x and $x + an^\mu$ which under gauge transformations transforms as

$$U_\mu(x) \to V(x + an^\mu)\,U_\mu(x)V^\dagger(x). \tag{2.2}$$

Also we require that $U_{-\mu}(x + an^\mu) = U_\mu^\dagger(x)$. For compact Lie groups U(1) and SU(N), we can write $U_{\boldsymbol{n},k}$ in the exponential form

$$U_\mu(x) = \exp\left(\mathrm{i}\Lambda_\mu^a(x)t^a\right), \tag{2.3}$$

where $\Lambda_\mu^a(x)$ are parameters. The infinitesimal form is

$$U_\mu(x) = \mathbb{I} + \mathrm{i}\Lambda_\mu^a(x)t^a + \mathcal{O}\left(\Lambda^2\right). \tag{2.4}$$

Comparing to the continuous case, we have

$$\frac{1}{ag}\Lambda_\mu^a(x) \to A_\mu^a(x)\,. \tag{2.5}$$

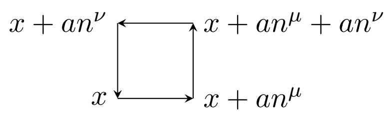


Figure 1. A plaquette p with boundary ∂p.

A plaquette is a smallest square on the lattice, denoted by p, see Figure 1. The (oriented) boundary of a plaquette p is denoted by ∂p. The comparators over a plaquette is defined as

$$U(\partial p) \equiv U_\nu^\dagger(x) U_\mu^\dagger(x + an^\nu) U_\nu(x + an^\mu) U_\mu(x). \tag{2.6}$$

Under a local gauge transformation, $U(\partial p)$ becomes $V(x)U(\partial p)V^\dagger(x)$, and hence the trace is invariant. A gauge invariant, local action should be constructed by the trace of $U(\partial p)$'s. We can require that the continuum limit restores the Yang-Mills Lagrangian (1.19). For

$$\Lambda_\mu^a\left(x + an^\nu\right) = \Lambda_\mu^a\left(x\right) + a\partial_\nu \Lambda_\mu^a(x) + \mathcal{O}\left(a^2\right), \tag{2.7}$$

expanding the $U(\partial p)$ using Baker-Campbell-Hausdorff formula, we have

$$\begin{aligned} U(\partial p) &= \exp\Big(\mathrm{i}a\left(\partial_\mu \Lambda_\nu^a(x) - \partial_\nu \Lambda_\mu^a(x)\right) t^a - \mathrm{i} f^{abc} \Lambda_\nu^b(x) \Lambda_\mu^c(x) t^a + \mathcal{O}\left(a^3\right)\Big) \\ &= \exp\left(\mathrm{i}a^2 g F_{\mu\nu}^a t^a + \mathcal{O}\left(a^3\right)\right). \end{aligned} \tag{2.8}$$

Note that $\Lambda \sim \mathcal{O}(a)$ in the continuum limit. The trace is then

$$\operatorname{tr} U(\partial p) = \operatorname{tr}\mathbb{I} - \frac{1}{4} a^4 g^2 \sum_a \left(F_{\mu\nu}^a\right)^2 \tag{2.9}$$

Furthermore, in the continuum limit, we perform the replacement

$$a^d \sum_x \to \int \mathrm{d}^4 x. \tag{2.10}$$

Therefore, we can take the action as[1]

$$S[U] = -\frac{a^{d-4}}{2g^2} \sum_p \left(\operatorname{tr} U(\partial p) - \operatorname{tr}\mathbb{I}\right), \tag{2.11}$$

where 1/2 comes from the fact that two orients of the boundary of a plaquette doubles the result.

The partition function is defined by

$$Z = \int \prod_{x,\mu} \mathrm{d}U_\mu(x)\, \mathrm{e}^{-S[U]}. \tag{2.12}$$

And the average for a gauge-invariant functional of $U_\mu(x)$ is

$$\langle F[U] \rangle = Z^{-1} \int \prod_{x,\mu} \mathrm{d}U_\mu(x)\, \mathrm{e}^{-S[U]} F[U]. \tag{2.13}$$

[1]The minus sign comes from the Wick rotation to form Euclidean field theory.

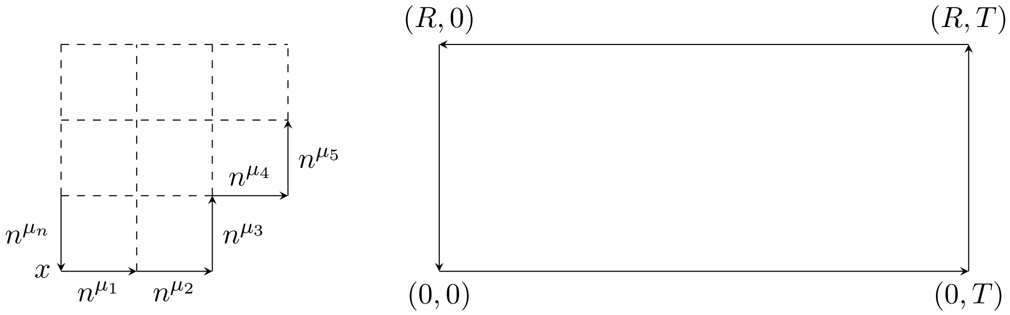


Figure 2. Left: An illustration of a closed loop C. Note that the loop is not necessarily 2D. Right: Rectangular loop of size $R \times T$.

Here we make a mathematical remark about the integral. Assume that the gauge group is a Lie group. The $\mathrm{d}U$ is a left-invariant volume element called Haar measure, which is also right-invariant is the Lie group is compact. For a Lie group G with a parameter space $\mathbb{R}^r$, the general expression for Haar measure is [6]

$$\mathrm{d}\mu(g) = \det{}^{-1}\left(\frac{\partial f^j(\boldsymbol{g},\boldsymbol{x})}{\partial x^i}\right)\bigg|_{\boldsymbol{x}=0} \mathrm{d}^r g, \tag{2.14}$$

where $g \in G$, $\mathrm{d}^r g$ is the Euclidean volume element in the parameter space, and f is the function that $g(\boldsymbol{x})g(\boldsymbol{y}) = g\left(f(\boldsymbol{x},\boldsymbol{y})\right)$. For example, for two n-dimensional matrices A and B,

$$\frac{\partial\,(AB)_{ij}}{\partial B_{kl}} = A_{ik}\delta_{jl}, \tag{2.15}$$

so if we arrange ij to a 1-dimensional array as $11, 21, \cdots n1, 12, 22, \cdots, n2, \cdots nn$, the Jacobi matrix is block diagonal

$$\left(\frac{\partial(AB)}{\partial B}\right)_{ij}^{kl} = \begin{pmatrix} A & 0 & \cdots & 0 \\ 0 & A & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A \end{pmatrix}. \tag{2.16}$$

Therefore for group $U(N)$ the Haar measure is

$$\int \mathrm{d}U \;\cdots = \int_{-\infty}^{+\infty} \prod_{i,j} \mathrm{d}\,\mathrm{Re}\,U_{ij}\,\mathrm{d}\,\mathrm{Im}\,U_{ij}\,\delta^{(N^2)}\left(UU^\dagger - \mathbb{I}\right)\cdots. \tag{2.17}$$

2.1.2 Wilson loops, strong-coupling expansion, area law and confinement

Define a loop

$$C = \{x; \mu_1, \ldots, \mu_n\}, \tag{2.18}$$

that $n^{\mu_1} + \cdots + n^{\mu_n} = 0$, see Figure 2. The lattice phase factor $U(C)$ is given by

$$U(C) = U_{\mu_n}\left(x + an^{\mu_1} + \cdots + an^{\mu_{n-1}}\right)\cdots U_{\mu_2}\left(x + an^{\mu_1}\right)U_{\mu_1}(x). \tag{2.19}$$

The trace of $U(C)$, which is gauge invariant, is called the Wilson loop. The average of a Wilson loop is then

$$\begin{aligned} W(C) &\equiv \langle \operatorname{tr} U(C) \rangle \\ &= Z^{-1} \int \prod_{x,\mu} \mathrm{d}U_\mu(x)\, \mathrm{e}^{-S[U]} \operatorname{tr} U(C), \end{aligned} \tag{2.20}$$

where N is the row/column number of the matrix U.

Consider a rectangular loop shown as the right part of Figure 2. Taking the gauge that $\Lambda_4 = 0$ so that only the vertical segments of the rectangle in Figure 2 contributes. Denoting

$$\Psi_{ij}(t) = \left(\mathcal{P} \exp \left(\mathrm{i} \int_0^R \mathrm{d}z_1\, \Lambda_1^a(z_1, \ldots, t) t^a \right) \right)_{ij}, \tag{2.21}$$

then we have

$$W\left(R \times T\right) = \left\langle \operatorname{tr} \Psi^\dagger(0) \Psi(T) \right\rangle. \tag{2.22}$$

Inserting a complete set of energy eigen states we have

$$W\left(R \times T\right) = \sum_n \left| \langle \Psi_{ij}(0) | n \rangle \right|^2 \mathrm{e}^{-E_n T}, \tag{2.23}$$

where E_n is the eigen energy of $|n\rangle$. As $T \to \infty$, only the lowest energy survive, and we have

$$W\left(R \times T\right) \to \mathrm{e}^{-E_0 T}, \quad \text{as } T \to \infty. \tag{2.24}$$

Here E_0 can be interpreted as the interacting potential energy of a quark-antiquark pair separated by a distance R. To be further discussed.

