Lecture Notes for

The Fargues–Fontaine Curve

Or: "The Fundamental Curve of p-adic Hodge Theory"

Lecturer

Johannes Anschütz

Notes typed by

Ferdinand Wagner

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This text consists of notes on the lecture Selected Topics in Algebra (The Fargues–Fontaine Curve), taught at the University of Bonn by Dr. Johannes Anschütz in the winter term (Wintersemester) 2019/20.

Some changes and some additions have been made by the author. To distinguish them from the lecture's actual contents, they are labelled with an asterisk. So any *Lemma** or *Remark** or *Proof** that the reader might encounter are wholly the author's responsibility.

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Introduction and Motivation

LECTURE 0
16^{th} Oct, 2019

The 0^{th} lecture had a lot of hard theorems and deep facts thrown at us—for purely motivational purposes! That is, none of the following is a prerequisite for this lecture; rather it shows where we're going, and parts of it will be discussed in detail.

Fix a prime p and a finite extension $K/\mathbb{Q}_p$. Let C be the completion of an algebraic closure $\overline{K}$ of K. We put $G_K = \mathrm{Gal}(\overline{K}/K)$. Note that the G_K-action on $\overline{K}$ can be continuously extended to C.

0.0.1. Theorem (Faltings, Tsuji, ...)**.** — *Let X/K be a proper smooth scheme. For $n \geqslant 0$ there exists a natural G_K-equivariant "Hodge–Tate decomposition"*

$$H^n_{\text{ét}}\big(X_{\overline{K}}, \mathbb{Q}_p\big) \otimes_{\mathbb{Q}_p} C \cong \bigoplus_{i+j=n} H^i\big(X, \Omega^j_{X/K}\big) \otimes_K C(-j)$$

0.0.2. Remark. — There are a *lot* of things in Theorem 0.0.1 that demand clarification.

(1) $H^n_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p)$ is the p-adic étale cohomology, defined as

$$H^n_{\text{ét}}\big(X_{\overline{K}}, \mathbb{Q}_p\big) \coloneqq \Big(\lim_{k\geqslant 0} H^n_{\text{ét}}\big(X_{\overline{K}}, \mathbb{Z}/p^k\mathbb{Z}\big)\Big) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p\,.$$

(2) G_K acts diagonally on the left-hand side and via $C(-j)$ on the right-hand side. Here, $M(-j)$ is a *Tate twist*. In general this is defined as $M(j) \coloneqq M \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(1)^{\otimes j}$, where

$$\mathbb{Z}_p(1) = \lim_{k\geqslant 0} \mu_{p^k}(C)\,,$$

equipped with its natural G_K-action.

(3) Theorem 0.0.1 got its name from the analogous assertion in complex Hodge theory: If Y is a compact Kähler manifold, then

$$H^n(Y, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C} \cong \bigoplus_{i+j=n} H^i\big(Y, \Omega^j_{Y/\mathbb{C}}\big)\,.$$

(4) The Tate twists are necessary to get G_K-invariance of the decomposition. To see this, take for example $X = \mathbb{P}^1_K$, $n = 2$. As $\mathbb{G}_{m,\overline{K}}$ is $\mathbb{P}^1_{\overline{K}} \setminus \{\text{two points}\}$, the left-hand side can be calculated as

$$\begin{aligned} H^2_{\text{ét}}\big(\mathbb{P}^1_{\overline{K}}, \mathbb{Q}_p\big) \cong H^1_{\text{ét}}\big(\mathbb{G}_{m,\overline{K}}, \mathbb{Q}_p\big) &\cong \mathrm{Hom}\left(\pi_1^{\text{ét}}(\mathbb{G}_{m,\overline{K}}), \mathbb{Q}_p\right) \\ &\cong \mathrm{Hom}\left(\mathbb{Z}_p(1), \mathbb{Q}_p\right) \\ &\cong \mathbb{Q}_p(-1) \end{aligned}$$

On the right-hand side, the only non-vanishing summand is $H^1(X, \Omega^1_{X/K}) \cong K$. So far, everything is ok as both sides in Theorem 0.0.1 are one-dimensional C-vector spaces. However, there can't be an G_K-equivariant isomorphism $C(-1) \cong C$, as can be seen from the following theorem.

0.0.3. Theorem (Tate). — *Let $H^*_{\mathrm{cts}}(G_K, -)$ denote continous group cohomology/Galois cohomology. With notation as above, we have*

(1) $H^*_{\mathrm{cts}}(G_K, C(j)) = 0$ *for all* $j \neq 0$.

(2) $K \cong H^0_{\mathrm{cts}}(G_K, C) \cong H^1_{\mathrm{cts}}(G_K, C)$. *In particular,* $K \cong C^{G_K}$ *(and not even this is trivial).*

0.0.4. Corollary. — *For all* $n \geqslant 0$ *and* $j \geqslant 0$ *we have*

$$H^{n-j}\big(X, \Omega^j_{X/K}\big) \cong \Big(H^n_{\text{ét}}\big(X_{\overline{K}}, \mathbb{Q}_p\big) \otimes_{\mathbb{Q}_p} C(j)\Big)^{G_K}.$$

0.0.5. Counterexample. — As a slogan, Corollary 0.0.4 shows that "p-adic étale cohomology knows Hodge cohomology". The converse, however, is not true, and in fact, it fails almost always. Here are two counterexamples.

(1) If X is an elliptic curve over K, then the Hodge–Tate decomposition shows

$$H^1_{\text{ét}}\big(X_{\overline{K}}, \mathbb{Q}_p\big) \cong C \oplus C(-1),$$

independent of X. However, the G_K-action on $H^1_{\text{ét}}\big(X_{\overline{K}}, \mathbb{Q}_p\big)$ knows if X has good or semistable reduction. So this is not seen by Hodge cohomology.

(2) If $X = \operatorname{Spec} L$, where L/K is finite, then

$$H^0_{\text{ét}}\big(X_{\overline{K}}, \mathbb{Q}_p\big) \cong \prod_{L \hookrightarrow \overline{K}} \mathbb{Q}_p,$$

on which G_K acts by permuting the factors. This action determines X. However, $H^0_{\text{ét}}\big(X_{\overline{K}}, \mathbb{Q}_p\big) \otimes_{\mathbb{Q}_p} \cong C^{[L:K]}$ only knows $[L : K]$ and not L.

A nice application of Theorem 0.0.1 and Corollary 0.0.4 is the following theorem.

0.0.6. Theorem (Ito, Veys, Kontsevich, ...). — *Let Y, Y' be smooth minimal models (i.e., smooth projective schemes over $\mathbb{C}$ with nef canonical bundle). If Y, Y' are birational, then*

$$\dim_{\mathbb{C}} H^i\big(Y, \Omega^j_{Y/\mathbb{C}}\big) = \dim_{\mathbb{C}} H^i\big(Y', \Omega^j_{Y'/\mathbb{C}}\big) \quad \textit{for all } i, j \geqslant 0.$$

Idea of the proof. It's well-known that if Y, Y' are birational and smooth minimal models, then they are K*-equivalent*. That is, there exists a diagram

$$\begin{tikzcd} & Z \arrow[dl, "f"'] \arrow[dr, "g"] & \\ Y & & Y' \end{tikzcd} \tag{0.1}$$

such that Z is proper and smooth over $\mathbb{C}$, the morphisms f and g are proper and birational, and $f^* K_Y \cong g^* K_{Y'}$ holds for the respective canonical bundles (or rather canonical divisors in this notation).

Now we *spread out* over some finitely generated $\mathbb{Z}$-algebra $A \subseteq \mathbb{C}$. This means the following: all data—the schemes Y, Y', Z together with the morphisms f and g—can be described by finitely many polynomials. Taking $A = \mathbb{Z}[\{\text{all their finitely many coefficients}\}]$

we see that all these polynomials are already defined over A. Hence also the corresponding schemes are already defined over A. To make this precise: there is a diagram

$$\begin{tikzcd} & \mathcal{Z} \arrow[dl, "\widetilde{f}"'] \arrow[dr, "\widetilde{g}"] & \\ \mathcal{Y} & & \mathcal{Y}' \end{tikzcd} \tag{0.2}$$

of schemes over A, such that (0.1) is the base-change of (0.2) along $\operatorname{Spec}\mathbb{C} \to \operatorname{Spec} A$. Since Hodge numbers are constant for proper smooth morphisms in characteristic 0, we can replace A by some suitable localization. Hence we may assume $A = \mathcal{O}_F[N^{-1}]$ for some number field $F/\mathbb{Q}$. By a p-adic integration black box we have $\mathcal{Y}(\mathbb{F}_{\ell^k}) = \mathcal{Y}'(\mathbb{F}_{\ell^k})$ for all primes ℓ such that $(\ell, N) = 1$ and all $k \geqslant 1$. Fix a prime p. If $(p, N) = 1$, then

$$H^*_{\text{ét}}\big(\mathcal{Y}_{\overline{\mathcal{F}}_\ell}, \mathbb{Q}_p\big)^{\text{ss}} \cong H^*_{\text{ét}}\big(\mathcal{Y}'_{\overline{\mathcal{F}}_\ell}, \mathbb{Q}_p\big)^{\text{ss}}$$

are isomorphic as Galois representations for all primes ℓ such that $(\ell, pN) = 1$. This is somehow implied by the Weil conjectures. Also $(-)^{\text{ss}}$ denotes semisimplification. By Chebotarev's density theorem we thus obtain

$$H^*_{\text{ét}}\big(\mathcal{Y}_{\overline{F}}, \mathbb{Q}_p\big)^{\text{ss}} \cong H^*_{\text{ét}}\big(\mathcal{Y}'_{\overline{F}}, \mathbb{Q}_p\big)^{\text{ss}}\,.$$

Now pick a prime ideal $\mathfrak{p} \mid p$ in $\mathcal{O}_F$ and put $K = F_{\mathfrak{p}}$. Then also

$$H^*_{\text{ét}}\big(\mathcal{Y}_{\overline{K}}, \mathbb{Q}_p\big)^{\text{ss}} \cong H^*_{\text{ét}}\big(\mathcal{Y}'_{\overline{K}}, \mathbb{Q}_p\big)^{\text{ss}}\,.$$

Finally, the Hodge decomposition from Theorem 0.0.1 together with Corollary 0.0.4 and a "small argument ε" (to get rid of the semisimplifications) implies

$$\dim_K H^i\big(\mathcal{Y}_K, \Omega^j_{\mathcal{Y}_K/K}\big) \cong \dim_K H^i\big(\mathcal{Y}'_K, \Omega^j_{\mathcal{Y}'_K/K}\big) \quad \text{for all } i, j \geqslant 0\,.$$

Base-changing (in a zig-zag) back to $\mathbb{C}$ finally proves the assertion. $\square$

Another nice application is the degeneration of the *Hodge–de Rham spectral sequence*. Let Y/k be a proper smooth scheme over a field k. The *de Rham cohomology* of Y is defined as the (hyper-)cohomology of the de Rham complex $\Omega^\bullet_{Y/k}$,

$$H^n_{\text{dR}}(Y/k) = H^n\left(0 \longrightarrow \mathcal{O}_Y \xrightarrow{\text{d}} \Omega^1_{Y/k} \xrightarrow{\text{d}} \Omega^2_{Y/k} \xrightarrow{\text{d}} \dots\right)\,.$$

Then, more or less by definition, there is a spectral sequence

$$E_1^{i,j} = H^j\big(Y, \Omega^i_{Y/k}\big) \Longrightarrow H^{i+j}_{\text{dR}}(Y/k)\,,$$

called *Hodge–de Rham spectral sequence*. This sequence is degenerate, which can be proved by similar methods as Theorem 0.0.6.

0.0.7. Question. — Again, one can ask whether in our original situation $H^n_{\text{ét}}\big(X_{\overline{K}}, \mathbb{Q}_p\big)$ "knows" $H^n_{\text{dR}}(X/K)$ including its Hodge filtration? This question is in part answered by the following theorem.

0.0.8. Theorem (Faltings, Tsuji, ...)**.** — *For $n \geqslant 0$ there exists a natural G_K-equivariant filtered "de Rham comparison" isomorphism*

$$H^n_{\text{ét}}\big(X_{\overline{K}}, \mathbb{Q}_p\big) \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}} \cong H^n_{\mathrm{dR}}(X/K) \otimes_K B_{\mathrm{dR}}\,.$$

0.0.9. Remark. — Again, a lot of clarifications need to be done.

(1) B_{dR} is Fontaine's field of *p-adic periods* and comes with a G_K-action. It is the fraction field of some complete DVR B^+_{dR} with residue field C (thus, abstractly, $B^+_{\mathrm{dR}} \cong C[\![t]\!]$, but this isomorphism is *not* G_K-equivariant). We have a natural filtration $\operatorname{Fil}^j B_{\mathrm{dR}} = \xi^j B^+_{\mathrm{dR}}$, where $\xi \in B^+_{\mathrm{dR}}$ is a uniformizer. The associated graded object is

$$B_{\mathrm{HT}} \coloneqq \operatorname{gr} B_{\mathrm{dR}} = \bigoplus_{j\in\mathbb{Z}} C(j)\,.$$

Thus, the de Rham comparison (Theorem 0.0.8) implies the Hodge–Tate decomposition (Theorem 0.0.1).

(2) The G_K-action is diagonally on the left-hand side and via B_{dR} on the right-hand side. Conversely, the filtration on the right-hand side is diagonally, whereas on the left-hand side it comes from B_{dR}.

(3) If $X = \mathbb{P}^1_K$ and $n = 2$, we obtain $\mathbb{Q}_p(-1) \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}} \cong B_{\mathrm{dR}}$ (we use the calculations from Remark 0.0.2(1)). Hence there exists a canonical G_K-stable line $\mathbb{Q}_p t \subseteq B_{\mathrm{dR}}$ such that G_K acts via a cyclotomic character $\chi_{\mathrm{cycl}}\colon G_K \to \mathbb{Z}_p^\times$ (i.e. $\mathbb{Q}_p t \cong \mathbb{Q}_p(1)$).
For some $\varepsilon \in \mathbb{Z}_p(1) \setminus \{0\}$ we thus get $t = \log\,[\varepsilon] \in B_{\mathrm{dR}}$. Such an element is also called "Fontaine's $2\pi\mathrm{i}$".

From now on, we will talk about stuff that will be the actual contents of the lecture. Assume that, additionally to the usual assumptions, X has *good reduction*. That is, $X = \mathfrak{X}_K$ for some smooth proper $\mathfrak{X} \to \operatorname{Spec} \mathcal{O}_K$. Let $\mathfrak{X}_0$ be the special fibre. Then we get refinement of the de Rham comparison theorem (Theorem 0.0.8):

0.0.10. Theorem (Faltings, Niziol, Tsuji)**.** — *For $n \geqslant 0$ there exists a natural G_K-equivariant filtered φ-equivariant isomorphism*

$$H^n_{\text{ét}}\big(X_{\overline{K}}, \mathbb{Q}_p\big) \otimes_{\mathbb{Q}_p} B_{\mathrm{cris}} \cong H^n_{\mathrm{cris}}\big(\mathfrak{X}_0/\mathcal{O}_{K_0}\big) \otimes_{\mathcal{O}_{K_0}} B_{\mathrm{cris}}\,.$$

0.0.11. Remark. — As usual, we should explain a lot of notation.

(1) Here, $K_0 \subseteq K$ is the maximal subextension that is unramified over $\mathbb{Q}_p$ (so p is a uniformizer of $\mathcal{O}_{K_0}$). There exists a (unique) Frobenius lift φ, which acts on $\mathcal{O}_{K_0}$.

(2) $H^n_{\mathrm{cris}}(\mathfrak{X}_0/\mathcal{O}_{K_0})$ is the *crystalline cohomology* of $\mathfrak{X}_0$ over $\mathcal{O}_{K_0}$. Roughly, this is the "de Rham cohomology of a smooth lift". It has the Frobenius φ acting on it. Moreover,

$$\Big(H^n_{\mathrm{cris}}\big(\mathfrak{X}_0/\mathcal{O}_{K_0}\big)\big[\tfrac{1}{p}\big], \varphi, \operatorname{Fil}^\bullet\Big)$$

is a *filtered φ-module* (or *Frobenius isocrystal*), that is, a finite-dimensional K_0-vector space D, with an automorphism $\varphi_D\colon D \to D$ that satisfies $\varphi_D(\lambda d) = \varphi(\lambda)\varphi_D(d)$ for all $\lambda \in K_0$, $d \in D$ (this is called *φ-semilinear*), and a filtration $\operatorname{Fil}^\bullet(D_K)$ (coming from the Hodge filtration) on $D_K \coloneqq D \otimes_{K_0} K$.

(3) B_{cris} is Fontaine's ring of *crystalline p-adic periods*. It is constructed as follows. Let

$$\mathbb{A}_{\mathrm{cris}} \coloneqq H^0_{\mathrm{cris}}\big((\mathcal{O}_C/p\mathcal{O}_C)/\mathbb{Z}_p\big)\,,$$

with a Frobenius action φ on it. Put $B^+_{\mathrm{cris}} \coloneqq \mathbb{A}_{\mathrm{cris}}\big[\frac{1}{p}\big]$. Then B^+_{cris} is actually a G_K-stable subring of B^+_{dR}, and it contains $t = \log{[\varepsilon]}$ from Remark 0.0.9(3). Then we can finally define $B_{\mathrm{cris}} = B^+_{\mathrm{cris}}\big[\frac{1}{t}\big]$. Also note that $\varphi(t) = pt$.

One cool feature of the Fargues–Fontaine curve is that all these strange period rings appear as rings of functions on it.

(4) Theorem 0.0.10 is analogous to the following statement in ℓ-adic cohomology (where $\ell \neq p$ is a prime). Let $\mathfrak{X} \to \operatorname{Spec}\mathcal{O}_K$ be smooth proper, and $s, \eta \in \operatorname{Spec}\mathcal{O}_K$ the special resp. the generic point. Then there exists a G_K-equivariant isomorphism

$$H^*_{\mathrm{\acute{e}t}}(\mathfrak{X}_{\overline{\eta}}, \mathbb{Q}_\ell) \cong H^*_{\mathrm{\acute{e}t}}(\mathfrak{X}_{\overline{s}}, \mathbb{Q}_\ell)\,.$$

In particular, $H^*_{\mathrm{\acute{e}t}}(\mathfrak{X}_{\overline{\eta}}, \mathbb{Q}_\ell)$ is unramified.

(5) By Grothendieck's philosophy of "motives" we should expect that $H^n_{\mathrm{\acute{e}t}}(X_{\overline{K}}, \mathbb{Q}_p)$ and $H^n_{\mathrm{cris}}(\mathfrak{X}_0/\mathcal{O}_{K_0})\big[\frac{1}{p}\big]$ contain the "same information". More mysterious, however, is the question how to pass from G_K representations on finite-dimensional $\mathbb{Q}_p$-vector spaces to K_0-vector spaces with Frobenius and a filtration over K? This became known as "Grothendieck's question on the *mysterious functor*". This was resolved by Fontaine: There are functors

$$D_{\mathrm{cris}}\colon \operatorname{Rep}_{\mathbb{Q}_p} G_K \rightleftarrows \{\text{filtered } \varphi\text{-modules}\} \;\colon\! V_{\mathrm{cris}}$$

given by $D_{\mathrm{cris}}(V) = (V \otimes_{\mathbb{Q}_p} B_{\mathrm{cris}})^{G_K}$ and $V_{\mathrm{cris}}(D) = \operatorname{Fil}^0(D \otimes_{K_0} B_{\mathrm{cris}})^{\varphi=1}$. They satisfy the following theorem, which will be the main goal of the lecture.

0.0.12. Theorem (Colmez/Fontaine)**.** — *"Weakly admissible implies admissible". That is, D_{cris} and V_{cris} restrict to equivalences*

$$D_{\mathrm{cris}}\colon \left\{\begin{array}{c}\textit{crystalline } G_K\text{-}\\ \textit{representations}\end{array}\right\} \overset{\sim}{\longleftrightarrow} \left\{\begin{array}{c}\textit{weakly admissible}\\ \textit{filtered } \varphi\textit{-modules}\end{array}\right\} \colon\! V_{\mathrm{cris}}\,.$$

0.0.13. Remark. — (1) $V \in \operatorname{Rep}_{\mathbb{Q}_p} G_K$ is called *crystalline* if $\dim_{K_0} D_{\mathrm{cris}}(V) = \dim_{\mathbb{Q}_p} V$.

(2) Being *weakly admissible* has something to do with "the Newton polygon lying above the Hodge polygon".

(3) The essential ingredient in the proof of Theorem 0.0.12 will be the *Fargues–Fontaine curve* (duh!), together with the relation between its G_K-invariant vector bundles and $\operatorname{Rep}_{\mathbb{Q}_p} G_K$ resp. {filtered φ-modules}. We can already define it as

$$X_{\mathrm{FF}} \coloneqq \operatorname{Proj}\left(\bigoplus_{d\geqslant 0} (B^+_{\mathrm{cris}})^{\varphi=p^d}\right).$$

We will see that this is a Dedekind scheme over $\mathbb{Q}_p$, and the completions of the local rings at its closed points are B^+_{dR}.

Chapter 1.

1

Construction of the Fargues–Fontaine Curve

1.1. Ramified Witt Vectors

Lecture 1
23rd Oct, 2019

Let p be a prime, $E/\mathbb{Q}_p$ a finite extension with ring of integers $\mathcal{O}_E$. We fix a choice of uniformizer π and let $\mathbb{F}_q = \mathcal{O}_E/\pi\mathcal{O}_E$ be the residue field of $\mathcal{O}_E$, where $q = p^f$. The goal for today is to prove

1.1.1. Proposition. — *There is an equivalence of categories*

$$\left\{\begin{array}{c}\pi\text{-torsionfree } \pi\text{-adically complete } \mathcal{O}_E\text{-alge-}\\ \text{bras } A \text{ with perfect residue ring } A/\pi A\end{array}\right\} \xrightarrow{\sim} \{\text{perfect } \mathbb{F}_q\text{-algebras}\}$$
$$A \longmapsto R = A/\pi A\,.$$

For the proof, we will construct an inverse functor $R \mapsto W_{\mathcal{O}_E}(R)$ that somehow "reconstructs" A from $A/\pi A$.

1.1.2. Remark. — The most important case is the unramified one, i.e., $E = \mathbb{Q}_p$, in which case we obtain an equivalence

$$\left\{\begin{array}{c}p\text{-torsionfree } p\text{-adically complete rings}\\ A \text{ with perfect residue ring } A/pA\end{array}\right\} \xrightarrow{\sim} \{\text{perfect } \mathbb{F}_p\text{-algebras}\}$$
$$A \longmapsto R = A/pA\,.$$

We will see (in Corollary 1.1.18) that the general case can be reduced to this one. Also we put $W \coloneqq W_{\mathbb{Z}_p}$ for brevity.

Example. — We will see $W(\mathbb{F}_p) = \mathbb{Z}_p$ and $W(\mathbb{F}_q) = \mathcal{O}_{E_0}$ where E_0 is the maximal unramified subextension of $E/\mathbb{Q}_p$ (i.e., the unique unramified extension with residue field $\mathbb{F}_q$). Moreover, we will see

$$W\big(\mathbb{F}_p[\![T^{1/p^\infty}]\!]\big) = \mathbb{Z}_p[\![T^{1/p^\infty}]\!]\,.$$

1.1.1. The construction of $W_{\mathcal{O}_E}$

1.1.3. Lemma. — *Let A be any $\mathcal{O}_E$-algebra and $x, y \in A$ such that $x \equiv y \mod \pi$. Then*

$$x^{q^k} \equiv y^{q^k} \mod \pi^{k+1} \quad \textit{for all } k \geqslant 0\,.$$

Proof. By induction on k, this boils down to the following question: if $x \equiv y \mod \pi^k$, show $x^q \equiv y^q \mod \pi^{k+1}$. To see this, write $x = y + \pi^k a$ for some $a \in A$. As all binomial coefficients $\binom{q}{i}$ except for $i = 0, q$ are divisible by p, we obtain

$$x^q = (y + \pi^k a)^q = y^q + p\pi^k(\dots) + \pi^{kq}a^q\,.$$

As $\pi \mid p$, the assertions follows. □

1.1.4. Definition/Lemma. — Let A be a p-adically complete $\mathcal{O}_E$-algebra with $R = A/\pi A$ perfect. Let $a \in R$. Choose any sequence of lifts $\alpha_n \in A$ of $a^{1/q^n} \in R$. Then the sequence $(\alpha_n^{q^n})_{n\in\mathbb{N}}$ converges in A to a lift of a, which is independent of the choices of α_n. The map

$$\begin{aligned} [-]\colon R &\longrightarrow A \\ a &\longmapsto [a] \coloneqq \lim_{n\to\infty} \alpha_n^{q^n} \end{aligned}$$

is well-defined and called the *Teichmüller representative*. It defines a natural multiplicative section of $A \twoheadrightarrow R$.

Proof. We have $\alpha_{n+1}^q \equiv \alpha_n \mod \pi$, hence

$$\alpha_{n+1}^{q^{n+1}} \equiv \alpha_n^{q^n} \mod \pi^{n+1}$$

by Lemma 1.1.3. This shows convergence of the sequence in question. To show that it doesn't depend on the choice of lifts can be seen by a similar argument. Now if $a, b \in R$ are given together with a choice of lifts α_n and β_n, we can choose $\alpha_n\beta_n$ as lifts of $(ab)^1/q^n$, since the choice of lifts doesn't matter. From this argument, multiplicativity is clear. Naturality is similar. □

1.1.5. Lemma. — *In our usual situation, every $x \in A$ admits a unique representation*

$$x = \sum_{n=0}^{\infty} [x_n]\pi^n \quad \textit{for some } x_n \in R\,.$$

Proof. Let $x_0 \in R$ be the reduction of x. Then $x \equiv [x_0] \mod \pi$, so $x - [x_0] = \pi y_1$ for some $y_1 \in A$, which is unique as A is π-torsionfree. Now let $x_1 \in R$ be the reduction of y_1. Similar as above, write $y_1 = [x_1] + \pi y_2$. Now repeat this process to get a representation of the desired type. Uniqueness can be shown along the lines of the construction. □

Remark. — We can think of $A \to R$ in a similar way as we think about $R[\![T]\!] \to R$ with its canonical section $R \to R[\![T]\!]$ given by $a \mapsto a$. Since in our situation A has characteristic 0 but R has characteristic p, there is no way $[-]\colon R \to A$ can be additive. So it being multiplicative is really the best we could hope for.

At this point, Lemma 1.1.5 allows us to recover A as a *set* from $R = A/\pi A$. But what about the ring structure? Let's try! Say we have sequences $(x_n), (y_n) \in R^{\mathbb{N}}$ and we want to find the unique sequence $(s_n) \in R^{\mathbb{N}}$ such that

$$\sum_{n=0}^{\infty} [x_n]\pi^n + \sum_{n=0}^{\infty} [y_n]\pi^n = \sum_{n=0}^{\infty} [s_n]\pi^n\,.$$

One could naively assume that s_n is just $x_n + y_n$. Spoiler: *it's not.* For $n = 0$, we calculate modulo π. We should have $[x_0] + [y_0] = [s_0]$, hence $s_0 = x_0 + y_0$. That was easy! Now for $n = 1$. We calculate modulo π^2:

$$[x_0] + [x_1]\pi + [y_0] + [y_1]\pi \equiv [s_0] + [s_1]\pi \equiv [x_0 + y_0] + [s_1]\pi \mod \pi^2 .$$

Hence we want to put

$$\text{“}s_1 = x_1 + y_1 + \frac{[x_0] + [y_0] - [x_0 + y_0]}{\pi}\text{”},$$

except it's not clear at all how to define this formally. Here we use a trick: since R is perfect and $[-]$ is multiplicative, we have

$$[x_0] + [y_0] - [x_0 + y_0] = \big[x_0^{1/q}\big]^q + \big[y_0^{1/q}\big]^q - \big[x_0^{1/q} + y_0^{1/q}\big]^q .$$

Since $\big[x_0^{1/q}\big] + \big[y_0^{1/q}\big] \equiv \big[x_0^{1/q} + y_0^{1/q}\big] \mod \pi$, Lemma 1.1.3 shows

$$\big[x_0^{1/q}\big]^q + \big[y_0^{1/q}\big]^q \equiv \big[x_0^{1/q} + y_0^{1/q}\big]^q \mod \pi^2 .$$

Hence we can choose

$$s_1 = x_1 + y_1 - \sum_{i=1}^{q-1} \frac{1}{\pi}\binom{q}{i}\big[x_0^{1/q}\big]^i\big[y_0^{1/q}\big]^{q-i} ,$$

where the $\pi^{-1}\binom{q}{i}$ are considered as elements of $\mathcal{O}_E$. In the very unpleasant Germany of 1936, the mathematician and SA member Ernst Witt understood this pattern and extended it to higher n as follows.

1.1.6. Definition. — For $n \geqslant 0$, define the n^{th} *ghost component* as

$$W_n(X_0, \dots, X_n) = \sum_{i=0}^{n} X_i^{q^{n-i}}\pi^i \in \mathcal{O}_E[X_0, \dots, X_n] .$$

Remark. — The idea behind the W_n is that

$$\sum_{i=0}^{n}[a_i]\pi^i = W_n\Big(\big[a_0^{1/q^n}\big], \dots, \big[a_n^{1/q^0}\big]\Big) .$$

1.1.7. Proposition. — *There are unique sequences of polynomials $(S_n)_{n\in\mathbb{N}}$, $(P_n)_{n\in\mathbb{N}}$ in the polynomial ring $\mathcal{O}_E[X_0, \dots, X_n, Y_0, \dots, Y_n]$, such that*

$$\begin{aligned} W_n(X_0, \dots, X_n) + W_n(Y_0, \dots, Y_n) &= W_n(S_0, \dots, S_n) \\ W_n(X_0, \dots, X_n) \cdot W_n(Y_0, \dots, Y_n) &= W_n(P_0, \dots, P_n) . \end{aligned}$$

Proof. We show more generally that for any polynomial $\Phi \in \mathcal{O}_E[X, Y]$ there is a unique sequence $(\Phi)_{n\in\mathbb{N}}$ of polynomials $\Phi_n \in \mathcal{O}_E[X_0, \dots, X_n, Y_0, \dots, Y_n]$ such that

$$\Phi\big(W_n(X_0, \dots, X_n), W_n(Y_0, \dots, Y_n)\big) = W_n(\Phi_0, \dots, \Phi_n) .$$

We show this via induction on n. For $n = 0$ we have to take $\Phi_0(X_0, Y_0) = \Phi(X_0, Y_0)$. Now suppose $\Phi_0, \dots, \Phi_n$ are already constructed. We need to check that

$$\Phi\big(W_{n+1}(X_0, \dots, X_{n+1}), W_{n+1}(Y_0, \dots, Y_{n+1})\big) - W_{n+1}(\Phi_0, \dots, \Phi_n, 0) \tag{1.1.1}$$

is a polynomial divisible by π^{n+1}; for then $\pi^{-(n+1)} \cdot$ (this polynomial) is the unique choice for Φ_{n+1}. Note that

$$W_{n+1}(X_0, \dots, X_{n+1}) \equiv W_n\left(X_0^q, \dots, X_n^q\right) \mod \pi^{n+1}\,. \tag{1.1.2}$$

Using (1.1.1) together with the induction hypothesis, we obtain

$$\begin{aligned}\Phi\big(W_{n+1}(X_0, \dots, X_{n+1}), W_{n+1}(Y_0, \dots, Y_{n+1})\big) &\equiv \Phi\big(W_n(X_0^q, \dots, X_n^q), W_n(Y_0^q, \dots, Y_n^q)\big)\\ &\equiv W_n\big(\Phi_0^{(q)}, \dots, \Phi_n^{(q)}\big) \mod \pi^{n+1}\,,\end{aligned}$$

where $\Phi_i^{(q)}$ is the polynomial obtained from Φ_i by replacing every variable by its q^{th} power. Note that $\Phi_i^{(q)} \equiv \Phi_i^q \mod \pi$. Thus, using Lemma 1.1.3 we get

$$\pi^i\big(\Phi_i^{(q)}\big)^{q^{n-i}} \equiv \pi^i \Phi_i^{q^{n+1-i}} \mod \pi^{n+1}\,.$$

But this shows $W_n\big(\Phi_0^{(q)}, \dots, \Phi_n^{(q)}\big) \equiv W_{n+1}(\Phi_0, \dots, \Phi_n, 0) \mod \pi^{n+1}$. Now putting everything together shows that the polynomial in (1.1.1) is indeed divisible by π^{n+1}, as required. □

1.1.8. Corollary. — *Let $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ be sequences in $R^{\mathbb{N}}$, where $R = A/\pi A$. For all $n \geqslant 0$ put*

$$\begin{aligned}s_n &= S_n\left(x_0^{1/q^n}, \dots, x_n^{1/q^0}, y_0^{1/q^n}, \dots, y_n^{1/q^0}\right)\\ p_n &= P_n\left(x_0^{1/q^n}, \dots, x_n^{1/q^0}, y_0^{1/q^n}, \dots, y_n^{1/q^0}\right)\,.\end{aligned}$$

Then these sequences $(s_n)_{n\in\mathbb{N}}$ and $(p_n)_{n\in\mathbb{N}}$ satisfy

$$\begin{aligned}\sum_{n=0}^{\infty}[x_n]\pi^n + \sum_{n=0}^{\infty}[y_n]\pi^n &= \sum_{n=0}^{\infty}[s_n]\pi^n\\ \left(\sum_{n=0}^{\infty}[x_n]\pi^n\right)\cdot\left(\sum_{n=0}^{\infty}[y_n]\pi^n\right) &= \sum_{n=0}^{\infty}[p_n]\pi^n\,.\end{aligned}$$

Proof.* Again, we show the assertion more generally for an arbitrary $\Phi \in \mathcal{O}_E[X, Y]$ and its associated Witt polynomials $(\Phi_n)_{n\in\mathbb{N}}$ constructed in the proof of Proposition 1.1.7. The key observation is the following:

$(*)$ If $a_0, \dots, a_n$ and $a_0', \dots, a_n'$ are elements of A such that $a_i \equiv a_i' \mod \pi$, then

$$W_n(a_0, \dots, a_n) \equiv W_n(a_0', \dots, a_n') \mod \pi^{n+1}\,.$$

Indeed, if you think about it, this immediately follows from Lemma 1.1.3 and the definition of the W_n. Now fix some N and put

$$\begin{aligned}\phi_n &= \Phi_n\left(x_0^{1/q^n}, \dots, x_n^{1/q^0}, y_0^{1/q^n}, \dots, y_n^{1/q^0}\right)\\ \phi_n' &= \Phi_n\left(\big[x_0^{1/q^N}\big], \dots, \big[x_n^{1/q^{N-n}}\big], \big[y_0^{1/q^N}\big], \dots, \big[y_n^{1/q^{N-n}}\big]\right)\,.\end{aligned}$$

By construction of the Witt polynomials $(\Phi_n)_{n\in\mathbb{N}}$ (see the proof of Proposition 1.1.7) we immediately have

$$\Phi\left(W_N\left([x_0^{1/q^N}],\dots,[x_N^{1/q^0}]\right), W_N\left([y_0^{1/q^N}],\dots,[y_N^{1/q^0}]\right)\right) = W_N(\phi_0',\dots,\phi_N')\,.$$

But also $\phi_n' \equiv \left[\phi_n^{1/q^{N-n}}\right] \mod \pi$. Hence, by $(*)$, we obtain

$$W_N(\phi_0',\dots,\phi_N') \equiv W_N\left([\phi_0^{1/q^N}],\dots,[\phi_n^{1/q^0}]\right) \mod \pi^{N+1}\,.$$

Taking $N\to\infty$, this shows

$$\Phi\left(\sum_{n=0}^\infty [x_n]\pi^n, \sum_{n=0}^\infty [y_n]\pi^n\right) = \sum_{n=0}^\infty [\phi_n]\pi^n\,.$$

For $\Phi = X+Y$ resp. $\Phi = XY$ we retain the assertion of this corollary. $\square$

The upshot is that we can now reconstruct A as a ring from $R = A/\pi A$. The next goal is to start with an arbitrary R and construct an A in a functorial way. In particular, we will allow R to be an $\mathcal{O}_E$-algebra instead of an $\mathbb{F}_q$-algebra (recall that $\mathbb{F}_q = \mathcal{O}_E/\pi\mathcal{O}_E$). In the end, we will only be interested in the latter case, but allowing for rings of characteristic 0 too gives us some nice uniqueness properties.

1.1.9. Definition. — For any $\mathcal{O}_E$-algebra R write $W_{\mathcal{O}_E}(R) = R^{\mathbb{N}}$. Its elements (which are sequences) are denoted $x = [x_0, x_1, \dots]$.

1.1.10. Proposition. — *The functor from Definition 1.1.9 admits a unique factorization*

$$\begin{tikzcd} \mathrm{Alg}_{\mathcal{O}_E} \arrow[rr, "(-)^{\mathbb{N}}"] \arrow[dr, dashed, "W_{\mathcal{O}_E}(-)"'] & & \mathrm{Set} \\ & \mathrm{Alg}_{\mathcal{O}_E} \arrow[ur, "\text{forget}"'] & \end{tikzcd}$$

such that the natural transformation $\mathcal{W}$ given by

$$\begin{aligned} \mathcal{W}_R \colon W_{\mathcal{O}_E}(R) &\longrightarrow R^{\mathbb{N}} \\ [x_n]_{n\in\mathbb{N}} &\longmapsto \big(W_n(x_0,\dots,x_n)\big)_{n\in\mathbb{N}} \end{aligned}$$

is a morphism of $\mathcal{O}_E$-algebras. Here $R^{\mathbb{N}}$ is equipped with its natural component-wise $\mathcal{O}_E$-algebra structure.

Proof. We first construct a natural $\mathcal{O}_E$-algebra structure on $W_{\mathcal{O}_E}(R)$. If two sequences $x = [x_n]_{n\in\mathbb{N}}$ and $[y_n]_{n\in\mathbb{N}}$ are given, we define $x+y = [s_n]_{n\in\mathbb{N}}$ and $xy = [p_n]_{n\in\mathbb{N}}$, where—you might have guessed it—we put

$$s_n = S_n(x_0,\dots,x_n,y_0,\dots,y_n) \quad\text{and}\quad p_n = P_n(x_0,\dots,x_n,y_0,\dots,y_n)\,.$$

To see that this is determines a ring structure, the crucial thing to notice is that the proof of Proposition 1.1.7 works just the same if $\Phi\in\mathcal{O}_E[X_1,\dots,X_N]$ is a polynomial in arbitrary many variables instead of just $N=2$. So by choosing suitable Φ, we can verify all ring axioms.

