Floretions and Actions of S_3 (DRAFT ver. 20231203)

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December 3, 2023

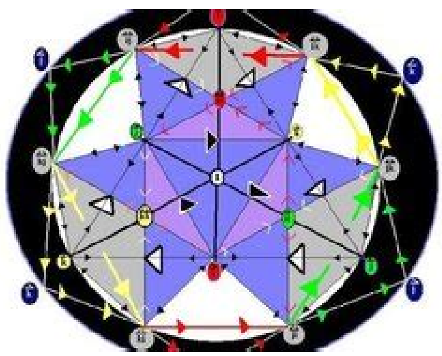

Figure 1: One of the first diagrams way back in 2003 (2nd order floretions) featuring an equilateral triangle. The connection with equilateral triangles for higher orders was made only recently, however.

Abstract

An equilateral triangle can be tiled into 4^n smaller equilateral triangles by iteratively joining the midpoints of each side. For any $n \in \mathbb{N}^+$, there are exactly 4^n (positive) floretion base vectors $b_1, b_2, \cdots, b_{4n}$ and each base vector can be associated with a specific tile in a certain order. Moreover, a base vector "knows" whether its triangle tile is facing up or down depending on whether $(b_i)^2 = \pm 1$ and whether n is odd or even. There are at least 3 different ways to characterize whether a floretion is symmetric about a triangle axis, two of these involving group actions via the dihedral group D_3 and symmetric group S_3 and one using floretion multiplication directly. An attempt to outline proofs of the equivalency of these group actions is made. We will want to know: if x and y are symmetric about an axis, under what conditions is $x \cdot y$ also symmetric? Moreover, questions along the lines of "What does it look like when we multiply the Sierpinski Gasket by a quaternion?" appear to make sense.

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1 Basic Introduction

Given two whole numbers, e.g. 791 and 636, two questions to ask first are 1. Are these floretions? 2. How is multiplication defined? A formal definition is given here A308496. An in depth paper for 2nd order only (as it was written before the general definition) which includes matrix representations is here. Assuming the reader is familar with quaternion multiplication, here is a short answer:

Step 1: Convert each number to octal: 791 → 1427, 636 → 1174

Step 2: Check that all digits in octal are from the set $S = \{1, 2, 4, 7\}$. If not, these are not floretions.

Step 3: Check that the order of each floretion is the same. The order is the number of digits in octal [1] . Both 791 and 636 have 4 octal digits. If they are not the same order, they are members of two separate groups and cannot be multiplied together.

Step 4: Associate digits with quaternions as: 1 ↔ i, 2 ↔ j, 4 ↔ k, 7 ↔ e where the latter is the unit. Thus 1427 becomes ikje and 1174 becomes iiek

[1] This is *not* the standard definition of the order of a group element, i.e. the smallest elment n such such that $g^n = unit$, which in our case would always be 1 or 2. See up/down parity, below

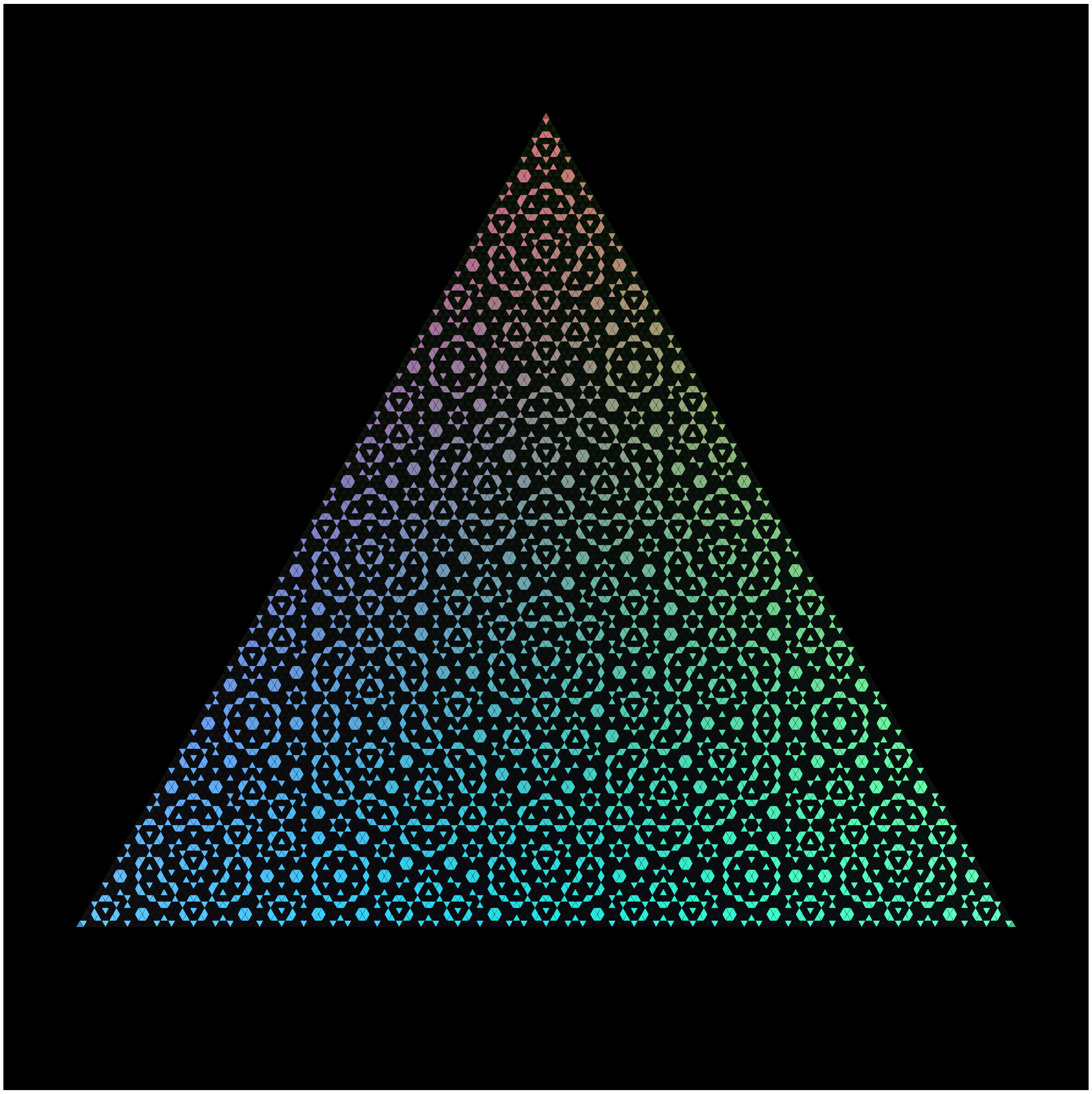


Figure 2: a 7-th order floretion.