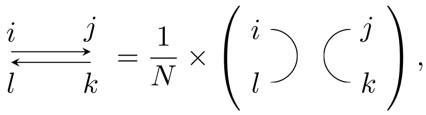
We can perform a strong coupling expansion, that is, expanding in powers of $\beta = a^{d-4}/2g^2$, to calculate the average of the Wilson loop. Normalize the Haar measure such that

$$\int \mathrm{d}U = 1. \tag{2.25}$$

We can also prove that the nonvanishing term should have the form

$$\int \mathrm{d}U U^i{}_j U^{\dagger k}{}_l = \frac{1}{N} \delta^i_l \delta^k_j. \tag{2.26}$$

The proof is to first consider parity, then to consider if $i = l$ and $k = j$ to determine the coefficient. This equation can be explained graphically,

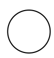


(2.27)

where

 $= \delta^i_j.$ (2.28)

Furthermore,

 $= \delta^i_i = N.$ (2.29)

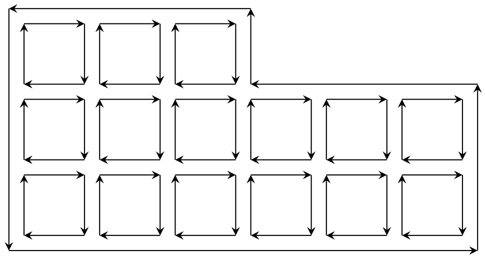


Figure 3. The lowest order term of a general Wilson loop.

For a plaquette, the average of the Wilson loop is

$$W\left(\partial p\right)=\frac{\int\prod_{x,\mu}\mathrm{d}U_\mu(x)\,\left(1+\beta\sum_{p'}\operatorname{tr}U\left(\partial p'\right)\right)\operatorname{tr}U\left(\partial p\right)}{\int\prod_{x,\mu}\mathrm{d}U_\mu(x)\,\left(1+\beta\sum_{p'}\operatorname{tr}U\left(\partial p'\right)\right)}+\mathcal{O}\left(\beta^2\right). \tag{2.30}$$

The constant term in the action has been suppressed. For the denominator, no terms like (2.26) occurs so the value is 1. For the numerator, the nonvanishing term, which is also the first order approximation of $W(\partial p)$, is

$$W\left(\partial p\right)=\beta\times\boxed{\boxed{\partial p'}}\;\partial p\;+\mathcal{O}\left(\beta^2\right)=\beta\frac{1}{N^4}\times\begin{pmatrix}\bigcirc & \bigcirc\\ \bigcirc & \bigcirc\end{pmatrix}+\mathcal{O}\left(\beta^2\right)=\beta+\mathcal{O}\left(\beta^2\right). \tag{2.31}$$

To get the first order approximation to a more general loop, expand the average (2.20) so that the loop is covered by plaquettes with a minimal area $A(C)$, see Figure 3. Here the area unit is taken as the area of a plaquette. The value is then

$$W(C)=\beta^{A(C)}\frac{A(C)!}{A(C)!}\approx\exp\left(-KA(C)\right), \tag{2.32}$$

where the string tension K is given by

$$K=\ln\frac{2g^2}{a^{d-4}}. \tag{2.33}$$

For the rectangular loop $R\times T$, the Wilson loop is

$$W(R\times T)=\mathrm{e}^{-KRT/a^2}, \tag{2.34}$$

which is exponential in R and T.

The argument for confinement is that in the continuous Yang-Mills theory, the weight for a quark path L contains a factor $\exp\left(\int_L \mathrm{d}x^\mu\, \mathrm{i}gA_\mu(x)\right)$. For generating and evolving a quark and an antiquark, this factor in the lattice formulation is nothing but the Wilson loop. The exponential decay with the distance R binds the quark-antiquark pair together. Another interpretation is that the potential energy of a quark-antiquark pair is $E(R)=KR/a^2$, which is a confined potential.

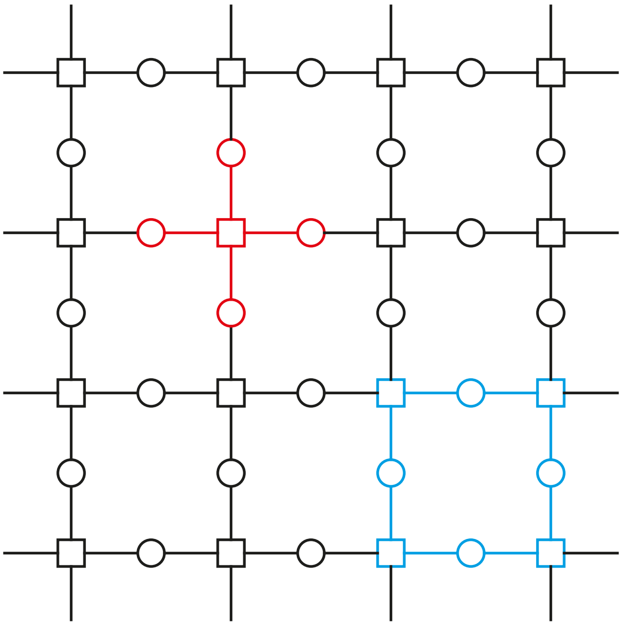


Figure 4. The spatial part of a 2+1 D lattice model. Fermion fields are located on the sites (squares), comparators U link the adjacent sites. The blue square is a plaquette. When doing the quantization, the squares are sites carrying fermionic Hilbert space, the circles are links carrying the gauge field Hilbert space. The red star is a vertex where a local gauge transformation is defined on.

2.2 Hamiltonian formulation of lattice gauge theory

Here we discuss the hamiltonian formulation of the lattice gauge field intruduced by Kogut and Susskind [7], which is more convenient for quantum simulation and tensor network (TN).

A $d+1$ dimensional lattice consists of a d dimensional spatial lattice and a continuous or discrete time dimension. The spatial part of a lattice is shown in Figure 4. Suppose, though we have not pointed out the true meaning yet, that there is a matter field $\psi_{\boldsymbol{n}}$ on each site, transforming under local gauge transformation as

$$\psi_{\boldsymbol{n}} \to V_{\boldsymbol{n}} \psi_{\boldsymbol{n}}. \tag{2.35}$$

Like the comparator in continuous case, define the $U_{\boldsymbol{n},k}$ linking the sites $\boldsymbol{n}$ and $\boldsymbol{n}+e_k$ which under gauge transformations transforms as

$$U_{\boldsymbol{n},k} \to V_{\boldsymbol{n}+e_k} U_{\boldsymbol{n},k} V_{\boldsymbol{n}}^{\dagger}. \tag{2.36}$$

Also we require that $U_{\boldsymbol{n}+e_k,-k} = U_{\boldsymbol{n},k}^{\dagger}$. For compact Lie groups U(1) and SU(N), we can write $U_{\boldsymbol{n},k}$ in the exponential form

$$U_{\boldsymbol{n},k} = \exp\left(\mathrm{i}\Lambda_{\boldsymbol{n},k}^{a} t^{a}\right), \tag{2.37}$$

where $\Lambda_{\boldsymbol{n},k}^{a}$ are parameters. The infinitesimal form is

$$U_{\boldsymbol{n},k} = \mathbb{I} + \mathrm{i}\Lambda_{\boldsymbol{n},k}^{a} t^{a} + \mathcal{O}\left(\Lambda^2\right). \tag{2.38}$$

Comparing to the continuous case, we have

$$\frac{1}{ag}\Lambda^a_{\boldsymbol{n},k} \to A^a_k\left(\boldsymbol{x}\right). \tag{2.39}$$

Now we consider the colour magnetic field Hamiltonian. A gauge invariant term is the trace of the product of $U_{\boldsymbol{n},k}$ around a closed loop. Locality leads us to make the smallest loop possible, which are plaquettes on the lattice, denoted by p. See Figure 4. We have

$$U_p \equiv U_{\boldsymbol{n}+e_j,-j}U_{\boldsymbol{n}+e_k+e_j,-k}U_{\boldsymbol{n}+e_k,j}U_{\boldsymbol{n},k} = U^\dagger_{\boldsymbol{n},j}U^\dagger_{\boldsymbol{n}+e_j,k}U_{\boldsymbol{n}+e_k,j}U_{\boldsymbol{n},k}. \tag{2.40}$$

To see the continuous limit, expand

$$\begin{aligned}\Lambda^a_{\boldsymbol{n}+e_k,j} &= \Lambda^a_{\boldsymbol{n},j} + a\partial_k\Lambda^a_{\boldsymbol{n},j} + \mathcal{O}\left(a^2\right)\\ \Lambda^a_{\boldsymbol{n}+e_j,k} &= \Lambda^a_{\boldsymbol{n},k} + a\partial_j\Lambda^a_{\boldsymbol{n},k} + \mathcal{O}\left(a^2\right)\end{aligned} \tag{2.41}$$

Up to the first order in a,

$$\begin{aligned}\operatorname{tr} U_p &= \operatorname{tr}\exp\left(\mathrm{i}a\left(\partial_k\Lambda^a_{\boldsymbol{n},j} - \partial_j\Lambda^a_{\boldsymbol{n},k}\right)t^a - \mathrm{i}f^{abc}\Lambda^b_{\boldsymbol{n},j}\Lambda^c_{\boldsymbol{n},k}t^a\right)\\ &= \operatorname{tr}\exp\left(\mathrm{i}a^2gF^a_{kj}t^a\right) \quad \text{at } a\to 0.\end{aligned} \tag{2.42}$$

Then at the continuous limit

$$\operatorname{tr} U_p = \operatorname{tr}\mathbb{I} - \frac{1}{4}a^4g^2\sum_a\left(F^a_{kj}\right)^2. \tag{2.43}$$

Therefore we can construct the magnetic Hamiltonian as

$$H_{\mathrm{mag}} = \frac{a^{d-4}}{g^2}\sum_p\left(\operatorname{tr}\mathbb{I} - \frac{1}{2}\left(\operatorname{tr} U_p + \text{h.c.}\right)\right) \tag{2.44}$$

If we are going to construct the Lagrangian form of a four-dimensional Euclidean lattice gauge theory with the time dimension also discrete, what we have done is enough. However, since we are going to construct the Hamiltonian form with a continuous time, we need to restrict the magnetic field to the spatial dimensions, and construct another electric field by canonical quantization. We also need to point out the Hilbert space the operators acting on [8].