For example, $\Phi = -X_1$ constructs additive inverses, $\Phi = (X_1 + X_2) + X_3 = X_1 + (X_2 + X_3)$ shows the associativity law of addition, $\Phi = X_1(X_2+X_3) = X_1X_2+X_1X_3$ shows distributivity, and so on. Also, if $\alpha \in \mathcal{O}_E$, then $\Phi = \alpha X_1$ defines multiplication by α on $W_{\mathcal{O}_E}(R)$, turning it into an $\mathcal{O}_E$-algebra.

This provides a factorization through $\mathrm{Alg}_{\mathcal{O}_E}$. It is clear from the construction that $\mathcal{W}_R$ is an $\mathcal{O}_E$-algebra morphism. So it remains to show that this factorization is unique. If R is π-torsionfree, then $\mathcal{W}_R \colon W_{\mathcal{O}_E}(R) \to R^{\mathbb{N}}$ is easily seen to be injective, hence the $\mathcal{O}_E$-algebra structure on $W_{\mathcal{O}_E}(R)$ is uniquely determined by the one on $R^{\mathbb{N}}$. In general, every R admits a surjection $R' \twoheadrightarrow R$ from a π-torsionfree $\mathcal{O}_E$-algebra; e.g., $R' = \mathcal{O}_E\left[T_a \mid a \in R\right]$ does it. Then $W_{\mathcal{O}_E}(R') \twoheadrightarrow W_{\mathcal{O}_E}(R)$ uniquely determines the $\mathcal{O}_E$-algebra structure on $W_{\mathcal{O}_E}(R)$. This shows uniqueness. □

Remark. — (1) For the uniqueness part it was crucial to have "enough" π-torsionfree $\mathcal{O}_E$-algebras. If we had worked with $\mathbb{F}_q$-algebras, where $\pi = 0$, this wouldn't have been possible. In this case, $W_n(x_0, \dots, x_n)$ is just $x_0^{q^n}$. Hence the name "ghost components".

(2) Also, Proposition 1.1.10 gives the functor $W_{\mathcal{O}_E}(-)$ the structure of a ring scheme.

1.1.11. Lemma. — *The natural map (which we will also call "Teichmüller lift")*

$$\begin{aligned}[-]\colon R &\longrightarrow W_{\mathcal{O}_E}(R)\\ x &\longmapsto [x, 0, 0, \dots]\end{aligned}$$

is multiplicative.

Proof.* It's easy to see $P_0(X_0, Y_0) = X_0Y_0$. So to prove the assertion it suffices to check that $P_n(X_0, 0, \dots, 0, Y_0, 0, \dots, 0) = 0$ for all $n > 0$. But

$$W_n(X_0, 0, \dots, 0) \cdot W_n(Y_0, 0, \dots, 0) = X_0^{q^n} Y_0^{q^n} = W_n(X_0Y_0, 0, \dots, 0)\,,$$

so this is easy to check by induction on n (and using that polynomial rings over $\mathcal{O}_E$ are π-torsionfree). □

1.1.2. Frobenius and Verschiebung

If R happens to be an $\mathbb{F}_q$-algebra, then we have the Frobenius $(-)^q$ on R. By functoriality, it extends to an endomorphism $F \colon W_{\mathcal{O}_E}(R) \to W_{\mathcal{O}_E}(R)$. The next lemma shows that F actually exists for arbitrary R and can be explicitly described.

1.1.12. Lemma. — (1) *There is a unique natural transformation $F \colon W_{\mathcal{O}_E}(-) \to W_{\mathcal{O}_E}(-)$ of $\mathcal{O}_E$-algebras making the following diagram commute:*

$$\begin{tikzcd} W_{\mathcal{O}_E}(R) \arrow[r, "\mathcal{W}"] \arrow[d, "F"'] & R^{\mathbb{N}} \arrow[d] & (x_n)_{n\in\mathbb{N}} \arrow[d, mapsto] \\ W_{\mathcal{O}_E}(R) \arrow[r, "\mathcal{W}"] & R^{\mathbb{N}} & (x_{n+1})_{n\in\mathbb{N}} \end{tikzcd}$$

(2) *If R is an $\mathbb{F}_q$-algebra, then F is given by $F([x_0, x_1, \dots]) = [x_0^q, x_1^q, \dots]$ and it is induced by the Frobenius on R.*

Proof.* We first construct a sequence $(F_n)_{n\in\mathbb{N}}$ of polynomials $F_n \in \mathcal{O}_E[X_0,\dots,X_{n+1}]$ satisfying $W_{n+1}(X_0,\dots,X_{n+1}) = W_n(F_0,\dots,F_n)$ and that $F_n \equiv X_n^q \mod \pi$. This is done by induction on n, the case $n=0$ being trivial. Suppose $F_0,\dots,F_{n-1}$ have already been constructed and have the required property. If we could prove that

$$W_{n+1}(X_0,\dots,X_{n+1}) - W_n(F_0,\dots,F_{n-1},0) \equiv \pi^n X_0^q \mod \pi^{n+1}\,, \tag{1.1.3}$$

this would show existence of F_n and $F_n \equiv X_n^q \mod \pi$ at once. To prove (1.1.3), we may equivalently show

$$\begin{aligned} 0 &\equiv W_{n+1}(X_0,\dots,X_{n-1},0,0) - W_n(F_0,\dots,F_{n-1},0) \\ &\equiv W_{n-1}\big(X_0^{q^2},\dots,X_{n-1}^{q^2}\big) - W_{n-1}\left(F_0^q,\dots,F_{n-1}^q\right) \mod \pi^{n+1}\,. \end{aligned} \tag{1.1.4}$$

But $F_i \equiv X_i^q \mod \pi$ shows $F_i^q \equiv X_i^{q^2} \mod \pi^2$ by Lemma 1.1.3, hence the bottom line of (1.1.4) is indeed 0 modulo π^{n+1} by another application of Lemma 1.1.3.

Thus we can construct a sequence $F = (F_n)_{n\in\mathbb{N}}$ with the required properties. By construction, F makes the diagram in (1) commute and satisfies (2). So it remains to show that F is unique with this property and a morphism of $\mathcal{O}_E$-algebras. This can be done by the same argument as in the proof of Proposition 1.1.10. If R is π-torsionfree, $W_{\mathcal{O}_E}(R)$ injects into $R^{\mathbb{N}}$, hence it is uniquely determined and an $\mathcal{O}_E$-algebra morphism. In general, we take a surjection $R' \twoheadrightarrow R$ from a π-torsionfree $\mathcal{O}_E$-algebra. $\square$

1.1.13. Lemma. — *There is a natural transformation $V\colon W_{\mathcal{O}_E}(-) \to W_{\mathcal{O}_E}(-)$ of $\mathcal{O}_E$-modules that makes the following diagram commute:*

$$\begin{array}{ccccc} [x_0,x_1,\dots] & W_{\mathcal{O}_E}(R) & \xrightarrow{\mathcal{W}} & R^{\mathbb{N}} & (x_n)_{n\in\mathbb{N}} \\ \downarrow & {\scriptstyle V}\downarrow & & \downarrow & \downarrow \\ [0,x_0,x_1,\dots] & W_{\mathcal{O}_E}(R) & \xrightarrow{\mathcal{W}} & R^{\mathbb{N}} & (\pi x_{n-1})_{n\in\mathbb{N}} \end{array},$$

where we put $x_{-1}=0$. Moreover, V is unique with this property.

Proof.* It's immediately clear that V as constructed makes the diagram commute. To show that V is unique, we use the usual trick: for π-torsionfree $\mathcal{O}_E$-algebras R, this is clear; in general, consider a surjection $R' \twoheadrightarrow R$ where R' is π-torsionfree. $\square$

Remark. — The letter V stands for the German word "Verschiebung". In contrast to F, V is no ring endomorphism and it does depend on the choice of π.[1]

1.1.14. Lemma. — *The following identities hold for F and the Verschiebung V.*

(1) $FV = \pi$.

(2) $V(F(x)y) = xV(y)$ *for all* $x,y \in W_{\mathcal{O}_E}(R)$.

(3) $\pi F(x)y = F(xV(y))$ *for all* $x,y \in W_{\mathcal{O}_E}(R)$.

Proof. If R is π-torsionfree, these can be checked in $R^{\mathbb{N}}$. In general, take a surjection $R' \twoheadrightarrow R$ where R' is π-torsionfree to reduce everything to the π-torsionfree case. $\square$

[1] Well, W_n and thus $\mathcal{W}$ depend on π too, so we cannot really say that F is "independent" of π. But at least its image in $R^{\mathbb{N}}$ is, in contrast to the image of V in $R^{\mathbb{N}}$.

1.1.15. Lemma. — (1) *For all $n \in \mathbb{N}$, the image of V^n is an ideal in $W_{\mathcal{O}_E}(R)$.*

(2) *We have $W_{\mathcal{O}_E}(R) \cong \lim_{n\in\mathbb{N}} W_{\mathcal{O}_E}(R)/\operatorname{im} V^n$.*

(3) *Every $x \in W_{\mathcal{O}_E}(R)$ admits a unique representation*

$$x = \sum_{n=0}^{\infty} V^n[x_n]$$

for some $x_n \in R$, where $[-]\colon R \to W_{\mathcal{O}_E}(R)$ is the Teichmüller lift from Lemma 1.1.11. In fact, the x_n are determined by $x = [x_n]_{n\in\mathbb{N}}$.

*Proof**. Since V is $\mathcal{O}_E$-linear, $\operatorname{im} V^n$ is a subgroup of $W_{\mathcal{O}_E}(R)$. Moreover, Lemma 1.1.14(1) shows $xV^n(y) = V^n(F^n(x)y)$ for all $x, y \in W_{\mathcal{O}_E}(R)$, hence $\operatorname{im} V^n$ is closed under scalar multiplication. This shows (1).

Now part (2). We claim that the canonical map of sets $W_{\mathcal{O}_E}(R) \to R^N$ given by $[x_n]_{n\in\mathbb{N}} \mapsto (x_0, \dots, x_{N-1})$ descends to a bijection

$$W_{\mathcal{O}_E}(R)/\operatorname{im} V^N \xrightarrow{\sim} R^N \,.$$

Let's first check that it is well-defined. Let $y = [y_n]_{n\in\mathbb{N}}$ be in the image of V^n, i.e., $y_n = 0$ for all $n < N$. Let $x + y = [s_n]_{n\in\mathbb{N}}$. Then what we need to show is that $s_n = x_n$ for all $n < N$. Thus, it suffices to check the polynomial identity

$$S_n(X_0, \dots, X_n, 0, \dots, 0) = X_n \,.$$

However, this is easily seen from induction and the trivial identity

$$W_n(X_0, \dots, X_n) + W_n(0, \dots, 0) = W_n(X_0, \dots, X_n)\,.$$

Since $W_{\mathcal{O}_E}(R)/\operatorname{im} V^N \to R^N$ is automatically surjective, it remains to show injectivity. So let $x, y \in W_{\mathcal{O}_E}(R)$ be such that $x_n = y_n$ for all $n < N$. Let $x - y = [\delta_n]_{n\in\mathbb{N}}$. To show that δ is in the image of V^n, we need to check $\delta_n = 0$ for $n < N$. Thus, it suffices to check the polynomial identity

$$\Delta_n(X_0, \dots, X_n, X_0, \dots, X_n) = 0\,,$$

where $\Delta = X - Y \in \mathcal{O}_E[X, Y]$ and $(\Delta_n)_{n\in\mathbb{N}}$ are the associated Witt polynomials constructed in the proof of Proposition 1.1.7. This can be done in the same way as above.

Now since $R^{\mathbb{N}} \cong \lim_{n\in\mathbb{N}} R^n$, the bijection $W_{\mathcal{O}_E}(R)/\operatorname{im} V^n \cong R^n$ for all $n \in \mathbb{N}$ shows that $W_{\mathcal{O}_E}(R) \cong \lim_{n\in\mathbb{N}} W_{\mathcal{O}_E}(R)/\operatorname{im} V^n$ is true as a limit of sets. However, the limit in the category of $\mathcal{O}_E$-algebras can be taken on the level of sets. This shows (2).

Finally, we show (3). First we prove that for all $N \in \mathbb{N}$ we have

$$\sum_{n=0}^{N} V^n[x_n] = [x_0, \dots, x_N, 0, 0, \dots]\,. \tag{1.1.5}$$

We use induction on N. The case $N = 0$ is trivial. Now suppose the assertion is true for $N - 1$. To prove it for N, it suffices to check the following polynomial identity: if $(X_n)_{n\in\mathbb{N}}$ and $(Y_n)_{n\in\mathbb{N}}$ are sequences of variables such that $X_N = 0$ and $Y_n = 0$ for all $n \neq N$, then

$$S_n(X_0, \dots, X_n, Y_0, \dots, Y_n) = \begin{cases} Y_N & \text{if } n = N \\ X_n & \text{else} \end{cases}\,.$$

For $n < N$, we obtain an identity that was already seen in the proof of (2). For $n \geqslant N$, this easily follows by induction on n, using the identity

$$W_n(X_0, \dots, X_n) + W_n(Y_0, \dots, Y_n) = W_n(X_0, \dots, X_{N-1}, Y_N, X_{N+1}, \dots, X_n)\,.$$

This shows (1.1.5). Now let $x - [x_0, \dots, x_N, 0, 0, \dots] = \delta = [\delta_n]_{n \in \mathbb{N}}$. As in the proof of (2) we see that $\delta_n = 0$ for $n \leqslant N$. Hence $\delta \in \operatorname{im} V^n$. This shows (3) except for the uniqueness part. But uniqueness is also clear from (1.1.5). □

Remark*. — Lemma 1.1.15 holds for arbitrary R, despite what was claimed in the lecture. We leave it as an exercise to relate this error to the lecture's overall rushed style.

Now that the general theory of $W_{\mathcal{O}_E}(-)$ is set up, we restrict ourselves to the case where R has characteristic p, i.e., $\pi = 0$ on R and R is an $\mathbb{F}_q$-algebra.

1.1.16. Lemma. — *Suppose $\pi = 0$ on R. Then the following hold:*

(1) *For $x = [x_n]_{n\in\mathbb{N}} \in R$ we have $x = \sum_{n=0}^\infty V^n[x_n]$.*

(2) *$VF = \pi$. Hence V and F commute.*

(3) *$F\big(\sum_{n=0}^\infty V^n[x_n]\big) = \sum_{n=0}^\infty V^n[x_n^q]$.*

Proof.* Part (1) was already seen in Lemma 1.1.15(3). Now (3) is an immediate consequence of (1) and Lemma 1.1.12(2). For (2), note that VF sends $[x_0, x_1, \dots]$ to $[0, x_0^q, x_1^q, \dots]$. Thus, it suffices to show that the Witt polynomials $(\Pi_n)_{n\in\mathbb{N}}$ associated to $\Pi = \pi X \in \mathcal{O}_E[X]$ satisfy

$$\Pi_n(X_0, \dots, X_n) \equiv X_{n-1}^q \mod \pi \quad \text{for } n \geqslant 1$$

and $\Pi_0 \equiv 0 \mod \pi$. We show this by induction on n, the case $n = 0$ being trivial. Now suppose the assertions holds up to n. Then $\Pi_i \equiv X_{i-1}^q \mod \pi$ for all $i \leqslant n$ shows, by Lemma 1.1.3, that

$$W_{n+1}(\Pi_1, \dots, \Pi_{n+1}) \equiv \pi X_0^{q^{n+1}} + \dots + \pi^n X_{n-1}^{q^2} + \pi^{n+1}\Pi_{n+1} \mod \pi^{n+2}\,.$$

However, the left-hand side can, by definition, be computed as

$$\begin{aligned} W_{n+1}(\Pi_1, \dots, \Pi_{n+1}) &\equiv \pi W_{n+1}(X_0, \dots, X_{n+1}) \\ &\equiv \pi X_0^{q^{n+1}} + \dots + \pi^n X_{n-1}^{q^2} + \pi^{n+1} X_n^q \mod \pi^{n+2}\,. \end{aligned}$$

This shows indeed $\Pi_{n+1} \equiv X_n^q \mod \pi$, as claimed. □

1.1.17. Lemma. — *If R is a perfect $\mathbb{F}_q$-algebra, then $W_{\mathcal{O}_E}(R)$ is π-adically complete, and if $x = [x_n]_{n\in\mathbb{N}}$, then*

$$x = \sum_{n=0}^\infty \big[x_n^{1/q^n}\big]\pi^n\,.$$

Proof.* Since R is perfect, the Frobenius is an automorphism, hence the same is true for F on $W_{\mathcal{O}_E}(R)$. Thus Lemma 1.1.16(2) shows that the image of V^n is the image of π^n. Thus Lemma 1.1.15(2) proves that $W_{\mathcal{O}_E}(R)$ is π-adically complete.

To see the second assertion, note that by Lemma 1.1.16(2) we have

$$[x_n]\pi^n = V^n F^n \big[x_n^{1/q^n}\big] = V^n[x_n]\,,$$

and use Lemma 1.1.15(3). □

Finally we have everything together to prove Proposition 1.1.1.

Proof of Proposition 1.1.1. We claim that $W_{\mathcal{O}_E}(-)$ defines an inverse functor. If R is a perfect $\mathbb{F}_q$-algebra, Lemma 1.1.17 shows $W_{\mathcal{O}_E}(R)/\pi W_{\mathcal{O}_E}(R) = W_{\mathcal{O}_E}(R)/\operatorname{im} V$. The right-hand side is isomorphic R as an $\mathcal{O}_E$-algebra. On the level of sets this was seen in the proof of Lemma 1.1.15(2). Rs $\mathcal{O}_E$-algebra this follows from $S_0 = X_0 + Y_0$, $P_0 = X_0 Y_0$, and $(aX)_0 = aX_0$ for all $a \in \mathcal{O}_E$.

Thus, the image of $W_{\mathcal{O}_E}(-)$ is as desired. It remains to provide a natural isomorphism between A and $W_{\mathcal{O}_E}(R)$ if $R = A/\pi A$. We define it via

$$\begin{aligned} W_{\mathcal{O}_E}(R) &\longrightarrow A \\ \sum_{n=0}^{\infty}[x_n]\pi^n &\longmapsto \sum_{n=0}^{\infty}[x_n]\pi^n \,. \end{aligned}$$

By Lemma 1.1.5 and Lemma 1.1.17, it is a natural bijection. By Corollary 1.1.8 it is $\mathcal{O}_E$-linear. We are done. $\square$

1.1.18. Corollary. — *Let E_0 be the maximal unramified subextension of $E/\mathbb{Q}_p$ (or in other words, the unique unramified extension of $\mathbb{Q}_p$ with residue field $\mathbb{F}_q$). Then there is a natural isomorphism*

$$W(R) \otimes_{\mathcal{O}_{E_0}} \mathcal{O}_E \xrightarrow{\sim} W_{\mathcal{O}_E}(R)\,.$$

Proof.* Since p is a uniformizer of $\mathcal{O}_{E_0}$, the Witt vectors $W(R)$ taken over $\mathbb{Z}_p$ are the same as if they were taken over $\mathcal{O}_{E_0}$. Now the diagram

$$\begin{tikzcd}
\left\{\begin{array}{c} p\text{-torsionfree } p\text{-adically complete} \\ \mathcal{O}_E\text{-algebras } A \text{ s.th. } A/pA \text{ is perfect} \end{array}\right\} \arrow[dd, "-\otimes_{\mathcal{O}_{E_0}}\mathcal{O}_E"'] \arrow[dr, "-/p-"] & \\
& \{\text{perfect } \mathbb{F}_q\text{-algebras}\} \arrow[ul, dotted, bend right, "W(-)"'] \arrow[dl, dotted, bend left, "W_{\mathcal{O}_E}(-)"] \\
\left\{\begin{array}{c} \pi\text{-torsionfree } \pi\text{-adically complete} \\ \mathcal{O}_E\text{-algebras } A \text{ s.th. } A/\pi A \text{ is perfect} \end{array}\right\} \arrow[ur, "-/\pi-"] &
\end{tikzcd}$$

of functors between categories commutes. Hence the diagram formed by the vertical arrow and the two dotted quasi-inverses commutes up to natural isomorphism, which is precisely what we want to show. $\square$

1.1.19. Example*. — Now we can easily verify the examples given at the beginning of the section. To prove

$$W(\mathbb{F}_p) = \mathbb{Z}_p\,, \quad W(\mathbb{F}_q) = \mathcal{O}_{E_0}\,, \quad \text{and} \quad W\big(\mathbb{F}_p[\![T^{1/p^\infty}]\!]\big) = \mathbb{Z}_p[\![T^{1/p^\infty}]\!]\,,$$

it suffices to see that the respective right-hand sides are p-complete, p-torsionfree and that modding out p gives $\mathbb{F}_p$, $\mathbb{F}_q$, and $\mathbb{F}_p[\![T^{1/p^\infty}]\!]$ respectively. This is easy to check.

1.2. The Ring $\mathbb{A}_{\mathrm{inf}}$

LECTURE 2
30^{th} Oct, 2019

Apparently, $\mathbb{A}_{\mathrm{inf}}$ is so awesome that Pierre Colmez titled it "The One Ring to rule them all" (somewhat related). For example, it already determines B_{cris} and B_{dR}.

Throughout this section, let p be a prime, $E/\mathbb{Q}_p$ a finite extension, $\pi \in \mathcal{O}_E$ a uniformizer and $\mathbb{F}_q = \mathcal{O}_E/\pi\mathcal{O}_E$ for $q = p^f$. Moreover, let $F/\mathbb{F}_q$ be a non-archimedean algebraically closed extension. For us, *non-archimedean* always means that F is complete with respect to a non-archimedean non-trivial valuation $|\ \ |\colon F \to \mathbb{R}_{\geqslant 0}$. As usual, the *ring of integers* $\mathcal{O}_F$ is defined as

$$\mathcal{O}_F = \{x \in F \mid |x| \leqslant 1\}\ .$$

Note that $\mathcal{O}_F$ is local with maximal ideal $\mathfrak{m}_F = \{x \in F \mid |x| < 1\}$.

1.2.1. Definition. — In the above setting, we define

$$\mathbb{A}_{\mathrm{inf}} = \mathbb{A}_{\mathrm{inf},E,F} \coloneqq W_{\mathcal{O}_E}(\mathcal{O}_F)\,.$$

1.2.2. Remark. — (1) $\mathbb{A}_{\mathrm{inf}}$ should be thought of a "power series ring over $\mathcal{O}_F$ in the indeterminate π". So its equal characteristic analogue should be $\mathcal{O}_F[\![z]\!]$.

(2) $\mathbb{A}_{\mathrm{inf}}$ has a natural Frobenius action φ, given by the Witt vector Frobenius, which is, in turn, given by the Frobenius on $\mathcal{O}_F$.

In the proof of Proposition 1.1.1 we have seen that $W_{\mathcal{O}_E}(-)$ is a quasi-inverse to $-/\pi-$ on some suitable category. In general, $W_{\mathcal{O}_E}(-)$ still possesses an adjoint, the *tilt functor*.

1.2.3. Definition. — Let A be a π-complete $\mathcal{O}_E$-algebra. Then the *tilt of* A is

$$A^\flat \coloneqq \lim_{x \mapsto x^q} A/\pi A = \left\{ (a_0, a_1, \dots) \in \prod_{n\in\mathbb{N}} A/\pi A \ \middle|\ a_i^q = a_{i-1} \text{ for all } i > 0 \right\}\ .$$

Note that $A^\flat$ is always a perfect $\mathbb{F}_q$-algebra (in fact, that's a purely category-theoretical statement): the Frobenius on $A^\flat$ is given by $\mathrm{Frob}_{q,A^\flat}(a_0, a_1, \dots) = (a_0^q, a_0, a_1, \dots)$ and it has an inverse defined by $\mathrm{Frob}_{q,A^\flat}^{-1}(a_0, a_1, \dots) = (a_1, a_2, \dots)$.

1.2.4. Proposition. — *There is an adjunction*

$$W_{\mathcal{O}_E}(-)\colon \{\pi\text{-}complete\ \mathcal{O}_E\text{-}algebras\} \rightleftarrows \{perfect\ \mathbb{F}_q\text{-}algebras\} :(-)^\flat\,.$$

1.2.5. Remark. — Before we sketch a proof of Proposition 1.2.4, let us leave two remarks.

(1) If R is a perfect $\mathbb{F}_q$-algebra, then the unit $R \to W_{\mathcal{O}_E}(R)^\flat$ of the adjunction is given by $r \mapsto (r, r^{1/q}, r^{1/q^2}, \dots)$. Thus it is an isomorphism. In particular, this shows that $W_{\mathcal{O}_E}(-)$ is fully faithful by abstract nonsense. However, we have already seen that in the proof of Proposition 1.1.1, where moreover the essential image of $W_{\mathcal{O}_E}(-)$ was identified as the class of π-complete π-torsionfree $\mathcal{O}_E$-algebras A such that $A/\pi A$ is perfect.

(2) The counit $\theta\colon W_{\mathcal{O}_E}(A^\flat) \to A$ is usually called *Fontaine's map*.

Sketch of a proof of Proposition 1.2.4. First we state the following slightly more general form of the key Lemma 1.1.3 (actually, this proof only uses the previous formulation, but for future use the more general version will be handy). It can be proved in the exact same way as Lemma 1.1.3.

1.2.6. Lemma (“q-power map is π-adically contracting”)**.** — *Let B be any $\mathcal{O}_E$-algebra and $I \subseteq B$ an ideal such that $\pi \in I$. If $x, y \in B$ such that $x \equiv y \mod I$, then*

$$x^{q^n} \equiv y^{q^n} \mod I^{n+1} \quad \textit{for all } n \geqslant 0\,.$$

We construct the counit θ as follows. Fix $n > 0$. By $W_{\mathcal{O}_E,n}(A)$ we denote the truncated Witt vectors of length $n+1$. These are obtained by cutting off everything after the first $n+1$ components. In other words, $W_{\mathcal{O}_E,n}(A) = W_{\mathcal{O}_E}(A)/\operatorname{im} V^{n+1}$. Consider the map

$$\begin{aligned} \mathcal{W}_n \colon W_{\mathcal{O}_E,n}(A) &\longrightarrow A/\pi^{n+1}A \\ [a_0,\dots,a_n] &\longmapsto W_n(a_0,\dots,a_n) \mod \pi^{n+1}\,. \end{aligned}$$

If $a_i \equiv 0 \mod \pi$ for all $i = 0,\dots,n$, then Lemma 1.2.6 shows $W_n(a_0,\dots,a_n) \equiv 0 \mod \pi^{n+1}$. Thus, we get an induced map $\theta_n \colon W_{\mathcal{O}_E,n}(A/\pi A) \to A/\pi^{n+1}A$. We check that the diagram

$$\begin{tikzcd} W_{\mathcal{O}_E,n+1}(A/\pi A) \arrow[r, "\theta_{n+1}"] \arrow[d, "F"'] & A/\pi^{n+2}A \arrow[d] \\ W_{\mathcal{O}_E,n}(A/\pi A) \arrow[r, "\theta_n"] & A/\pi^{n+1}A \end{tikzcd}$$

commutes. Indeed, given $[\overline{a}_0,\dots,\overline{a}_{n+1}] \in W_{\mathcal{O}_E,n+1}(A/\pi A)$ with lifts $[a_0,\dots,a_{n+1}]$, we have

$$W_{n+1}(a_0,\dots,a_{n+1}) \equiv W_n(a_0^q,\dots,a_n^q) \mod \pi^{n+1}\,,$$

which is precisely what we want. Passing to the limit, we obtain a map

$$\theta \colon W_{\mathcal{O}_E}(A^\flat) \cong \lim_F W_{\mathcal{O}_E,n}(A/\pi A) \longrightarrow \lim_{n\in\mathbb{N}} A/\pi^{n+1}A \cong A\,.$$

The isomorphism on the left is easy to check, and the isomorphism on the right follows from A being π-complete. In the lecture, that was the end of the proof sketch. In these notes we will finish the proof, but only after we understand the map θ a little better. $\square$

Another application of Lemma 1.2.6 is the following.

1.2.7. Proposition. — *Let A be a π-complete $\mathcal{O}_E$-algebra. Let $I \subseteq A$ be an ideal containing I, such that A is also I-complete. Then the canonical map*

$$\lim_{x\mapsto x^q} A \xrightarrow{\sim} (A/I)^\flat$$

is an isomorphism. In particular, the left-hand side (which is a priori only a multiplicative monoid) inherits a natural ring structure.

Proof. Let $x = (\overline{x}_0, \overline{x}_1, \dots) \in (A/I)^\flat$. For every $n \geqslant 0$ choose a lift $x_n \in A$ of $\overline{x}_n$. By Lemma 1.2.6, $(x_n^{q^n})_{n\in\mathbb{N}}$ is a Cauchy sequence in the I-adic topology. Put

$$x^\sharp = \lim_{n\to\infty} x_n^{q^n}\,.$$

As in the proof of Definition/Lemma 1.1.4, $x^\sharp$ is independent of the choice of lifts and $(-)^\sharp$ is multiplicative. Now it's easy to see that the map

$$\begin{aligned} (A/I)^\flat &\longrightarrow \lim_{x\mapsto x^q} A \\ x &\longmapsto \bigl(x^\sharp, (x^{1/q})^\sharp, \dots\bigr) \end{aligned}$$

is a multiplicative inverse of the map in question. This proves the assertion. $\square$

1.2.8. Lemma. — *The counit $\theta\colon W_{\mathcal{O}_E}(A^\flat) \to A$ can be explicitly described as*

$$\sum_{n=0}^{\infty} [a_n]\pi^n \longmapsto \sum_{n=0}^{\infty} a_n^\sharp \pi^n \,.$$

*Proof**. Let us first describe the isomorphism $W_{\mathcal{O}_E}(A^\flat) \cong \lim_F W_{\mathcal{O}_E,n}(A/\pi A)$ that is part of the definition of θ. The underlying set of $W_{\mathcal{O}_E}(A^\flat)$ consists of sequences $[a_0, a_1, \dots]$, where each $a_n \in A^\flat$ is itself a sequence $a_n = (\overline{a}_{n,i})_{i\in\mathbb{N}}$ in $A/\pi A$ such that $\overline{a}_{n,i}^q = \overline{a}_{n,i-1}$. The underlying set of $\lim_F W_{\mathcal{O}_E,n}(A/\pi A)$ consists of sequences $([\overline{a}_{0,0}], [\overline{a}_{0,1}, \overline{a}_{1,1}], [\overline{a}_{0,2}, \overline{a}_{1,2}, \overline{a}_{2,2}], \dots)$ that are compatible under F. The isomorphism in question is given by

$$\begin{aligned} W_{\mathcal{O}_E}(A^\flat) &\xrightarrow{\sim} \lim_F W_{\mathcal{O}_E,n}(A/\pi A) \\ [\overline{a}_0, \overline{a}_1, \overline{a}_2, \dots] &\longmapsto \big([\overline{a}_{0,0}], [\overline{a}_{0,1}, \overline{a}_{1,1}], [\overline{a}_{0,2}, \overline{a}_{1,2}, \overline{a}_{2,2}], \dots\big)\,. \end{aligned}$$

Indeed, it is clear that this defines a bijection on set level and one may check that it is also compatible with the ring structures on either side.

Now let $a = [a_0, a_1, \dots] \in W_{\mathcal{O}_E}(A^\flat)$ be as above. We unwind what $\theta(a)$ actually is. By definition of the map $\theta_N\colon W_{\mathcal{O}_E,N}(A/\pi A) \to A/\pi^{N+1}A$, we have

$$\theta_N[\overline{a}_{0,N}, \dots, \overline{a}_{N,N}] \equiv \sum_{n=0}^{N} a_{n,N}^{q^{N-n}} \pi^n \mod \pi^{N+1}\,,$$

where the $a_{n,N}$ are arbitrary lifts of $\overline{a}_{n,N}$. Thus, the coefficient of π^n in $\theta(a)$ is given by

$$\lim_{N\to\infty} a_{n,N}^{q^{N-n}} = \big(a_n^{1/q^n}\big)^\sharp\,.$$

The exponent $1/q^n$ seems off at first glance, but according to Lemma 1.1.17 this is exactly what we want. □

*End of proof of Proposition 1.2.4**. Let A be a π-complete $\mathcal{O}_E$-algebra and R a perfect $\mathbb{F}_q$-algebra. By Proposition 1.1.1 we have a bijection

$$\mathrm{Hom}(R, A^\flat) \cong \mathrm{Hom}\big(W_{\mathcal{O}_E}(R), W_{\mathcal{O}_E}(A^\flat)\big)\,,$$

so it suffices to see that every $\mathcal{O}_E$-algebra morphism $\alpha\colon W_{\mathcal{O}_E}(R) \to A$ factors uniquely over θ. Let such an α be given. Modulo π we get an induced morphism $\overline{\alpha}\colon R \to A/\pi A$. Since R is perfect, $R^\flat \cong R$. Also $A^\flat \cong (A/\pi A)^\flat$. Hence we get an induced morphism $\overline{\alpha}^\flat\colon R \to A^\flat$. We claim that

$$\begin{tikzcd} W_{\mathcal{O}_E}(R) \arrow[r, "\alpha"] \arrow[d, "W_{\mathcal{O}_E}(\overline{\alpha}^\flat)"'] & A \\ W_{\mathcal{O}_E}(A^\flat) \arrow[ur, "\theta"'] & \end{tikzcd}$$

commutes. In view of Lemma 1.2.8 we only need to check that $\alpha[x] = \overline{\alpha}^\flat(x)^\sharp$ for all $x \in R$. By construction, $\overline{\alpha}^\flat(x)$ is the sequence $(\overline{\alpha}(x), \overline{\alpha}(x^{1/q}), \dots) \in A^\flat$. Moreover, $\alpha[x^{1/q^n}]$ is a lift of $\overline{\alpha}(x^{1/q^n})$ for all $n \in \mathbb{N}$. Raising $\overline{\alpha}(x^{1/q^n})$ to the $(q^n)^{\mathrm{th}}$ power gives $\alpha[x]$ back, since both α and the Teichmüller lift $[-]$ are multiplicative. This shows indeed $\alpha[x] = \overline{\alpha}^\flat(x)^\sharp$.

To finish the proof, it's left to see why $\overline{\alpha}^\flat$ is the only choice. Suppose $\beta\colon R \to A^\flat$ leads to a commutative diagram as above. Reducing modulo π we see that the composition of β with $A^\flat \to A/\pi$ must coincide with $\overline{\alpha}$. In other words, the 0^{th} component of $\beta\colon R \to A^\flat$ must be given by $\overline{\alpha}$. By naturality of the Witt vector Frobenius, the diagram

$$\begin{tikzcd} W_{\mathcal{O}_E}(R) \arrow[r, "F^{-1}"] \arrow[d, "W_{\mathcal{O}_E}(\beta)"'] & W_{\mathcal{O}_E}(R) \arrow[r, "\alpha"] \arrow[d, "W_{\mathcal{O}_E}(\beta)"'] & A \\ W_{\mathcal{O}_E}(A^\flat) \arrow[r, "F^{-1}"] & W_{\mathcal{O}_E}(A^\flat) \arrow[ur, "\theta"'] & \end{tikzcd}$$

commutes as well. Reducing modulo π and walking around the perimeter, we see that the 1^{st} component of $R \to A^\flat$ must be given by $\overline{\alpha}((-)^{1/q})$. Repeating this argument, we see that $\beta = \overline{\alpha}^\flat$, as desired. $\square$

1.2.1. Perfectoid $\mathcal{O}_E$-Algebras

1.2.9. Definition. — (1) A *perfect prism* over $\mathcal{O}_E$ is a pair $(W_{\mathcal{O}_E}(R), I)$, where R is a perfect $\mathbb{F}_q$-algebra, $I \subseteq W_{\mathcal{O}_E}(R)$ is a principal ideal generated by an element d such that

$$\frac{F(d) - d^q}{\pi} \in W_{\mathcal{O}_E}(R)^\times$$

(such d is called *distinguished*), and such that $W_{\mathcal{O}_E}(R)$ is (π, I)-adically complete.

(2) An $\mathcal{O}_E$-algebra A is a *perfectoid $\mathcal{O}_E$-algebra* if it can be written as $A \cong W_{\mathcal{O}_E}(R)/I$ for some perfect prism $(W_{\mathcal{O}_E}(R), I)$ over $\mathcal{O}_E$.

1.2.10. Remark. — (1) To see that $F(d) - d^q$ is always divisible by π, note that F is the lift of the Frobenius on R. In particular, F and $(-)^q$ become equal after reducing modulo π.

(2) An element $d = \sum_{n=0}^\infty [r_n]\pi^n \in W_{\mathcal{O}_E}(R)$ is distinguished iff $r_1 \in R^\times$. Indeed, by Lemma 1.1.12(2) and Lemma 1.1.17 we have $F(d) \equiv [r_0^q] + [r_1^q]\pi \mod \pi^2$ and from the key Lemma 1.2.6 we get $d^q \equiv [r_0^q] \mod \pi^2$. Hence

$$\frac{F(d) - d^q}{\pi} \equiv [r_1^q] \mod \pi\,.$$

By π-completeness, an element $x \in W_{\mathcal{O}_E}(R)$ is invertible iff its modulo-π reduction is invertible. And $r_1^q \in R$ is invertible iff so is r_1. Moreover, $W_{\mathcal{O}_E}(R)$ is (π, d)-adically complete iff R is r_0-complete. Since this seems rather non-trivial to me, we give it a proper proof in Lemma* 1.2.11 below.

(3) Perfect rings are perfectoid. Indeed, if R is perfect, we have $R \cong W_{\mathcal{O}_E}(R)/\pi W_{\mathcal{O}_E}(R)$, and $(W_{\mathcal{O}_E}(R), \pi)$ is clearly a perfect prism (by (2) for example). Conversely, if an algebra A over $\mathbb{F}_q = \mathcal{O}_E/\pi\mathcal{O}_E$ is perfectoid, then it is also perfect. This too was not trivial for me, so we prove it in Lemma* 1.2.12 below.

(4) If A is perfectoid, say, $A \cong W_{\mathcal{O}_E}(R)/I$, then

$$A^\flat \cong (W_{\mathcal{O}_E}(R)/I)^\flat \cong \big(W_{\mathcal{O}_E}(R)/(\pi, I)\big)^\flat \cong (R/IR)^\flat \cong R^\flat \cong R\,.$$

The only non-obvious step is $(R/IR)^\flat \cong R^\flat$. To see this, first note that IR is an ideal containing the image of π in R since this image is 0. Moreover, R is IR-adically complete by Lemma* 1.2.11. Hence the isomorphism follows from Proposition 1.2.7.

