Topics on Group Theory

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Abstract

This note is about several topics on group theory. Completeness is not expected.
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1 Direct product and Semidirect product

1.1 Direct product

First define the direct product of two groups [1].

Definition 1.1 (Direct product of groups)**.** *Given groups G and H, the direct group $G \times H$ is defined as*

DP1 The underlying set is the Cartesian product $G \times H$. The elements are denoted as (g, h) where $g \in G$ and $h \in H$.

DP2 The operation is given by

$$(g_1, h_1) \cdot (g_2, h_2) = (g_1 g_2, h_1 h_2) \,. \tag{1}$$

1.2 Semidirect product

A more complex and common structure is the semidirect product [2]. We can first prove that the following statements are equivalent.

Theorem 1.1. *Given a group G with identity element e, a subgroup H and a normal subgroup $N \lhd G$, the following statements are equivalent:*

SDP1 G is the product of subgroups, $G = NH$, and these subgroups have trivial intersection: $N \cap H = \{e\}$.

SDP2 For every $g \in G$, there are unique $n \in N$ and $h \in H$ such that $g = nh$.

SDP3 For every $g \in G$, there are unique $h \in H$ and $n \in N$ such that $g = hn$.

SDP4 The composition $\pi \circ i$ of the natural embedding $i : H \to G$ with the natural projection $\pi : G \to G/N$ is an isomorphism between H and the quotient group G/N.

SDP5 There exists a homomorphism $G \to H$ that is the identity on H and whose kernel is N.

Proof. SDP1 $\to$ SDP2: Since $G = NH$, there exist $n \in N$ and $h \in H$ such that $g = nh, \forall g \in G$. If there are $n' \in N, h' \in H$ such that $nh = n'h'$, we have $n'^{-1}n = h'h^{-1} = e$ and therefore $n = n'$ and $h = h'$ since $N \cap H = \{e\}$.

SDP2 $\to$ SDP3: Since N is normal, there exist $n' = h^{-1}nh \in N$ such that $nh = hh^{-1}nh = hn'$ for $n \in N$ and $h \in H$. If there exist $n'' \in N$ and $h'' \in H$ such that $nh = hn' = h''n''$, we have $nh = h''n''h''^{-1}h''$. Since the decomposition $g = hn$ is unique, $h'' = h$ and $n'' = h^{-1}nh = n'$, i.e., the decomposition $g = hn$ is also unique.

SDP3 $\to$ SDP4: Clearly that $\pi \circ i(h) = hN$. Since N is normal, the map is homomorphism. For any $g \in G$ we have $h \in H$ and $n \in N$ such that $gN = hnN = hN$, so the map is surjective. For $h, h' \in H$, $hN = h'N$ if and only if there exist $g \in G$, $n, n' \in N$ such that $hn = h'n' = g$. However, due to SDP3 $h = h'$, so the map is injective. Therefore the map $\pi \circ i$ is isomorphism.

SDP4 $\to$ SDP5: Define a map $\varphi : G \mapsto H$ such that $\varphi(g) = h$ if and only if $g \in \pi \circ i(h)$ defined in SDP4. Since cosets of a subgroup is either the same or nonintersecting, this definition is reasonable. Clearly that φ is homomorphism and is the identity on H and has a kernal N directly due to SDP4.

SDP5 $\to$ SDP1: Suppose the homomorphism is φ. For an arbitrary $g \in G$, suppose that $\varphi(g) = h \in H$, then we have $\varphi\left(gh^{-1}\right) = e$ and therefore $gh^{-1} \in N$. That is, $G = NH$. For any $h \in H$, $h \in N$ if and only if $\varphi(h) = e$, that is, $h = e$. $\square$

With the above theorem, we can define and immediately get several properties of the inner semidirect product of a group.

Definition 1.2 (Inner semidirect product)**.** *Given a group G with identity element e, a subgroup H and a normal subgroup $N \lhd G$, if any of the conditions among SDP1 to SDP5 holds, G is called the semidirect product of N and H, written*

$$G = N \rtimes H \text{ or } G = H \ltimes N. \tag{2}$$

Since every element in $G = N \rtimes H$ can be decomposited as nh with $n \in N$ and $h \in H$, consider the product of two elements $g_1 = n_1h_1$ and $g_2 = n_2h_2$,

$$g_1g_2 = n_1h_1n_2h_2 = n_1h_1n_2h_1^{-1}h_1h_2 = n_1\varphi_{h_1}\left(n_2\right)h_1h_2, \tag{3}$$

where $\varphi_h\left(n\right) = hnh^{-1}$. Due to SDP2, the decomposition is unique, so we can denote $g \in G$ as an ordered pair (n, h), with the product law

$$\left(n_1, h_1\right) \cdot \left(n_2, h_2\right) = \left(n_1\varphi_{h_1}\left(n_2\right), h_1h_2\right), \tag{4}$$

where φ_h is defined above.