Step 5: Multiply ikje $\cdot$ iiek quaternion by quaternion, bringing any negative signs out to the front:

$$\begin{aligned}
\mathrm{ikje} \cdot \mathrm{iiek} &= (\mathrm{i} \cdot \mathrm{i})(\mathrm{k} \cdot \mathrm{i})(\mathrm{j} \cdot \mathrm{e})(\mathrm{e} \cdot \mathrm{k}) \\
&= (-\mathrm{e})(\mathrm{j})(\mathrm{j})(\mathrm{k}) \\
&= -\mathrm{ejjk}
\end{aligned}$$

Note the above is just to calculate the result of a single multiplication *without* the help of a computer. For general cases this is error prone: generally it's better to use bitwise logic operators[2], in particular ***XNOR*** for multiplication disregarding signs and ***AND*** to incorporate signs. Let's start with the first 4 elements of the sequence A308496: 1, 2, 4, 7 and examine these as binary triplets: 001, 010, 100, 111. The claim is that these are the (positive) quaternions in disguise. To see this, note first that if (abc) is any such binary triplet, any two digits determine other since XOR(a,b,c) = 1. Thus all elements take the form $x = ab$ where ab 00, 01, 10, 11. That the following formal definition actually works is surprising.

Definition 1.1: XNOR Definition of Floretion Multiplication

Given two binary representations $x = ab$ and $y = cd$ for quaternion elements, $x \cdot y$ is defined as:

$$x \cdot y = (ab)(cd) = (-1)^{m+1}(a \odot c)(b \odot d)$$

where $m = b \wedge c + (a \odot b) \wedge d + a \wedge (c \odot d)$, and $\odot$, $\wedge$ represent the bitwise XNOR and AND operators, respectively.

An example is in order:

[2] Or in theory some optimized matrix multiplication routines, given that n-th order floretions also have a matrix representation.

Example 1.1: XNOR Multiplication

Consider the multiplication of two floretions k and j, represented as $k = (10)$ and $j = (01)$, then their product is computed as follows:

$$k \cdot j = (10) \cdot (01) = -i$$

The base vector is obtained by performing the bitwise XNOR on each corresponding pair of digits:

$$\begin{aligned}(XNOR(1,0), XNOR(0,1)) &= (1 \odot 0, 0 \odot 1) \\ &= (0,0) \\ &= i\end{aligned}$$

The sign is determined by computing the bitwise AND of the XNOR results, and the original digits:

$$\begin{aligned}AND(0,1) &= 0 \wedge 1 \\ AND(XNOR(1,0),1) &= (1 \odot 0) \wedge 1 = 0 \wedge 1 \\ AND(1, XNOR(0,1)) &= 1 \wedge (0 \odot 1) = 1 \wedge 0\end{aligned}$$

These operations result in $0, 0, 0$ respectively. Since the total number of 1's is zero, which is an even number, the final result is negative. Thus, we have:

$$k \cdot j = -i$$

For a Python implementation of floretions using bitwise logic operators, see Github: Floretions. Is the definition just used for fast multiplication? The short answer is *no*- it points towards deeper results, as in the "Centralizer Decomposition for Base Vectors", below.

Definition 1.2: Symbols Used

For $n \in \mathbb{N}^+$, let $b_1, b_2, \cdots, b_{4^n}$ be all positive floretion base vectors of order n.

1. $\mathcal{F}_n = \{\pm b_1, \pm b_2, \cdots \pm b_{4^n}\}$ This is the group of floretions.

2. $\Delta_n = \{b_1, b_2, \cdots, b_{4^n}\}$ The set of (positive) base vectors only, not a group. To each vector corresponds a well-defined equilateral triangle tile in a larger equilateral triangle.

3. $\mathcal{A}(\Delta_n) = \{q_1 b_1 + \cdots + q_{4^n} b_{4^n} | q_i \in \mathbb{R}, b_n \in \Delta_n\}$ The algebra over the reals generated by the floretion base vectors of order n. Each element can be considered a set of tiles weighted by the real coefficients in some way (e.g. graphically in 2D by coloring or in 3D by height above or below zero). May also be referred to simply as *floretions*.

4. Dihedral Group D_3: $\{R_0, R_{120}, R_{240}, \sigma_I, \sigma_J, \sigma_K\}$ (rotations and reflections)

5. Symmetric Group S_3: $\{(12), (24), (41), (124), (142), (7)\}$ (permutations of the set $S = \{1, 2, 4, 7\}$)

Remark 1.1: Change of Notation

In general, it may be preferable to represent a floretion using decimal, octal, binary, quaternionic notation, actual triangle tiles, or real/complex matrices (see e.g. Integer Sequences with Musical Properties, where 2nd order floretions are defined using matrices), depending on what kind of symmetry is to be exploited. This should be clear from the application. E.g. if we are talking about even/odd parity (see example, below), we want to look at the floretion written in decimal. If the reference is to tiles on an equilateral triangle, we most likely want to use octal or quaternionic notation. On the other hand, if we have a deep result in the theory of matrices and need to apply that to floretions **or** we need to "abstract out" that any floretions were used in the first place, possibly because we only want to emphasize a certain result and not how we got there, use an isomorphic space of real matrices. An example of this is Number of white pearls in chest together with a discussion with M. Alekseyev Notes on A113166. Here I actually avoided mentioning "floretions" to avoid distractions and used matrices instead, although these were originally found looking at specfic floretions of the form $x^2 = 0$ which I considered particularly aesthetic. The group of 2nd order floretions is isomorph to the quaternion factor space $Q \times Q/\{1, -1\}$, so we could, of course, just work on that space with elements of the form (i, j), etc. It's reminiscent of how one patient teacher explained the equality of the fractions, e.g. $\frac{3}{2}$ and $1\frac{1}{2}$ to us in the 6th grade: *It's the same dude, it's just that he sometimes prefers to dress in his 3 piece suit*[a]. Let it be known that floretions have a whole chest full of clothes! One final note: in this paper, octal and quaternionic notation is used almost interchangeably without explicitely stating, e.g. an example may refer to base vector **ijke** and the accompanying image may show **1247** without further comment.

[a] My follow up question was "What **is** a 3-piece suit?"

Remark 1.2: Floretions are well-ordered

Floretion base vectors are well-ordered, i.e. for any $x, y \in \Delta_n$ we have $x < y$, $y > x$ or $x = y$. Note multiplication is not *not* transitive in general.

Example 1.2: Even/Odd Parity

If $x, y \in \Delta_n$ are given as decimals and "$\cdot$" is floretion multiplication, then $x \cdot y$ is odd if x and y are both even or both odd, otherwise $x \cdot y$ is even. This is a property that follows directly from the **XNOR** definition of multiplication.

