Lecture Notes for

Étale Cohomology I

Lecturer

Jens Franke

Notes typed by

Ferdinand Wagner

Winter Term 2019/20

University of Bonn

This text consists of notes on the lecture Advanced Topics in Algebraic Geometry (Étale Cohomology I), taught at the University of Bonn by Professor Jens Franke in the winter term (Wintersemester) 2019/20.

Some changes and some additions have been made by the author. To distinguish them from the lecture's actual contents, they are labelled with an asterisk. So any *Lemma** or *Remark** or *Proof** that the reader might come across are wholly the author's responsibility.

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Contents

List of Lectures

Preface

Organizational stuff. — As a result of a democratic decision in the preliminary meeting, the lecture will take place on Mondays from from 18:00 to 20:00 and on Fridays from 16:00 to 18:00, in the "Großer Hörsaal".

Recommended prerequisites to this lecture are

- flat morphisms and faithfully flat descent,
- abelian varieties, in particular, the Jacobian of a curve.

Nevertheless, Professor Franke promised to give a quick reminder on flat and étale morphisms in the first lecture. Moreover, typed lecture notes are available for Professor Franke's lecture on Jacobians of curves held in the winter term 2018/19 (see [Jac]).

The goal of this lecture is to eventually define the *ℓ-adic cohomology* of a scheme X, where $\ell \neq p$ is a prime different from the characteristic p of X. These groups will be constructed as

$$H^i(X_{\text{ét}}, \mathbb{Z}_\ell) = \lim_{n \geqslant 1} H^i(X_{\text{ét}}, \mathbb{Z}/\ell^n\mathbb{Z})\,.$$

Along the way, we will come across sheaves on the étale site, the relation between étale and Galois cohomology, cohomology of curves, and proper base change.

Professor Franke offered a Q&A session, especially for those of you taking the exam. It will take place on Monday 3^{rd} at 18:00 in the "Großer Hörsaal".

Sequel on Weil I and seminar next semester. — Professor Franke has confirmed there will be a follow-up lecture on Deligne's first proof of the Weil conjectures. It will take place on Tuesdays from 16:00 to 18:00 in a room tba, and on Fridays from 16:00 to 18:00 in the "Großer Hörsaal". The plan is to present Deligne's proof in the very beginning, and then to work backwards through the proofs of the hard technical results required. So hopefully, the end of the sequel and the end of the current lecture will meet.

Professor Franke additionally intends to offer a seminar about étale cohomology. This provides the opportunity to present proofs that we would otherwise not have time for in the Weil I lecture, and reduce the overall amount of blackboxing. *The preliminary meeting is on Friday 7^{th} February at 16:00 in the "Großer Hörsaal".*

Author's note. — In these notes, the modern meaning of the word *scheme* is used. That is, a scheme in these lecture notes is what Professor Franke would call a *prescheme*, and what he would call a *scheme* will be called a *separated scheme* in here.

Also I will not follow Franke's numbering scheme[1], as I believe this document is easier to navigate if propositions/lemmas/etc. are numbered consecutively rather than independent of each other.

[1] . . . if not to say, numbering *prescheme*.

Chapter 1.

Motivation and Basic Definitions

1

1.1. Motivation

Lecture 1
18^{th} Oct, 2019

1.1.1. Problem. — For a scheme X, we would like to have cohomology groups $H^\bullet(X, \mathbb{Z})$ with properties similar to the ones familiar from algebraic topology. For example, if $f \colon X \to X$ is a continuous map of a topological space into itself, then (under some sensible conditions) the *Lefschetz trace formula* says

$$\#\{\text{fixed points of } f, \text{ counted with multiplicity}\} = \sum_{i=0}^{\dim X} (-1)^i \operatorname{Tr}\left(f^* \middle| H^i(X, \mathbb{Q})\right) .$$

Now assume that X is some variety over $k = \mathbb{F}_q$, where $q = p^n$, and $f = \mathrm{Frob}_q$ is the Frobenius on X. Then the fixed points of f are precisely the k-valued points of X. As the "derivative" $\mathrm{d}f$ vanishes, $(\mathrm{id} - \mathrm{d}f)$ should be invertible, so all fixed points of f ought to have multiplicity one (bear in mind that all of this is purely motivational and has no ambition of being a formal argument). Hence we could hope that

$$\#X(k) = \sum_{i=0}^{2\dim X} (-1)^i \operatorname{Tr}\left(f^* \middle| H^i(X_{\overline{k}}, \text{🎩})\right) ,$$

where $H^i(X_{\overline{k}}, \text{🎩})$ is a mysterious cohomology group of the base change of X to an algebraic closure of k. Also the sum ranging up to $2\dim X$ accounts for the fact that the "cohomological dimension" of X should be twice its Krull dimension, same as the topological dimension of a complex manifold is twice its $\mathbb{C}$-dimension.

Such a mystery cohomology theory with sufficiently nice properties is called a *Weil cohomology theory*[1], named after André Weil, who noticed that such a cohomology theory would solve most of his conjectures on varieties over finite fields, that became famously known as the *Weil conjectures*.

1.1.2. Counterexample. — A natural candidate for a Weil cohomology theory is *de Rham cohomology*. For a variety X over a field k it is defined as

$$H^\bullet_{\mathrm{dR}}(X/k) = H^\bullet(X, \Omega^\bullet_{X/k}) ,$$

i.e., as the (hyper-)cohomology of the de Rham complex $\Omega^\bullet_{X/k}$. By de Rham's famous theorem, the de Rham cohomology of a real manifold coincides with its singular cohomology over $\mathbb{R}$, so it makes sense to hope that $H^\bullet_{\mathrm{dR}}(X/k)$ still works as a replacement of singular

[1] In fact, one can formulate a series of axioms to properly define the notion of *Weil cohomology theory*, but we didn't do that in the lecture.

cohomology for varieties over an arbitrary field k. But on second glance, this can't be true: even if we are lucky and a Lefschetz-like equation holds for de Rham cohomology in characteristic $p > 0$, then it would still only be a congruence modulo p, since the $H^*_{\mathrm{dR}}(X/k)$ are $\mathbb{F}_p$-vector spaces in this case, so the traces take values in $\mathbb{F}_p$ as well.

1.1.3. Counterexample. — It is also impossible to find a Weil cohomology with coefficients in $\mathbb{Q}$, nor in $\mathbb{Z}$, nor in $\mathbb{Z}_p$ if we work in characteristic p. Professor Franke sketched a counterexample, which I'm trying my best to reproduce here (but it may well be I got it wrong—however, this won't be needed for the lecture). For a *supersingular elliptic curve* E the endomorphism ring $\operatorname{End}(E)$ can have the property that $D = \operatorname{End}(E) \otimes \mathbb{Q}$ is a quaternion algebra over $\mathbb{Q}$ with the properties that

$$\operatorname{inv}_v(D) = \begin{cases} \frac{1}{2} & \text{if } v = p \text{ or } v = \infty \\ 0 & \text{else} \end{cases}$$

However, if we had a Weil cohomology theory with coefficients in $\mathbb{Q}$, then $H^0(E, \mathbb{Q}) \oplus H^2(E, \mathbb{Q})$ would be a two-dimensional representation of D. But this can only be true if D is split over $\mathbb{Q}$, which can't be true as $\operatorname{inv}_v(D) = \frac{1}{2}$ for $v \in \{p, \infty\}$.

1.1.4. Solutions. — The following approaches (might) lead to suitable Weil cohomology theories.

(*a*) Étale cohomology $H^\bullet_{\text{ét}}(X, \mathbb{Z}/\ell^n\mathbb{Z})$ for a prime $\ell \neq p$. This will lead to ℓ-adic cohomology, which has coefficients in $\mathbb{Z}_\ell$ resp. in $\mathbb{Q}_\ell$. It can still be defined for $\ell = p$, but gives the wrong results in this case.

(*b*) Crystalline cohomology, with coefficients in the *Witt ring* $W(k)$.

(*c*) Constructing a $\mathbb{C}$-valued Weil cohomology theory seems hard. For example, there ought to be an anti-linear endomorphism σ of $H^i(X, \mathbb{C})$ that satisfies $\sigma^2 = (-1)^i$. Still some people (e.g. Connes) try this. See for example Peter Scholze's survey at the ICM 2018 in Rio de Janeiro.

In this lecture we will stick with approach (*a*). Grothendieck's construction of étale cohomology is to relax the usual notion of a *topology* on a topological space. In a *Grothendieck topology*, the "open subsets" no longer need to form a partially ordered set, but rather more general categories are allowed. Then étale cohomology can be introduced as sheaf cohomology for such a generalized topology.

1.1.5. Example. — Here is an example why we would want a topology that is finer than the usual Zariski topology. On a complex manifold X with its sheaf $\mathcal{O}_X$ of C^∞-functions we have the short exact sequence

$$0 \longrightarrow 2\pi\mathrm{i}\mathbb{Z} \longrightarrow \mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^\times \longrightarrow 0\,.$$

For a scheme X, there is a similar sequence

$$0 \longrightarrow \mu_n \longrightarrow \mathcal{O}_X^\times \xrightarrow{(-)^n} \mathcal{O}_X^\times \longrightarrow 0\,,$$

where μ_n is the sheaf of n^{th} roots of unity on X. This has but one flaw: it is usually not exact. Indeed, for $(-)^n\colon \mathcal{O}_X^\times \to \mathcal{O}_X^\times$ to be an epimorphism, we would have to take "local n^{th} roots" in $\mathcal{O}_X^\times$, which is usually not possible. Instead, taking a "local n^{th} root" corresponds to some quasi-finite morphism $X' \to X$, which need not be an open immersion in the Zariski topology—that's the point! *But if* morphisms like $X' \to X$ would count as a open subsets in some topology, then the above sequence might well be exact in that topology!

In the étale topology, *étale morphisms* (i.e. those that are flat and unramified) play the role of open subsets. It will turn out that the above sequence is exact as a sequence of étale sheaves. So after all it should come as no surprise that every étale morphism is also quasi-finite.

1.2. Reminder on Flat Morphisms

This section is really just a crash course. Professor Franke gave a much more detailed introduction to flat morphisms in his Jacobians of curves lecture, so be sure to have a look at [Jac, Chapter 2].

1.2.1. Definition/Proposition. — An A-module M is *flat* if $- \otimes_A M \colon \mathrm{Mod}_A \to \mathrm{Mod}_A$ is an exact functor, or equivalently, if $\mathrm{Tor}_i^A(-, M) = 0$ for all $i > 0$.

1.2.2. Definition/Proposition. — Let $f \colon X \to Y$ a morphism of schemes and $\mathcal{F}$ a quasi-coherent $\mathcal{O}_X$-module. Then $\mathcal{F}$ is called *flat over* $\mathcal{O}_Y$ if the following equivalent conditions hold.

(a) For all affine open subsets $U \subseteq X$, $V \subseteq Y$ such that $f(U) \subseteq V$, $\Gamma(U, \mathcal{F})$ is a flat $\Gamma(V, \mathcal{O}_Y)$-module.

(b) It is possible to cover X with affine opens U and Y with affine opens V such that the above holds.

(c) If $x \in X$ and $y = f(x)$, then $\mathcal{M}_x$ is a flat $\mathcal{O}_{Y,y}$-module.

In the case where $\mathcal{O}_X$ itself is flat over $\mathcal{O}_Y$, the morphism f is called a *flat morphism*.

1.2.3. Remark. — (a) The property of being a flat morphism is local on source and target and stable under composition and base-change. That is, if $f \colon X \to Y$ is flat and $Y' \to Y$ is any morphism, then the base change

$$f' \colon X \times_Y Y' \longrightarrow Y'$$

is flat again.

(b) When f is flat, the pullback functor $f^* \colon \mathrm{Mod}_{\mathcal{O}_X} \to \mathrm{Mod}_{\mathcal{O}_Y}$ is exact.

1.2.4. Proposition (Flat base change)**.** — *Consider the following pullback diagram of morphisms of schemes*

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, "g'"'] \arrow[dr, phantom, "\lrcorner"] & Y' \arrow[d, "g"] \\ X \arrow[r, "f"] & Y \end{tikzcd},$$

where f is quasi-compact separated and g is flat. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module.

(a) *Assume $Y = \mathrm{Spec}\, A$ and $Y = \mathrm{Spec}\, A'$ are affine (so A' is a flat A-algebra). Then there is a natural isomorphism*

$$H^i(X, \mathcal{F}) \otimes_A A' \xrightarrow{\sim} H^i(X', g'^* \mathcal{F}) \quad \textit{for all } i \geqslant 0\,.$$

(*b*) *For arbitrary Y and Y' there is a natural isomorphism*

$$g^* R^i f_* \mathcal{F} \xrightarrow{\sim} R^i f'_*(g'^* \mathcal{F}) \quad \textit{for all } i \geqslant 0\,.$$

Sketch of a proof. Note that the cohomology of quasi-coherent sheaves on quasi-compact separated schemes can be computed as the Čech cohomology of an affine open cover. This easily shows (*a*). Part (*b*) can be checked locally, hence it can be reduced to (*a*). For more details, check out [Jac, Subsection 2.1.1]. □

1.2.5. Remark*. — Proposition 1.2.4 already holds if f is quasi-compact and quasi-separated (but in this case Čech cohomology no longer computes sheaf cohomology). To prove this, one uses the Čech-to-derived spectral sequence to reduce the quasi-separated case to the separated case (check out [Stacks, Tag 02KH] for details).

1.2.6. Definition. — A morphism $f\colon X \to Y$ is called *faithfully flat* if it is flat and surjective (as a map on underlying sets).

1.2.7. Notation. — Before we give the next definition, let's fix once and for all the following notation. Let morphisms $X_i \to Y$ for $i = 1, \dots, n$ be given (usually, they will all be the same). Then for all $i_1, \dots, i_k \in \{1, \dots, n\}$,

$$\mathrm{pr}_{i_1,\dots,i_k/n}\colon \underbrace{X_1 \times_Y \dots \times_Y X_n}_{n \text{ factors}} \longrightarrow \underbrace{X_{i_1} \times_Y \dots \times_Y X_{i_k}}_{k \text{ factors}}$$

denotes the canonical projection. If no ambiguities can occur (in other words, everywhere except in Definition 1.2.8 ☺), we drop the subscript $_{-/n}$ and just write $\mathrm{pr}_{i_1,\dots,i_k}$.

1.2.8. Definition. — Let $f\colon X \to Y$ be a morphism of schemes. A *descent datum* for f is a pair $(\mathcal{F}, \mu)$, where $\mathcal{F}$ is a quasi-coherent $\mathcal{O}_X$-module and μ is an isomorphism

$$\mu\colon\ \mathrm{pr}_{1/2}^* \mathcal{F} \xrightarrow{\sim} \mathrm{pr}_{2/2}^* \mathcal{F}\,,$$

such that the following diagram commutes:

$$\begin{tikzcd}
\mathrm{pr}_{1/3}^* \mathcal{F} \arrow[rr, "\sim", "\mathrm{pr}_{1,2/3}^*(\mu)"'] \arrow[dr, "\sim", "\mathrm{pr}_{1,3/3}^*(\mu)"'] & & \mathrm{pr}_{2/3}^* \mathcal{F} \arrow[dl, "\sim"', "\mathrm{pr}_{2,3/3}^*(\mu)"] \\
& \mathrm{pr}_{3/3}^* \mathcal{F} &
\end{tikzcd}\,. \tag{1.2.1}$$

A *morphism of descent data* $\varphi\colon (\mathcal{F}, \mu) \to (\mathcal{F}', \mu')$ is a morphism of $\mathcal{O}_X$-modules $\varphi\colon \mathcal{F} \to \mathcal{F}'$ such that the diagram

$$\begin{tikzcd}
\mathrm{pr}_{1/2}^* \mathcal{F} \arrow[r, "\sim", "\mu"'] \arrow[d, "\mathrm{pr}_{1/2}^*(\varphi)"'] & \mathrm{pr}_{2/2}^* \mathcal{F} \arrow[d, "\mathrm{pr}_{2/2}^*(\varphi)"] \\
\mathrm{pr}_{1/2}^* \mathcal{F}' \arrow[r, "\sim", "\mu'"'] & \mathrm{pr}_{2/2}^* \mathcal{F}'
\end{tikzcd}$$

commutes. One thus obtains a *category of descent data* for f, which is denoted $\mathrm{Desc}_{X/Y}$.

1.2.9. Remark*. — You might have seen a different definition of descent data, which, instead of a single morphism $f\colon X \to Y$ and a single $\mathcal{F}$, considers a family of morphisms $\{X_i \to Y\}_{i \in I}$ and for each $i \in I$ an $\mathcal{O}_{X_i}$-module $\mathcal{F}_i$. For example, this is the definition used in [Stacks, Tag 023A]. On taking $X = \coprod_{i \in I} X_i$ this definition becomes equivalent to Definition 1.2.8. Under this equivalence, (1.2.1) becomes the infamous *cocycle condition*.

1.2.10. Remark. — (*a*) The notion of a *descent datum* can be defined in a purely abstract way, as soon as one has suitable "pullback functors" f^*. The abstract framework to do are *fibred categories*. See [SGA$_1$, Exposé VI].

(*b*) There is a functor $f^*\colon \mathrm{QCoh}_{\mathcal{O}_Y} \to \mathrm{Desc}_{X/Y}$ that assigns to a quasi-coherent $\mathcal{O}_Y$-module $\mathcal{G}$ the pair $(f^*\mathcal{G}, \mu_{\mathcal{G}})$, where $\mu_{\mathcal{G}}$ is the canonical isomorphism

$$\mu_{\mathcal{G}}\colon\ \mathrm{pr}_{1/2}^* f^*\mathcal{G} \xrightarrow{\sim} (f\,\mathrm{pr}_{1/2})^*\mathcal{G} = (f\,\mathrm{pr}_{2/2})^* \xrightarrow{\sim} \mathrm{pr}_{2/2}^* f^*\mathcal{G}\,.$$

1.2.11. Proposition. — *If $f\colon X \to Y$ is faithfully flat and quasi-compact (which we abbreviate as "fpqc" in the following, from French "fidélement plat et quasi-compact"), then the functor $f^*\colon \mathrm{QCoh}_{\mathcal{O}_Y} \to \mathrm{Desc}_{X/Y}$ from Remark 1.2.10(b) is an equivalence of categories.*

Sketch of a proof. The proof consists of two essentially independent steps and a third step that combines the first two. Step 1 is to prove the assertion under the assumption that f has a section $\sigma\colon Y \to X$ (this is pretty much straightforward). Step 2 is to construct a right-adjoint $R\colon \mathrm{Desc}_{X/Y} \to \mathrm{QCoh}_{\mathcal{O}_Y}$ of f^*.[2]

In Step 3 we show that R (and thus f^*) is an equivalence of categories. This boils down to checking that unit and counit of the adjunction are natural isomorphisms. However, a map being an isomorphism can be checked after faithfully flat base change. Base-changing by f itself, we end up in a situation where a section σ exists—the diagonal $\Delta\colon X \to X \times_Y X$. So Step 1 can be applied, which concludes the proof. For more details check out [Jac, Theorem 7]. □

Lecture 2
21st Oct, 2019

1.2.12. Corollary. — *If $f\colon X \to Y$ is fpqc, then for all open subset $U \subseteq Y$ we have a natural isomorphism*

$$\Gamma(U, \mathcal{O}_Y) \xrightarrow{\sim} \left\{\lambda \in \Gamma(f^{-1}(U), \mathcal{O}_X)\ \middle|\ \mathrm{pr}_1^*\lambda = \mathrm{pr}_2^*\lambda \text{ in } \Gamma(p^{-1}(U), \mathcal{O}_{X\times_Y X})\right\}\,.$$

Here, $p\colon X \times_Y X \to Y$ denotes the natural morphism, so that $p = f\,\mathrm{pr}_1 = f\,\mathrm{pr}_2$.

At this point, Professor Franke recalls the notion of mono-/epimorphism and their *effective* variants. We refer to [AG$_2$, Appendix A.1] for the relevant definitions and to [AG$_1$, Subsection 1.3.1] for a construction of equalizers in the category of schemes.

1.2.13. Proposition. — *An fpqc morphism is an effective epimorphism in the category of schemes.*

Sketch of a proof. One first shows that Y carries the quotient topology with respect to X. To prove this, use [SGA$_1$, Exposé VIII Théorème 4.1]. Alternatively you can look up the proof in [Jac, Proposition 2.5.3]. This shows the topological part of the assertion. For the algebraic part, use Corollary 1.2.12.

If you are looking for a more detailed proof than this extremely brief sketch, check out [Jac, Corollary 2.6.2]. □

1.2.14. Proposition. — *If $f\colon X \to Y$ is flat a morphism of locally finite type between locally noetherian schemes, then f is an open map on underlying topological spaces.*

Proof.* See [Jac, Corollary 2.5.1]. □

[2] Professor Franke emphasizes that this should be in every mathematicians bag of tricks: if you are to show that some functor is an equivalence, look for a right- or left-adjoint!

1.2.15. Remark*. — One can generalize Proposition 1.2.14 to flat morphisms of *locally finite presentation* between arbitrary (i.e. not necessarily locally noetherian) schemes. The key idea in the proof is an ingenious trick that reduces everything to the noetherian case. You can find a very nice exposition of this in Akhil Mathews blog, see https://amathew.wordpress.com/2010/12/26/!

1.3. Grothendieck Topologies, the fpqc Topology, and related ones

You might have already seen Grothendieck topologies defined via *covering families*. However, this a priori only gives a Grothendieck *pre*topology, as one has to pass to equivalence classes afterwards. Thus, Professor Franke prefers the later approach via *sieves* (which he attributes to Giraud). Of course, both approaches are equivalent.

1.3.1. Definition. — Let $\mathcal{C}$ be a category. A *sieve* over an object $x \in \mathcal{C}$ is a class $\mathcal{S}$ of morphisms $u \to x$ such that whenever $(u \to x) \in \mathcal{S}$, then also $(v \to u \to x) \in \mathcal{S}$ for all $(v \to u) \in \operatorname{Hom}_{\mathcal{C}}(x, y)$.

1.3.2. Example. — Let $\mathcal{C}$ be the partially ordered set of open subsets of a topological space X, and $\mathcal{U} = \{U_i\}_{i \in I}$ be any family of open subsets (not necessarily covering X). Then

$$\mathcal{S} = \{V \subseteq X \mid V \text{ is open and there exists an } i \in I \text{ such that } V \subseteq U_i\}$$

is a sieve over $X \in \mathcal{C}$.

1.3.3. Definition. — A *Grothendieck topology* on a category $\mathcal{C}$ is given by specifying a collection C_x of sieves over x for all $x \in \mathcal{C}$, called the *covering sieves*, which are subject to the following conditions.

(*a*) The all-sieve (containing all morphisms $u \to x$) is a covering sieve of x.

(*b*) If $p\colon y \to x$ is a morphism in $\mathcal{C}$ and $\mathcal{S} \in C_x$ a covering sieve of x, then $p^*\mathcal{S} \in C_y$ is a covering sieve of y. Here, we define

$$p^*\mathcal{S} = \left\{ u \to y \;\middle|\; (u \to y \xrightarrow{p} x) \in \mathcal{S} \right\} .$$

(*c*) Let $\mathcal{S}, \mathcal{T}$ be sieves over x such that $\mathcal{S} \in C_x$. If for all $(p\colon y \to x) \in \mathcal{S}$ we have $p^*\mathcal{T} \in C_y$, then also $\mathcal{T} \in C_x$.

A category $\mathcal{C}$ together with a fixed Grothendieck topology is called a *site*.

1.3.4. Remark. — (*a*) One can interpret Definition 1.3.3(*c*) as saying that being a covering is a local property (and can thus be tested on another covering).

(*b*) If $\mathcal{T} \in C_x$ and $\mathcal{S} \supseteq \mathcal{T}$, then also $\mathcal{S} \in C_x$ (as one would expect that if a subsieve of $\mathcal{S}$ is already sufficient to "cover" the element x, then a fortiori the same is true for the whole sieve $\mathcal{S}$). Indeed, the condition in Definition 1.3.3(*c*) is then trivially satisfied, because if $(p\colon y \to x) \in \mathcal{T}$, then $p^*\mathcal{S} \supseteq p^*\mathcal{T}$. However, the right-hand side is the all-sieve in this case, hence so is the left-hand side.

1.3.5. Example. — Let $\mathcal{C}$ be again the partially ordered set of open subsets of a topological space X. Define $\mathcal{S} \in C_U$ iff $U = \bigcup_{(V \subseteq U) \in \mathcal{S}} V$. Then this defines a Grothendieck topology on the category $\mathcal{C}$.

1.3.6. Construction. — Let S be a scheme. We make the category Sch/S of schemes over S into a site $(\mathrm{Sch}/S)_{\mathrm{Zar}}$ by defining a Grothendieck topology as follows: A sieve $\mathcal{S}$ over some S-scheme X is a covering sieve iff there is a Zariski-open covering $X = \bigcup_{i\in I} U_i$ such that all morphisms $Y \to X$ that factor over some $U_i \hookrightarrow X$ belong to $\mathcal{S}$.

The next thing to do is to introduce the fpqc topology and the fppf topology on Sch/S, and then finally the étale topology. But before we do this, we prove the very abstract and technical Proposition 1.3.8, which in the end will save us some work in proving that certain equivalent descriptions of our topologies are indeed equivalent.

1.3.7. Remark*. — Before we dive into the horrible technical nightmare of Proposition 1.3.8, let us motivate some of the things that happen there. First of all, the three topologies we are going to look at are all somehow generated by a certain class of morphisms: the fpqc morphisms, the fppf morphisms, and the étale morphisms respectively. This role is played by $\mathcal{C}$ in Proposition 1.3.8. In the first two cases, $\mathcal{C}_{\mathrm{fpqc}}$ and $\mathcal{C}_{\mathrm{fppf}}$ are precisely the classes of fpqc and fppf morphisms. For the étale topology, we would take $\mathcal{C}_{\text{ét}}$ to be the class of étale and surjective morphisms.

Second, our topologies are, of course, generated by sieves. This is what $\mathcal{S}$ stands for in Proposition 1.3.8. The purpose of said proposition is to establish two equivalent characterizations of these sieves—essentially, we will show that everything can be chosen affine if we wish to.

Third, it might become reasonable to define our topology only on a nice subcategory of Sch/S: for example, on the full subcategory of locally noetherian S-schemes. However, this might get us into trouble. The problem is that if we use Definition 1.3.3(b) to its full potential, then it suddenly spawns fibre products. But fibre products need not preserve noetherianness. For example, $\operatorname{Spec}\mathbb{C}$ and $\operatorname{Spec}\mathbb{Q}$ are perfectly fine noetherian schemes, but $\operatorname{Spec}\mathbb{C}\times_{\operatorname{Spec}\mathbb{Q}}\operatorname{Spec}\mathbb{C}\cong\operatorname{Spec}(\mathbb{C}\otimes_{\mathbb{Q}}\mathbb{C})$ is a non-noetherian abomination. That's where the property $\mathcal{P}$ comes in (and in particular, that's why we need $\mathcal{P}$ to be preserved under morphisms in $\mathcal{C}$). It turns out that it's possible to restrict the étale and fppf topology to the full subcategory of locally noetherian S-schemes, so in this case $\mathcal{P}_{\text{ét}}$ and $\mathcal{P}_{\mathrm{fppf}}$ could be the property of being locally noetherian. In the fpqc case however, such a restriction is impossible. So $\mathcal{P}_{\mathrm{fpqc}}$ will necessarily be the empty property (that is satisfied by every scheme) in this case. Of course, $\mathcal{P}_{\text{ét}}$ and $\mathcal{P}_{\mathrm{fppf}}$ can also be chosen to be the empty property.

1.3.8. Proposition. — *Let S be a scheme and $\mathcal{C}$ a class of quasi-compact morphisms of S-schemes, which has the following properties.*

(a) *$\mathcal{C}$ is closed under composition, finite coproducts, and base-change.*

(b) *If $U=\bigcup_{i=1}^n U_i$ is an affine open cover of an affine*[3] *scheme U over S, then the canonical morphism $\coprod_{i=1}^n U_i \to U$ is an element of $\mathcal{C}$.*[4]

Now let $\mathcal{P}$ be a local property of S-schemes, such that if $X'\to X$ is a morphism in $\mathcal{C}$ and X has property $\mathcal{P}$, then X' has $\mathcal{P}$ as well.[5] *Let $(\mathrm{Sch}/S)^{\mathcal{P}}\subseteq\mathrm{Sch}/S$ be the full subcategory of all objects with $\mathcal{P}$. Then for an object $X\in(\mathrm{Sch}/S)^{\mathcal{P}}$ and any sieve $\mathcal{S}$ over X, the following conditions on $\mathcal{S}$ are equivalent.*

[3] This means that U is affine, and an S-scheme, and *not* that $U\to S$ is an affine morphism.

[4] Note that in the lecture we also had the requirement that id_U is an element of $\mathcal{C}$. However, this trivially follows from (b).

[5] In the lecture we required $\mathcal{P}$ to be "stable under base change ...", but didn't define what this was supposed to mean for a property of schemes (rather than morphisms). This is (at least equivalent to) what Franke had in mind.

(1) *There are an open cover* $X = \bigcup_{i \in I} U_i$*, together with finite sets* J_i *for all* $i \in I$*, and morphisms* $U_{i,j} \to U_i$ *for all* $j \in J_i$*, such that all* $U_{i,j}$ *satisfy* $\mathcal{P}$*, the coproduct* $\coprod_{j \in J_i} U_{i,j} \to U_i$ *is in* $\mathcal{C}$ *for all* $i \in I$*, and all compositions* $U_{i,j} \to U_i \to X$ *are in* $\mathcal{S}$*.*

(2) *The same as* (1)*, but now all* U_i *and* $U_{i,j}$ *are required to be affine.*

Moreover, these sieves define a Grothendieck topology on $(\mathrm{Sch}/S)^{\mathcal{P}}$*.*

LECTURE 3
25[th] Oct, 2019

Proof. It's clear that (2) implies (1). For the converse we basically need the observations that quasi-compact schemes admit finite affine open covers and that affine schemes are quasi-compact, together with the fact that all morphisms from $\mathcal{C}$ are quasi-compact by assumption. However, writing this up is quite a pain, so we leave it as an exercise.

We are left to check the conditions for a Grothendieck topology on $(\mathrm{Sch}/S)^{\mathcal{P}}$. To see that the all-sieve is covering, take any affine open cover $X = \bigcup_{i \in I} U_i$, $J_i = \{i\}$ for all i and $U_{i,i} = U_i$. Then $\bigcup_{j \in J_i} U_{i,j} \to U_i$ is the identity on U_i, which is in $\mathcal{C}$ by (b). Also $U_{i,i} \to U_i \to X$ is obviously part of the all-sieve. This shows that the all-sieve is covering.