When doing the quantization, we know and will discuss intensively in soon that fermions are located on the sites. The natural choice of the Hilbert spaces for comparators to act on are the Hilbert spaces located on the links, see Figure 4. The states in these Hilbert spaces are labelled by representation of the gauge group $|jmn\rangle$ corresponding to $\left(U^j\right)_{mn}$, where the superscript j indicates the j'th representation. For these Hilbert spaces on links we can define the generators for gauge transformations whose effects are left and right actions for the comparator. That is, for generator $L_{\boldsymbol{n},k}$, we have

$$\exp\left(-\mathrm{i}\lambda^aL^a_{\boldsymbol{n},k}\right)\left(U^j_{\boldsymbol{n},k}\right)_{\mu\nu}\exp\left(\mathrm{i}\lambda^aL^a_{\boldsymbol{n},k}\right) = \left(\exp\left(\mathrm{i}\lambda^at^a\right)\right)^j_{\mu\rho}\left(U^j_{\boldsymbol{n},k}\right)_{\rho\nu}, \tag{2.45}$$

and for generator $R^a_{\boldsymbol{n},k}$, we have

$$\exp\left(-\mathrm{i}\lambda^a R^a_{\boldsymbol{n},k}\right)\left(U^j_{\boldsymbol{n},k}\right)_{\mu\nu}\exp\left(\mathrm{i}\lambda^a R^a_{\boldsymbol{n},k}\right)=\left(U^j_{\boldsymbol{n},k}\right)_{\mu\rho}\left(\exp\left(-\mathrm{i}\lambda^a t^a\right)\right)^j_{\rho\nu}. \tag{2.46}$$

The physical meaning is that the effect of the right generator is the same as generating a gauge transformation on site $\boldsymbol{n}$, and the effect of the left generator is the same as generating a transformation on site $\boldsymbol{n}+e_k$. Using Baker-Hausdorff formula we have the commutation relations

$$\begin{aligned}\left[L^a_{\boldsymbol{n},k},\left(U^j_{\boldsymbol{n}',k'}\right)_{\mu\nu}\right]&=-\delta_{\boldsymbol{n}\boldsymbol{n}'}\delta_{kk'}\left(T^a\right)^j_{\mu\rho}\left(U^j_{\boldsymbol{n},k}\right)_{\rho\nu},\\ \left[R^a_{\boldsymbol{n},k},\left(U^j_{\boldsymbol{n}',k'}\right)_{\mu\nu}\right]&=\delta_{\boldsymbol{n}\boldsymbol{n}'}\delta_{kk'}\left(U^j_{\boldsymbol{n},k}\right)_{\mu\rho}\left(T^a\right)^j_{\rho\nu},\end{aligned} \tag{2.47}$$

where T^a is the representation of the Lie algebra and we use indices i,j,l to make clear that it is a matrix product. The commutation relations between the generators are *set* as

$$\begin{aligned}\left[L^a_{\boldsymbol{n},k},L^b_{\boldsymbol{n}',k'}\right]&=\mathrm{i}\delta_{\boldsymbol{n}\boldsymbol{n}'}\delta_{kk'}f^{abc}L^c_{\boldsymbol{n},k},\\ \left[R^a_{\boldsymbol{n},k},R^b_{\boldsymbol{n}',k'}\right]&=\mathrm{i}\delta_{\boldsymbol{n}\boldsymbol{n}'}\delta_{kk'}f^{abc}R^c_{\boldsymbol{n},k},\\ \left[L^a_{\boldsymbol{n},k},R^b_{\boldsymbol{n}',k'}\right]&=0.\end{aligned} \tag{2.48}$$

The quadratic Casimir operator $\boldsymbol{J}^2_{\boldsymbol{n},k}\equiv\sum_a L^a_{\boldsymbol{n},k}L^a_{\boldsymbol{n},k}=\sum_a R^a_{\boldsymbol{n},k}R^a_{\boldsymbol{n},k}$. In the continuous model, the electric field $E^{a,k}=F^{a,0k}=-\pi^{a,k}$ fulfils the commutation relation

$$\left[E^{a,k}(\boldsymbol{x}),U(y+hn,y)\right]=-\delta\left(\boldsymbol{y}-\boldsymbol{x}\right)ghn^k t^a U(y+nh,y). \tag{2.49}$$

By comparing this equation with the commutation relation between $L_{\boldsymbol{n},k}$ and $U_{\boldsymbol{n},k}$, we have the that in the continuous limit

$$a^{1-d}gL_{\boldsymbol{n},k}\to E^{a,k}\left(\boldsymbol{x}\right). \tag{2.50}$$

Therefore, the colour-electric energy is given by

$$H_{\mathrm{el}}=\frac{a^{2-d}g^2}{2}\sum_{\boldsymbol{n},k}\boldsymbol{J}^2_{\boldsymbol{n},k}. \tag{2.51}$$

Now let's take the SU(2) gauge group as an example to show how the electric field is defined. The complete realization will be discussed in later sections.

Example For the link $\boldsymbol{n},k$, the ground state is defined as $|0\rangle$ such that $L|0\rangle=R|0\rangle=0$, the $\boldsymbol{n},k$ subscript is suppressed for simplicity. We take the representation matrix for Lie algebra being J_0, $J_{1/2}$, $J_1\cdots$, where J_j is the angular momentum vector with $(J_j)^2=j(j+1)$. The matrix is taken in the eigenbasis of J^z_j, $|j,m\rangle$, with $m=-j,-j+1,\cdots,j$. Therefore $\left(J^z_j\right)_{lm}=\delta_{lm}m$. Define the state $U^j_{lr}|0\rangle=|jlr\rangle$, then we have

$$\begin{aligned}R^2|jlr\rangle=L^2|jlr\rangle&=\sum_{a\in\{x,y,z\}}\left[L^a,\left[L^a,U^j_{lr}\right]\right]|0\rangle\\ &=\left((J_j)^2\right)_{lm}U^j_{mr}|0\rangle\quad j\text{ not summed}\\ &=j(j+1)|jlr\rangle.\end{aligned} \tag{2.52}$$

And also

$$\begin{aligned}
L^z \left|jlr\right\rangle &= \left[L^z, U^j_{lr}\right] \left|0\right\rangle \\
&= -\left(J^z_j\right)_{lm} U^j_{mr} \left|0\right\rangle \quad j \text{ not summed} \\
&= -l \left|jlr\right\rangle .
\end{aligned} \tag{2.53}$$

Similarly we have $R\left|jlr\right\rangle = r\left|jlr\right\rangle$.

Quark confinement can also be discussed in the Hamiltonian form [7, 9]. Put a quark-antiquark pair on $\boldsymbol{n}$ and $\boldsymbol{n}+\boldsymbol{m}$, which is generated by operators $\psi^\dagger_{\boldsymbol{n}}$ and $\psi_{\boldsymbol{n}+\boldsymbol{m}}$ and a string of U matricis between them,

$$\psi^\dagger_{\boldsymbol{n}} \left(\prod_{\text{from } \boldsymbol{n} \text{ to } \boldsymbol{m}} U\right) \psi_{\boldsymbol{n}+\boldsymbol{m}}. \tag{2.54}$$

The vacuum before placing the pair is the ground state that

$$E\left|0\right\rangle = 0, \tag{2.55}$$

where E is either L or R. Taking SU(2) as an example, after putting the pair, the electric energy can be evaluated by

$$\boldsymbol{J}^2 U^j_{lr} \left|0\right\rangle = j\left(j+1\right) U^j_{lr} \left|0\right\rangle . \tag{2.56}$$

Choosing the fundamental representation, the electric energy is

$$V(R) = \frac{a^{2-d} g^2}{2} \frac{3}{4} \frac{R}{a}, \tag{2.57}$$

which is also linear in the distance $R = |\boldsymbol{m}|$.

2.3 Lattice fermions

2.3.1 The doubling problem and the staggered fermion

The trivial discretization of Hamiltonian (1.26) causes problems. Let us illustrate this problem in the simplest $1+1$ dimensional free massless fermion case. The Hamiltonian of such fermions is

$$H = \int \mathrm{d}x\, \psi^\dagger\left(x\right) \left(-\mathrm{i}\gamma^0\gamma^1\partial_x\right) \psi\left(x\right). \tag{2.58}$$

The equation of motion $\mathrm{i}\partial_t\psi(x) = [\psi(x), H]$ is simple the Dirac equation

$$\partial_t\psi(x) = -\gamma^0\gamma^1\partial_x\psi(x). \tag{2.59}$$

Using the Weyl representation for Dirac matrices

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^5 = \gamma^0\gamma^1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{2.60}$$

and the plain wave ansatz $\psi(x) = \exp\left(\mathrm{i}(Et - px)\right) u(p, E)$, we have the eigenvalue problem

$$\begin{pmatrix} p & 0 \\ 0 & -p \end{pmatrix} u(p, E) = Eu(p, E). \tag{2.61}$$

The solution is the usual dispersion relation $E = \pm p$ with corresponding eigenvectors $u_{\pm}(p, E)$, which are the eigenvectors of γ^5 with well defined chirality ± 1.