Example 1.3: Up/Down Parity

If x is a base floretion, then $x^2 = \pm 7 \ldots 7$. This is a property that follows directly from the **XNOR** definition of multiplication. The up/down labeling will become imporant once we have associated a floretion base vector with a triangle and need to know if it is oriented up or down. Note that, given only the information that x^2 is positive or negative, one cannot say whether the associated triangle tile is oriented up or down as this also depends on whether the order of the group is even or odd, as discussed below.

2 Floretions and Equilateral Triangles

2.1 Main Centroid Algorithm

This is main algorithm associating floretions with equilateral triangles. Note there is no actual drawing of triangles yet- that is a subsequent step. Instead, the coordinates of the triangle centers (centroids) along with the heights associated with them are returned. Up/down parity then determines whether the triangle is pointing up or down. For each octal digit encountered (with the total number of digits corresponding to the order of the floretion), the triangle height halves. Each time a '7' is encountered as a digit in a base vector, the position stays the same but the direction changes by 180 degrees and the distance still decreases by half for the following digit, provided it isn't already the last one. Seeing that this directional change is absolutely necessary was the "missing link" to solving the stubborn issue of why only certain base vectors were being mapped correctly.

```

    def place_base_vecs(self, base_vector):
        """
        For each base vector of a given order, returns the
        coordinates of the center (centroid) of the
        equilateral triangle associated with it, along
        with the final distance that determines
        the triangle height.

        Args:
            base_vector (str):
          The base vector in octal representation.

        Returns:
        tuple: x, y coordinates, final distance
        """

```

```
        # Initialize x and y here at (0,0)
        x, y = 0,0

        dist = self.height // self.distance_scale_fac
        sign_dist = 1


        for digit in base_vector:

            if digit == '7':
                sign_dist *= -1
            else:
                if digit == '1':
                    angle = 210

                elif digit == '2':
                    angle = 90

                elif digit == '4':
                    angle = 330
                else:
                    print(f"Invalid digit {digit}")
                    return

                # Only excecuted when digit not 7
                x += np.cos(np.radians(angle))*dist*sign_dist
                y += np.sin(np.radians(angle))*dist*sign_dist

            # Halve the distance for the next iteration
            dist /= 2

        return x, y, 2 * dist
```

2.2 Properties of the centroid algorithm

Claim 2.1: Main Triangle Corners

If a base vector $b \in \Delta_n$ has all octal digits in exactly one of the sets $\{1\}$, $\{2\}$, or $\{4\}$, then it must be one of the 3 corner triangles of the main triangle.

Proof of Main Triangle Corners

If a base vector consists solely of a single digit 1, 2, or 4, the function will move in only one of the specified angles (210, 90, or 330 degrees, respectively). Thus, it will reach one of the corners as it can only move in a straight line and travels the furthest possible distance from the center.

Claim 2.2: Main Triangle Primary Axes

If a base vector $b \in \Delta_n$ has all octal digits in exactly one the sets $\{1,7\}$, $\{2,7\}$, or $\{4,7\}$, then it must lie at an angle 90, 210, or 330 degrees from the center, i.e. along one of the 3 primary axes.

Proof of Main Triangle Primary Axes

This is by simple inspection of the algorithm.

Definition 2.3: Actions on Floretions

Consider two group actions on $A(F_n)$:

1. $D_3 \times A(F_n) \to A(F_n)$ Rotation and reflection of the corresponding centroids in the plane.

2. $S_3 \times A(F_n) \to A(F_n)$ Permutation of the octal digits.

Example 2.1: Action Stabilizers

$n = 3$ and $x = 111 + 222$:

1. Stabilizer under $D_3 : \mathrm{Stab}_{D3}(x) = \{\mathrm{id}, \sigma_K\}$

2. Stabilizer under $S_3 : \mathrm{Stab}_{S3}(x) = \{(7), (12)\}$

Claim 2.4: Equivalence of Dihedral and Symmetric Group Actions

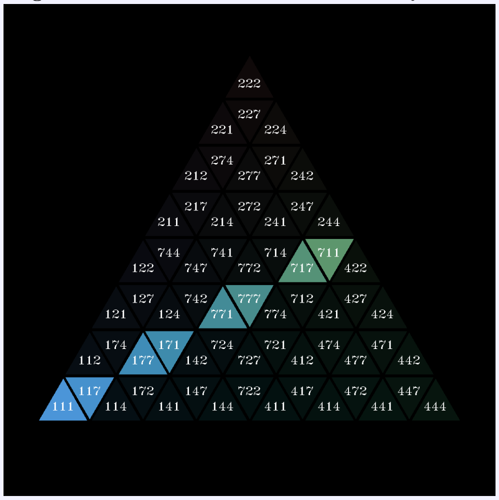
Swapping all digits octal 2 and 4 $b \in \Delta_n$ (e.g. $1274 \to 1472$) is equivalent to performing the operation $\sigma_I b$. Similar statements hold for swapping digits 1 and 4: $\sigma_J b$ and digits 1 and 2: $\sigma_K b$

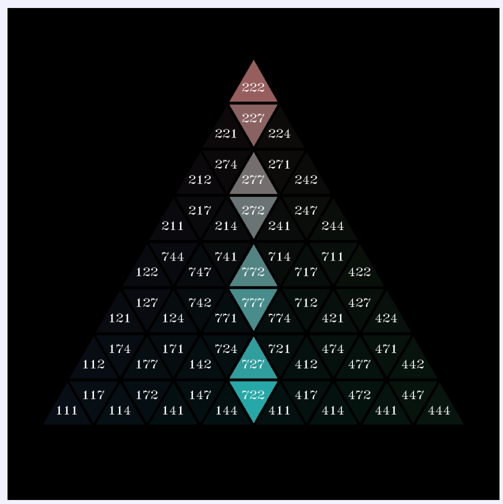
Proof of Equivalence of Dihedral and Symmetric Group Actions

This is the least obvious of the three claims. Note that the digits in a base vector essentially determine a "path" that the algorithm takes. It must be shown that the operation of swapping digits creates a mirror path. This has been brought up on Mathoverflow, see Equivalence of dihedral and symmetric group actions on a specialized real algebra