Now let $\mathcal{S}$ be a sieve on X satisfying the equivalent properties (1), (2). Let $p\colon Y \to X$ be any morphism in $(\mathrm{Sch}/S)^{\mathcal{P}}$. We need to show that $p^*\mathcal{S}$ satisfies the equivalent properties as well. To this end, let $X = \bigcup_{i \in I} U_i$, J_i and $U_{i,j}$ for all $i \in I$, $j \in J_i$ witness the property (2) for $\mathcal{S}$. Now consider

$$V_i = Y \times_X U_i \quad \text{and} \quad V_{i,j} = Y \times_X U_{i,j}\,.$$

Then the morphism $\coprod_{j \in J_i} V_{i,j} \to V_i$ is in $\mathcal{C}$ because it is a base change of $\coprod_{j \in J_i} U_{i,j} \to U_i$, which is in $\mathcal{C}$, and $\mathcal{C}$ is stable under base change by (a). Moreover, the $V_{i,j}$ all satisfy $\mathcal{P}$. Indeed, since $\mathcal{P}$ is local, it suffices to show that $\coprod_{j \in J_i} V_{i,j}$ has $\mathcal{P}$, and since $\coprod_{j \in J_i} V_{i,j} \to V_i$ is in $\mathcal{C}$, it suffices to show that V_i has $\mathcal{P}$ (by our assumption on $\mathcal{P}$ and $\mathcal{C}$). However, V_i is an open subset of Y, which has $\mathcal{P}$, so V_i has $\mathcal{P}$ as well since $\mathcal{P}$ is local. It remains to see $(V_{i,j} \to Y) \in p^*\mathcal{S}$. But the $V_{i,j}$ fit into pullback diagrams

$$\begin{tikzcd} V_{i,j} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & Y \arrow[d, "p"] \\ U_{i,j} \arrow[r] & X \end{tikzcd},$$

so $V_{i,j} \to Y \to X$ factors over a morphism in $\mathcal{S}$. Thus $(V_{i,j} \to Y) \in p^*\mathcal{S}$ holds by definition.

Last but not least we prove locality of covering sieves. Let $\mathcal{S} \in C_X$ be a covering sieve of X and let U_i, J_i and $U_{i,j}$ witness (2) for $\mathcal{S}$. Let $\mathcal{T}$ be another sieve over X such that $p^*\mathcal{T} \in C_Y$ for any $(p\colon Y \to X) \in \mathcal{S}$. In particular, we can apply this to $(\sigma_{i,j}\colon U_{i,j} \to X) \in \mathcal{S}$. Thus, there are an affine open cover $U_{i,j} = \bigcup_{k \in K_{i,j}} V_{i,j,k}$ and finite sets $L_{i,j,k}$ for all $k \in K_{i,j}$ together with morphisms $V_{i,j,k,l} \to V_{i,j,k}$, such that $\coprod_{l \in L_{i,j,k}} V_{i,j,k,l} \to V_{i,j,k}$ is in $\mathcal{C}$, all $V_{i,j,k,l}$ have $\mathcal{P}$, and $V_{i,j,k,l} \to U_{i,j}$ is an element of $\sigma_{i,j}^*\mathcal{T}$ (up to now that was just unraveling of definitions). Since the $U_{i,j}$ are affine, we may assume that the $K_{i,j}$ are finite sets as well. Thus

$$\coprod_{j \in J_i} \coprod_{k \in K_{i,j}} \coprod_{l \in L_{i,j,k}} V_{i,j,k,l} \longrightarrow U_i$$

is a finite coproduct. This morphism is also in $\mathcal{C}$, since it can be factored as

$$\coprod_{j \in J_i} \coprod_{k \in K_{i,j}} \coprod_{l \in L_{i,j,k}} V_{i,j,k,l} \longrightarrow \coprod_{j \in J_i} \coprod_{k \in K_{i,j}} V_{i,j,k} \longrightarrow \coprod_{j \in J_i} U_{i,j} \longrightarrow U_i\,.$$

The left-most arrow is in $\mathcal{C}$ since it is a finite coproduct of $\coprod_{l\in L_{i,j,k,l}} V_{i,j,k,l} \to V_{i,j,k}$, which is in $\mathcal{C}$ by assumption, and $\mathcal{C}$ is stable under finite coproducts by (a). The middle arrow is in $\mathcal{C}$, because it is a finite coproduct of $\coprod_{k\in K_{i,j}} V_{i,j,k} \to U_{i,j}$, which are in $\mathcal{C}$ by (b). Finally, the right-most arrow is in $\mathcal{C}$ by assumption.

This finally shows that $\mathcal{T}$ is a covering sieve of X. Thus, we indeed get a Grothendieck topology of $(\mathrm{Sch}/S)^{\mathcal{P}}$. □

1.3.9. Remark. — This can be found in the 4th issue of the *Séminaire de Géométrie Algébrique du Bois Marie (SGA)* publications. Professor Franke outlines the contents of the various SGAs.

[SGA_1] Flat descent, the étale fundamental group.

[SGA_2] Local cohomology.

[SGA_4] This consists of three parts: in [$\mathrm{SGA}_{4/1}$], [$\mathrm{SGA}_{4/2}$] the general theory of topoi and the étal topos of a scheme are introduced. The third part [$\mathrm{SGA}_{4/3}$] proves hard theorems in étale cohomology.

[$\mathrm{SGA}_{4\frac{1}{2}}$] This is a very good reference besides [Mil80] and [FK88]. Especially the "Arcata" part is very recommendable.

[SGA_5] ℓ-adic cohomology.

You should be able to read French though (*author's note*: personal experience shows that Google Translate is usually sufficient).

1.3.10. Definition. — (a) Let $\mathcal{P}_{\mathrm{fpqc}}$ be the trivial property and $\mathcal{C}_{\mathrm{fpqc}}$ be the class of fpqc morphisms. Then the Grothedieck topology constructed in Proposition 1.3.8 is called the *fpqc topology*. The corresponding site $(\mathrm{Sch}/S)_{\mathrm{fpqc}}$ is called the *big fpqc site*.

(b) Let $\mathcal{P}_{\mathrm{fppf}}$ be either the trivial property or $\mathcal{P}_{\mathrm{fppf}} = \{\text{locally noetherian schemes}\}$. Let $\mathcal{C}_{\mathrm{fppf}}$ be the class of faithfully flat and finitely presented morphisms. Then the Grothendieck topology from Proposition 1.3.8 is called the *fppf topology*. The corresponding site $(\mathrm{Sch}/S)_{\mathrm{fppf}}$ is called the *big fppf site*.

1.3.11. Remark. — (a) We cannot choose $\mathcal{P}_{\mathrm{fpqc}} = \{\text{locally noetherian schemes}\}$. For example, take the counterexample from Remark* 1.3.7: $\operatorname{Spec}\mathbb{C} \to \operatorname{Spec}\mathbb{Q}$ is an fpqc morphism, hence so is its base change $\operatorname{Spec}(\mathbb{C}\otimes_{\mathbb{Q}}\mathbb{C}) \to \operatorname{Spec}\mathbb{C}$. However, $\mathbb{C}\otimes_{\mathbb{Q}}\mathbb{C}$ is non-noetherian.

To see this, let I be the kernel of the multiplication map $\mathbb{C}\otimes_{\mathbb{Q}}\mathbb{C} \to \mathbb{C}$. If $\mathbb{C}\otimes_{\mathbb{Q}}\mathbb{C}$ was noetherian, then I/I^2 would be a finitely generated module over $(\mathbb{C}\otimes_{\mathbb{Q}}\mathbb{C})/I \cong \mathbb{C}$. However, $I/I^2 \cong \Omega_{\mathbb{C}/\mathbb{Q}}$, whose $\mathbb{C}$-dimension is the cardinality of the continuum.

(b) However, this works for the fppf topology since being locally noetherian is preserved under finitely presented morphisms. The abbreviation fppf comes from French "fidélement plat et de présentation finie". For fppf covering sieves, Professor Franke briefly mentioned some equivalent characterizations, which we summarize in the following lemma.

1.3.12. Lemma*. — *Let X be a scheme over S and let $\mathcal{S}$ be a sieve over X. Then the following are equivalent:*

(a) *$\mathcal{S}$ is an fppf-covering sieve.*

(b) *We find an affine open cover $X = \bigcup_{i\in I} U_i$ and fppf morphisms $V_i \to U_i$ such that $(V_i \to U_i \to X) \in \mathcal{S}$ for all $i \in I$.*

(c) *Same as (b), but the $V_i \to U_i$ are quasi-finite in addition to being fppf.*

*Sketch of a proof**. Clearly (c) implies (b) implies (a). To see (a) ⇒ (b), the crucial thing to note is that if $X = \bigcup_{i\in I} U_i$, K_i, and $U_{i,j}$ are as in Proposition 1.3.8(2), then the $U_{i,j} \to U_i$ are open maps by Proposition 1.2.14. The rest is purely formal.

However, (b) ⇒ (c) is not so easy to see. This needs Cohen–Macaulay properties and we refer to [Stacks, Tag 056X]. □

Having defined Grothendieck topologies and seen some examples, the next step is to define sheaves on sites.

1.3.13. Definition. — Let $\mathcal{C}$ be an arbitrary category.

(a) A *presheaf* on $\mathcal{C}$ (with values in sets, groups, rings, ...) is a functor from $\mathcal{C}^{\mathrm{op}}$ to the categories of groups, sets, rings, ...

(b) Suppose $\mathcal{C}$ is equipped with a Grothendieck topology defined by collections C_x of covering sieves for all $x \in \mathcal{C}$. Then a presheaf $\mathcal{F}$ is called a *sheaf* if for all $x \in \mathcal{C}$ and all $\mathcal{S} \in C_x$ the following condition holds: the morphisms $v^*\colon \mathcal{F}(x) \to \mathcal{F}(u)$ for $(v\colon u \to x) \in \mathcal{S}$ induce a bijection
$$\mathcal{F}(x) \xrightarrow{\sim} \lim_{v\in\mathcal{S}} \mathcal{F}(u)$$

1.3.14. Remark. — (a) If only injectivity of the above map is assumed, the presheaf $\mathcal{F}$ is called *separated*.

(b) If $\mathcal{F}$ has values in sets, groups, rings, ..., then the limit on the right-hand side of Definition 1.3.13(b) can be explicitly described as follows:
$$\lim_{v\in\mathcal{S}} \mathcal{F}(u) = \left\{ (f_v)_{v\in\mathcal{S}} \in \prod_{v\in\mathcal{S}} \mathcal{F}(u) \;\middle|\; \begin{array}{c} \text{if } (v\colon u\to x), (v'\colon u'\to x) \in \mathcal{S} \text{ and} \\ \pi\colon u \to u' \text{ is any morphism such} \\ \text{that } v = v'\pi, \text{ then } f_v = \pi^* f_{v'} \end{array} \right\}$$

(c) I usually write $\Gamma(u,\mathcal{F})$ instead of $\mathcal{F}(u)$ to avoid awkward notation (believe me, you wouldn't want to write stuff like "$(\operatorname{colim}_{\mathcal{I}/\alpha} \pi_\beta^*\mathcal{F}_\beta)(X_\beta)$" in Proposition 2.4.12).

1.3.15. Proposition. — *For the Grothendieck topologies of Proposition 1.3.8, a presheaf $\mathcal{F}$ on $(\mathrm{Sch}/S)^{\mathcal{P}}$ is a sheaf iff its restriction to Zariski-open subsets of any $X \in (\mathrm{Sch}/S)^{\mathcal{P}}$ is an ordinary sheaf on X, and for every morphism $(X' \to X) \in \mathcal{C}$ the sequence*
$$\Gamma(X,\mathcal{F}) \longrightarrow \Gamma(X',\mathcal{F}) \underset{\mathrm{pr}_2^*}{\overset{\mathrm{pr}_1^*}{\rightrightarrows}} \Gamma(X'\times_X X',\mathcal{F})$$
establishes the left arrow as an equalizer of the double arrow on the right. Here we denote by $\mathrm{pr}_1, \mathrm{pr}_2\colon X'\times_X X' \to X'$ the canonical projections, as introduced in Notation 1.2.7.

*Sketch of a proof**. Let's first assume that $\mathcal{F}$ is a sheaf. Since every sieve over X generated by a Zariski-open cover is indeed a covering sieve (since the condition from Proposition 1.3.8(2) is obviously satisfied), we see that $\mathcal{F}$ restricts to a Zariski-sheaf on X. Moreover, if the morphism $X' \to X$ is in $\mathcal{C}$, then the sieve $\mathcal{S}$ of all morphisms $v\colon U \to X$ that factor through X' is a covering sieve. Indeed, the condition from Proposition 1.3.8(1) is clearly satisfied. Hence
$$\Gamma(X,\mathcal{F}) \xrightarrow{\sim} \lim_{v\in\mathcal{S}} \Gamma(U,\mathcal{F})\,.$$

Now every $\Gamma(X,\mathcal{F}) \to \Gamma(U,\mathcal{F})$ factors through $\Gamma(X,\mathcal{F}) \to \Gamma(X',\mathcal{F})$. Moreover, if $U \to X$ factors in two different ways through X', then this induces a unique map $U \to X' \times_X X'$, and thus a map $\mathcal{F}(X' \times_X X') \to \Gamma(U,\mathcal{F})$. If you think about it, this shows that

$$\Gamma(X',\mathcal{F}) \underset{\mathrm{pr}_2^*}{\overset{\mathrm{pr}_1^*}{\rightrightarrows}} \Gamma(X' \times_X X',\mathcal{F})$$

is a coinitial subdiagram of the diagram given by $\{\Gamma(U,\mathcal{F})\}_{v\in\mathcal{S}}$. Hence the limit over the latter diagram is the same as the equalizer of pr_1^* and pr_2^*, so $\Gamma(X,\mathcal{F})$ mapping isomorphically to that limit means that $\Gamma(X,\mathcal{F})$ is said equalizer, as claimed.

Now for the converse. Assume that $\mathcal{F}$ is a presheaf with the required property and let $\mathcal{S}$ be a covering sieve over X. Let $X = \bigcup_{i\in I} U_i$, J_i, and $U_{i,j} \to U_i$ be the associated data. For all $i \in I$, let $\mathcal{S}_i \subseteq \mathcal{S}$ be the subsieve of all morphisms $(v\colon U \to X)$ that factor through some $U_{i,j}$. We first show that we have an isomorphism

$$\Gamma(U_i,\mathcal{F}) \xrightarrow{\sim} \lim_{v\in\mathcal{S}_i} \Gamma(U,\mathcal{F})$$

To see this, note that the subdiagram spanned by all $\Gamma(U_{i,j},\mathcal{F})$ and all $\Gamma(U_{i,j} \times_X U_{i,k},\mathcal{F})$ for $j,k \in J_i$, together with the projection morphisms between them, is a coinitial subdiagram of the whole $\{\Gamma(U,\mathcal{F})\}_{v\in\mathcal{S}_i}$. Indeed, that's basically the same argument as in the proof above (if $U \to X$ factors through $U_{i,j}$ and $U_{i,k}$, then also through $U_{i,j} \times U_{i,k}$). So we may as well take the limit over that subdiagram. But taking into account that $\mathcal{F}$ takes disjoint unions to products (because it restricts to an ordinary Zariski sheaf), the limit over said subdiagram is given by the equalizer of

$$\Gamma\left(\coprod_{j\in J_i} U_{i,j},\mathcal{F}\right) \underset{\mathrm{pr}_2^*}{\overset{\mathrm{pr}_1^*}{\rightrightarrows}} \Gamma\left(\coprod_{j\in J_i} U_{i,j} \times_X \coprod_{k\in J_i} U_{i,k},\mathcal{F}\right).$$

Since $\coprod_{j\in J_i} U_{i,j} \to U_i$ is in $\mathcal{C}$, our assumption on $\mathcal{F}$ shows that the above equalizer is just $\Gamma(U_i,\mathcal{F})$, as claimed.

For $i,i' \in I$ let $\mathcal{S}_{i,i'} \subseteq \mathcal{S}$ be the subsieve of all $U \to X$ that factor through some $U_{i,j} \times_X U_{i',j'}$ for $j \in J_i$, $j' \in J_{i'}$. In the same way as above we find an isomorphism

$$\Gamma(U_i \times_X U_{i'},\mathcal{F}) \xrightarrow{\sim} \lim_{v\in\mathcal{S}_{i,i'}} \Gamma(U,\mathcal{F}).$$

Now let $\mathcal{S}' \subseteq \mathcal{S}$ be the subsieve of all $U \to X$ that factor through $U_{i,j}$ for some $i \in I$, $j \in J_i$. By the above considerations, we find that the limit over the diagram $\{\Gamma(U,\mathcal{F})\}_{v\in\mathcal{S}'}$ is the same as the limit over

$$\prod_{i\in I} \Gamma(U_i,\mathcal{F}) \underset{\mathrm{pr}_2^*}{\overset{\mathrm{pr}_1^*}{\rightrightarrows}} \prod_{i,i'\in I} \Gamma(U_i \times_X U_{i'},\mathcal{F}),$$

which is just $\Gamma(X,\mathcal{F})$ by the usual Zariski sheaf axiom for X. So it remains to show that replacing $\mathcal{S}$ by $\mathcal{S}'$ doesn't change the limit. To see this, let $p\colon Y \to X$ be an element of $\mathcal{S}$ and let $V_i = Y \times_X U_i$, $V_{i,j} = Y \times_X U_{i,j}$. Repeating the above steps with Y instead of X, we see that $\Gamma(Y,\mathcal{F})$ is already determined by the $\Gamma(V_{i,j},\mathcal{F})$. However, each $V_{i,j} \to X$ factors over $U_{i,j}$, i.e., lies in $\mathcal{S}'$. This shows that indeed it doesn't matter whether the limit is taken over all $v \in \mathcal{S}$ are all $v \in \mathcal{S}'$. □

1.3.16. Example. — Let F be any S-scheme. Then $\mathrm{Hom}_{\mathrm{Sch}/S}(-,F)\colon (\mathrm{Sch}/S)^{\mathrm{op}} \to \mathrm{Set}$ is a presheaf on Sch/S. We claim that it is actually an fpqc sheaf, i.e., a sheaf on the site $(\mathrm{Sch}/S)_{\mathrm{fpqc}}$ (and then the same is true for $(\mathrm{Sch}/S)_{\mathrm{fppf}}$).

To prove this, we use Proposition 1.3.15 of course. It is easy to see that $\mathrm{Hom}_{\mathrm{Sch}/S}(-,F)$ is a sheaf in the Zariski topology (since morphisms can be glued). So it's left to check the second condition, i.e.,

$$\mathrm{Hom}_{\mathrm{Sch}/S}(X,F) \xrightarrow{\sim} \left\{\varphi \in \mathrm{Hom}_{\mathrm{Sch}/S}(X',F) \;\middle|\; \varphi\,\mathrm{pr}_1 = \varphi\,\mathrm{pr}_2 \text{ in } \mathrm{Hom}_{\mathrm{Sch}/S}(X' \times_X X', F)\right\}$$

whenever $X' \to X$ is an fpqc morphism. For fixed $X' \to X$, the condition that this holds for all F is precisely the definition for $X' \to X$ being the coequalizer

$$\mathrm{Coeq}\left(X' \times_X X' \underset{\mathrm{pr}_2}{\overset{\mathrm{pr}_1}{\rightrightarrows}} X'\right).$$

But then again this is equivalent to $X' \to X$ being an effective epimorphism, which we proved in Proposition 1.2.13.

1.4. Étale Morphisms

1.4.1. Basic Definitions and Properties

Lecture 4
28th Oct, 2019

Most of the results of this section have already been featured in Professor Franke's lecture on Jacobians of curves. So check out [Jac, Section 2.7] for more detailed proofs.

1.4.1. Proposition. — *Let $f\colon X \to Y$ be a morphism of locally finite type between arbitrary schemes. Then the following conditions are equivalent for all points $x \in X$:*

(a) *We have $(\Omega_{X/Y})_x = 0$.*

(b) *The diagonal $\Delta_{X/Y}\colon X \to X \times_Y X$ is an open embedding on some neighbourhood of x.*

(c) *If $y = f(x)$, then $\mathfrak{m}_{X,x} = \mathfrak{m}_{Y,y}\mathcal{O}_{X,x}$, and the extension $\kappa(x)/\kappa(y)$ on residue fields is separable.*

If f is separated (so that $\Delta_{X/Y}$ is a closed embedding), these are moreover equivalent to

(d) *If $\mathcal{J} \subseteq \mathcal{O}_{X\times_Y X}$ is the sheaf of ideals defining the closed embedding $\Delta_{X/Y}$, then $\mathcal{J}_w = 0$ for $w = \Delta_{X/Y}(x)$.*

Sketch of a proof. Since the assertion is local, we may assume $X = \operatorname{Spec} B$ and $Y = \operatorname{Spec} A$. In this case, f is automatically separated. Then (b) $\Leftrightarrow$ (d) follows basically from the fact that $\mathcal{J}$ is locally finitely generated (see Remark* 1.4.2 below). The equivalence with (a) follows from the following Proposition 1.4.3 as follows. If $\mathcal{J}$ vanishes at w, then so does $\Omega_{X/Y} \cong \Delta_{X/Y}^*(\mathcal{J}/\mathcal{J}^2)$ at x. Conversely, if $(\Omega_{X/Y})_x = 0$, then $(\mathcal{J}/\mathcal{J}^2)_w = 0$, hence $\mathcal{J}_w = \mathcal{J}_w^2$, hence $\mathcal{J}_w = 0$ by Nakayama. We won't prove the equivalence with (c) here, but you can find it in [Jac, Lemma 2.7.2]. $\square$

1.4.2. Remark*. — In the lecture we had the assumption that X and Y be locally noetherian, but in fact this is not needed! The only critical point is the application of the Nakayama lemma, which needs that $\mathcal{J}$ is locally finitely generated. But if B is of finite type over A, with A-algebra generators $b_1, \dots, b_n$ say, then the kernel of $B \otimes_A B \to B$ is generated by the finitely many elements $b_i \otimes 1 - 1 \otimes b_i$.

I believe Professor Franke adds these noetherianness assumptions for simplicity. In these notes I try to make things work in the non-noetherian cases as well whenever possible.

1.4.3. Proposition. — *Let $f\colon X \to Y$ be a separated morphism of schemes and let $\mathcal{J} \subseteq \mathcal{O}_{X\times_Y X}$ the sheaf of ideals defined by the closed embedding $\Delta_{X/Y}\colon X \to X \times_Y X$. Then we have canonical isomorphisms*

$$\Omega_{X/Y} \cong \Delta_{X/Y}^*\mathcal{J} \cong \Delta_{X/Y}^*(\mathcal{J}/\mathcal{J}^2)\,.$$

Sketch of a proof. The assertion is local on X and Y, hence it can be reduced to the affine case, where it follows from Lemma 1.4.4 below. □

1.4.4. Lemma. — *Let B be an algebra over A. Let I be the kernel of the multiplication map $B \otimes_A B \to B$. Then, canonically,*

$$I/I^2 \cong \Omega_{B/A}\,.$$

Sketch of a proof. In fact, for any B-module M we obtain a canonical bijection

$$\operatorname{Hom}_B(I/I^2, M) \xrightarrow{\sim} \operatorname{Der}_A(B, M)\,,$$

sending a morphism $\varphi\colon I/I^2 \to M$ to the A-linear derivation $d\colon B \to M$ defined by $d(b) = \varphi(1\otimes b - b\otimes 1)$, and conversely a derivation d to a morphism φ defined by $\varphi(b_1 \otimes b_2) = b_1 d(b_2)$. Lots of things are to check here actually, but we leave it like that since this is also a pretty well-known fact. □

1.4.5. Definition. — Let $f\colon X \to Y$ be a morphism of locally finite type between arbitrary schemes.

(a) If the equivalent properties of Proposition 1.4.1 are satisfied, the morphism f is called *unramified at x*. If f is unramified at every $x \in X$, we call f *unramified*.

(b) Suppose f is of locally finite presentation and flat at x. Then f is called *étale at x*. If f is étale at every $x \in X$, we call f *étale*.

(c) The morphism f is called an *étale covering* if it is finite and étale (see Lemma* 1.5.5 to justify this terminology).

1.4.6. Fact. — *Let $f\colon X \to Y$ and $g\colon Y \to Z$ be morphisms of schemes.*

(a) *The class of étale morphisms is stable under composition and base change, and being étale is a local property on source and target. The same holds for unramified morphisms.*

(b) *If $g \circ f$ is étale and g is unramified, then f is étale.*

(c) *If f is étale and a closed embedding, then f is also an open embedding. In fact, this holds already when f is flat and locally of finite presentation.*

Proof. Part (a). It is clear from the definitions that being unramified is local on source and target. Moreover, from Proposition 1.4.1(a) and the base change properties of Kähler differentials it is evident that being unramified is preserved under base change, and from Proposition 1.4.1(c) we easily see that compositions of unramified morphisms are unramified again. Then the same follows for étale morphisms, since flat morphisms also have all these properties. This shows (a).

Part (b). We factor f as

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "j=(\mathrm{id}_X, f)"'] & Y \\ & X \times_Z Y \arrow[u, "p"'] \end{tikzcd},$$

where p is étale since it is a base change of $gf\colon X \to Z$. By Proposition 1.4.1(b), the diagonal $\Delta_{Y/Z}\colon Y \to Y \times_Z Y$ is an open embedding. Hence so is j, since it is the base change of $\Delta_{X/Y}$ with respect to $(f, \mathrm{id}_Y)\colon X \times_Y Z \to Y \times_Z Y$. Hence j is étale too (see Example 1.4.8(b) below), which by (a) proves that f is étale as well.

Part (c). Suppose f is a flat closed embedding of locally finite presentation. Locally, f looks like $\operatorname{Spec} A/I \hookrightarrow \operatorname{Spec} A$ for some finitely generated ideal $I \subseteq A$. As A/I is flat over A, we have $I \otimes_A A/I \cong IA/I$. But the right-hand side is 0, hence $I/I^2 = 0$. As I is finitely generated, this implies $I_\mathfrak{p} = 0$ for all prime ideals $\mathfrak{p} \in V(I)$ by Nakayama. But then for all such $\mathfrak{p}$ there is an $g \notin \mathfrak{p}$ such that already $I_g = 0$. Hence $D(g) \subseteq V(I)$, proving that $\operatorname{Spec} A/I \hookrightarrow \operatorname{Spec} A$ is also an open embedding. $\square$

1.4.7. Fact. — *Let $f\colon X \to Y$ be a morphism of locally finite type between arbitrary schemes. Let $x \in X$, $y = f(x)$. Then f is unramified at x iff the fibre $f^{-1}\{y\}$ is unramified at x over $\kappa(y)$. If, in addition, f is flat at x, then it is étale at x iff $f^{-1}\{y\} \to \operatorname{Spec} \kappa(y)$ is étale at x.*

Proof. The residue fields of x and y don't change upon passing to $f^{-1}\{y\} \to \operatorname{Spec} \kappa(y)$, and likewise the condition $\mathcal{O}_{X,x}/\mathfrak{m}_{Y,y}\mathcal{O}_{X,x} \cong \kappa(x)$ is preserved. Hence Proposition 1.4.1(c) shows that f is unramified at x iff $f^{-1}\{y\} \to \operatorname{Spec} \kappa(y)$ is unramified at x. Since flatness is preserved under base change, the second assertion follows at once. $\square$

1.4.8. Example. — (a) Let k be a field and $f\colon X \to \operatorname{Spec} k$ a morphism of finite type. Then f is étale at $x \in X$ iff $\mathcal{O}_{X,x}$ is a finite separable field extension of k. This is a straightforward consequence of Proposition 1.4.1(c).

(b) Every open or closed embedding is unramified (this is clear from Proposition 1.4.1(c)). Hence every open embedding is étale.

1.4.9. Lemma. — *Let A be a finite-dimensional algebra over a field k. Then the following are equivalent:*

(a) *A is étale over k.*

(b) *We can write $A \cong \prod_{i=1}^n \ell_i$, where the ℓ_i are finite separable field extensions of k.*

(c) *The trace form $(a, b) \mapsto \operatorname{Tr}_{A/k}(ab)$ is a perfect pairing[6] on $A \times A$.*

*Proof**. We prove (a) $\Leftrightarrow$ (b). Since over a field everything is flat, the only question is whether A is unramified. Since A is finite-dimensional over k and thus an artinian ring, we have

$$A \cong \prod_{i=1}^n A_{\mathfrak{m}_i}$$

[6] To avoid ambiguity, we use the term *perfect pairing* rather than *non-degenerate pairing* for bilinear forms $\langle -, - \rangle\colon P \times Q \to R$, with P and Q finite projective R-modules, that induce isomorphisms $P \xrightarrow{\sim} \operatorname{Hom}_R(Q, R)$ and $Q \xrightarrow{\sim} \operatorname{Hom}_R(P, R)$. Actually it can be shown that if either of these morphisms is an isomorphism, then so is the other).

where $\{\mathfrak{m}_1, \dots, \mathfrak{m}_n\}$ are the finitely many prime ideals of A (see e.g. [Eis95, Corollary 2.16] for a proof). By Example 1.4.8(a), A is unramified at $\mathfrak{m}_i$ over k iff $A_{\mathfrak{m}_i}$ is a finite separable field extension of k. This easily shows equivalence of (a) and (b).

If ℓ/k is a finite field extension, then a well-known assertion from classical field theory shows that $\mathrm{Tr}_{\ell/k}\colon \ell \times \ell \to k$ is perfect iff ℓ/k is separable. This immediately shows (b) $\Rightarrow$ (c). For the converse, we only need to verify that all $A_{\mathfrak{m}_i}$ are fields. But if $x \in \mathfrak{m}_i A_{\mathfrak{m}_i}$, then x is nilpotent in $A_{\mathfrak{m}_i}$, hence $a \mapsto \mathrm{Tr}_{A_{\mathfrak{m}_i}/k}(ax)$ is identically 0 as nilpotent maps have vanishing trace. Thus $x = 0$ as the trace pairing $\mathrm{Tr}_{A/k}$ is assumed perfect. $\square$

1.4.10. Proposition. — *Let $f\colon X \to Y$ be a finite and finitely presented flat morphism of schemes; so $\mathcal{B} = f_*\mathcal{O}_X$ is a vector bundle on Y in addition to being an $\mathcal{O}_Y$-algebra, and $X = \underline{\mathrm{Spec}}\,\mathcal{B}$. Then the following are equivalent:*

(a) *f is étale.*

(b) *The trace pairing $\mathrm{Tr}_{\mathcal{B}/Y}\colon \mathcal{B} \times \mathcal{B} \to \mathcal{O}_Y$, which is “locally” (i.e., on small enough affine open subsets) given by $(a, b) \mapsto \mathrm{Tr}_{\Gamma(U,\mathcal{B})/\Gamma(U,\mathcal{O}_Y)}(ab)$, is perfect.*

Sketch of a proof.* Since both assertions are local, we may assume that $X = \mathrm{Spec}\,B$ and $Y = \mathrm{Spec}\,A$ are affine, and moreover that B is a finite free A-module. The trick is—of course—to reduce everything to fibres and apply Lemma 1.4.9. For condition (a) this is straightforward: since B is already flat (even free) over A, étaleness can be checked on fibres by Fact 1.4.7. So it suffices to transform (b) into a fibre-wise condition. To show that the map $B \to \mathrm{Hom}_A(B, A)$ induced by $\mathrm{Tr}_{B/A}$ is an isomorphism, it suffices to check that it is a locally split injection, since both sides are free A-modules of the same rank. But being a locally split injection in a neighbourhood of a prime $\mathfrak{p} \in \mathrm{Spec}\,A$ can be tested after tensoring with $\kappa(\mathfrak{p})$. This is a very nice lemma that can be found in [$\mathrm{EGA}_{\mathrm{IV}/1}$, Ch. 0 (19.1.12)]. Thus also (b) can be tested on fibres, so everything reduces to Lemma 1.4.9. For more details, check out [Jac, Proposition 2.7.2]. $\square$

1.4.11. Corollary. — *In the situation of Proposition 1.4.10, suppose X and Y are locally noetherian and let $U \subseteq Y$ be an open subset such that every irreducible component of $Y \setminus U$ has codimension $\geqslant 2$. If the restriction $f|_{f^{-1}(U)}\colon f^{-1}(U) \to U$ is étale, then also f is étale.*

Proof.* Working locally, we may assume that $X = \mathrm{Spec}\,B$ and $Y = \mathrm{Spec}\,A$ are affine and B is a finite free A-module. By Proposition 1.4.10 it suffices to show that $\mathrm{Tr}_{B/A}$ induces an isomorphism $B \xrightarrow{\sim} \mathrm{Hom}_A(B, A)$. Since B and $\mathrm{Hom}_A(B, A)$ are finite free A-modules of the same rank, this morphism is given by some square matrix C with coefficients in A. Thus is suffices to show that $\det C$ is invertible in A. Since $B \to \mathrm{Hom}_A(B, A)$ is an isomorphism over U by assumption, we see that $V(\det C)$ must be contained in $Y \setminus U$. But every irreducible component of $V(\det C)$ has codimension $\leqslant 1$ by Krull's principal ideal theorem. Thus $V(\det C) = \emptyset$, whence $\det C$ is indeed invertible. $\square$

1.4.12. Remark. — In the case of a separated noetherian regular scheme, something much stronger is true: by a theorem of Zariski–Nagata, the *étale fundamental group* of a scheme doesn't change when a closed subscheme of codimension $\geqslant 2$ is removed. We will sketch the proof in Theorem 1.5.18.