A trivial discretization is defined by the maps

$$\psi(x) \to \frac{\widehat{\psi}_n}{\sqrt{a}}, \quad \partial_x \psi(x) \to \frac{1}{2a^{3/2}} \left(\widehat{\psi}_{n+1} - \widehat{\psi}_{n-1} \right), \quad \int \mathrm{d}x \to a \sum_n. \tag{2.62}$$

The resulting lattice Hamiltonian is

$$\widehat{H} = \sum_n \widehat{\psi}_n^{\dagger} \left(-\mathrm{i}\gamma^0 \gamma^1 \frac{1}{2a} \left(\delta_{k,n+1} - \delta_{k,n-1} \right) \right) \widehat{\psi}_k, \tag{2.63}$$

and the equation of motion

$$\partial_t \widehat{\psi}_n = -\gamma^0 \gamma^1 \frac{1}{2a} \left(\widehat{\psi}_{n+1} - \widehat{\psi}_{n-1} \right). \tag{2.64}$$

Still using the plain wave ansatz $\widehat{\psi}_n = \exp\left(\mathrm{i}\left(Et - pna\right)\right) \widehat{u}(p, E)$, we have the eigenvalue equation

$$\begin{pmatrix} \sin(pa)/a & 0 \\ 0 & -\sin(pa)/a \end{pmatrix} \widehat{u}(p, E) = E\widehat{u}(p, E), \tag{2.65}$$

with the momentum restricted to the first Brillouin zone $-\pi/a \leqslant p \leqslant \pi/a$. With the same eigenstates as the continuous case, the corresponding dispersion relation becomes $E = \pm \sin(pa)/a$. Comparing to the continuous case, additional low energy states appear at $p = \pi/a$, and for each chirality there exist a left moving mode and a right moving mode. These new modes do not vanish at the limit $a \to 0$. In general, for a Dirac theory with d spatial dimensions, there occur 2^d zeros in the first Brillion zone, that is, for each dimension the zeros are doubled. This is the so called doubling problem.

Furthermore, it is shown by Nielsen and Ninomiya [10–12] that any local, translationally invariant, hermitian lattice formulation of a chiral fermion gives rise to a equal number of left and right movers. There are several solutions to this problem from which we introduce the staggered fermion approach for the convenience of quantum simulaton and TN.

Contrary to the naive Hamiltonian (2.63), consider a Hamiltonian with a single component fermionic field $\widehat{\phi}_n$ on each lattice site n,

$$\widehat{H} = -\frac{\mathrm{i}}{2a} \sum_n \left(\widehat{\phi}_n^{\dagger} \widehat{\phi}_{n+1} - \widehat{\phi}_{n+1}^{\dagger} \widehat{\phi}_n \right). \tag{2.66}$$

The fields still fulfil the anticommutation relation $\left\{ \widehat{\phi}_n, \widehat{\phi}_m \right\} = 0$, $\left\{ \widehat{\phi}_n^{\dagger}, \widehat{\phi}_m \right\} = \delta_{mn}$. The equation of motion is therefore

$$\partial_t \widehat{\phi}_n = \frac{1}{2a} \left(\widehat{\phi}_{n+1} - \widehat{\phi}_{n-1} \right), \tag{2.67}$$

which is the same as lower component of the spinor in Eq. (2.64) and as a result yields a single left and a single right moving solution. Label field $\widehat{\phi}_N$ as $\widehat{\psi}_{\mathrm{u},n}$ for even $N = 2n$ and $\widehat{\psi}_{\mathrm{l},n}$ for odd $N = 2n+1$, we have

$$\partial_t \widehat{\psi}_{\mathrm{u},n} = \frac{1}{2a}\left(\widehat{\psi}_{\mathrm{l},n} - \widehat{\psi}_{\mathrm{l},n-1}\right), \quad \partial_t \widehat{\psi}_{\mathrm{l},n} = \frac{1}{2a}\left(\widehat{\psi}_{\mathrm{u},n+1} - \widehat{\psi}_{\mathrm{u},n}\right). \tag{2.68}$$

Taking the continuum limit $a \to 0$, and taking the representation

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \gamma^5 = \gamma^0\gamma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{2.69}$$

we recover the dirac equation (2.59) with the map

$$\lim_{a\to 0} \begin{pmatrix} \widehat{\psi}_{\mathrm{u},n}/\sqrt{a} \\ \widehat{\psi}_{\mathrm{l},n}/\sqrt{a} \end{pmatrix} \to \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \end{pmatrix} = \psi(x). \tag{2.70}$$

The recovery of continuum model at the limit $a \to 0$ can also be checked by the Hamiltonian directly.

The mass term can be added as follows. The mass term

$$\int \mathrm{d}x\, m\overline{\psi}(x)\,\psi(x) = \int \mathrm{d}x\, m\left(\psi_1^\dagger(x)\,\psi_1(x) - \psi_2^\dagger(x)\,\psi_2(x)\right) \tag{2.71}$$

can be written in the discretized version

$$m\sum_n \left(\widehat{\psi}^\dagger_{\mathrm{u},n}\widehat{\psi}_{\mathrm{u},n} - \widehat{\psi}^\dagger_{\mathrm{l},n}\widehat{\psi}_{\mathrm{l},n}\right) = m\sum_n (-1)^n \widehat{\phi}^\dagger_n \widehat{\phi}_n. \tag{2.72}$$

Finally, the Hamiltonian for d spatial dimensional lattice reads

$$\widehat{H} = \epsilon_d \sum_{\boldsymbol{n}} \sum_{k=1}^{d} \left(\widehat{\phi}^\dagger_{\boldsymbol{n}+e_k}\widehat{\phi}_{\boldsymbol{n}} \mathrm{e}^{\mathrm{i}\theta_k} + \mathrm{h.c.}\right) + m\sum_{\boldsymbol{n}} (-1)^{\sum_{k=1}^d n_k}\, \widehat{\phi}^\dagger_{\boldsymbol{n}}\widehat{\phi}_{\boldsymbol{n}}, \tag{2.73}$$

where $\boldsymbol{n} \in \mathbb{Z}^d$ is a lattice vector with components n_k and unit vector e_k, ϵ_d is a dimension dependent prefactor and $\exp(\mathrm{i}\theta_k)$ are direction dependent phase factors. This approach cannot remove the doubling solutions completely, for example, in three dimensions two flavours of Dirac fermions in the continuum limit are recovered instead of 8. To consider the colour degrees of freedom one just need to replace the field with a colour spinor.

Up to now the fermion Hamiltonian is not invariant under local gauge trnasformations. To recover the gauge invariance, we simply add the comparator between the creation and annihilation operators in the hopping term, that is,

$$\psi^\dagger_{\boldsymbol{n}+e_k}\psi_{\boldsymbol{n}} \to \psi^\dagger_{\boldsymbol{n}+e_k} U_{\boldsymbol{n},k}\psi_{\boldsymbol{n}}. \tag{2.74}$$

From now on, we use $\psi_{\boldsymbol{n}}$ instead of $\widehat{\phi}_{\boldsymbol{n}}$ for convenience. We state again the commutation relations to quantize the theory,

$$\{\psi_{\boldsymbol{n}}, \psi_{\boldsymbol{m}}\} = 0, \quad \left\{\psi^\dagger_{\boldsymbol{n}}, \psi_{\boldsymbol{m}}\right\} = \delta_{mn}. \tag{2.75}$$

2.3.2 The Kogut-Susskind Hamiltonian and Gauss's law

Combining the gauge parts and the fermion parts discussed above, we have the local gauge invariant Hamiltonian containing both fermions and gauge fields, which reads

$$H = \epsilon_d \sum_{\boldsymbol{n},k} \left(\psi^\dagger_{\boldsymbol{n}+e_k} U_{\boldsymbol{n},k} \psi_{\boldsymbol{n}} \mathrm{e}^{\mathrm{i}\theta_k} + \text{h.c.}\right) + m \sum_{\boldsymbol{n}} (-1)^{\sum_k n_k} \psi^\dagger_{\boldsymbol{n}} \psi_{\boldsymbol{n}} + \frac{a^{d-4}}{g^2} \sum_p \left(\operatorname{tr} \mathbb{I} - \frac{1}{2} (\operatorname{tr} U_p + \text{h.c.})\right) + \frac{a^{2-d} g^2}{2} \sum_{\text{links}} \boldsymbol{J}^2. \tag{2.76}$$

Finally we demonstrate that the effect of local gauge transformation is performed on vertices, which includes the site, d ingoing links and d outgoing links, see Figure 4. The transformation is

$$V_{\boldsymbol{n}} (\lambda^a) = \sum_i \exp (\mathrm{i}\lambda^a L^a_i) \sum_o \exp (\mathrm{i}\lambda^a R^a_o) \exp (\mathrm{i}\lambda^a Q^a_{\boldsymbol{n}}), \tag{2.77}$$

where i indicates 'incoming', o indicates 'outcoming' and Q^a is the generator for gauge transformation of fermion fields, fulfilling

$$(\exp (-\mathrm{i}\lambda^a Q^a))^j \psi_\mu (\exp (\mathrm{i}\lambda^a Q^a))^j = D^j_{\mu\nu} (\lambda^a) \psi_\nu. \tag{2.78}$$