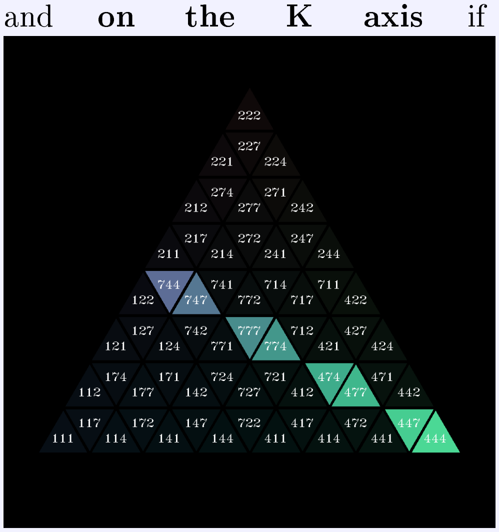
Definition 2.5: On the I-J-K Axis

A floretion $x \in \mathcal{A}(\Delta_n) = \{q_1 b_1 + \cdots + q_{4^n} b_{4^n} | q_i \in \mathbb{R}, b_n \in \Delta_n\}$ is **on the I axis** if all octal digits from each base vector b_i are in the set $\{1, 7\}$. Third order example:





It is **on the J axis** if all octal digits are in the set $\{2, 7\}$ and **on the K axis** if all octal digits are in the set $\{4, 7\}$



Definition 2.6: Floretions Symmetric about a Primary Axis

A floretion $x = q_1 b_1 + \cdots + q_{4^n} b_{4^n} \in \mathcal{A}(\Delta_n)$ is said to be **symmetric about the axis-I** if and only if for each base vector b_i either b_i is on the I-axis or there exists another base vector b_j of x such that $|q_i| = |q_j|$ and $|b_i \cdot b_j|$ is on the I-axis. **Symmetric about the J-axis and K-axis** are defined similarly.

Example 2.2: Three Characterizations of "Symmetry About an Axis"

We have now accumulated three different ways of characterizing symmetry about an axis. For example, $x = 111 + 222$ is symmetric about the K-axis because

1. $\sigma_c \in \mathrm{Stab}_{D3}(x)$

2. $(12) \in \mathrm{Stab}_{S3}(x)$

3. $111 \cdot 222 = 444$ is on the K-axis (by def 2.5) so is symmetric about the K-axis (by def 2.6)

An obvious question is: are these equivalent?

Proposition 2.7: Symmetric Floretions are Geometrically Symmetric

All three above characterizations are equivalent, i.e. if one holds, the other two follow.

Proof

Ongoing.

Proposition 2.8: Corner Symmetry Principle

For an equilateral triangle tiling of order n, consider the sets of triangles parallel to any two corners of the three corners i . . . i, j . . . j, k . . . k (each repeated n times). Then the product of all elements in each set (ignoring signs) in any order yields the other corner triangle.

Example 2.3: Corner Symmetry Principle

For order $n = 2$ and the corners ii $-$ kk we have 4 rows:

$$\{\mathrm{jj}\}, \{\mathrm{ji}, \mathrm{je}, \mathrm{jk}\}, \{\mathrm{ij}, \mathrm{ek}, \mathrm{ee}, \mathrm{ei}, \mathrm{kj}\}, \{\mathrm{ii}, \mathrm{ie}, \mathrm{ik}, \mathrm{ej}, \mathrm{ki}, \mathrm{ke}, \mathrm{kk}\}$$

and

$$\mathrm{jj} = |\mathrm{ji} \cdot \mathrm{je} \cdot \mathrm{jk}| = |\mathrm{ij} \cdot \mathrm{ek} \cdot \mathrm{ee} \cdot \mathrm{ei} \cdot \mathrm{kj}| = |\mathrm{ii} \cdot \mathrm{ie} \cdot \mathrm{ik} \cdot \mathrm{ej} \cdot \mathrm{ki} \cdot \mathrm{ke} \cdot \mathrm{kk}|$$

Proof of Corner Symmetry Principle

By induction. For quaternions, we have

$$i = j \cdot e \cdot k, \quad j = k \cdot e \cdot i, \quad k = i \cdot e \cdot j,$$

and this proves the case for $n = 1$.

Assume the claim holds for n. Let $T(n+1)$ be the triangle tiling associated with order $n+1$ base vectors. Consider the four main subtriangles: $T(n+1, i)$, $T(n+1, j)$, $T(n+1, k)$, $T(n+1, e)$. Assume we have a row parallel to side i-k. Then there are only two options:

1. The row passes entirely through $T(n+1, j)$.
2. The row passes through all three subtriangles $T(n+1, i)$, $T(n+1, e)$, $T(n+1, k)$.

Case 1: If the row passes through $T(n+1, j)$, it is easy to see that the first digit of the product of all base vectors from the row must begin with 2. Why? Because there are always an odd number of base vectors in each row, and each base vector must have a 2 as its most significant digit. So we have an odd number of 2's multiplied by each other, which must also yield 2 (for the same reason that $|j^m| = j$ for odd m). Let's remove the first significant digit from all the base vectors in the row—now we have $T(n)$! Moreover, this row is also parallel to side i-k for $T(n)$, so by induction the result must be $j \dots j$ (n-times). Including the most significant digit from above, we have $j \dots j$ ($n+1$ times), thus the claim holds.

Case 2: If the row passes through $T(n+1, i)$, $T(n+1, e)$, $T(n+1, k)$, let us write the product of base vectors $b_1 \cdot b_2 \cdot \ldots \cdot b_m$ as $(b_1 \cdot \ldots \cdot b_q) \cdot (b_{q+1} \cdot \ldots \cdot b_{2q}) \cdot (b_{2q+1} \cdot \ldots \cdot b_{3q})$ where all the base vectors from the first parenthesis come from $T(n+1, i)$, and the others from $T(n+1, e)$, $T(n+1, k)$. Using a similar consideration as with case 1, we know the most significant digit of each subproduct is i, e and k respectively. But $i \cdot e \cdot k = j$, so again the most significant digit must be 2. We can continue as in case 1 removing the most significant digit from each base vector. Then we have 3 copies of $T(n)$ and by induction we get $(j \dots j) \cdot (j \dots j) \cdot (j \dots j)$ with $3 \cdot n$ j's. But $|(j \dots j) \cdot (j \dots j) \cdot (j \dots j)| = (j \dots j)$. As we just as easily could have assumed the row was parallel to sides i-j or sides j-k, we are done. ■

(the following is just a placeholder name until I can think of something better)

Proposition 2.9: Beautiful Animals, Beautiful Offspring Principle

If $x, y \in \mathcal{A}(\Delta_n)$ are both simultaneously symmetric about the same axis, then so is $[x, y] = x \cdot y - y \cdot x$ and $x \cdot y + y \cdot x$. A direct corallary is that if x, y commute, then symmetry carries over to $x \cdot y$. There is also a good chance that, if the tiling associated with $x \cdot y$ visually appears "almost symmetric" to an axis, then x and y "almost" commute and viceversa.

Proof

This is perhaps one of the harder proofs *pour moi*, though it is possible it can also be tackled inductively.