1.4.13. Proposition. — *Let $f\colon X \to Y$ be an étale morphism between locally noetherian S-schemes. Then we have a canonical isomorphism*

$$f^*\Omega_{Y/S} \xrightarrow{\sim} \Omega_{X/S}\,.$$

Proof. Surjectivity follows from the well-known short exact sequence (sometimes called the *cotangent sequence*)

$$f^*\Omega_{Y/S} \longrightarrow \Omega_{X/S} \longrightarrow \Omega_{X/Y} \longrightarrow 0\,,$$

in which $\Omega_{X/Y} = 0$ by Proposition 1.4.1 as f is unramified. For injectivity, first note that the assertion is local on X, Y, and S. Hence without restriction they are all affine. Now consider the following diagram:

$$\begin{tikzcd}
X \arrow[r, hook, "\Delta_{X/Y}"] \arrow[rr, hook, bend left, "\Delta_{X/S}"] \arrow[dr, "f"'] & X \times_Y X \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "(\boxtimes)"] & X \times_S X \arrow[d, "p"] \\
& Y \arrow[r, hook, "\Delta_{Y/S}"] & Y \times_S Y
\end{tikzcd}$$

Since X, Y, and S are affine and thus separated, the diagonaly $\Delta_{X/S}$, $\Delta_{Y/S}$, and $\Delta_{X/Y}$ are closed embeddings. Moreover, $\Delta_{X/Y}$ is also an open embedding by Proposition 1.4.1(b). Moreover, it's easy to see that $(\boxtimes)$ is a pullback square. Hence also $X \times_Y X \hookrightarrow X \times_S X$ is a closed embedding. Moreover, the pullback is taken along p, which is a flat since it factors into a composition $X \times_S X \to X \times_S Y \to Y \times_S Y$ of base changes of the flat morphism $f\colon X \to Y$.

Now let $\mathcal{J}_Y \subseteq \mathcal{O}_{Y\times_S Y}$ be the ideal defined by $\Delta_{Y/S}$. Then commutativity of the diagram together with Proposition 1.4.3 shows

$$f^*\Omega_{Y/S} \cong f^*\Delta_{Y/S}^*\mathcal{J}_Y \cong \Delta_{X/S}^* p^*\mathcal{J}_Y\,.$$

Since $\Omega_{X/S} \cong \Delta_{X/S}^*\mathcal{J}_X$, where $\mathcal{J}_X$ defines the closed embedding $\Delta_{X/S}$, it suffices to identify $\mathcal{J}_X$ with $p^*\mathcal{J}_Y$ (we will immediately see why this is not quite true, but at least that's the spirit). Since p is flat and $(\boxtimes)$ a pullback square, $p^*\mathcal{J}_Y \subseteq \mathcal{O}_{X\times_S X}$ is the ideal defined by the closed embedding $X \times_Y X \hookrightarrow X \times_S X$. Moreover, $\Delta_{X/Y}$ is an open-closed embedding. Hence $p^*\mathcal{J}_Y \subseteq \mathcal{J}_X$, and while they might not coincide, their pullbacks to X are certainly equal. This shows indeed $f^*\Omega_{Y/S} \cong \Omega_{X/S}$, as claimed. □

1.4.14. Proposition. — *Let $f\colon X \to Y$ be a morphism of locally finite type between locally noetherian schemes. If f is étale at $x \in X$ and $y = f(x)$, then $\mathcal{O}_{X,x}$ is regular iff so is $\mathcal{O}_{Y,y}$.*

Lecture 5
4th Nov, 2019

Proof. Recall that if R is a noetherian local ring with maximal ideal $\mathfrak{m}$, then the numbers $\dim_{\kappa(\mathfrak{m})} \mathfrak{m}^n/\mathfrak{m}^{n+1}$ are given by the *Hilbert–Samuel polynomial* $H_{\mathfrak{m}}(n)$ for $n \gg 0$. Moreover, $\dim R = 1 + \deg H_{\mathfrak{m}}$ (here the degree of the zero polynomial is -1 by convention). See [Eis95, Chapter 12] or [Alg_2, Theorem 20] for proofs.

Now since $\mathcal{O}_{X,x}$ is flat over $\mathcal{O}_{Y,y}$ and $\mathfrak{m}_{X,x} = \mathfrak{m}_{Y,y}\mathcal{O}_{X,x}$ by Proposition 1.4.1(c), we easily derive

$$\mathfrak{m}_{X,x}^n/\mathfrak{m}_{X,x}^{n+1} \cong \mathfrak{m}_{Y,y}^n/\mathfrak{m}_{Y,y}^{n+1} \otimes_{\kappa(y)} \kappa(x)$$

for all n. Comparing Hilbert–Samuel polynomials, we get $\dim \mathcal{O}_{X,x} = \dim \mathcal{O}_{Y,y}$. But the above isomorphism also shows

$$\dim_{\kappa(x)} \mathfrak{m}_{X,x}/\mathfrak{m}_{X,x}^2 = \dim_{\kappa(y)} \mathfrak{m}_{Y,y}/\mathfrak{m}_{Y,y}^2\,.$$

This immediately shows that $\mathcal{O}_{X,x}$ is regular iff so is $\mathcal{O}_{Y,y}$. □

1.4.15. Remark*. — Another (slightly more general) way to see $\dim \mathcal{O}_{X,x} = \dim \mathcal{O}_{Y,y}$ is as follows: let $X_y = f^{-1}\{y\} \to \operatorname{Spec} \kappa(y)$ be the fibre of f over y. By [Stacks, Tag 00ON], the inequality

$$\dim \mathcal{O}_{X,x} \leqslant \dim \mathcal{O}_{Y,y} + \dim \mathcal{O}_{X_y,x} ,$$

is actually an equality as $\mathcal{O}_{X,x}$ is flat over $\mathcal{O}_{Y,y}$. Moreover, $X_y \to \operatorname{Spec} \kappa(y)$ is étale at x, hence $\dim \mathcal{O}_{X_y,x} = 0$ by Example 1.4.8(a). This shows $\dim \mathcal{O}_{X,x} = \dim \mathcal{O}_{Y,y}$ and we conclude as above.

1.4.2. The Lifting Criterion and the Jacobian Criterion

The formulation of the following Proposition 1.4.16 was a bit messy in the lecture. I tried my best to fix the presentation conditions in (c) and (f) and make them as strong as possible (also please tell me if I got something wrong). This results in some minor changes in the proof.

1.4.16. Proposition ([SGA$_{4\frac{1}{2}}$, Arcata II Def. (1.1)]). — *Let $R \to S$ be a map of finite type between noetherian rings (or, more generally, a map of finite presentation between arbitrary rings). Then the following conditions are equivalent:*

(a) *Let A be an R-algebra with a nilpotent ideal $I \subseteq A$. Then there is a canonical isomorphism*

$$\operatorname{Hom}_{\mathrm{Alg}_R}(S, A) \cong \operatorname{Hom}_{\mathrm{Alg}_R}(S, A/I) .$$

In other words, for every solid "lifting problem" as below, there exists a unique dashed "solution":

$$\begin{tikzcd} R \arrow[r] \arrow[d] & A \arrow[d] \\ S \arrow[r] \arrow[ur, dashed, "\exists !"] & A/I \end{tikzcd} \quad . \tag{1.4.1}$$

(b) *Same as (a), but $I^2 = 0$ rather than I being just nilpotent.*

(c) *Same as (b), but only for local rings A.*

(d) *S is flat over R and $\Omega_{S/R} = 0$.*

(e) *There is a presentation $S \cong R[X_1, \dots, X_n]/J$ with the following property: the ideal J has generators $f_1, \dots, f_n$ such that the "Jacobian determinant"*

$$\Delta = \det\left(\partial f_i / \partial X_j\right)$$

maps to a unit in S.

(f) *Let $S \cong R[X_1, \dots, X_n]/J$ be an arbitrary presentation. Then there are elements $f_1, \dots, f_n \in J$ and $f \in R[X_1, \dots, X_n]$ such that $V(J) \subseteq D(f)$, the localization J_f is generated by $f_1, \dots, f_n$, and the "Jacobian determinant" Δ as in (e) maps to a unit in S.*

Proof. Brace yourselves, for this proof is going to take long. Also note that some parts have been omitted in the lecture

Proof of equivalence of (a), (b), and (c). It's clear that (a) $\Rightarrow$ (b) $\Rightarrow$ (c). For the converse, let's show (b) implies (a). Let $I \subseteq A$ such that $I^n = 0$. Using (a) repeatedly, we see that $S \to A/I$ lifts uniquely to some $S \to A/I^2$, which in turn lifts uniquely to some $S \to A/I^4$

etc. Inductively, we get a unique lift to $S \to A/I^{2^m}$ for all $m \geqslant 0$. Choosing m such that $2^m \geqslant n$ provides a unique lift to $A = A/I^{2^m}$, as desired.

Next, we show $(c) \Rightarrow (a)$. Suppose we are given a lifting problem as in (1.4.1). Condition (c) provides unique lifts $S \to A_{\mathfrak{p}}$ of $S \to A_{\mathfrak{p}}/IA_{\mathfrak{p}}$ for every prime $\mathfrak{p} \in \operatorname{Spec} A$. But S is of finite presentation over R, so an easy argument shows that $S \to A_{\mathfrak{p}}$ already factors over A_f for some $f \notin \mathfrak{p}$. Since this can be done for any prime $\mathfrak{p}$, we end up with a bunch of maps $S \to A_{f_\lambda}$, or equivalently $D(f_\lambda) \to \operatorname{Spec} S$, where λ ranges over some indexing set Λ, and $\operatorname{Spec} A = \bigcup_{\lambda \in \Lambda} D(f_\lambda)$. Note that $D(f_\lambda) \to \operatorname{Spec} S$ and $D(f_\mu) \to \operatorname{Spec} S$ coincide on $D(f_\lambda) \cap D(f_\mu) = D(f_\lambda f_\mu)$. Indeed, this follows from the fact that for any prime $\mathfrak{q} \in D(f_\lambda f_\mu)$ the induced map $S \to A_{\mathfrak{q}}$ is uniquely determined as a lift of $S \to A_{\mathfrak{q}}/IA_{\mathfrak{q}}$, by the uniqueness condition of (c). Thus, the maps $D(f_\lambda) \to \operatorname{Spec} S$ determine a unique morphism $\operatorname{Spec} A \to \operatorname{Spec} S$ (in fancy words: here we used that $\operatorname{Hom}_{\mathrm{Sch}}(-, \operatorname{Spec} S)$ is a sheaf in the Zariski topology). Therefore we get a map $S \to A$ with the desired properties.

Proof of $(a) \Rightarrow (d)$. Since R and S are assumed noetherian, there is actually a very quick argument for flatness. Write $S \cong T/J$, where $T = R[X_1, \ldots, X_n]$ is a polynomial ring over R. By (a), $S \xrightarrow{\sim} T/J$ lifts to unique maps $S \to T/J^n$ for all $n > 0$. Hence, if $\widehat{T}$ denotes the J-adic completion of T, then $\widehat{T} \twoheadrightarrow S$ has a unique split $S \to \widehat{T}$. In particular, S is a direct summand of $\widehat{T}$. But $\widehat{T}$ is flat over T, which is flat over R, so S too is flat over R.

Now we show $\Omega_{S/R} = 0$. Consider any lifting problem like (1.4.1), where $I^2 = 0$. Then $\overline{\varphi}\colon S \to A/I$ has a unique lift $\varphi\colon S \to A$. Suppose $d\colon S \to I$ is an R-linear derivation. An easy calculation shows that $\varphi + d\colon S \to A$ is a morphism of R-algebras. But then $\varphi + d$ is another lift of $\overline{\varphi}$! Thus $d = 0$ and we conclude $0 = \operatorname{Der}_R(S, I) = \operatorname{Hom}_S(\Omega_{S/R}, I)$. Now consider $A = S \oplus \Omega_{S/R}$, with its natural S-module structure. We can extend this to a natural graded ring structure via $\omega_1\omega_2 = 0$ for all $\omega_1, \omega_2 \in \Omega_{S/R}$. Thus, A is an S-algebra with an ideal $I = \Omega_{S/R}$ that satisfies $I^2 = 0$. Applying our previous considerations, we see $0 = \operatorname{Hom}_S(\Omega_{S/R}, \Omega_{S/R})$, whence $\Omega_{S/R} = 0$, as desired.

Proof of $(d) \Rightarrow (f)$. Put $X = \operatorname{Spec} S$ and $Y = \operatorname{Spec} R$, since Professor Franke's opinion is that the argument is best understood geometrically. Let $S \cong T/J$ be any presentation, where $T = R[X_1, \ldots, X_n]$ is a polynomial ring over R. Consider the conormal sequence

$$J/J^2 \longrightarrow \Omega_{T/R} \otimes_T S \longrightarrow \Omega_{S/R} \longrightarrow 0\,.$$

Note that $\Omega_{T/R} \otimes_T S$ is a free S-module generated by $\mathrm{d}x_1, \ldots, \mathrm{d}x_n$. Since $\Omega_{S/R} = 0$, we find elements $f_1, \ldots, f_n \in I$ that map to a basis of $\Omega_{T/R} \otimes_T S$. In particular, this implies that the Jacobian determinant

$$\Delta = \det\left(\partial f_i / \partial X_j\right)$$

is invertible in S, since the Jacobian matrix is precisely the change of basis matrix between the $\mathrm{d}x_j$ and the images of the f_i.

Now let $J' \subseteq J$ be the ideal generated by $(f_1, \ldots, f_n)$. Put $S' = T/J'$ and $X' = \operatorname{Spec} S'$. Then $X \to Y$ factors over the closed embedding $X \hookrightarrow X'$. We claim that $X' \to Y$ is unramified at all points $x \in X$. Indeed, let $\mathfrak{q} \subseteq R[X_1, \ldots, X_n]$ be the prime ideal corresponding to the image of x in $\mathbb{A}^n_R$. Then $\mathfrak{q}$ contains J. Consider the conormal sequence

$$J'/J'^2 \longrightarrow \Omega_{T/R} \otimes_T S' \longrightarrow \Omega_{S'/R} \longrightarrow 0\,.$$

We know that $\Omega_{T/R} \otimes_T S \cong \Omega_{T/R}/J\Omega_{T/R}$ is generated by the images of the f_i. By Nakayama and $\mathfrak{q} \supseteq J$ we see that $(\Omega_{T/R})_{\mathfrak{q}}$ is generated by the f_i too. Thus, the left arrow in the above sequence becomes surjective upon localizing at $\mathfrak{q}$. Thus $(\Omega_{S'/R})_{\mathfrak{q}} = 0$, so by Proposition 1.4.1(a) $X' \to Y$ is indeed unramified at x.

Let U be the subset of points $x' \in X'$ where $X' \to Y$ is unramified, i.e., the set of points where $\Omega_{S'/R}$ vanishes. Since $\Omega_{S'/R}$ is a finite S'-module (as follows from the conormal sequence above), the set U is open, and it contains X as seen above. By Fact 1.4.6(b), $X \to U$ is étale as well. Then also $X \hookrightarrow X'$ is étale as $U \subseteq X'$ is open. But then $X \hookrightarrow X'$ must be an open-closed embedding by Fact 1.4.6(c)! Thus, there is an $f \in T$ such that $S \cong S'_f = (T/J')_f$. This immediately shows (f).

Proof of (f) $\Rightarrow$ (e). We use notation as above. If f has the property from (f), then f is invertible in S. Hence $S \cong S_f \cong T_f/J_f \cong T_f/J'_f \cong S'_f$. Observe that $S'_f \cong S[t]/(1-tf)$. Hence we have a presentation

$$S \cong R[X_1, \dots, X_n, t]/(f_1, \dots, f_n, 1-tf)\,.$$

We claim that this new presentation has the required properties. Indeed, the new Jacobian matrix has only zeros in its last column, as $(\partial/\partial t)f_i = 0$, except for the bottom entry $(\partial/\partial t)(1-tf) = -f$, which is invertible in S by construction. Thus, the new Jacobian determinant is $-f\Delta$, hence invertible in S.

Proof of (e) $\Rightarrow$ (b). Let $S \cong R[X_1, \dots, X_n]/(f_1, \dots, f_n)$ be a presentation of S such that the associated Jacobian matrix D has invertible determinant in S. Let A be an A-algebra with an ideal $I \subseteq A$ such that $I^2 = 0$. Consider the function $f\colon A^n \to A^n$ which is given component-wise by by the f_i, and let $\overline{f}\colon (A/I)^n \to (A/I)^n$ be its reduction modulo I. Then the set of R-algebra morphisms $S \to A$ is in bijection with the set of solutions $x \in A^n$ of $f(x) = 0$. Similarly, R-algebra morphisms $S \to A/I$ are in bijection with solutions $\overline{x} \in (A/I)^n$ of $\overline{f}(\overline{x}) = 0$. Thus it suffices to show that for any solution $\overline{x}$ there is a unique $x^* \in A^n$ satisfying $f(x^*) = 0$ and $x^* \equiv \overline{x} \mod I$.

Now comes the funny part: existence and uniqueness of x^* follows from Newton's method—you know, this thing from calculus! Indeed, let at first $x \in A^n$ be any lift of $\overline{x}$. Then $f(x) = 0$ is not necessarily true, but at least $f(x)$ is an element of I. Let $\delta \in A^n$ be a vector with entries in I. Then

$$f(x+\delta) = f(x) + D\delta$$

by "Taylor expansion" and $I^2 = 0$. Since D is invertible, there is a unique δ^* such that $x^* = x + \delta^*$ satisfies $f(x^*) = 0$; more precisely, δ^* is given by $\delta^* = -D^{-1}f(x)$. We are done, at last! $\square$

1.4.17. Remark. — [$\text{SGA}_{4\frac{1}{2}}$, Arcata] has only conditions (a) and (d), and (f). Moreover, R doesn't need to be noetherian; instead it is assumed that S is finitely presented over R. Then the equivalent conditions are used as a definition *étale ring maps*.

1.4.18. Remark*. — In the lecture we presented a different, and admittedly more messy proof of flatness (in the noetherian case). For the sake of completeness, we outline the argument.

Without restriction, R is local. Flatness can be tested after completion at the maximal ideal $\mathfrak{m}_R$, so we base change to the completion $\widehat{R}$. So now R is noetherian complete local. It can be shown that (a) still holds for artinian local R-algebras. Using arguments as in [Jac, pp. 15–17] we may reduce to a situation where R is noetherian complete local and S is a finite local $\mathfrak{m}_R S$-complete R-algebra such that $S/\mathfrak{m}_R S$ is a finite separable extension of $R/\mathfrak{m}_R$.

Let $\overline{\beta} \in S/\mathfrak{m}_R S$ be a primitive element of the field extension and $P \in R[T]$ a lift of its minimal polynomial. Note that $P'(\beta) \neq 0$ in $S/\mathfrak{m}_R S$ by separability. Hence by Hensel's lemma we may lift $\overline{\beta}$ to a root $\beta \in S$ of P. Let $B' = A[t]/(P)$. Then there is a unique $S' \to S$ sending $t \mapsto \beta$. Also S' clearly satisfies (e), hence also (a). However, applying (a) we get unique maps $S \to S'/\mathfrak{m}_R^n S'$ for all n, hence a map $S \to S'$. Using uniqueness in (a), we see that this map is actually an inverse to $S' \to S$. Hence $S \cong S'$. But it's easy to check that S' is flat over R.

1.4.19. Remark*. — Our of Proposition 1.4.16 works for the non-noetherian case as well—except, unfortunately, for a tiny detail: completions of non-noetherian rings need not be flat, so the proof of flatness is not complete! Here we present a way to circumvent this argument.

As usual, write $S \cong T/J$, and let $\mathfrak{r} \in \operatorname{Spec} S$ be a prime with preimages $\mathfrak{q} \in \operatorname{Spec} T$ and $\mathfrak{p} \in \operatorname{Spec} R$. It suffices to show that $S_\mathfrak{r}$ is flat over $R_\mathfrak{p}$. Our goal is to show that $J_\mathfrak{q}$ can be generated by elements $f_1, \dots, f_c$, whose images in $T_\mathfrak{q}/\mathfrak{p}T_\mathfrak{q}$ form a regular sequence. Then [Stacks, Tag 0470] can be applied to see that $S_\mathfrak{r} \cong T_\mathfrak{q}/J_\mathfrak{q}$ is flat over $R_\mathfrak{p}$.

Let $\overline{T} = T/\mathfrak{p}T$ and let $\overline{\mathfrak{q}} = \mathfrak{q}\overline{T}_\mathfrak{q}$ be the maximal ideal of $\overline{T}_\mathfrak{q}$. Then $\kappa(\overline{\mathfrak{q}}) = \kappa(\mathfrak{q})$. Tensoring $J_\mathfrak{q} \to \overline{\mathfrak{q}}$ with $\kappa(\mathfrak{q})$ gives a map $J_\mathfrak{q}/\mathfrak{q}J_\mathfrak{q} \to \overline{\mathfrak{q}}/\overline{\mathfrak{q}}^2$. Our first goal is to show that this map is injective. To see this, first note that

$$J/J^2 \longrightarrow \Omega_{T/R} \otimes_T S$$

is split injective. Indeed, this follows from [Eis95, Proposition 16.12], because the projection map $T/J^2 \to T/J \cong S$ admits a splitting by Proposition 1.4.16(a). In particular, the above map stays injective under tensoring with $- \otimes_R \kappa(\mathfrak{p})$ and then with $- \otimes_{\overline{T}} \kappa(\mathfrak{q})$. That is,

$$J_\mathfrak{q}/\mathfrak{q}J_\mathfrak{q} \longrightarrow \Omega_{\overline{T}_\mathfrak{q}/\kappa(\mathfrak{p})} \otimes_{\overline{T}_\mathfrak{q}} \kappa(\mathfrak{q})$$

is still injective. But this map factors through $\overline{\mathfrak{q}}/\overline{\mathfrak{q}}^2$ via the conormal sequence

$$\overline{\mathfrak{q}}/\overline{\mathfrak{q}}^2 \longrightarrow \Omega_{\overline{T}_\mathfrak{q}/\kappa(\mathfrak{p})} \otimes_{\overline{T}_\mathfrak{q}} \kappa(\mathfrak{q}) \longrightarrow \Omega_{\kappa(\mathfrak{q})/\kappa(\mathfrak{p})} \longrightarrow 0\,,$$

hence $J_\mathfrak{q}/\mathfrak{q}J_\mathfrak{q} \to \overline{\mathfrak{q}}/\overline{\mathfrak{q}}^2$ is indeed injective. Now choose a basis $(\overline{f}_1, \dots, \overline{f}_c)$ of the $\kappa(\mathfrak{q})$-vector space $J_\mathfrak{q}/\mathfrak{q}J_\mathfrak{q}$ and extend it to a basis $(\overline{f}_1, \dots, \overline{f}_d)$ of $\overline{\mathfrak{q}}/\overline{\mathfrak{q}}^2$. For all $i = 1, \dots, c$ choose lifts $f_i \in J_\mathfrak{q}$ of $\overline{f}_i$. By Nakayama's lemma, the f_i generate $J_\mathfrak{q}$. Moreover, their images in $\overline{T}_\mathfrak{q}$ are part of a minimal generating system of the maximal ideal $\overline{\mathfrak{q}} \subseteq \overline{T}_\mathfrak{q}$. Indeed, just choose lifts $f_j \in \overline{T}_\mathfrak{q}$ of the $\overline{f}_j$ for $j = c+1, \dots, d$ to get a complete minimal generating system of $\overline{\mathfrak{q}}$, by a well-known Nakayama argument.

But $\overline{T}_\mathfrak{q}$ is a regular local ring because it is a localization of $\kappa(\mathfrak{p})[X_1, \dots, X_n]$ at some prime ideal. Hence any minimal generating system of its maximal ideal form a regular sequence (see [Stacks, Tag 00NQ] or the proof of [Hom, Proposition 2.2.1]). This shows that $f_1, \dots, f_c$ have the required property and we are done.

1.4.20. Proposition. — *Let X be a scheme and $X_0 \subseteq X$ a closed subscheme defined by a locally nilpotent quasi-coherent sheaf of ideals $\mathcal{I}$. Let $\text{Ét}/X$ denote the full subcategory of* Sch/X *spanned by the étale X-schemes $U \to X$. Then the canonical functor*

$$\begin{aligned} - \times_X X_0 \colon \text{Ét}/X &\longrightarrow \text{Ét}/X_0 \\ U &\longmapsto U_0 = U \times_X X_0 \end{aligned}$$

is an equivalence of categories.

Sketch of a proof. Before we start, note that any morphism $f\colon U \to U'$ in $\text{Ét}/X$ is étale by Fact 1.4.6(b). One first shows that $-\times_X X_0$ is fully faithful. By gluing morphisms in the Zariski topology, we can readily reduce this to the affine case. Then Proposition 1.4.16(a) can be applied.

It remains to show essential surjectivity. Since we already know $-\times_X X_0$ is fully faithful, we can check essential surjectivity affine-locally. Let R be a ring with a nilpotent ideal I. Put $R_0 = R/I$ and let S_0 be an étale R_0-algebra. By Proposition 1.4.16(c) we can write $S_0 \cong R_0[x_1, \dots, x_n]/(\overline{f}_1, \dots, \overline{f}_n)$, where the Jacobian determinant $\det(\partial \overline{f}_i/\partial x_j)$ is invertible in S_0. Let $S = R[X_1, \dots, X_n]/(f_1, \dots, f_n)$, where the f_i are arbitrary lifts of the $\overline{f}_i$. Then $S_0 \cong S/I$ and we are done if we can show that S is étale over R. It suffices to show that $\Delta \det(\partial f_i/\partial x_j)$ is invertible in S. But its reduction modulo I is invertible in S_0 and I is nilpotent, hence Δ is invertible as well. $\square$

1.4.21. Remark. — Note that any base change $X_0' = X' \times_X X_0 \hookrightarrow X'$ of $X_0 \hookrightarrow X$ is also defined by a locally nilpotent sheaf of ideals. Hence it is a homeomorphism of Zariski topologies.

In general, a morphism such that all of its basechanges are homeomorphisms is called a *universal homeomorphism*. To study universal homeomorphisms, we start with *universal bijections*, i.e., morphisms $f\colon X \to Y$ such that all base changes $f'\colon X' = X \times_Y Y' \to Y'$ are bijections too. Let $|\ \ |\colon \mathrm{Sch} \to \mathrm{Top}$ denote the forgetful functor sending a scheme to its underlying topological space. With the obvious terminology, a morphism is universally bijective iff it is universally injective and universally surjective.

First observe that every surjection $f\colon X \to Y$ of schemes is automatically a universal surjection. Indeed, for all schemes $Y' \to Y$ the canonical map

$$|X \times_Y Y'| \longrightarrow |X| \times_{|Y|} |Y'|$$

is surjective (see [AG_1, Corollary 1.3.2]), so $f'\colon X' = X \times_Y Y' \to Y'$ is indeed surjective again. This shows that surjections are automatically universal.

Now let $f\colon X \to Y$ be morphism of schemes which is injective on points. Then the following conditions are equivalent:

(a) The morphism f is universally injective.

(b) For every $x \in X$ with image $y = f(x)$ the residue field extension $\kappa(x)/\kappa(y)$ is algebraic and purely inseparable.

Such morphisms are also called *radiciel* in [$\mathrm{EGA}_{\mathrm{I}}$, Ch. I §3.5]. Since injectivity is a fibre-wise condition, we can easily reduce equivalence of (a) and (b) to the case of a field extension ℓ/k. Now a well-known characterization of ℓ/k being algebraic and purely inseparable is that for any field extension K/k the ring $\ell \otimes_k K$ is a local ring (with only one prime ideal), which is exactly what we want.

At this point we can effectively describe universally bijective morphisms. Now for a morphism $f\colon X \to Y$ of finite type between locally noetherian schemes or a morphism of locally finite presentation between arbitrary schemes. Then the following conditions are equivalent.

(1) The morphism f is a universal homeomorphism.

(2) The morphism f is proper and universally bijective.

(3) The morphism f is finite, bijective and satisfies the equivalent conditions (a) and (b).

Clearly (1) ⇔ (2) as proper morphism are universally closed by definition. To see equivalence with (3), note that f is necessarily quasi-finite if it injective on points, and quasi-finite proper morphisms are finite by Zariski's main theorem (see [Jac, Theorem 2(a)] and use a base change argument for the non-noetherian case).

The cool thing about universal homeomorphisms is that they allow a striking generalization of Proposition 1.4.20!

1.4.22. Proposition. — *If $X_0 \to X$ is a universal homeomorphism, then the functor*

$$- \times_X X_0 \colon \mathrm{\acute{E}t}/X \longrightarrow \mathrm{\acute{E}t}/X_0$$

defined as in Proposition 1.4.20 is an equivalence of categories.

The easy part of the proof. Let $f, f' \colon U \to U'$ be a pair of parallel morphisms between étale X-schemes U and U'. The equalizer $\mathrm{Eq}(f,f')$ sits in a pullback diagram

$$\begin{tikzcd} \mathrm{Eq}(f,f') \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & U' \arrow[d, "\Delta_{U'/X}"] \\ U \arrow[r, "{(f,f')}"] & U' \times_X U' \end{tikzcd}$$

Since $\Delta_{U'/X}$ is an open embedding by Proposition 1.4.1(b), $\mathrm{Eq}(f,f')$ is an open subscheme of U. Now suppose the base changes f_0 and f_0' agree, i.e., $\mathrm{Eq}(f_0,f_0') = U_0$. Since equalizers commute with base change and $X_0 \to X$ is a universal homeomorphism, we get $\mathrm{Eq}(f,f') = U$ as topological spaces. But since $\mathrm{Eq}(f,f')$ is an open subscheme, this equality also holds on the level of schemes. Thus, $f = f'$, whence $- \times_X X_0$ is a faithful functor.