Here $D^j (\lambda^a)$ is the j'th representation for transforming by λ^a. It can be proved that [8]

$$(\exp (\mathrm{i}\lambda^a Q^a))^j = \exp \left(\mathrm{i}\psi^\dagger_\mu q_{\mu\nu} \psi_\nu\right) \det \left(D^j (-\lambda^a)\right)^N, \tag{2.79}$$

where $q_{\mu\nu}$ is the component of the matrix q^j such that $D^j (\lambda^a) = \exp \left(\mathrm{i}q^j\right)$ and that $N = 0$ for a vertex in the even sublattice and $N = 1$ for the odd one. The determinant term is related to the staggering of fermions. How and why? Explain it. Then we have immediately that for SU(N) group the generator for gauge transformation of fermion fields is

$$(Q^a)^j_{\mathrm{SU}(N)} = \psi^\dagger_\mu (T^a)^j_{\mu\nu} \psi_\nu, \tag{2.80}$$

and for U(1) group the generator is

$$Q_{\mathrm{U}(1)} = \psi^\dagger \psi - \frac{1}{2} \left(1 - (-1)^N\right). \tag{2.81}$$

Since our model is invariant by the local gauge transformations, the physical states $|\text{phy}\rangle$ satisfy

$$V_{\boldsymbol{n}} (\lambda^a) |\text{phy}\rangle = \mathrm{e}^{\mathrm{i}\lambda^a q^a} |\text{phy}\rangle, \tag{2.82}$$

where q_a is just a real number and the transformation comes to a global phase shift. The generator for local gauge transformation reads

$$G^a_{\boldsymbol{n}} = \sum_i L^a_i + \sum_o R^a_o + Q^a_{\boldsymbol{n}}. \tag{2.83}$$

Since the Hamiltonian is gauge invariant, it commutes with $G^a_{\boldsymbol{n}}$. The Gauss's law is therefore recovered

$$G^a_{\boldsymbol{n}} |q^a\rangle = q^a |q^a\rangle. \tag{2.84}$$

Here q^a can be viewed as some static, external charge distribution.

3 Mapping LGT to finite dimensional models

In this section we discuss several schemes to map the lattice gauge theory (2.76) to finite dimensional models.

3.1 A general scheme based on group representation

In this subsection the summation is written explicitely and Einstein convention is not assumed.

First consider a general realization [8] given in the group representation space. The group representation space is spanned by orthonormal basis $|jmn\rangle$ corresponding to the unitary representation matrix elements D^j_{mn}. Since the sum of the square of the dimension of all inequivalent representations equals to the order of the group (in the case of finite group, and the proposition can be generalized to the Lie groups), the number of the basis $|jmn\rangle$ is the same as the number of the basis labelled by group elements $|g\rangle$. Define the transformation relation

$$\langle g|jmn\rangle = \sqrt{\frac{\dim(j)}{|G|}} D^j_{mn}(g). \tag{3.1}$$

Note that the coefficient ensures that

$$\langle g|h\rangle = \sum_{j,m,n} \frac{\dim(g)}{|G|} D^j_{mn}(g) D^{j*}_{mn}(h) = \delta_{gh}. \tag{3.2}$$

Before discussing the realization let us review the definition and properties of the Clebsch-Gordan coefficients. If a state $|j,\mu\rangle$ transforms under the group action $\widehat{g}$ by a representation j like

$$\widehat{g}\,|j,\mu\rangle = \sum_\rho D^j_{\rho\mu}(g)\,|j,\rho\rangle\,, \tag{3.3}$$

and state $|k,\nu\rangle$ transforms under representation k in a similar way. Then the tensor product of the states transforms as

$$\widehat{g}\,|j\mu,k\nu\rangle = \sum_{\rho,\lambda} D^j_{\rho\mu}(g) D^k_{\lambda\nu}(g)\,|j\rho,k\lambda\rangle\,, \tag{3.4}$$

which is also a representation of the group, though generally not irreducible. This representation can be decomposed into irreducible representations by inserting a complete basis $|J,M\rangle$ to both sides, yielding

$$\sum_{J,M,N} |J,N\rangle\,\langle J,M|j\mu,k\nu\rangle\, D^J_{NM}(g) = \sum_{J,M,\rho,\lambda} |J,M\rangle\,\langle J,M|j\rho,k\lambda\rangle\, D^j_{\rho\mu}(g) D^k_{\lambda\nu}(g). \tag{3.5}$$

Multiplying $\langle J,N|$ to both sides we have

$$\sum_M D^J_{NM}(g)\,\langle JM|j\mu,k\nu\rangle = \sum_{\rho,\lambda} D^j_{\rho\mu}(g) D^k_{\lambda\nu}(g)\,\langle JN|j\rho,k\lambda\rangle\,. \tag{3.6}$$

The inner product of the form $\langle JM|j\mu,k\nu\rangle$ is called the Clebsch-Gordan coefficient, abbreviated as CG coefficient.

Now we map the link Hilbert space to a Fock space with the vacuum state denoted by $|\Omega\rangle$. The creation operator is defined as

$$a^{j\dagger}_{mn}|\Omega\rangle = |jmn\rangle\,. \tag{3.7}$$

The total number $N = \sum_{jmn} a^{j\dagger}_{mn}a^{j}_{mn}$ is restricted to unity be requiring that the Hamiltonian commutes with the number operator. Note that this creation operator can be either fermion or boson.

Define the U^j operator as

$$U^j_{mm'} = \sum_{J,M,M',K,N,N'} \sqrt{\frac{\dim(J)}{\dim(K)}} \langle JM, jm|KN\rangle \langle KN'|JM', jm'\rangle\, a^{K\dagger}_{NN'} a^{J}_{MM'}. \tag{3.8}$$

Under the unit filling constraint, we have the matrix element

$$\langle KNN'|\, U^j_{mm'}\, |JMM'\rangle = \sqrt{\frac{\dim(J)}{\dim(K)}} \langle JM, jm|KN\rangle \langle KN'|JM', jm'\rangle\,. \tag{3.9}$$

To evaluate the matrix element in the group element basis, first using the inner product (3.1)

$$\langle g|\, U^j_{mn}\, |h\rangle = \sum_{J,M,M',K,N,N'} \frac{\dim(J)}{|G|} \langle JM, jm|KN\rangle \langle KN'|JM', jn\rangle\, D^K_{NN'}(g) D^{J*}_{MM'}(h). \tag{3.10}$$

Using the property of the CG coefficient (3.6),

$$\begin{aligned}\langle g|\, U^j_{mn}\, |h\rangle = &\sum_{J,M,M',K,N,M'',m''} \frac{\dim(J)}{|G|} \\ &\times \langle JM, jm|KN\rangle \langle KN|JM'', jm''\rangle\, D^J_{M''M'}(g) D^j_{m''n}(g) D^{J*}_{MM'}(h).\end{aligned} \tag{3.11}$$

Summ over K, N gives an identity and hence $\delta_{MM''}\delta_{mm''}$, therefore we have

$$\langle g|\, U^j_{mn}\, |h\rangle = D^j_{mn} \sum_{J,M,M'} \frac{\dim(J)}{|G|} D^J_{MM'}(g) D^{J*}_{MM'}(h). \tag{3.12}$$

Finally from the typical result in group theory, we have

$$\langle g|\, U^j_{mn}\, |h\rangle = D^j_{mn}\delta_{gh}. \tag{3.13}$$

The left and right transformation operators are defined as

$$\widehat{g}_L = \sum_{j,m,n,l} a^{j\dagger}_{ml} D^{j*}_{mn}(g) a^j_{nl} \tag{3.14}$$

First we can verify that this is actually a group action. Consider group elements g and h,

$$\begin{aligned}\widehat{g}_L\widehat{h}_L &= \sum_{j,m,n,l}\sum_{j',m',n',l'} a^{j\dagger}_{ml} D^j_{nm}\left(g^{-1}\right) a^j_{nl} a^{j'\dagger}_{m'l'} D^{j'}_{n'm'}\left(h^{-1}\right) a^{j'}_{n'l'} \\ &= \sum_{j,m,n,l,n'} a^{j\dagger}_{ml} D^j_{nm}\left(g^{-1}\right) D^j_{n'n}\left(h^{-1}\right) a^j_{n'l} \\ &= \sum_{j,m,n',l} a^{j\dagger}_{ml} D^j_{n'm}\left((gh)^{-1}\right) a^j_{n'l} \\ &= \widehat{gh}_L.\end{aligned} \tag{3.15}$$

Here we have considered that for we are restricted to the unit filling subspace, two annihilation operators in succession will give the result zero, so for either bosons or fermions the operator $a_A a_B^\dagger a_C$ equals to $\delta_{AB} a_C$. We also used the condition that the representation is unitary. We should also verify that this definition coincides with Equation (2.45), that is, we should verift that

$$\left\langle KNN' \right| \widehat{g}_L^\dagger U^j_{mn} \widehat{g}_L \left| JMM' \right\rangle = \left\langle KNN' \right| \sum_k D^j_{mk}(g) U^j_{kn} \left| JMM' \right\rangle . \tag{3.16}$$

The right hand side is

$$\text{R.H.S.} = \sqrt{\frac{\dim(J)}{\dim(K)}} \sum_k D^j_{mk}(g) \left\langle JM, jk | KN \right\rangle \left\langle KN' | JM', jn \right\rangle . \tag{3.17}$$

And the left hand side is

$$\begin{aligned} \text{L.H.S.} = \left\langle KNN' \right| \sum_{j',s,t,l} a^{j'\dagger}_{tl} D^{j'}_{st}(g) a^{j'}_{sl} \sum_{\widetilde{J},\widetilde{M},\widetilde{M}',\widetilde{K},\widetilde{N},\widetilde{N}'} \sqrt{\frac{\dim\left(\widetilde{J}\right)}{\dim\left(\widetilde{K}\right)}} \\ \times \left\langle \widetilde{J}\widetilde{M}, jm \right| \left. \widetilde{K}\widetilde{N} \right\rangle \left\langle \widetilde{K}\widetilde{N}' \right| \left. \widetilde{J}\widetilde{M}', jn \right\rangle a^{\widetilde{K}\dagger}_{\widetilde{N}\widetilde{N}'} a^{\widetilde{J}}_{\widetilde{M}\widetilde{M}'} \sum_{j'',p,q,r} a^{j''\dagger}_{qr} D^{j''}_{pq}\left(g^{-1}\right) a^{j''}_{pr} \left| JMM' \right\rangle \end{aligned} \tag{3.18}$$