2.3 Equilateral Triangle Tiles

Since we know the positions of the centroids of every base vector tile and it's corresponding height, we can draw the triangles once we know their orientation. Mathematically, this is equivalent to defining a set of three vertices $V = \{(x_1, y_1), (x_2, y_2), (x_3, y_3)\}$, where (x, y) is the previously calculated center, and the height h is determined by the distance factor.
The vertices are calculated based on trigonometric relations:

$$x_{i+1} = x + \sin\left(\frac{\pi}{3}\right) \times h, \quad y_{i+1} = y \pm \frac{h}{2}$$

where the $\pm$ depends on the orientation as determined by the Orientation function.
The last step is to consider the up/down parity of the base vector in order to set the orientation of tiles. Each tile within the tiling is an equilateral triangle and can have two orientations: either pointing upwards or downwards. The orientation of a tile corresponding to a base vector is determined by the parity of the digits and the order of the floretion group to which the base vector belongs.

For $b \in \Delta_n$ $b^2 = 1$, then

$$\text{Orientation}(b) = \begin{cases} \text{up} & \text{if } n \text{ even} \\ \text{down} & \text{if } n \text{ odd} \end{cases}$$

If $b^2 = -1$

$$\text{Orientation}(b) = \begin{cases} \text{down} & \text{if } n \text{ even} \\ \text{up} & \text{if } n \text{ odd} \end{cases}$$

Proposition 2.10: Triangle Orientation Determined by Up/Down Parity of Floretion

The triangle orientation determined by up/down parity of a floretion does not lead to any geometric contradictions, i.e. it "works" and does not lead to overlapping triangles.

Proof

Author's note: Unsure about the difficulty of this one.

Once we have a tiling of the equilateral triangle to represent all base vectors, we can then represent $x = q_1 f_1 + \cdots + q_n f_n \in \mathcal{A}(\Delta_n)$ as a "scuplture" of tiles. This does introduce a slight problem, however, because just by looking where white is "on" and black is "off", there is no way to tell whether the coefficients are positive or negative. If it becomes necessary to distinguish between positive and negative coefficients, there are many options- for example showing blue if a coefficient is positive and red if negative where the brightness is determined by the size of the coefficient or placing an even smaller triangle inside to denote negativity. Another unexplored option in three dimensions would be to give the triangles a positive or negative height.

3 Specific Tiles

3.1 Centralizer Tiles

It is easy to see that the conjugacy classes of floretions of any order are always of the form $\{g, -g\}$. Allowing $\mathcal{F}_n$ to act on itself by conjugation, by the Orbit-Stabilizer Theorem we have

$$|\mathcal{F}_n| = |\mathrm{Orb}(g)| \cdot |\mathrm{Stab}(g)| = |\mathrm{cl}_{\mathcal{F}_n}(g)| \cdot |C_{\mathcal{F}_n}(g)| = 2 \cdot |C_{\mathcal{F}_n}(g)|$$

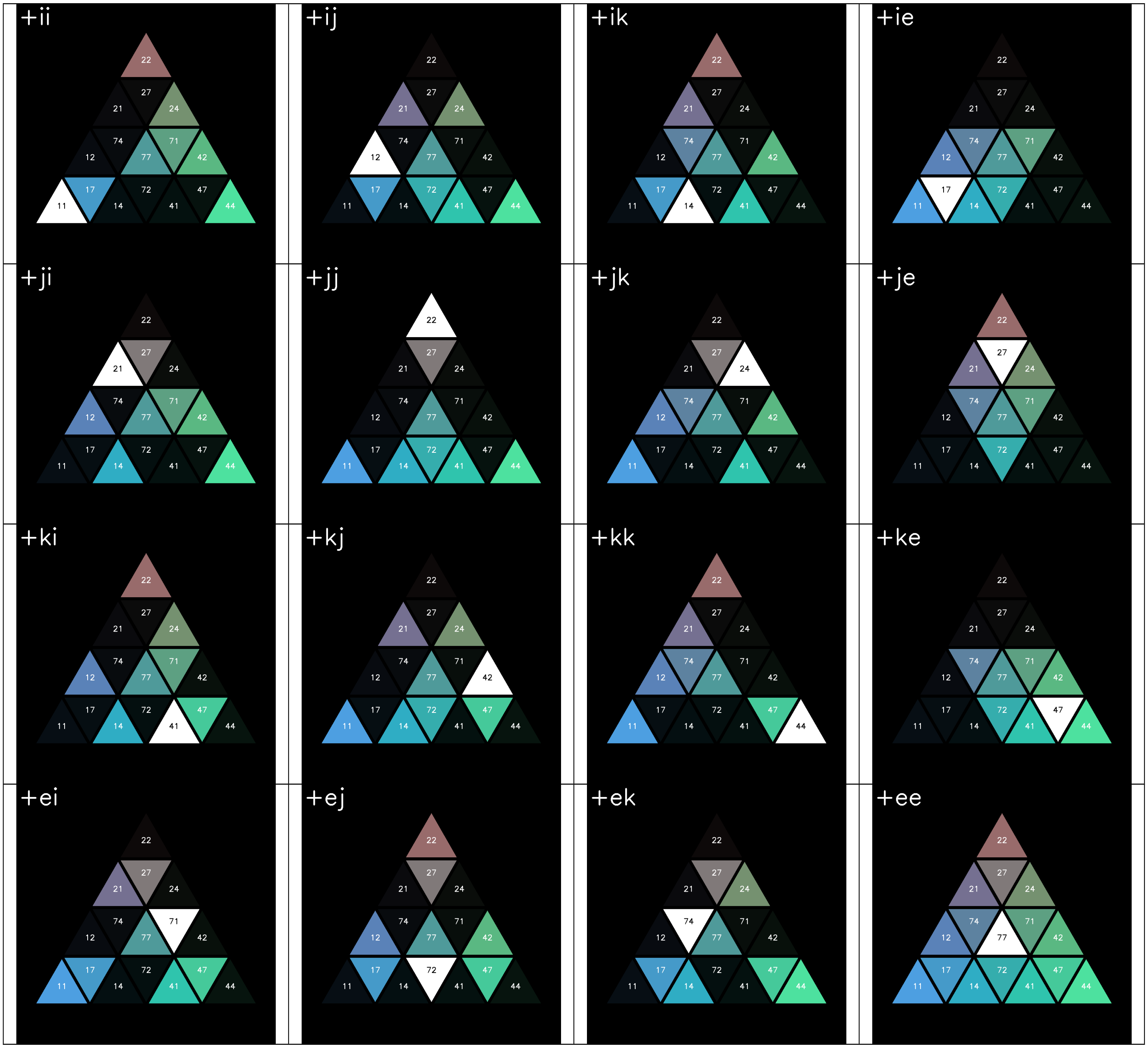
In other words, the size of the centralizer of an element is always half the size of the group, which we already know to be $2 \cdot 4^n$ (as a reminder: the factor 2 is because we are also counting negative base vectors), thus we have shown