1.2.11. Lemma*. — *Let R be a perfect $\mathbb{F}_q$-algebra and $d = \sum_{n=0}^{\infty}[r_n]\pi^n$ be an element of $W = W_{\mathcal{O}_E}(R)$. Then W is (π, d)-adically complete iff R is r_0-complete.*

Proof.* Let's first assume W is (π, d)-complete. Then R being r_0-complete is equivalent to R being (π, d)-complete too. By [Stacks, Tag 031A], we need to check that

$$\pi W = \bigcap_{n\geqslant 1}\left(\pi W + (\pi, d)^n\right).$$

Suppose some $w \in W$ is contained in $\pi W + (\pi, d)^n$ for all $n \in \mathbb{N}$. Then its image $\overline{w} \in R$ is divisible by r_0^n for all $n \geqslant 0$, hence also $[\overline{w}]$ is divisible by $[r_0]^n$ for all $n \geqslant 0$. By a well-known argument, W being (π, d)-complete is equivalent to W being complete with respect to the ideals $\{(\pi^n, d^n)\}_{n\geqslant 1}$. By abstract nonsense, we may replace this family of ideals by $\{(\pi^{n+1}, d^{q^n})\}_{n\geqslant 1}$. But $d^{q^n} \equiv [r_0]^{q^n} \mod \pi^{n+1}$ by the key Lemma 1.2.6, hence W is also complete with respect to the ideals $\{(\pi^{n+1}, [r_0]^{q^n})\}_{n\geqslant 1}$. Since $[\overline{w}]$ lies in all of them by assumption, we get $\overline{w} = 0$, hence $w \in \pi W$, as required.

Now assume R is r_0-complete. It suffices to show that W is complete with respect to the ideals $\{(\pi^n, d^n)\}_{n\geqslant 1}$. By an abstract nonsense argument, this is equivalent to W being complete with respect to $\{(\pi^n, d^m)\}_{n,m\geqslant 1}$. Since W is π-complete, it thus suffices to show that $W/\pi^n W$ is d-complete for all $n \geqslant 1$. The key Lemma 1.2.6 shows $d^{q^m} \equiv [r_0]^{q^m} \mod \pi^n$ for all $m \geqslant n-1$. Thus we may equivalently show that $W/\pi^n W$ is $[r_0]$-complete.

We argue by induction over n. The case $n = 1$ is just the assumption. Now assume the assertion holds up to n. Consider the short exact sequence

$$0 \longrightarrow W/\pi^n W \xrightarrow{\pi} W/\pi^{n+1}W \longrightarrow R \longrightarrow 0.$$

Suppose $x \in W/\pi^n W$ has the property that $\pi x \in W/\pi^{n+1}W$ is divisible by $[r_0]^m$, say, $\pi x = [r_0^m]y$. Write $x = [x_0] + [x_1]\pi + \cdots + [x_{n-1}]\pi^{n-1}$ and $y = [y_0] + [y_1]\pi + \cdots + [y_n]\pi^n$. Then

$$[x_0]\pi + [x_1]\pi^2 + \cdots + [x_{n-1}]\pi^n = [r_0^m y_0] + [r_0^m y_1]\pi + \cdots + [r_0^m y_n]\pi^n.$$

By uniqueness of these representations, we get $0 = r_0^m y_0$, $x_0 = r_0^m y_1$ and so on up to $x_{n-1} = r_0^m y_n$. In particular, $x = [r_0]^m([y_1] + \cdots + [y_n]\pi^{n-1})$ is divisible by $[r_0]^m$! We conclude that the sequence

$$0 \longrightarrow W/(\pi^n, [r_0]^m) \xrightarrow{\pi} W/(\pi^{n+1}, [r_0]^m) \longrightarrow R/r_0^m R \longrightarrow 0$$

is exact again. Taking limits over m we obtain a diagram

$$\begin{tikzcd}
0 \arrow[r] & W/\pi^n W \arrow[r, "\pi"] \arrow[d, "\wr"] & W/\pi^{n+1}W \arrow[r] \arrow[d] & R \arrow[r] \arrow[d, "\wr"] & 0 \\
0 \arrow[r] & \lim_{m\geqslant 1} W/(\pi^n, [r_0]^m) \arrow[r, "\pi"] & \lim_{m\geqslant 1} W/(\pi^{n+1}, [r_0]^m) \arrow[r] & \lim_{m\geqslant 1} R/r_0^m R \arrow[r] & 0
\end{tikzcd}$$

in which the outer vertical arrows are isomorphisms by the induction hypothesis. Thus the middle vertical arrow is an isomorphism as well by the five lemma (note that the bottom sequence is exact by the Mittag-Leffler condition, but this isn't even needed for the argument). □

1.2.12. Lemma*. — *If an algebra A over $\mathbb{F}_q = \mathcal{O}_E/\pi\mathcal{O}_E$ is perfectoid, then A is already a perfect $\mathbb{F}_q$-algebra.*

Proof.* Write $A \cong W_{\mathcal{O}_E}(R)/I$. Since π vanishes on A, we have $\pi \in A$. By Remark 1.2.10(4), $A^\flat \cong R \cong W_{\mathcal{O}_E}(R)/\pi W_{\mathcal{O}_E}(R)$. Hence it suffices to prove that I is generated by π, since then $A \cong A^\flat$ is perfect.

The argument that follows is stolen from [BMS18, Lemma 3.10]. Write $\pi = dw$, where $d \in I$ is a distinguished generator and $w = \sum_{n=0}^{\infty}[w_n]\pi^n$ is some element of $W_{\mathcal{O}_E}(A^\flat)$. The Witt polynomial P_1 is given by $P_1(X,Y) = X_0^q Y_1 + X_1 Y_0^q + \pi X_1 Y_1$. Thus $\pi = dw$ yields

$$1 = r_0^q w_1 + r_1 w_0^q$$

(note that $\pi r_1 w_1$ vanishes in $A^\flat$). We claim that $r_1 w_0^q = 1 - r_0^q w_1$ is a unit in $A^\flat$. It suffices to check that it is mapped to a unit under the projection $A^\flat \to A/\pi A = A$ to the 0^{th} component. But $A \cong A^\flat / r_0 A^\flat$, hence $1 - r_0^q w_1$ is mapped to $1 \in A$, which is indeed a unit. Thus also r_1 and w_0 are units in $A^\flat$. But w_0 being a unit implies that w itself is a unit in $W_{\mathcal{O}_E}(A^\flat)$, hence π is indeed a generator of I. $\square$

The following fact wasn't mentioned in the lecture, making it hard for me to read some of the literature that uses the "old" definition of perfectoid rings. So we prove it here.

1.2.13. Lemma*. — *Let $(W_{\mathcal{O}_E}(R), I)$ be a perfect prism over $\mathcal{O}_E$ and $A = W_{\mathcal{O}_E}(R)/I$.*

(*a*) *If ξ is a distinguished generator of I, then ξ is a non-zero divisor in $W_{\mathcal{O}_E}(R)$.*

(*b*) *A is π-complete.*

Proof.* Put $W = W_{\mathcal{O}_E}(R)$ for convenience. Both (*a*) and (*b*) are based on the following observation.

($*$) Let $(x_n)_{n\in\mathbb{N}}$ be a sequence such that $\xi x_n \equiv 0 \mod \pi^n$. Then the x_n converge to 0 in the (π,ξ)-adic topology.

Claim ($*$) immediately implies (*a*). Also (*b*) is not far: by [Stacks, Tag 031A], we need to check that

$$\xi W = \bigcap_{n\geqslant 1}(\xi W + \pi^n W)\,.$$

So suppose y lies in the intersection and choose $(x_n)_{n\in\mathbb{N}}$ such that $y \equiv \xi x_n \mod \pi^n$. Then $\xi(x_{n+1} - x_n) \equiv 0 \mod \pi^n$. Thus the $(x_{n+1}-x_n)$ converge to 0 in the (π,ξ)-adic topology. Hence $(x_n)_{n\in\mathbb{N}}$ converges to some $x \in W$ satisfying $y = \xi x$. This shows (*a*).

It remains to show ($*$). Write $\xi = [r_0] + \pi u$, where $u \in W$ is a unit. If $\xi x_n \equiv 0 \mod \pi^n$, then also $([r_0]^s + \pi^s u^s)x_n \equiv 0 \mod \pi^n$ for all odd s, since $[r_0] + \pi u$ divides $[r_0]^s + \pi^s u^s$ for odd s. Now $\pi^s x_n \equiv -[r_0]^s u^{-s} x_n \mod \pi^n$ shows that the first n coefficients in π-adic expansion of $\pi^s x_n$ must be divisible by r_0^s. In other words, we can write

$$x_n = [r_0^s y_0] + [r_0^s y_1]\pi + \dots + [r_0^s y_{n-s-1}]\pi^{n-s-1} + \pi^{n-s} z\,.$$

Thus, $x_n \in (\pi^{n-s}, [r_0]^s)$ for all odd s. Choosing s roughly equal to $n/2$, we see that $(x_n)_{n\in\mathbb{N}}$ converges with respect to the ideals $\{(\pi^m, [r_0]^m)\}_{m\geqslant 1}$. But these ideals generate (π,ξ)-adic topology, as seen in the proof of Lemma* 1.2.11. $\square$

Remark 1.2.10(4) suggests the following definition.

1.2.14. Definition. — Let R be a perfect $\mathbb{F}_q$-algebra. An *untilt* of R is a pair (A,ι), where A is a perfectoid $\mathcal{O}_E$-algebra and ι an isomorphism $\iota\colon R \xrightarrow{\sim} A^\flat$.

Again by Remark 1.2.10(4) we get a bijection

$$\left\{\begin{array}{c}\text{isomorphism classes of}\\ \text{untilts } (A,\iota) \text{ of } R\end{array}\right\} \overset{\sim}{\longleftrightarrow} \left\{\begin{array}{c}\text{ideals } I \subseteq W_{\mathcal{O}_E}(R) \text{ such that}\\ (W_{\mathcal{O}_E}(R), I) \text{ is a perfect prism over } \mathcal{O}_E\end{array}\right\}$$

1.2.15. Exercise (Tilting equivalence)**.** — If A is a perfectoid $\mathcal{O}_E$-algebra, then there is an equivalence of categories

$$\begin{aligned}\{\text{perfectoid } A\text{-algebras}\} &\overset{\sim}{\longleftrightarrow} \left\{\text{perfect(oid) } A^\flat\text{-algebras}\right\}\\ B &\longmapsto B^\flat\\ W_{\mathcal{O}_E}(S) \otimes_{W_{\mathcal{O}_E}(A^\flat)} A &\longleftarrow S\end{aligned}$$

(on the left-hand side, A gets a $W_{\mathcal{O}_E}(A^\flat)$-algebra structure via θ).

*Disproof**. The assertion as stated is wrong. Take $A^\flat = \mathbb{F}_p[\![T^{1/p^\infty}]\!]$ and A comes from the perfect prism $(W(A^\flat), T - p)$. This works by Lemma* 1.2.11 since $T - p$ is clearly distinguished and $A^\flat$ is T-complete. We claim that there is a perfect $A^\flat$-algebra S such that $W(S) \otimes_{W(A^\flat)} A$ is not perfectoid. Indeed, for it to be perfectoid, $(W(S), (T-p)W(S))$ would need to be a perfect prism, which again needs S to be T-complete by Lemma* 1.2.11 again. However, there are perfect $A^\flat$ algebras S which are not T-complete; for example, the Laurent series ring $S = \mathbb{F}_p(\!(T^{1/p^\infty})\!)$. □

1.2.16. Corrected exercise* (The actual tilting equivalence)**.** — By a *perfectoid $A^\flat$-algebra* S we don't just understand an $A^\flat$-algebra that is perfectoid. The topology on S must also be induced by the topology on $A^\flat$, i.e., S must be (π, I)-complete, where I is the kernel of $\theta\colon W_{\mathcal{O}_E}(A^\flat) \to A$ (so that $(W_{\mathcal{O}_E}(A^\flat), I)$ is a perfect prism that gives A). Then there is an equivalence of categories

$$\{\text{perfectoid } A\text{-algebras}\} \overset{\sim}{\longleftrightarrow} \left\{\text{perfect } A^\flat\text{-algebras}\right\}$$

as in Exercise 1.2.15.

*Proof**. Put $W_A = W_{\mathcal{O}_E}(A^\flat)$ and $W_S = W_{\mathcal{O}_E}(S)$ for convenience. Let ξ be a distinguished generator of I. First note that $W_S \otimes_{W_A} A \cong W_S/\xi W_S$ is again perfectoid. Indeed, we need to check that $(W_S, \xi W_S)$ is a perfect prism. Clearly ξW_S is a distinguishedly generated ideal. Also S is (π, I)-complete and hence ξ-complete, so W_S is $(\pi, \xi W_S)$-complete by Lemma* 1.2.11. This shows that $(W_S, \xi W_S)$ is a perfect prism, as required. Now the calculation from Remark 1.2.10(4) shows $(W_S/\xi W_S)^\flat \cong S$.

Conversely, we have to show that for a perfectoid A-algebra B we get $B \cong W_B \otimes_{W_A} A$, where $W_B = W_{\mathcal{O}_E}(B^\flat)$ for brevity, and that $B^\flat$ is (π, I)-complete. Write $B \cong W_B/J$. Then $(W_A, I) \to (W_B, J)$ is a morphism of perfect prisms in the sense that it is a $\mathcal{O}_E$-algebra morphism that maps I into J. An argument analogous to the stolen one from the proof of Lemma* 1.2.12 (hint: replace 1 be the coefficient of π in ξ, which is still a unit) shows that actually $J = IW_B$. But this immediately shows $B \cong W_B \otimes_{W_A} A$ and we are done. □

1.2.17. Example ⚠. — If C/E is a non-archimedean (recall that this requires C to be complete) algebraically closed field extension, then the ring of integers $\mathcal{O}_C$ is a perfectoid $\mathcal{O}_E$-algebra.

Proof. We first formulate two claims which together will imply the assertion.

(1) Let $\{\pi^{1/q^n}\}_{n\geqslant 0}$ be a compatible system of $(q^n)^{\mathrm{th}}$ roots of π in $\mathcal{O}_C$. They define an element $\pi^\flat = (\pi, \pi^{1/q}, \dots) \in \mathcal{O}_C^\flat$. Then

$$\mathcal{O}_C^\flat/\pi^\flat\mathcal{O}_C^\flat \cong \mathcal{O}_C/\pi\mathcal{O}_C\,.$$

(2) The kernel of $\theta\colon W_{\mathcal{O}_E}(\mathcal{O}_C^\flat) \to \mathcal{O}_C$ is generated by $\pi - [\pi^\flat]$.

We start with (1). Note that by Proposition 1.2.7 we may write $\mathcal{O}_C^\flat \cong \lim_{x\mapsto x^q}\mathcal{O}_C$. Now let $y = (y_0, y_1, \dots) \in \mathcal{O}_C^\flat$. Then $\pi^\flat \mid y$ iff $\pi^{1/q^n} \mid y_n$ for all $n \geqslant 0$. Since $\mathcal{O}_C$ is a valuation ring, this is equivalent to $|\pi|^{1/q^n} \geqslant |y_n| = |y_0|^{1/q^n}$. Thus, $\pi^\flat \mid y$ is equivalent to the single condition $y_0 \equiv 0 \mod \pi$. Therefore, the kernel of $(-)^\sharp\colon \mathcal{O}_C^\flat \to \mathcal{O}_C/\pi\mathcal{O}_C$ is generated by $\pi^\flat$. However, $\mathcal{O}_C^\flat \twoheadrightarrow \mathcal{O}_C/\pi\mathcal{O}_C$ is clearly surjective (since C is algebraically closed), hence indeed

$$\mathcal{O}_C^\flat/\pi^\flat\mathcal{O}_C^\flat \cong \mathcal{O}_C/\pi\mathcal{O}_C\,.$$

For (2), Lemma 1.2.8 shows $\theta(\pi - [\pi^\flat]) = \pi - (\pi^\flat)^\sharp = \pi - \pi = 0$. So $\pi - [\pi^\flat] \in \ker\theta$. Conversely, let $x = \sum_{n=0}^\infty [x_n]\pi^n$ be an element of $\ker\theta$. Hence

$$0 \equiv \theta(x) \equiv \sum_{n=0}^\infty x_n^\sharp \pi^n \equiv x_0^\sharp \mod \pi\,.$$

From (1) we get $\pi^\flat \mid x_0$, say, $x_0 = \pi^\flat y$. Write $z^{(0)} = \sum_{n=1}^\infty [x_n]\pi^{n-1}$ and $x^{(1)} = [y] + z^{(0)}$. Then $x = [\pi^\flat]x^{(1)} + (\pi - [\pi^\flat])z^{(0)}$. We obtain

$$0 = \theta(x) = \theta\big([\pi^\flat]x^{(1)}\big) = \pi\theta\big(x^{(1)}\big)\,,$$

hence also $\theta(x^{(1)}) = 0$ since $\mathcal{O}_C$ is π-torsionfree. Repeating this process with $x^{(1)}$ and iterating, we get an expression

$$x = \xi\big(z^{(0)} + [\pi^\flat]z^{(1)} + \cdots\big)\,,$$

where $\xi = \pi - [\pi^\flat]$. This shows that x lies in the ideal generated by ξ, proving (2).

It remains to see that $\theta\colon W_{\mathcal{O}_E}(\mathcal{O}_C^\flat) \to \mathcal{O}_C$ is surjective and that $(W_{\mathcal{O}_E}(\mathcal{O}_C^\flat), \xi)$ is a perfect prism. The first assertion is because $(-)^\sharp\colon \mathcal{O}_C^\flat \to \mathcal{O}_C$ is surjective since C is algebraically closed. For the second assertion, $\xi = \pi - [\pi^\flat]$ is clearly distinguished by Remark 1.2.10(2), so it remains to show that $\mathcal{O}_C^\flat$ is $\pi^\flat$-complete. Observe that for all $c \geqslant 0$ the c^{th} component of $(\pi^\flat)^{q^n}$ is 0 for all $n \geqslant c$. From this observation, $\pi^\flat$-completeness of $\mathcal{O}_C^\flat$ easily follows. □

Next time we proof the first half of the following Lemma 1.2.18 (see Lemma 1.2.23). The other half will have to wait until the 4^{th} lecture.

1.2.18. Lemma. — *Let A be a perfectoid $\mathcal{O}_E$-algebra. Then A is isomorphic to $\mathcal{O}_C$ for some non-archimedean algebraically closed extension C/E if and only if $A^\flat$ is isomorphic to $\mathcal{O}_F$ for some non-archimedean algebraically closed extension $F/\mathbb{F}_q$.*

1.2.19. Remark. — Recall that for F as in Lemma 1.2.18 we put $\mathbb{A}_{\mathrm{inf}} = W_{\mathcal{O}_E}(\mathcal{O}_F)$.

(1) $\mathbb{A}_{\mathrm{inf}}$ is a local integral domain. This is in fact true for any $W_{\mathcal{O}_E}(R)$ if R itself is a local integral domain over $\mathbb{F}_q$ (this follows from Lemma 1.1.17 for example).

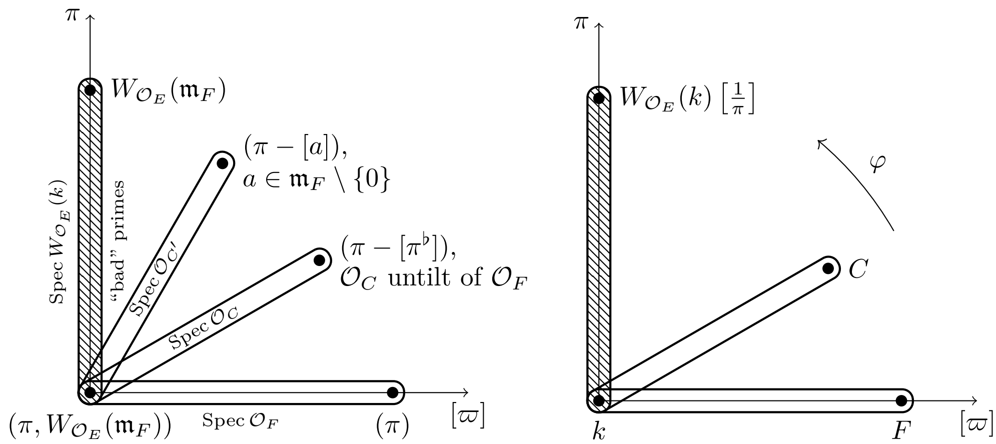
(2) $\mathbb{A}_{\mathrm{inf}}$ is $(\pi, [\varpi])$-complete for any $\varpi \in \mathfrak{m}_F \setminus \{0\}$. Indeed, this follows from Remark 1.2.10(2) as $\mathcal{O}_F$ is easily seen to be ϖ-complete. Such ϖ is called a *pseudo-uniformizer*.

(3) By a theorem of Ludwig–Lang, $\mathbb{A}_{\mathrm{inf}}$ has infinite Krull dimension (and is, in particular, non-noetherian). We can actually see by hand that $\mathbb{A}_{\mathrm{inf}}$ is at least three-dimensional: there is a chain

$$0 \subsetneq \bigcup_{x \in \mathfrak{m}_F} [x]\mathbb{A}_{\mathrm{inf}} \subsetneq W_{\mathcal{O}_E}(\mathfrak{m}_F) \subsetneq (\pi, W_{\mathcal{O}_E}(\mathfrak{m}_F))$$

of prime ideals. Also note that $(\pi, W_{\mathcal{O}_E}(\mathfrak{m}_F))$ is the unique maximal ideal of $\mathbb{A}_{\mathrm{inf}}$ since an element of $\mathbb{A}_{\mathrm{inf}}$ is invertible iff its image in $\mathbb{A}_{\mathrm{inf}}/\pi\mathbb{A}_{\mathrm{inf}} \cong \mathcal{O}_F$ is invertible.

Despite Remark 1.2.19(3), we should think of $\mathbb{A}_{\mathrm{inf}}$ as a two-dimensional ring, except for some "bad" primes. Here's a "picture" of $\operatorname{Spec} \mathbb{A}_{\mathrm{inf}}$. The left picture shows a select choice of prime ideals of $\mathbb{A}_{\mathrm{inf}}$. In the right picture the corresponding residue fields are shown and the Frobenius action φ is indicated.



We put $k = \mathcal{O}_F/\mathfrak{m}_F$ for convenience. We will see next time that $\operatorname{Spec} \mathbb{A}_{\mathrm{inf}}$ is indeed "two-dimensional away from $[\varpi] = 0$". More precisely, we will show the following: let $(\mathcal{O}_C, \iota)$ be an untilt of $\mathcal{O}_F$ and ξ a generator of $\ker(\theta\colon \mathbb{A}_{\mathrm{inf}} \to \mathcal{O}_C)$. Put

$$B_{\mathrm{dR}}^+ = \mathbb{A}_{\mathrm{inf}}\left[\tfrac{1}{\pi}\right]_\xi^\wedge .$$

Then B_{dR}^+ is always a DVR and the same is true for $\mathbb{A}_{\mathrm{inf},(\pi-[\pi^\flat])}$ (see Lemma 1.2.24 below). Moreover, in the lecture after the next one we will show that all $(\pi - [a])$ for $a \in \mathfrak{m}_F \setminus \{0\}$ are prime ideals, and in fact $\mathbb{A}_{\mathrm{inf}}/(\pi - [a])$ is isomorphic to another untilt $\mathcal{O}_{C'}$ of $\mathcal{O}_F$ (as indicated in the left picture), with C'/E an algebraically closed non-archimedean extension.

Lecture 3
6th Nov, 2019

1.2.20. Side remark. — Why this setup? Let $K/\mathbb{Q}_p$ be a discretely valued non-archimedean field extension with perfect residue field and let X/K be a smooth proper scheme. The objects of interest in p-adic hodge theory are the p-adic cohomology groups $H^*_{\mathrm{\acute{e}t}}(X_{\overline{K}}, \mathbb{Q}_p)$. We will replace $\mathbb{Q}_p$ by E and $\overline{K}$ by $C = \widehat{\overline{K}}$, with $F = C^\flat = \operatorname{Frac}(\mathcal{O}_C^\flat)$.

1.2.21. Definition. — An element $x = \sum_{n=0}^\infty [x_n]\pi^n$ of $\mathbb{A}_{\mathrm{inf}}$ is called *primitive* if $x_0 \neq 0$ and there exists a $d \geqslant 0$ such that $x_d \in \mathcal{O}_F^\times$. If x is primitive, the smallest such d is called the *degree* of x. The set primitive elements of degree d is denoted Prim_d.

1.2.22. Example. — We have $\mathrm{Prim}_0 = \mathbb{A}_{\mathrm{inf}}^{\times}$. Moreover, any element $x \in \mathrm{Prim}_1$ is distinguished. The converse is true iff $[x_0] \neq 0$.

Next time we will see that if $a \in \mathrm{Prim}_1$, then $a\mathbb{A}_{\mathrm{inf}}$ is a prime ideal and $\mathbb{A}_{\mathrm{inf}}/a\mathbb{A}_{\mathrm{inf}} \cong \mathcal{O}_C$ for some non-archimedean algebraically closed extension C/E (which generalizes the claim about the $(\pi - [a])$ above). For now let C/E be such an extension and $|\ |\colon C \to \mathbb{R}_{\geqslant 0}$ its norm. Recall that Proposition 1.2.7 provides an isomorphism

$$\mathcal{O}_C^\flat \cong \lim_{x \mapsto x^q} \mathcal{O}_C\,,$$

sending an element $x \in \mathcal{O}_C^\flat$ of the left-hand side to $(x^\sharp, (x^{1/q})^\sharp, \dots)$ contained in the right-hand side.

1.2.23. Lemma. — *Assume we are in the above situation.*

(1) *The map $|\ |^\flat\colon \mathcal{O}_C^\flat \to \mathbb{R}_{\geqslant 0}$ given by $x \mapsto |x^\sharp|$ is a norm on $\mathcal{O}_C^\flat$. Moreover, $\mathcal{O}_C^\flat$ is complete with respect to the topology induced by $|\ |^\flat$.*

(2) *$C^\flat = \mathrm{Frac}(\mathcal{O}_C^\flat)$ is a non-archimedean algebraically closed extension of $\mathbb{F}_q$.*

Proof. It is clear that $|\ |^\flat$ is multiplicative, that $|1|^\flat = 1$, and that $|x|^\flat = 0$ iff $x = 0$. So only the triangle inequality remains. We calculate

$$\begin{aligned} |x+y|^\flat = |(x+y)^\sharp| &= \lim_{n\to\infty} \left|\Big(\big(x^{1/q^n}\big)^\sharp + \big(y^{1/q^n}\big)\Big)^{q^n}\right| \\ &= \lim_{n\to\infty} \max\left\{\big|\big(x^{1/q^n}\big)^\sharp\big|^{q^n}, \big|\big(y^{1/q^n}\big)^\sharp\big|^{q^n}\right\} \\ &= \lim_{n\to\infty} \max\left\{|x^\sharp|, |y^\sharp|\right\} \\ &= \max\left\{|x|^\flat, |y|^\flat\right\} \end{aligned}$$

This shows that $|\ |^\flat$ is a norm in $\mathcal{O}_C^\flat$. To show that $\mathcal{O}_C^\flat$ is complete, we claim that the topology generated by $|\ |^\flat$ is the inverse limit topology on $\mathcal{O}_C^\flat \cong \lim_{x\mapsto x^q} \mathcal{O}_C$. A neighbourhood basis of 0 in the topology generated by $|\ |^\flat$ is given by the sets

$$\left\{x \;\middle|\; |x|^\flat < \varepsilon\right\} \quad \text{for all } \varepsilon > 0\,.$$

In the inverse limit topology, a neighbourhood basis of 0 is given by the sets

$$\left\{x \in \mathcal{O}_C^\flat \;\middle|\; \big|\big(x^{1/q^n}\big)^\sharp\big| < \delta\right\} \quad \text{for all } \delta > 0,\ n \geqslant 0\,.$$

But $|(x^{1/q^n})^\sharp| = (|x|^\flat)^{1/q^n}$, so its easy to see that these topology bases not only generate the same topology, but even coincide on the nose.

For (2), it remains to show that $C^\flat$ is algebraically closed, and for this it suffices to show that $\mathcal{O}_C^\flat$ is integrally closed. So let $f \in \mathcal{O}_C^\flat[T]$ be a monic polynomial. Write $f(T) = T^d + a_{d-1}T^{d-1} + \dots + a_0$. For all $n \geqslant 0$ put

$$f_n(T) = T^d + \big(a_{d-1}^{1/q^n}\big)^\sharp T^{d-1} + \dots + \big(a_0^{1/q^n}\big)^\sharp \in \mathcal{O}_C[T]\,.$$

Then $f_{n+1}(T)^q \equiv f_n(T^q) \mod \pi$. Now fix $n \geqslant 0$ and let $x \in \mathcal{O}_C$ be a zero of f_n, which exists as $\mathcal{O}_C$ is integrally closed. Choose $y \in \mathcal{O}_C$ such that $y^q = x$. Although y need not be a root of f_{n+1}, we certainly have $|f_{n+1}(y)| \leqslant |\pi|^{1/q}$. Let $z_1, \dots, z_n \in \mathcal{O}_C$ be the actual roots of f_{n+1}. Then

$$|f_{n+1}(y)| = \prod_{i=1}^{d} |y - z_i| \leqslant |\pi|^{1/q} .$$

hence there exists an index i such that $|y - z_i| \leqslant |\pi|^{1/dq}$, or equivalently $|y - z_i|^q \leqslant |\pi|^{1/d}$. Then also $|x - z_i^q| \leqslant |\pi|^{1/d}$ as all other terms in the expansion of $(y - z_i)^q$ are divisible by π. By induction, we obtain a sequence $(x_n)_{n\in\mathbb{N}}$ such that $x_n \in \mathcal{O}_C$, $f_n(x_n) = 0$, and the x_n are "close" to being q-power compatible in the sense that $|x_{n+1} - x_n^q| \leqslant |\pi|^{1/d}$. But this is actually sufficient! Indeed, put $\mathfrak{a} = \left\{y \in \mathcal{O}_C \;\middle|\; |y| \leqslant |\pi|^{1/d}\right\}$. Then $x = (x_n)_{n\in\mathbb{N}}$ is an element of

$$\lim_{x\mapsto x^q} \mathcal{O}_C/\mathfrak{a} \cong \lim_{x\mapsto x^q} \mathcal{O}_C/\pi\mathcal{O}_C = \mathcal{O}_C^\flat ,$$

where we use Proposition 1.2.7 to obtain the isomorphism on the left. Hence x corresponds to an element $x \in \mathcal{O}_C^\flat$, which clearly satisfies $f(x) = 0$. □

1.2.24. Lemma. — *Let $\mathcal{O}_C$ be an untilt of $\mathcal{O}_F$ and let ξ be a distinguished generator of the kernel of $\theta\colon \mathbb{A}_{\mathrm{inf}} \to \mathcal{O}_C$. As above, we put $B_{\mathrm{dR}}^+ = \mathbb{A}_{\mathrm{inf}}\left[\frac{1}{\pi}\right]_\xi^\wedge$. Then the following holds.*

(1) *The canonical map $\mathbb{A}_{\mathrm{inf}} \hookrightarrow B_{\mathrm{dR}}^+$ is an injection.*

(2) *B_{dR}^+ and $\mathbb{A}_{\mathrm{inf},(\xi)}$ are discrete valuation rings.*

Proof. Since we are not in a noetherian setting, we need to be careful with completion. As (ξ) is obviously a finitely generated ideal, [Stacks, Tag 05GG] shows that B_{dR}^+ is ξ-complete. Moreover,

$$B_{\mathrm{dR}}^+/(\xi^n) \cong \mathbb{A}_{\mathrm{inf}}\left[\tfrac{1}{\pi}\right]/(\xi^n) \cong \mathbb{A}_{\mathrm{inf}}/(\xi^n)\left[\tfrac{1}{\pi}\right] ,$$

by exactness of localization. We claim that $\mathbb{A}_{\mathrm{inf}}/(\xi^n) \hookrightarrow \mathbb{A}_{\mathrm{inf}}/(\xi^n)\left[\frac{1}{\pi}\right]$ is injective for all n. To show this, we need to check that $\mathbb{A}_{\mathrm{inf}}/(\xi^n)$ is π-torsionfree. We use induction on n. For $n = 1$ we get $\mathbb{A}_{\mathrm{inf}}/(\xi) \cong \mathcal{O}_C$, which is π-torsionfree. Now suppose $\pi x = \xi^n y$ for some $x, y \in \mathbb{A}_{\mathrm{inf}}$. By the $n = 1$ case we see that x must be divisible by ξ, say, $x = \xi x'$. Since $\mathbb{A}_{\mathrm{inf}}$ is a domain this implies $\pi x' = \xi^{n-1} y$. But then the induction hypothesis shows that x' itself must be divisible by ξ^{n-1}, proving the claim. Now since limits are left exact, we see that

$$\mathbb{A}_{\mathrm{inf}} \cong \lim_{n\in\mathbb{N}} \mathbb{A}_{\mathrm{inf}}/(\xi^n) \longrightarrow \lim_{n\in\mathbb{N}} \left(\mathbb{A}_{\mathrm{inf}}/(\xi^n)\left[\tfrac{1}{\pi}\right]\right) \cong B_{\mathrm{dR}}^+$$

is injective, as required. The isomorphism on the left-hand side uses that $\mathbb{A}_{\mathrm{inf}}$ is ξ-complete by [Stacks, Tag 09OT] and the fact that $\mathbb{A}_{\mathrm{inf}}$ is (π, ξ)-complete. This shows (1).

For (2), first note that $B_{\mathrm{dR}}^+/(\xi) \cong \mathcal{O}_C\left[\frac{1}{\pi}\right] \cong C$. Hence [Stacks, Tag 05GH] implies that B_{dR}^+ is noetherian. Moreover, we know that B_{dR}^+ is local with maximal ideal (ξ), because it is ξ-adically complete and its quotient by ξ is C, which is a field. This implies $\dim B_{\mathrm{dR}}^+ \leqslant 1$. Moreover, we are done once we show $\dim B_{\mathrm{dR}}^+ \geqslant 1$, since then B_{dR}^+ is a one-dimensional noetherian local ring whose maximal ideal is principal, hence regular, hence a DVR.

For $\dim B_{\mathrm{dR}}^+ \geqslant 1$ it suffices to see that B_{dR}^+ is a domain, since then $0 \subsetneq (\xi)$ is a chain of prime ideals. From (1) and the fact that $\mathbb{A}_{\mathrm{inf}}$ is a domain, it's easy to see that B_{dR}^+ is ξ-torsionfree. Now if $xy = 0$ for $x, y \in B_{\mathrm{dR}}^+$, then x or y must be divisible by ξ as $B_{\mathrm{dR}}^+/(\xi) \cong C$. Say $x = \xi x'$. Then B_{dR}^+ being ξ-torsionfree shows $x'y = 0$. Iterating the argument shows $x = 0$ or $y = 0$ as B_{dR}^+ is ξ-complete. This finishes the proof that B_{dR}^+ is indeed a DVR.

Now for $\mathbb{A}_{\mathrm{inf},(\xi)}$. Take any prime ideal $\mathfrak{p} \subseteq \mathbb{A}_{\mathrm{inf},(\xi)}$ such that $\xi \notin \mathfrak{p}$. Still $\mathfrak{p} \subseteq (\xi)$ as (ξ) is the maximal ideal of $\mathbb{A}_{\mathrm{inf},(\xi)}$. Hence, if $a \in \mathfrak{p}$, then $a = b\xi$. But since $\mathfrak{p}$ is prime and $\xi \notin \mathfrak{p}$, this implies $b \in \mathfrak{p}$. Thus $\xi\mathfrak{p} = \mathfrak{p}$. Now let $\mathfrak{q} = \mathfrak{p}B_{\mathrm{dR}}^+$. Then $\xi\mathfrak{q} = \mathfrak{q}$ shows $\mathfrak{q} = 0$ as B_{dR}^+ is a DVR. But $\mathbb{A}_{\mathrm{inf},(\xi)} \hookrightarrow B_{\mathrm{dR}}^+$ is injective by (1) as localizations of injections stay injective. This shows $\mathfrak{p} = 0$.

What we have shown is that $\operatorname{Spec} \mathbb{A}_{\mathrm{inf},(\xi)}$ has exactly two points, namely $\{0, (\xi)\}$. But then all prime ideals of $\mathbb{A}_{\mathrm{inf}}$ are finitely generated, which implies that $\mathbb{A}_{\mathrm{inf}}$ is noetherian by the rather obscure fact [Stacks, Tag 05KG]. Now it's clear that $\mathbb{A}_{\mathrm{inf},(\xi)}$ is one-dimensional and regular, hence a DVR. □

Have you ever wondered what the "inf" in $\mathbb{A}_{\mathrm{inf}}$ actually means? It stands for *infinitesimal*. In fact, this leads to a description of $\mathbb{A}_{\mathrm{inf}}$ as a universal thickening of $\mathcal{O}_C$!

1.2.25. Definition. — Let R be a π-complete $\mathcal{O}_E$-algebras. A *π-adic pro-infinitesimal thickening of R* is a surjection $D \twoheadrightarrow R$ of $\mathcal{O}_E$-algebras with kernel I such that D is (π, I)-adically complete.

1.2.26. Example. — For $R \in \{\mathcal{O}_C, \mathcal{O}_C/\pi\mathcal{O}_C\}$, the natural map $\mathbb{A}_{\mathrm{inf}} \twoheadrightarrow R$ is a π-adic pro-infinitesimal thickening. Indeed, its kernel is given by (ξ) and (π, ξ) respectively. Actually, $\mathbb{A}_{\mathrm{inf}}$ is the universal π-adic pro-infinitesimal thickening of R, as shown in the following lemma!

1.2.27. Lemma. — *Let $R \in \{\mathcal{O}_C, \mathcal{O}_C/\pi\mathcal{O}_C\}$ and let $D \twoheadrightarrow R$ be a π-adic pro-infinitesimal thickening. Then it factors uniquely as*

$$\begin{tikzcd} \mathbb{A}_{\mathrm{inf}} \arrow[r, two heads] \arrow[d, dashed, "\exists!"'] & R \\ D \arrow[ur, two heads] & \end{tikzcd}$$

Sketch of a proof. By Proposition 1.2.7 we have $\lim_{x\mapsto x^q} D \cong (D/(\pi, I))^\flat \cong R^\flat$. By the same argument $\lim_{x\mapsto x^q} D \cong D^\flat$. Hence $D^\flat \cong R^\flat$. Thus, the Witt-tilting adjunction (Proposition 1.2.4) provides a unique map

$$\mathbb{A}_{\mathrm{inf}} \cong W_{\mathcal{O}_E}(R^\flat) \longrightarrow D\,.$$

It's easily verified that this map has the required properties. □

1.2.2. p-adic PD-thickenings and $\mathbb{A}_{\mathrm{cris}}$

From now on, we restrict our attention to the case $E = \mathbb{Q}_p$ and $\pi = p$. As above, let $R \in \{\mathcal{O}_C, \mathcal{O}_C/p\mathcal{O}_C\}$.