Motivated by the inner semidirect product, we can define the outer semidirect product in a similar way. Given two groups N and H and a group homomorphism $\varphi : H \mapsto \mathrm{Aut}(N)$[1], define a product $\cdot$ on the Cartesian product of N and H,

$$\left(n_1, h_1\right) \cdot \left(n_2, h_2\right) = \left(n_1\varphi\left(h_1\right)\left(n_2\right), h_1, h_2\right). \tag{5}$$

[1]The group of automorphisms on N. A simple example is that if $N \lhd G$, $n \to gng^{-1}$ with $n \in N$ and $g \in G$ is an automorphism.

We can verify whether it forms a group.

First, for $(n_1, h_1), (n_2, h_2), (n_3, h_3) \in N \times H$, we have the associativity

$$\begin{aligned}((n_1, h_1) \cdot (n_2, h_2)) \cdot (n_3, h_3) &= (n_1 \varphi(h_1)(n_2), h_1 h_2) \cdot (n_3, h_3) \\ &= (n_1 \varphi(h_1)(n_2) \varphi(h_1 h_2)(n_3), h_1 h_2 h_3) \\ &= (n_1 \varphi(h_1)(n_2 \varphi(h_2)(n_3)), h_1 h_2 h_3) \\ &= (n_1, h_1) \cdot ((n_2, h_2) \cdot (n_3, h_3)).\end{aligned} \tag{6}$$

Second, for every $(n, h) \in N \times H$,

$$(e, e) \cdot (n, h) = (n, h). \tag{7}$$

So (e, e) is the identity element.

Third, for every $(n, h) \in N \times H$, we have

$$\left(\varphi\left(h^{-1}\right)\left(n^{-1}\right), h^{-1}\right) \cdot (n, h) = (e, e). \tag{8}$$

Therefore we can define a group as follows.

Definition 1.3. *Given two groups N and H and a group homomorphism $\varphi : H \mapsto \mathrm{Aut}(N)$, we can construct a new group $N \rtimes_\varphi H$, called the outer semidirect product of N and H with respect to φ, defined as*

OSP1 The underlying set is the Cartesian product $N \times H$.

OSP2 The group operation is defined as

$$(n_1, h_1) \cdot (n_2, h_2) = (n_1 \varphi(h_1)(n_2), h_1, h_2), \tag{9}$$

for $n_1, n_2 \in N$ and $h_1, h_2 \in H$.

To see the connection between the outer semidirect product and the inner semidirect product, consider the group $G = N \rtimes_\varphi H$. Define two subgroups (easy to verify) $\widetilde{N} = \{(n, e) \in G | n \in n\}$ and $\widetilde{H} = \{(e, h) \in G | h \in H\}$. For any $g = (n, h) \in G$ and $\widetilde{n} = (n', e) \in \widetilde{N}$,

$$g \widetilde{n} g^{-1} = (n, h) \cdot \left(n' \varphi\left(h^{-1}\right)\left(n^{-1}\right), h^{-1}\right) = \left(n \varphi(h)(n') n^{-1}, e\right) \in \widetilde{N}. \tag{10}$$

We have immediately that $\widetilde{N}$ is a normal subgroup and that we can define $\varphi_{\widetilde{h}}(\widetilde{n}) = \widetilde{h} \widetilde{n} \widetilde{h}^{-1}$ for $\widetilde{h} = (e, h) \in \widetilde{H}$ and $\widetilde{n} = (n, e) \in \widetilde{N}$ such that G is the inner semidirect product of $\widetilde{N}$ and $\widetilde{H}$ with the same group operation.

1.3 Examples

All translations and rotations in $\mathbb{R}^3$ forms a group which is the semidirect product of translation group and rotation group, $\mathbb{R}^3 \rtimes_\varphi \mathrm{SO}(3)$, where $\varphi : \mathrm{SO}(3) \mapsto \mathrm{Aut}\left(\mathbb{R}^3\right)$ is defined as

$$\varphi(R)\left(T(\boldsymbol{x})\right) = T\left(R\boldsymbol{x}\right) \tag{11}$$

with $T\left(\boldsymbol{x}\right) \in \mathbb{R}^3$ the translation by $\boldsymbol{x}$ and $R \in \mathrm{SO}(3)$ a rotation. The general element of the group is $(T(\boldsymbol{x}), R)$, with the product

$$\left(T\left(\boldsymbol{x_1}\right), R_1\right) \cdot \left(T\left(\boldsymbol{x_2}\right), R_2\right) = \left(T\left(\boldsymbol{x_1}\right) T\left(R_1 \boldsymbol{x_2}\right), R_1 R_2\right). \tag{12}$$

If we interprete $\left(T\left(\boldsymbol{x}\right), R\right)$ as rotation by R followed a translation by $\boldsymbol{x}$, the product above has the proper physical meaning. Of course we can view (T, R) as a product TR in the whole group, with the latter interpreted as the inner semidirect product of $\mathbb{R}^3$ and $\mathrm{SO}(3)$.

A widely use generalisation of the above example is the Poincaré group

$$\mathbb{R}^{1,3} \rtimes \mathrm{O}(1,3), \tag{13}$$

which forms the foundation of relativistic quantum field theory.

References

[1] Direct product of groups. https://en.wikipedia.org/wiki/Direct_product_of_groups. Accessed: 2021-7-23.

[2] Semidirect product. https://en.wikipedia.org/wiki/Semidirect_product. Accessed: 2021-7-23.