Proving that $- \times_X X_0$ is fully faithful is only slightly harder. But proving that it is essentially surjective is hard as f*ck, so we leave the rest of the proof to [SGA_1, Exposé IX Théorème 4.10]. □

1.4.23. Remark. — Suppose X is a scheme over $\mathbb{F}_q$. Then Proposition 1.4.22 may be applied to the *absolute Frobenius* $\mathrm{Frob}_X \colon X \to X$, whenever it is finitely presented. Frob_X is defined as the identity on points and $(-)^q$ on the structure sheaf.

Likewise, if X is a projective variety over a field $k/\mathbb{F}_q$, then we have the *relative Frobenius* sending projective coordinates $[x_1, \dots, x_n]$ to $[x_1^q, \dots, x_n^q]$. This too is a universal homeomorphism.

1.4.3. The Étale Topology and the Pro-Étale Topology

Lecture 6
8th Nov, 2019

Finally we are ready to define the étale topology on a scheme!

1.4.24. Definition. — Let X be an arbitrary scheme.

(a) The *étale topology* on the category $\mathrm{\acute{E}t}/X$ of étale X-schemes is the Grothendieck topology with covering sieves as follows: a sieve $\mathcal{S}$ over an étale X-scheme U is covering iff there are étale morphisms $\{V_i \to X\}_{i \in I} \subseteq \mathcal{S}$ whose images cover X. The corresponding site is called the *small étale site* $X_{\mathrm{\acute{e}t}}$.

(b) The *étale topology* on the category Sch/X of all X-schemes (or all locally noetherian X-schemes if X is locally noetherian) has covering sieves as in (a). The corresponding site is called the *big étale site* $X_{\mathrm{\acute{E}t}}$, or $(\mathrm{Sch}/X)_{\mathrm{\acute{e}t}}$ to be consistent with the notation in Definition 1.3.10.

1.4.25. Remark. — The big étale site is the same as the site defined by Proposition 1.3.8 with respect to the class $\mathcal{C}_{\text{ét}}$ of étale surjective morphisms and the trivial property $\mathcal{P}_{\text{ét}}$ (or $\mathcal{P}_{\text{ét}} = \{\text{locally noetherian } X\text{-schemes}\}$). Indeed, the reason is basically that étale maps are open by Proposition 1.2.14 (compare the argument in Lemma* 1.3.12). By the same argument, we may also require all morphisms to be quasi-compact in addition to being étale.

Similarly, the small étale site can be obtained as a special case of Proposition 1.3.8 too. In this case, $\mathcal{C}_{\text{ét}}$ is again the class of étale surjective morphisms, but $\mathcal{P}_{\text{ét}}$ is the property of being étale over X (which is obviously local and compatible with $\mathcal{C}_{\text{ét}}$).

1.4.26. Lemma. — *Let $f\colon X \to Y$ and $g\colon Y \to Z$ be morphisms of schemes. If gf and f are étale and $y \in Y$ is in the image of f, then also g is étale y.*

Sketch of a proof.* Let $x \in X$ be a preimage and $z \in Z$ the image of y. Then $\mathcal{O}_{X,x}$ is flat over $\mathcal{O}_{Y,y}$. But flat local morphisms of local rings are faithfully flat, hence $\mathcal{O}_{X,x}$ is faithfully flat over $\mathcal{O}_{Y,y}$! Since $\mathcal{O}_{X,x}$ is flat over $\mathcal{O}_{Z,z}$, this shows that $\mathcal{O}_{Y,y}$ is too.

To show that g is unramified at y, we use Proposition 1.4.1(d). That $\mathfrak{m}_{Z,z}\mathcal{O}_{Y,y} \subseteq \mathfrak{m}_{Y,y}$ is an equality can be tested after tensoring with the faithfully flat $\mathcal{O}_{Y,y}$-algebra $\mathcal{O}_{X,x}$. But then it becomes $\mathfrak{m}_{Z,z}\mathcal{O}_{X,x} = \mathfrak{m}_{X,x} = \mathfrak{m}_{Y,y}\mathcal{O}_{X,x}$, using that gf and f are étale at x. Finally, $\kappa(y)/\kappa(z)$ is a subextension of $\kappa(x)/\kappa(z)$, hence finite separable too. □

To finish the section, Professor Franke would like to give some hints on the *pro-étale topology*. Although some of this already appeared in old papers of Olivier (1972) and Gabber/Ramero, the actual developments have happened very recently in the paper [BS15] by Bhatt and Scholze. To start things off, we introduce a relaxation of étale morphisms.

1.4.27. Definition. — A morphism $X \to Y$ of schemes is called *weakly étale* if it is flat and the diagonal $\Delta_{X/Y} \to X \times_Y X$ is also flat.

1.4.28. Theorem (Bhatt/Scholze). — *Let B be a weakly étale A-algebra. Then there exists a weakly étale and faithfully flat B-algebra $\overline{B}$ which is ind-étale as an A-algebra (i.e., a filtered colimit of étale A-algebras).*

Theorem 1.4.28 can be roughly viewed as saying that the topology defined by the ind-étales in the affine case is the same as the topology defined by the weakly étales. Also note that noetherianness is not assumed here! Now the pro-étale topology can be defined in the obvious way.

1.4.29. Definition. — The *small* and *big pro-étale site* $X_{\text{proét}}$ and $(\mathrm{Sch}/X)_{\text{proét}}$ are obtained from Proposition 1.3.8, where $\mathcal{C}_{\text{proét}}$ is the class of weakly étale, quasi-compact, and faithfully flat morphisms, and $\mathcal{P}_{\text{proét}}$ is the property of being weakly étale over X resp. the trivial property.

1.5. The Étale Fundamental Group

1.5.1. Geometric Points and the Fundamental Group

1.5.1. Definition. — Let X be a scheme. A *geometric point* $\overline{x}$ of X is a morphism $\overline{x}\colon \operatorname{Spec} k \to X$, where k is a separably closed field. In other words, a geometric point is an ordinary point $x \in X$ together with an embedding $\kappa(x) \hookrightarrow k$ of its residue field into a separably closed field k.

We consider geometric points since as in Algebraic Topology, the étale fundamental group will of course depend on a choice of base point.

1.5.2. Definition. — Let $\pi\colon Y \to X$ be an étale covering in the sense of Definition 1.4.5(c), and let $\overline{x}\colon \operatorname{Spec} k \to X$ be a geometric point.

(a) We define the *fibre over* $\overline{x}$ as

$$\operatorname{Fib}_{\overline{x}}(Y) = \{\overline{y}\colon \operatorname{Spec} k \to Y \mid \pi(\overline{y}) = \overline{x}\} \ .$$

(b) Consider $\operatorname{Fib}_{\overline{x}}$ as a functor $\{\text{étale coverings of } X\} \to \mathrm{Set}$, where the left-hand side is considered as a full subcategory of $\acute{\mathrm{E}}\mathrm{t}/X$. The *étale fundamental groupoid* $\Pi_1^{\text{ét}}(X)$ is defined as follows: its objects are the geometric points $\overline{x}$ of X, and its morphisms are functor isomorphisms $\operatorname{Fib}_{\overline{x}} \xrightarrow{\sim} \operatorname{Fib}_{\overline{y}}$ for geometric points $\overline{x}, \overline{y} \in \Pi_1^{\text{ét}}(X)$.

(c) The automorphism group of $\overline{x}$ in $\Pi_1^{\text{ét}}(X)$ is denoted $\pi_1^{\text{ét}}(X, \overline{x})$ and called *étale fundamental group of X with basepoint x*.

1.5.3. Remark. — The étale fundamental group $\pi_1^{\text{ét}}(X, \overline{x})$ can be given a canonical topology (generalizing the Krull topology on infinite Galois groups) as follows: for an étale covering $Y \to X$ put

$$\Omega_Y = \left\{\sigma \in \pi_1^{\text{ét}}(X, \overline{x}) \;\middle|\; \sigma \text{ acts identically on } \operatorname{Fib}_{\overline{x}}(Y)\right\}$$

Then a neighbourhood basis of the $\mathrm{id}_{\overline{x}} \in \pi_1^{\text{ét}}(X, \overline{x})$ is given by

$$\{\Omega_Y \mid Y \to X \text{ is an étale covering}\} \ .$$

General morphism sets $\operatorname{Hom}_{\Pi_1^{\text{ét}}(X)}(\overline{x}, \overline{y})$ are given a topology such that composition is continuous.

The topology on $\pi_1^{\text{ét}}(X, \overline{x})$ turns it into a *profinite group* (to prove this we would need to show that there are "enough normal coverings $Y \to X$" in the sense that Ω_Y is a normal subgroup of $\pi_1^{\text{ét}}(X, \overline{x})$). Moreover, $\Pi_1^{\text{ét}}(X)$ is connected (as a groupoid) if X is connected—in other words, if X is connected, then $\pi_1^{\text{ét}}(X, \overline{x})$ doesn't depend (up to non-canonical isomorphism) on the choice of base point $\overline{x}$. Proofs can be found in [SGA_1, Exposé V]; we will give detailed references and sketch some of the proofs in Theorem 1.5.10 below.

1.5.4. Remark. — For smooth proper varieties over a field of characteristic 0, you could consider the set of pairs $(\mathcal{V}, \nabla)$, where $\mathcal{V}$ is a vector bundle on X and ∇ a connection on $\mathcal{V}$ with vanishing curvature (this actually gives a *Tannakian category*). Then instead of $\operatorname{Fib}_{\overline{x}}$ one could consider the functor $\mathcal{V} \mapsto \mathcal{V}(x)$ for $x \in X$.

1.5.2. "Étale Covering Theory" and G-Principal Bundles

Lecture 7
11th Nov, 2019

This lecture was a bit of a special one. Despite our limited time, Professor Franke tried to at least mention the most important facts about étale fundamental groups, ultimately culminating in a sketch of the proof of the Zariski–Nagata theorem. I tried my best to provide the missing proofs where possible (we will use results from Sections 1.6 and 2.1 freely), and to give references where not.

We start with a technical lemma (that was not in the lecture) to gain a better understanding of étale coverings.

1.5.5. Lemma*. — *Let X be a scheme and* $\mathrm{FÉt}/X$ *the category of étale coverings of X in the sense of Definition 1.4.5(c).*

(*a*) *A morphism $Y \rightarrow X$ is an étale covering iff there is an étale cover (without "-ing") $\{U_i \rightarrow X\}_{i\in I}$ such that each $Y \times_X U_i \rightarrow U_i$ is isomorphic to the canonical projection $\mathrm{pr}_2\colon S_i \times U_i \rightarrow U_i$ where S_i is a finite discrete set. In other words, étale coverings are precisely the "covering spaces in the étale topology".*

(*b*) *The category* $\mathrm{FÉt}/X$ *has all finite limits and colimits.*

(*c*) *Every morphism $Y \rightarrow Y'$ in* $\mathrm{FÉt}/X$ *has a factorization into $Y \twoheadrightarrow Y'' \hookrightarrow Y'$, where $Y \twoheadrightarrow Y''$ is an effective epimorphism and $Y'' \hookrightarrow Y'$ the inclusion of an open-closed subscheme. Moreover, a morphism $Y \rightarrow Y'$ is an epimorphism iff it is surjective.*

Proof.* We start with (*a*). The proof uses the étale structure sheaf and strictly henselian rings, so you might want to read Section 1.6 first. Suppose $Y \rightarrow X$ is an étale covering. Let $\overline{x}$ be a geometric point of X. Then as $\mathcal{O}_{X_{\text{ét}},\overline{x}}$ is strictly henselian and any base change of $Y \rightarrow X$ is still finite étale, we see that

$$Y \times_X \operatorname{Spec} \mathcal{O}_{X_{\text{ét}},\overline{x}} \cong \coprod_{i=1}^{n} \operatorname{Spec} \mathcal{O}_{X_{\text{ét}},\overline{x}}$$

is isomorphic to a finite disjoint union of copies of $\operatorname{Spec} \mathcal{O}_{X_{\text{ét}},\overline{x}}$, or equivalently, isomorphic to $S_{\overline{x}} \times \operatorname{Spec} \mathcal{O}_{X_{\text{ét}},\overline{x}}$ for some discrete n-element set $S_{\overline{x}}$. Also note that n is necessarily the degree of the finite locally free morphism $Y \rightarrow X$ at the ordinary point $x \in X$ underlying $\overline{x}$. Let $e_1, \dots, e_n$ be the idempotent global sections of $Y \times_X \operatorname{Spec} \mathcal{O}_{X_{\text{ét}},\overline{x}}$ corresponding to the n copies of the second factor. Since

$$\mathcal{O}_{X_{\text{ét}},\overline{x}} = \operatorname*{colim}_{(U,\overline{u})} \Gamma(U, \mathcal{O}_U)$$

can be written as a filtered colimit over affine étale neighbourhoods $(U, \overline{u})$ of $\overline{x}$, the e_i must already occur as idempotent global sections of some $Y \times_X U$. Thus

$$Y \times_X U \cong Y_0 \sqcup \coprod_{i=1}^{n} Y_i\,,$$

where Y_i corresponds to e_i. However, $Y \times_X U \rightarrow U$ is still finite locally free, and its degree must be equal to n. Thus, restricting U to some suitable open subset (still containing the underlying point of $\overline{u}$) if necessary, we obtain $Y_i \cong U$ for all $i = 1, \dots, n$ and $Y_0 = \emptyset$.

As $\overline{x}$ varies, let $U_{\overline{x}}$ denote the étale X-scheme U constructed as above. Then $\{U_{\overline{x}} \rightarrow X\}_{\overline{x}}$ with $\overline{x}$ ranging over all geometric points is an étale cover of X (see Proposition 1.6.3(*a*)). Moreover, every $Y \times_X U_{\overline{x}} \rightarrow U_{\overline{x}}$ is, by construction, isomorphic to the canonical projection $\mathrm{pr}_2\colon S_{\overline{x}} \times U_{\overline{x}} \rightarrow U_{\overline{x}}$ for some finite discrete set $S_{\overline{x}}$. This proves the first half of (*a*).

Conversely, assume $Y \rightarrow X$ is a morphism of schemes for which there is an étale cover (in fact, an fpqc cover would suffice) $\{U_i \rightarrow X\}_{i\in I}$ such that each $Y \times_X U_i \rightarrow U_i$ is a *split étale covering*, i.e., isomorphic to $\mathrm{pr}_2\colon S_i \times U_i \rightarrow U_i$ for some finite discrete set S_i. Then $Y \rightarrow X$ is finite flat and finitely presented, because all these properties can be checked fpqc-locally. Now $Y \rightarrow X$ is an étale covering iff the trace pairing from Proposition 1.4.10(*b*) is perfect, which is a question about a certain morphism of quasi-coherent modules being an isomorphism, which again can be checked fpqc-locally by faithfully flat descent.

Now for (b). To have finite limits in $\mathrm{F\acute{E}t}/X$, it suffices to construct products and equalizers. It's straightforward to see that for étale coverings $Y \to X$ and $Y' \to X$ the fibre product $Y \times_X Y' \to X$ is an étale covering again. For equalizers, let $f, f'\colon Y \to Y'$ be a pair of parallel morphism between étale coverings. As in the proof of Proposition 1.4.22, $\mathrm{Eq}(f, f')$ is an open subscheme of Y. But it is also closed because $Y' \to X$ is finite, hence separated. Hence $\mathrm{Eq}(f, f')$ is an open-closed subscheme of Y and therefore an étale covering of X as well.

Analogously, to construct finite colimits it suffices to have finite disjoint unions (which is trivial) and coequalizers. We omit the construction of the latter (but see [Stacks, Tag 0BN9]), since we only need the special case of the quotient by a finite group action, which is in Proposition 1.5.7 below.

Finally, we prove (c). As noted before, every morphism $Y \to Y'$ in $\mathrm{F\acute{E}t}/X$ is étale, hence an open map on underlying topological spaces. But $Y \to Y'$ is also clearly finite, hence proper, hence closed. Thus, it's image Y'' is an open-closed subscheme of Y'. Moreover, $Y \to Y''$ is finite étale and surjective, hence fpqc, hence an effective epimorphism by Proposition 1.2.13. This yields a factorization $Y \twoheadrightarrow Y'' \hookrightarrow Y'$ as required.

Conversely, suppose $Y \to Y'$ is an epimorphism. Write $Y' = Y'' \sqcup Y_0'$ with Y'' as above. If $Y_0' \neq \emptyset$, then the two natural inclusions $j_1, j_2\colon Y'' \sqcup Y_0' \hookrightarrow Y'' \sqcup Y_0' \sqcup Y_0'$ would be distinct from each other. However, they become equal after composing with $Y \to Y'$, contradicting the fact that this is an epimorphism. $\square$

1.5.6. Definition/Lemma. — Let G be a finite group acting on an étale covering $Y \to X$ (i.e., G acts on Y by morphisms of X-schemes). We say that Y is a *G-principal homogeneous space* if the following equivalent conditions hold.

(a) Fppf-locally (or fpqc-locally) on X there is a G-equivariant isomorphism $Y \cong X \times G$ of X-schemes.

(b) The canonical mophism $Y \times G \to Y \times_X Y$ given by $(y, g) \mapsto (y, gy)$ is an isomorphism, and $Y \to X$ is faithfully flat (i.e., Y is non-empty as long as X is).

Proof of equivalence.* Let's suppose (a), i.e., there is an fppf or fpqc cover $\{U_i \to X\}_{i \in I}$ such that each $Y \times_X U_i$ is G-equivariantly isomorphic to $U_i \times G$. Checking that $Y \times G \to Y \times_X Y$ is an isomorphism can be done fpqc-locally. But $(Y \times G) \times_X U_i \cong U_i \times G \times G$ and likewise $(Y \times_X Y) \times_X U_i \cong U_i \times G \times G$, so the morphism in question is indeed fpqc-locally an isomorphism. This proves the implication $(a) \Rightarrow (b)$.

The reverse implication $(b) \Rightarrow (a)$ is trivial: indeed, if $Y \to X$ is finite étale and surjective, then it is fpqc, hence $\{Y \to X\}$ is already an fpqc cover with the required property. $\square$

1.5.7. Proposition. — *Let G be a finite group acting on an étale covering $Y \to X$ (where "acting on" means the same as in Definition/Lemma 1.5.6).*

(a) *The quotient Y/G exists and $Y/G \to X$ is an étale covering again.*

(b) *Suppose Y has constant degree n over X and G acts fixed-point free on Y, i.e., $\mathrm{Eq}(\mathrm{id}_Y, g) = \emptyset$ for all $g \in G \setminus \{1\}$. Then $\#G \leqslant n$ and the following conditions are equivalent.*

  (1) $\#G = n$.

  (2) *The canonical morphism $Y/G \to X$ is an isomorphism.*

  (3) *Y is a G-principal homogeneous space in the sense of Definition/Lemma 1.5.6*

Proof.* We start with (a) (which was not in the lecture). For Y/G to exist we need that every G-orbit is contained in an G-invariant affine open subset (see e.g. [Jac, Theorem 11]). But G acts by morphisms of X-schemes, hence every G-orbit is contained in a single fibre over some point $x \in X$. Now take any affine open neighbourhood U of x, then its preimage in Y is the required G-invariant affine open subset.

It remains to prove that $Y/G \to X$ is an étale covering again. This can be checked étale-locally (even fpqc-locally, but we won't need that). In fact, we will construct an étale cover $\{U_i \to X\}$ with the following property:

$(*)$ *$Y \to X$ "splits G-equivariantly" over $\{U_i \to X\}$. That is, every $Y \times_X U_i \to U_i$ is isomorphic to $\mathrm{pr}_2 \colon S_i \times U_i \to U_i$ for some finite discrete set S_i, and G acts on $Y \times_X U_i$ via S_i, i.e., the G-action on $S_i \times U_i$ only permutes the copies of U_i.*

Let's first see how $(*)$ implies (a). Since $-/G$ is a finite colimit, it commutes with flat base change, so $(Y \times_X U_i)/G \cong Y/G \times_X U_i$. But since G acts on $Y \times_X U_i \cong S_i \times U_i$ through S_i, we see $Y/G \times_X U_i \cong S_i/G \times U_i$. Hence $Y/G \times_X U_i \to U_i$ is an étale covering for trivial reasons, finishing the proof of (a).

For $(*)$, we refine the method from Lemma* 1.5.5. Let $\overline{x}$ be a geometric point of X. As seen before, $Y \times_X \operatorname{Spec} \mathcal{O}_{X_{\text{ét}},\overline{x}} \cong \coprod_{i=1}^n \operatorname{Spec} \mathcal{O}_{X_{\text{ét}},\overline{x}}$. We claim that G only permutes the copies of $\operatorname{Spec} \mathcal{O}_{X_{\text{ét}},\overline{x}}$. Indeed, let $e_1, \dots, e_n$ be the idempotent global sections corresponding to the n copies. Since the action of $g \in G$ is uniquely determined by $(g^*(e_1), \dots, g^*(e_n))$, it suffices to show this sequence is a permutation of $(e_1, \dots, e_n)$. Observe that the e_i are "orthogonal" in the sense that $e_i e_j = 0$ for $i \neq j$. Since $g \in G$ acts as a ring automorphism, we see that $g^*(e_1), \dots, g^*(e_n)$ is a collection of non-zero orthogonal idempotents again. But since the local ring $\mathcal{O}_{X_{\text{ét}},\overline{x}}$ has no non-trivial idempotents, the idempotent global sections of $\coprod_{i=1}^n \operatorname{Spec} \mathcal{O}_{X_{\text{ét}},\overline{x}}$ are precisely those of the form $\varepsilon_1 e_1 + \dots + \varepsilon_n e_n$ with $\varepsilon_i \in \{0,1\}$. Thus, a set of non-zero orthogonal idempotents can have cardinality at most n, since each e_i can occur as a summand at most once; and in particular, equality holds precisely for $\{e_1, \dots, e_n\}$. Thus, the sequence $(g^*(e_1), \dots, g^*(e_n))$ is indeed a permutation of $(e_1, \dots, e_n)$.

Choosing a sufficiently small étale neighbourhood $(U, \overline{u})$ of $\overline{x}$, we deduce as in the proof of Lemma* 1.5.5 that $Y \times_X U \cong \coprod_{i=1}^n U$, with the n copies of U corresponding to $e_1, \dots, e_n$. Choosing U even smaller, we may moreover achieve that g^* only permutes the e_i for all $g \in G$, since this is true after taking the colimit, as seen above, and G is finite. Thus, G only permutes the copies of U in $\coprod_{i=1}^n U$. Therefore, denoting U by $U_{\overline{x}}$ and letting $\overline{x}$ vary, we obtain an étale covering $\{U_{\overline{x}} \to X\}_{\overline{x}}$ satisfying the property from $(*)$.

Having proved $(*)$, the proof of (b) becomes pretty easy. Throughout we fix an étale cover $\{U_i \to X\}_{i \in I}$ as in $(*)$. Then G acts still fixed-point free on $Y \times_X U_i \cong S_i \times U_i$ since equalizers commute with base change. If $\#G > \#S_i$, then the G-action on S_i would necessarily have a fixed point for some $g \in G \setminus \{1\}$, hence $\#G \leqslant \#S_i = n$, as claimed. Moreover, equality holds iff the G-action on S_i is simply transitive. This is equivalent to $S_i \times U_i \cong G \times U_i$, which immediately shows (1) $\Leftrightarrow$ (3). Moreover, since we already know that G acts fixed-point free on S_i, the action is simply transitive iff $S_i/G = \{*\}$ is a single point. Since checking whether $Y/G \to X$ is an isomorphism can be done étale-locally and $Y/G \times_X U_i \cong S_i/G \times U_i$, we finally see that (2) is equivalent to (1), (3) as well. □

1.5.8. Definition. — An étale covering $Y \to X$ in which X and Y are both connected is called a *Galois covering* if it satisfies the equivalent conditions from Proposition 1.5.7(b) with $G = \operatorname{Aut}(Y/X)$.

1.5.9. Remark*. — Note that if X and Y are connected, then $\operatorname{Aut}(Y/X)$ acts fixed-point free on Y, so the assumptions of Proposition 1.5.7(b) are met. Indeed, as was noted in the proof of Lemma* 1.5.5(b), the equalizer $\operatorname{Eq}(\mathrm{id}_Y, g)$ is always an open-closed subscheme of Y, hence either $\emptyset$ or Y if Y is connected.

1.5.10. Theorem. — *Let X be a scheme which is the disjoint union of its connected components (in other words, X is locally connected, which holds for example when X is locally noetherian).*

(a) *Suppose moreover that X is connected. Then for every geometric point $\overline{x}$ there is an equivalence of categories*

$$\begin{aligned}\{\textit{étale coverings of } X\} &\xrightarrow{\sim} \{\textit{finite discrete sets with a continuous } \pi_1^{\text{ét}}(X,\overline{x})\textit{-action}\}\\ Y &\longmapsto \operatorname{Fib}_{\overline{x}}(Y)\,.\end{aligned}$$

(b) *In general, there is an equivalence of categories*

$$\begin{aligned}\{\textit{étale coverings of } X\} &\xrightarrow{\sim} \left\{\begin{array}{c}\textit{functors } F\colon \Pi_1^{\text{ét}}(X) \to \{\textit{finite discrete sets}\} \textit{ s.th.}\\ F(\overline{x}) \times \operatorname{Hom}_{\Pi_1^{\text{ét}}(X)}(\overline{x},\overline{y}) \to F(\overline{y}) \textit{ is continuous}\end{array}\right\}\\ Y &\longmapsto \operatorname{Fib}(Y)\,,\end{aligned}$$

where $\operatorname{Fib}(Y) = \operatorname{Fib}_{(-)}(Y)$ *is given by* $\operatorname{Fib}(Y)(\overline{x}) = \operatorname{Fib}_{\overline{x}}(Y)$.

Moreover, under the above conditions $\pi_1^{\text{ét}}(X,\overline{x})$ is a pro-finite group and the connected components of $\Pi_1^{\text{ét}}(X)$ correspond to the connected components of X.

Sketch of a proof.* Part (a) was not mentioned in the lecture, but it shouldn't be missing in a theorem about *Grothendieck's Galois theory* (and we will need it later). We will only roughly outline the proof of (a); a full proof is in [SGA$_1$, Exposé V.4].[7] Grothendieck's approach is, of course, to prove a vast generalization of (a). He proves that given a category $\mathcal{C}$ equipped with a functor $F\colon \mathcal{C} \to \mathrm{Set}$ satisfying a certain list of six properties, one can construct a pro-finite group $\pi = \operatorname{Aut}(F)$ (the "fundamental group") such that F defines an equivalence of categories between $\mathcal{C}$ and the category of finite discrete sets with a continuous π-action. The proof proceeds as follows: one first shows that F is *strictly pro-representable*, i.e., there is a cofiltered system $\widetilde{X} = (\widetilde{X}_i)_{i\in I}$ (the "universal covering") in $\mathcal{C}$ whose transition maps are epimorphisms, such that

$$F(Y) = \operatorname{Hom}_{\operatorname{Pro}(\mathcal{C})}\left(\widetilde{X}, Y\right) = \operatorname*{colim}_{i\in I} \operatorname{Hom}_{\mathcal{C}}\left(\widetilde{X}_i, Y\right).$$

Next, one shows that $\widetilde{X}$ has a cofinal subsystem of $\widetilde{X}_i$ which are *Galois* in the sense that $F(\widetilde{X}_i) = \operatorname{Hom}_{\operatorname{Pro}(\mathcal{C})}(\widetilde{X}, \widetilde{X}_i) = \operatorname{Aut}_{\mathcal{C}}(\widetilde{X}_i)$. After that, one can finally construct an inverse functor $\widetilde{X} \times_\pi -$ from finite discrete π-sets to $\mathcal{C}$. One can show that $\widetilde{X} \times_\pi -$ is an adjoint of F and an equivalence of categories, so F is an equivalence of categories as well.

[7] In case you decide to read Grothendieck's original proof—which I absolutely recommend, it is really beautiful—beware that there is a mistake in the second paragraph of (e). It is claimed that $F(P_j) \to F(P_i)$ is surjective because $P_j \to P_i$ is an epimorphism; however, the functor F is only assumed to transform *effective* epimorphisms (or *épimorphismes stricts* in French) into surjections. This issue can be fixed as follows: we already know that $P_j \to P_i = A \sqcup B$ factors over A. In particular, $P_j \to P_i$ equalizes the two canonical morphisms $j_1, j_2\colon P_i \to A \sqcup B \sqcup B$. Then $j_1 = j_2$ as $P_j \to P_i$ is an epimorphism. However, this implies $F(j_1) = F(j_2)$, which can only happen if $F(B) = \emptyset$, hence $B = \emptyset_{\mathcal{C}}$ by the argument from the first paragraph.

In the concrete situation of (a), this is applied to $\mathcal{C} = \mathrm{F\acute{E}t}/X$ and $F = \mathrm{Fib}_{\overline{x}}$. It is either trivial or follows rather easily from Lemma* 1.5.5 that these satisfy Grothendieck's six properties, so (a) follows as a special case.

To prove (b) and the additional assertions, first note that if $\overline{x}$ and $\overline{y}$ are geometric points belonging to distinct connected components of X, then there is no functor isomorphism $\mathrm{Fib}_{\overline{x}} \cong \mathrm{Fib}_{\overline{y}}$, or in other words, $\mathrm{Hom}_{\Pi_1^{\text{ét}}(X)}(\overline{x}, \overline{y}) = \emptyset$. Indeed, we can find an étale covering $Y \to X$ which is, say, one-sheeted on the connected component of $\overline{x}$ and two-sheeted on the connected component of $\overline{y}$, so $\mathrm{Fib}_{\overline{x}}(Y) \ncong \mathrm{Fib}_{\overline{y}}(Y)$. Conversely, if $\overline{x}$ and $\overline{y}$ belong to the same connected component, then $\mathrm{Fib}_{\overline{x}}$ and $\mathrm{Fib}_{\overline{y}}$ are isomorphic; in fact, [SGA_1, Exposé V Corollaire 5.7] shows that any two *fundamental functors* F and F' of a *Galois category* $\mathcal{C}$ are isomorphic. This shows that the connected components of X and $\Pi_1^{\text{ét}}(X)$ are in canonical correspondence.

Now write $X = \coprod_{i \in I} X_i$, choose geometric points $\overline{x}_i$ in X_i, and put $\pi_i = \pi_1^{\text{ét}}(X, \overline{x}_i)$. From part (a) we get $\mathrm{F\acute{E}t}/X \cong \prod_{i \in I} \mathrm{F\acute{E}t}/X_i \cong \prod_{i \in I} \pi_i\text{-FSet}$, where π_i-FSet is the category of finite discrete sets with a continuous π_i-action. Note that π_i-FSet is equivalent to the category of functors from π_i, considered as a groupoid with only one object, to the category of finite sets, such that a continuity condition similar to the one in (b) holds. Thus, it's easy to see that π_i-FSet is equivalent to functors from $\Pi_1^{\text{ét}}(X_i)$ to finite sets with the desired continuity condition. This proves (b), more or less. $\square$

1.5.3. "Kummer Coverings" and "Artin–Schreier Coverings"

The next point on our agenda is to study "cyclic coverings", i.e., étale coverings which are $\mathbb{Z}/n\mathbb{Z}$-principal homogeneous spaces for some $n \in \mathbb{N}$. A particular special case is the case of cyclic Galois extensions in Galois theory. As is well-known from the classical theory, there are two basic types of them:

(1) *Kummer extensions*, i.e., extensions of the form $K\left(\sqrt[n]{a}\right)/K$ for some $a \in K$, provided that K contains a primitive n^{th} root of unity and that the characteristic $\operatorname{char} K$ does not divide n.