1.2.28. Definition. — A *p-adic PD-thickening* of R is a triple $(D, D \twoheadrightarrow R, (\gamma_n)_{n\in\mathbb{N}})$, where D is p-complete and $(\gamma_n)_{n\in\mathbb{N}}$ a PD-structure on $J = \ker(D \twoheadrightarrow R)$ which is compatible with the canonical PD-structure on pR.

1.2.29. Remark. — (1) If D is p-torsionfree, then necessarily $\gamma_n(x) = x^n/n!$.

(2) Normalize $|\ \ |\colon C \to \mathbb{R}_{\geqslant 0}$ such that $|p| = p^{-1}$. Then a well-known calculation shows $|n!| \geqslant p^{(n-1)/(p-1)}$ for all $n \in \mathbb{N}$. In fact, the 1 in $n-1$ can be replaced by the digit sum

of the p-adic expansion of n. Thus, it's easy to check that $|x^n/n!| \leqslant 1$ for all $n \in \mathbb{N}$ iff $|x| < p^{-1/(p-1)}$. Moreover, one may check that

$$\left\{x \in \mathcal{O}_C \;\middle|\; |x| < p^{-1/(p-1)}\right\}$$

is the largest ideal in $\mathcal{O}_C$ admitting divided powers.

1.2.30. Definition. — The ring $\mathbb{A}_{\mathrm{cris}}$ denotes the universal p-adic PD-thickening of $\mathcal{O}_C$, or equivalently, of $\mathcal{O}_C/p\mathcal{O}_C$. In fancy words,

$$\mathbb{A}_{\mathrm{cris}} = H^0_{\mathrm{cris}}(\mathcal{O}_C/\mathbb{Z}_p) \cong H^0_{\mathrm{cris}}\big((\mathcal{O}_C/p\mathcal{O}_C)/\mathbb{Z}_p\big)\,.$$

Concretely, $\mathbb{A}_{\mathrm{cris}}$ is the p-adic divided power envelope of $\ker\theta = (\xi) \subseteq \mathbb{A}_{\mathrm{inf}}$. This follows more or less from Lemma 1.2.27, but this requires an additional argument, since a p-adic PD-thickening D of R need not be (p, J)-complete, so $D \twoheadrightarrow R$ need not be a p-adic pro-infinitesimal thickening. But the conclusion of that lemma is still true: we get a unique map $\mathbb{A}_{\mathrm{inf}} \to D$ over R, and then its formal to see that $\mathbb{A}_{\mathrm{cris}}$ can be described as above.

So where does the map $\mathbb{A}_{\mathrm{inf}} \to D$ come from? A closer inspection of the proof of Lemma 1.2.27 shows that we only need to show that $D^\flat \to R^\flat$ is an isomorphism. We can't use Proposition 1.2.7 to prove this. However, we can still construct an inverse $R^\flat \to D^\flat$ in the same way as in the proof of that proposition. This is based on the following observation, that serves as a replacement for Lemma 1.2.6.

1.2.31. Lemma*. — *If $x, y \in D$ such that $x \equiv y \mod (p, J)$, then $(x^{p^n} - y^{p^n})_{n\in\mathbb{N}}$ converges to 0 in the p-adic topology.*

Proof.* Observe that for $d \in J$ we have $d^t = t!\gamma_t(d)$, so d^t is divisible by $p^{v_p(t!)}$. Now put $x = y + pz + d$, where $z \in R$ and $d \in J$. Then a typical term in the multinomial expansion of $x^{p^n} - y^{p^n}$ looks like

$$\binom{p^n}{r, s, t} y^r (pz)^s d^t\,,$$

where $r + s + t = p^n$. Fix some $N > 0$. If $t > p^N$, then the above consideration shows that d^t is at least divisible by p^N (we are very permissive here). If $t \leqslant p^N$, then the multinomial coefficient is at least divisible by p^{n-N}. Hence if $n \geqslant 2N$, every term will at least be divisible by p^N, and we're done. □

Now that we know $\mathbb{A}_{\mathrm{cris}}$ is the p-adic divided power envelope of (ξ), we can write it down explicitly as

$$\mathbb{A}_{\mathrm{cris}} \cong \mathbb{A}_{\mathrm{inf}}\left[\frac{\xi^n}{n!} \;\middle|\; n \in \mathbb{N}\right]^\wedge_p \cong \mathbb{A}_{\mathrm{inf}} \mathbin{\widehat{\otimes}}_{\mathbb{Z}[x]} D_{\mathbb{Z}[x]}(x)\,,$$

using that ξ is a non-zero divisor in $\mathbb{A}_{\mathrm{inf}}$. Also $- \mathbin{\widehat{\otimes}}_{\mathbb{Z}[x]} -$ refers to the p-adic completed tensor product, with $\mathbb{Z}[x] \to \mathbb{A}_{\mathrm{inf}}$ sending $x \mapsto \xi$. Finally, the tensor factor on the right is defined as

$$D_{\mathbb{Z}[x]}(x) = \mathbb{Z}\langle x\rangle = \bigoplus_{n\in\mathbb{N}} \mathbb{Z}\left\{\frac{x^n}{n!}\right\}\,.$$

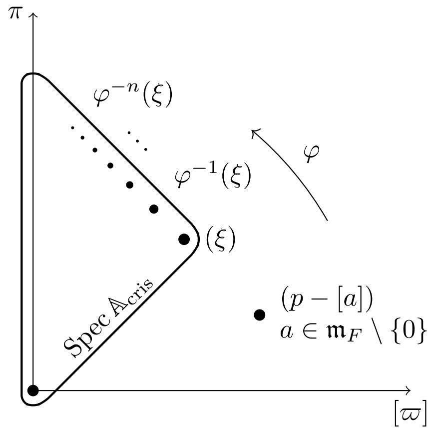
Then

$$D_{\mathbb{Z}[x]}(x)^\wedge_p \cong \big(\mathbb{Z}[y_0, y_1, \dots]/(y_0 - x, y_n^p - p y_{n+1} \text{ for } n \in \mathbb{N})\big)^\wedge_p\,.$$

In particular, we can calculate

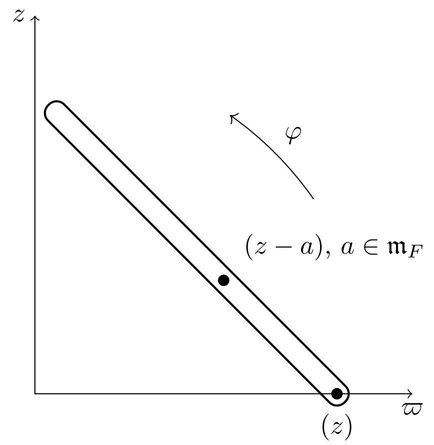
$$\mathbb{A}_{\mathrm{cris}} \cong \mathcal{O}_C/p\mathcal{O}_C \otimes_{\mathbb{F}_p} \mathbb{F}_p[y_1, y_2, \dots]/(y_1^p, y_2^p, \dots)\,.$$

Some intuition: the image of $\operatorname{Spec} \mathbb{A}_{\mathrm{cris}}$ in $\operatorname{Spec} \mathbb{A}_{\mathrm{inf}}$ is roughly described by the following picture.



Note that $\varphi^{-1}(\xi) = (p - [p^\flat]^{1/p})$. Concretely, if $a \in \mathfrak{m}_F \setminus \{0\}$ such that $|a| \leqslant |p^\flat|^p = |p|^p$, then $(p - [a])\mathbb{A}_{\mathrm{cris}} = (p)$. We may think of this as "$1 - [a]/p \in \mathbb{A}_{\mathrm{cris}}$". And if $a = \varphi^{-n}(p^\flat)$ for some $n \in \mathbb{N}$, then $\mathbb{A}_{\mathrm{inf}} \twoheadrightarrow \mathbb{A}_{\mathrm{inf}}/(p - [a])$ factors over $\mathbb{A}_{\mathrm{cris}}$.

Recall that $\mathbb{A}_{\mathrm{inf}}$ should be thought of as a mixed characteristic analogue of $\mathcal{O}_F[\![z]\!]$. In fact, we see a similar picture for $\mathcal{O}_F[\![z]\!]$.



The surrounded area may be described as $\operatorname{Prim}_1 /\mathcal{O}_F[\![z]\!]^\times \cong \mathfrak{m}_F = \{x \in F \mid |x| < 1\}$. This is also the "open rigid-analytic disc" $\mathbb{D}_F$. It contains the "punctured disc" $\mathbb{D}_F^* = \mathfrak{m}_F \setminus \{0\}$. Then the equal characteristic analogue of the Fargues–Fontaine curve is the quotient $\mathbb{D}_F^*/\varphi^{\mathbb{Z}}$.

However, for $\mathbb{A}_{\mathrm{inf}}$ the canonical map $\mathfrak{m}_F \twoheadrightarrow \operatorname{Prim}_1 /\mathbb{A}_{\mathrm{inf}}^\times$ sending $a \in \mathfrak{m}_F$ to $(\pi - [a])$ is not bijective! For example, $(\pi - [\pi^\flat])$ depends on choices of $(q^n)^{\mathrm{th}}$ roots of π to get $\pi^\flat = (\pi, \pi^{1/q}, \dots)$.

1.3. Newton Polygons and Factorizations

1.3.1. The Power Series Case

Lecture 4
13th Nov, 2019

Let K be a non-archimedean field, $v\colon K \to \mathbb{R} \cup \{\infty\}$ its valuation (written additively). Let $f = a_0 + a_1 T + \dots + a_n T^n \in K[T]$ be a polynomial.

1.3.1. Definition. — The *Newton polygon* $\mathrm{Newt}_{\mathrm{poly}}(f)$ is the largest convex polygon below $\{(i, v(a_i))\}_{i\in\mathbb{Z}}$, where we put $a_i = 0$ for $i \notin \{0, 1, \dots, n\}$ by convention.

There is a better description via the *Legendre transform*. To set this up, we introduce the notations $\overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$ and $\mathcal{F} = \{\varphi\colon \overline{\mathbb{R}} \to \overline{\mathbb{R}}\}$.

1.3.2. Definition. — We define the *Legendre transform* $\mathcal{L}\colon \mathcal{F} \to \mathcal{F}$ and the *inverse Legendre transform* $\widetilde{\mathcal{L}}\colon \mathcal{F} \to \mathcal{F}$ via

$$\begin{aligned} \mathcal{L}\varphi(\lambda) &= \inf_{x\in\mathbb{R}}\{\varphi(x) + \lambda x\} \\ \widetilde{\mathcal{L}}\varphi(x) &= \sup_{x\in\mathbb{R}}\{\varphi(\lambda) - \lambda x\}\,. \end{aligned}$$

Note that $\widetilde{\mathcal{L}}\varphi = -\mathcal{L}(-\varphi)$. As a Slogan: "$\mathcal{L}$ and $\widetilde{\mathcal{L}}$ interchange x-coordinates and slopes".

1.3.3. Example. — If $\varphi(x) = ax + b$, then one easily verifies

$$\mathcal{L}\varphi(\lambda) = \begin{cases} b & \text{if } \lambda = -a \\ -\infty & \text{else} \end{cases}\,.$$

Also note that $\widetilde{\mathcal{L}}\mathcal{L}\varphi = \varphi$ in this case.

1.3.4. Lemma. — *Let $\varphi, \psi \in \mathcal{F}$. Then the following hold.*

(1) *The Legendre transform $\mathcal{L}\varphi$ is always concave, i.e., it satisfies the inequality*

$$\mathcal{L}\varphi(a\lambda + b\mu) \geqslant a\mathcal{L}\varphi(\lambda) + b\mathcal{L}\varphi(\mu)$$

for all $a, b \geqslant 0$ such that $a + b = 1$. Similarly $\widetilde{\mathcal{L}}\varphi$ is convex.

(2) *If $\varphi \leqslant \psi$, then $\mathcal{L}\varphi \leqslant \mathcal{L}\psi$ and $\widetilde{\mathcal{L}}\varphi \leqslant \widetilde{\mathcal{L}}\psi$.*

(3) *We have $\widetilde{\mathcal{L}}\mathcal{L}\varphi \leqslant \varphi \leqslant \mathcal{L}\widetilde{\mathcal{L}}\varphi$.*

(4) *If φ admits a supporting line at x of slope λ, i.e., $\varphi(y) \geqslant \varphi(x) + \lambda(y - x)$ for all y, then $\mathcal{L}\varphi$ admits a capping line at $-\lambda$ of slope x.*

(5) *$\widetilde{\mathcal{L}}\mathcal{L}\varphi$ is the largest convex function below φ, and likewise $\mathcal{L}\widetilde{\mathcal{L}}\varphi$ is the smallest concave function over φ.*

(6) *$\mathcal{L}$ and $\widetilde{\mathcal{L}}$ define inverse bijections $\{\text{convex } \varphi\colon \mathbb{R} \to \overline{\mathbb{R}}\} \xleftrightarrow{\sim} \{\text{concave } \psi\colon \mathbb{R} \to \overline{\mathbb{R}}\}$.*

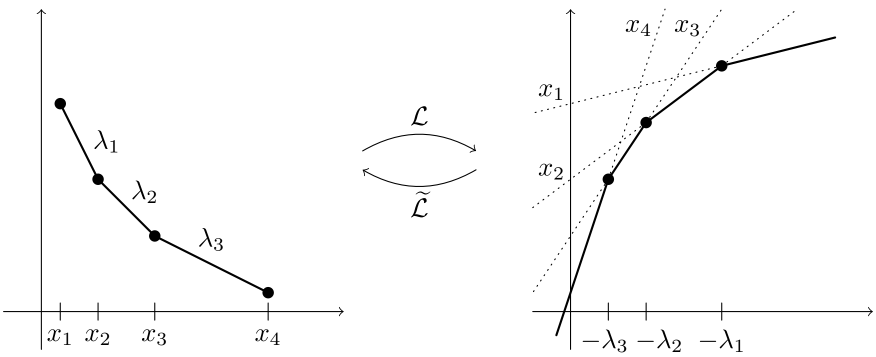
Proof. Parts (1) to (4) are straightforward from the definitions. We only prove (5) and (6). For $a, b \in \mathbb{R}$ put $\psi_{a,b}(x) = ax + b$ and consider the set $M \coloneqq \{(a, b) \mid \psi_{a,b} \leqslant \varphi\}$. Then $\psi = \sup_M \{\psi_{a,b}\}$ is the largest convex function below φ. By (1) and (3) we get $\widetilde{\mathcal{L}}\mathcal{L}\varphi \leqslant \psi$. Moreover, Example 1.3.3 and (2) show

$$\psi_{a,b} = \widetilde{\mathcal{L}}\mathcal{L}\psi_{a,b} \leqslant \widetilde{\mathcal{L}}\mathcal{L}\varphi\,,$$

hence $\psi \leqslant \widetilde{\mathcal{L}}\mathcal{L}\varphi$. This proves that $\widetilde{\mathcal{L}}\mathcal{L}\varphi$ is the largest convex function below φ. The second part of (5) is completely analogous.

Part (6) is a formal consequence of (3) and (5). We know that $\mathcal{L}\colon (\mathcal{F}, \leqslant) \rightleftarrows (\mathcal{F}, \leqslant) : \widetilde{\mathcal{L}}$ define an adjoint pair of functors. Hence $\mathcal{L}$ and $\widetilde{\mathcal{L}}$ induce inverse bijections between the full subcategories on which the unit resp. the counit of this adjunction is a natural isomorphism. Now $\{\varphi \mid \widetilde{\mathcal{L}}\mathcal{L}\varphi = \varphi\} = \{\varphi \text{ convex}\}$ and likewise $\{\psi \mid \psi = \mathcal{L}\widetilde{\mathcal{L}}\psi\} = \{\psi \text{ concave}\}$. $\square$

1.3.5. Example ⚠. — $\mathcal{L}$ and $\widetilde{\mathcal{L}}$ preserve piece-wise linear functions. In pictures, this looks roughly as follows.



Back to Newton polynoms for polygonials. Let $f \in K[T]$ be as above. As $\operatorname{Newt}_{\mathrm{poly}}(f)$ is the largest convex function below $\{(i, v(a_i))\}_{i\in\mathbb{Z}}$, we have

$$\mathcal{L}\operatorname{Newt}_{\mathrm{poly}}(f)(r) = \inf_{i\in\mathbb{Z}} \{v(a_i) + ri\} \eqqcolon v_r(f)\,.$$

The expression on the right-hand side has a nice geometric interpretation for $r \in v(\overline{K})$. In this case we have

$$v_r(f) \coloneqq \inf_{i\in\mathbb{Z}} \{v(a_i) + ri\} = \inf\left\{v(f(x)) \;\middle|\; x \in \overline{K} \text{ and } v(x) = r\right\}\,.$$

Indeed, as $v(a_i x^i) = v(a_i) + ri$, it's clear that "$\geqslant$" holds. For the converse, choose $x_0 \in \overline{K}$ with $v(x_0) = r$. We want to find a $y \in \overline{K}$ such that $v(y) = 0$ and $v(f(x_0 y)) = v_r(f)$. Let $b \in \overline{K}$ such that $v(b) = -v_r(f)$ and put $b_i = b a_i x_0^i$. Then $v(b_i) \geqslant 0$. Let $n_0 \leqslant n$ be the largest index such that $v(b_{n_0}) = 0$. Let $c \in \overline{K}$ such that $v(c) = 0 = v(b_0 - c)$, which exists as $\mathcal{O}_{\overline{K}}/\mathfrak{m}_{\overline{K}}$ is an algebraically closed field, hence has at least two non-zero elements. Now let $y \in \overline{K}$ be a solution of $c = b_0 + \dots + b_{n_0} y^{n_0}$. As $v(b_{n_0}) = 0$, $y \in \mathcal{O}_{\overline{K}}$, and then a simple inspection shows $v(y) = 0$. Now it's easy to check that indeed $v(f(x_0 y)) = v_r(f)$, hence we are done.

1.3.6. Exercise. — For all polynomials $f, g \in K[T]$ and all $r \in \mathbb{R}$ we have

$$v_r(fg) = v_r(f) + v_r(g)\,.$$

Proof.* Let (a_i) and (b_j) be the coefficients of f and g, and (c_k) the coefficients of fg. Then $c_k = \sum_{i+j=k} a_i b_j$. Hence $v(c_k) + rk \geqslant \inf\{v(a_i) + ri + v(b_j) + rj\}$ by the strong triangle inequality. This shows "$\geqslant$".

For the converse, let $s \geqslant 0$ and $t \geqslant 0$ be minimal indices such that $v(a_s) + rs = v_r(f)$ and $v(b_t) + rt = v_r(g)$. For all $(i, j) \neq (s, t)$ satisfying $i + j = s + t$, we have

$$v(a_i) + ri + v(b_j) + rj > v(a_s) + rs + v(b_t) + rt\,,$$

hence $v(a_i) + v(b_j) > v(a_s) + v(b_t)$, hence $v(c_{s+t}) = v(a_s) + v(b_t)$. This finally proves $v(c_{s+t}) + r(s + t) = v_r(f) + v_r(g)$ and we are done. □

1.3.7. Remark. — For $f = a_0 + a_1T + \cdots + a_nT^n$ let φ_f be the piece-wise linear function connecting the $\{(i, v(a_i))\}_{i\in\mathbb{Z}}$. Then Exercise 1.3.6 shows $\mathcal{L}\varphi_{fg} = \mathcal{L}\varphi_f + \mathcal{L}\varphi_g$. As a slogan, we "concatenate the concave piece-wise linear functions to a new one".

1.3.8. Definition. — Let $\varphi, \psi \in \mathcal{F}$ such that $-\infty \notin \operatorname{im}\varphi \cup \operatorname{im}\psi$. Then we define the *convolution* $\varphi * \psi\colon \mathbb{R} \to \overline{\mathbb{R}}$ as

$$(\varphi * \psi)(x) = \inf_{y+z=x} \{\varphi(y) + \psi(z)\}\ .$$

1.3.9. Lemma. — *Let $\varphi, \psi \in \mathcal{F}$ such that $-\infty \notin \operatorname{im}\varphi \cup \operatorname{im}\psi$.*

(1) *If φ and ψ are convex, then so is $\varphi * \psi$.*

(2) *We have $\mathcal{L}(\varphi * \psi) = \mathcal{L}\varphi + \mathcal{L}\psi$.*

Proof.* We start with (1). Let $x_1, x_2 \in \mathbb{R}$ and $a, b \geqslant 0$ such that $a + b = 1$. We need to show

$$(\varphi * \psi)(ax_1 + bx_2) \leqslant a(\varphi * \psi)(x_1) + b(\varphi * \psi)(x_2)\,.$$

Let $y_1, y_2, z_1, z_2 \in \mathbb{R}$ such that $x_1 = y_1 + z_1$ and $x_2 = y_2 + z_2$. Put $x = ax_1 + bx_2$, $y = ay_1 + bz_1$, and $z = ay_2 + by_2$. Then $x = y + z$ and

$$\varphi(y) + \psi(z) \leqslant a(\varphi(y_1) + \psi(y_2)) + b(\varphi(z_1) + \psi(z_2))$$

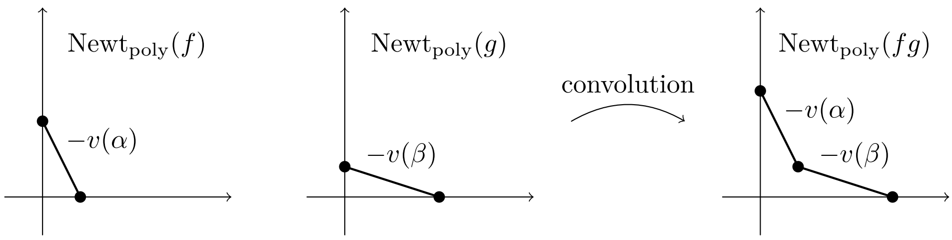
by convexity of φ and ψ. Taking infima shows the required inequality. This shows (1). Part (2) follows straight from the definitions. □

1.3.10. Corollary. — *If $f, g \in K[T]$ are polynomials, then*

$$\mathrm{Newt}_{\mathrm{poly}}(fg) = \mathrm{Newt}_{\mathrm{poly}}(f) * \mathrm{Newt}_{\mathrm{poly}}(g)\,.$$

Proof. Both sides are convex by definition, and agree after applying $\mathcal{L}$ (use Remark 1.3.7 and Lemma 1.3.9), so they are already equal by Lemma 1.3.4. □

1.3.11. Example. — Let's illustrate Corollary 1.3.10 for $f = T - \alpha$ and $g = T - \beta$:



In particular, if $f \in K[T]$ is a polynomial of degree n and $\alpha_1, \dots, \alpha_n \in \overline{K}$ its zeros counted with multiplicity, then induction on n shows that $\mathrm{Newt}_{\mathrm{poly}}(f)$ has exactly $-v(\alpha_1), \dots, -v(\alpha_n)$ as its slopes, with the same multiplicities.

Now we want to define Newton polygons for power series. So let $f \in \mathcal{O}_K[\![T]\!]$ be a power series, say, $f = \sum_{i=0}^\infty a_i T^i$.

1.3.12. Definition. — The *Newton polygon* $\mathrm{Newt}(f)$ is the largest *decreasing* convex function below $\{(i, v(a_i))\}_{i \in \mathbb{Z}}$, where $a_i = 0$ for $i < 0$ by convention. In other words,

$$\mathcal{L}\,\mathrm{Newt}(f)(r) = \begin{cases} v_r(f) \coloneqq \inf_{i \in \mathbb{Z}}\{v(a_i) + ri\} & \text{if } r \geqslant 0 \\ -\infty & \text{if } r < 0 \end{cases}.$$

1.3.13. Remark. — (1) A power series f defines a function on the "open rigid-analytic unit disc" $\mathbb{D} = \{x \mid |x| < 1\}$. In fact, one can handle general $f \in K[\![T]\!]$ as functions if one is careful with domains of convergence.

(2) For $f \in \mathcal{O}_K[T]$, $\mathrm{Newt}(f)$ omits the positive slopes from $\mathrm{Newt}_{\mathrm{poly}}(f)$, as these correspond to zeros outside of $\mathbb{D}$. In particular, $\mathrm{Newt}(f) \neq \mathrm{Newt}_{\mathrm{poly}}(f)$ in general.

Similar to the polynomial case in Example 1.3.11, there is a connection between slopes of the Newton polygon of a power series f and zeros of f (but we won't prove this).

1.3.14. Theorem (Lazard). — *Let $f \in \mathcal{O}_K[\![T]\!]$. Let $\lambda \neq 0$ be a slope of* $\mathrm{Newt}(f)$. *Then there exists a zero $\alpha \in \widehat{\overline{K}}$ of f with $v(\alpha) = -\lambda$. In other words, in $\mathcal{O}_{\widehat{\overline{K}}}[\![T]\!]$ the power series f an be factored as $f = (T - \alpha)g$.*

1.3.15. Remark. — (1) Suppose $\mathrm{Newt}(f)$ is eventually constant (this is e.g. the case if K is discretely valued), i.e., $f = a\widetilde{f}$ for $a \in \mathcal{O}_K$ and $\widetilde{f}$ is primitive of some degree d. Then the Weierstraß preparation theorem shows $f = aPg$, where $a \in \mathcal{O}_K$, P is a monic polynomial of degree d, and $g \in \mathcal{O}_K[\![T]\!]^\times$ is a unit.

(2) Suppose $\operatorname{char} K = 0$. Then $\log_p(1-x) = \sum_{i=0}^\infty (-1)^{i-1} x^i / i$ has zeros precisely at $\mu_{p^\infty}(\overline{K})$. Draw the Newton polygon of $\log_p(1-x)$ and prove this!

1.3.2. Newton Polygons in $\mathbb{A}_{\mathrm{inf}}$

With notation as usual we put $\mathbb{A}_{\mathrm{inf}} = W_{\mathcal{O}_E}(\mathcal{O}_F)$. We want to introduce an analogue of Newton polygons of power series for $\mathbb{A}_{\mathrm{inf}}$. As a side note, there is no "subring of polynomials in $\mathbb{A}_{\mathrm{inf}}$": $\left\{\sum_{i=0}^n [a_i]\pi^i \;\middle|\; a_i \in \mathcal{O}_F\right\}$ is not closed under addition. So there's no sensible generalization of Newton polygons of polynomials to $\mathbb{A}_{\mathrm{inf}}$.

1.3.16. Definition. — Let $f = \sum_{i=0}^\infty [a_i]\pi^i \in \mathbb{A}_{\mathrm{inf}}$. Then the *Newton polygon* $\mathrm{Newt}(f)$ *of* f is the largest decreasing convex function below $\{(i, v(a_i))\}_{i \in \mathbb{Z}}$. In other words,

$$\mathcal{L}\,\mathrm{Newt}(f)(r) = \begin{cases} v_r(f) \coloneqq \inf_{i \in \mathbb{Z}}\{v(a_i) + ri\} & \text{if } r \geqslant 0 \\ -\infty & \text{if } r < 0 \end{cases}.$$

1.3.17. Lemma ⚠. — *For all $r \geqslant 0$, $v_r \colon \mathbb{A}_{\mathrm{inf}} \to \mathbb{R} \cup \{\infty\}$ is a valuation. In particular, for $f, g \in \mathbb{A}_{\mathrm{inf}}$ we have*

$$\mathrm{Newt}(fg) = \mathrm{Newt}(f) * \mathrm{Newt}(g)\,.$$

Proof.* It's clear that $v_r(f)=\infty$ iff $f=0$. We first establish the strong triangle inequality. This is essentially straightforward, but still technical. Define "twisted Witt polynomials"

$$\widetilde{S}_n(X_0,\dots,X_n,Y_0,\dots,Y_n)=S_n\left(X_0^{q^0},\dots,X_n^{q^n},Y_0^{q^0},\dots,Y_n^{q^n}\right).$$

Using the inductive construction of the S_n, it's easy to check that $\widetilde{S}_n$ is homogeneous of degree q^n. If $\alpha X_0^{s_0}\cdots X_n^{s_n}Y_0^{t_0}\cdots Y_n^{t_n}$ is a monomial of total degree q^n, we define its *weight* as $q^{-n}\sum_{i=0}^n i(s_i+t_i)$. We claim:

($*$) There is a polynomial T_n such that $\widetilde{S}_n\equiv T_n \mod \pi$, no coefficient of T_n is divisible by π, and every monomial of T_n has weight $\leqslant n$.

Let's first see why ($*$) implies the strong triangle inequality for v_r. Let $f,g\in\mathbb{A}_{\mathrm{inf}}$, say, $f=\sum_{n=0}^\infty[a_n]\pi^n$, $g=\sum_{n=0}^\infty[b_n]\pi^n$, and let $f+g=\sum_{n=0}^\infty[c_n]\pi^n$. Since $\pi=0$ in $\mathcal{O}_F$, we have $c_n=T_n(a_0,\dots,a_n,b_0,\dots,b_n)^{1/q^n}$. Now let $\alpha X_0^{s_0}\cdots X_n^{s_n}Y_0^{t_0}\cdots Y_n^{t_n}$ be a monomial of T_n. Since this monomial has weight $\leqslant n$, we have

$$q^{-n}v\big(\alpha a_0^{s_0}\cdots a_n^{s_n}b_0^{t_0}\cdots b_n^{t_n}\big)+rn\geqslant q^{-n}\sum_{i=0}^n\big(s_i(v(a_i)+ri)+t_i(v(b_i)+ri)\big).$$

Since the right-hand side is a convex combination of $v(a_i)+ri$ and $v(b_i)+ri$ for $i=0,\dots,n$, it is bounded below by the minimal value of these guys. This shows

$$v(c_n)+rn\geqslant\min_{i=0,\dots,n}\{v(a_i)+ri,v(b_i)+ri\}.$$

If you think about this a bit, this is enough to prove the strong triangle inequality.

We prove ($*$) by induction on n. The case $n=0$ is clear. Now assume ($*$) holds up to $n-1$. Revisiting the proof of Proposition 1.1.7, we see that

$$\widetilde{S}_n=\pi^{-n}\left(\sum_{i=0}^n\pi^i\left(X_i^{q^n}+Y_i^{q^n}\right)-\sum_{i=0}^{n-1}\pi^i\widetilde{S}_i^{q^{n-i}}\right).$$

Clearly, all monomials of the first sum have weight at most n. For the second sum, our key Lemma 1.2.6 implies $\widetilde{S}_i^{q^{n-i}}\equiv T_i^{q^{n-i}}\mod\pi^{n-i+1}$, and multiplying by π^i, we get a congruence modulo π^{n+1}, as usual. Moreover, since all monomials of T_i have weight at most i, the same is true for $T_i^{q^{n-i}}$. This shows that T_n can be defined in an appropriate way.

It remains to show $v_r(fg)=v_r(f)+v_r(g)$. In his notes, Johannes Anschütz writes that $v_r(fg)\geqslant v_r(f)+v_r(g)$ is trivial, but I don't quite agree on that one. It's certainly trivial for power series, but multiplication in $\mathbb{A}_{\mathrm{inf}}$ is more complicated than that. So here we sketch a proof: introduce "twisted Witt polynomials" $\widetilde{P}_n$ as above. An easy induction shows that $\widetilde{P}_n$ is homogeneous of degree $2q^n$, and moreover, that it can be written as a polynomial in $Z_{i,j}$, where we put $Z_{i,j}=X_iY_j$ (and then this polynomial has degree q^n). As above, we introduce a notion of *weight* of a monomial $\alpha Z_{i_0,j_0}^{s_0}\cdots Z_{i_t,j_t}^{s_t}$ of total degree q^n. It is defined as $q^{-n}\sum_{k=0}^t(i_k+j_k)s_k$. We claim:

($\boxtimes$) There is a polynomial Q_n such that $\widetilde{P}_n\equiv Q_n\mod\pi$, no coefficient of Q_n is divisible by π, and every monomial of Q_n has weight $\leqslant n$.

Claim ($\boxtimes$) can be proved in the exact same way as ($*$). Likewise, we can adapt the above arguments to see that ($\boxtimes$) indeed implies $v_r(fg)\geqslant v_r(f)+v_r(g)$.

To show equality, it suffices to consider the case $r > 0$, since it is easy to see that $v_0(f) = \lim_{r\to 0} v_r(f)$, so the $r = 0$ case will follow automatically. Let $f = \sum_{n=0}^{\infty}[a_n]\pi^n$, $g = \sum_{n=0}^{\infty}[b_n]\pi^n$, and $fg = \sum_{n=0}^{\infty}[c_n]\pi^n$. Since $r > 0$, there are minimal indices m, n such that $v(a_m) + rm = v_r(f)$ and $v(b_n) + rn = v_r(g)$. Put $N = m + n$. We will show that $v(c_N) = v(a_m) + v(b_n)$, which will conclude the proof.

We claim that $(1 + \pi\beta)Z_{m,n}^{q^N}$ is a monomial in Q_N for some $\beta \in \mathcal{O}_E$. To see this, we use the proof of Proposition 1.1.7 to get

$$\widetilde{P}_N = \pi^{-N}\left(\sum_{i=0}^{N}\pi^i X_i^{q^N}\cdot\sum_{j=0}^{N}\pi^i Y_i^{q^N} - \sum_{i=0}^{N-1}\pi^i\widetilde{P}_i^{q^{N-i}}\right).$$

Clearly, $\pi^N Z_{m,n}^{q^N}$ occurs as a summand if we expand the product of the two sums. So we only need to show that $Z_{m,n}^{q^N}$ doesn't occur, up to multiples of π^{N+1}, in the right-most sum. For that, note that $Z_{m,n}^{q^N}$ doesn't occur, up to multiples of π^{N-i+1}, in any $Q_i^{q^{N-i}}$ for $i \leqslant N-1$. This follows from (⊠) as $Z_{m,n}^{q^N}$ has weight N. But $\widetilde{P}_i^{q^{N-i}} \equiv Q_i^{q^{N-i}} \mod \pi^{N-i+1}$ by our key Lemma 1.2.6. This does it.

As in the proof of the strong triangle inequality, $c_N = Q_N(a_0,\dots,a_N,b_0,\dots,b_N)^{1/q^N}$. If $\alpha Z_{i_0,j_0}^{s_0}\cdots Z_{i_t,j_t}^{s_t}$ is a monomial of Q_N different from $(1+\pi\beta)Z_{m,n}^{q^N}$, then

$$q^{-N}v\big(\alpha(a_{i_0}b_{j_0})^{s_0}\cdots(a_{i_t}b_{j_t})^{s_t}\big) + rN \geqslant q^{-N}\sum_{k=0}^{t}s_k\big((v(a_{i_k}) + ri_k) + (v(b_{j_k}) + rj_k)\big)\,,$$

using that the monomial in question has weight $\leqslant N$. The sum on the right-hand side is a convex combination of terms $(v(a_{i_k}) + ri_k) + (v(b_{j_k}) + rj_k)$ for $k = 0,\dots,t$, each of which is at least $v_r(f) + v_r(g)$. So the right-hand side is $\geqslant v_r(f) + v_r(g)$. But since m and n are minimal with the property that $v(a_m) + rm = v_r(f)$ and $v(b_n) + rn = v_r(g)$ and since the weight is $\leqslant N$, some terms will be strictly greater than $v_r(f) + v_r(g)$. Thus, the right-hand side is $> v_r(f) + v_r(g) = v(a_m) + v(b_n) + rN$. In particular, $(1+\pi\beta)(a_m b_n)^{q^N}$ is the unique summand with minimal valuation, proving indeed $v(c_N) = v(a_m) + v(b_n)$. □

Let now $a = [a_0] - u\pi$ be an element of Prim_1, where $u \in \mathbb{A}_{\mathrm{inf}}^{\times}$ is a unit and $a_0 \in \mathfrak{m}_F \setminus \{0\}$. Put $D = \mathbb{A}_{\mathrm{inf}}/a\mathbb{A}_{\mathrm{inf}}$. Then we have Fontaine's map $\theta\colon \mathbb{A}_{\mathrm{inf}} \to D$.

1.3.18. Proposition. — *Suppose we are in the above situation.*

(1) *D is π-complete and π-torsionfree.*

(2) *We have $D^\flat \cong \mathcal{O}_F$.*

(3) *The p^{th} power map $D \to D$, $x \mapsto x^p$ is surjective. In particular, every element of D is the of the form $\theta([x])$ for some $x \in \mathcal{O}_F$.*

*Proof**. Part (1) and (2) are easy: a is distinguished and $\mathcal{O}_F$ is a_0-complete as $a \in \mathfrak{m}_F$. By Remark 1.2.10, $(\mathbb{A}_{\mathrm{inf}}, a)$ is thus a perfect prism, hence π-complete by Lemma* 1.2.13. In particular, D is perfectoid, so (2) already follows from Remark 1.2.10(4). For (1), it remains to show that D is π-torsionfree. Suppose $x, y \in \mathbb{A}_{\mathrm{inf}}$ are such that $\pi x = ay$. Put $x = \sum_{n=0}^{\infty}[x_n]\pi^n$ and $y = \sum_{n=0}^{\infty}[y_n]\pi^n$. Then $[a_0y_0] = 0$, hence $y_0 = 0$ as $a_0 \in \mathfrak{m}_F \setminus \{0\}$. So $y = \pi y'$, and since $\mathbb{A}_{\mathrm{inf}}$ is π-torsionfree, we obtain $x = ay'$. This shows that D is indeed π-torsionfree.