Claim 3.1: Size of centralizer for any element g of order n

$$|C_{\mathcal{F}_n}(g)| = 4^n$$

This means if we define x as the sum of all (positive!) elements from the centralizer of some base vector of a floretion of any order and respresent this as a set of tiles, then exactly half of these tiles, i.e. $4^n/2$ will be "on". That is indeed what we find (note the solid white triangle denotes the tile of the base vector):

Table 1: Centralizers of floretion base vectors. These determine one dimensional character tables (see Appendix): **Light tiles** $\leftrightarrow +1$ **Dark tiles** $\leftrightarrow -1$



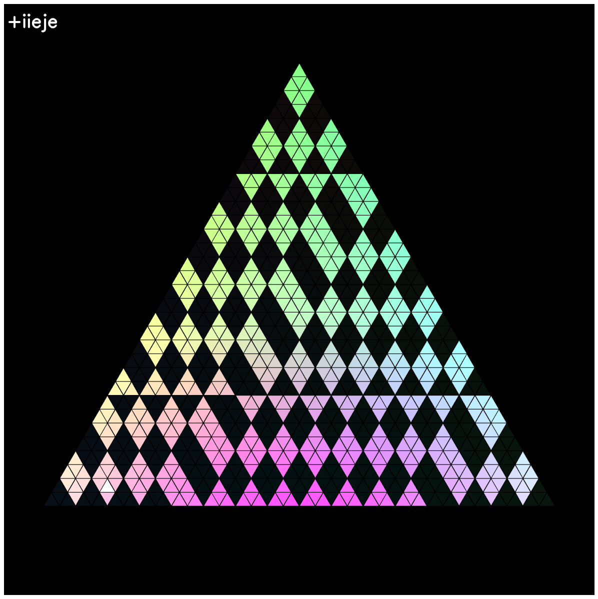


Figure 3: Sum of positive elements from $C_{\mathcal{F}_5}(11727)$

Assume we want to know if

$$\tilde{b} \cdot b = b \cdot \tilde{b}$$

for two base vectors. We know by the XNOR definition of multiplication (Def 1.1) that multiplication breaks down into two parts: the unsigned part and the signed part $m = b \wedge c + (a \odot b) \wedge d + a \wedge (c \odot d)$. Convince yourself that the unsigned part is always the same for $\tilde{b} \cdot b$ and $b \cdot \tilde{b}$, thus, we only need to test the signed parts: if $\tilde{b} \cdot b$ and $b \cdot \tilde{b}$ are both the same sign, they must commute. This means the above triangle representations all break down into the intersection of two other representations, one "positive" and one "negative".

Theorem 3.2: Centralizer Decomposition for Base Vectors

$$\begin{aligned} C_{\mathcal{F}_n}(b) = &\{\tilde{b} \in \mathcal{F}_n \mid \tilde{b} \cdot b = b \cdot \tilde{b}\} \\ = &\Big(\{\tilde{b} \in \mathcal{F}_n \mid \tilde{b} \cdot b > 0\} \bigcap \{\tilde{b} \in \mathcal{F}_n \mid b \cdot \tilde{b} > 0\}\Big) \\ \bigcup &\Big(\{\tilde{b} \in \mathcal{F}_n \mid \tilde{b} \cdot b < 0\} \bigcap \{\tilde{b} \in \mathcal{F}_n \mid b \cdot \tilde{b} < 0\}\Big) \end{aligned}$$

To see this in action, consider squaring the sum of the set of elements of the centralizer of 11111:

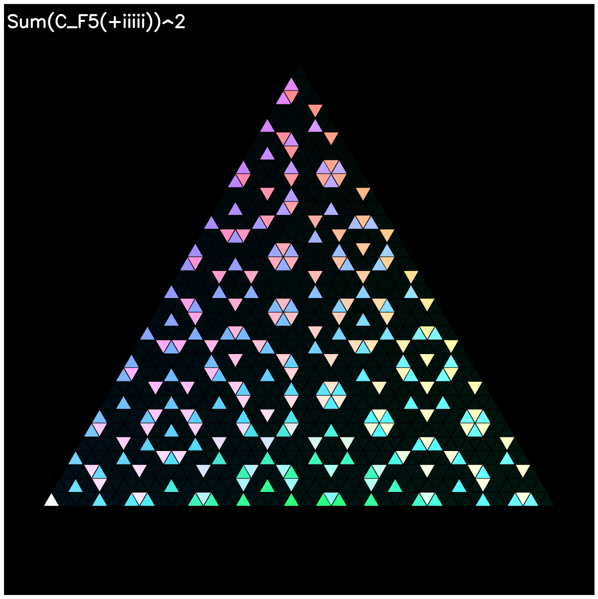


Figure 4: $\left(\sum\{\tilde{b} \in \mathcal{F}_n \mid \tilde{b} \cdot b = b \cdot \tilde{b}\}\right)^2$

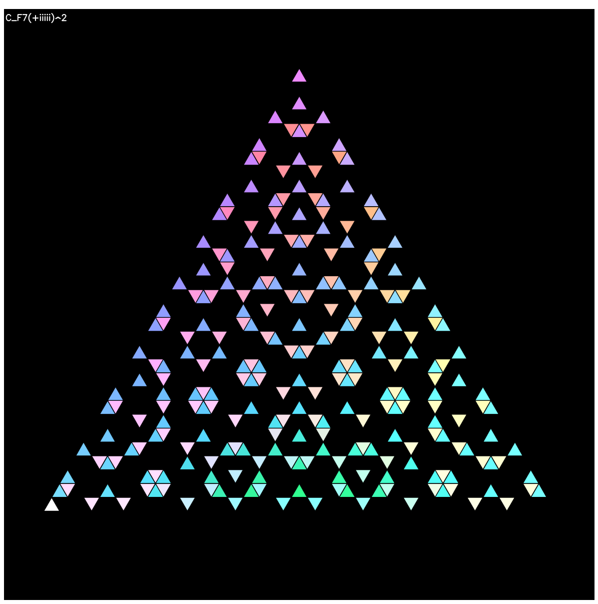


Figure 5: $\left(\sum\{\tilde{b} \in \mathcal{F}_n \mid \tilde{b} \cdot b > 0\} \bigcap \{\tilde{b} \in \mathcal{F}_n \mid b \cdot \tilde{b} > 0\}\right)^2$