(2) In characteristic $p > 0$, the Kummer extensions are accompanied by *Artin–Schreier extensions*, i.e., extensions of the form $K(\alpha)/K$, where α satisfies $\alpha^p - \alpha + a = 0$ for some $a \in K$.

It turns out that a similar pattern can be found at a global scale.

1.5.11. Proposition ("Kummer coverings"). — *Let X be a scheme on which $n \in \mathbb{N}$ is invertible and such that there is a primitive n^{th} root of unity $\zeta \in \Gamma(X, \mu_n)$. Let $\mathcal{L}$ be a line bundle on X and $\tau \colon \mathcal{L}^{\otimes n} \xrightarrow{\sim} \mathcal{O}_X$ be a trivialization. Consider the functor*

$$\begin{aligned} F \colon (\mathrm{Sch}/X)^{\mathrm{op}} &\longrightarrow \mathrm{Set} \\ (f \colon Y \to X) &\longmapsto \left\{ \lambda \in \Gamma(Y, f^*\mathcal{L}) \;\middle|\; f^*\tau(\lambda^{\otimes n}) = 1 \right\} . \end{aligned}$$

We construct a $\mathbb{Z}/n\mathbb{Z}$-action on F as follows: write $\mathbb{Z}/n\mathbb{Z}$ multiplicatively and fix once and for all a generator $\sigma \in \mathbb{Z}/n\mathbb{Z}$. Then σ^k acts via $\sigma^{k,*}(\lambda) = \zeta^k \lambda$.

(a) F is representable by an étale covering $\Lambda \to X$ which is a $\mathbb{Z}/n\mathbb{Z}$-principal homogeneous space. Moreover, every $\mathbb{Z}/n\mathbb{Z}$-principal homogeneous space has this form, and $(\mathcal{L}, \tau)$ is determined by Λ up to isomorphism.

(*b*) *If X is connected, then for any geometric point $\overline{x}$ there is a long exact sequence*

$$\begin{aligned}0 \longrightarrow \Gamma(X_{\mathrm{Zar}}, \mu_n) \longrightarrow \Gamma(X_{\mathrm{Zar}}, \mathcal{O}_X^\times) \xrightarrow{(-)^n} \Gamma(X_{\mathrm{Zar}}, \mathcal{O}_X^\times) \longrightarrow \\ \longrightarrow \operatorname{Hom}_{\mathrm{cont}}\left(\pi_1^{\text{ét}}(X, \overline{x}), \mathbb{Z}/n\mathbb{Z}\right) \longrightarrow \operatorname{Pic}(X) \xrightarrow{(-)^{\otimes n}} \operatorname{Pic}(X) \longrightarrow \dots .\end{aligned}$$

Proof. Step 1.* We first construct Λ and prove that it represents F. Put $\mathcal{A} = \bigoplus_{l=0}^{n-1} \mathcal{L}^{\otimes l}$. Identifying $\mathcal{L}^{\otimes n}$ with $\mathcal{O}_X$ via τ, we can introduce a multiplication on $\mathcal{A}$, given by $- \otimes -$. Under this multiplication, $\mathcal{A}$ becomes a coherent $\mathcal{O}_X$-algebra. Put $\Lambda = \underline{\operatorname{Spec}}\,\mathcal{A}$. We will show that $\operatorname{Hom}_{\mathrm{Sch}/X}(-, \Lambda)$ is isomorphic to the given functor F.

Let $f \colon Y \to X$ be an X-scheme. By the universal property of the $\underline{\operatorname{Spec}}$ functor, we find that

$$\operatorname{Hom}_{\mathrm{Sch}/X}(Y, \Lambda) \cong \operatorname{Hom}_{\mathrm{Alg}_{\mathcal{O}_X}}(\mathcal{A}, f_*\mathcal{O}_Y)\,.$$

Fix a (Zariski-)open cover $X = \bigcup_{i \in I} U_i$ such that each $\mathcal{L}|_{U_i}$ trivializes, say, with a generator $\lambda_i \in \Gamma(U_i, \mathcal{L})$. Suppose we are given a morphism $Y \to \Lambda$, or equivalently, a morphism $\mathcal{A} \to f_*\mathcal{O}_Y$. Let α_i be the image of λ_i. By construction, $\alpha_i^n = \tau(\lambda_i^{\otimes n})$ is given by some section $\varepsilon_i \in \Gamma(U_i, \mathcal{O}_X)$. Note that ε_i and thus α_i must be invertible, as $\lambda_i^{\otimes n}$ is a local generator of $\mathcal{L}^{\otimes n}$ and τ is an isomorphism. Now consider $\alpha_i^{-1} \otimes \lambda_i \in \Gamma(f^{-1}(U_i), f^*\mathcal{L})$. Since λ_i and λ_j only differ by a unit $\varepsilon_{i,j} \in \Gamma(U_i \cap U_j, \mathcal{O}_X^\times)$, and sections from $\mathcal{O}_X$ can be moved around in a tensor product, we see that $\alpha_i^{-1} \otimes \lambda_i$ and $\alpha_j^{-1} \otimes \lambda_j$ coincide on $f^{-1}(U_i \cap U_j)$. Thus, these elements define a unique global section $\lambda \in \Gamma(Y, f^*\mathcal{L})$, and by construction it's clear that $f^*\tau(\lambda^{\otimes n}) = 1$. This defines a map $\operatorname{Hom}_{\mathrm{Sch}/X}(Y, \Lambda) \to F(Y)$.

Conversely, let $\lambda \in F(Y)$. Since $f^*\tau(\lambda^{\otimes n}) = 1$, we see that λ must be a global generator of $f^*\mathcal{L}$. Since λ_i is a local generator of $f^*\mathcal{L}$ on $f^{-1}(U_i)$, we can write $\lambda|_{f^{-1}(U_i)} = \alpha_i^{-1} \otimes \lambda_i$. Now define $\mathcal{A} \to f_*\mathcal{O}_Y$ by mapping λ_i to $\alpha_i \in \Gamma(f^{-1}(U_i), \mathcal{O}_Y) = \Gamma(U_i, f_*\mathcal{O}_Y)$. Reading the above argument backwards, we obtain that this induces a well-defined morphism of $\mathcal{O}_X$-algebras. Thus, we obtain a map $F(Y) \to \operatorname{Hom}_{\mathrm{Sch}/X}(Y, \Lambda)$ which is, by construction, inverse to the map constructed in the previous paragraph. Up to checking functoriality (which we gladly omit), this shows that Λ represents F.

Step 2. We check that Λ is a $\mathbb{Z}/n\mathbb{Z}$-principal homogeneous space. With notation as before, have

$$\mathcal{A}|_{U_i} \cong \mathcal{O}_{U_i}\left[\sqrt[n]{\varepsilon_i}\right] \cong \mathcal{O}_X[T]/(T^n - \varepsilon_i)\,.$$

This is a finite étale $\mathcal{O}_X$-algebra, which can be seen for example via Proposition 1.4.16(*e*): indeed, the derivative $\partial(T^n - \varepsilon_i)/\partial T = nT^{n-1}$ is mapped to $n\lambda_i^{\otimes(n-1)} \in \Gamma(U_i, \mathcal{A})$, which is a unit as n is invertible on X by assumption and $\lambda_i^{\otimes(n-1)}$ has the inverse $\varepsilon_i^{-1}\lambda_i$. This shows that $\Lambda \to X$ is an étale covering.

Let $\sigma^{k,*} \colon \mathcal{A} \to \mathcal{A}$ be the automorphism of $\mathcal{A}$, which is defined on the l^{th} component as the multiplication map $\zeta^{kl} \colon \mathcal{L}^{\otimes l} \to \mathcal{L}^{\otimes l}$. This $\sigma^{k,*}$ defines a morphism $\sigma^k \colon \Lambda \to \Lambda$. Thus we have constructed a $\mathbb{Z}/n\mathbb{Z}$-action on Λ. If $k \neq 0$, then the coequalizer of the two morphisms $\mathrm{id}_{\mathcal{A}}, \sigma^{k,*} \colon \mathcal{A} \to \mathcal{A}$ is 0. Indeed, the ideal quotiented out contains the sections $(1 - \zeta^k)\lambda_i$, and these are units in $\mathcal{A}$ because λ_i is a unit and $\prod_{k \neq 0}(1 - \zeta^k) = n$ is invertible too, using that ζ is a primitive n^{th} root of unity. Thus $\operatorname{Eq}(\mathrm{id}_X, \sigma^k) = \emptyset$ and $\mathbb{Z}/n\mathbb{Z}$ acts fixed-point free. Moreover, $\mathcal{A}$ has rank $n = \#\mathbb{Z}/n\mathbb{Z}$ over $\mathcal{O}_X$, so Proposition 1.5.7(*b*) shows that Λ is indeed a $\mathbb{Z}/n\mathbb{Z}$-principal homogeneous space.

Step 3. We show that all $\mathbb{Z}/n\mathbb{Z}$-principal homogeneous spaces $\Lambda = \underline{\operatorname{Spec}}\,\mathcal{A}$ arise in the form constructed in Step 1. To this end, we will construct a suitable line bundle $\mathcal{L}$ via faithfully flat descent (or actually *étale descent*, if you want). Let $\{V_j \to X\}_{j\in J}$ be an étale covering such that there is a $\mathbb{Z}/n\mathbb{Z}$-equivariant isomorphism $\Lambda \times_X V_j \cong \mathbb{Z}/n\mathbb{Z} \times V_j$. Equivalently, if $\mathcal{A}_j$ denotes the pullback of $\mathcal{A}$ to V_j, we have

$$\mathcal{A}_j \cong \prod_{l=0}^{n-1} \mathcal{O}_{V_j}\,,$$

and $\sigma^{k,*}$ acts by a cyclic shift of factors. Let $e_l \in \Gamma(V_j, \mathcal{A}_j)$ be the idempotent global section corresponding to the l^{th} factor. Then $e_{k+l} = \sigma^{k,*}(e_l)$. For all $\sigma^k \in \mathbb{Z}/n\mathbb{Z}$ consider the element[8]

$$R_k = \sum_{l=0}^{n-1} \zeta^{k-l} e_l = \sum_{l=0}^{n-1} \zeta^{k-l} \sigma^{l,*}(e_0)\,.$$

Let $\mathcal{L}_j \subseteq \mathcal{A}_j$ be the $\mathcal{O}_{V_j}$-submodule generated by the elements R_k. Then $\mathcal{L}_j$ is actually free of rank 1. Indeed, the R_k are invertible (in fact, so is every coefficient ζ^{k-l} in their vector expression), and $R_{k+l} = \zeta^l R_k$, so they differ only by a unit. Moreover, $\mathcal{L}_i$ and $\mathcal{L}_j$ coincide after pullback to $V_i \times_X V_j$. The reason is that $R_{k+l} = \sigma^{l,*}(R_k)$, so the *set* $\{R_0, \dots, R_{n-1}\}$ (but not its particular order) only depends on the $\mathbb{Z}/n\mathbb{Z}$-action and no other choices. Thus, by faithfully flat descent, the $\mathcal{L}_j$ define an $\mathcal{O}_X$-submodule $\mathcal{L} \subseteq \mathcal{A}$ which is a line bundle. Since $R_k^n = 1$ for all k (again, this is clear from their vector expressions), we can further construct an isomorphism $\tau\colon \mathcal{L}^{\otimes n} \xrightarrow{\sim} \mathcal{O}_X$ via faithfully flat descent again. Finally, we claim that the canonical morphism $\bigoplus_{l=0}^{n-1} \mathcal{L}^{\otimes l} \xrightarrow{\sim} \mathcal{A}$ is an isomorphism. Again, this can be checked on the locally on the étale cover $\{V_j \to X\}_{j\in J}$. Since $\mathcal{L}_j$ is generated by R_0, what we need to prove is that $(1, R_0, R_0^2, \dots, R_0^{n-1})$ is a basis of $\mathcal{A}_j$. But this is—literally (!)—just a discrete Fourier transformation of the standard basis $(e_0, \dots, e_{n-1})$, hence a indeed a basis too (here we use again that n is invertible on X, because a wild n^{-1} occurs in the inverse discrete Fourier transformation).

Step 4. We verify that $(\mathcal{L}, \tau)$ is determined by Λ up to isomorphism. In fact, Step 3 provides a way to reconstruct a suitable pair $(\mathcal{L}, \tau)$, so we only need to check that this procedure is inverse to $(\mathcal{L}, \tau) \mapsto \mathcal{A} = \bigoplus_{l=0}^{n-1} \mathcal{L}^{\otimes l}$. We have already seen in Step 3 that starting with $\mathcal{A}$ and constructing $(\mathcal{L}, \tau)$ as described gives us $\mathcal{A}$ again. Conversely, let's start with $(\mathcal{L}, \tau)$ and verify that the pair $(\mathcal{L}', \tau')$ constructed from $\mathcal{A} = \bigoplus_{l=0}^{n-1} \mathcal{L}^{\otimes l}$ is isomorphic to the original pair $(\mathcal{L}, \tau)$. Let $\mathcal{A}_j$, $(\mathcal{L}_j, \tau_j)$, and $(\mathcal{L}_j', \tau_j')$ denote the respective pullbacks to V_j. We would like to show that $\mathcal{L}_j$ is generated by the R_k. Since $V_j \times_X \Lambda \cong \mathbb{Z}/n\mathbb{Z} \times V_j$, we get an embedding $V_j \hookrightarrow V_j \times_X \Lambda$ identifying V_j with $\{\sigma^0\} \times V_j$. Thus, after composition with the natural projection to Λ we obtain a morphism $V_j \to \Lambda$. By definition of F, this defines an element $\lambda_j \in F(V_j)$, which is a global generator of $\mathcal{L}_j$ satisfying $\lambda_j^{\otimes n} = 1$ in $\mathcal{A}_j$. In particular, the sequence $(1, \lambda_j, \dots, \lambda_j^{\otimes(n-1)})$ is a basis of $\mathcal{A}_j$. Moreover, we have $\sigma^{k,*}(\lambda_j) = \zeta^k \lambda_j$ for all k.[9] Now consider the elements

$$e_k' = \sum_{l=0}^{n-1} \zeta^{kl} \lambda_j^{\otimes l} \in \Gamma(V_j, \mathcal{A}_j)\,.$$

[8] In case you wondered: Professor Franke's intent when he gave the cryptic hint to use "Lagrange resolvents" was to look at expressions of that form.

[9] This looks like a triviality, but that's the result of carefully chosen abuse of notation. By definition of the $\mathbb{Z}/n\mathbb{Z}$-action on F, the element $\zeta^k\lambda_j$ is the image of λ_j under the action of σ^k, which luckily coincides with the image under $\sigma^{k,*}\colon \mathcal{A}_j \to \mathcal{A}_j$, because the image of λ_j under the action of σ^k corresponds to the morphism $V_j \to \Lambda$ coming from $\{\sigma^k\} \times V_j \hookrightarrow V_j \times_X \Lambda$.

We claim that $n^{-1}e'_0, \dots, n^{-1}e'_{n-1}$ are equal to $e_0, \dots, e_{n-1}$ from Step 3 (not necessarily in that order though, but the order is at most off by a cyclic shift). To prove this, first note that $\sigma^{k,*}(e'_l) = e'_{k+l}$ by our above observation. Moreover, we calculate

$$e'_{k_1} e'_{k_2} = \sum_{l=0}^{n-1} \sum_{l_1+l_2=l} \zeta^{k_1 l_1 + k_2 l_2} \lambda_j^{\otimes l} = \sum_{l=0}^{n-1} \sum_{l_1=0}^{n-1} \zeta^{k_1 l} \zeta^{(k_1-k_2)l_1} \lambda_j^{\otimes l} = \begin{cases} n e'_{k_1} & \text{if } k_1 = k_2 \\ 0 & \text{else} \end{cases} .$$

Thus, the $n^{-1}e'_k$ are mutually "orthogonal" idempotents. But then they must coincide (up to cyclic shift) with the e_k! Indeed, the e_k are mutually orthogonal idempotents too, and satisfy $\sigma^{k,*}(e_l) = e_{k+l}$, and by the classification of idempotents of $\Lambda \times_X \operatorname{Spec} \mathcal{O}_{X_{\text{ét}},\overline{x}}$ for any geometric point $\overline{x}$ of X (see the proof of $(*)$ in Proposition 1.5.7), there's only one such set of idempotents.

Now, by construction, the sequence $(n^{-1}e_0, \dots, n^{-1}e_{n-1})$ is the inverse Fourier transformation of the sequence $(1, \lambda_j, \dots, \lambda_j^{\otimes(n-1)})$. Likewise, by construction, the sequence $(1, R_0, \dots, R_0^{n-1})$ is the Fourier transformation of the sequence $(e_0, \dots, e_{n-1})$, which coincides with $(n^{-1}e_0, \dots, n^{-1}e_{n-1})$ up to cyclic shift, i.e., up to some $\sigma^{l,*}$. Thus, $R_0 = \sigma^{l,*}(\lambda_j) = \zeta^l \lambda_j$. So $\mathcal{L}_j$ is indeed generated by R_0, and we get an identification $\mathcal{L}_j \cong \mathcal{L}'_j$. Observe that this also identifies τ_j and τ'_j since they are defined via $R_0^n = 1$ and $\lambda_j^{\otimes n} = 1$ respectively. Thus, $(\mathcal{L}_j, \tau_j)$ and $(\mathcal{L}'_j, \tau'_j)$ are isomorphic. By faithfully flat descent, this shows that the pairs $(\mathcal{L}, \tau)$ and $(\mathcal{L}', \tau')$ are isomorphic, and we have finally proved (a).

Step 5. For (b), recall that by Theorem 1.5.10(a) the functor $\mathrm{Fib}_{\overline{x}}$ defines an equivalence between $\mathrm{F\acute{E}t}/X$ and the category of finite discrete continuous $\pi_1^{\text{ét}}(X, \overline{x})$-sets. By Proposition 1.5.7(b), the $\mathbb{Z}/n\mathbb{Z}$-principal bundles correspond precisely to those sets S with an additional simply transitive $\mathbb{Z}/n\mathbb{Z}$-action (so that S becomes isomorphic to $\mathbb{Z}/n\mathbb{Z}$), which, by functoriality, must commute with the $\pi_1^{\text{ét}}(X, \overline{x})$-action. Since $\mathbb{Z}/n\mathbb{Z}$ is cyclic, this entails that $\pi_1^{\text{ét}}(X, \overline{x})$ acts through $\mathbb{Z}/n\mathbb{Z}$ on S. Thus, the set $\operatorname{Hom}_{\text{cont}}(\pi_1^{\text{ét}}(X, \overline{x}), \mathbb{Z}/n\mathbb{Z})$ of continuous group homomorphisms is in bijection with the set of isomorphism classes of $\mathbb{Z}/n\mathbb{Z}$-principal bundles. Using (a), we thereby obtain a bijection

$$\operatorname{Hom}_{\text{cont}}\left(\pi_1^{\text{ét}}(X, \overline{x}), \mathbb{Z}/n\mathbb{Z}\right) \cong \{\text{isomorphism classes of } (\mathcal{L}, \tau)\} .$$

But moreover, both sides come equipped with a group structure: the one on the left is given by addition of two group homomorphisms, on the right it is induced by the tensor product. We claim that then the above bijection is even a group isomorphism.

Indeed, the identity element $(\mathcal{O}_X, \mathrm{id}_{\mathcal{O}_X})$ on the right-hand side corresponds to the étale covering

$$\Lambda_0 = \underline{\operatorname{Spec}}\left(\mathcal{O}_X[T]/(T^n - 1)\right) \cong \coprod_{k=0}^{n-1} \underline{\operatorname{Spec}}\left(\mathcal{O}_X[T]/\left(T - \zeta^k\right)\right) ,$$

which is split. Split coverings correspond to finite discrete sets S with trivial $\pi_1^{\text{ét}}(X, \overline{x})$-action (because there is a section $X \hookrightarrow \Lambda_0$ for every sheet of Λ_0, hence for every point $s \in S$ there is a $\pi_1^{\text{ét}}(X, \overline{x})$-equivariant map $\{s\} \hookrightarrow S$), thus $(\mathcal{O}_X, \mathrm{id}_{\mathcal{O}_X})$ is sent to the 0-morphism $0\colon \pi_1^{\text{ét}}(X, \overline{x}) \to \mathbb{Z}/n\mathbb{Z}$, which is exactly what we want.

Now consider two elements $(\mathcal{L}', \tau')$ and $(\mathcal{L}'', \tau'')$ and let $(\mathcal{L}, \tau) = (\mathcal{L}' \otimes \mathcal{L}'', \tau' \otimes \tau'')$; also let $\mathcal{A}'$, $\mathcal{A}''$, and $\mathcal{A}$ be as in Step 1. Then $\mathcal{A}$ is a direct summand of

$$\mathcal{A}' \otimes_{\mathcal{O}_X} \mathcal{A}'' \cong \bigoplus_{k,l=0}^{n-1} \left(\mathcal{L}'^{\otimes k} \otimes_{\mathcal{O}_X} \mathcal{L}''^{\otimes l}\right)$$

In particular, if we define $\Lambda = \underline{\operatorname{Spec}}\,\mathcal{A}$ and Λ', Λ'' similarly, then we obtain a natural morphism $\Lambda' \times_X \Lambda'' \to \Lambda$. Moreover, it's straightforward to check that for every k and l, the automorphism $\sigma^{k,*} \otimes \sigma^{l,*}$ on $\mathcal{A}' \otimes \mathcal{A}''$ restricts to an automorphism of $\mathcal{A}$, which coincides with $\sigma^{k+l,*}\colon \mathcal{A} \to \mathcal{A}$. Since the group $\pi_1^{\text{ét}}(X, \overline{x})$ acts through $\mathbb{Z}/n\mathbb{Z}$ as noted above, this shows that Λ indeed corresponds to the sum of the two continuous morphisms $\pi_1^{\text{ét}}(X, \overline{x}) \to \mathbb{Z}/n\mathbb{Z}$ given by Λ' and Λ''. Therefore we get a group isomorphism, as claimed.

Step 6. We put the pieces together: the morphism $\operatorname{Hom}_{\text{cont}}(\pi_1^{\text{ét}}(X, \overline{x}), \mathbb{Z}/n\mathbb{Z}) \to \operatorname{Pic}(X)$ sends a pair $(\mathcal{L}, \tau)$ to $\mathcal{L}$. It is immediately clear that its image coincides with the kernel of $(-)^{\otimes n}\colon \operatorname{Pic}(X) \to \operatorname{Pic}(X)$, so the sequence is exact at the first $\operatorname{Pic}(X)$. Given $\mathcal{L}$, the isomorphism $\tau\colon \mathcal{L}^{\otimes n} \xrightarrow{\sim} \mathcal{O}_X$ is unique up to a global section of $\mathcal{O}_X^\times$, whence we get exactness at $\operatorname{Hom}_{\text{cont}}(\pi_1^{\text{ét}}(X, \overline{x}), \mathbb{Z}/n\mathbb{Z})$. But $\mathcal{L}$ too is only determined up to isomorphism, i.e., up to a global section of $\mathcal{O}_X^\times$, and scaling $\mathcal{L}$ by $\varepsilon \in \Gamma(X_{\text{Zar}}, \mathcal{O}_X^\times)$ means scaling τ by ε^n. Hence the sequence is exact at the second $\Gamma(X_{\text{Zar}}, \mathcal{O}_X^\times)$, and thus it is exact after all. This finishes the proof of (b). $\square$

1.5.12. Proposition ("Artin–Schreier coverings")**.** — *Let p be a prime and X a scheme over $\mathbb{F}_p$. Let $\mathcal{T}$ be an $\mathcal{O}_X$-torsor (in the Zariski topology). Let $\varphi = \operatorname{Frob}_X$ denote the absolute Frobenius on X and let $\tau\colon \varphi^*\mathcal{T} \xrightarrow{\sim} \mathcal{T}$ be an isomorphism. Consider the functor*

$$\begin{aligned} F\colon (\mathrm{Sch}/X)^{\mathrm{op}} &\longrightarrow \mathrm{Set} \\ (f\colon Y \to X) &\longmapsto \{t \in \Gamma(Y, \mathcal{T}) \mid f^*\tau(t) = t\}\ , \end{aligned}$$

and note that $\mathbb{Z}/p\mathbb{Z}$ acts on F via $t \mapsto k + t$ for $k \in \mathbb{Z}/p\mathbb{Z}$.

(a) *F is representable by a $\mathbb{Z}/p\mathbb{Z}$-principal homogeneous space Λ. Moreover, there is a bijection between isomorphism classes of $\mathbb{Z}/p\mathbb{Z}$-principal homogeneous spaces and isomorphism classes as of $(\mathcal{T}, \tau)$ similar to Proposition 1.5.11.*

(b) *If X is connected, then for any geometric point $\overline{x}$ there is a long exact sequence*

$$\begin{aligned} 0 \longrightarrow \mathbb{Z}/p\mathbb{Z} \longrightarrow \Gamma(X_{\text{Zar}}, \mathcal{O}_X) &\xrightarrow{\varphi^* - \mathrm{id}} \Gamma(X_{\text{Zar}}, \mathcal{O}_X) \longrightarrow \\ \longrightarrow \operatorname{Hom}_{\text{cont}}\left(\pi_1^{\text{ét}}(X, \overline{x}), \mathbb{Z}/p\mathbb{Z}\right) \longrightarrow H^1(X_{\text{Zar}}, \mathcal{O}_X) &\xrightarrow{\varphi^* - \mathrm{id}} H^1(X_{\text{Zar}}, \mathcal{O}_X) \longrightarrow \dots\ . \end{aligned}$$

Sketch of a proof.* Since the proof is very similar to Proposition 1.5.11, we will only highlight the differences. We begin with a somewhat subtle one: at the heart of the proof of Proposition 1.5.11 was the theorem of faithfully flat descent, allowing us to work locally in the étale topology. But faithfully flat descent is a statement about quasi-coherent modules, not about torsors. However, we are lucky, and étale descent still works for $\mathcal{O}_X$-torsors. In fact, Zariski $\mathcal{O}_X$-torsors are parametrized by $\check{H}^1(X_{\text{Zar}}, \mathcal{O}_X)$. Likewise, étale $\mathcal{O}_{X_{\text{ét}}}$-torsors are parametrized by the first *étale Čech cohomology* $\check{H}^1(X_{\text{ét}}, \mathcal{O}_{X_{\text{ét}}})$. And these two cohomology groups happen to be isomorphic! Indeed, Čech cohomology coincides with sheaf cohomology in degree 1 (both in the Zariski and the étale topology), and for quasi-coherent sheaves, Zariski and étale cohomology coincide (see [Stacks, Tag 03OY] for a proof). Thus we do indeed have étale descent for $\mathcal{O}_X$-torsors. For more about torsors, check out Section 2.3

Next, given $(\mathcal{T}, \tau)$, we construct $\Lambda = \underline{\operatorname{Spec}}\,\mathcal{A}$ as follows: let $\mathcal{O}_X[\mathcal{T}]$ be the polynomial algebra generated by the sections of $\mathcal{T}$ as free variables, modulo the "obvious relations". That is, if $U \subseteq X$ is open, $t, t' \in \Gamma(U, \mathcal{T})$ and $a \in \Gamma(U, \mathcal{O}_X)$ are sections over U such that $t + a = t'$, then the same relation should hold in $\mathcal{O}_X[\mathcal{T}]$. Define $\theta\colon \mathcal{T} \to \mathcal{O}_X$ via $\theta(t) = \tau(t) - t$ and let $\mathcal{I} \subseteq \mathcal{O}_X[\mathcal{T}]$ be the ideal generated by $t^p - t - \theta(t)$ for all sections t of $\mathcal{T}$ (here t^p is to be read as the p^{th} power of the formal variable t). Then we put $\mathcal{A} = \mathcal{O}_X[\mathcal{T}]/\mathcal{I}$. Thus, on small enough affine open subsets $U \subseteq X$ (such that $\mathcal{T}$ trivializes) we have $\mathcal{A}|_U \cong \mathcal{O}_U[\alpha]$, where α satisfies $\alpha^p - \alpha - a = 0$ for some $a \in \Gamma(U, \mathcal{O}_X)$. So $\Lambda = \underline{\operatorname{Spec}}\, \mathcal{A} \to X$ is finite étale by Proposition 1.4.16(*e*).

To show that Λ really represents F, we use a similar construction as in Step 1 of the proof of Proposition 1.5.11. To reconstruct $(\mathcal{T}, \tau)$ from Λ, we proceed as in Step 3 and 4, but this time we replace the "Lagrange resolvents" by

$$R_k = \sum_{l=0}^{p-1} (l-k)e_l = \sum_{l=0}^{p-1} (l-k)\sigma^{l,*}(e_0)\,.$$

Then $R_k + l = \sigma^{l,*}(R_k) = R_{k+l}$. To show that $(1, R_0, \dots, R_0^{p-1})$ is a basis, we use that the Vandermonde determinant doesn't vanish instead of the discrete Fourier transform being invertible.

Finally, to prove (*b*), we first identify $\operatorname{Hom}_{\text{cont}}(\pi_1^{\text{ét}}(X, \overline{x}), \mathbb{Z}/p\mathbb{Z})$ with the set of isomorphism classes of $(\mathcal{T}, \tau)$ and an argument analogous to Step 6 works. However, some extra care is needed to ensure exactness at $\mathbb{Z}/p\mathbb{Z}$, since this becomes wrong if X is not connected (in contrast to $\Gamma(X, \mu_n) \to \Gamma(X, \mathcal{O}_X^\times)$ from Proposition 1.5.11(*b*), which is still injective for non-connected X). So suppose $a \in \Gamma(X, \mathcal{O}_X)$ is in the kernel of $\varphi^* - \mathrm{id}$, i.e., $a^p - a = 0$. Then

$$X = V(a^p - a) = V\left(\prod_{k \in \mathbb{F}_p} (a-k)\right) = \bigcup_{k \in \mathbb{F}_p} V(a-k)\,.$$

Observe that $V(a-k) \cap V(a-l) \subseteq V(k-l) = \emptyset$ for $k \neq l$, since then $k-l$ is invertible in $\mathbb{F}_p$. Thus the union above is a disjoint union, which can only happen if $V(a-k) = X$ for some k and $V(a-l) = \emptyset$ for all $l \neq k$. But then all $(a-l)$ are units, hence $0 = a^p - a = (a-k)\prod_{l \neq k}(a-l)$ shows $a = k$. Thus the kernel of $\varphi^* - \mathrm{id}$ is indeed given by $\mathbb{Z}/p\mathbb{Z}$. $\square$

1.5.13. Remark. — The long exact sequences from Proposition 1.5.11(*b*) and Proposition 1.5.12(*b*) are, in fact, long exact sequences of étale cohomology groups, associated to the short exact sequences

$$0 \longrightarrow \mu_n \longrightarrow \mathcal{O}_{X_{\text{ét}}}^\times \xrightarrow{(-)^n} \mathcal{O}_{X_{\text{ét}}}^\times \longrightarrow 0$$
$$0 \longrightarrow \mathbb{Z}/p\mathbb{Z} \longrightarrow \mathcal{O}_{X_{\text{ét}}} \xrightarrow{\varphi^* - \mathrm{id}} \mathcal{O}_{X_{\text{ét}}} \longrightarrow 0\,.$$

They are not exact as sequences of Zariski sheaves, but as étale sheaves they are. As a side effect, we find that $H^1(X, \mathcal{O}_X^\times) \cong \operatorname{Pic}(X) \cong H^1(X_{\text{ét}}, \mathcal{O}_{X_{\text{ét}}}^\times)$ and $H^1(X, \mathcal{O}_X) = H^1(X_{\text{ét}}, \mathcal{O}_{X_{\text{ét}}})$. The first chain of isomorphisms is basically equivalent to the fact that étale descent works for line bundles. The second chain of isomorphism says the same about étale descent of torsors, as noted in the proof of Proposition 1.5.12.