Proving that the p^{th} power map is surjective is a technical nightmare, but I still want to sketch the proof here. The key to the proof is the following claim:

($*$) For all $x \in \mathbb{A}_{\text{inf}}$ with non-zero image in D and all $N \geqslant 1$ there is an $m \geqslant 0$ and $y = \sum_{n=0}^{\infty}[y_n]\pi^n \in \mathbb{A}_{\text{inf}}$ such that $v(y_0) < v(a_0)$, $y_n = 0$ for all $n = 1, \dots, N-1$, and
$$x \equiv [a_0]^m y \mod a\,.$$

We first describe how ($*$) implies (3). Let e be the ramification index of π, i.e., $\pi^e\mathcal{O}_E = p\mathcal{O}_E$, and normalize the valuation v of $\mathcal{O}_F$ in such a way that $v(a_0) = 1/e$. To show that the image of x in D admits a p^{th} root, choose y as above such that $N/e > 1 + v(a_0) + 1/(p-1)$. It suffices to construct a p^{th} root of the image of y, since $[a_0]$ already admits a p^{th} root in $\mathbb{A}_{\text{inf}}$. Consider the "Taylor series expansion of $\sqrt[p]{y}$ around $[y_0]$", i.e., the series
$$\sum_{n=0}^{\infty}\prod_{k=0}^{n}\left(\frac{1-kp}{p}\right)\frac{[y_0]^{(-p(n-1)+1)/p}}{n!}(y-[y_0])^n$$
(right now this doesn't make sense at all, but soon it will). We claim that in D this sum can be rewritten into a converging series. It is well-known that $v_p(n!) < n/(p-1)$. Hence the denominator $p^n n!$ divides $\pi^{e(n+n/(p-1))}$ in $\mathcal{O}_E$ and thus also in $\mathbb{A}_{\text{inf}}$. Likewise, since $v(y_0) < v(a_0)$ and since $[a_0]$ and π only differ by a unit in D, we see that $[y_0]^{(p(n-1)-1)/p}$ divides π^n in D. Since $y - [y_0]$ is, by assumption on y, divisible by π^N and $N/e > 1 + v(a_0) + 1/(p-1)$, all terms in the above sum can be interpreted as elements of D, and moreover they converge to 0 in the π-adic topology. By (1), D is π-complete, so we can indeed represent $\sqrt[p]{y}$ as a convergent series in D.

It suffices to show ($*$). If $v(x_0) \geqslant v(a_0)$, we may subtract a suitable multiple of a to kill the π^0-term of x. In other words, we find x' such that $x \equiv \pi x' \equiv [a_0]u^{-1}x' \mod a$. Now iterate this argument for $u^{-1}x'$. If this doesn't end at some point, the image of x in D is divisible by arbitrary powers of π, hence 0 by π-completeness.

So let's assume $v(x_0) < v(a_0)$. Then it's easy to check that $v(y_0) < v(a_0)$ for all $y \in \mathbb{A}_{\text{inf}}$ satisfying $x \equiv y \mod a$. Therefore it suffices to find some $b \in \mathbb{A}_{\text{inf}}$ such that $y = x + ab$ satisfies $y_n = 0$ for all $n = 1, \dots, N-1$. We will see that this amounts to a system of polynomial equations for $b_0, \dots, b_{N-1}$, which has a solution in $\mathcal{O}_F$. To get $y_n = 0$, we would like to have
$$S_n\left(x_0^{q^0}, \dots, x_n^{q^n}, Q_0, \dots, Q_n\right) = 0 \quad \text{for all } n = 1, \dots, N-1\,,$$
where $Q_n = Q_n(a_0, \dots, a_n, b_0, \dots, b_n)$ is defined as in the proof of Lemma 1.3.17. If $b_0, \dots, b_{n-1}$ are known, then $S_n = 0$ uniquely determines b_n. Indeed, using the recursive definition of the S_n (compare this to the proof of Lemma 1.3.17), we see that b_n only occurs in the summand Q_n. And in Q_n, b_n only occurs as $a_0^{q^n}b_n^{q^n}$. This shows that $b_n^{q^n} = -$some polynomial in $b_0, \dots, b_{n-1})/a_0^{q^n}$ is uniquely determined, and then b_n is unique since the Frobenius is an automorphism of $\mathcal{O}_F$. Moreover, we need to ensure that the value of the polynomial in question is divisible by $a_0^{q^n}$, to get $b_n \in \mathcal{O}_F$.

Our goal now is to use the above observation to eliminate $b_1, \dots, b_{N-2}$ from the equations. We know that b_1^q is a polynomial (with coefficients not in $\mathcal{O}_F$, but in $a_0^{-1}\mathcal{O}_F$) in b_0. Replacing all subsequent polynomial equations by their q^{th} powers, which we may do since the Frobenius is an automorphism, we may substitute each b_1^q by the polynomial in b_0 to eliminate b_1 everywhere. Now repeat this procedure with $b_2, \dots, b_{N-2}$. What we obtain in the end is a huge polynomial $\Psi(b_0)$ with coefficients in $a_0^{-K}\mathcal{O}_F$ for some very large K. We want to choose b_0 to be a root of Ψ, to get that $b_{N-1}^{q^M} = -\Psi(b_0)/a_0^{q^M} = 0$ (for some very large M) is in $\mathcal{O}_F$ (actually, it would suffice to choose b_0 in such a way that $\Psi(b_0)$ is divisible by a very large power of a_0). To show that the roots of Ψ are in $\mathcal{O}_F$, we claim that Ψ is *quasi-monic*, which is to say that all coefficients of Ψ are in $a_0^{-K}\mathcal{O}_F$ for some K and its leading coefficient has the form εa_0^{-K}, where $\varepsilon \in \mathcal{O}_F^\times$ is a unit. For a quasi-monic polynomial, $-K$ as above is called its *valuation*.

Let $b_n^{q^{M_n}} = -\Psi_n(b_0)$ be the polynomial equation we get by the above procedure. We claim that all Ψ_n are quasi-monic, and that the valuation of Ψ_n is smaller than that of Ψ_{n-1}. This can be seen by induction on n. The key part in the inductive step is that Q_n contains the monomial $a_1^{q^n} b_{n-1}^{q^n}$ up to multiples of π (this was seen in the proof of Lemma 1.3.17). Moreover, any other monomial containing $b_{n-1}^{q^n}$ must also contain a_0 since its weight is $\leqslant n$. Now $a_1^{q^n} \in \mathcal{O}_F^\times$ and $a_0 \in \mathfrak{m}_F \setminus \{0\}$ by assumption on a, so if we group all monomials containing $b_{n-1}^{q^n}$ we get a coefficient in $\mathcal{O}_F^\times$. This being the key idea, we omit the details of the induction.

In particular, this shows that Ψ is quasi-monic since it is a the product of Ψ_{N-1} with some power of a_0. So $b_0, b_{N-1} \in \mathcal{O}_F$ by construction. For $n = 1, \dots, N-2$ we define $b_n = -\Psi_n(b_0)$. By construction, $(b_0, \dots, b_{N-1})$ satisfy all polynomial equations, so it suffices to see $b_n \in \mathcal{O}_F$ for all $n = 1, \dots, N-2$. If, for the sake of contradiction, n is a minimal index such that $0 > v(b_n)$, then an induction as above shows $v(b_n) > v(b_{n+1}) > \dots > v(b_{N-1})$, contradicting $b_{N-1} \in \mathcal{O}_F$.

This shows that D admits p^{th} roots. To finish the proof of (3), we need to show that every $d \in D$ is of the form $\theta([x])$. Using Lemma 1.2.8, it suffices to show that $(-)^\sharp \colon \mathcal{O}_F \to D$ is surjective. But (2) together with Proposition 1.2.7 shows $\mathcal{O}_F \cong D^\flat \cong \lim_{d \mapsto d^q} D$. Since D admits p^{th} roots, it's clear that the right-hand side surjects onto D. $\square$

1.3.19. Corollary. — *Suppose we are in the above situation.*

(1) *D is a complete valuation ring, with well-defined valuation $v \colon D \to \mathbb{R} \cup \{\infty\}$ constructed as follows: for $d = \theta([x])$ we put $v(d) = v_F(x)$, where v_F denotes the valuation of F.*

(2) *The fraction field $C = \operatorname{Frac}(D)$ is algebraically closed.*

Proof. We start with (1). The first thing to show is that v is well-defined. Suppose $x, y \in \mathcal{O}_F$ satisfy $x^\sharp = \theta([x]) = \theta([y]) = y^\sharp$ (the outer equalities are due to Lemma 1.2.8). Suppose $v_F(x) \geqslant v_F(y)$, so w.l.o.g. $x = yz$. Then $z^\sharp = 1$, and it suffices to show $z \in \mathcal{O}_F^\times$. Write $z = (z_0, z_1, \dots) \in \lim_{d \mapsto d^q} D$, where $z_0 = 1$. Then each z_n is invertible in D, so z is invertible in $D^\flat \cong \mathcal{O}_F$.

We proceed to show that D is an integral domain. Assume $de = 0$ with $d, e \in D$. Write $d = x^\sharp$ and $e = y^\sharp$. Then $[xy] = az$ for some $z \in \mathbb{A}_{\text{inf}}$. But $\operatorname{Newt}([xy])$ is a horizontal line, whereas $\operatorname{Newt}(az) = \operatorname{Newt}(a) * \operatorname{Newt}(z)$ contains $-v(a_0)$ as a slope if $z \neq 0$.

Now it is completely formal to see that D is a valuation ring. The map v clearly extends to $C = \operatorname{Frac}(D)$ via $v(d/e) = v_F(x) - v_F(y)$ if $d = x^\sharp$ and $e = y^\sharp$. This v is multiplicative and satisfies $v(d/e) = \infty$ iff $d/e = 0$. Moreover, an element $d/e \in C$ lies in D precisely iff $v(d/e) > 0$. Indeed, if x and y are as above, then $v_F(x) \geqslant v_F(y)$, hence $x = yz$ because $\mathcal{O}_F$ is a valuation ring. Then $d/e = z^\sharp \in D$. This already implies the strong triangle inequality: if $d, e \in D$ and $v(d) \geqslant v(e)$, then $d/e \in D$, hence $v(1 + d/e) \geqslant 0$, hence $v(d + e) = v(e) + v(1 + d/e) \geqslant v(e)$. This proves (1).

For (2), let $P(T) = T^n + b_{n-1}T^{n-1} + \cdots + b_0 \in D[T]$ be an irreducible polynomial. Since C is a non-archimedean field (which, by our convention, always implies completeness), the Newton polygon $\mathrm{Newt}_{\mathrm{poly}}(P)$ is a single line. This follows from the classical theory of Newton polygons (e.g. [Neu92, Ch. II (6.4)]), or from Example 1.3.11 if one shows that all roots of P have the same valuation, which boils down to the fact that v extends uniquely to any finite extension of C. Choose $c_0 \in D$ such that $nv(c_0) = v(b_0)$ (such a c_0 exists by construction of v and the fact that $\mathcal{O}_F$ is integrally closed). Since $\mathrm{Newt}_{\mathrm{poly}}(P)$ is a single line, $c_0^{-n}P(c_0T)$ is monic and has coefficients in D again, where Replacing P by $c_0^{-n}P(c_0T)$ we may thus assume $v(b_0) = 0$.

Now let $Q_0 \in \mathcal{O}_F[T]$ be a monic polynomial such that the images of P and Q_0 in the polynomial ring $(D/\pi D)[T] \cong (\mathcal{O}_F/a_0\mathcal{O}_F)[T]$ coincide. Let $y_0 \in \mathcal{O}_F$ be a zero of Q_0. Then $P(T + y_0^\sharp)$ is still irreducible and its constant coefficient is divisible by π. Now choose $c_1 \in D$ such that $nv(c_1) = v(P(y_0^\sharp)) \geqslant v(\pi)$. As above, $P_1(T) = c_1^{-n}P(c_1T + y_0^\sharp)$ is monic, has coefficients in D, and its constant coefficient is invertible. Now choose Q_1 and y_1 as above and iterate the argument. The series $y_0 + c_1y_1 + \cdots$ converges to a zero of P. □

We can now finally prove Lemma 1.2.18, a result that was already announced long ago in the 2[nd] lecture.

Proof of Lemma 1.2.18*. Combining Lemma 1.2.23 and Corollary 1.3.19 immediately shows this result. More generally, these results show that there is a bijection

$$\{\text{iso. classes of } C/E \text{ non-arch. alg. closed s.th. } \mathcal{O}_C^\flat \cong \mathcal{O}_F\} \xleftrightarrow{\sim} \mathrm{Prim}_1 /\mathbb{A}_{\mathrm{inf}}^\times .$$

If C/E is as on the left-hand side, then $\mathcal{O}_C$ is perfectoid by Example 1.2.17, hence the kernel of $\theta\colon \mathbb{A}_{\mathrm{inf}} \to \mathcal{O}_C$ is generated by an element of Prim_1, which is unique up to $\mathbb{A}_{\mathrm{inf}}$. Conversely, if $a \in \mathrm{Prim}_1$ is given, then $D = \mathbb{A}_{\mathrm{inf}}/a\mathbb{A}_{\mathrm{inf}}$ and $C = \mathrm{Frac}(D)$ define an element of the left-hand side. These maps induce inverse bijections as required. □

1.3.3. The Space $|Y|$ and Factorizations

Lecture 5
20[th] Nov, 2019

Let notation be as usual and recall our construction of the Newton polygon for elements of $\mathbb{A}_{\mathrm{inf}}$ in Definition 1.3.16. The goal for today is to prove the following analogue of Lazard's Theorem 1.3.14.

1.3.20. Theorem (Fargues–Fontaine). — *If $f \in \mathbb{A}_{\mathrm{inf}}$ and $\lambda \neq 0$ is a slope of* $\mathrm{Newt}(f)$, *then there exists an $a \in \mathcal{O}_F$ such that $v(a) = -\lambda$ and $f = (\pi - [a])g$ for some $g \in \mathbb{A}_{\mathrm{inf}}$.*

Crucial to the proof of Theorem 1.3.20 will be to interpret $\mathbb{A}_{\mathrm{inf}}$ as "functions on the punctured open unit disc in mixed characteristic". This leads to the following definition.

1.3.21. Definition. — We define the space

$$\begin{aligned}|Y| &\coloneqq \{\mathfrak{p} \in \operatorname{Spec} \mathbb{A}_{\mathrm{inf}} \mid \mathfrak{p} \text{ is generated by a primitve element of degree } 1\}\\ &\cong \mathrm{Prim}_1 /\mathbb{A}_{\mathrm{inf}}^\times\\ &\cong \{\text{iso. classes of } C/E \text{ non-arch. alg. closed s.th. } \mathcal{O}_C^\flat \cong \mathcal{O}_F\}\end{aligned}$$

Note that $\mathfrak{m}_F \setminus \{0\}$ surjects onto $|Y|$ via $a \mapsto (\pi - [a])$ (to prove this, we must show that any $[a_0] - u\pi \in \mathrm{Prim}_1$ can be multiplied by a suitable unit to become of the form $\pi - [a]$; the coefficients of such a unit can be constructed inductively), but this need not be a bijection. The notation suggests that $|Y|$ should be thought of the underlying space of some Y. This should not be taken too literally, but in some sense this is indeed the case. More about this in Remark 1.3.23

1.3.22. Notation. — (1) For $y \in |Y|$, let $\mathfrak{p}_y$ denote the corresponding prime ideal, C_y its residue field which is a non-archimedean algebraically closed extension of E with valuation $v_y\colon C_y \to \mathbb{R} \cup \{\infty\}$, and finally let $\theta_y\colon \mathbb{A}_{\mathrm{inf}} \to \mathcal{O}_{C_y}$ denote Fontaine's map.

(2) For $f \in \mathbb{A}_{\mathrm{inf}}$, let $f(y)$ denote the class of f in C_y (think of this as "$f \in \Gamma(|Y|, \mathcal{O}_{|Y|})$"), and for $y \in |Y|$ we put $v(f(y)) = v_y(f)$.

(3) For $y_1, y_2 \in |Y|$, we put $d(y_1, y_2) = v_{y_1}(\theta_{y_1}(\xi_{y_2}))$, where $\xi_{y_2} \in \mathfrak{p}_{y_2}$ is a distinguished generator. We will see in Lemma 1.3.24 below that $d(-,-)$ defines a ultra-metric on $|Y|$. In particular, $d(y,0) = v_y(\pi(y))$ is in some sense the "distance to the origin".

1.3.23. Remark. — (1) One can define $\mathcal{Y} = \operatorname{Spf}(\mathbb{A}_{\mathrm{inf}})^{\mathrm{ad}} \setminus V(\pi[\varpi])$ as an adic space, where $\varpi \in \mathfrak{m}_f \setminus \{0\}$. Then $|Y| \subseteq \mathcal{Y}$ is the set of classical points, and $\mathbb{A}_{\mathrm{inf}} \subseteq \Gamma(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$.

(2) There is a properly discontinuous action $\varphi \curvearrowright \mathcal{Y}$ (think of this as "$d(\varphi(y), 0) = \frac{1}{q} d(y, 0)$"). One can define $\mathcal{X} = \mathcal{Y}/\varphi^{\mathbb{Z}}$, the "adic Fargues–Fontaine curve".

Now we can reformulate Theorem 1.3.20 as follows: if $f \in \mathbb{A}_{\mathrm{inf}}$ and $\lambda \neq 0$ is a slope of $\operatorname{Newt}(f)$, then there exists a $y \in |Y|$ such that $v(\pi(y)) = -\lambda$ and $f(y) = 0$. This is the form in which we will prove it below.

1.3.24. Lemma. — *For $r \geqslant 0$ let $\mathfrak{a}_r = \{x \in \mathbb{A}_{\mathrm{inf}} \mid v_0(x) \geqslant r\}$. Then for all $y_1, y_2 \in |Y|$,*

$$d(y_1, y_2) = \sup\left\{r \mid \mathfrak{p}_{y_1} + \mathfrak{a}_r = \mathfrak{p}_{y_2} + \mathfrak{a}_r\right\}\,.$$

In particular, $d\colon |Y| \times |Y| \to \mathbb{R} \cup \{\infty\}$ is an ultra-metric.[2] *That is:*

(1) $d(y_1, y_2) = d(y_2, y_1)$.

(2) *For all $y_3 \in |Y|$, $d(y_1, y_2) \geqslant \min\{d(y_1, y_2), d(y_2, y_3)\}$.*

(3) $d(y_1, y_2) = \infty$ *iff* $y_1 = y_2$.

Proof. Let $\mathfrak{p}_{y_i} = (\xi_{y_i})$ and write $\xi_{y_1} = \sum_{n=0}^{\infty} [x_n] \xi_{y_2}^n$. The coefficients x_n exist and can be constructed inductively as follows: since $(-)^\sharp\colon \mathcal{O}_F \cong \mathcal{O}_{C_{y_2}}^\flat \twoheadrightarrow \mathcal{O}_{C_{y_2}}$ is surjective by Proposition 1.3.18(3), the image of ξ_{y_1} in $\mathcal{O}_{C_{y_2}}$ has the form $\theta_{y_2}([x_0])$. As $(\xi_{y_2}) = \ker \theta_{y_2}$, $\xi' = \xi_{y_1} - [x_0]$ is divisible by ξ_{y_2}. Now iterate the argument for $\xi_{y_2}^{-1} \xi'$. Then

$$d(y_2, y_1) = v_{y_2}\big(\theta_{y_2}(\xi_{y_1})\big) = v(x_0)\,.$$

Applying θ_{y_1}, we see $0 = \theta_{y_1}(\xi_{y_1}) = \sum_{n=0}^{\infty} \theta_{y_1}([x_n]) \theta_{y_1}(\xi_{y_2}^n)$, hence

$$\theta_{y_1}([x_0]) = \theta_{y_1}(\xi_{y_2}) \left(\sum_{n=1}^{\infty} \theta_{y_1}([x_n]) \theta_{y_1}(\xi_{y_2})^{n-1} \right).$$

Note that the sum on the left-hand side is convergent in $\mathcal{O}_{C_{y_1}}$ because in $\xi_{y_2} = [a_{y_2}] - u_{y_2}\pi$ both $a_{y_2} \in \mathfrak{m}_F$ and π have positive valuation. This shows

$$d(y_2, y_1) = v(x_0) = v_{y_1}\big(\theta_{y_1}([x_0])\big) \geqslant v_{y_1}\big(\theta_{y_1}(\xi_{y_2})\big) = d(y_1, y_2)\,,$$

[2] As a slogan, an ultra-metric is related to a metric in the same way a valuation is related to a (non-archimedean) norm.

with equality iff $v_{y_1}(\theta_{y_1}([x_1])) = 0$, which is fulfilled iff $x_1 \in \mathcal{O}_F^\times$. But by symmetry we also get $d(y_1, y_2) \geqslant d(y_2, y_1)$, hence equality must hold.

In particular $x_1 \in \mathcal{O}_F^\times$ is indeed true. Thus, $\xi_{y_1} = [x_0] + u\xi_{y_2}$ for some unit $u \in \mathbb{A}_{\mathrm{inf}}^\times$. This shows $\mathfrak{p}_{y_1} + \mathfrak{a}_{v(x_0)} = \mathfrak{p}_{y_2} + \mathfrak{a}_{v(x_0)}$. Therefore

$$d(y_1, y_2) \leqslant \sup \{r \mid \mathfrak{p}_{y_1} + \mathfrak{a}_r = \mathfrak{p}_{y_2} + \mathfrak{a}_r\} .$$

It remains to show the converse inequality. Let $r \geqslant 0$ such that $\mathfrak{p}_{y_1} + \mathfrak{a}_r = \mathfrak{p}_{y_2} + \mathfrak{a}_r$. Applying θ_{y_2}, we see the ideal $\theta_{y_2}(\xi_{y_1})\mathcal{O}_{C_{y_2}} = \theta_{y_2}([x_0])\mathcal{O}_{C_{y_2}}$ is continued in $\left\{c \in \mathcal{O}_{C_{y_2}} \;\middle|\; v_{y_2}(c) \geqslant r\right\}$. Thus $v(x_0) = v_{y_2}(\theta_{y_2}([x_0])) \geqslant r$. This shows $d(y_1, y_2) \geqslant r$, as required.

Properties (1) and (2) are now clear. For (3), we observe that $\mathcal{O}_{C_{y_1}} \cong \mathbb{A}_{\mathrm{inf}}/\mathfrak{p}_{y_1}$ is complete in its valuative topology, which is the topology induced by the images of the $\mathfrak{a}_r$. Thus, $\mathfrak{p}_{y_1} = \bigcap_{r\geqslant 0}(\mathfrak{p}_{y_1} + \mathfrak{a}_r)$. The same is true for $\mathfrak{p}_{y_2}$. Now $\dim(y_1, y_2) = \infty$ implies $\bigcap_{r\geqslant 0}(\mathfrak{p}_{y_1} + \mathfrak{a}_r) = \bigcap_{r\geqslant 0}(\mathfrak{p}_{y_2} + \mathfrak{a}_r)$. Therefore $\mathfrak{p}_{y_1} = \mathfrak{p}_{y_2}$, as required. ☐

1.3.25. Definition. — For $r > 0$ let $|Y_r| = \{y \in |Y| \mid d(y, 0) = r\}$ denote the *"circle of radius r"*.

1.3.26. Proposition ⚠. — *For $r > 0$, the space $|Y_r|$ is complete with respect to the ultra-metric d.*

Proof. Let $(y_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $|Y_r|$. We need to show that $(y_n)_{n\in\mathbb{N}}$ converges. To this end we claim:

($*$) For all $s > 0$ the sequence $(\mathfrak{p}_{y_n} + \mathfrak{a}_s)_{n\in\mathbb{N}}$ of ideals is constant for $n \gg 0$.

Indeed, there exists a n_0 such that $d(y_n, y_m) > s$ for all $n, m \geqslant n_0$. By Lemma 1.3.24 this implies $\mathfrak{p}_{y_n} + \mathfrak{a}_s = \mathfrak{p}_{y_m} + \mathfrak{a}_s$ and ($*$) is proven.

Set $I_s = (\mathfrak{p}_{y_n} + \mathfrak{a}_s)/\mathfrak{a}_s \subseteq \mathbb{A}_{\mathrm{inf}}/\mathfrak{a}_s$, where $n \gg 0$ so that the eventual constant value is reached. Put $I = \lim_{s\geqslant 0} I_s$. This is an ideal in $\lim_{s\geqslant 0} \mathbb{A}_{\mathrm{inf}}/\mathfrak{a}_s \cong \mathbb{A}_{\mathrm{inf}}$ (here we use that $\mathbb{A}_{\mathrm{inf}}$ is v_0-complete; to prove this, you can use the first part of the proof of Lemma* 1.2.11). Then $I_s = (I + \mathfrak{a}_s)/\mathfrak{a}_s$. We claim:

(⊠) The ideal I is a prime ideal generated by a primitive element of degree 1, and $(\mathfrak{p}_{y_n})_{n\in\mathbb{N}}$ converges to I (and then automatically $I \in |Y_r|$; indeed, if the sequence is to converge in $|Y|$ at all, then the limit will be in $|Y_r|$ since this is a closed subspace).

To prove this, fix $s > r$ and $n \gg 0$ such that $\mathfrak{p}_{y_n} + \mathfrak{a}_s = I + \mathfrak{a}_s$. Writing $\mathfrak{p}_{y_n} = (\xi_{y_n})$, we see that there exists $x \in \mathfrak{a}_s$ such that $a \coloneqq \xi_{y_n} + x$ is an element of I. Then $a \in \mathrm{Prim}_1$. Indeed, writing $\xi_{y_n} = [\xi_0] + [\xi_1]\pi + \cdots$ and $x = [x_0] + [x_1]\pi + \cdots$, we get $v(\xi_0) = r$ and $v(\xi_1) = 0$ because ξ_{y_n} is a distinguished generator of $\mathfrak{p}_{y_n} \in |Y_r|$, and $v(x_0), v(x_1) \geqslant s > r$ because $x \in \mathfrak{a}_s$. Then using the explicit descriptions for the first two coefficients of a one easily confirms $a \in \mathrm{Prim}_1$.

We claim that a generates I. Clearly $(a) \subseteq I$, so assume this inclusion is not an equality. Since $\mathbb{A}_{\mathrm{inf}}/(a)$ is a valuation ring by Corollary 1.3.19(1), there exists $r_0 > 0$ such that $(a) + \mathfrak{a}_{r_0} \subseteq I$. Let $t > \max\{r_0, s\}$ and choose m such that $I + \mathfrak{a}_t = \mathfrak{p}_{y_m} + \mathfrak{a}_t$. But since $\mathfrak{a}_{r_0} \subseteq I \subseteq \mathfrak{p}_{y_m} + \mathfrak{a}_t$ we get $r_0 \geqslant t$ (after applying θ_{y_m}), a contradiction!

Now that we know $I \in |Y|$, it's clear that $(\mathfrak{p}_{y_n})_{n\in\mathbb{N}}$ converges to I as $n \to \infty$, because $\mathfrak{p}_{y_n} + \mathfrak{a}_s = I + \mathfrak{a}_s$ for all $s > 0$ and $n \gg 0$. This proves (⊠) and we are done. ☐

Sketch of a proof of Theorem 1.3.20. Step 1. We reduce to the case where $f \in \mathbb{A}_{\mathrm{inf}}$ is primitive of some degree d. So assume the assertion is proved in this case and write

$f = \sum_{i=0}^{\infty}[x_i]\pi^i$, $f_n = \sum_{i=0}^{n}[x_i]\pi^i$. Each f_n is primitive of some degree up to multiplying by a Teichmüller element, so the theorem holds for the f_n. Choose n_0 such that for all $n \geqslant n_0$, λ occurs as a slope in $\operatorname{Newt}(f_n)$ with the same multiplicity it does in $\operatorname{Newt}(f)$. Let $Y_n = \{y \in |Y| \mid f(y) = 0 \text{ and } d(y,0) = -\lambda\}$. It suffices to show that we can find a Cauchy sequence $(y_n)_{n\in\mathbb{N}}$ such that $y_n \in Y_n$. Indeed, by Proposition 1.3.26, this sequence converges to a limit $y \in |Y_{-\lambda}|$. We claim that $f(y) = 0$. To see this, it suffices to show $v_y(f) \geqslant r$ for all $r > 0$. Choose N such that $-\lambda N > r$ and N' such that $d(y_n, y) > r$ for all $n \geqslant N'$. Now if $n \geqslant \max\{N, N'\}$ we have $v_{y_n}(f) \geqslant (n+1)v_y(\pi) = -\lambda(n+1) > r$ since $f_n(y_n) = 0$ and $f - f_n$ is divisible by π^{n+1}. Hence $f \in \mathfrak{p}_{y_n} + \mathfrak{a}_r$. But $d(y_n, y) > r$ implies $\mathfrak{p}_{y_n} + \mathfrak{a}_r = \mathfrak{p}_y + \mathfrak{a}_r$ by Lemma 1.3.24, hence also $v_y(f) \geqslant r$.

So it remains to construct the Cauchy sequence. We will only hint on how to do that. Let $m < n$ be very large and let y_n be a zero of f_n. We wish to find a zero y_m of f_m that is close to y_n. Using the theorem repeatedly on f_m, we can factor it as $f_m = \xi_1 \cdots \xi_\ell \cdot [a]u$, where $\xi_1, , \dots, \xi_\ell$ are distinguished elements corresponding to the roots of f_m inside $|Y|$, $a \in \mathcal{O}_F$ is some element, and $u \in \mathbb{A}_{\mathrm{inf}}^\times$ is a unit. Clearly $v_{y_n}(u) = 0$ and $v_{y_n}([a]) = v(a) \leqslant v_0(f_m)$. Let $x_i \in |Y|$ be the zero corresponding to ξ_i. There are only a bounded number, say at most N, of x_i with $d(x_i, 0) > -\lambda$, since these correspond to smaller slopes of $\operatorname{Newt}(f)$, and the strong triangle inequality gives $d(x_i, y_n) = -\lambda$ in this case. Similarly, there are at most M indices such that $d(x_i, 0) = -\lambda$ (these are the interesting ones). All the rest satisfies $d(x_i, 0) < -\lambda$, hence $d(x_i, y_n) \leqslant -\lambda'$ by the strong triangle inequality, where $\lambda' > \lambda$ is the next slope after λ in $\operatorname{Newt}(f)$. Thus,

$$\begin{aligned} v_{y_n}(f_m) &= \sum_{i=0}^{\ell} v_{y_n}(\xi_i) + v_{y_n}([a]u) = \sum_{i=0}^{\ell} d(x_i, y_n) + v_{y_n}([a]u) \\ &\leqslant N\lambda + \sum_{d(x_i,0)=-\lambda} d(x_i, y_n) + (\ell - M - N)\lambda' + v_0(f)\,. \end{aligned}$$

However, $v_{y_n}(f_m) \geqslant (m+1)v_{y_n}(\pi) = -\lambda(m+1)$ since $f_n(y_n) = 0$ and $f_n - f_m$ is divisible by π^{m+1}. This shows that $\sum_{d(x_i,0)=-\lambda} d(x_i, y_n)$ must be quite large. But these are at most M summands, so we find a summand such that $d(x_i, y_n)$ is pretty large. Now take $y_m = x_i$. Up to some technical stuff we will omit, this allows us to construct the desired Cauchy sequence.

Step 2. Having done the reduction to $f \in \operatorname{Prim}_d$ for some d, we may moreover assume that λ is the maximal slope of $\operatorname{Newt}(f)$ (i.e., the least steep, since all slopes are negative), by factorizing f and using induction. We claim that there exists a sequence $(y_n)_{n\in\mathbb{N}}$ in $|Y|$ satisfying

(*a*) $v_{y_n}(f) \geqslant -\lambda(d+n)$,

(*b*) $d(y_n, y_{n+1}) \geqslant -\lambda(d+n)/d$, and

(*c*) $d(y_n, 0) = -\lambda$ for all $n \in \mathbb{N}$.

This will immediately imply the theorem, since (*b*) and (*c*) show that $(y_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $|Y_\lambda|$, hence convergent by Proposition 1.3.26, and (*a*) ensures that the limit y is a zero of f by a similar argument as above. We will construct this sequence inductively.

Write $f = \sum_{n=0}^{\infty}[x_n]\pi^n$, where $x_d \in \mathcal{O}_F^\times$. Let $z \in \mathcal{O}_F$ be a zero of $\sum_{i=0}^{d} x_i T^i \in \mathcal{O}_F[T]$ such that $v(z) = -\lambda$, using that F is non-archimedean algebraically closed (and for $v(z) = -\lambda$ we use the classical Newton polygon theory, as in Example 1.3.11). As $\lambda < 0$ is the maximal slope, we have $v(x_i) \geqslant \lambda(d-i)$ for all $i = 0, \dots, d$. Thus $x_i z^i = w_i z^d$ for some $w_i \in \mathcal{O}_F$. Now put $\mathfrak{p}_{y_1} = (\pi - [z])$. Clearly y_1 satisfies (c). Moreover,

$$f(y_1) = \theta_{y_1}(f) = \sum_{i=0}^{d} \theta_{y_1}([x_i z^i]) + \pi^{d+1} \sum_{i=d+1}^{\infty} \theta_{y_1}([x_i])\pi^{i-(d+1)}\,.$$

To show (a) for y_1, it suffices to check that the first sum is divisible by π^{d+1}. As $x_i z^i = w_i z^d$ and $\pi = [z]$ in $\mathcal{O}_{C_{y_1}}$, we get

$$\sum_{i=0}^{d} \theta_{y_1}([x_i z^i]) = \pi^d \sum_{i=0}^{d} \theta_{y_1}([w_i])\,.$$

By construction of z and the w_i we have $\sum_{i=0}^d w_i = 0$. Hence $\sum_{i=0}^d [w_i] \in \pi\mathbb{A}_{\mathrm{inf}}$, whence we conclude that (a) holds for y_1. Since we were cut short by a sudden evaluation of the lecture, we will finish the induction next time . . .

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. . . that is, right now. Assume y_n has been constructed. As in the proof of Lemma 1.3.24, we can write $f = \sum_{i=0}^\infty [a_i]\xi_{y_n}^i$ with $a_i \in \mathcal{O}_F$. Let $z \in F$ be a zero of $\sum_{i=0}^d a_i T^i \in \mathcal{O}_F[T]$ of maximal valuation. We claim $z \in \mathcal{O}_F$. To see this, it suffices to check $a_d \in \mathcal{O}_F^\times$, i.e., that a_d maps to a unit in $k = \mathcal{O}_F/\mathfrak{m}_F$. Consider $\mathbb{A}_{\mathrm{inf}} \to W_{\mathcal{O}_E}(k)$, where $k = \mathcal{O}_F/\mathfrak{m}_F$. Since we can choose ξ_{y_n} to be of the form $\pi - [\varpi]$ with $\varpi \in \mathfrak{m}_F$, its image in $W_{\mathcal{O}_E}(k)$ coincides with the image of π. Thus, reducing $f = \sum_{i=0}^\infty [a_i]\xi_{y_n}^i$ modulo $\mathfrak{m}_F$ shows $a_d \equiv x_d \mod \mathfrak{m}_F$, so $a_d \in \mathcal{O}_F^\times$ is indeed a unit. Now we claim that $\mathfrak{p}_{y_{n+1}} = (\xi_{y_n} - [z])$ works.

First of all, since z has maximal valuation among the d zeros of $\sum_{i=0}^d a_i T^i$, we get $v(z) \geqslant v(a_0)/d = v_{y_n}(f)/d \geqslant -\lambda(d+n)/d$. In particular, $v(z) > -\lambda$, so by the explicit descriptions of S_0 and S_1 it's easy to check that $\xi_{y_n} - [z_{n+1}]$ is primitive of degree 1, hence $y_{n+1} \in |Y|$. Moreover, $d(y_n, y_{n+1}) = v_{y_n}(-[z]) = v(z) \geqslant -\lambda(d+n)/d$, as required. By the strong triangle inequality this also implies $d(y_{n+1}, 0)$, so (b) and (c) hold. It remains to check (a). Since $v(z)$ is maximal among the zeros of $\sum_{i=0}^d a_i T^i$, $-v(z)$ is the minimal (i.e., steepest) slope in the Newton polygon of that polynomial. In other words, $v(a_i) \geqslant v(a_0) - iv(z)$. Thus we may write $a_i z^i = a_0 b_i$. Calculating in a similar way as above, we obtain

$$f(y_{n+1}) = \theta_{y_{n+1}}(f) = \theta_{y_{n+1}}([a_0]) \sum_{i=0}^{d} \theta_{y_{n+1}}([b_i]) + \xi_{y_n}^{d+1} \sum_{i=d+1}^{\infty} \theta_{y_{n+1}}([a_i])\xi_{y_n}^{i-(d+1)}\,.$$

By construction we have $\sum_{i=0}^d b_i = 0$ (or $a_0 = 0$, but in this case $f(y_n) = 0$ and we are already done), hence $\sum_{i=0}^d [b_i] \in \pi\mathbb{A}_{\mathrm{inf}}$. Thus, the first term has valuation at least $v(a_0) + v_{y_{n+1}}(\pi) = v(a_0) + d(y_{n+1}, 0) \geqslant -\lambda(d+n+1)$. The second term has valuation at least $(d+1)v_{y_{n+1}}(\xi_{y_n}) = (d+1)d(y_n, y_{n+1}) \geqslant -\lambda(d+1)(d+n)/d > -\lambda(d+n+1)$. This shows

$$v_{y_{n+1}}(f) \geqslant -\lambda(d+n+1)\,,$$

hence (a) holds and the induction is complete. $\square$

1.3.27. Exercise. — We have seen that there is a surjection $\mathfrak{m}_F \setminus \{0\} \twoheadrightarrow |Y|$ sending a to $(\pi - [a])$, but this doesn't tell much. The goal of this exercise is to work out a better description in the special case $E = \mathbb{Q}_p$.

(1) Let $\varepsilon \in (1+\mathfrak{m}_F) \setminus \{1\}$ and put

$$u_\varepsilon = \frac{[\varepsilon]-1}{[\varepsilon^{1/p}]-1} = 1 + [\varepsilon^{1/p}] + \dots + [\varepsilon^{(p-1)/p}]\,.$$

Show that u_ε is primitive of degree 1!

(2) Show that $(1+\mathfrak{m}_F) \setminus \{1\} \twoheadrightarrow |Y|$ given by $\varepsilon \mapsto (u_\varepsilon)$ is surjective! (*Hint:* If $C/\mathbb{Q}_p$ is non-archimedean algebraically closed together with an isomorphism $\iota\colon \mathcal{O}_C^\flat \xrightarrow{\sim} \mathcal{O}_F$, and if $\varepsilon = (1, \zeta_p, \dots)$ is an element of $\mathcal{O}_C^\flat \cong \mathcal{O}_F$ where $\zeta_p \neq 1$ is a p^{th} root of unity, show that the kernel of $\theta\colon \mathbb{A}_{\inf} \to \mathcal{O}_C$ is generated by u_ε, using that for $a, b \in \mathrm{Prim}_1$ we have $a \in (b)$ iff $(a) = (b)$.)

(3) Let $\mathbb{Z}_p^\times$ act on $(1+\mathfrak{m}_F) \setminus \{1\}$ via exponentiation, i.e., as $(a, \varepsilon) \mapsto \varepsilon^a = \sum_{i=0}^\infty \binom{a}{i}(\varepsilon-1)^i$. Show that

$$|Y| \cong \big((1+\mathfrak{m}_F) \setminus \{1\}\big)/\mathbb{Z}_p^\times\,!$$

That is, show if $(u_\varepsilon) = (u_{\varepsilon'})$, then $\varepsilon' = \varepsilon^a$ for some $a \in \mathbb{Z}_p^\times$! (*Hint:* Let $C = \mathbb{A}_{\inf}/(u_\varepsilon)\big[\frac{1}{p}\big]$. Then $\varepsilon \in \mathcal{O}_F \cong \mathcal{O}_C^\flat$ is a generator of $T_pC^\times = \big\{a = (a_0, a_1, \dots) \in \mathcal{O}_C^\flat \;\big|\; a_0 = 1\big\}$. If $(u_\varepsilon) = (u_{\varepsilon'})$, show $\varepsilon' \in T_pC^\times$.)