Using the unit filling condition, i.e., $a_A a_B^\dagger a_C = \delta_{AB} a_C$, we can simplify the equation above as

$$\begin{aligned} \text{L.H.S.} = \left\langle KNN' \right| \sum_{\widetilde{K},\widetilde{N},\widetilde{N}',t,\widetilde{J},\widetilde{M},\widetilde{M}',p} a^{\widetilde{K}\dagger}_{t\widetilde{N}'} D^{\widetilde{K}}_{\widetilde{N}t}(g) \sqrt{\frac{\dim\left(\widetilde{J}\right)}{\dim\left(\widetilde{K}\right)}} \\ \times \left\langle \widetilde{J}\widetilde{M}, jm \right| \left. \widetilde{K}\widetilde{N} \right\rangle \left\langle \widetilde{K}\widetilde{N}' \right| \left. \widetilde{J}\widetilde{M}', jn \right\rangle D^{\widetilde{J}}_{p\widetilde{M}}\left(g^{-1}\right) a^{\widetilde{J}}_{p\widetilde{M}'} \left| JMM' \right\rangle . \end{aligned} \tag{3.19}$$

Calculating the operators and the states we have $\delta_{K\widetilde{K}} \delta_{Nt} \delta_{N'\widetilde{N}'} \delta_{\widetilde{J}J} \delta_{pM} \delta_{\widetilde{M}'M'}$, and the equation reduces to

$$\text{L.H.S.} = \sqrt{\frac{\dim(J)}{\dim(K)}} \sum_{\widetilde{N},\widetilde{M}} D^K_{\widetilde{N}N}(g) D^J_{M\widetilde{M}}\left(g^{-1}\right) \left\langle J\widetilde{M}, jm \right| \left. K\widetilde{N} \right\rangle \left\langle KN' \right| \left. JM', jn \right\rangle . \tag{3.20}$$

Using Equation (3.6) we have

$$\begin{aligned} \sum_{\widetilde{N}} D^K_{\widetilde{N}N}(g) \left\langle J\widetilde{M}, jm \right| \left. K\widetilde{N} \right\rangle &= \left(\sum_{\widetilde{N}} D^K_{N\widetilde{N}}\left(g^{-1}\right) \left\langle K\widetilde{N} \right| \left. J\widetilde{M}, jm \right\rangle \right)^* \\ &= \sum_{\widetilde{M}',k} D^J_{\widetilde{M}\widetilde{M}'}(g) D^j_{mk}(g) \left\langle J\widetilde{M}', jk \right| \left. KN \right\rangle . \end{aligned} \tag{3.21}$$

Finally,

$$\begin{aligned} \text{L.H.S.} = \sqrt{\frac{\dim(J)}{\dim(K)}} \sum_{\widetilde{M},\widetilde{M}',k} D^j_{mk}(g) D^J_{M\widetilde{M}}\left(g^{-1}\right) D^J_{\widetilde{M}\widetilde{M}'}(g) \\ \times \left\langle J\widetilde{M}', jk \right| \left. KN \right\rangle \left\langle KN' \right| \left. JM', jn \right\rangle = \text{R.H.S.} \end{aligned} \tag{3.22}$$

Similarly, we can define

$$\widehat{g}_R = \sum_{j,m,n,l} a^{j\dagger}_{lm} D^j_{mn}(g) a^j_{ln}, \tag{3.23}$$

and verify that Equation (2.46) holds as

$$\langle KNN'| \widehat{g}^\dagger_R U^j_{mn} \widehat{g}_R |JMM'\rangle = \langle KNN'| \sum_k U^j_{mk} D^j_{kn}\left(g^{-1}\right) |JMM'\rangle . \tag{3.24}$$

For compact Lie groups the left and right transformation operators can be written as

$$\widehat{g}_L = \sum_{j,m,n,l} a^{j\dagger}_{ml} \left(\exp\left(-\mathrm{i}\lambda^a T^a\right)\right)^j_{mn} a^j_{nl} = \sum_j \exp\left(-\mathrm{i}\lambda^a \sum_{m,n,l} a^{j\dagger}_{ml} \left(T^a\right)^j_{mn} a^j_{nl}\right) . \tag{3.25}$$

The second equal sign is valid under the unit filling constraint as we have argued. Therefore the electric field can be expressed as

$$L^a = -\sum_{j,m,n,l} a^{j\dagger}_{ml} \left(T^a\right)^j_{mn} a^j_{nl}. \tag{3.26}$$

Note that the sum over the representation index j is interpreted as the direct sum of operators over different invariant subspaces. Similarly we have

$$R^a = \sum_{j,m,n,l} a^{j\dagger}_{lm} \left(T^a\right)^j_{mn} a^j_{ln}. \tag{3.27}$$

The commutation relations (2.48) can be verified straightforwardly.

Now let's consider in the group element basis the magnetic energy (2.44) under the j's representation subspace. First calculate

$$\mathrm{Tr}\, U_p = \sum_{g_1,g_2,g_3,g_4} \left|g_4^{-1} g_3^{-1} g_2 g_1\right\rangle \left\langle g_4^{-1} g_3^{-1} g_2 g_1\right| \mathrm{Tr}\, D^j\left(g_4^{-1} g_3^{-1} g_2 g_1\right) . \tag{3.28}$$

Denote the conjugate class as C, define the projection operator $\Pi_C \equiv \sum_{g\in C} |g\rangle \langle g|$. We can then rewrite the trace as

$$\mathrm{Tr}\, U_p = \sum_C \chi^j(C) \Pi_C, \tag{3.29}$$

where $\chi^j(C)$ is the character of the j'th representation of the conugate class C. The magnetic part of the Hamiltonian is

$$H_{\mathrm{mg}} = \frac{a^{d-4}}{2g^2} \sum_p \left(1 - \sum_C \chi^j(C) \Pi_{C,p}\right) . \tag{3.30}$$

The ground state of the magnetic Hamiltonian is the configuration that for every plaquette $g = e$, for $\max_C \chi^j(C) = j$ achieved at $C = \{e\}$.

Since every U^j_{mn}, L and R conserves the total occupation number, they are suitable for constructing the Kogut-Susskind Hamiltonian. Sometimes we need not use all representations, and a truncation is often introduced. Take SU(2) gauge theory as an example.

Example We truncate the SU(2) model to only two lowest representations, $j = 0, 1/2$. We have

$$\boldsymbol{L} = -\frac{1}{2} a^{1/2\dagger}_{ml} \boldsymbol{\sigma}_{mn} a^{1/2}_{nl}, \quad \boldsymbol{R} = \frac{1}{2} a^{1/2\dagger}_{lm} \boldsymbol{\sigma}_{mn} a^{1/2}_{ln}, \tag{3.31}$$

where $\boldsymbol{\sigma}$ is the vector of the Pauli matrices. The U matrix reads

$$U^0_{00} = \sum_{mn} a^{1/2\dagger}_{mn} a^{1/2}_{mn}, \tag{3.32}$$

and

$$\begin{aligned} U^{1/2}_{mm'} &= \frac{1}{\sqrt{2}} \left\langle 00, \frac{1}{2} m \middle| \frac{1}{2} m \right\rangle \left\langle \frac{1}{2} m' \middle| 00, \frac{1}{2} m' \right\rangle a^{1/2\dagger}_{mm'} a^0_{00} \\ &\quad + \sqrt{2} \left\langle \frac{1}{2} (-m), \frac{1}{2} m \middle| 00 \right\rangle \left\langle 00 \middle| \frac{1}{2} \left(-m'\right), \frac{1}{2} m' \right\rangle a^{0\dagger}_{00} a^{1/2}_{(-m)(-m')}. \\ &= \frac{1}{\sqrt{2}} \left(a^{1/2\dagger}_{mm'} a^0_{00} + \left(2\delta_{mm'} - 1\right) a^{0\dagger}_{00} a^{1/2}_{(-m)(-m')} \right). \end{aligned} \tag{3.33}$$

All the operators above act on a 5-dimensional Hilbert space.

3.2 Quantum link model

Quantum link models (QLM) [13, 14] replace the continuous comparator U by discrete quantum degrees of freedom.

For U(1) model, denote $R = -L = E$ and we have $[E, U] = U$, $\left[E, U^\dagger\right] = ([U, E])^\dagger = -U^\dagger$ at the same link. We can choose the quantum spin operator S^i, $i \in \{x, y, z\}$ and map the link operators to spins,

$$U \mapsto S^+ \equiv S^x + \mathrm{i}S^y, \quad U^\dagger \mapsto S^- \equiv S^x - \mathrm{i}S^y, \quad E \mapsto S^z. \tag{3.34}$$

This can be rewritten using Schwinger algebra constructed from two boson species a and b,

$$S^+ = a^\dagger b, \quad S^z = \frac{1}{2} \left(a^\dagger a - b^\dagger b \right). \tag{3.35}$$

This representation is called truncated cQED [2] model in some literature [15].