1.5.14. Remark. — If k is a field of characteristic $p > 0$, one has $\pi_1^{\text{ét}}(\mathbb{A}_k^n, \overline{x}) \neq 1$ for any base point x, which comes perhaps a bit counterintuitive. This holds even in the case where k is separably or even algebraically closed (for non-separably closed k we shouldn't expect $\pi_1^{\text{ét}}(\mathbb{A}_k^1, \overline{x})$ to be trivial, since the absolute Galois group $\operatorname{Gal}(k^{\text{sep}}/k) = \pi_1^{\text{ét}}(\operatorname{Spec} k)$ should somehow come into play); the reason is that there are Artin–Schreier coverings.

1.5.4. The Zariski–Nagata Purity Theorem

Before we give a rough sketch of the proof of the theorem in the title, we prove a proposition that supplements Remark 1.5.14.

1.5.15. Proposition. — *Let k be an algebraically (or just separably) closed field.*

(*a*) $\pi_1^{\text{ét}}(\mathbb{P}_k^n, \overline{x}) = 1$ *for all base points.*

(*b*) *If X is a regular scheme of finite type over k, and $\pi\colon \widetilde{X} \to X$ the blow-up of a closed point, then $\pi_*\colon \pi_1^{\text{ét}}(\widetilde{X}, \overline{x}) \xrightarrow{\sim} \pi_1^{\text{ét}}(X, \overline{x})$ is an isomorphism for all base points.*

Sketch of a proof. We prove (*a*) and (*b*) simultaneously by induction on the dimension. Part (*a*) is equivalent to the condition that all étale coverings of $\mathbb{P}_k^n$ split, and this is equivalent to the condition that all étale coverings $\underline{\operatorname{Spec}}\,\mathcal{A} \to \mathbb{P}_k^n$ have $\mathcal{A} \cong \prod_{i=1}^d \mathcal{O}_{\mathbb{P}^n}$ for some d. For $n = 1$ we have

$$\mathcal{A} \cong \bigoplus_{i=1}^{d} \mathcal{O}(d_i)$$

by the Grothendieck–Birkhoff theorem. We first claim that there are no positive d_i. Indeed, let $\mathcal{A}_{\geqslant n} = \bigoplus_{d_i \geqslant n} \mathcal{O}(d_i)$ and consider the multiplication map $\mu\colon \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ given by the $\mathcal{O}_X$-algebra structure. Since $\mathcal{O}(n) \otimes \mathcal{O}(m) \cong \mathcal{O}(n+m)$ and for $i > j$ there is no non-zero morphism $\mathcal{O}(i) \to \mathcal{O}(j)$, we see that μ maps $\mathcal{A}_{\geqslant n} \otimes \mathcal{A}_{\geqslant m} \to \mathcal{A}_{\geqslant n+m}$. In particular, all $\mathcal{O}(d_i)$ with $d_i > 0$ must be contained in $\operatorname{nil}(\mathcal{A})$. But $\underline{\operatorname{Spec}}\,\mathcal{A}$ is regular by Proposition 1.4.14, hence $\operatorname{nil}(\mathcal{A}) = 0$.

So there are no positive d_i. Since $\mathcal{A}$ is self-dual by Proposition 1.4.10, there are no negative d_i as well, hence $\mathcal{A} \cong \mathcal{O}_{\mathbb{P}^1}^{\oplus d}$. We are not done yet, since this is just an isomorphism of *modules* over $\mathcal{O}_{\mathbb{P}^1}$, not of algebras. Let $x \in \mathbb{P}^1$ be any k-rational point (e.g., the point at infinity). Since $\Gamma(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) = k = \kappa(x)$ and $\mathcal{A}$ is a trivial vector bundle, we see that the canonical morphism $\Gamma(\mathbb{P}^1, \mathcal{A}) \to \mathcal{A}_x \otimes \kappa(x)$ is an isomorphism of k-algebras. But $\mathcal{A}_x \otimes \kappa(x)$ is étale over $\kappa(x) = k$, which is separably closed, hence $\mathcal{A}_x \otimes \kappa(x)$ is a product of d copies of k by Lemma 1.4.9(*b*). Then the same is true for $\Gamma(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1})$, so we can find d non-zero orthogonal idempotent global sections $e_1, \dots, e_d$. Thus $\mathcal{A} \cong \prod_{i=1}^d e_i\mathcal{A} \cong \prod_{i=1}^d \mathcal{O}_{\mathbb{P}^1}$ holds as algebras too. An alternative proof of the $\mathbb{P}_k^1$ case uses the Hurwitz formula.

Part (*b*) is trivial when $\dim X = 1$, since then blowing up a closed point doesn't change anything. Now assume (*a*) holds in dimension n. To prove (*b*) in dimension $n+1$, we would like to show that the functor

$$\begin{aligned} \pi^*\colon \{\text{étale coverings of } X\} &\xrightarrow{\sim} \{\text{étale coverings of } \widetilde{X}\} \\ \left(\underline{\operatorname{Spec}}\,\mathcal{B} \to X\right) &\longmapsto \left(\underline{\operatorname{Spec}}\,\pi^*\mathcal{B} \to \widetilde{X}\right) \end{aligned}$$

is an equivalence of categories. Our strategy is to show that $\mathcal{A} \mapsto \pi_*\mathcal{A}$ is a quasi-inverse. So let $\underline{\operatorname{Spec}}\,\mathcal{A} \to \widetilde{X}$ be an étale covering of $\widetilde{X}$ and let d be its degree. Since π is proper, $\pi_*\mathcal{A}$ is at least a coherent $\mathcal{O}_X$-module. Moreover, if $\widetilde{X}_0$ denotes the unique non-trivial fibre of π, and $\widetilde{X}_m$ its infinitesimal thickenings, then $\mathcal{A}|_{\widetilde{X}_0}$ is split by the induction assumption because $\widetilde{X}_0 \cong \mathbb{P}_{\kappa(x)}^n$, hence the $\mathcal{A}|_{\widetilde{X}_m}$ are split by Proposition 1.4.20. In other words, $\mathcal{A}|_{\widetilde{X}_m} \cong \prod_{i=1}^d \mathcal{O}_{\widetilde{X}_m}$. Let $y \in X$ be the blown-up point. Then the theorem of formal functions shows

$$(\pi_*\mathcal{A})^\wedge_y \cong \lim_{m \in \mathbb{N}} \Gamma\left(\widetilde{X}_m, \mathcal{A}|_{\widetilde{X}_m}\right) \cong \prod_{i=1}^d \lim_{m \in \mathbb{N}} \Gamma\left(\widetilde{X}_m, \mathcal{O}_{\widetilde{X}_m}\right) \cong \prod_{i=1}^d \lim_{m \in \mathbb{N}} \mathcal{O}_{X,y}/\mathfrak{m}_{X,y}^{m+1} \cong \prod_{i=1}^d \widehat{\mathcal{O}}_{X,y}\,.$$

Note that the third isomorphism is actually not that trivial since for $m > 0$ the thickened fibres $\widetilde{X}_m$ are no longer isomorphic to the projective space $\mathbb{P}^n_R$ over the ring $R = \mathcal{O}_{X,y}/\mathfrak{m}_{X,y}^{m+1}$; see Lemma* 1.5.16 below.

The above calculation shows that $(\pi_*\mathcal{A})_y$ is flat over $\mathcal{O}_{X,y}$. Moreover, it's clear that $\pi_*\mathcal{A}$ is a vector bundle on $X \setminus \{x\}$ since π is just the identity on that open subset. Thus $\pi_*\mathcal{A}$ is indeed a vector bundle. Finally, $\underline{\operatorname{Spec}}\, \pi_*\mathcal{A} \to X$ is an étale covering because it's restriction to $X \setminus \{x\}$ is finite étale for obvious reasons, and the closed point $\{x\}$ has codimension $\geqslant 2$ as $\dim X \geqslant 2$, so Corollary 1.4.11 can be applied.

This shows that $\pi_*\colon \mathrm{F\acute{E}t}/\widetilde{X} \to \mathrm{F\acute{E}t}/X$ is a functor in the reverse direction. To show that π^* and π_* are quasi-inverse, we must show that the canonical morphisms $\alpha\colon \pi^*\pi_*\mathcal{A} \to \mathcal{A}$ and $\beta\colon \mathcal{B} \to \pi_*\pi^*\mathcal{B}$ are isomorphisms. We will only prove this for α; the argument for β is very similar. It's clear that α is a morphism of vector bundles of the same rank, so it suffices to prove that it is an epimorphism. This is obvious over $X \setminus \{x\}$ since π is just the identity there. It remains show that α is an epimorphism at every point lying over the blown-up point y. Since we are only interested in having an epimorphism, this can be tested after pullback to the fibre $\widetilde{X}_0$. But $\mathcal{A}|_{\widetilde{X}_0}$ is a trivial vector bundle as noted before, and $\pi^*\pi_*\mathcal{A}|_{\widetilde{X}_0}$ must be trivial as well, hence the pullback of α is an isomorphism, hence α is an epimorphism and thus an isomorphism as well. This proves the inductive step for (b).

For (a) with $n \geqslant 2$ we may replace $X = \mathbb{P}^n_k$ by the blow-up $\widetilde{X}$ of a k-rational point, since (b) has already been established in dimension n. Now we claim that there is a morphism $\pi\colon \widetilde{X} \to \mathbb{P}^{n-1}_k$ whose fibres over k-rational points are isomorphic to $\mathbb{P}^1_k$. We will only sketch how π looks like on k-rational points and leave it to the reader to work out the scheme-theoretic construction. Without restriction let $0 = [0 : \ldots : 0 : 1]$ be the blown-up point. Then we obtain a morphism $X \setminus \{0\} \to \mathbb{P}^{n-1}_k$ by sending k-rational points $0 \neq [t_0 : \ldots : t_n]$ to $[t_0 : \ldots : t_{n-1}]$. It's easy to see that fibres over k-rational points are isomorphic to $\mathbb{A}^1_k$. Now the fibre $\widetilde{X}_0$ over the blown-up point 0 is isomorphic to $\mathbb{P}^{n-1}_k$. Mapping $\widetilde{X}_0$ identically to $\mathbb{P}^{n-1}_k$ induces the map $\pi\colon \widetilde{X} \to \mathbb{P}^{n-1}_k$. Its fibres over k-rational points are $\mathbb{P}^1_k$ because they comprise the fibre $\mathbb{A}^1_k$ of $X \setminus \{0\} \to \mathbb{P}^{n-1}_k$ plus one additional point from $\widetilde{X}_0$.

Now that $\pi\colon \widetilde{X} \to \mathbb{P}^{n-1}_k$ has been constructed, we can show via the formal function theorem as in (b) that

$$\pi^*\colon \left\{\text{étale coverings of } \mathbb{P}^{n-1}_k\right\} \xrightarrow{\sim} \{\text{étale coverings of } \widetilde{X}\}$$

is an equivalence of categories, with quasi-inverse π_*. So $\pi_1^{\mathrm{\acute{e}t}}(\mathbb{P}^n_k, \overline{x}) \cong \pi_1^{\mathrm{\acute{e}t}}(\widetilde{X}, \overline{x}) \cong \pi_1^{\mathrm{\acute{e}t}}(\mathbb{P}^{n-1}_k, \overline{y})$ for some base point $\overline{x}$ that is mapped to $\overline{y}$. There's one caveat though: this time we can't use Corollary 1.4.11 to show that $\underline{\operatorname{Spec}}\, \pi_*\mathcal{A} \to \mathbb{P}^{n-1}_k$ is étale again. Instead we can do an argument as in the proof of Proposition 1.4.10 to reduce this to a question on fibres, and then use that the fibres over k-rational points look like $\mathbb{P}^1_k$ and that the k-rational points are dense in $\mathbb{P}^{n-1}_k$. □

1.5.16. Lemma*. — *Let X be a noetherian scheme and $y \in X$ a regular closed point. Let $\pi\colon \widetilde{X} \to X$ be the blow-up of y, $\widetilde{X}_0$ the fibre over y, and $\widetilde{X}_m$ its infinitesimal thickenings. Then, for all $m \geqslant 0$,*

$$\Gamma\left(\widetilde{X}_m, \mathcal{O}_{\widetilde{X}_m}\right) \cong \mathcal{O}_{X,y}/\mathfrak{m}_{X,y}^{m+1}\,.$$

Sketch of a proof.* Our strategy is to mimic the calculation of cohomology of twisting sheaves on $\mathbb{P}^n$ via comparison of the Čech complex and the Koszul complex (see [AG_2, Theorem 2] for example). Fix $m \geqslant 0$ and let $R = \bigoplus_{i \geqslant 0} \mathfrak{m}_{X,y}^i/\mathfrak{m}_{X,y}^{i+m+1}$. It's straightforward to check that $\widetilde{X}_m \cong \operatorname{Proj} R$. Since $\mathcal{O}_{X,y}$ is regular, its maximal ideal $\mathfrak{m}_{X,y}$ can be generated by a regular sequence $(t_1, \dots, t_n)$. In particular, every quotient $\mathcal{O}_{X,y}/(t_1, \dots, t_i)$ is regular again. Henceforth, the sequence $(t_1, \dots, t_n)$ will be considered as elements R that have homogeneous degree 1, i.e., as elements of $R_1 = \mathfrak{m}_{X,y}/\mathfrak{m}_{X,y}^{m+1}$!!! We first claim that $(t_1, \dots, t_n)$ is again a regular sequence in R. Using induction on $n = \dim \mathcal{O}_{X,y}$, we only need to check that multiplication with t_1 is injective on R. For $m = 0$ this is trivial because $R \cong \operatorname{gr}_{\mathfrak{m}_{X,y}}(\mathcal{O}_{X,y}) \cong k[t_1, \dots, t_n]$ by a well-known result about regular rings. The general case can be deduced by another induction.

Now let $\mathcal{U}\colon \operatorname{Proj} R = \bigcup_{i=1}^n D_+(t_i)$ be the standard affine open cover associated to the generators t_i of R_1. As for every graded ring R that is generated by R_0 and R_1, we have sheaves $\mathcal{O}(d)$ on $\operatorname{Proj} R$ for all $d \in \mathbb{Z}$. Consider $\mathcal{O} = \bigoplus_{d \in \mathbb{Z}} \mathcal{O}(d)$. For all $1 \leqslant i_0 < \dots < i_l \leqslant n$ we have

$$\Gamma\left(D_+(t_{i_0} \cdots t_{i_l}), \mathcal{O}\right) \cong R\big[(t_{i_0} \cdots t_{i_l})^{-1}\big] \cong \operatorname{colim}\left(R \xrightarrow{t_{i_0} \cdots t_{i_l}} R \xrightarrow{t_{i_0} \cdots t_{i_l}} \dots\right).$$

For all $j \geqslant 1$ let $t^j = (t_1^j, \dots, t_n^j)$ and let $K^\bullet(t^j, R)$ denote the corresponding cohomological Koszul complex. Moreover, let $\check{C}^\bullet_{\mathrm{alt}}(\mathcal{U}, \mathcal{O})$ be the alternating Čech complex associated to the open cover $\mathcal{U}$. Then there is a canonical map

$$K^{l+1}(t^j, R) \cong \bigwedge^{l+1} R^n \longrightarrow \prod_{1 \leqslant i_0 < \dots < i_l \leqslant n} R\big[(t_{i_0} \cdots t_{i_l})^{-1}\big] \cong \check{C}^l_{\mathrm{alt}}(\mathcal{U}, \mathcal{O})$$

sending the basis vectors $e_{i_0} \wedge \dots \wedge e_{i_l}$ of $\bigwedge^{l+1} R^n$ to $(0, \dots, (t_{i_0} \cdots t_{i_l})^{-j}, \dots, 0)$. Up to checking various compatibilities, these maps assemble into an isomorphism of complexes

$$\operatorname*{colim}_{j \geqslant 1} K^\bullet(t^j, R)[-1] \xrightarrow{\sim} \left(R \to \check{C}^\bullet(\mathcal{U}, \mathcal{O})\right)$$

(the R on the right-hand side is placed in degree -1). But t is a regular sequence as shown above, hence the sequences t^j are regular too, thus the colimit of Koszul complexes is exact in low degrees. Therefore, $R \to \check{C}^\bullet(\mathcal{U}, \mathcal{O})$ is exact in low degrees too, hence $H^0(\check{C}^\bullet(\mathcal{U}, \mathcal{O})) \cong R$. After some unraveling, this proves the assertion. $\square$

1.5.17. Remark. — If $X \to Y$ is a morphism satisfying some suitable assumptions, and if $\overline{x}$ is a geometric point of X that is mapped to a geometric point $\overline{y}$ of Y, and $y \to Y$ there is an exact sequence

$$\pi_1^{\text{ét}}(X_y, \overline{x}) \longrightarrow \pi_1^{\text{ét}}(X, \overline{x}) \longrightarrow \pi_1^{\text{ét}}(Y, \overline{y}) \longrightarrow \pi_0(X_y)\,,$$

which can be established by a similar application of the formal function theorem.

We now reach the highlight of this section about the étale fundamental group: the Zariski–Nagata purity theorem (or actually a version of it). A full proof is in [SGA_1, Exposé X Corollaire 3.3]

1.5.18. Theorem (Zariski/Nagata). — *Let X be a separated noetherian regular scheme and $j\colon U \hookrightarrow X$ an open subscheme such that every irreducible component of $X \setminus U$ has codimension $\geqslant 2$. Then restriction to U is an equivalence of categories*

$$j^*\colon \{\text{étale coverings of } X\} \xrightarrow{\sim} \{\text{étale coverings of } U\}\,.$$

In particular, there is an isomorphism $j_*\colon \pi_1^{\text{ét}}(U, \overline{x}) \xrightarrow{\sim} \pi_1^{\text{ét}}(X, \overline{x})$ for every base point $\overline{x}$.

1.5.19. Remark. — Before we start with the proof, let's recall the conditions R_k and S_k, since they are going to be used in the proof. For a coherent $\mathcal{O}_X$-module $\mathcal{F}$ on a locally noetherian scheme X, we say $\mathcal{F}$ *satisfies condition* S_k' if for all $x \in X$ we have $\operatorname{depth} \mathcal{F}_x \geqslant \min\{k, \dim \mathcal{O}_x\}$.[10] For small k we have some nice equivalent reformulations:

(1) *$\mathcal{F}$ satisfies S_1' iff $\Gamma(U, \mathcal{F}) \hookrightarrow \Gamma(V, \mathcal{F})$ is injective whenever V is a dense open subset of the open subset $U \subseteq X$.*

(2) *$\mathcal{F}$ satisfies S_2' iff it satisfies S_1' and $\Gamma(U, \mathcal{F}) \xrightarrow{\sim} \Gamma(V, \mathcal{F})$ whenever $V \subseteq U$ are open subsets of X such that* $\operatorname{codim}(Z, U) \geqslant 2$ *for every connected component Z of $U \setminus V$.*

For proofs, see the appendix, Lemma* A.2.3. Moreover, we say X *satisfies* R_k iff $\mathcal{O}_{X,x}$ is regular whenever $\dim \mathcal{O}_{X,x} \leqslant k$. Again, for small k there are some famous reformulations:

(a) *X is reduced iff it satisfies R_0 and S_1.*

(b) *X is normal iff it satisfies R_1 and S_2. This is known as* Serre's normality criterion*.*

Sketch of a proof of Theorem 1.5.18. The idea is to construct a quasi-inverse functor. Suppose $\underline{\operatorname{Spec}}\, \mathcal{A} \to U$ is an étale covering of U. We need to extend it to all of X. A straightforward candidate for an extension is $\underline{\operatorname{Spec}}\, j_* \mathcal{A} \to X$. By elementary scheme theory, $j_*\mathcal{A}$ is at least a quasi-coherent $\mathcal{O}_X$-algebra. If we could show that it is a vector bundle, then Corollary 1.4.11 would automatically imply that $\underline{\operatorname{Spec}}\, j_*\mathcal{A} \to X$ is étale. To prove that the ensuing functor j_* is a quasi-inverse of j^*, it suffices to prove that the natural morphisms $j^* j_* \mathcal{A} \to \mathcal{A}$ and $\mathcal{B} \to j_* j^* \mathcal{B}$ are isomorphisms when $\mathcal{A}$ and $\mathcal{B}$ are vector bundles over U and X respectively. The former is trivial, and the latter follows from Remark 1.5.19(2) as vector bundles over a regular scheme satisfy S_2' for trivial reasons.

By a finiteness theorem for the cohomology of open subsets (see [SGA$_2$, Exposé VIII Proposition 3.2], but only a rather trivial special case is used), $j_*\mathcal{A}$ is coherent. It is still hard to show that it is a vector bundle.

If dimension $\dim X \leqslant 2$ however, this is rather easy. Note that $\mathcal{F}$ is locally free iff every $\mathcal{F}_x$ has projective dimension 0. By the Auslander–Buchsbaum formula (which is applicable because regular rings local rings have finite projective dimension) we have

$$\operatorname{pr.dim} \mathcal{F}_x + \operatorname{depth} \mathcal{F}_x = \operatorname{depth} \mathcal{O}_{X,x} = \dim \mathcal{O}_{X,x}\,,$$

using that $\mathcal{O}_{X,x}$ is regular. Since $\dim \mathcal{O}_{X,x} \leqslant 2$, an easy argument shows that $\operatorname{pr.dim} \mathcal{F}_x = 0$ iff $\mathcal{F}$ has S_2' at x. Thus, in dimension $\leqslant 2$ a coherent module is locally free iff it has S_2'. Now $\mathcal{F} = j_*\mathcal{A}$ has S_2' by Remark 1.5.19(2) and the fact that $\mathcal{A}$ already has S_2' because it is a vector bundle over the regular scheme U.

The general case of $j_*\mathcal{A}$ being a vector bundle (or equivalently being flat) is dealt with by noetherian and ordinary induction. Using noetherian induction, we may assume the assertion is true for all proper closed subsets $Z' \subsetneq Z$. Let η be a generic point of an irreducible component of Z. If we can show that $j_*\mathcal{A}$ is flat at η, then it is flat over some neighbourhood U' of η too. Then U and Z may be replaced by $U \cup U'$ and $Z \setminus U'$ and we can apply the induction hypothesis.

To show that $j_*\mathcal{A}$ is flat at η, it suffices that $(j_*\mathcal{A})_\eta^\wedge$ is flat over $\widehat{\mathcal{O}}_{X,\eta}$. Thus, we may replace $j \colon U \hookrightarrow X$ by the flat base change $U \times_X \operatorname{Spec} \mathcal{O}_{X,\eta} \hookrightarrow \operatorname{Spec} \mathcal{O}_{X,\eta}$ (flatness is crucial to make j_* commute with pullbacks). Thus, we may assume that $X = \operatorname{Spec} A$ is a complete

[10] In the lecture, Professor Franke just called this *condition* S_k, but this doesn't seem to be the standard definition (see Definition* A.2.1(b) instead), However, with the standard definition, (1) becomes wrong: see Warning* A.2.2.

regular local ring with maximal ideal $\mathfrak{m} = \mathfrak{m}_{X,\eta}$, and $U = X \setminus \{\mathfrak{m}\}$. Let $t \in \mathfrak{m} \setminus \mathfrak{m}^2$ and let $X_n = V(t^n)$, $U_n = V(t^n|_U)$. Now comes a rather strange argument: since $\dim A \geqslant 3$ and $\mathcal{A}$ is a vector bundle on U, we may apply [SGA$_2$, Exposé IX Proposition 1.4] to get

$$\Gamma(U, \mathcal{A}) \xrightarrow{\sim} \lim_{n \in \mathbb{N}} \Gamma(U_n, \mathcal{A}|_{U_n}) . \tag{1.5.1}$$

Where does this "$\dim A \geqslant 3$" come from? Conditions (a) and (c) from *loc. cit.* are trivially satisfied, so we only need to check (b). Since we are only interested in global sections, i.e., H^0, what we need to check is $\operatorname{depth}(j_*\mathcal{A})_x \geqslant 2$ for all $x \in X$ satisfying $\operatorname{codim}(\overline{\{x\}} \cap \{\mathfrak{m}\}, \overline{\{x\}}) = 1$. Since every closed subset contains $\{\mathfrak{m}\}$, such points x correspond to prime ideals $\mathfrak{p} \in \operatorname{Spec} A$ satisfying $1 = \operatorname{codim}(\{\mathfrak{m}\}, V(\mathfrak{p})) = \dim A/\mathfrak{p}$. Since A is regular, thus catenary, we get $\dim A_\mathfrak{p} = \dim A - 1 \geqslant 2$. But $j_*\mathcal{A}$ satisfies S_2' as noted before, hence $\dim A_\mathfrak{p} \geqslant 2$ implies $\operatorname{depth}(j_*\mathcal{A})_x \geqslant 2$, as needed. So in some sense we are lucky that we found an argument that works precisely for $\dim X \leqslant 2$ and one that works only for $\dim X \geqslant 3$.

Since $t \in \mathfrak{m} \setminus \mathfrak{m}^2$, the ring A/tA is regular again, and has dimension $\dim A - 1$. Using induction on the dimension, we may thus assume that the theorem is true for X_1 and U_1 as above. Thus, $\underline{\operatorname{Spec}}\, \mathcal{A}|_{U_1} \to U_1$ can be extended to a finite étale covering $\underline{\operatorname{Spec}}\, \mathcal{B}_1 \to X_1$. Using Proposition 1.4.20, we obtain étale coverings $\underline{\operatorname{Spec}}\, \mathcal{B}_n \to X_n$ for all $n \geqslant 1$, and these guys satisfy $\mathcal{B}_n|_{X_{n-1}} = \mathcal{B}_{n-1}$ and $\mathcal{B}_n|_{U_n} = \mathcal{A}_n = \mathcal{A}|_{U_n}$ by functoriality. The thickenings X_n for $n \geqslant 2$ are no longer regular, but it's quite easy to see that they are still S_2; in fact they are even Cohen–Macaulay. Then $\mathcal{B}_n$ has property S_2 (or S_2', equivalently). Therefore we may apply "Hartog's theorem" (this is a fancy name for Remark 1.5.19(2)) to get $\Gamma(U_n, \mathcal{A}_n) \cong \Gamma(X_n, \mathcal{B}_n)$. But since finite projective modules over the local ring A/t^nA are free, we see that $\Gamma(X_n, \mathcal{B}_n) \cong (A/t^nA)^{\oplus d}$ for some $d \geqslant 0$. Moreover, we have $\Gamma(X_n, \mathcal{B}_n)/t\,\Gamma(X_n, \mathcal{B}_n) \cong \Gamma(X_{n-1}, \mathcal{B}_{n-1})$ by compatibility of the $\mathcal{B}_n$. Then all of this is still true for the $\Gamma(U_n, \mathcal{A}_n)$; more precisely, there are isomorphisms

$$\begin{tikzcd} \Gamma(U_n, \mathcal{A}_n) \arrow[r] \arrow[d, "\wr"] & \Gamma(U_m, \mathcal{A}_m) \arrow[d, "\wr"] \\ (A/t^nA)^{\oplus d} \arrow[r] & (A/t^mA)^{\oplus d} \end{tikzcd}$$

for $n \geqslant m$. This shows that the limit on the right-hand side of (1.5.1) is a finite free A-module. Thus $\Gamma(U, \mathcal{A}) = \Gamma(X, j_*\mathcal{A})$ is finite free as well, which finally proves that $j_*\mathcal{A}$ is a vector bundle. We are done! □

Lecture 8
15th Nov, 2019

1.5.20. Remark. — In the pro-étale topology of Bhatt/Scholze (see [BS15]), there is actually a *pro-étale fundamental group* $\pi_1^{\text{proét}}(X, \overline{x})$ defined in the same way as $\pi_1^{\text{ét}}(X, \overline{x})$, for schemes X whose underlying topological space is locally noetherian. They consider locally constant sheaves of sets on the pro-étale site, and also étale X-schemes satisfying the valuation criterion for properness.

1.5.21. Example. — For most cases (like $\mathbb{C} \setminus \{0\}$, elliptic curves $\mathbb{C}/\Gamma$ for some lattice Γ, or curves $\mathbb{H}/\Gamma$) the topological universal covering admits no algebraic definition.

Let k be an algebraically closed field, and X be the topological space obtained from $\mathbb{P}^1_k$ by identifying 0 and ∞. Let $\mathbb{P}^1_k \twoheadrightarrow X$ be the canonical projection. We can make X into a scheme with structure sheaf $\mathcal{O}_X$ defined via

$$\Gamma(U, \mathcal{O}_X) = \left\{ f \in \Gamma(\pi^{-1}(U), \mathcal{O}_{\mathbb{P}^1}) \;\middle|\; f(0) = f(\infty) \text{ if } 0 \in \pi^{-1}(U) \right\}$$

(note that if $0 \in \pi^{-1}(U)$ then also $\infty \in \pi^{-1}(U)$). This is an example of a *non-unibranch* scheme. Here we call a local ring R *unibranch* if $R/\operatorname{nil}(R)$ is a domain whose normalization S is local. R is called *geometrically unibranch* if in addition the residue field extension $\kappa(\mathfrak{m}_S)/\kappa(\mathfrak{m}_R)$ is purely inseparable. A scheme being unibranch then means the obvious thing, i.e., that all its local rings are unibranch.

Our X fails to be unibranch at its singular point $[0] = [\infty] \in X$. There are étale coverings

$$X_N \longrightarrow X\,,$$

where X_N is the quotient of $\coprod_{i\in\mathbb{Z}/N\mathbb{Z}} \mathbb{P}^1_k$ upon identifying $(i, 0)$ with $(i + 1, \infty)$ for all $i \in \mathbb{Z}/N\mathbb{Z}$. The cyclic group $\mathbb{Z}/N\mathbb{Z}$ acts on X_N by permuting the components of the disjoint union. This defines morphisms $\pi_1^{\text{ét}}(X, \overline{x}) \to \mathbb{Z}/N\mathbb{Z}$. Since it can be shown that every étale covering of X is a finite disjoint union of X_N (use that k is algebraically closed), these morphisms assemble to an isomorphism

$$\pi_1^{\text{ét}}(X, \overline{x}) \xrightarrow{\sim} \widehat{\mathbb{Z}} = \lim_{N\in\mathbb{N}} \mathbb{Z}/N\mathbb{Z}\,.$$

But in the pro-étale topology, we can also form an "infinite cyclic covering" X_∞ by replacing $\mathbb{Z}/N\mathbb{Z}$ by $\mathbb{Z}$. This defines an isomorphism

$$\pi_1^{\text{proét}}(X, \overline{x}) \xrightarrow{\sim} \mathbb{Z}\,.$$

Note that for $k = \mathbb{C}$, in fact, $X_\infty(\mathbb{C})$ happens to be a universal covering for $X(\mathbb{C})$.