A similar description can be given for arbitrary E actually, but the general case needs Lubin–Tate group laws.

1.4. The Ring B

As usual, let p be a prime, $E/\mathbb{Q}_p$ a finite extension with uniformizer $\pi \in \mathcal{O}_E$ and residue field $\mathbb{F}_q = \mathcal{O}_E/\pi\mathcal{O}_E$, and $F/\mathbb{F}_q$ an algebraically closed non-archimedean field extension (i.e., F is complete with respect to a non-archimedean valuation $v\colon F \to \mathbb{R} \cup \{\infty\}$).

To warm up for the construction of B, we first define its "bounded version" B^b as follows: for $\varpi \in \mathfrak{m}_F \setminus \{0\}$ we put

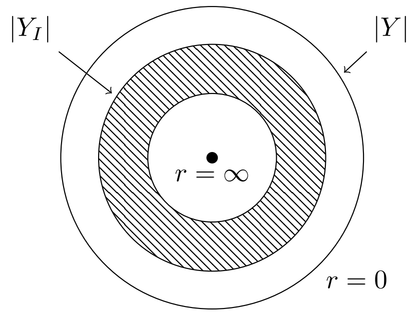
$$B^b = \mathbb{A}_{\inf}\big[\tfrac{1}{\pi}, \tfrac{1}{[\varpi]}\big] = \left\{ \sum_{i \gg -\infty}^{\infty} [x_i]\pi^i \;\middle|\; x_i = 0 \text{ for } i \ll 0,\ \inf_{i\in\mathbb{Z}} v(x_i) > -\infty \right\}.$$

Here we allow the notation $[x]$ also for elements $x \in F$ that need not be in $\mathcal{O}_F$. This works as follows: for sufficiently large n, we have $\varpi^n x \in \mathcal{O}_F$. Then we put $[x] = [\varpi^n x]/[\varpi]^n$, which is indeed an element of B^b. And since the Teichmüller lift $[-]$ is multiplicative, it's clear that $[x]$ is independent of the choice of n, thus well-defined. Also B^b is clearly independent of the choice of ϖ.

Recall the valuations $v_r\colon \mathbb{A}_{\inf} \to \mathbb{R} \cup \{\infty\}$ from Lemma 1.3.17. Then v_r and the construction of the Newton polygon can be extended to B^b; v_r can now attain negative values, and the Newton polygon of an element $f \in \mathrm{Newt}(f)$ need not be contained in the first quadrant $[0,\infty) \times [0,\infty) \subseteq \mathbb{R}^2$, but in $[x,\infty) \times [y,\infty)$ for some $x, y \in \mathbb{R}$.

1.4.1. Definition. — Let $I \subseteq (0,\infty)$ be an intervall. We define the *ring* B_I to be the completion of B^b with respect to the family of valuations $\{v_r\}_{r\in I}$.

The intuition behind this is that "$B_I = \Gamma(|Y_I|, \mathcal{O}_{|Y|})$", where $|Y_I|$ denotes the "annulus" $\{y \in |Y| \mid d(y,0) \in I\}$ that is depicted below.



1.4.2. Remark. — If R is a topological ring such that 0 has a fundamental system $\mathcal{F}$ of neighbourhoods which are open subgroups, then

$$\widehat{R} = \lim_{U \in \mathcal{F}} R/U$$

is the completion of R. Here R/U is a priori only an abelian group since U need not be an ideal. However, by continuity of multiplication in R, the completion $\widehat{R}$ becomes a ring again in a canonical way.

For B_I the situation is a bit different, since there is no single topology on B^b, but one for every valuation v_r (and in fact these are incompatible for different values of r). In this case we take the family $\mathcal{F} = \left\{\bigcap_{i=1}^n v_{r_i}^{-1}[m,\infty) \;\middle|\; m,n \in \mathbb{N} \text{ and } r_i \in I\right\}$ and then define

$$B_I = \lim_{U \in \mathcal{F}} B^b/U$$

as above. In the case where $I = [a,b]$ is compact, this is indeed the "smallest" ring in which all sequences that are Cauchy with respect to every $r \in I$ are convergent. Indeed, every $r \in I$ can be written as $r = \lambda a + (1-\lambda)b$ for $0 \leqslant \lambda \leqslant 1$, and if $v_a(f), v_b(f) \geqslant m$, then also $v_r(f) \geqslant \lambda m + (1-\lambda)m = m$. For non-compact I, the situation is not that easy, but at least the above construction shows

$$B_I = \lim_{J \subseteq I} B_J\,,$$

where the limit is taken over all compact subintervalls of $J \subseteq I$. In particular, this applies to the most important special case $I = (0,\infty)$.

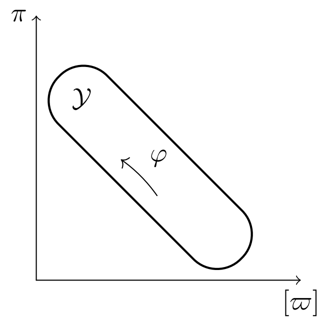
1.4.3. Definition. — The *ring* B is defined as $B = B_{(0,\infty)}$.

So B can be viewed as the "ring of global sections of $\mathcal{O}_{|Y|}$". Note that the Frobenius $\varphi \curvearrowright B^b$ extends, by continuity, to an automorphism of B. More generally, φ induces an isomorphism $\varphi\colon B_I \xrightarrow{\sim} B_{qI}$ for $I \subseteq (0,\infty)$. For every $d \in \mathbb{Z}$, let $B^{\varphi=\pi^d}$ be the eigenspace of φ with respect to the eigenvalue π^d.

1.4.4. Definition ⚠. — The *schematic Fargues–Fontaine curve* (with respect to E and F) is the scheme

$$X = X_{\mathrm{FF}} \coloneqq \operatorname{Proj}\left(\bigoplus_{d \geqslant 0} B^{\varphi=\pi^d}\right).$$

1.4.5. Remark. — As already remarked in Remark 1.3.23, there is an adic analogue of the Fargues–Fontaine curve: put $\mathcal{Y} = \operatorname{Spf}(\mathbb{A}_{\mathrm{inf}})^{\mathrm{ad}} \setminus V(\pi[\varpi])$ and let $\mathcal{X} = \mathcal{Y}/\varphi^{\mathbb{Z}}$. Then $B \cong \Gamma(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$.



One can show that $\pi^{-1}\colon \varphi^*\mathcal{O}_{\mathcal{Y}} \xrightarrow{\sim} \mathcal{O}_{\mathcal{Y}}$ defines the descent datum for a line bundle $\mathcal{O}(1)$ on $\mathcal{X}$. Moreover, one has

$$H^0\big(\mathcal{X}, \mathcal{O}(1)^{\otimes d}\big) \cong B^{(\pi^{-1}\varphi)^d = 1} = B^{\varphi = \pi^d}\,.$$

The goal of the next few lectures is to understand the scheme X_{FF}, and in particular the rings B and B^b. We start with a lemma that essentially says that the eigenspaces of φ on B^b, i.e., before completion, are rather boring.

1.4.6. Lemma. — *For $d \in \mathbb{Z}$ let $(B^b)^{\varphi=\pi^d} \subseteq B^b$ be the eigenspace of φ with respect to the eigenvalue π^d. Then*

$$(B^b)^{\varphi=\pi^d} = \begin{cases} E & \text{if } d = 0 \\ 0 & \text{else} \end{cases}.$$

Proof. For $d = 0$, let $f = \sum_{i \gg -\infty}^{\infty} [x_i]\pi^i \in B^b$. If f is fixed under φ, then $\varphi(x_i) = x_i$ for all i, hence $x_i \in \mathbb{F}_q \subseteq \mathcal{O}_F$. This shows $f \in W_{\mathcal{O}_E}(\mathbb{F}_q)\big[\frac{1}{\pi}\big] \cong E$. The converse can be shown in the same way. Now let $d \neq 0$. If $f \in (B^b)^{\varphi=\pi^d}$ and $x \in \mathbb{R}$, then

$$q\,\mathrm{Newt}(f)(x) = \mathrm{Newt}(\varphi(f))(x) = \mathrm{Newt}(\pi^d f)(x) = \mathrm{Newt}(f)(x-d)\,.$$

Iterating gives $q^n\,\mathrm{Newt}(f)(x) = \mathrm{Newt}(f)(x - dn)$ for all $n \geqslant 1$. For $d > 0$, we have $\mathrm{Newt}(f)(x - dn) = +\infty$ for $n \gg 0$, hence already $\mathrm{Newt}(f)(x) = +\infty$. This implies $f = 0$. For $d < 0$ pick $x_0 \gg 0$ with $\mathrm{Newt}(f)(x_0) = +\infty$. Since $\mathrm{Newt}(f)$ is decreasing, for all $x \in \mathbb{R}$ there exists an n such that $\mathrm{Newt}(f) \geqslant \mathrm{Newt}(x_0 - nd) = q^n\,\mathrm{Newt}(f)(x_0) = +\infty$. Thus $f = 0$ follows as before. $\square$

1.4.7. Lemma. — *Let $(x_n)_{n\in\mathbb{Z}}$ be a sequence in F such that $\lim_{|n|\to\infty} v(x_n) + rn = \infty$ for all $r \in (0,\infty)$. Then $\sum_{n\in\mathbb{Z}}[x_n]\pi^n$ converges in B.*

Proof. It suffices to show that $v_r([x_n]\pi^n) \to \infty$ as $|n| \to \infty$ for all $r \in (0,\infty)$. But $v_r([x_n]\pi^n) = v(x_n) + rn$, so this holds by assumption. $\square$

1.4.8. Remark. — (1) For all $a \in \mathfrak{m}_F$, the series $f_a \coloneqq \sum_{i\in\mathbb{Z}}[a^{q^{-i}}]\pi^i$ coverges in B, and $f_a \in B^{\varphi=\pi}$. To prove this we need to check $q^{-i}v(a) + ri = v(a^{q^{-i}}) + ri \to \infty$ as $|i| \to \infty$ for all $r \in (0,\infty)$, using Lemma 1.4.7. This is clear as $v(a) > 0$. Also one immediately checks $\varphi(f_a) = \pi f_a$, hence indeed $f_a \in B^{\varphi=\pi}$. In fact, we will prove later that the above construction gives a bijection $\mathfrak{m}_F \cong B^{\varphi=\pi}$. This should seem a bit weird at first since the right-hand side $B^{\varphi=\pi}$ is an E-vector space, so the left-hand side better be one as well. One can indeed construct a E-vector space structure on $\mathfrak{m}_F$ by Lubin–Tate theory.

(2) In general, it is not known whether elements in B can be written as $\sum_{n\in\mathbb{Z}}[x_n]\pi^n$ for $[x_n] \in F$. So we need different tools to study B.

The goal for the next few lectures is to prove that $X = X_{\mathrm{FF}}$ is indeed a curve.

1.4.9. Main Theorem (Fargues–Fontaine). — *The Fargues–Fontaine curve X_{FF} is a Dedekind scheme. More precisely, for each $t \in B^{\varphi=\pi}$, the open subset $D_+(t) \cong \operatorname{Spec} B\left[\frac{1}{t}\right]^{\varphi=1}$ is the spectrum of a principal ideal domain.*

The first ingredient in the proof of Main Theorem 1.4.9 is to construct Newton polygons for elements in B_I, where $I \subseteq (0,\infty)$ is an ideal. Note that for all $r \in I$, the valuation $v_r \colon B^b \to \mathbb{R} \cup \{\infty\}$ extends to B_I by continuity.

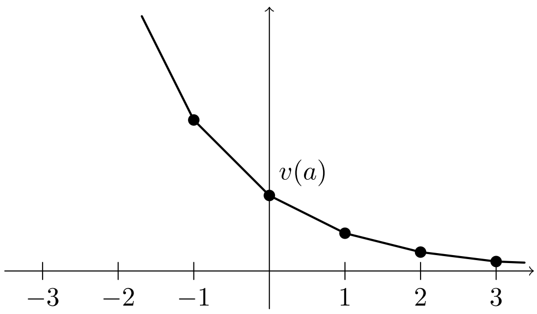
1.4.10. Definition. — Assume $I \subseteq (0,\infty)$ is an open intervall, and $f \in B_I$. Let $\mathrm{Newt}_I^0(f)$ be the decreasing convex function whose Legendre transform is

$$\mathcal{L}\,\mathrm{Newt}_I^0(f)(r) = \begin{cases} v_r(f) & \text{if } r \in I \\ -\infty & \text{else} \end{cases}.$$

The *Newton polygon* $\mathrm{Newt}_I(f) \subseteq \mathbb{R}^2$ is the subset of the graph of $\mathrm{Newt}_I^0(f)$ with slopes in $-I$.

1.4.11. Remark. — If $K \subseteq I$ is compact and $(f_n)_{n\in\mathbb{N}}$ a sequence in B^b converging to $f \neq 0$, then there exists an N such that for all $n \geqslant N$ we have $v_r(f_n) = v_r(f)$ for all $r \in K$. In particular, $\mathcal{L}\,\mathrm{Newt}_I^0(f)$ is a concave piece-wise linear function with integral slopes. Thus, $\mathrm{Newt}_I^0(f)$ is a decreasing convex polygon with integral breakpoints.

1.4.12. Remark. — (1) If $a \in \mathfrak{m}_F$ and f_a is as above, then $\mathrm{Newt}_{(0,\infty)}(f_a)$ is a "polygon version" of the exponential function $i \mapsto v(a)q^{-i}$:



Note that there is no $x \in \mathbb{R}$ where $\mathrm{Newt}_{(0,\infty)}(f_a)(x) = +\infty$.

(2) If $f \in B$ and λ_i is the slope of $\mathrm{Newt}_{(0,\infty)}(f)$ on $[i, i+1]$, then

$$\lambda_i \leqslant 0\,, \quad \lim_{i\to\infty} \lambda_i = 0\,, \quad \text{and } \lim_{i\to-\infty} \lambda_i = -\infty\,.$$

So far we have defined Newton polygons for open intervalls I. In the case of compact intervalls $I = [a,b]$, the above Definition 1.4.10 doesn't work any more and we need slightly more complicated one.

1.4.13. Definition. — Let $I = [a, b]$ be a compact intervall and $0 \neq f \in B_I$. We define $\mathrm{Newt}_I^0(f)$ to be the decreasing convex function whose Legendre transform is

$$\mathcal{L}\,\mathrm{Newt}_I^0(f)(r) = \begin{cases} v_r(f) & \text{if } r \in I \\ -\infty & \text{if } r < 0 \\ v_a(f) + (r-a)\partial_- v_a(f) & \text{if } r < a \\ v_b(f) + (r-b)\partial_+ v_b(f) & \text{if } r \geqslant b \end{cases},$$

and again $\mathrm{Newt}_I(f) \subseteq \mathbb{R}$ is the subset of the graph of $\mathrm{Newt}_I^0(f)$ with slopes in $-I$.

1.4.14. Remark. — (1) If $(f_n)_{n\in\mathbb{N}}$ is a sequence in B^b converging to f, then

$$\partial_+ v_r(f) = \lim_{n\to\infty} \partial_+ v_r(f_n) \quad \text{and} \quad \partial_- v_r(f) = \lim_{n\to\infty} \partial_- v_r(f_n)\,.$$

Here $\partial_+ v_r(f)$ denotes the right-derivative of the function $s \mapsto v_s(f)$ at $s = r$. Likewise ∂_- denotes left-derivatives.

(2) For $f \in B^b$ and λ a slope of $\mathrm{Newt}(f)$, then $\partial_+ v_r(f) - \partial_- v_r(f)$ is precisely the multiplicity of λ in $\mathrm{Newt}(f)$.

1.5. Proof that the Fargues–Fontaine Curve is a Curve

1.5.1. The Graded Algebra P

Lecture 7
11th Dec, 2019

As usual, let p be a prime, $E/\mathbb{Q}_p$ a finite extension with ring of integers $\mathcal{O}_E$ and residue field $\mathcal{O}_E/\pi\mathcal{O}_E \cong \mathbb{F}_q$. Let $F/\mathbb{F}_q$ be a non-archimedean algebraically closed extension and $\varpi \in \mathfrak{m}_F \setminus \{0\}$. Last time we defined the ring $B = B_{(0,\infty)}$. Now let

$$P \coloneqq \bigoplus_{d\geqslant 0} P_d\,, \quad \text{where} \quad P_d = B^{\varphi=\pi^d}\,,$$

so that $X_{\mathrm{FF}} = \operatorname{Proj} P$ is the Fargues–Fontaine curve as defined in Definition 1.4.4. The goal for today is to prove the following theorem, working towards Main Theorem 1.4.9, i.e. that X_{FF} is indeed a curve.

1.5.1. Theorem (Fargues–Fontaine)**.** — *P is graded factorial with irreducible elements of degree 1, i.e., the multiplicative monoid*

$$\bigcup_{d\geqslant 0} (P_d \setminus \{0\})/E^\times$$

is free on $(P_1 \setminus \{0\})/E^\times$. In particular, if $d \geqslant 1$ and $x \in P_d$, then there exist $t_1, \ldots, t_d \in P_1$ (unique up $E^\times$ and order) such that $x = t_1 \cdots t_d$.

Mind that Theorem 1.5.1 does *not* imply $P \cong \mathrm{Sym}^*_{\mathbb{Q}_p} P_1$. In fact, the right-hand side has non-noetherian Proj, whereas $\operatorname{Proj} P = X_{\mathrm{FF}}$ will turn out to be noetherian. For the proof we need

1.5.2. Theorem. — *Assume $I \subseteq (0, \infty)$ is compact. Then B_I is a PID, and* $\operatorname{Spec} B_I \setminus \{0\}$ *is in canonical bijection with $|Y_I|$.*

Proof. We use the easy to prove fact that an integral domain A is a PID iff A is factorial and each (non-invertible) irreducible element generates a maximal ideal. Thus, it suffices to prove the following three claims:

(1) For $y \in |Y_I|$ the map $\theta_y \colon B^b \twoheadrightarrow C_y$ has a unique extension to a continuous morphism $\theta'_y \colon B_I \twoheadrightarrow C_y$. Moreover, if $\ker \theta_y = (\xi_y)$, then ξ_y is also a generator of $\ker \theta_y$.

(2) If $f \in B_I \setminus \{0\}$ such that $\mathrm{Newt}_I(f) = \emptyset$, then $f \in B_I^\times$ is a unit.

(3) If $f \in B_I$ and λ is a slope of $\mathrm{Newt}_I(f)$, then there exists a $y \in |Y_{-\lambda}|$ such that $f = \xi_y g$ for some $g \in B_I$. Note that by (1) this is equivalent to the existence of some $y \in |Y_{-\lambda}|$ such that $f(y) \coloneqq \theta'_y(f) = 0$.

We first deduce the theorem from these two claims. Note that since I is compact and the slopes of $\mathrm{Newt}_I^0(f)$ approach 0, only finitely many of them can be contained in I. Hence $\mathrm{Newt}_I(f)$ has only finitely many segments. Using (3) and (2) and induction on the number of segments, we see that any non-zero f can be decomposed into a product $f = u\xi_{y_1} \cdots \xi_{y_n}$ of a unit $u \in B_I^\times$ and prime elements ξ_{y_i}. This shows that B_I is factorial. By (1), every ξ_y generates a maximal ideal, so B_I is indeed a PID by the fact cited in the beginning. Moreover, we remark that this implies

$$(B_I)^\wedge_{\xi_y} \cong B^+_{\mathrm{dR},y} \cong (B^b)^\wedge_{\xi_y} \,.$$

The isomorphism on the right-hand side is due to $\mathbb{A}_{\inf}\left[\frac{1}{\pi}\right]/(\xi_y^n) \cong B^b/(\xi_y^n)$ for all $n \geqslant 1$, which follows from the easy fact that $[\varpi]$ is already invertible in $\mathbb{A}_{\inf}\left[\frac{1}{\pi}\right]/(\xi_y^n)$. Hence we get a morphism $B^+_{\mathrm{dR},y} \to (B_I)^\wedge_{\xi_y}$ of complete DVRs. This induces an isomorphism on residue fields C_y and ξ_y is a uniformizer on both sides, thus is indeed an isomorphism.

Now we prove the three claims, beginning with (1). Let $y \in |Y_I|$ and $r = d(y, 0) = v_y(\pi)$, so that $r \in I$. We claim:

$(*)$ The map $\theta_y \colon B^b \twoheadrightarrow C_y$ is continuous for the v_r-topology on B^b.

Indeed, if $x = \sum_{i \gg -\infty}^{\infty} [x_i]\pi^i \in B^b$, then $\theta_y(x) = \sum_{i \gg -\infty}^{\infty} \theta_y([x_i])\pi^i$, hence

$$v_y\big(\theta_y(x)\big) \geqslant \inf_{i\in\mathbb{Z}}\big\{v_y\big(\theta_y([x_i])\big) + i v_y(\pi)\big\} = v_r(x)\,,$$

using $r = v_y(\pi)$. This immediately implies continuity of θ_y, so $(*)$ is proved. Since every element of B_I can be written as a sequence of elements of B^b which is a Cauchy sequence in the v_r-topology (in fact, even a Cauchy sequence in the v_s-topology for all $s \in I$), we see that θ_y has indeed a unique continuous extension $\theta'_y \colon B_I \to C_y$.

In the lecture it was claimed to be a "general fact" that $\ker \theta'_y = \overline{\ker \theta_y}$, the closure being taken in B_I. I don't see what fact that should be (please enlighten me), so here's a proof. Since θ'_y is continuous and $0 \in C_y$ is closed, the inclusion "$\supseteq$" is clear. For the converse, let $f \in \ker \theta'_y$ and $(f_n)_{n\in\mathbb{N}}$ a Cauchy sequence in B^b converging to f. Every f_n can be written as $f_n = [x_n] + \xi_y g_n$ for some $x_n \in F$ and $g_n \in B^b$. Indeed, we may assume $\xi_y = \pi - [a]$. For $N \gg 0$ we have $\pi^N\theta_y(f) \in \mathcal{O}_{C_y}$, hence by Proposition 1.3.18(3) we may write $\pi^N\theta_y(f) = [z_n]$ for some $z_n \in \mathcal{O}_F$. Then $x_n = z_n a^{-N}$ does it. Since $v_y(\theta_y(f_n)) = v_F(x_n) = v_s([x_n])$ for all $s \in (0,\infty)$, we see that $([x_n])_{n\in\mathbb{N}}$ is a Cauchy sequence in the v_s-topology for all $s \in (0,\infty)$. Thus, $(\xi_y g_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in the v_s-topology for all $s \in I$, and converges to f. This proves $f \in \overline{\ker \theta_y}$.

The fact that $\ker\theta'_y = (\xi_y)$ is now an easy consequence: suppose $f \in \overline{\ker\theta_y}$ and write f as the limit of a Cauchy sequence $(f_n)_{n\in\mathbb{N}}$ such that $f_n = \xi_y g_n$. For all $s \in I$ we have

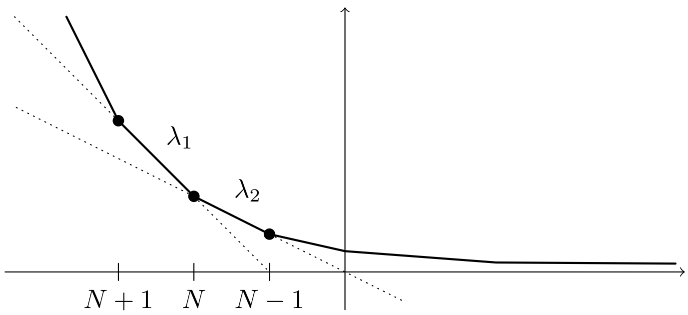
$$v_s(g_n - g_m) = v_s(f_n - f_m) - v_s(\xi_y)\,,$$

hence $(g_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in the v_s-topology as $v_s(\xi_y) \neq \infty$. Thus $g = \lim_{n\to\infty} g_n$ exists in B_I and satisfies $f = \xi_y g$. This proves (1).

For (2), let $I = [a,b]$ and let $(f_n)_{n\in\mathbb{N}}$ be an v_s-Cauchy sequence for all $s \in I$ converging to f. Since $f \neq 0$ and $\operatorname{Newt}_I(f) = \emptyset$, we deduce that already $\operatorname{Newt}_I(f_n) = \emptyset$ for all $n \gg 0$ (this follows from Remark 1.4.11 for example). So it suffices to consider the case $f \in B^b$ (indeed, if the f_n are units for $n \gg 0$, then $(f_n^{-1})_{n\gg 0}$ is a Cauchy sequence—here we critically use $f \neq 0$ again—and it converges to an inverse of f). Now we can write

$$f = \sum_{n\gg-\infty}^{\infty} [x_n]\pi^n = \sum_{n\leqslant N} [x_n]\pi^n + \sum_{n>N} [x_n]\pi^n =: f_- + f_+\,.$$

Here N is chosen in such a way that each slope of $\operatorname{Newt}(f)$ on $(-\infty, N]$ is $< -b$ and each slope on $[N,\infty)$ is $> -a$. This works, since by assumption $\operatorname{Newt}(f)$ has no slopes in I.



Let λ_1 and λ_2 be the slopes in the picture, so that $-\lambda_1 > b > a > -\lambda_2$. Looking at the dotted lines we derive inequalities

$$\begin{aligned} v(x_n) &\geqslant (n-N)\lambda_1 + v(x_N) \quad \text{for all } n \leqslant N\,,\\ v(x_n) &\geqslant (n-N)\lambda_2 + v(x_N) \quad \text{for all } n \geqslant N\,. \end{aligned}$$

Let $f = f_- + f_+$ be the above sum decomposition and write

$$f_- = [x_N]\pi^N \left(1 + \sum_{n<N} [x_n x_N^{-1}]\pi^{n-N}\right)\,.$$

Writing the second factor as $1+g$, we claim that g is topologically nilpotent in B_I. Indeed, let $r \in I$. We compute

$$v_r(g) = \inf_{n<N}\left\{v(x_n) - v(x_N) + r(n-N)\right\} \geqslant \inf_{n<N}\left\{(\lambda_1 + r)(n-N)\right\} = -\lambda_1 - r > 0\,,$$

proving that g is topologically nilpotent, as claimed. Thus $(1+g) \in B_I^\times$. The element $[x_N]\pi^N$ is already invertible in B^b, hence $f_- = [x_N]\pi^N(1+g) \in B_I^\times$. To show that f is a unit too, write $f = f_-(1 + f_-^{-1} f_+)$. Similar as above we show $v_r(f_+) > v_r([x_N]\pi^N)$ for all $r \in I$, hence $f_-^{-1} f_+$ is topologically nilpotent in B_I, so f is indeed a unit.

We omit the proof of (3), since it is very similar to the proof of Theorem 1.3.20 (approximate f by a Cauchy sequence $(f_n)_{n \in \mathbb{N}}$ in B^b, show that$-\lambda$ occurs as a slope in $\mathrm{Newt}_I(f_n)$ for all $n \gg 0$, find a Cauchy sequence $(y_n)_{n \gg 0}$ of zeros $y_n \in |Y_{-\lambda}|$ of f_n, and use that $|Y_{-\lambda}|$ is complete by Proposition 1.3.26). ☐

1.5.2. Divisors on $|Y|$

1.5.3. Definition. — Let $I \subseteq (0, \infty)$ be any intervall. The *monoid of effective divisors on* $|Y_I|$ is the partially ordered monoid $\mathrm{Div}^+(|Y_I|)$ of formal sums

$$\sum_{y \in |Y_I|} n_y y\,, \quad n_y \in \mathbb{N}\,,$$

such that for each compact intervall $J \subseteq I$ the set $\{y \in |Y_J| \mid n_y \neq 0\}$ is finite (so in particular, if I is not compact, the above sums need not be finite).

1.5.4. Example. — if I is compact, we have $\mathrm{Div}^+(|Y_I|) = \mathbb{N}^{|Y_I|}$. In general, for arbitrary intervalls I we have $\mathrm{Div}^+(|Y_I|) \cong \lim_J \mathrm{Div}^+(|Y_J|)$, where the limit is taken over all compact subintervalls $J \subseteq I$.

1.5.5. Definition. — For all y we denote by $\mathrm{ord}_y \colon B_{\mathrm{dR},y}^+ \to \mathbb{N} \cup \{\infty\}$ the valuation of $B_{\mathrm{dR},y}^+$. For $f \in B_I \setminus \{0\}$, let

$$\mathrm{div}(f) = \sum_{y \in |Y_I|} \mathrm{ord}_y(f) y \in \mathrm{Div}^+(|Y_I|)$$

be the *principal divisor associated to* f. Since B_J is a PID for all compact $J \subseteq I$ by Theorem 1.5.2, the map $\mathrm{div} \colon B_I \setminus \{0\} \to \mathrm{Div}^+(|Y_I|)$ is well-defined, multiplicative, and vanishes on units.

1.5.6. Proposition. — *If $I \subseteq (0, \infty)$ is an intervall, then the map*

$$\mathrm{div} \colon \big(B_I \setminus \{0\}\big)/B_I^\times \longrightarrow \mathrm{Div}^+(|Y_I|)$$

is injective, and bijective if I is compact. Moreover, $\mathrm{div}(f) \geqslant \mathrm{div}(g)$ *iff* $f \in gB_I$.

Proof. The assertion is clear if I is compact, since in this case B_I is a PID by Theorem 1.5.2, whose primes are precisely (up to units) the ξ_y for $y \in |Y_I|$. In general, write $B_I \cong \lim_J B_J$ and $\mathrm{Div}^+(|Y_I|) \cong \lim_J \mathrm{Div}^+(|Y_J|)$ and use that limits preserve injective maps. The second assertion can be seen in a similar way. ☐

1.5.7. Lemma. — *Recall that* $P_d = B^{\varphi = \pi^d}$. *Then*

$$P_d = \begin{cases} E & \text{if } d = 0 \\ 0 & \text{if } d < 0 \\ \text{complicated} & \text{if } d > 0 \end{cases}\,.$$

Proof. Similar as for B^b (see Lemma 1.4.6), using $\mathbb{A}_{\mathrm{inf}} = \big\{f \in B \bigm| \mathrm{Newt}_{(0,\infty)}(f) \subseteq \mathbb{R}^2_{\geqslant 0}\big\}$. This equality is left as an exercise (a hard one, though). ☐

We have a canonical Frobenius action of $\operatorname{Div}^+(|Y|)$ defined as follows: for $y \in |Y|$ let $\varphi^*(y) \in |Y|$ be the point associated to the prime ideal $\varphi^{-1}(\mathfrak{p}_y)$. Then we put

$$\varphi^*\left(\sum_{y\in|Y|} n_y y\right) = \sum_{y\in|Y|} n_y \varphi^*(y)\,.$$

Since $d(\varphi^*(y),0) = q^{-1}d(y,0)$, it is easily established that the right-hand side is an element of $\operatorname{Div}^+(|Y|)$ again, so we get indeed an action $\varphi \curvearrowright \operatorname{Div}^+(|Y|)$.

1.5.8. Definition. — We define $\operatorname{Div}^+(|Y|/\varphi^{\mathbb{Z}}) \coloneqq \operatorname{Div}^+(|Y|)^{\varphi^{\mathbb{Z}}}$ to be the monoid of *effective divisors on the Fargues–Fontaine curve.*

1.5.9. Remark. — Let $a > 0$ be arbitrary and $I = [a, qa)$. Then $\operatorname{Div}^+(|Y|/\varphi^{\mathbb{Z}})$ is in canonical bijection with $\operatorname{Div}^+_{\mathrm{fin}}(|Y_I|)$, where the subscript $_{\mathrm{fin}}$ denotes the subset of those divisors which are actually finite sums. Indeed, since the Frobenius action is a "contraction with factor q^{-1}" in the sense that $d(\varphi^*(y),0) = q^{-1}d(y,0)$, exactly one of the $\{(\varphi^n)^*(y)\}_{n\in\mathbb{Z}}$ will be contained in $|Y_I|$. Now the claimed bijection comes from the fact that every divisor $D \in \operatorname{Div}^+(|Y|/\varphi^{\mathbb{Z}})$ can be decomposed into a finite sum of divisors of the form $\sum_{n\in\mathbb{Z}}(\varphi^n)^*(y)$ (do induction in the finite number of $y \in Y_I$ occurring with non-zero coefficient in D), and this decomposition is unique.

1.5.10. Theorem. — *The principal divisors map* div *from Definition 1.5.5 induces an isomorphism*

$$\operatorname{div}\colon \bigcup_{d\geqslant 0} \big(P_d\setminus\{0\}\big)/E^\times \xrightarrow{\ \sim\ } \operatorname{Div}^+\big(|Y|/\varphi^{\mathbb{Z}}\big)\,.$$

In particular, this implies Theorem 1.5.1.

Proof. To derive Theorem 1.5.1, use that every effective divisor $D \in \operatorname{Div}^+(|Y|/\varphi^{\mathbb{Z}})$ decomposes uniquely into a finite sum of divisors of the form $\sum_{n\in\mathbb{Z}}(\varphi^n)^*(y)$ as in Remark 1.5.9. If you think about it, the isomorphism div translates this into the assertion that the monoid on the left-hand side is free on $(P_d\setminus\{0\})/E^\times$.

To prove the theorem, we first check well-definedness. For $x \in P_d\setminus\{0\}$ we have $\varphi^*(\operatorname{div}(x)) = \operatorname{div}(\varphi^{-1}(x)) = \operatorname{div}(\pi^{-d}x) = \operatorname{div}(x)$, since π^{-d} is a unit in B. Hence div has indeed image in $\operatorname{Div}^+(|Y|)^{\varphi^{\mathbb{Z}}}$.

For injectivity, let $x \in P_d\setminus\{0\}$ and $x' \in P_{d'}\setminus\{0\}$ such that $\operatorname{div}(x) = \operatorname{div}(y)$. Without restriction $d' \geqslant d$. From the second part of Proposition 1.5.6 we get $x = ux'$ for some $u \in B^\times$. Then $u \in B^{\varphi=\pi^{d-d'}}$. But then Lemma 1.5.7 allows only $d = d'$. In this case $B^{\varphi=1} = E$, so $u \in E^\times$, proving that x and x' represent the same element. This shows injectivity.

To prove surjectivity, it suffices to show that $\sum_{n\in\mathbb{Z}}(\varphi^n)^*(y)$ is in the image of $P_1\setminus\{0\}$. We may assume $\xi_y = \pi - [a]$. Put

$$x = \prod_{n\geqslant 0}\left(1 - \frac{[a]^{q^n}}{\pi}\right) = \prod_{n\leqslant 0}\frac{\varphi^n(\xi_y)}{\pi}\,.$$

This x is well-defined as $[a]^{q^n}$ converges to 0 for $n\to\infty$, see Remark 1.4.8(1). Moreover, $\varphi(x) = \prod_{n\geqslant 1}(\varphi^n(\xi_y)/\pi) = (\xi_y/\pi)^{-1}x$ and $\operatorname{div}(x) = \sum_{n\leqslant 0}(\varphi^n)^*(y)$. Applying Lemma 1.5.11 below to ξ_y provides an element $z \neq 0$ such that $\varphi(z) = \xi_y z$. Then

$$\operatorname{div}(z) = \operatorname{div}\big(\xi_y\varphi^{-1}(z)\big) = y + \varphi^*\big(\operatorname{div}(z)\big) = y + \varphi^*(y) + (\varphi^2)^*(y) + \cdots\,.$$

Hence $\operatorname{div}(x\varphi^{-1}(z)) = \sum_{n\in\mathbb{Z}}(\varphi^n)^*(y)$. Moreover, $\varphi(x\varphi^{-1}(z)) = (\xi_y/\pi)^{-1}xz = \pi x\varphi^{-1}(z)$. Hence $t \coloneqq x\pi^{-1}(z)$ is an element of $P_1 \setminus \{0\}$ mapping to $\sum_{n\in\mathbb{Z}}(\varphi^n)^*(y)$. $\square$

1.5.11. Lemma. — *Let $\beta \in B^b \cap W_{\mathcal{O}_E}(F)^\times$ (for example, ξ_y lies in this intersection). Then we have*

$$\dim_E (B^b)^{\varphi=\beta} = 1\,.$$

Proof. Proving "$\leqslant 1$" is easy: if $f, f' \in (B^b)^{\varphi=\beta}$ are two non-zero eigenvectors, then

$$f/f' \in \left(W_{\mathcal{O}_E}(F)\left[\tfrac{1}{\pi}\right]\right)^{\varphi=1} = W_{\mathcal{O}_E}(\mathbb{F}_q)\left[\tfrac{1}{\pi}\right] = E\,,$$

which has dimension 1 over E.

It remains to prove that $(B^b)^{\varphi=\beta}$ is non-zero. Without restriction let $\beta \in \mathbb{A}_{\mathrm{inf}} \setminus \pi\mathbb{A}_{\mathrm{inf}}$ (a general $\beta \in B^b \cap W_{\mathcal{O}_E}(F)^\times$ can be written as $[z^{-1}]\beta'$ for $\beta' \in \mathbb{A}_{\mathrm{inf}} \setminus \pi\mathbb{A}_{\mathrm{inf}}$ and $z \in \mathcal{O}_F$, and it's easy to construct an eigenvector with eigenvalue $[z^{-1}]$). To obtain a non-zero eigenvector of β, we construct a converging sequence $(x_n)_{n\in\mathbb{N}}$ in $\mathbb{A}_{\mathrm{inf}}$ with the properties

(1) $x_1 \notin \pi\mathbb{A}_{\mathrm{inf}}$,

(2) $x_n \equiv x_1 \mod \pi$ for all $n \geqslant 1$, and

(3) $\varphi(x_n) \equiv \beta x_n \mod \pi^n$ for all $n \geqslant 1$.

