Transformations in Physics

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1 Hubbard-Stratonovich transformation

Condider the identity

$$\begin{aligned}&\int \mathrm{d}[\kappa]\exp\left\{-\frac{1}{2\hbar}\int_0^{\hbar\beta}\mathrm{d}\tau\mathrm{d}\tau'\left(\kappa(\tau)-\kappa_0(\tau)\right)M\left(\tau,\tau'\right)\left(\kappa\left(\tau'\right)-\kappa_0\left(\tau'\right)\right)\right\}\\&\times\exp\left\{\frac{1}{2}\mathrm{Tr}[\log(M/\hbar)]\right\}=1,\end{aligned} \tag{1}$$

or for the complex or Grassmann field,

$$\begin{aligned}&\int \mathrm{d}\left[\kappa^*\right]\mathrm{d}[\kappa]\exp\left\{-\frac{1}{2\hbar}\int_0^{\hbar\beta}\mathrm{d}\tau\mathrm{d}\tau'\left(\kappa^*(\tau)-\kappa_0^*(\tau)\right)M\left(\tau,\tau'\right)\left(\kappa\left(\tau'\right)-\kappa_0\left(\tau'\right)\right)\right\}\\&\times\exp\left\{\frac{\zeta}{2}\mathrm{Tr}[\log(M/\hbar)]\right\}=1,\end{aligned} \tag{2}$$

where for complex variable $\zeta = 1$ and for Grassmann variable $\zeta = -1$, by choosing proper M and κ_0, the undesired term in a functional integral can be cancelled.

As an example, consider Bose-Hubbard model in which we want to treat the hopping term as perturbation. The grand-canonical partition function is given by

$$Z=\mathrm{Tr}\left[\mathrm{e}^{-\beta\hat{H}}\right]=\int \mathrm{d}\left[a^*\right]\mathrm{d}[a]\exp\left\{-\frac{1}{\hbar}S\left[a^*,a\right]\right\}, \tag{3}$$

with the action

$$\begin{aligned}S\left[a^*,a\right]=\int_0^{\hbar\beta}\mathrm{d}\tau&\left\{\sum_i a_i^*(\tau)\left(\hbar\frac{\partial}{\partial\tau}-\mu\right)a_i(\tau)\right.\\&\left.-\sum_{\langle i,j\rangle}t_{ij}a_i^*(\tau)a_j(\tau)+\frac{U}{2}\sum_i a_i^*(\tau)a_i^*(\tau)a_i(\tau)a_i(\tau)\right\}.\end{aligned} \tag{4}$$

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With $M_{ij}(\tau, \tau') = 2t_{ij}\delta(\tau - \tau')$, $\kappa_{0,i}(\tau) = a_i(\tau)$ and $\kappa_i(\tau) = \psi_i(\tau)$, the action becomes

$$\begin{aligned} S\left[a^*, a, \psi^*, \psi\right] = &\int_0^{\hbar\beta} \mathrm{d}\tau \sum_i \left\{ a_i^*(\tau)\left(\hbar\frac{\partial}{\partial\tau} - \mu\right) a_i(\tau) + \frac{U}{2} a_i^*(\tau) a_i^*(\tau) a_i(\tau) a_i(\tau) \right\} \\ &+ \int_0^{\hbar\beta} \mathrm{d}\tau \sum_{i,j} \left\{ -t_{ij}\left(a_i^*(\tau)\psi_j(\tau) + \psi_i^*(\tau) a_j(\tau)\right) + t_{ij}\psi_i^*(\tau)\psi_j(\tau) \right\}. \end{aligned} \tag{5}$$

Note that this transformation adds a field to the action which corresponds to the superfluid order parameter introduced in the mean-field theory.