An SU(2) quantum link model is constructed in [13] by ten generators of SO(5). Docomposite U into real and imaginary part

$$U = U^0 + \mathrm{i}\boldsymbol{U} \cdot \boldsymbol{\sigma}, \tag{3.36}$$

where $\boldsymbol{\sigma}$ is the vector of Pauli matrices. The result is that

$$\boldsymbol{R} = \begin{pmatrix} \boldsymbol{\tau} & 0 \\ 0 & 0 \end{pmatrix}, \quad \boldsymbol{L} = \begin{pmatrix} 0 & 0 \\ 0 & \boldsymbol{\tau} \end{pmatrix}, \quad U^0 = \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}, \quad \boldsymbol{U} = \begin{pmatrix} 0 & -\mathrm{i}\boldsymbol{\tau} \\ \mathrm{i}\boldsymbol{\tau} & 0 \end{pmatrix}. \tag{3.37}$$

Here $\mathbf{1}$ is the 2×2 unit matrix, τ is the vector of Pauli matrices. We use $\boldsymbol{\tau}$ instead of $\boldsymbol{\sigma}$ to emphasize that they act on different spaces. This representation is similar to the truncated model discussed in the last part of the previous subsection.

[2] The letter ‘c’ here stands for ‘compact’, see later sections for details.

To be explained further. See [14] for details. A general construction of quantum link model of SU(N) gauge group is as follows. The real and imaginary parts of the N^2 matrix elements of U are represented by $2N^2$ Hermitian generators of the embedding algebra SU($2N$). The $2\left(N^2-1\right)$ generators L^a and R^a also belong to the SU($2N$) algebra. An additional Abelian U(1) gauge transformation is related to the generator E. The total number of generators is precisely $2N^2+2\left(N^2-1\right)+1=4N^2-1$, which is the number of generators of SU($2N$).

4 Experimental proposals and realisations

Although local gauge invariance is a natural constraint in the real world, it is challenging to engineer such constraint in a synthetic quantum system. In a general experiment, gauge fields and matter fields are represented by different atoms or different states of an atom, that is, the LGT Hamiltonian is mapped to the many-body Hamiltonian of the system. For a system with laser and atoms interacting, the operator that gauge transformation generator maps to and the Hamiltonian does not necessarily commute, which means that local gauge invariance is not preserved in general. Therefore, the key to the simulation of LGT is to suppress the gauge-variant terms in the system Hamiltonian. The most popular proposal to suppress such terms is the energy-penalty scheme, in which the undesired processes gain a high energy cost and therefore are less possible to happen.

4.1 Experimental realisations of $\mathbb{Z}_2$ or truncated U(1) model

Among all the gauge groups, $\mathbb{Z}_2$ and truncated two-dimensional U(1) are the simplest ones. Here we introduce recent experiments about such models.

A double well realisation based on Floquet approach Quantum simulation of $\mathbb{Z}_2$ gauge theory is demonstrated in a double well cold atom system by Floquet approach [16]. The target Hamiltonian is

$$\widehat{H}_{\mathbb{Z}_2}=-\sum_j J_a\left(\widehat{\tau}^z_{\langle j,j+1\rangle}\widehat{a}^\dagger_j\widehat{a}_{j+1}+\text{ h.c. }\right)-\sum_j J_f\widehat{\tau}^x_{\langle j,j+1\rangle}. \tag{4.1}$$

Here $\widehat{a}^\dagger_j$ generates a matter particle on lattice site j and the Pauli operators $\widehat{\tau}_{\langle j,j+1\rangle}$ defined on the links encode the gauge degrees of freedom. The matter field has a charge $\widehat{Q}_j=\mathrm{e}^{\mathrm{i}\pi\widehat{n}^a_j}$. The energy scale of the dynamics of the matter field coupled with $\widehat{\tau}^z$ is J_a and the energy scale of the electric field is J_f. The generator of the local gauge transformation on site j is

$$\widehat{G}_j=\widehat{Q}_j\prod_{i:\langle i,j\rangle}\widehat{\tau}^x_{\langle i,j\rangle},\quad\left[\widehat{H},\widehat{G}_j\right]=0\quad\forall j. \tag{4.2}$$

The eigenvalues of $\widehat{G}_j$ is $g_j=\pm1$ and the dynamics of the model is constrained by Gauss' law $\widehat{G}_j\left|\psi\right\rangle=g_j\left|\psi\right\rangle$.

In the experiment, the matter and gauge fields are implemented using two different species denoted a and f, which are realised by two states of the ^{87}Rb, $|a\rangle=|F=1,m_F=-1\rangle$ and $|f\rangle=|F=1,m_F=+1\rangle$. The matter field is associated with the a particle and the

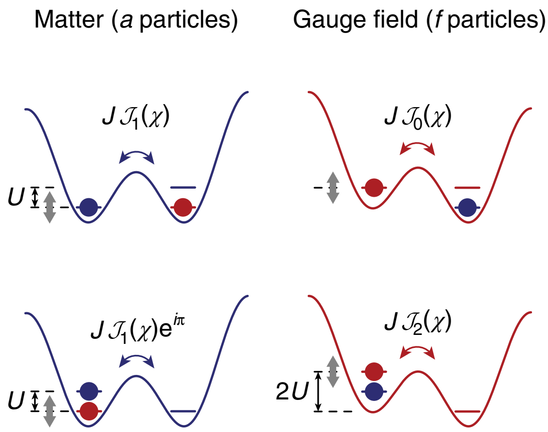


Figure 5. Effective tunnelling processes for the matter-field (blue, a) and gauge-field (red, f) particle. For $\phi = 0$, hopping of a particles occurs for resonant one-photon processes at $\hbar\omega \approx U$ with an effective amplitude $J\mathcal{J}_1(\chi)$, where U is the interspecies on-site interaction. Depending on the position of the f particle, the a particle acquires a phase shift of π, which realizes the matter-gauge coupling. Tunnelling of the f particle is renormalized by zero- or induced via two-photon processes, with amplitudes $J\mathcal{J}_0(\chi)$ and $J\mathcal{J}_2(\chi)$ depending on the a particle's position.

gauge field corresponds to the number imbalance of f particle as $\widehat{\tau}^z_{\langle j,j+1\rangle} = \widehat{n}^f_{j+1} - \widehat{n}^f_j$ or the tunnelling of f as $\widehat{\tau}^x_{\langle j,j+1\rangle} = \widehat{f}^\dagger_j \widehat{f}_{j+1} + \widehat{f}^\dagger_{j+1}\widehat{f}_j$. Making use of the opposite magnetic moments of the two species we can implement a double well with an energy offset only seen by f with a magnetic-field gradient. The author prepared exactly one a and one f in the double well and the Hamiltonian can be written as

$$\widehat{H}(t) = -J\left(\widehat{a}^\dagger_2\widehat{a}_1 + \widehat{f}^\dagger_2\widehat{f}_1 + \text{ h.c. }\right) + U\sum_{j=1,2}\widehat{n}^a_j\widehat{n}^f_j + \Delta_f\widehat{n}^f_1 + A\cos(\omega t+\phi)\left(\widehat{n}^a_1 + \widehat{n}^f_1\right), \quad (4.3)$$

where the Floquet term satisfies $\hbar\omega = \sqrt{U^2 + 4J^2} \approx U$ and $\hbar\omega \gg J$ (which is also the strong interaction limit).

To the lowest order, the effective Hamiltonian takes the form

$$\widehat{H}_{\text{eff}} = -J_a\widehat{\tau}^z\left(\widehat{a}^\dagger_2\widehat{a}_1 + \widehat{a}^\dagger_1\widehat{a}_2\right) - \widehat{J}_f\widehat{\tau}^x, \quad (4.4)$$

where $\widehat{J}_f$ depends on the density of the a particle,

$$\widehat{J}_f = J\mathcal{J}_0(\chi)\widehat{n}^a_1 + J\mathcal{J}_2(\chi)\widehat{n}^a_2. \quad (4.5)$$

To avoid such a dependence, the driving parameter $\chi = A/(\hbar\omega)$ can be chosen that the Bessel functions satisfy $\mathcal{J}_0(\chi) = \mathcal{J}_2(\chi)$. A more direct inllustration of the suppressing mechanism can is shown in Fig. 5.

The authors measured the dynamics of the system with two initial states $|\psi^x_0\rangle = |a,0\rangle \otimes (|f,0\rangle + |0,f\rangle)/\sqrt{2}$ and $|\psi^z_0\rangle = |a,0\rangle \otimes |0,f\rangle$. They compared the experiment result with the analytical one and proved the existence of local gauge invariance. The authors also noticed the difference between experimental and analytical results and evaluated the effect

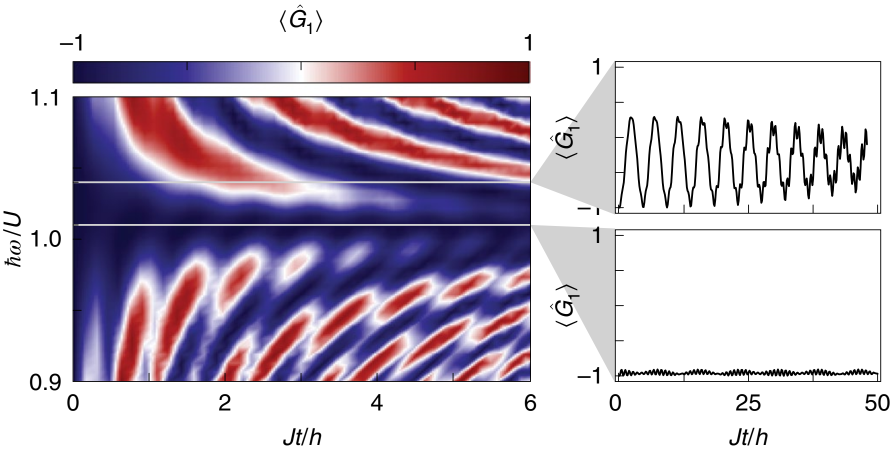


Figure 6. Stroboscopic dynamic of the expectation value $\langle \widehat{G}_1 \rangle$ for different ω. The right panels show examples of the time traces for $\hbar\omega \approx 1.04U$ (top) and $\hbar\omega = 1.01U$ (bottom).

of the inhomogeneous tilt distribution $\Delta(x, y, z)$. After considering such inhomogenity, the experiment agrees well with the numerical analysis.