Bhatt/Scholze give the pro-étale fundamental group a topology that makes it a *Noohi group* (see [BS15, Definition 7.1.1]). Moreover, by Lemma 7.4.3 and 7.4.10 of *loc. cit.*, the pro-finite completion of $\pi_1^{\text{proét}}(X, \overline{x})$ gives back the good old $\pi_1^{\text{ét}}(X, \overline{x})$, and for schemes X which are geometrically unibranch there is an isomorphism $\pi_1^{\text{proét}}(X, \overline{x}) \cong \pi_1^{\text{ét}}(X, \overline{x})$.

1.6. Stalks at Geometric Points and Henselian Rings

1.6.1. Stalks of Sheaves on $X_{\text{ét}}$

In general, there is no good notion of "stalks of sheaves on arbitrary sites". This makes stuff like sheafification harder to understand in general. But for the étale site over a scheme, we are lucky and a suitable notion of stalks does exist!

1.6.1. Definition. — Let $\overline{x}\colon \operatorname{Spec} k \to X$ be a geometric point of X. An *étale neighbourhood* of x is a pair $(U, \overline{u})$, where $U \to X$ is étale and the diagram

$$\begin{tikzcd} \operatorname{Spec} k \arrow[r, "\overline{u}"] \arrow[dr, "\overline{x}"'] & U \arrow[d] \\ & X \end{tikzcd}$$

commutes. A *morphism of étale neighbourhoods* $(U, \overline{u}) \to (V, \overline{v})$ is a morphism $U \to V$ of X-schemes that makes $(U, \overline{u})$ into an étale neighbourhood of the geometric point $\overline{v}$ of V. Finally, if $\mathcal{F}$ is a presheaf on $X_{\text{ét}}$ one puts

$$\mathcal{F}_{\overline{x}} = \operatorname*{colim}_{(U,\overline{u})} \Gamma(U, \mathcal{F})\,,$$

where the colimit is taken over all étale neighbourhoods $(U, \overline{u})$ of x. This is called the *stalk of $\mathcal{F}$ at $\overline{x}$*.

1.6.2. Fact. — *The colimit in Definition 1.6.1 is in fact a filtered colimit.*

Proof. It is clear that the category of étale neighbourhoods of $\overline{x}$ is non-empty, as $(X, \overline{x})$ is trivially an element. It remains to check the other conditions of cofilteredness (and keep in mind that $\Gamma(-, \mathcal{F})$ is contravariant, so the category needs to be cofiltered for the colimit to be filtered).

(*a*) For all étale neighbourhoods $(U, \overline{u})$ and $(V, \overline{v})$, there is an étale neighbourhood $(W, \overline{w})$ with morphisms $(W, \overline{w}) \to (U, \overline{u})$ and $(W, \overline{w}) \to (V, \overline{v})$.

(*b*) If $\alpha, \beta\colon (U, \overline{u}) \to (V, \overline{v})$ is a pair of parallel morphisms, there is an étale neighbourhood $(W, \overline{w})$ with a morphism $(W, \overline{w}) \to (U, \overline{u})$ equalizing α and β.

For (*a*), consider $W = U \times_X V$ with its geometric point $w = (u, v)$. For (*b*), take $W = \operatorname{Eq}(\alpha, \beta)$. The equalizer is an open subscheme of U by the argument from Proposition 1.4.22, hence étale over X, and $\overline{u}\colon \operatorname{Spec} k \to U$ factors over W. □

In the following we will tacitly work with $X_{\text{ét}}$, but the big étale site $(\mathrm{Sch}/X)_{\text{ét}}$ would work as well. Also we silently assume that all sheaves we consider are sheaves of sets, (abelian) groups, rings, or modules.

1.6.3. Proposition. — *Let $U \to X$ be an étale X-scheme and $\mathcal{F}$ a presheaf on $X_{\text{ét}}$.*

(*a*) *A sieve $\mathcal{S}$ over U is covering iff for every geometric point $\overline{u}$ of U there is some $(V \to U) \in \mathcal{S}$ over which $\overline{u}$ factors.*

(*b*) *The presheaf $\mathcal{F}$ is separated iff the canonical map $\Gamma(U, \mathcal{F}) \to \prod_{\overline{u}} \mathcal{F}_{\overline{u}}$ is injective for all U, the product being taken over all geometric points of U.*

(*c*) *Define a presheaf $\mathcal{F}^{\mathrm{Sh}}$ on $X_{\text{ét}}$ by*

$$\Gamma\big(U, \mathcal{F}^{\mathrm{Sh}}\big) = \left\{ (f_{\overline{u}}) \in \prod_{\overline{u}} \mathcal{F}_{\overline{u}} \;\middle|\; \begin{array}{c} \textit{the sieve of all } j\colon V \to U\textit{, for which there exists} \\ f_V \in \Gamma(V, \mathcal{F}) \textit{ such that } f_{j(\overline{v})} \textit{ is the image of } f_V \\ \textit{in } \mathcal{F}_{\overline{v}} \textit{ for all geometric points } \overline{v} \textit{ of } V\textit{, is covering} \end{array} \right\}.$$

Then $\mathcal{F}^{\mathrm{Sh}}$ is a sheaf and $(-)^{\mathrm{Sh}}\colon \mathrm{PSh}(X_{\text{ét}}) \to \mathrm{Sh}(X_{\text{ét}})$ is a left-adjoint of the forgetful functor $\mathrm{Sh}(X_{\text{ét}}) \to \mathrm{PSh}(X_{\text{ét}})$.

(*d*) *The unit $\mathcal{F} \to \mathcal{F}^{\mathrm{Sh}}$ of the adjunction induces an isomorphism on stalks.*

(*e*) *When $\mathcal{F}$ and $\mathcal{G}$ are sheaves on $X_{\text{ét}}$, a morphism $\mathcal{F} \to \mathcal{G}$ is an isomorphism iff $\mathcal{F}_{\overline{x}} \to \mathcal{G}_{\overline{x}}$ is an isomorphism at all geometric points $\overline{x}$.*

1.6.4. Remark. — If X is Jacobson, it suffices to check the conditions from Proposition 1.6.3 on geometric points $\overline{u}$ whose *support* $u = |\overline{u}|$ (i.e., the unique point in the image of $\overline{u}\colon \operatorname{Spec} k \to U$) is closed. This is proved in the proof of Definition 1.4.24.

Lecture 9
18th Nov, 2019

1.6.5. Remark. — In Proposition 1.6.3(*c*) we have implicitly used the diagram

$$\begin{tikzcd} \Gamma(U, \mathcal{F}) \arrow[r] \arrow[d] & \mathcal{F}_{\pi(\overline{v})} \arrow[d] \\ \Gamma(V, \mathcal{F}) \arrow[r] & \mathcal{F}_{\overline{v}} \end{tikzcd}.$$

This uses the fact that the étale neighbourhoods of $\pi(\overline{v})$ in U (resp. $\overline{v}$ in V) are cofinal in the étale neighbourhoods of $\sigma\overline{v}$, where $\sigma\colon V \to X$ denotes the structure morphism of the X-scheme V.

1.6.6. Remark. — So far we considered general geometric points $\overline{x}\colon \operatorname{Spec} k \to X$. From now on, let $\kappa(\overline{x})$ denote that k. Let $x = |\overline{x}|$. Let k' be the separable closure of the residue field $\kappa(x)$ of $\mathcal{O}_{X,x}$ in $\kappa(\overline{x})$ and $\overline{x}'\colon \operatorname{Spec} k' \to X$ denote the corresponding morphism.

If $\rho\colon U \to X$ is étale and $u \in U$ such that $\rho(u) = x$, then $\kappa(u)$ is a finite separable extension of $\kappa(x)$. Therefore, if $\overline{u}\colon \operatorname{Spec} \kappa(\overline{x}) \to U$ is a geometric point of U such that $\rho\overline{u} = \overline{x}$, thus turning $(U, \overline{u})$ into an étale neighbourhood of $\overline{x}$, then the image of $\kappa(u) \to \kappa(\overline{x}) = \kappa(\overline{u})$ is contained in k'. Thus, there is a unique $\overline{u}'\colon \operatorname{Spec} k' \to U$ such that $\rho\overline{u}' = \overline{x}'$ and such that

$$\begin{tikzcd} & U \\ \operatorname{Spec} \kappa(x) \arrow[ur, "\overline{u}"] \arrow[r] & \operatorname{Spec} k' \arrow[u, "\overline{u}'"'] \end{tikzcd}$$

commutes. Thus, $\overline{x}$ and $\overline{x}'$ have the same étale neighbourhoods, in the sense that there is a canonical equivalence of categories. Therefore we can replace $\overline{x}$ by $\overline{x}'$ without changing the statements and constructions of the proposition.

This is actually crucial to avoid set-theoretical difficulties! Indeed, if we took the product $\prod_{\overline{u}} \mathcal{F}_{\overline{u}}$ from Proposition 1.6.3 at face value, this would be a monstrous abomination, due to the fact that the set of geometric points of U is no set, but a proper class. The above arguments solve one half of that problem: the half that is caused by separably closed extensions of $\kappa(x)$ becoming arbitrarily large. The other half is that there is a full class of field extensions of $\kappa(x)$ that are isomorphic to k'. To fix this issue, we fix a choice of separable closure $\kappa(x)^{\mathrm{sep}}$ and allow only those geometric points $\overline{x}$ with $|\overline{x}| = x$ that satisfy $\kappa(\overline{x}) = \kappa(x)^{\mathrm{sep}}$.

Sketch of a proof of Proposition 1.6.3. Once the characterization of covering sieves is shown, the proofs from ordinary sheaf theory can be copied, so we show (a), and then very briefly sketch the rest.

If $\mathcal{S}$ is a covering sieve over $U \in X_{\text{ét}}$, then the members of $\mathcal{S}$ are jointly surjective just by Definition 1.4.24(a). For a geometric point $\overline{u}$ of U, there are thus a morphism $(\sigma\colon V \to U) \in \mathcal{S}$ and an ordinary point $v \in V$ such that $\sigma(v) = u = |\overline{u}|$. Then $\kappa(v)$ is a finite separable extension of $\kappa(u)$. But $\kappa(\overline{u})$ is separably closed, so there exists a (not necessarily unique) extension $\kappa(v) \to \kappa(\overline{u})$ of $\kappa(u) \to \kappa(\overline{u})$. This defines a geometric point $\overline{v}$ of V such that $\sigma(\overline{v}) = \overline{u}$.

The opposite direction is merely trivial: if a geometric point $\overline{u}$ factors over some morphism $(V \to U) \in \mathcal{S}$, then its support u is in the image of $V \to U$. Since this is to be true for every geometric point, we see that the maps from $\mathcal{S}$ are jointly surjective, hence $\mathcal{S}$ is covering straight from Definition 1.4.24. This shows (a).

Adressing Remark 1.6.4: if X is Jacobson, then so is U as it is of finite type over X. Hence the closed points of U are dense in any closed subset. But the joint image of the maps from $\mathcal{S}$ is open by Proposition 1.2.14, thus it is U if it contains all closed points.

For (b), assume there are $f, f' \in \Gamma(U, \mathcal{F})$ whose images in $\mathcal{F}_{\overline{u}}$ coincide for every geometric point $\overline{u}$ of U. By Definition 1.6.1 and Fact 1.6.2, for every such $\overline{u}$, there is an étale neighbourhood $(V_{\overline{u}}, \overline{v}_{\overline{u}})$ such that the restrictions of f and f' coincide in $\Gamma(V_{\overline{u}}, \mathcal{F})$. By (a), the sieve generated by all such $V_{\overline{u}} \to U$ is covering. As $\mathcal{F}$ was assumed separated, this shows $f = f'$ and thus (b).

For (d), we have an obvious morphism $\mathcal{F} \to \mathcal{F}^{\mathrm{Sh}}$ of presheaves. Let $\overline{x}$ be a geometric point of $U \in X_{\text{ét}}$, let $(V, \overline{y})$ be an étale neighbourhood of $\overline{x}$, and let $f = (f_{\overline{v}}) \in \Gamma(V, \mathcal{F}^{\mathrm{Sh}})$. Put

$\operatorname{res}_{V,\overline{y}}(f) = f_{\overline{y}} \in \mathcal{F}_{\overline{y}} \cong \mathcal{F}_{\overline{x}}$. It is easy to see that the $\operatorname{res}_{V,\overline{y}}$ induce a unique map $\mathcal{F}^{\mathrm{Sh}}_{\overline{x}} \to \mathcal{F}_x$ via the universal property of colimits. This map is inverse to the previous map $\mathcal{F}_{\overline{x}} \to \mathcal{F}^{\mathrm{Sh}}_{\overline{x}}$.

It is also easy to see that $\mathcal{F}^{\mathrm{Sh}}$ is an étale sheaf, and if $\mathcal{F} \to \mathcal{G}$ is a morphism of presheaves inducing isomorphisms on stalks, then $\mathcal{F}^{\mathrm{Sh}} \to \mathcal{G}^{\mathrm{Sh}}$ is an isomorphism. Also $\mathcal{F} \xrightarrow{\sim} \mathcal{F}^{\mathrm{Sh}}$ in the case where $\mathcal{F}$ is already a sheaf is not hard: apply the sheaf axiom to the sieve occuring in the coherence condition in the definition of $\mathcal{F}^{\mathrm{Sh}}$ in Proposition 1.6.3(c), using (b) to show that the f_V are unique and assemble to an element of $\lim_{\mathcal{S}} \Gamma(V, \mathcal{F}) \cong \Gamma(U, \mathcal{F})$.

Finally, if $\mathcal{G}$ is a sheaf and $\mathcal{F}$ a presheaf, then any morphism $\varphi\colon \mathcal{F} \to \mathcal{G}$ factors as

$$\begin{tikzcd} \mathcal{F} \arrow[r, "\varphi"] \arrow[d] & \mathcal{G} \arrow[d, "\wr"] \\ \mathcal{F}^{\mathrm{Sh}} \arrow[r, "\varphi^{\mathrm{Sh}}"] & \mathcal{G}^{\mathrm{Sh}} \end{tikzcd} .$$

The rest of the adjunction from (c) and the proof of (e) are easy. □

1.6.2. Henselian Rings

1.6.7. Proposition ([Mil80, Thm. I.4.3]). — *Let A be a local ring with maximal ideal $\mathfrak{m}$ and residue field k. Let $\overline{f}$ denote the image of a polynomial $f \in A[T]$ under $A[T] \to k[T]$. Then the following conditions are equivalent.*

(a) *If $f \in A[T]$ and $\overline{f} = g_0 h_0$, where g_0 is monic and $\gcd(g_0, h_0) = 1$ in $k[T]$, then there is a unique decomposition $f = gh$ in $A[T]$, such that g is monic and $\overline{g} = g_0$, $\overline{h} = h_0$.*

(b) *The same as (a), but f, h_0 and h have to be monic.*

(c) *Put $X = \operatorname{Spec} A$. If $Y \to X$ is a finite morphism, then $Y = \coprod_{i=1}^n \operatorname{Spec} B_i$, where each B_i is a local finite A-algebra.*

(d) *Put $X = \operatorname{Spec} A$. If $Y \to X$ is a quasi-finite and separated morphism of finite presentation, then $Y = Y_0 \sqcup \coprod_{i=1}^n \operatorname{Spec} B_i$, with B_i as in (c) and $\mathfrak{m}$ is not contained in the image of Y_0.*

(e) *Put $X = \operatorname{Spec} A$. If $U \to X$ is étale, then any lift $\operatorname{Spec} k \to U$ of $\operatorname{Spec} k \to X$ extends to a unique section $X \to U$ of $U \to X$.*

(f) *If $f_1, \dots, f_n \in A[X_1, \dots, X_n]$ are polynomials and $x_0 \in k^n$ a common zero of the $\overline{f}_j$ such that $\det(\partial f_i/\partial X_j)(x_0) \neq 0$, then there is a unique $x \in A^n$ which is a common zero of the f_j whose image in k^n is x_0.*

Lecture 10
22nd Nov, 2019

Proof. Clearly (a) ⇒ (b). We continue with (b) ⇒ (c). As a first step, consider $B = A[T]/(f)$, where $f \in A[T]$ is a monic polynomial. Decompose $\overline{f} = \prod_{i=1}^n \varphi_i^{e_i}$ into (monic) prime powers in the PID $k[T]$. Using (b), we get a decomposition $f = \prod_{i=1}^n f_i$, where $\overline{f}_i = \varphi_i^{e_i}$. Note that for $i \neq j$ we have $(f_i, f_j) = A[T]$. Indeed, $C = A[T]/(f_i, f_j)$ is finite over A, and $C \otimes_A k \cong k[T]/(\varphi_i^{e_i}, \varphi_j^{e_j}) \cong 0$ vanishes as φ_i and φ_j are coprime, so C vanishes already by Nakayama. Putting $B_i = A[T]/(f_i)$, the Chinese remainder theorem shows

$$A[T]/(f) \cong \prod_{i=1}^n B_i \,,$$

and it suffices to show that the B_i are local, since then $\operatorname{Spec} B = \coprod_{i=1}^n \operatorname{Spec} B_i$ has the desired form. Because $A \subseteq B_i$ is a finite ring extension, the going-up theorem shows that every maximal ideal of B_i lies over the maximal ideal of A, i.e., contains $\mathfrak{m}B_i$. But $B_i/\mathfrak{m}B_i \cong k[T]/(\varphi_i^{e_i})$ is an artinian local ring, hence B_i is local as well.

For the general case, let $Y = \operatorname{Spec} B$ be finite over X. We may decompose Y into finitely many affine connected components.[11] Thus, without losing generality, let $\operatorname{Spec} B$ be connected, or equivalently, B have no non-trivial idempotents. We must show that B is local. As above, every maximal ideal of B contains $\mathfrak{m}B$.[12] So it suffices that $\overline{B} = B/\mathfrak{m}B$ is local. But this is an artinian ring, being finite over k, hence it suffices to show that $\operatorname{Spec} \overline{B}$ is connected (see [Eis95, Corollary 2.16] for example). Assume the contrary, and choose $e \in B$ such that $\overline{e} \in \overline{B}$ is a non-trivial idempotent. Let $f \in A[T]$ be a monic polynomial such that $f(e) = 0$ (which exists as B is finite over A) and let $C = A[T]/(f)$. We have a morphism $C \to B$ sending the image of T to e.

Put $\overline{C} = C/\mathfrak{m}C$. Now $\overline{f}$ is divisible by the minimal polynomial μ of $\overline{e}$ over k, i.e., the monic generator of the ideal $I \subseteq k[T]$ of polynomials vanishing on $\overline{e}$. Since $\overline{e} \neq 0, 1$ is an idempotent, we must have $\mu = T^2 - T$. Factoring $\overline{f} = T^m(T-1)^n\overline{f}_0$ for some $\overline{f}_0 \in k[T]$ coprime to T and $T-1$, we get $\overline{C} \cong k[T]/(T^m(T-1)^n) \times \overline{C}_0$, where $\overline{C}_0 = k[T]/(\overline{f}_0)$. Moreover, since T is mapped to $\overline{e}$, which is already idempotent, the morphism $\overline{C} \to \overline{B}$ factors over

$$\begin{tikzcd} \overline{C} \arrow[r] \arrow[d, two heads] & \overline{B} \\ k[T]/(T^m(T-1)^n) \arrow[ur, dashed] & \end{tikzcd},$$

where the vertical morphism on the left is the canonical projection that forgets the factor $\overline{C}_0$. Now since T is idempotent in $k[T]/(T(T-1))$, it can be lifted to an idempotent $\tau \in k[T]/(T^m(T-1)^n)$; this is a consequence of Hensel's lemma since $k[T]/(T^m(T-1)^n)$ is obviously complete with respect to the nilpotent ideal $(T(T-1))$. Now $\overline{c} = (\tau, 0) \in \overline{C}$ is a non-trivial idempotent that is mapped to $\overline{e}$, by construction.

Using the special case that was already proved, we see that $\overline{c}$ can be lifted to an idempotent $c \in C$. The image of c in B is a non-trivial idempotent, since it is mapped to $\overline{e} \neq 0, 1$ in $\overline{B}$. This shows that $\operatorname{Spec} B$ is not connected, contradicting our assumption. This finally finishes the proof of $(b) \Rightarrow (c)$.

Now $(c) \Rightarrow (d)$ follows from Zariski's main theorem (see [Jac, Theorem 2(b)] for example) and the fact that if $\operatorname{Spec} B$ is the spectrum of a local ring, then every open subset containing the unique closed point is already $\operatorname{Spec} B$. However, Professor Franke points out that this veils a substantial technical detail; more about that in Remark 1.6.8.

For $(d) \Rightarrow (e)$, let $\pi\colon U \to X$ and $\iota\colon \operatorname{Spec} k \to U$ be as in the statement of this proposition. Clearly we may assume that U is affine, so π is separated and (d) is applicable to U. Thus we may even assume $U = \operatorname{Spec} B$, where B is a local finite étale A-algebra. From Proposition 1.4.1(c) we get that $\mathfrak{m}B$ is the maximal ideal of B and $\kappa(B) = B/\mathfrak{m}B$ is a finite separable field extension of k, which is mapped to k via ι^*. Thus $\kappa(B) \cong k$, hence B is a quotient of A by Nakayama. Since B is also finite flat over A and A is local, we get $B \cong A$ and everything is clear.

Next we prove $(e) \Rightarrow (f)$. Let $B = A[X_1, \dots, X_n]/(f_1, \dots, f_n)$ and let $\Delta = \det(\partial f_i/\partial X_j)$ be the Jacobian determinant. An easy application of Proposition 1.4.16(e) shows that

[11] This works without noetherianness: since every decomposition $Y = Y_1 \sqcup Y_2$ gives rise to a decomposition $B = B_1 \times B_2$, an easy Nakayama argument shows that the number of connected components can be at most $\dim_k B \otimes_A k < \infty$.

[12] The going-up argument works even though $\alpha\colon A \to B$ need not be an inclusion: we can just replace A by $A/\ker\alpha$ to see that every maximal ideal of B lies over $\mathfrak{m}/\ker\alpha$.

$B[\Delta^{-1}]$ is étale over A. Now (e) can be applied to $\operatorname{Spec} B[\Delta^{-1}] \to \operatorname{Spec} A$. This shows that the morphism $\operatorname{Spec} k \to \operatorname{Spec} B[\Delta^{-1}]$ determined by $x_0 \in k^n$ can be lifted to a unique section $\operatorname{Spec} A \to \operatorname{Spec} B[\Delta^{-1}] \hookrightarrow \operatorname{Spec} B$. Now $x \in A^n$ can be chosen to be the image of $(X_1, \dots, X_n)$ under the corresponding ring morphism $B \to A$.

Finally we prove $(f) \Rightarrow (a)$. Consider the system of equations for the coefficients g_i, h_j determining $f = gh$. One shows that the Jacobian determinant of this system is the *resultant* $\operatorname{res}(g, h)$ of the polynomials g and h (this follows straight from the definition e.g. in the Wikipedia article), which modulo $\mathfrak{m}$ is $\operatorname{res}(g_0, h_0) \neq 0$ since g_0 and h_0 are coprime by our assumption. $\square$

1.6.8. Remark. — We will later be forced to apply Proposition 1.6.7 in the non-noetherian case as well. The only point in the proof where this gets hairy is Zariski's main theorem. It turns out that it is indeed true in the non-noetherian case as well (as usual, finite type needs to be replaced by finite presentation), and in fact the general case can be reduced to the noetherian case.

The general idea is to write the quasi-finite separated morphism $f\colon X \to S$ in question as a base change of a morphism $f_0\colon X_0 \to S_0$ between noetherian schemes, as hinted in Remark* 1.2.15. But there is still a technical problem: f_0 need not be quasi-finite, even though its base change f is. The solution is to write X and S as cofiltered limits of noetherian schemes $\{X_\lambda\}$, $\{S_\lambda\}$, and f as a cofiltered limit over $\{f_\lambda\colon X_\lambda \to S_\lambda\}$, and to show that already some "finite" stage must be quasi-finite. The details can be found in [$\text{EGA}_{\text{IV}/3}$, Théorème (8.10.5)], but here is the rough idea: the quasi-finite locus $U_\lambda \subseteq X_\lambda$ is always open by [Jac, Theorem 2(c)], and its preimage in X is all of X by assumption. Choosing a coherent ideal $\mathcal{I}_\lambda \subseteq \mathcal{O}_{X_\lambda}$ cutting out $X_\lambda \setminus U_\lambda$, we see that the ideal pullback of $\mathcal{I}_\lambda$ in $\mathcal{O}_X$ vanishes. But since $\mathcal{I}_\lambda$ is coherent, its pullbacks must already vanish at some "finite" stage. More about this kind of arguments can be found in the appendix, Appendix A.1.

1.6.9. Definition. — A local ring A satisfying the equivalent conditions of Proposition 1.6.7 is called *henselian*. If in addition the residue field k is separably closed, A is called *strictly henselian*.

There is also a notion of being *henselian in an ideal* I, which only depends on the radical $\sqrt{I}$, so one can define what it means for a scheme to be henselian in a closed subset. But we won't need that here.

1.6.10. Proposition. — *Let* $X = \operatorname{Spec} A$ *be the spectrum of a henselian ring with residue field* k *and let* $X_0 = \operatorname{Spec} k$. *Then there is an equivalence of categories*

$$\begin{aligned} \{\textit{finite étale } X\textit{-schemes}\} &\longrightarrow \{\textit{finite étale } X_0\textit{-schemes}\} \\ Y &\longmapsto Y_0 = Y \times_X X_0\,. \end{aligned}$$

Proof. For essential surjectivity we may assume Y_0 to be connected. Then $Y_0 = \operatorname{Spec} \ell$, where ℓ is a finite separable field extension of k. Galois theory tells us that ℓ is generated by a primitive element, say, $\ell \cong k[T]/(f_0)$ for some monic irreducible polynomial f_0. Let $f \in A[T]$ be a monic lift of f_0. Then $Y = A[T]/(f)$ is étale over X and lifts Y_0. That the functor in question is fully faithful follows from Lemma 1.6.11 below. $\square$

1.6.11. Lemma. — *Let X_0 be a closed subscheme of the noetherian scheme X, with the property that for every finite étale X-scheme Y, the map $\pi_0(Y_0) \to \pi_0(Y)$ is bijective. Here $Y_0 = Y \times_X X_0$, and π_0 denotes the set of connected components. Then the functor*

$$\begin{aligned} \{\textit{finite étale } X\textit{-schemes}\} &\longrightarrow \{\textit{finite étale } X_0\textit{-schemes}\} \\ Y &\longmapsto Y_0 \end{aligned}$$

is fully faithful.

Proof. If $f, f' \colon Y \to Y'$ are morphisms between finite étale X-schemes, then $\mathrm{Eq}(f, f')$ is an open-closed subscheme of Y (to see that it's open, use the argument from the proof of Proposition 1.4.22, to see closedness, use that f and f' are necessarily separated). Thus the equalizer equals Y iff it contains Y_0, since Y_0 intersects all connected components of Y by assumptions. This shows that the functor in question is faithful.

For fullness, let $f \colon Y \to Y'$ be a morphism of finite étale X-schemes. Let Γ_f be the *graph* of f, i.e., the image of the open-closed immersion $(\mathrm{id}_Y, f) \colon Y \to Y \times_X Y'$ (this is open-closed since it is the equalizer of the two morphisms $Y \times_X Y' \to Y \times_X Y'$ given by id_Y and $(\mathrm{pr}_1, f\,\mathrm{pr}_1)$). The association $f \mapsto \Gamma_f$ defines a bijection between $\mathrm{Hom}_{\mathrm{Ét}/X}(Y, Y')$ and the set of open-closed subschemes $\Gamma \subseteq Y \times_X Y'$ such that the projection restricts to an isomorphism $\mathrm{pr}_1 \colon \Gamma \xrightarrow{\sim} Y$. Since that projection is finite étale, it is an isomorphism iff it has degree 1 (see Fact 1.6.12 below). The set of all open-closed subschemes of $Y \times_X Y'$ is precisely the set of all subsets of $\pi_0(Y \times_X Y')$. By assumption, $\pi_0(Y_0 \times_{X_0} Y_0') \xrightarrow{\sim} \pi_0(Y \times_X Y')$. Moreover, Y_0 meets every connected component of Y, hence $\Gamma \to Y$ has degree 1 iff the same is true for $\Gamma_0 \to Y_0$. These consideration show that the set of $\Gamma \subseteq Y \times_X Y'$ with the required properties is in canonical bijection with the set of $\Gamma_0 \subseteq Y_0 \times_{X_0} Y_0'$ with the same properties. This shows fullness. □

1.6.12. Fact. — *A finite étale morphism (in fact, any finite and finitely presented flat morphism) is an isomorphism iff it has degree* 1.

Proof. Locally, a finite and finitely presented flat morphism is of the form $\operatorname{Spec} B \to \operatorname{Spec} A$, where B is finite free as an A-module; and the degree is just the rank of B over A. Clearly $A \cong B$ iff the rank is 1. □

1.6.13. Corollary. — *Assume we are in the situation of Proposition* 1.6.10.

(a) *For every geometric point $\overline{x}$ of X_0, we have isomorphisms*

$$\mathrm{Gal}(k^{\mathrm{sep}}/k) \cong \pi_1^{\text{ét}}(X_0, \overline{x}) \cong \pi_1^{\text{ét}}(X, \overline{x})\,.$$

(b) *The following conditions are equivalent:*

(1) *A is strictly henselian.*

(2) *Every étale covering of X splits (i.e., is a disjoint union of copies of X).*

(3) $\pi_1^{\text{ét}}(X, \overline{x}) = 1$.

(4) *If $Y \to X$ is a surjective étale morphism (or equivalently, by Proposition* 1.6.7(d)*, an étale morphism with the closed point in its image), then there is a section $Y \to X$ of this morphism.*

Proof. Part (a) follows from Lemma 1.4.9 and Proposition 1.6.10. For (b), (1) ⇔ (2) follows from (a), (2) ⇔ (3) follows from Definition/Lemma 1.5.6, (3) ⇒ (4) is trivial and (4) ⇒ (3) follows from Proposition 1.6.7(d). □

1.6.3. Henselization

We are now going to define the *henselization* and the *strict henselization* of a local ring A. These are going to be characterized by universal properties of course. The *category of henselian A-algebras* has local morphisms $A \to S$ as objects, where S is a henselian. Morphisms in this category are just morphisms of A-algebras.

Fix an embedding $\eta_0\colon k \to k^{\mathrm{sep}}$ into a separable closure. We define a *category of strictly henselian A-algebras with respect to η_0* as follows: its objects consist of the following data: a local morphism $A \to S$, where S is strictly henselian, together with a morphism $k^{\mathrm{sep}} \to \kappa(S)$ such that the diagram

$$\begin{tikzcd} A \arrow[rr] \arrow[d] & & S \arrow[d] \\ k \arrow[r, "\eta_0"] & k^{\mathrm{sep}} \arrow[r] & \kappa(S) \end{tikzcd}$$

commutes. Morphisms in this category are morphisms of A-algebras $S \to S'$, such that the obvious diagram involving k^{sep}, $\kappa(S)$, and $\kappa(S')$ commutes.