We do this by induction on n. For $n = 1$, take $x_1 = [a]$, where $a \in \mathcal{O}_F \setminus \{0\}$ is a non-zero solution of $a^q = \overline{\beta}a$, where $\overline{\beta} \in \mathcal{O}_F$ is the reduction of β modulo π. Such a exists as F is algebraically closed. Now let $n \geqslant 1$ and assume x_n has already been constructed. We use the ansatz $x_{n+1} = x_n + [u]\pi^n$. Then (2) is automatically satisfied. For (3) write $\varphi(x_n) \equiv \beta x_n + [z]\pi^n \mod \pi^{n+1}$ and compute

$$\begin{aligned}\varphi(x_n + [u]\pi^n) &\equiv \beta(x_n + [u]\pi^n) - \beta[u]\pi^n + [u^q]\pi^n + [z]\pi^n\\ &\equiv \beta(x_n + [u]\pi^n) - [\overline{\beta}u - u^q - z]\pi^n \mod \pi^{n+1}\,.\end{aligned}$$

Thus it suffices to choose u such that $\overline{\beta}u - u^q - z = 0$, which is always possible as F is algebraically closed. $\square$

1.5.3. Proof of the Main Result

Lecture 8
18th Dec, 2019

The aim for today is to show Main Theorem 1.4.9, asserting that $X = \operatorname{Proj} P$ is indeed a "curve", i.e., a Dedekind scheme. In some sense, X is similar to $\mathbb{P}^1_E$. In fact, if E is replaced by $\mathbb{C}$ or $\mathbb{R}$, then the analogs of X should be $\mathbb{P}^1_{\mathbb{C}}$ and $\widetilde{\mathbb{P}}^1_{\mathbb{R}} = V_+(x^2+y^2+z^2) \subseteq \mathbb{P}^2_{\mathbb{R}}$.

We have seen in the proof of Theorem 1.5.1 that for every $y \in |Y|$ there is an element $t \in B^{\varphi=\pi}$ such that $\operatorname{div}(t) = \sum_{n\in\mathbb{Z}}(\varphi^n)^*(y)$. We denote this element by $\Pi(\xi_y)$.

1.5.12. Theorem ("Fundamental exact sequence of p-adic Hodge theory"). — *Let $y \in |Y|$ and $t \coloneqq \Pi(\xi_y) \in B^{\varphi=\pi}$. Then for all $d \geqslant 0$ the following sequence is exact:*

$$0 \longrightarrow E\cdot t^d \longrightarrow B^{\varphi=\pi^d} \longrightarrow B^+_{\mathrm{dR},y}/\xi_y^d B^+_{\mathrm{dR},y} \longrightarrow 0\,.$$

Proof. Injectivity on the left is clear. By construction of $\operatorname{div}(t)$, we see that $\operatorname{ord}_y(t) = 1$, hence the image of $E\cdot t^d$ is contained in the kernel of $B^{\varphi=\pi^d} \to B^+_{\mathrm{dR},y}/\xi_y^d B^+_{\mathrm{dR},y}$. Conversely, if $x \in B^{\varphi=\pi} \cap \xi_y^d B^+_{\mathrm{dR},y}$, then $\operatorname{div}(x) \geqslant dy$. But $\operatorname{div}(x)$ is φ-invariant, hence we even get $\operatorname{div}(x) \geqslant d\operatorname{div}(t) = \operatorname{div}(t^d)$, so $x \in E\cdot t^d$. This shows exactness at $B^{\varphi=\pi^d}$.

So it remains to show surjectivity of the third arrow. We claim that it suffices to deal with the case $d = 1$. Indeed, suppose $B^{\varphi=\pi} \twoheadrightarrow C_y$ is surjective. For all $c \in C_y$ let $t_c \in B^{\varphi=\pi}$ be a preimage; in particular, t_1 maps to 1. Now suppose $x \in B_{\mathrm{dR},y}^+/\xi_y^d B_{\mathrm{dR},y}$ is given. If c_0 is the image of x in C_y, then $t_{c_0} \cdot t_1^{d-1} \in B^{\varphi=\pi^d}$ is an element whose image approximates x up to a multiple of ξ_y, say,

$$x - t_{c_0} \cdot t_1^{d-1} \equiv c_1 \xi_y \mod \xi_y^2 B_{\mathrm{dR},y}^+$$

for some $c_1 \in C_y$. By construction we have $\mathrm{ord}_y(t) = 1$, hence $t \equiv u\xi_y \mod \xi_y^2 B_{\mathrm{dR},y}^+$ for some $u \in C_y \setminus \{0\}$. Now $t_{u^{-1}c_1} \cdot t \cdot t_1^{d-2}$ is an element of $B^{\varphi=\pi^d}$ and satisfies

$$x - t_{c_0} \cdot t_1^{d-1} - t_{u^{-1}c_1} \cdot t \cdot t_1^{d-2} \equiv c_2 \xi_y^2 \mod \xi_y^3 B_{\mathrm{dR},y}^+$$

for some $c_2 \in C_y$. Continuing in this fashion, we obtain the desired surjectivity.

Thus we may assume $d = 1$. For simplicity, we finish the proof of surjectivity only for the case $E = \mathbb{Q}_p$ (the general case needs Lubin–Tate theory). In this case the assertion follows from Lemma 1.5.13 below. □

1.5.13. Lemma. — *Suppose $E = \mathbb{Q}_p$. Let $\varepsilon = (1, \zeta_p, \zeta_{p^2}, \dots) \in \mathcal{O}_{C_y}^\flat \cong \mathcal{O}_F$ be a compatible system of non-trivial $(p^n)^{th}$ roots of unity. Let $t = \log[\varepsilon] \in B^{\varphi=p}$. Then there exists a commutative diagram*

$$\begin{tikzcd}
1 \arrow[r] & \varepsilon^{\mathbb{Q}_p} \arrow[r] \arrow[d, "\wr"] & 1 + \mathfrak{m}_F \arrow[r] \arrow[d, "\log[-]"', "\wr"] & C_y \arrow[r] \arrow[d, equal] & 0 \\
1 \arrow[r] & \mathbb{Q}_p \cdot t \arrow[r] & B^{\varphi=p} \arrow[r, "\theta_y"] & C_y \arrow[r] & 0
\end{tikzcd}$$

Proof. In the above diagram, $\log[-]$ denotes the power series defined as in Remark 1.3.15(2) by $\log(1-x) = \sum_{i=1}^\infty (-1)^{i-1} x^i/i$. An element $u \in 1 + \mathfrak{m}_F$ satisfies $v_r(1-u) > 0$ for all $r \in (0,\infty)$, so its easy to check that the partial sums of this power series form a Cauchy sequence in the v_r-topology on B^b. Thus the series converges. Moreover, we check $\varphi \log(u) = \log(\varphi(u)) = \log(u^p) = p\log(u)$, so $\log[-]\colon 1 + \mathfrak{m}_F \to B^{\varphi=p}$ is well-defined.

In view of Theorem 1.5.12 have to show surjectivity of $\theta_y\colon B^{\varphi=p} \to C_y$. Consider the commutative diagram

$$\begin{tikzcd}
1 + \mathfrak{m}_F \arrow[r, "\log[-]"] \arrow[d, two heads, "(-)^\sharp"'] & B^{\varphi=p} \arrow[d, "\theta_y"] \\
1 + \mathfrak{m}_{C_y} \arrow[r, two heads, "\log"] & C_y
\end{tikzcd}.$$

The left vertical arrow is surjective since C_y is algebraically closed (so $(-)^\sharp\colon \mathcal{O}_F \twoheadrightarrow \mathcal{O}_C$ is surjective, and it's easy to check that $1 + \mathfrak{m}_F$ is the preimage of $1 + \mathfrak{m}_{C_y}$). The bottom horizontal arrow is surjective as its image is p-divisible (because C_y admits p^{th} roots) and open (because for all $c \in C_y$ in the image, the power series defining exp converges on $c + p^n\mathcal{O}_{C_y}$ for sufficiently large n, so exp defines a local inverse). This shows surjectivity of $\theta_y\colon B^{\varphi=p} \to C_y$, which is all we need for Theorem 1.5.12.

Nevertheless, to finish the proof of Lemma 1.5.13, we also need to show exactness of the top row. But the top row is the inverse limit of

$$1 \longrightarrow \mu_{p^\infty}(C_y) \longrightarrow 1 + \mathfrak{m}_{C_y} \xrightarrow{\log} C_y \longrightarrow 0$$

along multiplication/exponentiation by p. Here we use $\lim_{x\mapsto x^p}\mu_{p^\infty}(C_y)\cong \varepsilon^{\mathbb{Q}_p}$, the left-hand side being isomorphic to $\mathbb{Q}_p/\mathbb{Z}_p$ and the right-hand side to $\mathbb{Q}_p$. □

1.5.14. Corollary. — *Let $t=\Pi(\xi_y)$ as in Theorem 1.5.12. Then we have an isomorphism of graded rings*

$$P/tP\cong S:=\{f\in C_y[T]\mid f(0)\in E\}\ .$$

In particular, $\operatorname{Proj} P/tP=\{0\}$ *is a single point. We denote by ∞_t its image in* $\operatorname{Proj} P=X$.

Proof. As usual, let $\theta_y\colon B\to C_y$ denote the canonical map (or rather the continuous extension of the canonical map, constructed as in Claim (1) in the proof of Theorem 1.5.2). Then we obtain a morphism of graded rings $P\to S$ by sending

$$\sum_{d\geqslant 0}x_d\longmapsto \sum_{d\geqslant 0}\theta_y(x_d)T^d$$

(the left-hand side denotes a decomposition of an element of $P=\bigoplus_{d\geqslant 0}P_d$ into homogeneous components). As $\theta_y(t)=0$, this descends to a graded ring morphism $P/tP\to S$. By Lemma 1.5.7 this is an isomorphism in degree 0, and surjective by Theorem 1.5.12.

To see injectivity, suppose $x\in P_d$ satisfies $\theta_y(x)=0$. Then x is divisible by ξ_y in $B^+_{\mathrm{dR},y}$. Using $\operatorname{ord}_y(t)=1$ and the surjectivity part of the fundamental sequence, we see that we can write $x\equiv t't \mod \xi_y^dB^+_{\mathrm{dR},y}$ for some $t'\in P_{d-1}$. By the fundamental sequence again, we obtain $x-t't\in Et^d$. Hence $x\in tP$, proving injectivity.

It remains to show $\operatorname{Proj} P/tP=\{0\}$. Suppose $\mathfrak{p}$ is a homogeneous prime ideal of P/tP and $cT^d\in\mathfrak{p}$ for some $c\in C_y\setminus\{0\}$, $d\geqslant 1$. Then $T^{d+1}=c^{-1}T\cdot cT^d$ is an element of $\mathfrak{p}$ since the first factor is an element of S and the second is in $\mathfrak{p}$. Thus $T\in\mathfrak{p}$, so $\mathfrak{p}\notin\operatorname{Proj} S$. □

We know P is generated by P_1. Hence for all $n\in\mathbb{Z}$ there are canonical line bundles $\mathcal{O}_X(n)$ on X. These are obtained as the quasi-coherent sheaves associated to the graded modules $P[n]$ defined by $P[n]_d=P_{d+n}$.

1.5.15. Lemma. — *For all $n\in\mathbb{Z}$ we have an isomorphism $H^0(X,\mathcal{O}_X(n))\cong B^{\varphi=\pi^n}$.*

Proof. We have a canonical morphism $B^{\varphi=\pi^d}=P_d\to H^0(X,\mathcal{O}_X(n))$. Since P is graded factorial, it is easy to check that this is an isomorphism. □

Now we can restate and prove our main result, Main Theorem 1.4.9. This finally justifies calling the Fargues–Fontaine curve a *curve*.

1.5.16. Main Theorem (Fargues–Fontaine)**.** — *For any non-zero $t\in P_1=B^{\varphi=\pi}$, the ring $B_t:=P\left[\frac{1}{t}\right]_0=B\left[\frac{1}{t}\right]^{\varphi=1}$ is a PID, and on underlying sets*

$$X=\operatorname{Proj} P=D_+(t)\sqcup V_+(t)=\operatorname{Spec} B_t\sqcup\{\infty_t\}\,.$$

In particular, X is noetherian and regular of Krull dimension 1.

Proof. As in the proof of Theorem 1.5.2, it suffices that B_t is factorial and that each (non-invertible) irreducible element generates a maximal ideal. If $x\in B_t$, then for some $d\geqslant 0$ we have $x=t'/t^d$ with $t'\in P_d$. By Theorem 1.5.1, we can factor $t'=t_1\cdots t_d$ with $t_i\in P_1$. By the previous Corollary 1.5.14, the vanishing set of each t_i/t is either empty or a single closed point. Hence t_i/t is a unit or generates a maximal ideal.

Now pick $t,t'\in P_1\setminus\{0\}$ non-E-collinear (these exist by Theorem 1.5.10 for example). Then $X=\operatorname{Spec} B_t\cup\operatorname{Spec} B_{t'}$ and the theorem follows. □

1.5.17. Lemma. — *Let $|X|$ denote the set of closed points of X. Then—as for $\mathbb{P}^1_E$—there exists bijections*

$$|X| \cong |Y|/\varphi^{\mathbb{Z}} \cong (P_1 \setminus \{0\})/E^\times .$$

Moreover, if $y \in |Y|$ corresponds to $x \in |X|$, then $B^+_{\mathrm{dR},x} \coloneqq \widehat{\mathcal{O}}_{X,x} \cong B^+_{\mathrm{dR},y}$.

Proof.* By Theorem 1.5.10, we get a bijection $|Y|/\varphi^{\mathbb{Z}} \cong (P_1 \setminus \{0\})/E^\times$, as the left-hand side are precisely the "indecomposable" divisors in $\mathrm{Div}^+(|Y|/\varphi^{\mathbb{Z}})$. By Main Theorem 1.5.16 and Corollary 1.5.14, we see that the closed points of X are precisely the points of the form ∞_t for $t \in P_1 \setminus \{0\}$, and moreover t is unique up to $E^\times$. This shows the asserted bijection.

Choose $t \in P_1$ corresponding to y and let $t' \in P_1$ be such that $X = \operatorname{Spec} B_t \cup \operatorname{Spec} B_{t'}$. Then $\mathrm{ord}_y(t) = 1$ and $\mathrm{ord}_y(t') = 0$, so the canonical map $B \to B^+_{\mathrm{dR},y}$ induces maps $B_{t'}/(t/t')^d B_{t'} \to B^+_{\mathrm{dR},y}/\xi_y^d B^+_{\mathrm{dR},y}$ for all $d \geqslant 1$. Taking limits (and using that (t/t') is a maximal ideal in $B_{t'}$) gives a map

$$\widehat{\mathcal{O}}_{X,x} \longrightarrow B^+_{\mathrm{dR},y} .$$

We claim that this is an isomorphism. Indeed, it is a morphism of DVRs mapping the uniformizer t/t' to some uniformizer of $B^+_{\mathrm{dR},y}$, so we only need to check that the induced morphism on residue fields is an isomorphism. But a non-zero map of fields is always injective, so surjectivity suffices. By the fundamental sequence (Theorem 1.5.12), $P_1 \twoheadrightarrow C_y$ surjects onto the residue field of $B^+_{\mathrm{dR},y}$. Since t' maps to a unit in $B^+_{\mathrm{dR},y}$, we see that $t'^{-1}P_1 \twoheadrightarrow C_y$ is still surjective, and $t'^{-1}P_1 \subseteq B_{t'}$. $\square$

1.5.18. Definition. — As usual, let $\mathrm{Div}(X)$ denote the group of *divisors* of X, i.e., the free abelian group on the set of closed points $|X|$.

(1) We define the *degree map* $\deg\colon \mathrm{Div}(X) \to \mathbb{Z}$ by $\deg\left(\sum_{x\in|X|} n_x x\right) = \sum_{x\in|X|} n_x$.

(2) If $f \in K(X)^\times$ is a non-zero element of the function field (i.e., the stalk at the generic point) of X, we put $\mathrm{div}(f) = \sum_{x\in|X|} \mathrm{ord}_x(f)$, where ord_x denotes the valuation of $B^+_{\mathrm{dR},x}$.

1.5.19. Remark. — In the lecture it was pointed out that Definition 1.5.18(1) is actually a rather odd choice of degree map. To see where this comes from, recall that the "real" analogue of the Fargues–Fontaine curve should be $\widetilde{\mathbb{P}}^1_{\mathbb{R}}$. Now for a divisor on $\widetilde{\mathbb{P}}^1_{\mathbb{R}}$ consisting of a single point x, there are two ways to define its degree:

(1) we could put $\deg x = [\kappa(x) : \mathbb{R}]$,

(2) or just $\deg x = 1$.

Option (1) is the one we would expect to be the canonical choice, since it comes from $\widetilde{\mathbb{P}}^1_{\mathbb{R}}$ considered as a curve over $\mathbb{R}$. But in Definition 1.5.18 we actually go with option (2).

1.5.20. Proposition. — *For $f \in K(X)^\times$ we have $\deg(\mathrm{div}(f)) = 0$ (so heuristically speaking "X is proper"). Moreover, the induced map*

$$\deg\colon \mathrm{Pic}(X) \xrightarrow{\sim} \mathbb{Z}$$

is an isomorphism, with inverse given by $n \mapsto \mathcal{O}_X(n)$.

Proof. Without restriction assume $f = t'/t$, with $t', t \in P_1$. Indeed, since X is locally a PID (Main Theorem 1.5.16) and P is graded factorial (Theorem 1.5.1), every element in the function field can be decomposed into a product of elements of the form t/t'. In this case we have $\operatorname{div}(f) = \infty_{t'} - \infty_t$, which is clearly of degree 0.

To see the second assertion, use the short exact sequence

$$0 \longrightarrow \mathbb{Z}\{\mathcal{O}_X(1)\} \longrightarrow \operatorname{Pic}(X) \longrightarrow \operatorname{Pic}(\operatorname{Spec} B_t) \longrightarrow 0$$

(this is rather easy to derive) and the fact that $\operatorname{Pic}(\operatorname{Spec} B_t) = 0$ as B_t is factorial. □

1.5.21. Proposition. — *The cohomology of the twisting sheaves $\mathcal{O}_X(n)$ is given by*

$$H^i\big(X, \mathcal{O}_X(n)\big) = \begin{cases} B^{\varphi = \pi^n} & \text{if } i = 0 \\ 0 & \text{if } i \geqslant 2 \\ 0 & \text{if } i = 1,\ n \geqslant 0 \\ B_{\mathrm{dR},x}^+/(\operatorname{Fil}^{-n} B_{\mathrm{dR},x}^+ + E) & \text{if } i = 1,\ n < 0 \end{cases},$$

where x may be any closed point of X and $\operatorname{Fil}^d B_{\mathrm{dR},x}^+ = t^d B_{\mathrm{dR},x}^+$ for t corresponding to x under the bijection from Lemma 1.5.17.

*Proof**. The case $i = 0$ was done in Lemma 1.5.15. The case $i \geqslant 2$ follows from Grothendieck's theorem on cohomological dimension and the fact that X is one-dimensional, or alternatively via Čech cohomology, using that X can be covered by two affine opens.

For $i = 1$, $n \geqslant 0$, we claim that it suffices to show $H^1(X, \mathcal{O}_X) = 0$. Indeed, choose any $t \in P_1 \setminus \{0\}$ and let $B_{\mathrm{dR}}^+ = B_{\mathrm{dR},x}^+$ for the corresponding point $x = \infty_t$. Then we have an exact sequence

$$0 \longrightarrow \mathcal{O}_X \xrightarrow{t^n} \mathcal{O}_X(n) \longrightarrow t^{-n} B_{\mathrm{dR}}^+/B_{\mathrm{dR}}^+ \longrightarrow 0\,. \tag{1.5.1}$$

The term on the right-hand side is abuse of notation for the corresponding skyscraper sheaf supported on ∞_t. Since a sheaf supported only at a closed point as vanishing higher cohomology, we find that $H^1(X, \mathcal{O}_X) \twoheadrightarrow H^1(X, \mathcal{O}_X(n))$ is surjective, so $H^1(X, \mathcal{O}_X) = 0$ is indeed sufficient to deduce $H^1(X, \mathcal{O}_X(n)) = 0$ as well.

To see $H^1(X, \mathcal{O}_X) = 0$, let $j\colon \operatorname{Spec} B_t \hookrightarrow X$ denote the corresponding open embedding. Now look at the exact sequence

$$0 \longrightarrow \mathcal{O}_X \longrightarrow j_*\mathcal{O}_{\operatorname{Spec} B_t} \longrightarrow B_{\mathrm{dR}}/B_{\mathrm{dR}}^+ \longrightarrow 0\,, \tag{1.5.2}$$

where $B_{\mathrm{dR}} \coloneqq \operatorname{Frac}(B_{\mathrm{dR}}^+)$ denotes the fraction field. Exactness of this sequence can be seen by writing $j_*\mathcal{O}_{\operatorname{Spec} B_t} \cong \operatorname{colim}_{d \geqslant 0} \mathcal{O}_X(d \cdot \infty_t)$ and $B_{\mathrm{dR}}/B_{\mathrm{dR}}^+ \cong \operatorname{colim}_{d \geqslant 0} t^{-d} B_{\mathrm{dR}}^+/B_{\mathrm{dR}}^+$. Since $X = \operatorname{Proj} P$ is separated, the inclusion $j\colon \operatorname{Spec} B_t \hookrightarrow X$ is affine. Hence $H^1(X, j_*\mathcal{O}_{\operatorname{Spec} B_t}) \cong H^1(\operatorname{Spec} B_t, \mathcal{O}_{\operatorname{Spec} B_t}) = 0$. Therefore, taking the long exact cohomology sequence associated to (1.5.2) gives

$$0 \longrightarrow E \longrightarrow B_t \longrightarrow B_{\mathrm{dR}}/B_{\mathrm{dR}}^+ \longrightarrow H^1(X, \mathcal{O}_X) \longrightarrow 0$$

(using $H^0(X, \mathcal{O}_X) \cong P_0 \cong E$ by Lemma 1.5.15 and Lemma 1.5.7). Now the fundamental exact sequence implies that $t^{-d} P_d \twoheadrightarrow t^{-d} B_{\mathrm{dR}}^+/B_{\mathrm{dR}}^+$ is surjective. Since $B_t = \bigoplus_{d \geqslant 0} t^{-d} P_d$ and $B_{\mathrm{dR}}/B_{\mathrm{dR}}^+ \cong \operatorname{colim}_{d \geqslant 0} t^{-d} B_{\mathrm{dR}}^+/B_{\mathrm{dR}}^+$ and surjectivity behaves well under colimits, wee see that $B_t \twoheadrightarrow B_{\mathrm{dR}}/B_{\mathrm{dR}}^+$ must be surjective as well. Thus $H^1(X, \mathcal{O}_X) = 0$, as required.

Now let $n < 0$. In this case we use the short exact sequence

$$0 \longrightarrow \mathcal{O}_X(n) \xrightarrow{t^{-n}} \mathcal{O}_X \longrightarrow B_{\mathrm{dR}}^+/t^{-n} B_{\mathrm{dR}}^+ \longrightarrow 0\,, \tag{1.5.3}$$

which can be derived analogous to (1.5.1). Using $H^1(X,\mathcal{O}_X)=0$ and $H^0(X,\mathcal{O}_X)\cong E$ and $H^0(X,\mathcal{O}_X(n))\cong P_n=0$, we find that the induced sequence on cohomology looks like

$$0\longrightarrow E\longrightarrow B_{\mathrm{dR}}^+/t^{-n}B_{\mathrm{dR}}^+\longrightarrow H^1\big(X,\mathcal{O}_X(n)\big)\longrightarrow 0\,.$$

This immediately implies $H^1(X,\mathcal{O}_X(n))=B_{\mathrm{dR}}^+/(\mathrm{Fil}^{-n}B_{\mathrm{dR}}^+ + E)$, as required. □

1.5.4. The Fargues–Fontaine Curve and $\mathbb{A}_{\mathrm{cris}}$

Assume $E=\mathbb{Q}_p$. In this case we can express X in terms of the crystalline period ring $B_{\mathrm{cris}}^+=\mathbb{A}_{\mathrm{cris}}\big[\frac{1}{p}\big]$. Let's recall the construction of $\mathbb{A}_{\mathrm{cris}}$ from Subsection 1.2.2 first. Fix a non-archimedean algebraically closed extension $C/\mathbb{Q}_p$. We have seen in the proof of Example 1.2.17 that the kernel of $\theta\colon\mathbb{A}_{\mathrm{inf}}\rightarrow\mathcal{O}_C$ is generated by $\xi=p-[p^\flat]$, where $p^\flat=(p,p^{1/p},\dots)\in\mathcal{O}_C^\flat\cong\mathcal{O}_F$. Then

$$\mathbb{A}_{\mathrm{cris}}=\mathbb{A}_{\mathrm{inf}}\left[\frac{[p^\flat]^n}{n!}\;\middle|\;n\in\mathbb{N}\right]_p^\wedge$$

(in our original definition of $\mathbb{A}_{\mathrm{cris}}$ we adjoined divided powers ξ instead; however, p already has divided powers in $\mathbb{A}_{\mathrm{inf}}$, so we may equivalently adoin divided powers of $[p^\flat]$).

1.5.22. Definition. — We define the following variations of the ring B.

(1) Let $B^{b,+}\coloneqq\mathbb{A}_{\mathrm{inf}}\big[\frac{1}{p}\big]$.

(2) For an interval $I\subseteq(0,\infty)$, let B_I^+ denote the completion of $B^{b,+}$ with respect to the valuations $(v_r)_{r\in I}$. This coincides with the closure of $B^{b,+}$ in B_I.

(3) For $I=\{r\}$, we put $B_r^+\coloneqq B_{\{r\}}^+$ for convenience.

(4) Let $B^+=B_{(0,\infty)}^+$ be the ring of "functions on $|Y|$ that extend to the boundary".

1.5.23. Remark. — Note that if $s\leqslant r$, then $v_s(x)\geqslant\frac{s}{r}v_r(x)$ for $x\in B^{b,+}$. This would be false for B^b, but in $B^{b,+}$ it works because $B^{b,+}$ consist of p-power series whose Teichmüller coefficients have non-negative valuation. Thus every v_s-Cauchy sequence is a v_r-Cauchy sequence too, and we get a canonical inclusion $B_s^+\subseteq B_r^+$. In particular, we obtain $B_r^+=B_{(0,r]}^+$.

1.5.24. Lemma. — *Let $a\in\mathfrak{m}_F\setminus\{0\}$ and $r=v_F(a)$. Then*

$$B_r^+=\mathbb{A}_{\mathrm{inf}}\left[\tfrac{[a]}{p}\right]_p^\wedge\left[\tfrac{1}{p}\right]$$

*Proof**. Let A denote the right-hand side. We must show that A is v_r-complete (or more precisely, complete with respect to the obvious continuous extension of v_r to A) and that every v_r-continuous map $B^{b,+}\rightarrow A'$ into a complete topological ring extends uniquely to a map $A\rightarrow A'$.

The latter is quite easy to see: elements $\alpha\in A$ can be (non-uniquely) written as p-Laurent series $\alpha=\sum_{n\gg-\infty}^\infty a_np^n$ for $a_n\in\mathbb{A}_{\mathrm{inf}}\,[[a]/p]$. Now if $B^{b,+}\rightarrow A'$ is given, then the images of a_np^n are determined, so it suffices to see that $(a_np^n)_{n\gg-\infty}$ is a v_r-null sequence (since then the partial sums converge in A', hence we can take their limit as the image of α). But $v_r([a]/p)=0$, hence $v_r(b)\geqslant0$ for all $b\in\mathbb{A}_{\mathrm{inf}}\,[[a]/p]$. Thus $v_r(a_np^n)\geqslant rn$ and we get indeed a v_r-null sequence.

To see that every v_r-Cauchy sequence in A converges, it's enough to check that every series whose terms form a v_r-null sequence is convergent. This will be an immediate consequence of the following claim:

$(*)$ If $\alpha \in A$ is an element such that $v_r(\alpha) \geqslant rn$ for some $n \geqslant 0$, then $\alpha \in p^n \mathbb{A}_{\mathrm{inf}}\,[[a]/p]_p^{\widehat{\ }}$.

To prove $(*)$, write α as a p-Laurent series as above. Also, without restriction, $r(n+1) > v_r(\alpha)$. All terms $a_m p^m$ with $m \geqslant n+1$ may be ignored, so we may assume that $\alpha = bp^{-N}$ for some $b \in \mathbb{A}_{\mathrm{inf}}\,[[a]/p]$ and some $N \geqslant 0$. Increasing N if necessary we may even assume $b \in \mathbb{A}_{\mathrm{inf}}$. Write $b = \sum_{i=0}^{\infty} [b_i] p^i$. As $v_r(bp^{-N}) \geqslant rn$, we obtain $v_F(b_i) \geqslant r(N+n-i)$. As $v_F(a) = r$, we may write $b_i = a^{N+n-i} c_i$ for some $c_i \in \mathcal{O}_F$. Then

$$\alpha = bp^{-N} = p^n \sum_{i=0}^{N+n} [c_i] \left(\frac{[a]}{p}\right)^{N+n-i} + p^{n+1} \sum_{i=N+n+1}^{\infty} [b_i] p^{i-(N+n+1)}\,.$$

Both sums are elements of $\mathbb{A}_{\mathrm{inf}}[[a]/p]$. This shows that α is indeed divisible by p^n, and thus $(*)$ is proved. ☐

The Frobenius φ on $B^{b,+}$ induces an isomorphism $\varphi\colon B_r^+ \xrightarrow{\sim} B_{pr}^+ \subseteq B_r^+$. Moreover, if $v_C\colon C \to \mathbb{R} \cup \{\infty\}$ denotes the valuation of C, then the following lemma holds.

1.5.25. Lemma. — *Let $r = v_C(p)$. Then $B_{pr}^+ \subseteq B_{\mathrm{cris}}^+ \subseteq B_r^+$. Moreover, we have*

$$B^+ = \bigcap_{n=1}^{\infty} \varphi^n B_{\mathrm{cris}}^+ = \bigcap_{n=1}^{\infty} \varphi^n B_r^+\,.$$

In particular, B^+ is the largest subring of B_{cris}^+ on which φ is bijective (or, equivalently, surjective, since it's easy to check that φ is injective on B_{cris}^+).

Proof. Note that $r = v_C(p) = v_F(p^\flat)$. Hence B_{pr}^+ and B_r^+ can be described via Lemma 1.5.24. Concretely, we obtain a chain of inclusions

$$\mathbb{A}_{\mathrm{inf}}\left[\frac{[p^\flat]^p}{p}\right] \subseteq \mathbb{A}_{\mathrm{inf}}\left[\frac{[p^\flat]^n}{n!}\ \middle|\ n \in \mathbb{N}\right] \subseteq \mathbb{A}_{\mathrm{inf}}\left[\frac{[p^\flat]}{p}\right].$$

After p-completion and localization at p (both operations preserve inclusions) this becomes $B_{pr}^+ \subseteq B_{\mathrm{cris}}^+ \subseteq B_r^+$, as required. This already shows that it doesn't matter whether we take the intersection over $\varphi^n B_{\mathrm{cris}}^+$ or $\varphi^n B_r^+$.

Moreover, $\varphi^n\colon B_r^+ \to B_r^+$ has image $B_{p^n r}^+$. Using the observation from Remark 1.5.23, we thus obtain

$$\bigcap_{n=1}^{\infty} \varphi^n B_r^+ = \lim_{n \geqslant 1} B_{p^n r} = B^+\,,$$

where the limit in the middle is taken along the canonical inclusions $B_{p^{n+1}r}^+ \subseteq B_{p^n r}^+$. This finishes the proof. ☐

1.5.26. Proposition. — *We have canonical isomorphisms*

$$P = \bigoplus_{d \geqslant 0} B^{\varphi = p^d} \cong \bigoplus_{d \geqslant 0} (B^+)^{\varphi = p^d} \cong \bigoplus_{d \geqslant 0} (B_{\mathrm{cris}}^+)^{\varphi = p^d}\,.$$

Sketch of a proof. One first checks that if $x \in B^{\varphi=p^d}$, then $\operatorname{Newt}_{(0,\infty)}(x) \geqslant 0$. Indeed, since $\varphi(x) = p^d x$, scaling $\operatorname{Newt}_{(0,\infty)}$ along the y-axis with factor p is the same as a translation by d along the x-axis. Now if $\operatorname{Newt}_{(0,\infty)}(x)$ is not always strictly positive, then it has to cross the x-axis somewhere. But then it must be identically zero by the above symmetry observation.

Moreover, one can show that

$$B^+ = \left\{x \in B \;\middle|\; \operatorname{Newt}_{(0,\infty)}(x) \geqslant 0\right\} .$$

Thus $B^{\varphi=p^d} = (B^+)^{\varphi=p^d}$. Moreover, $B^+_{\mathrm{cris}} = \mathbb{A}_{\mathrm{cris}}\left[\frac{1}{p}\right]$ is p-divisible, hence φ is bijective on the subring $\bigoplus_{d\geqslant 0}(B^+_{\mathrm{cris}})^{\varphi=p^d}$. But since B^+ is the largest subring with this property by Lemma 1.5.25, hence it contains all $(B^+_{\mathrm{cris}})^{\varphi=p^d}$. This shows $(B^+)^{\varphi=p^d} = (B^+_{\mathrm{cris}})^{\varphi=p^d}$ and we are done. $\square$

1.5.27. Remark. — Let $\varepsilon = (1, \zeta_p, \zeta_{p^2}, \dots) \in \mathcal{O}_C^\flat \cong \mathcal{O}_F$ and $t = \log[\varepsilon]$ as in Lemma 1.5.13. Put $B_{\mathrm{cris}} = B^+_{\mathrm{cris}}\left[\frac{1}{t}\right]$ and let $B_e = (B_{\mathrm{cris}})^{\varphi=1}$. Then on underlying topological spaces we can write

$$X = \text{“}\operatorname{Spec} B_e \cup_{\operatorname{Spec} B_{\mathrm{dR}}} \operatorname{Spec} B^+_{\mathrm{dR}}\text{”} .$$

That is, the Fargues–Fontaine curve is obtained by “gluing” the open subset $D_+(t)$ together with the spectrum of the DVR $B^+_{\mathrm{dR}} = B^+_{\mathrm{dR},\infty_t}$ (which has only two points) along their generic points.

CHAPTER 2.

Classification of Vector Bundles on the Fargues–Fontaine Curve

2

2.1. The Vector Bundles $\mathcal{O}_X(\lambda)$

LECTURE 9
8th Jan, 2020

As usual, let $E/\mathbb{Q}_p$ be a finite extension with uniformizer π and residue field $\mathcal{O}_E/\pi\mathcal{O}_E = \mathbb{F}_q$, and let $F/\mathbb{F}_q$ a non-archimedean algebraically closed extension.

2.1.1. The Harder–Narasimhan Formalism

Henceforth, the names Harder–Narasimhan will be abbreviated as HN. Let $\mathcal{C}$ be an exact category; roughly speaking, this is an additive category together with a notion of short exact sequences (for example, the categories $\mathrm{Bun}_{\mathbb{P}^1_k}$ and Bun_X of vector bundles on $\mathbb{P}^1_k$ and on the Fargues–Fontaine curve X respectively). Moreover, we assume there are:

(a) a function $\mathrm{rk}\colon \mathrm{Ob}(\mathcal{C}) \to \mathbb{N}_{\geqslant 0}$ (the *rank function*). In the case where $\mathcal{C}$ is a category of vector bundles on $\mathbb{P}^1_k$ or X, this is just what one would expect.

(b) a function $\deg\colon \mathrm{Ob}(\mathcal{C}) \to \mathbb{Z}$ (the *degree function*). For vector bundles, we would take $\deg\mathcal{E} = \deg(\bigwedge^{\mathrm{rk}\,\mathcal{E}}\mathcal{E})$. We have seen last time in Proposition 1.5.20 that for the Fargues–Fontaine curve X, $\deg\colon \mathrm{Pic}(X) \xrightarrow{\sim} \mathbb{Z}$ is an isomorphism.

Both rk and deghave to be additive on short exact sequences in $\mathcal{C}$. Moreover, we require that there is an exact and faithful functor $F\colon \mathcal{C} \to \mathcal{A}$ (the *generic fibre functor* in the case where $\mathcal{C}$ equals $\mathrm{Bun}_{\mathbb{P}^1_k}$ or Bun_X) into an abelian category $\mathcal{A}$, such that for all $\mathcal{E} \in \mathcal{C}$, the functor F induces a bijection

$$F\colon \{\text{strict subobjects of } \mathcal{E}\} \xrightarrow{\sim} \{\text{subobjects of } F(\mathcal{E})\}\ .$$

Here a *strict subobjects* means a monomorphism $\mathcal{E}' \hookrightarrow \mathcal{E}$ that is part of a short exact sequence in $\mathcal{C}$. Finally, we assume

(1) $\mathrm{rk}\colon \mathrm{Ob}(\mathcal{C}) \to \mathbb{N}_{\geqslant 0}$ is the restriction of another function $\mathrm{rk}\colon \mathrm{Ob}(\mathcal{A}) \to \mathbb{N}_{\geqslant 0}$ along F, which again has to be additive on short exact sequences and satisfies $\mathrm{rk}\,V = 0$ iff $V = 0$ for all $V \in \mathcal{A}$.

(2) If $u\colon \mathcal{E} \to \mathcal{E}'$ is a morphism in $\mathcal{C}$ such that $F(u)$ is an isomorphism, then $\deg\mathcal{E} \leqslant \deg\mathcal{E}'$ with equality iff u is an isomorphism.

2.1.1. Definition. — Let $\mathcal{E} \in \mathcal{C}$ be an arbitrary object.

(1) The *slope of* $\mathcal{E}$ is the (possibly infinite) number $\mu(\mathcal{E}) \coloneqq \deg(\mathcal{E})/\,\mathrm{rk}(\mathcal{E}) \in \mathbb{Q} \cup \{\infty\}$.

(2) $\mathcal{E}$ is called *semistable* if for all non-zero strict subobjects $\mathcal{F} \subseteq \mathcal{E}$ we have $\mu(\mathcal{F}) \leqslant \mu(\mathcal{E})$.

2.1.2. Example. — If $\mathcal{C} = \mathrm{Bun}_X$, then the twisting sheaves $\mathcal{O}_X(n)$ for $n \in \mathbb{Z}$ are semistable line bundles. Moreover, $\mathcal{E} = \mathcal{O}_X(m) \oplus \mathcal{O}_X(n)$ is semistable iff $m = n$. Indeed, its slope is $\mu(\mathcal{E}) = (m+n)/2$. Thus for $n \neq m$, either $\mathcal{O}_X(m)$ or $\mathcal{O}_X(n)$ is a subbundle of higher slope. Conversely, for $m = n$, every non-trivial strict subobject of $\mathcal{E}$ is of the form $\mathcal{O}_X(m') \hookrightarrow \mathcal{O}_X(m)$ for $m' \leqslant m$, hence has slope at most m.