To investigate the Floquet effect to the local gauge invariance, the authors numerically calculated the dynamics of $|\psi_0^x\rangle$ under the full Hamiltonian (4.3). The result is shown in Fig. 6. The optimal frequency is $\hbar\omega = 1.01U$.

A single-site realisation based on spin changing collisions Using interspecies spin-changing collisions in an atomic mixture, gauge invariant interactions between matter and gauge fields with spin- and species-independent trapping potentials can be achieved [17]. In their scheme, the authors implemented gauge degrees of freedom and particles in a single site.

In the experiment scheme, there are two-component matter fields 'p' and 'v'. The Hamiltonian

$$\widehat{H} = \sum \left[\widehat{H}_n + \hbar\Omega \left(\widehat{b}^\dagger_{n,\mathrm{p}} \widehat{b}_{n,\mathrm{v}} + \widehat{b}^\dagger_{n,\mathrm{v}} \widehat{b}_{n,\mathrm{p}} \right) \right] \tag{4.6}$$

of the extended system is composed of the building-block Hamiltonian $\widehat{H}_n$ and Raman-assited tunnelling. The building-blok Hamiltonian reads

$$\widehat{H}_n/\hbar = \chi \widehat{L}_z^2 + \frac{\Delta}{2} \left(\widehat{b}^\dagger_\mathrm{p} \widehat{b}_\mathrm{p} - \widehat{b}^\dagger_\mathrm{v} \widehat{b}_\mathrm{v} \right) + \lambda \left(\widehat{b}^\dagger_\mathrm{p} \widehat{L}_- \widehat{b}_\mathrm{v} + \widehat{b}^\dagger_\mathrm{v} \widehat{L}_+ \widehat{b}_\mathrm{p} \right), \tag{4.7}$$

where $\widehat{b}_{n+1,\mathrm{p}}$ is written as $\widehat{b}_\mathrm{p}$ and that $\widehat{b}_{n,\mathrm{v}}$ is written as $\widehat{b}_\mathrm{v}$. The ladder operator $\widehat{L}_\pm = \widehat{L}_x \pm \mathrm{i}\widehat{L}_y$.

To implement the building block Hamiltonian, the authors used an optical dipole trap such that the trap potential is spin insensitive for both species. An external magnetic field $B = 2\,\mathrm{G}$ suppressed any spin change energetically, such that only two Zeeman levels $m_F = 0$ and 1 of $F = 1$ hyperfine states are populated in the experiment. The matter fields 'p' and 'v' are realised by ^{7}Li through $|\mathrm{p}\rangle = |m_F = 0\rangle$ and $|\mathrm{v}\rangle = |m_F = 1\rangle$. The spin operator $\widehat{\boldsymbol{L}}$ associated to the gauge field acts on the ^{23}Na states $|\uparrow\rangle = |m_F = 0\rangle$ and $|\downarrow\rangle = |m_F = 1\rangle$.

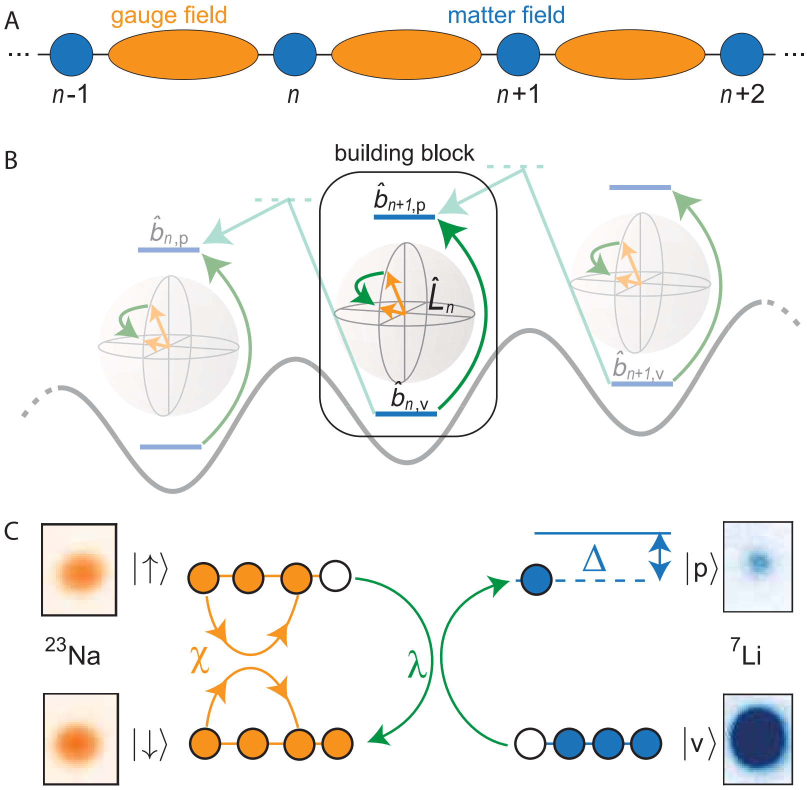


Figure 7. (**A**) The general implementation of lattice gauge theory. (**B**) The extended system and the building block. Inside a building block there are gauge fields represented by long spins and different matter states, while the building blocks are connected via Raman-assited tunnelling. (**C**) Experimental realisation of the building block with bosonic gauge (^{23}Na) and matter (^{7}Li) fields. The gauge invariant interaction is realised by heteronuclear spin-changing collisions.

The first term in Eq. (4.7) is the one-axis twisting Hamiltonian, and the second arises from the energy shift due to the external magnetic field. Finally, the third term is implemented by heteronuclear spin-changing interactions [18].

The authors observed the dynamics of the single block. With out gauge fields (Na atoms), the dynamics of the matter sector is nothing but noise. By contrast, with the gauge field the dynamics of particle production can be detected.

References

[1] S. Kühn, *Quantum and classical simulation of High Energy Physics*, Ph.D. thesis, Munich, Tech. U., 10, 2017.

[2] M. Srednicki, *Quantum Field Theory*. Cambridge University Press, 2007, 10.1017/CBO9780511813917.

[3] C. N. Yang and R. L. Mills, *Conservation of isotopic spin and isotopic gauge invariance*, *Phys. Rev.* **96** (Oct, 1954) 191–195.

[4] K. G. Wilson, *Confinement of quarks*, *Phys. Rev. D* **10** (Oct, 1974) 2445–2459.

[5] Y. Makeenko, *Methods of Contemporary Gauge Theory*. Cambridge Monographs on Mathematical Physics. Cambridge University Press, 2002, 10.1017/CBO9780511535147.

[6] S. Hassani, *Mathematical physics: a modern introduction to its foundations*. Springer Science & Business Media, 2013.

[7] J. Kogut and L. Susskind, *Hamiltonian formulation of wilson's lattice gauge theories*, *Phys. Rev. D* **11** (Jan, 1975) 395–408.

[8] E. Zohar and M. Burrello, *Formulation of lattice gauge theories for quantum simulations*, *Phys. Rev. D* **91** (Mar, 2015) 054506.

[9] J. B. Kogut and M. A. Stephanov, *The Hamiltonian version of lattice-gauge theory*, p. 136–157. Cambridge Monographs on Particle Physics, Nuclear Physics and Cosmology. Cambridge University Press, 2003. 10.1017/CBO9780511534980.006.

[10] H. Nielsen and M. Ninomiya, *A no-go theorem for regularizing chiral fermions*, *Physics Letters B* **105** (1981) 219–223.

[11] H. Nielsen and M. Ninomiya, *Absence of neutrinos on a lattice: (i). proof by homotopy theory*, *Nuclear Physics B* **185** (1981) 20–40.

[12] H. Nielsen and M. Ninomiya, *Absence of neutrinos on a lattice: (ii). intuitive topological proof*, *Nuclear Physics B* **193** (1981) 173–194.

[13] S. Chandrasekharan and U.-J. Wiese, *Quantum link models: A discrete approach to gauge theories*, *Nuclear Physics B* **492** (1997) 455–471.

[14] U.-J. Wiese, *Ultracold quantum gases and lattice systems: quantum simulation of lattice gauge theories*, *Annalen der Physik* **525** (2013) 777–796, [`https://onlinelibrary.wiley.com/doi/pdf/10.1002/andp.201300104`].

[15] S. Kühn, J. I. Cirac and M.-C. Bañuls, *Quantum simulation of the schwinger model: A study of feasibility*, *Phys. Rev. A* **90** (Oct, 2014) 042305.

[16] C. Schweizer, F. Grusdt, M. Berngruber, L. Barbiero, E. Demler, N. Goldman et al., *Floquet approach to $\mathbb{Z}_2$ lattice gauge theories with ultracold atoms in optical lattices*, *Nature Physics* **15** (11, 2019) 1168–1173.

[17] A. Mil, T. V. Zache, A. Hegde, A. Xia, R. P. Bhatt, M. K. Oberthaler et al., *A scalable realization of local U(1) gauge invariance in cold atomic mixtures.*, *Science (New York, N.Y.)* **367** (mar, 2020) 1128–1130.

[18] X. Li, B. Zhu, X. He, F. Wang, M. Guo, Z.-F. Xu et al., *Coherent heteronuclear spin dynamics in an ultracold spinor mixture*, *Phys. Rev. Lett.* **114** (Jun, 2015) 255301.