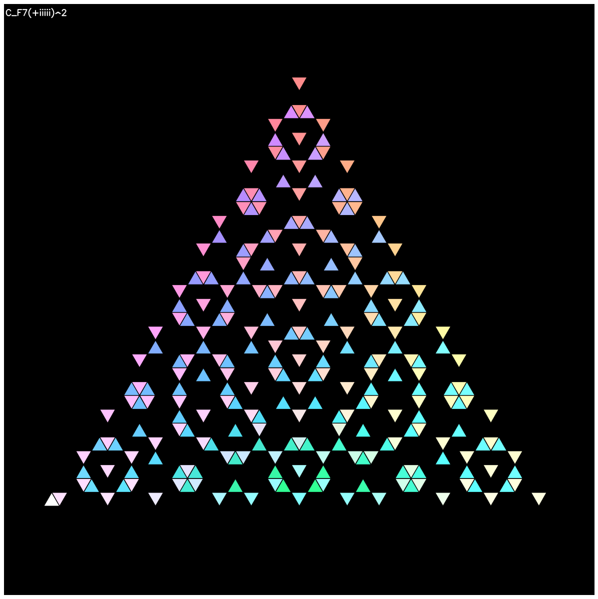


Figure 6: $\left(\sum\{\tilde{b} \in \mathcal{F}_n \mid \tilde{b} \cdot b < 0\} \bigcap \{\tilde{b} \in \mathcal{F}_n \mid b \cdot \tilde{b} < 0\}\right)^2$

3.2 The Sierpinski Gasket

Remark 3.1: Sierpinski Gasket

For Δ_n in the limit as $n \to \infty$, the Sierpinski Gasket is the set obtained by removing all triangles with digit '7'

We can do this for each order. In fact, we can go a step further by removing digits 1, 2, and 4. Below is a 5th order tiling example.
We can go even further at this point and as see what it means to multiply a floretion by the Sierpinski Gasket(s). Let's start with the easiest non trivial example: a quaternion times a 5th order Sierpinski Gasket. Now I can hear the interjection

> But you can't do that- a quaternion is first order and first order floretions cannot be multiplied by 5th order floretions!
>
> That is quite correct, but consider the fact that the quaternions are embedded (in the sense of being an isomorphic subgroup) in floretion groups of all orders ≥ 1 [3]. Specifically, $\{\pm 1, \pm 7\} \cong \{\pm 17\ldots 7, \pm 77\ldots 7\}$

[3] And more generally, being a 2-group, we have proper subgroups of the form $2 \cdot 4^m$ for $m < n$ by the First Sylow Theorem

Therefore, yes, we can do it, but will it lead to some new, finite group and, if so, what significance would this group have? I believe this is likely, but with two conditions. First, proper norming should be established. Emperical evidence suggests that the correct norming factor is $\left(\sqrt{\frac{1}{3}}\right)^{n}$ for each of the gaskets composed of floretions of order n. Secondly, the signs of the base vectors may play an important role and it may be necessary to understand what that is. Pictured below is a "quaternion" times the

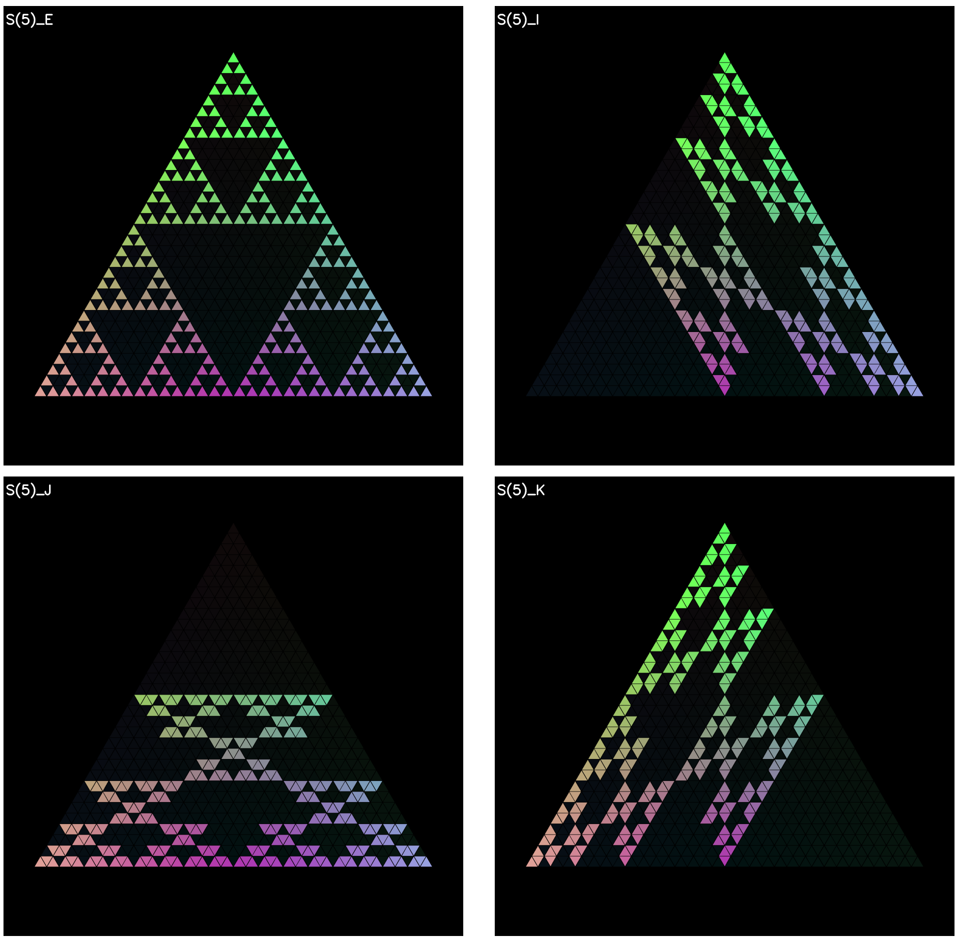


Figure 7: Removing all digits 7, 1, 2 and 4.