2.1.3. Lemma. — *Let $\mathcal{E}, \mathcal{E}' \in \mathcal{C}$ be semistable objects of slopes λ, λ' respectively. If $\lambda > \lambda'$, then we have*

$$\mathrm{Hom}_{\mathcal{C}}(\mathcal{E}, \mathcal{E}') = 0\,.$$

*Proof**. Let $\mathcal{E} \to \mathcal{E}'$ be a morphism in $\mathcal{C}$. Since $\mathcal{A}$ is abelian, we can splice $F(\mathcal{E}) \to F(\mathcal{E}')$ into short exact sequences $0 \to K \to F(\mathcal{E}) \to K' \to 0$ and $0 \to K' \to F(\mathcal{E}') \to K'' \to 0$. By our assumption on F, the objects K and K' correspond to strict subobjects $\mathcal{K} \subseteq \mathcal{E}$ and $\mathcal{K}' \subseteq \mathcal{E}'$. Since F is exact, these guys satisfy $F(\mathcal{E}/\mathcal{K}) \cong K' \cong F(\mathcal{K}')$ and $F(\mathcal{E}'/\mathcal{K}') \cong K''$. Moreover, we claim that $\mathcal{E} \to \mathcal{E}'$ factors over $\mathcal{E}/\mathcal{K}$. Indeed, what we need to prove is that $\mathcal{K} \to \mathcal{E}'$ is the zero morphism. Since F is faithful, this may be checked after applying F, and for $K \cong F(\mathcal{K}) \to F(\mathcal{E}')$ this is clearly true. In the same way we show that $\mathcal{E} \to \mathcal{E}'$ factors over $\mathcal{K}'$.

Hence we get a canonical morphism $\mathcal{E}/\mathcal{K} \to \mathcal{K}'$. By construction, this becomes an isomorphism after applying F, so $\deg(\mathcal{E}/\mathcal{K}) \leqslant \deg(\mathcal{K}')$ and $\mathrm{rk}(\mathcal{E}/\mathcal{K}) = \mathrm{rk}(\mathcal{K}')$ by the above properties. In particular, we have $\mu(\mathcal{E}/\mathcal{K}) \leqslant \mu(\mathcal{K}')$. But $\mathcal{E}'$ and $\mathcal{E}$ are semistable, hence $\mu(\mathcal{K}') \leqslant \lambda'$ and $\mu(\mathcal{K}) \leqslant \lambda$, except for $\mathcal{K}' = 0$ (in which case we are done) or $\mathcal{K} = 0$ (which leads to $\lambda = \mu(\mathcal{E}) = \mu(\mathcal{E}/\mathcal{K}) \leqslant \lambda'$, a contradiction).

So if no of these two special cases occurs, we get $\mu(\mathcal{K}) \leqslant \lambda$ and $\mu(\mathcal{E}/\mathcal{K}) < \lambda$. This however contradicts Lemma* 2.1.4 below. □

2.1.4. Lemma*. — *For any short exact sequence $0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to 0$ in $\mathcal{C}$ we have*

$$\min\{\mu(\mathcal{E}'), \mu(\mathcal{E}'')\} \leqslant \mu(\mathcal{E}) \leqslant \max\{\mu(\mathcal{E}'), \mu(\mathcal{E}'')\}\,.$$

For the left half, equality holds iff $\mu(\mathcal{E}') = \mu(\mathcal{E}'')$ or one of $\mathcal{E}'$, $\mathcal{E}''$ is zero. Equality on the right holds iff $\mu(\mathcal{E}') = \mu(\mathcal{E}'')$.

*Proof**. Put $d' = \deg(\mathcal{E}')$, $d'' = \deg(\mathcal{E}'')$ and $r' = \mathrm{rk}(\mathcal{E}')$, $r'' = \mathrm{rk}(\mathcal{E}'')$. By additivity of deg and rk on short exact sequences, we obtain

$$\mu(\mathcal{E}) = \frac{d' + d''}{r' + r''} = \frac{r'}{r' + r''} \cdot \mu(\mathcal{E}') + \frac{r''}{r' + r''} \cdot \mu(\mathcal{E}'')\,.$$

Thus, $\mu(\mathcal{E})$ is a convex combination of $\mu(\mathcal{E}')$ and $\mu(\mathcal{E}'')$ and the inequality as well as the discussion of equality cases follow rather easily. □

2.1.5. Theorem. — *Each $\mathcal{E} \in \mathcal{C}$ has a unique functorial filtration, called "HN-filtration", of the form*

$$0 = \mathcal{E}_0 \subsetneq \mathcal{E}_1 \subsetneq \cdots \subsetneq \mathcal{E}_r = \mathcal{E}$$

such that $\mathcal{E}_i/\mathcal{E}_{i-1}$ is semistable for $i = 1, \dots, r$ and the sequene of slopes $\mu(\mathcal{E}_i/\mathcal{E}_{i-1})$ is strictly decreasing.

Sketch of a proof. Before we start with the proof, we remark that Lemma* 2.1.4 shows $\mu(\mathcal{E}_1) > \mu(\mathcal{E}_2) > \ldots \geqslant \mu(\mathcal{E})$ for any HN-filtration of $\mathcal{E}$ as above.

If $F(\mathcal{E})$ is simple in $\mathcal{A}$, i.e., has no non-zero subobjects, then $\mathcal{E}$ has no non-zero strict subobjects by our assumption on F, hence $\mathcal{E}$ is semistable for trivial reasons. Then $\mathcal{E}$ is its own HN-filtration. Thus we may assume that $F(\mathcal{E})$ is non-simple, so there exists a short exact sequence $0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to 0$ with $\operatorname{rk}\mathcal{E}', \operatorname{rk}\mathcal{E}'' < \operatorname{rk}\mathcal{E}$. Using induction, we may assume that $\mathcal{E}'$ and $\mathcal{E}''$ have HN-filtrations. We claim:

$(*)$ The slopes of strict subobjects of $\mathcal{E}$ are bounded.

To prove $(*)$, we first use the above observation to see that it suffices to bound the slopes of stric semistable subobject, because if $\mathcal{F} \subseteq \mathcal{E}$ is a strict subobject, then it has an HN-filtration by the induction hypothesis, hence $\mathcal{F}_1$ is a strict semistable subobject of $\mathcal{E}$ satisfying $\mu(\mathcal{F}_1) \geqslant \mu(\mathcal{F})$.[1] So w.l.o.g. $\mathcal{F} = \mathcal{F}_1$. Let $0 = \mathcal{E}''_0 \subsetneq \cdots \subsetneq \mathcal{E}''_r = \mathcal{E}''$ be the HN-filtration of $\mathcal{E}''$. If $\mathcal{F} \to \mathcal{E} \to \mathcal{E}''_r/\mathcal{E}''_{r-1}$ is non-zero, then $\mu(\mathcal{F}) \leqslant \mu(\mathcal{E}''_r/\mathcal{E}''_{r-1})$ by Lemma 2.1.3. Otherwise, $\mathcal{F} \to \mathcal{E}''$ factors over $\mathcal{E}''_{r-1}$. Repeating this argument shows $\mu(\mathcal{F}) \leqslant \mu(\mathcal{E}''_{r-1}/\mathcal{E}''_{r-2})$ or $\mathcal{F} \to \mathcal{E}''$ factors over $\mathcal{E}''_{r-2}$, and so on. So all in all $\mu(\mathcal{F})$ is bounded by the slopes of the HN-filtration of $\mathcal{E}''$ or $\mathcal{F} \to \mathcal{E}''$ is zero. But in that case $\mathcal{F} \to \mathcal{E}$ factors over $\mathcal{E}'$. Then the argument can be repeated with $\mathcal{E}'$, showing that $\mu(\mathcal{F})$ is bounded by the slopes of the HN-filtration of $\mathcal{E}'$, or $\mathcal{F}$ itself is zero, which is of course excluded. This proves $(*)$.

Take $\mathcal{E}_1 \subseteq \mathcal{E}$ a strict subobject of maximal slope, whose rank is also maximal among all strict subobjects of maximal slope. Such an $\mathcal{E}_1$ exists since the strict subobjects of $\mathcal{E}$ can have rank at most $\operatorname{rk}\mathcal{E}$ by additivity of rk, so the denominators are bounded above. As noted above, $\mathcal{E}_1$ is necessarily semistable. Moreover, $\mathcal{E}/\mathcal{E}_1$ has a HN-filtration $0 = \mathcal{F}_0 \subsetneq \cdots \subsetneq \mathcal{F}_r = \mathcal{E}/\mathcal{E}_1$ by the induction hypothesis. For all $i \geqslant 2$ let $\mathcal{E}_i$ be the kernel of $\mathcal{E} \to \mathcal{F}_r/\mathcal{F}_{i-1}$.[2] Then $\mathcal{E}_1$ and $\mathcal{E}_{i+1}/\mathcal{E}_i \cong \mathcal{F}_i/\mathcal{F}_{i-1}$ for $i \geqslant 1$ are semistable and the slopes of the latter are strictly decreasing, so all that's left to prove is $\mu(\mathcal{E}_1) > \mu(\mathcal{E}_2/\mathcal{E}_1)$. By Lemma* 2.1.4 it suffices to show $\mu(\mathcal{E}_1) > \mu(\mathcal{E}_2)$. But $\mathcal{E}_2$ has larger rank than $\mathcal{E}_1 \subsetneq \mathcal{E}_2$, hence its slope must be strictly smaller by construction of $\mathcal{E}_1$.

It remains to show uniqueness and functoriality. Suppose $0 = \mathcal{E}_0 \subsetneq \cdots \subsetneq \mathcal{E}_r = \mathcal{E}$ and $0 = \mathcal{E}'_0 \subsetneq \cdots \subsetneq \mathcal{E}'_s = \mathcal{E}$ are different HN-filtrations. Without restriction $\mu(\mathcal{E}'_1) \geqslant \mu(\mathcal{E}_1)$. By Lemma 2.1.3, the morphism $\mathcal{E}'_1 \hookrightarrow \mathcal{E} \to \mathcal{E}_r/\mathcal{E}_{r-1}$ must be zero, hence $\mathcal{E}'_1 \hookrightarrow \mathcal{E}$ factors over $\mathcal{E}_{r-1}$. Iterating this argument we obtain that it even factors over $\mathcal{E}_1$. Thus $\mu(\mathcal{E}'_1) \leqslant \mu(\mathcal{E}_1)$ by Lemma* 2.1.4 again. So equality must hold we can apply the same argument to $\mathcal{E}_1$, ultimately obtaining that $\mathcal{E}_1 \hookrightarrow \mathcal{E}$ and $\mathcal{E}'_1 \hookrightarrow \mathcal{E}$ factor over each other. Then $\mathcal{E}_1 = \mathcal{E}'_1$. Now we can repeat the argument for $\mathcal{E}/\mathcal{E}_1$. I'm not so sure what "functoriality" means, but it certainly also follows from Lemma 2.1.3. □

2.1.6. Definition. — The *HN-polygon* $\operatorname{HN}(\mathcal{E})$ of an object $\mathcal{E} \in \mathcal{C}$ is the unique polygon in $\mathbb{R}^2$ with origin $(0,0)$ and slopes $\mu(\mathcal{E}_i/\mathcal{E}_{i-1})$ with multiplicity $\operatorname{rk}(\mathcal{E}_i/\mathcal{E}_{i-1})$. In particular, $\mathcal{E}$ is semistable iff the HN-polygon is a straight line.

The following theorem wasn't mentioned in the lecture. I thought it would fix a later argument that went quite wrong. Turns out it doesn't, but by that time I had already typed the proof.

2.1.7. Theorem*. — *If $\mathcal{F} \subseteq \mathcal{E}$ is a strict subobject, then the HN-polygon $\operatorname{HN}(\mathcal{F})$ lies below $\operatorname{HN}(\mathcal{E})$. In particular, $\operatorname{HN}(\mathcal{E})$ is the upper concave hull of the points $(\operatorname{rk}(\mathcal{F}), \deg(\mathcal{F})) \in \mathbb{R}^2$, where $\mathcal{F}$ ranges over all strict subobjects of $\mathcal{F}$.*

[1] By the way, here we use that compositions of strict subobjects are strict subobjects again. We didn't mention this in our "definition" of exact categories, but it's actually one of the axioms.

[2] This kernel exists because compositions of *strict quotients*, i.e., epimorphisms $\mathcal{E} \to \mathcal{E}''$ that are part of a short exact sequence, are strict quotients again. This is another axiom we weren't told.

Proof.* By additivity of deg and rk we see that the break points of $\mathrm{HN}(\mathcal{E})$ are precisely the points $(\mathrm{rk}(\mathcal{E}_i), \deg(\mathcal{E}_i))$ for $i = 0, \ldots, r$. We prove the theorem by induction on the length s of the HN-filtration $0 = \mathcal{F}_0 \subsetneq \cdots \subsetneq \mathcal{F}_s = \mathcal{F}$. The case $s = 0$ is trivial. Now assume $s \geqslant 1$ and let i be the minimal index such that $\mathcal{F}_{s-1} \to \mathcal{E}$ factors over $\mathcal{E}_i$. Let $j > i$ be minimal such that $\mathcal{F}_s \to \mathcal{E}$ factors over $\mathcal{E}_j$. Then $\mathcal{F}_s \to \mathcal{E}_j/\mathcal{E}_{j-1}$ is non-zero, and since the image of $\mathcal{F}_{s-1}$ is contained in $\mathcal{E}_i \subseteq \mathcal{E}_{j-1}$, we get a non-zero morphism $\mathcal{F}_s/\mathcal{F}_{s-1} \to \mathcal{E}_j/\mathcal{E}_{j-1}$. Hence $\mu(\mathcal{F}_s/\mathcal{F}_{s-1}) \leqslant \mu(\mathcal{E}_k/\mathcal{E}_{k-1})$ for all $k \leqslant j$ by Lemma 2.1.3 and the fact that the sequence of $\mu(\mathcal{E}_k/\mathcal{E}_{k-1})$ is strictly decreasing.

Now $\mathrm{HN}(\mathcal{F})$ is obtained by attaching a line of slope $\mu(\mathcal{F}_s/\mathcal{F}_{s-1})$ to $\mathrm{HN}(\mathcal{F}_{s-1})$. Moreover, its endpoint $(\mathrm{rk}(\mathcal{F}), \deg(\mathcal{F}))$ has x-coordinate $\mathrm{rk}(\mathcal{F}) \leqslant \mathrm{rk}(\mathcal{E}_j)$. So $\mathrm{HN}(\mathcal{F}_{s-1})$ lies below $\mathrm{HN}(\mathcal{E}_i)$, and the single segment that is attached to it has smaller slope than all segments of $\mathrm{HN}(\mathcal{E}_j)$. Thus $\mathrm{HN}(\mathcal{F})$ lies below $\mathrm{HN}(\mathcal{E}_j)$ and therefore also below $\mathrm{HN}(\mathcal{E})$.

In particular, we see that $(\mathrm{rk}(\mathcal{F}), \deg(\mathcal{F}))$ lies below $\mathrm{HN}(\mathcal{E})$. But the break points of $\mathrm{HN}(\mathcal{E})$ are of this form too as seen above, and $\mathrm{HN}(\mathcal{E})$ is concave by construction, thus it is indeed the upper concave hull of all points of the given form. This finishes the proof. ☐

2.1.8. Proposition. — *Let $\lambda \in \mathbb{Q}$. Then the full subcategory*

$$\mathcal{C}_\lambda^{\mathrm{sst}} = \{\mathcal{E} \in \mathcal{C} \mid \mathcal{E} \textit{ semistable}, \ \mu(\mathcal{E}) \in \{\lambda, \infty\}\}$$

is abelian and every object in it is of finite length.

Proof.* From Lemma* 2.1.4 we get that direct sums (and moreover, arbitrary extensions) of objects in $\mathcal{C}_\lambda^{\mathrm{sst}}$ are in $\mathcal{C}_\lambda^{\mathrm{sst}}$ again. Next we construct kernels and cokernels of morphisms $\mathcal{E} \to \mathcal{E}'$. This is trivial if $\mathcal{E} = 0$ or $\mathcal{E}' = 0$, so we may assume $\mu(\mathcal{E}) = \lambda = \mu(\mathcal{E}')$. Let $\mathcal{K}$ and $\mathcal{K}'$ be as in the proof of Lemma 2.1.3. As was observed there, we have $\mu(\mathcal{K}) \leqslant \lambda$ and $\mu(\mathcal{E}/\mathcal{K}) \leqslant \mu(\mathcal{K}') \leqslant \lambda$ (except in the special cases where one of them is zero, but these are easily handled). But then by Lemma* 2.1.4 equality must hold everywhere. In particular, since $\mathrm{rk}(\mathcal{E}/\mathcal{K}) = \mathrm{rk}(\mathcal{K}')$ we must also have $\deg(\mathcal{E}/\mathcal{K}) = \deg(\mathcal{K}')$, hence $\mathcal{E}/\mathcal{K} \xrightarrow{\sim} \mathcal{K}'$ is an isomorphism by assumption (2).

Therefore $\mathcal{K}$ and $\mathcal{E}/\mathcal{K} \cong \mathcal{K}'$ are strict subobjects of $\mathcal{E}$ and $\mathcal{E}'$ of the same slope, hence they are semistable too. So $\mathcal{K}, \mathcal{K}' \in \mathcal{C}_\lambda^{\mathrm{sst}}$. Another application of Lemma* 2.1.4 shows that $\mu(\mathcal{E}'/\mathcal{K}')$ must be λ or $\mathcal{E}'/\mathcal{K}' = 0$. In the latter case $\mathcal{E}'/\mathcal{K}'$ is an element of $\mathcal{C}_\lambda^{\mathrm{sst}}$ for trivial reasons. So assume the former is the case and let $\mathcal{F} \subseteq \mathcal{E}'/\mathcal{K}'$ be a strict subobject. Let $\mathcal{F}' \subseteq \mathcal{E}'$ be the kernel of $\mathcal{E} \to (\mathcal{E}'/\mathcal{K}')/\mathcal{F}$ (this exists by the argument from the proof of Theorem 2.1.5). Then $\mu(\mathcal{F}') \leqslant \lambda$. But now the short exact sequence[3] $0 \to \mathcal{K}' \to \mathcal{F}' \to \mathcal{F} \to 0$ together with $\mu(\mathcal{K}') = \lambda \geqslant \mu(\mathcal{F}')$ implies that $\mu(\mathcal{F}) \leqslant \mu(\mathcal{F}') \leqslant \lambda = \mu(\mathcal{E}'/\mathcal{K}')$ by Lemma* 2.1.4. This finally shows that $\mathcal{E}'/\mathcal{K}'$ is semistable.

Thus, $\mathcal{E} \to \mathcal{E}'$ has a kernel and a cokernel in $\mathcal{C}_\lambda^{\mathrm{sst}}$, and moreover the morphism from its coimage $\mathcal{E}/\mathcal{K}$ to its image $\mathcal{K}'$ is an isomorphism. We conclude that $\mathcal{C}_\lambda^{\mathrm{sst}}$ is abelian. It remains to show that any $\mathcal{E} \in \mathcal{C}_\lambda^{\mathrm{sst}}$ has finite length. In fact, we will show that $\mathcal{E}$ has length $\mathrm{rk}(\mathcal{E})$. So suppose $\mathcal{E}_1 \subsetneq \cdots \subsetneq \mathcal{E}_r \subsetneq \mathcal{E}$ is a chain of subobjects of length $r > \mathrm{rk}(\mathcal{E})$. By the pidgeonhole principle there must be an i with $\mathrm{rk}(\mathcal{E}_i) = \mathrm{rk}(\mathcal{E}_{i+1})$. Then $F(\mathcal{E}_i) \xrightarrow{\sim} F(\mathcal{E}_{i+1})$ must be an isomorphism by assumption (1). Then from assumption (2) we get $\deg(\mathcal{E}_i) = \deg(\mathcal{E}_{i+1})$. Since $\mathcal{E}_i$ and $\mathcal{E}_{i+1}$ have the same rank, hence the same slope (either λ or ∞), we get equality and $\mathcal{E}_i \subseteq \mathcal{E}_{i+1}$ must be an isomorphism, contradicting $\mathcal{E}_i \subsetneq \mathcal{E}_{i+1}$. ☐

[3] Here we are veiling a not so trivial detail (and we already did this in the proof of Theorem 2.1.5): that $\mathcal{K}'$ is indeed a strict subobject of $\mathcal{F}'$. This follows formally from the axioms (that were never given), but that's a bit fiddly.

2.1.9. Example. — Suppose $\mathcal{C}$ is the category of vector bundles on $\mathbb{P}^1_k$. By the Grothendieck–Birkhoff theorem, every vector bundle $\mathcal{E}$ can be written uniquely as

$$\mathcal{E} \cong \bigoplus_{i=1}^{r} \mathcal{O}(d_i)^{\oplus n_i}\,,$$

where $d_1 > d_2 > \cdots > d_j$ and all $n_i > 0$. Then the j^{th} piece $\mathcal{E}_j$ of the HN-filtration of $\mathcal{E}$ is given by $\bigoplus_{i=1}^{j} \mathcal{O}(d_i)^{\oplus n_i}$. Moreover, for $\lambda \in \mathbb{Z}$ the category $\mathcal{C}_\lambda^{\text{sst}}$ is the full subcategory of vector bundles isomorphic to a finite direct sum of copies of $\mathcal{O}(\lambda)$. In particular, $\mathcal{C}_\lambda^{\text{sst}}$ is equivalent to Vect_k.

2.1.2. φ-Modules and the Vector Bundles $\mathcal{O}_X(\lambda)$

2.1.10. Definition. — Let $\breve{E} = W_{\mathcal{O}_E}(\overline{\mathbb{F}}_q)\big[\frac{1}{\pi}\big]$, where $\overline{\mathbb{F}}_q$ is the algebraic closure of $\mathbb{F}_q$ inside F. Note that $\breve{E}$, being a localization of a ring of Witt vectors, comes equipped with a natural Frobenius action φ.

2.1.11. Remark. — We observe that $\breve{E}$ is a field. In fact, it is the completion of the maximal unramified extension of E (this follows more or less from Proposition 1.1.1). Moreover, since $\overline{\mathbb{F}}_q \subseteq F$, $\breve{E}$ is naturally a subring of B.

2.1.12. Definition. — Let A be a ring with an endomorphism $\varphi\colon A \to A$. A *φ-module* over A is a pair (M, φ_M), where M is a finite projective A-module and $\varphi_M\colon M \xrightarrow{\sim} M$ is φ-semilinear isomorphism. The category of φ-modules is denoted $\varphi\text{-}\mathrm{Mod}_A$.

In Definition 2.1.12, recall that a φ-semilinear isomorphism is an isomorphism of underlying abelian groups that satisfies $\varphi_M(am) = \varphi(a)\varphi_M(m)$ for all $a \in A$, $m \in M$. If M is even a free A-module and $e_1, \dots, e_n \in M$ a basis, one can write

$$\varphi_M(e_i \otimes 1) = \sum_{j=1}^{n} a_{i,j} e_j\,,$$

and we obtain a matrix $a = (a_{i,j}) \in \mathrm{GL}_n(A)$. Changing $e_1, \dots, e_n$ according to an invertible matrix $g \in \mathrm{GL}_n(A)$ transforms a into $ga\varphi(g)^{-1}$ (this operation is called "φ-conjugation"). Thus, we get a bijection

$$\{\text{iso. classes of free rank } n \ \varphi\text{-modules}\} \xrightarrow{\sim} \mathrm{GL}_n(A)/\varphi\text{-conj.}\,.$$

From now on, we consider the category $\mathcal{C} = \varphi\text{-}\mathrm{Mod}_{\breve{E}}$, where φ is the ordinary Frobenius (at least if $E/\mathbb{Q}_p$ is unramified, this is also known as the category of *isocrystals*). Since $\breve{E}$ is a field, all φ-modules over $\breve{E}$ are free. Hence the above bijection provides a map

$$\deg\colon \{\text{iso. classes of rank-1 } \varphi\text{-modules}\} \cong \breve{E}^\times/\varphi\text{-conj.} \xrightarrow{\sim} \mathbb{Z}\,;$$

the isomorphism on the right-hand side is induced by the valuation on $\breve{E}$ and it is an isomorphism because for $a, b \in \breve{E}$, trying to find a $g \in \mathcal{O}_{\breve{E}}^\times$ with $b = ga\varphi(g)^{-1}$ leads to a list of polynomial equations in the Teichmüller coefficients of g, which always have solutions in the algebraically closed field $\overline{\mathbb{F}}_q$ as long as a and b have the same valuation. For arbitrary $M \in \mathcal{C}$

$$\operatorname{rk} M = \dim_{\breve{E}} M \quad \text{and} \quad \deg M = \deg\left(\bigwedge^{\operatorname{rk} M} M\right)$$

Also let F be simply the identity functor on $\mathcal{C}$. Then one checks that all conditions from Subsection 2.1.1 are satisfied, so the HN-formalism is available for $(\mathcal{C}, \mathrm{rk}, \deg, \mathrm{id}_\mathcal{C})$! Moreover, since $\varphi\text{-Mod}_{\breve{E}}$ is already abelian F is the identity functor, it's easily checked that $(\mathcal{C}, \mathrm{rk}, -\deg, \mathrm{id}_\mathcal{C})$ satisfies the conditions too. Therefore we actually have two HN-structures on $\mathcal{C}$! This has interesting consequences.

(1) Every HN-filtration $0 = \mathcal{E}_0 \subsetneq \cdots \subsetneq \mathcal{E}_r = \mathcal{E}$ in $\mathcal{C}$ is canonically split, so that there is a canonical isomorphism $\mathcal{E} \cong \bigoplus_{i=1}^r \mathcal{E}_i/\mathcal{E}_{i-1}$. In fact, the splitting is induced by the second filtration $0 = \mathcal{E}_0' \subsetneq \cdots \subsetneq \mathcal{E}_r' = \mathcal{E}$ associated to the second HN-structure. That is, for all $j = 1, \dots, r$ we have $\mathcal{E}_j' = \bigoplus_{i=r-j+1}^r \mathcal{E}_i/\mathcal{E}_{i-1}$.

(2) If $\mathcal{E}$ and $\mathcal{E}'$ are semistable of different slopes, then $\mathrm{Hom}_\mathcal{C}(\mathcal{E}, \mathcal{E}') = 0$.

2.1.13. Warning*. — Neither (1) nor (2) are as easy as the lecture made them sound. For example, (2) seemingly follows immediately from Lemma 2.1.3, but the major obstacle here is to show that the *semistable objects are the same in both HN-structures*!

What saves our *sses here (and only here, we really need that we are in the particular case where $\mathcal{C} = \varphi\text{-Mod}_{\breve{E}}$!) is the *Dieudonné–Manin decomposition* (see Theorem 2.1.14 below). It can be checked by hand that this decompositions induces two split filtrations which have the property from Theorem 2.1.5 for the respective HN-structures. So all in all, (1) and (2) are a consequence of the Dieudonné–Manin decomposition, not the other way around.

Let $\lambda = d/r \in \mathbb{Q}$, where d and r are coprime integers and $r > 0$. Let $D(\lambda)$ be the φ-module over $\breve{E}$ whose underlying module is $\breve{E}^{\oplus r}$ and with associated matrix

$$\varphi_{D(\lambda)} = \begin{pmatrix} 0 & \cdots & 0 & \pi^d \\ 1 & & & 0 \\ \vdots & \ddots & & \vdots \\ 0 & \cdots & 1 & 0 \end{pmatrix}$$

2.1.14. Theorem (Dieudonné–Manin classification). — *The category $\mathcal{C} = \varphi\text{-Mod}_{\breve{E}}$ is semisimple and its simple objects are precisely those which are isomorphic to $D(\lambda)$ for $\lambda \in \mathbb{Q}$ (in particular, every φ-module M has a unique decomposition $M = \bigoplus_{i=1}^r D(\lambda_i)^{\oplus n_i}$). Moreover, the division algebra $\mathrm{End}_\mathcal{C}(D(\lambda))$ over E is central (i.e. its center is E) of invariant $\pm[\lambda] \in \mathrm{Br}\, E \cong \mathbb{Q}/\mathbb{Z}$.*

Sketch of a proof. Some technical arguments including the HN-formalism, passage to unramified coverings of E (thus replacing φ by φ^h) and twisting reduces the theorem to its essential part:

$(*)$ Every semistable φ-module D over $\breve{E}$ of slope 0 is a direct sum of copies of $D(0) = (\breve{E}, \varphi)$.

By inspection, $\mathrm{Ext}^1_\mathcal{C}(D(0), D(0)) \cong \breve{E}/(\varphi - \mathrm{id})\breve{E}$. But $\mathcal{O}_{\breve{E}}/\pi\mathcal{O}_{\breve{E}} = \overline{\mathbb{F}}_q$ is algebraically closed. Hence $\mathcal{O}_{\breve{E}}/(\varphi - \mathrm{id})\mathcal{O}_{\breve{E}}$ vanishes after reduction modulo π, hence by Nakayama it must vanish all along. Inverting π, we thus see that $\mathrm{Ext}^1_\mathcal{C}(D(0), D(0)) = 0$, so any self extension of $D(0)$ is split.

Therefore, given an arbitrary D as in $(*)$, it suffices to construct a non-zero morphism $D(0) \to D$. Indeed, such a morphism is necessarily a monomorphism because $D(0)$ is simple, and using induction on the rank we may assume that its cokernel $D/D(0)$ is already a direct sum of copies of $D(0)$. Then the above extension argument shows that D itself must be such a direct sum. To construct a non-zero morphism $D(0) \to D$, write $\varphi_D = a$ for some $a \in \mathrm{GL}_n(\breve{E})$. After performing row operations we may assume a is triangular (doing some row operations forces us to do the corresponding "φ-inverse" column operations to keep the φ-conjugacy property alive; however, these column operations won't stop us from making a upper triangular). Moreover $a_{1,1} \in \mathcal{O}_{\breve{E}}^\times$ because D is semistable of slope 0, so $\det(\varphi_D)$ should have valuation 0. As $\overline{\mathbb{F}}_q$ is algebraically closed, we may write $a_{1,1} = \varphi(x)/x$ for some $x \in \mathcal{O}_{\breve{E}}^\times$ (as usual, x has to be constructed Teichmüller coefficient-wise). Then we get $D(0) \cong (\breve{E}, a_{1,1}\varphi) \hookrightarrow D$, as desired. □

2.1.15. Definition. — Recall that $\breve{E} \subseteq B$ canonically. Let X denote the Fargues–Fontaine curve as usual. We construct a functor

$$\begin{aligned} \mathcal{E}_E(-)\colon \varphi\text{-Mod}_{\breve{E}} &\longrightarrow \mathrm{QCoh}_X \\ (D, \varphi_D) &\longmapsto \left(\bigoplus_{d \geqslant 0} (B \otimes_{\breve{E}} D)^{\varphi \otimes \varphi_D = \pi^d}\right)^{\sim}, \end{aligned}$$

where $(-)^\sim$ denotes the graded twiddleization.

2.1.16. Example/Warning. — If $n \in \mathbb{Z}$, then $D(n) = (\breve{E}, \pi^n\varphi)$ is sent to the twisting sheaf $\mathcal{E}_E(D(n)) = \mathcal{O}_X(-n)$. Indeed, its graded components are given by

$$(B \otimes_{\breve{E}} \breve{E})^{\varphi \otimes \pi^n \varphi = \pi^d} = B^{\varphi = \pi^{d-n}},$$

hence $\mathcal{E}_E(D(n))$ is the quasi-coherent module associated to the shift $P[-n]$, whence we catch a sign swap.

2.1.17. Lemma. — *For $h \geqslant 1$ let E_h/E be the unique unramified extension of degree h and $X_h = \operatorname{Proj}\left(\bigoplus_{d \geqslant 0} B^{\varphi^h = \pi^d}\right)$ the corresponding Fargues–Fontaine curve. Let (D, φ_D) be a φ-module over $\breve{E}$.*

(1) *For all $d \geqslant 0$ we have $E_h \otimes_E (B \otimes_{\breve{E}} D)^{\varphi \otimes \varphi_D = \pi^d} \cong (B \otimes_{\breve{E}} D)^{\varphi^h \otimes \varphi_D^h = \pi^{hd}}$.*

(2) *X_h is isomorphic to the base change $X \otimes_E E_h$ (this works in the ramified case as well).*

(3) *The following diagram commutes:*

$$\begin{array}{ccccc} (D, \varphi_D) & & \varphi\text{-Mod}_{\breve{E}} & \xrightarrow{\mathcal{E}_E(-)} & \mathrm{QCoh}_X \\ \big\downarrow & & \big\downarrow & & \big\downarrow {\scriptstyle - \otimes_E E_h} \\ (D, \varphi_D^h) & & \varphi^h\text{-Mod}_{\breve{E}} & \xrightarrow{\mathcal{E}_{E_h}(-)} & \mathrm{QCoh}_{X_h} \end{array} .$$

2.1.18. Remark. — Lemma 2.1.17 shows that $\mathcal{E}_E(-)$ takes values in vector bundles. Indeed, by Theorem 2.1.14, every $M \in \varphi\text{-Mod}_{\breve{E}}$ is a direct sum of $D(\lambda)$'s, so it suffices to check that every $\mathcal{E}_E(D(\lambda))$ is a vector bundle. This can be verified étale-locally. Writing $\lambda = d/r$, we see that $X_r \to X$ is an étale covering and by Lemma 2.1.17(3), $\mathcal{E}_E(D(\lambda)) \otimes_E E_h$ corresponds to

$$\left(D(\lambda), \varphi_{D(\lambda)}^r\right) \cong \bigoplus_{i=1}^r (\breve{E}, \pi^d\varphi),$$

which is sent to the vector bundle $\mathcal{O}_{X_h}(-d)^{\oplus r}$ under $\mathcal{E}_{E_h}(-)$ by Example/Warning 2.1.16. Henceforth we will write $\mathcal{O}_X(\lambda) \coloneqq \mathcal{E}_E(D(-\lambda))$. We have just seen that this is a vector bundle of rank r.

Proof of Lemma 2.1.17. For (1), note that $\mathbb{Z}/h\mathbb{Z} \cong \mathrm{Gal}(E_h/E)$ acts E_h-semilinearly on $M = (B \otimes_{\breve{E}} D)^{\varphi^h \otimes \varphi_D^h = \pi^{hd}}$ via $\pi^{-d}\varphi \otimes \varphi_D$. Moreover, the invariants of this action are $M^{\mathrm{Gal}(E_h/E)} = (B \otimes_{\breve{E}} D)^{\varphi \otimes \varphi_D = \pi^d}$. Then the claim follows from Hilbert 90 (in the form of Galois descent; see e.g. [SGA$_{4\frac{1}{2}}$, I (5.2)]).

Parts (2) and (3) are easy consequences of (1); for (2) we use that $\mathrm{Proj}\big(\bigoplus_{d\geqslant 0} B^{\varphi^h = \pi^d}\big)$ coincides with $\mathrm{Proj}\big(\bigoplus_{d\geqslant 0} B^{\varphi^h = \pi^{hd}}\big)$ because that's just how the Proj construction works. $\square$

A slightly different perspective on the vector bundles $\mathcal{O}_X(\lambda)$ is given by the following lemma.

2.1.19. Lemma. — *Let $\lambda = d/r \in \mathbb{Q}$ and let $f_r\colon X_r = X \otimes_E E_r \to X$. Then there is a canonical isomorphism*

$$\mathcal{O}_X(\lambda) \xrightarrow{\sim} f_{r,*}\mathcal{O}_{X_r}(d)$$

In particular, $\mathcal{O}_X(\lambda)$ is a semistable vector bundle of rank r and slope λ, and a simple object in $\mathrm{Bun}_{X,\lambda}^{\mathrm{sst}}$.

Proof.* The isomorphism follows easily by pulling back to X_r and comparing descent datas; details are left as an exercise. To prove the additional assertions, we claim:

$(*)$ The functors $f_{r,*}$ and f_r^* preserve semistable objects.

We first observe that f_r^* preserves rk and scales deg by r, because it is straightforward to check that $\mathcal{O}_X(n) \otimes_E E_r \cong \mathcal{O}_{X_r}(rn)$. Conversely, $f_{r,*}$ scales rk by r (this is straightforward) and preserves deg (this follows from the fact that both f_r^* and $f_r^* f_{r,*} \cong (-)^{\oplus r}$ scale deg by r). Now suppose $\mathcal{E}' \in \mathrm{Bun}_{X_r}$ is semistable and $\mathcal{E} \subseteq f_{r,*}\mathcal{E}'$ is a strict subobject. Then $f^*\mathcal{E} \subseteq f_r^* f_{r,*}\mathcal{E}' \cong \mathcal{E}'^{\oplus r}$ is a strict subobject and $\mathcal{E}'^{\oplus r}$ is semistable, hence

$$r \deg(\mathcal{E}) = \deg(f^*\mathcal{E}) \leqslant \deg(\mathcal{E}'^{\oplus r}) = r \deg(\mathcal{E}')\,,$$

proving that $f_{r,*}\mathcal{E}'$ is semistable as well. In the same way we prove that f_r^* preserves semistable objects. This proves $(*)$.

In particular, $\mathcal{O}_X(\lambda)$ is semistable of rank r and slope $d/r = \lambda$. It remains to show that it is simple in $\mathrm{Bun}_{X,\lambda}^{\mathrm{sst}}$. If $0 \neq \mathcal{E} \subseteq \mathcal{O}_X(\lambda)$, then $\mathrm{rk}(\mathcal{E}) \leqslant r$. But if $\mathcal{E}$ has slope λ, then equality must hold as d and r are coprime. Since $\mathrm{Bun}_{X,\lambda}^{\mathrm{sst}}$ is abelian by Proposition 2.1.8, we see that $\mathcal{O}_X(\lambda)/\mathcal{E}$ is a vector bundle again and of rank 0, hence $\mathcal{E} = \mathcal{O}_X(\lambda)$. $\square$

We are now ready to state our second main theorem: the classification of vector bundles on the Fargues–Fontaine curve. Its proof will occupy most of the remaining lectures.

2.1.20. Main Theorem (Fargues–Fontaine). — *The functor $\mathcal{E}_E(-)$ from Definition 2.1.15 induces a bijection*

$$\mathcal{E}_E(-)\colon \big\{\textit{iso. classes in } \varphi\text{-}\mathrm{Mod}_{\breve{E}}\big\} \xrightarrow{\sim} \{\textit{iso. classes in } \mathrm{Bun}_X\}\,.$$

In particular, every vector bundle $\mathcal{E}$ on X has a unique decomposition $\mathcal{E} \cong \bigoplus_{i=1}^r \mathcal{O}_X(\lambda_i)^{\oplus n_i}$ with $\lambda_1 > \dots > \lambda_r$ and all $n_i > 0$.

2.1.21. Warning. — Don't be fooled: $\mathcal{E}_E(-)$ will *not* be an equivalence of categories. Upon closer inspection this can't possibly be true, for φ-$\mathrm{Mod}_{\breve{E}}$ is an abelian category, but Bun_X is not.

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