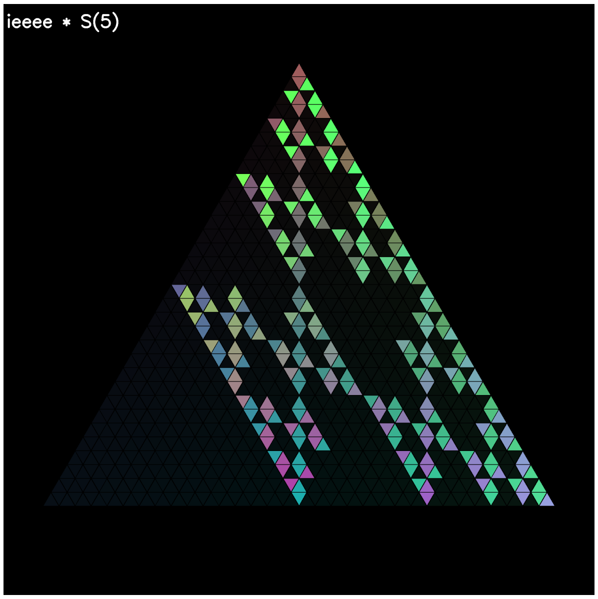


Figure 8: 17777 times the Sierpinski Gasket (no digit 7)

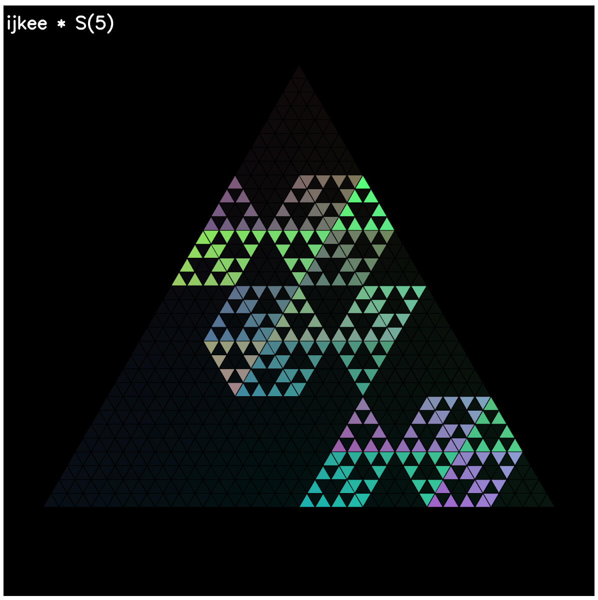


Figure 9: 12477 times the Sierpinski Gasket (no digit 7)

A Appendix

A.1 Character table of 1-dim irreps for 2nd order base vectors

Table 2: Character table of 1-dim irreps for 2nd order base vectors

Conjugacy Classes	{11, -11}	{12, -12}	{14, -14}	{17, -17}	{21, -21}	{22, -22}	{24, -24}	{27, -27}	{41, -41}	{42, -42}	{44, -44}	{47, -47}	{71, -71}	{72, -72}	{74, -74}	{77, -77}
+11 $\longleftrightarrow$ +ii	1	-1	-1	1	-1	1	1	-1	-1	1	1	-1	1	-1	-1	1
+12 $\longleftrightarrow$ +ij	-1	1	-1	1	1	-1	1	-1	1	-1	1	-1	-1	1	-1	1
+14 $\longleftrightarrow$ +ik	-1	-1	1	1	1	1	-1	-1	1	1	-1	-1	-1	-1	1	1
+17 $\longleftrightarrow$ +ie	1	1	1	1	-1	-1	-1	-1	-1	-1	-1	-1	1	1	1	1
+21 $\longleftrightarrow$ +ji	-1	1	1	-1	1	-1	-1	1	-1	1	1	-1	1	-1	-1	1
+22 $\longleftrightarrow$ +jj	1	-1	1	-1	-1	1	-1	1	1	-1	1	-1	-1	1	-1	1
+24 $\longleftrightarrow$ +jk	1	1	-1	-1	-1	-1	1	1	1	1	-1	-1	-1	-1	1	1
+27 $\longleftrightarrow$ +je	-1	-1	-1	-1	1	1	1	1	-1	-1	-1	-1	1	1	1	1
+41 $\longleftrightarrow$ +ki	-1	1	1	-1	-1	1	1	-1	1	-1	-1	1	1	-1	-1	1
+42 $\longleftrightarrow$ +kj	1	-1	1	-1	1	-1	1	-1	-1	1	-1	1	-1	1	-1	1
+44 $\longleftrightarrow$ +kk	1	1	-1	-1	1	1	-1	-1	-1	-1	1	1	-1	-1	1	1
+47 $\longleftrightarrow$ +ke	-1	-1	-1	-1	-1	-1	-1	-1	1	1	1	1	1	1	1	1
+71 $\longleftrightarrow$ +ei	1	-1	-1	1	1	-1	-1	1	1	-1	-1	1	1	-1	-1	1
+72 $\longleftrightarrow$ +ej	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1
+74 $\longleftrightarrow$ +ek	-1	-1	1	1	-1	-1	1	1	-1	-1	1	1	-1	-1	1	1
+77 $\longleftrightarrow$ +ee	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1

B History of Floretions

It started in 2002 after "reinventing" the quaternions with a pencil and paper at home. After coming to the realization this had been done many years before, I decided to try to expand the concept again while sitting under a tree at the windmill *Am Wall* in Bremen (which, according to Wikipedia, is featured in the Phil Collins video *Take Me Home*). These were floretions of second order. After showing this to a professor at my university, she initially replied "Well that may look nice to you, but I'll believe it's a group once you've proved it!" Then she looked at it a bit longer and said "I have a strong hunch what you've found is the quaternion factor group $Q \times Q/\{1, -1\}$". She was correct!

2003 - 2015

- Exploration of Fibonacci, Pell, Triangular numbers, and n-th order linear recurrence relations.
- Development of "Cyclic, sigma, and force" transforms and projections.

Links:

- R. Munafo: Sequences Related to "Floretions"
- R. J. Mathar: STRUCTURE OF THE FLORETION GROUP
- An older floretion sequence revisited- but now looking at which triangle tiles were activated in the definition: A115032

2006

- Necklaces and prime numbers.

Links:

- Number of white pearls in chest . Discussion of elements with $x^2 = 0$ in matrix form with M. Alekseyev Notes on A113166. Here I actually avoided mentioning "floretions" at all to avoid distractions, but the matrices were originally found looking at floretions I considered especially symmetric.

2018 - Today

- Applications in image processing, neural networks (some hints at ideas only), and cryptography.
- Exploration of higher-dimensional geometric shapes, fractals, and music.

Links:

- Graphics World Tests (YouTube)
- Fractal Structures (YouTube)
- Sounds of Sequences

2019 - Today

- Utilization of bitwise logical operations for fast multiplication.
- Development of electrical circuits and software.

Link:

- Conway's Game of Life with Minimum Age & Lifespan Info (YouTube)

2023 - Today

- Representations in terms of equilateral triangles (also leading to new perspectives on previous topics).
- Aggregate properties of the "big triangle".
- Exploring centralizer representations and Sierpinski Gasket Groups.
- Examining conditions for symmetry transfer in floretion multiplication.

Note: Generally, research in various areas was paused to explore new topics, always with the feeling of just scratching the surface. For instance, a floretion version of "Conway's Game of Life", where each cell also "knows" its lifespan, was set up for representation via floretions (including musical scales), but this research direction was shifted upon realizing the fundamental importance of equilateral triangle representations. This area has multiple branches that are extensive to investigate individually.