1.6.14. Definition. — Let A be a local ring. We define

(a) "the" *henselization* A^{h} of A to be an initial object in the category of henselian A-algebras (and we show that this exists in Proposition 1.6.15(c) below).

(b) "the" *strict henselization* A^{sh} of A with respect to η_0 to be an initial object in the category of strictly henselian A-algebras with respect to η_0, for which $k^{\mathrm{sep}} \to \kappa(A^{\mathrm{sh}})$ is an isomorphism (and we show that this exists in Proposition 1.6.15(c) below).

Lecture 11
25th Nov, 2019

1.6.15. Proposition. — *Let A be a local ring.*

(a) *If A is complete, then A is henselian.*

(b) *Every finite local algebra B over A is henselian again. If A is strictly henselian, then so is B.*

(c) *The henselization A^{h} exists and can be constructed as a filtered colimit over localizations of étale A-algebras with residue field k. Moreover, any such filtered colimit is already a henselization if it is henselian. Similarly, the strict henselization A^{sh} exists and is a filtered colimit over localizations of étale A-algebras (with no condition on the residue fields). Any such filtered colimit B is already a strict henselization with respect to $k \to \kappa(B)$ if it is strictly henselian.*

(d) *If B is a finite local A-algebra, then $B^{\mathrm{h}} \cong A^{\mathrm{h}} \otimes_A B$. Moreover, B^{sh} is a direct summand of $A^{\mathrm{sh}} \otimes_A B$. In particular, if $I \subseteq A$ is an ideal, then $(A/I)^{\mathrm{h}} \cong A^{\mathrm{h}}/IA^{\mathrm{h}}$ and $(A/I)^{\mathrm{sh}} \cong A^{\mathrm{sh}}/IA^{\mathrm{sh}}$.*

(e) *If A is noetherian, then A^{h} and A^{sh} are also noetherian, and $\widehat{A} \cong (A^{\mathrm{h}})^{\wedge}$. Moreover, $\dim A = \dim A^{\mathrm{h}} = \dim A^{\mathrm{sh}}$.*

(f) *If k is separably closed, then $A^{\mathrm{h}} \cong A^{\mathrm{sh}}$.*

(g) *If $\overline{x}$ is a geometric point of X, then $\mathcal{O}_{X,x}^{\mathrm{sh}} \cong \mathcal{O}_{X_{\mathrm{\acute{e}t}},\overline{x}}$, where the strict henselization is taken with respect to $\kappa(x) \to \kappa(\overline{x})$.*

1.6.16. Remark. — The *étale structure sheaf* $\mathcal{O}_{X_{\mathrm{\acute{e}t}}}$ is defined by $\Gamma(U, \mathcal{O}_{X_{\mathrm{\acute{e}t}}}) = \Gamma(U, \mathcal{O}_U)$ for all étale X-schemes $U \to X$. By faithfully flat descent (see Proposition 1.2.11)—or to put it in fancy words, by the fact that this functor can be represented by the affine line $\mathbb{A}^1_X$ and using Example 1.3.16—it follows that this is indeed an fpqc sheaf, hence an étale sheaf.

To prove noetherianness in Proposition 1.6.15(e), the strategy is to show that the completions of A^{h} and A^{sh} are noetherian and then use Lemma 1.6.17(a) below. However, there is a problem: the completion of a ring is, in general, only flat for noetherian rings, and noetherianness is just what we want to show. So we need some different flatness criteria.

1.6.17. Lemma. — *Let A be an arbitrary ring and M an A-module.*

(a) *If B a faithfully flat A-algebra which is noetherian, then A is already noetherian.*

(b) *M is flat iff the following condition holds: whenever $m_1, \dots, m_n \in M$ and $a_1, \dots, a_n \in A$ are chosen such that $\sum_{i=1}^n a_i m_i = 0$, there are a vector $(\mu_j) \in M^\ell$ and a matrix $(\alpha_{i,j}) \in A^{n\times\ell}$ satisfying*

$$m_i = \sum_{j=1}^{\ell} \alpha_{i,j}\mu_j \quad \textit{for } i = 1, \dots, n \quad \textit{and} \quad \sum_{i=1}^{n} a_i\alpha_{i,j} = 0 \quad \textit{for } j = 1, \dots, \ell\,.$$

(c) *If $A \cong \operatorname{colim}_{\lambda\in\Lambda} A_\lambda$ is a filtered colimit of rings A_λ over which M is flat, then M is flat over A.*

Proof.* For (a), let $I \subseteq A$ be an ideal. Since B is flat over A, the tensor product $I \otimes_A B$ is an ideal of B, hence generated by finitely many elements. Let $\beta_i = \sum_j a_{i,j} \otimes b_{i,j}$ be such generators, where $a_{i,j} \in I$ and $b_{i,j} \in B$. We claim that the $\{a_{i,j}\}$ already generate I. Consider the map $\varphi\colon \bigoplus_{i,j} A \to A$ of A-modules that sends the $(i,j)^{\mathrm{th}}$ basis vector on the left-hand side to $a_{i,j}$. By construction, $\varphi \otimes \mathrm{id}_B\colon \bigoplus_{i,j} B \to B$ is surjective. But B is faithfully flat over A, so φ must already be surjective.

For part (b) check out [Stacks, Tag 00HK] or [Jac, Lemma 4.2.1] (but we only proved one half there). Part (c) is an immediate consequence of (b) and the explicit description of filtered colimits. □

Proof of Proposition 1.6.15. *Proof of* (a), (b). Part (a) is well-known as "Hensel's lemma". The henselian part of (b) follows from Proposition 1.6.7(c). For the strictly henselian part, we observe that every finite extension of a separably closed field is separably closed as well. By the way, note that since B is finite over A, the going-up theorem implies that the ring morphism $A \to B$ is automatically local (even though it need not be an inclusion; see the second footnote on page 44), so the otherwise ambiguous term "local A-algebra" is actually not ambiguous in this case.

Proof of (c). We construct A^{h} and A^{sh} as colimits

$$A^{\mathrm{h}} = \operatorname*{colim}_{(U,\overline{u})\in\Lambda_0} \mathcal{O}_{U,u} \quad \text{and} \quad A^{\mathrm{sh}} = \operatorname*{colim}_{(U,\overline{u})\in\Lambda} \mathcal{O}_{U,u}\,. \tag{1.6.1}$$

For A^{sh}, the colimit is taken over the system Λ of all affine étale neighbourhoods $(U, \overline{u})$ of the geometric point $\overline{x}\colon \operatorname{Spec} k^{\mathrm{sep}} \to \operatorname{Spec} A$; as usual, u denotes the underlying ordinary point of $\overline{u}$. For A^{h} we restrict to the subsystem $\Lambda_0 \subseteq \Lambda$ of of affine étale neighbourhoods with trivial residue field extension $\kappa(u)/k$. It follows from Fact 1.6.2 that Λ is indeed a filtered system. Tweaking the arguments a bit shows that Λ_0 is filtered too. Moreover, we may equivalently write

$$A^{\mathrm{h}} = \operatorname*{colim}_{(U,\overline{u})\in\Lambda_0} \Gamma(U, \mathcal{O}_U) \quad \text{and} \quad A^{\mathrm{sh}} = \operatorname*{colim}_{(U,\overline{u})\in\Lambda} \Gamma(U, \mathcal{O}_U)\,, \tag{1.6.2}$$

since the stalk $\mathcal{O}_{U,u} \cong \operatorname{colim}_{u\in U'} \Gamma(U', \mathcal{O}_U)$ is itself a colimit over the sections on its affine open neighbourhoods U', and the U' are étale over X again. So much for the constructions, now we are going to prove that the universal properties are satisfied!

First of all, A^{h} and A^{sh} are local rings because they are filtered colimits of local rings along local ring maps. Moreover its clear that $\kappa(A^{\mathrm{h}}) \cong k$, since all local rings $\mathcal{O}_{U,u}$ in the colimit defining A^{h} have residue field k. To determine $\kappa(A^{\mathrm{sh}})$, note that in this case for all local rings $\mathcal{O}_{U,u}$ the residue field is a finite separable extension of k, hence $\kappa(A^{\mathrm{sh}}) \subseteq k^{\mathrm{sep}}$. To get $k^{\mathrm{sep}} \subseteq \kappa(A^{\mathrm{sh}})$, we need to check that every finite separable of k does occur as the residue field of some $\mathcal{O}_{U,u}$. So let ℓ/k be finite separable. Then ℓ is generated by a single monic separable irreducible polynomial $\varphi \in k[T]$. Let $f \in A[T]$ be a monic lift and put $B = A[T]/(f)$. By Proposition 1.4.16(e) the localization $B[(f')^{-1}]$ is étale over A, which easily provides an étale neighbourhood with the required properties.

Now we show that A^{h} is henselian, verifying Proposition 1.6.7(e). Let $A^{\mathrm{h}} = \operatorname{colim}_{\lambda\in\Lambda_0} B_\lambda$ be the colimit (1.6.2), i.e., Λ_0 is the filtered system of affine étale neighbourhoods $(U, \overline{u})$ of $\overline{x}$ with trivial residue field extension $\kappa(u)/k$ and $B_\lambda = \Gamma(U, \mathcal{O}_U)$. Let $\varphi\colon A^{\mathrm{h}} \to S$ be an étale A^{h}-algebra.[13] Then S is finitely presented over A^{h}. In particular, φ is already defined over some "finite stage", i.e., it is the base change of some $\varphi_\lambda\colon B_\lambda \to S_\lambda$. Moreover, using the characterization from Proposition 1.4.16(e) we may even assume that φ_λ is already étale (you might want to have a look Appendix A.1). Thus, restricting Λ_0 to a suitable cofinal subsystem Λ_0^+, we obtain that

$$\big(\varphi\colon A^{\mathrm{h}} \longrightarrow S\big) = \operatorname*{colim}_{\lambda\in\Lambda_0^+} \big(\varphi_\lambda\colon B_\lambda \longrightarrow S_\lambda\big)$$

is the colimit of étale ring morphisms φ_λ such that φ_μ is the base change of φ_λ for all $\lambda \leqslant \mu$. Assume there is a section $\operatorname{Spec} k \to \operatorname{Spec} S$, or equivalently, a compatible system of maps $S_\lambda \to k$. These maps define geometric points $\overline{s}_\lambda\colon \operatorname{Spec} k^{\mathrm{sep}} \to \operatorname{Spec} S_\lambda$ for all λ in such a way that $(\operatorname{Spec} S_\lambda, \overline{s}_\lambda)$ is an étale neighbourhood of $\overline{x}$ and the residue field $\kappa(s_\lambda)$ of the underlying point s_λ is isomorphic to k. In particular, every S_λ actually occurs as some B_μ in the colimit $A^{\mathrm{h}} = \operatorname{colim}_{\mu\in\Lambda_0} B_\mu$! Identifying each S_λ with the corresponding B_μ gives a canonical morphism

$$\operatorname*{colim}_{\lambda\in\Lambda_0^+} S_\lambda \longrightarrow \operatorname*{colim}_{\mu\in\Lambda_0} B_\mu\,.$$

It's straightforward to check that the ensuing morphism $S \to A^{\mathrm{h}}$ is a section of $\varphi\colon A^{\mathrm{h}} \to S$, and moreover that this section is unique. This proves that A^{h} is henselian by Proposition 1.4.16(e). In the exact same way one can prove that A^{sh} is henselian; moreover its residue field $\kappa(A^{\mathrm{sh}}) = k^{\mathrm{sep}}$ is separably closed, so A^{sh} is indeed strictly henselian.

It remains to show that A^{h} and A^{sh} are initial in their respective categories. Let $(U, \overline{u}) = \lambda \in \Lambda$ with $U \cong \operatorname{Spec} B_\lambda$ be an affine étale neighbourhood of x. Since B_λ occurs in the colimit (1.6.1), we have a morphism $\kappa(B_\lambda) \to \kappa(A^{\mathrm{sh}}) = k^{\mathrm{sep}}$. Now let $A \to S$ be a local morphism into a strictly henselian ring together with a morphism $k^{\mathrm{sep}} \to \kappa(S)$. To show that A^{sh} is initial, we need to construct a unique morphism $B_\lambda \to S$; then taking the colimit will give the required unique morphism $A^{\mathrm{sh}} = \operatorname{colim}_{\lambda\in\Lambda} B_\lambda \to S$. To construct this, note that $\kappa(B_\lambda) \to k^{\mathrm{sep}} \to \kappa(S)$ gives a morphism $B_\lambda \otimes_A S \to \kappa(S)$. Using Proposition 1.6.7(e) this lifts uniquely to an S-algebra morphism $B_\lambda \otimes_A S \to S$. From the adjunction

$$\operatorname{Hom}_{\mathrm{Alg}_S}(B_\lambda \otimes_A S, S) \cong \operatorname{Hom}_{\mathrm{Alg}_A}(B_\lambda, S)$$

[13] The U in Proposition 1.6.7(e) need not be affine, but the proof shows that it suffices to consider the affine case $U = \operatorname{Spec} S$.

we obtain a unique A-algebra morphism $B_\lambda \to S$. It's straightforward to check that this has the required property. In the exact same way one can show that A^{h} is indeed initial. Rewinding the argument moreover shows the additional assertion, i.e., that any colimit of the described kind is already an initial object in henselian resp. strictly henselian A-algebras provided they are elements of these categories at all. This finally finishes the proof of (c).

Proof of (d). To show $B^{\mathrm{h}} \cong A^{\mathrm{h}} \otimes_A B$, first note that the right-hand side is indeed local. Indeed, it is a finite product of finite local A^{h}-algebras by Proposition 1.6.7(c); moreover, modding out the maximal ideal $\mathfrak{m}A^{\mathrm{sh}} \subseteq A^{\mathrm{sh}}$ gives

$$A^{\mathrm{h}}/\mathfrak{m}A^{\mathrm{h}} \otimes_A B \cong B/\mathfrak{m}B\,,$$

which is a local ring since B is local. So $A^{\mathrm{h}} \otimes_A B$ is indeed local, and we even see that its residue field is $\kappa(B)$. Now (b) shows that $A^{\mathrm{h}} \otimes_A B$ is again henselian. Finally, since filtered colimits commute with tensor products, we see that $A^{\mathrm{h}} \otimes_A B$ is a colimit of the type considered in (c). Thus the additional assertion in (c) shows $B^{\mathrm{h}} \cong A^{\mathrm{h}} \otimes_A B$, as claimed. The same essentially works for B^{sh}, with some minor modifications. Here we are given a morphism $k^{\mathrm{sep}} \to \kappa(B^{\mathrm{sh}}) = \kappa(B)^{\mathrm{sep}}$ giving B^{sh} the structure of a strictly henselian A-algebra. Now $A^{\mathrm{sh}} \otimes_A B$ need not be local any more, but by Proposition 1.6.7(c) it's still a finite product of local algebra. We claim that B^{sh} is the factor B' corresponding to the maximal ideal forming the kernel of $A^{\mathrm{sh}} \otimes_A B \twoheadrightarrow \kappa(B)^{\mathrm{sep}}$ (induced by $k^{\mathrm{sep}} \to \kappa(B)^{\mathrm{sep}}$ and $\kappa(B) \to \kappa(B)^{\mathrm{sep}}$; surjectivity is an easy argument). Indeed, using (b) we see that this factor is strictly henselian. Finally, since $\operatorname{Spec} B'$ is an open subset of $\operatorname{Spec}(A^{\mathrm{sh}} \otimes_A B)$, B' can be written as a colimit as in (c), whence indeed $B' \cong B^{\mathrm{sh}}$.

The additional assertion $(A/I)^{\mathrm{h}} \cong A^{\mathrm{h}}/IA^{\mathrm{h}}$ is now clear. For $(A/I)^{\mathrm{sh}} \cong A^{\mathrm{sh}}/IA^{\mathrm{sh}}$, observe that $A^{\mathrm{sh}}/IA^{\mathrm{sh}}$ is already a local ring, since the same is true for A^{sh}. So every factor of $A^{\mathrm{sh}}/IA^{\mathrm{sh}}$ is already the ring itself.

Proof of (e). Let A be noetherian. We first show A^{h} is noetherian. By Lemma 1.6.17(a) it's enough to show that $(A^{\mathrm{h}})^\wedge$ noetherian and flat over A^{h} (since then it's automatically faithfully flat, as both are local rings). We claim that the canonical morphism

$$\widehat{A} \xrightarrow{\sim} (A^{\mathrm{h}})^\wedge \tag{1.6.3}$$

is an isomorphism! Indeed, let $B_\lambda = \mathcal{O}_{U,u}$ be a term in the first colimit defining A^{h}, where $\lambda = (U, \overline{u}) \in \Lambda$ is an affine étale neighbourhood of $x\colon \operatorname{Spec} k^{\mathrm{sep}} \to A$. Since A and B_λ have the same residue field and B_λ is étale over A, we have $A/\mathfrak{m}^n \cong B_\lambda/\mathfrak{m}^n B_\lambda$ for all $n \geqslant 1$ (this is a quite well-known property; see [Jac, Lemma A.4.2] for example). Taking colimits shows $A/\mathfrak{m}^n \cong A^{\mathrm{h}}/\mathfrak{m}^n A^{\mathrm{h}}$, thus their completions do indeed coincide. In particular, $(A^{\mathrm{h}})^\wedge$ is noetherian. Moreover, the above argument moreover shows $(A^{\mathrm{h}})^\wedge \cong \widehat{B}_\lambda$. Since each B_λ is noetherian, this shows that $(A^{\mathrm{h}})^\wedge$ is flat over B_λ, and thus also flat over A^{h} by Lemma 1.6.17(c). This finishes the proof that A^{h} is noetherian. Moreover, we see $\dim A = \dim A^{\mathrm{h}}$ since the dimension stays invariant under completion.

The proof that A^{sh} is noetherian is more involved. The first step is to see that it suffices to prove the assertion in the case where A is complete. Indeed, suppose we already know this special case. As above, our goal is to show that $(A^{\mathrm{sh}})^\wedge$ is noetherian and (faithfully) flat over A^{sh}. Observe that the canonical map

$$(A^{\mathrm{sh}})^\wedge \xrightarrow{\sim} \big(\widehat{A}^{\mathrm{sh}}\big)^\wedge \tag{1.6.4}$$

is an isomorphism. This follows from $A^{\mathrm{sh}}/\mathfrak{m}^n A^{\mathrm{sh}} \cong (A/\mathfrak{m}^n)^{\mathrm{sh}} \cong (\widehat{A}/\mathfrak{m}^n\widehat{A})^{\mathrm{sh}} \cong \widehat{A}^{\mathrm{sh}}/\mathfrak{m}^n\widehat{A}^{\mathrm{sh}}$ for all $n \geqslant 1$, using (d). Therefore, assuming the complete case has been proved already, we see that $(A^{\mathrm{sh}})^\wedge$ is noetherian. To see that $(A^{\mathrm{sh}})^\wedge$ is flat over A^{sh}, it suffices to prove flatness over all B_λ as above, because of Lemma 1.6.17(c). But B_λ is noetherian, hence $\widehat{B}_\lambda$ is flat over B_λ, so it suffices to show that $(A^{\mathrm{sh}})^\wedge$ is flat over $\widehat{B}_\lambda$. By our assumption, we know that $\widehat{B}_\lambda^{\mathrm{sh}}$ is noetherian, hence $(\widehat{B}_\lambda^{\mathrm{sh}})^\wedge$ is flat over $\widehat{B}_\lambda^{\mathrm{sh}}$. But the above isomorphism shows

$$\left(\widehat{B}_\lambda^{\mathrm{sh}}\right)^\wedge \cong (B_\lambda^{\mathrm{sh}})^\wedge \cong (A^{\mathrm{sh}})^\wedge\,,$$

so $(A^{\mathrm{sh}})^\wedge$ is flat over $\widehat{B}_\lambda^{\mathrm{sh}}$. All that's left to see is that $\widehat{B}_\lambda^{\mathrm{sh}}$ is flat over $\widehat{B}_\lambda$. But this is obvious, since $\widehat{B}_\lambda^{\mathrm{sh}}$ is a filtered colimit of étale $\widehat{B}_\lambda$-algebras. This finishes the reduction.

So from now on we may assume A is complete, hence henselian by (a). We do induction on $\dim A$ (in the lecture we did a noetherian induction, but this way we actually save ourselves a bit of work). So assume the assertion is true for rings of smaller dimension. We first show that we can moreover reduce to the case where A is a domain. So assume for the moment that the case of domains of dimension $\leqslant \dim A$ has been settled. By a result of Cohen, A^{sh} is already noetherian of only every prime ideal is finitely generated (see [Mat89, Theorem 3.4] for a proof and [Stacks, Tag 05K7] for the more general truth behind this fact). So let $\mathfrak{q} \in \operatorname{Spec} A^{\mathrm{sh}}$ be prime and $\mathfrak{p} = \mathfrak{q} \cap A$. By ($d$) and the assumption, $(A/\mathfrak{p})^{\mathrm{sh}} \cong A^{\mathrm{sh}}/\mathfrak{p}A^{\mathrm{sh}}$ is noetherian, hence $\mathfrak{q}/\mathfrak{p}A^{\mathrm{sh}}$ is finitely generated. Since $\mathfrak{p}$ too is finitely generated as A is noetherian, this shows that $\mathfrak{q}$ is finitely generated, as claimed.

Henceforth we assume A is a domain. If $\dim A = 0$, then $A = k$ is a field. It's easy to see that $A^{\mathrm{sh}} = k^{\mathrm{sep}}$ in this case, which is indeed noetherian. This settles the base case of the induction. So now assume $\dim A > 0$. Since A is complete, it is already henselian. Applying Proposition 1.6.7(d) to each B_λ in the colimit $A^{\mathrm{sh}} = \operatorname{colim}_{\lambda\in\Lambda} B_\lambda$ coming from (1.6.2), we see that A^{sh} can be written as a filtered colimit of finite local étale A-algebras. In particular, every element of A^{sh} is integral over A! Now let $\mathfrak{q} \in \operatorname{Spec} A^{\mathrm{sh}}$ be a prime ideal. Using Cohen's result as before, it suffices to show that $\mathfrak{q}$ is finitely generated. If $\mathfrak{q} = 0$, this is clear, otherwise choose $\alpha \in \mathfrak{q} \setminus \{0\}$. Then α is integral over A, say, $\alpha^n = a_0 + \dots + a_{n-1}\alpha^{n-1}$ for some $a_i \in A$. Clearly $a_0 \in \mathfrak{q}$. If $a_0 \neq 0$, then we are done! Indeed, since $\dim A/a_0A < \dim A$, we may apply the induction hypothesis to the prime ideal $\mathfrak{q}/a_0A^{\mathrm{sh}} \subseteq A^{\mathrm{sh}}/a_0A^{\mathrm{sh}} \cong (A/a_0A)^{\mathrm{sh}}$ which is thus finitely generated. Hence $\mathfrak{q}$ is finitely generated as well.

However, in general there is no reason why $a_0 \neq 0$, since A^{sh} can't be guaranteed to be a domain again, because this may already fail for the B_λ. But in the case where A is normal the argument works: since the conditions R_n and S_n are preserved under étale ring maps (Lemma* A.2.4), Serre's normality criterion shows that the B_λ are normal domains again, hence domains at all, which shows that A^{sh} is a domain again.

To finish the proof in general, let B be the normalization of A. Since complete local rings are universally Japanese, B is a finite A-algebra (see Lemma* A.3.8). In particular, $\dim A = \dim B$ and B is local again by going-up, hence the above inductive argument works for B (doing induction on the dimension spares us the somewhat delicate argument from the lecture) and we obtain that B^{sh} is noetherian. By (c), B^{sh} is a factor of $A^{\mathrm{sh}} \otimes_A B$, hence finite over A^{sh}. In the lecture we used the Eakin–Nagata theorem ([Mat89, Theorem 3.7]) to conclude that A^{sh} is noetherian too. But this needs that A^{sh} is a subring of B^{sh}, and I really don't see why that should be obvious. So we use a workaround here. We claim that $A^{\mathrm{sh}} \otimes_A B$ is a finite product of normal domains. If that was shown, we could conclude the proof as follows: it suffices to show that $A^{\mathrm{sh}} \otimes_A B$ is noetherian, since $A^{\mathrm{sh}} \hookrightarrow A^{\mathrm{sh}} \otimes_A B$ is injective, using that $A \subseteq B$ is a subring and A^{sh} is flat over A (since it is a filtered colimit of flat A-algebras), so we could apply the Eakin–Nagata theorem to $A^{\mathrm{sh}} \otimes_A B$ instead. Now let $\mathfrak{q} \subseteq \operatorname{Spec}(A^{\mathrm{sh}} \otimes_A B)$ be a prime ideal. If $\mathfrak{q} = 0$, then $\mathfrak{q}$ is finitely generated. Otherwise let

$\alpha \in \mathfrak{q} \setminus \{0\}$. Clearly $A^{\mathrm{sh}} \otimes_A B$ is integral over A, hence $\alpha^n = a_0 + \dots + a_{n-1}\alpha^{n-1}$ for some $\alpha_i \in A$. And now we can assume $a_0 \neq 0$! Indeed, if $A^{\mathrm{sh}} \otimes_A B \cong \prod_{i=1}^m B_i$ is the assumed decomposition into (normal) domains and $\alpha = (\alpha_1, \dots, \alpha_m)$ for $\alpha_i \in B_i$, then $P_i(\alpha_i) = 0$ for some monic polynomials with non-zero constant coefficients. Thus $P = \prod_{i=1}^m P_i$ is a monic polynomial satisfying $P(\alpha) = 0$, and its constant coefficient is non-zero because A is a domain. Now $\mathfrak{q}/(a_0)$ is a prime ideal of $(A^{\mathrm{sh}}/a_0A^{\mathrm{sh}}) \otimes_A B \cong (A/a_0A)^{\mathrm{sh}} \otimes_A B$. By the induction hypothesis, this ring is a finite algebra over the noetherian ring $(A/a_0A)^{\mathrm{sh}}$, hence noetherian itself, proving that $\mathfrak{q}/(a_0)$ and thus $\mathfrak{q}$ is finitely generated.

It remains to see the assertion about $A^{\mathrm{sh}} \otimes_A B$. The idea is straightforward: we have $A^{\mathrm{sh}} \otimes_A B \cong \operatorname{colim}_{\lambda \in \Lambda} B_\lambda \otimes_A B$, and each $B_\lambda \otimes_A B$ can be decomposed into a product of normal domains by Lemma* A.2.4 and Serre's normality criterion. So it will be enough to show that upon shrinking Λ these decompositions may be chosen in a compatible way. Since $A^{\mathrm{sh}} \otimes_A B$ is finite over the henselian ring A^{sh}, it can be decomposed into a product

$$A^{\mathrm{sh}} \otimes_A B \cong \prod_{i=1}^m B_i$$

of finite local A^{sh}-algebras B_i. Let $e_1, \dots, e_m \in A^{\mathrm{sh}} \otimes_A B$ be the corresponding idempotents, i.e., e_i maps to $1 \in B_i$ and to $0 \in B_j$ for all $j \neq i$. Since Λ is a filtered system, the e_i are already contained in some $B_{\lambda_0} \otimes_A B$, and $A^{\mathrm{sh}} \cong \operatorname{colim}_{\lambda_0 \leqslant \lambda} B_\lambda \otimes_A B$. Now every $\lambda_0 \leqslant \lambda$ has a decomposition

$$B_\lambda \otimes_A B \cong \prod_{i=1}^m (B_\lambda \otimes_A B)_{e_i}\,.$$

We claim that this is already the decomposition into normal domains ensured by Serre's normality criterion. Indeed, the only thing that could go wrong is that some factor $(B_\lambda \otimes_A B)_{e_i}$ can be further factored into some $B_\lambda' \times B_\lambda''$. But then

$$(B_\mu \otimes_A B)_{e_i} \cong (B_\lambda' \otimes_{B_\lambda} B_\mu) \times (B_\lambda'' \otimes_{B_\lambda} B_\mu)$$

holds for all $\lambda \leqslant \mu$ (and both factors are non-zero as B_μ is étale over B_λ), so from $A^{\mathrm{sh}} \otimes_A B \cong \operatorname{colim}_{\lambda \leqslant \mu} B_\mu \otimes_A B$ we would get a decomposition $B_i \cong B_i' \times B_i''$ as well, which is a contradiction. This finally shows that

$$A^{\mathrm{sh}} \otimes_A B \cong \prod_{i=1}^m \operatorname*{colim}_{\lambda_0 \leqslant \lambda} (B_\lambda \otimes_A B)_{e_i}$$

is indeed a finite product of normal domains, and the proof that A^{sh} is noetherian is finished. To see $\dim A = \dim A^{\mathrm{sh}}$, we first use $\dim A = \dim \widehat{A}$ and (1.6.4) to reduce to the case where A is complete. In this case, we've seen that A^{sh} is integral over A, so $\dim A^{\mathrm{sh}} \leqslant \dim A$ by going-up. However, if A is a domain, then $A \hookrightarrow A^{\mathrm{sh}}$ is injective. Indeed, for every $a \in A \setminus \{0\}$ the multiplication map $a\colon A \to A$ is injective, hence the same is true for $a\colon A^{\mathrm{sh}} \to A^{\mathrm{sh}}$ as A^{sh} is flat over A. Thus, going-up even shows $\dim A = \dim A^{\mathrm{sh}}$ in the case where A is a domain. For the general case, $\dim A = \min_{\mathfrak{p}} \dim A/\mathfrak{p}$, where $\mathfrak{p}$ ranges through the minimal prime ideals of A. Now $\dim A/\mathfrak{p} = \dim (A/\mathfrak{p})^{\mathrm{sh}} = \dim A^{\mathrm{sh}}/\mathfrak{p}A^{\mathrm{sh}} \leqslant \dim A^{\mathrm{sh}}$, proving $\dim A \leqslant \dim A^{\mathrm{sh}}$, whence equality must hold.

Proof of (f), (g). Assertion (f) is trivial, as A^{h} clearly satisfies the universal property of A^{sh} in this case. Part (g) is an immediate consequence of the construction of A^{sh} in (1.6.2). We are done! $\square$

1.6.4. G-Rings, Excellent Rings, and Universally Japanese Rings

Lecture 12
29th Nov, 2019

In the rest of the section we collect some more properties of (strict) henselization and relate them to the notions of *excellent* and *universally Japanese* rings. Along the way we will stumble upon Artin's famous *approximation theorem* and some more pieces of advanced commutative algebra.

1.6.18. Fact. — *The henselization and strict henselization of a local ring A has the following properties.*

(*a*) *A^{h} and A^{sh} are faithfully flat A-algebras.*

(*b*) *If A is noetherian, then A has the property R_k iff A^{h} has R_k iff A^{sh} has R_k. The same is true for property S_k.*

(*c*) *If A is noetherian and $\mathfrak{p} \in \operatorname{Spec} A^{\mathrm{h}}$ or $\mathfrak{p} \in \operatorname{Spec} A^{\mathrm{sh}}$, then $\mathfrak{p} \cap A$ is an associated prime of A iff $\mathfrak{p}$ is already an associated prime ideal of A^{h} resp. A^{sh}.*

(*d*) *The fibres of $\operatorname{Spec} A^{\mathrm{h}} \to \operatorname{Spec} A$ and $\operatorname{Spec} A^{\mathrm{sh}} \to \operatorname{Spec} A$ over some $\mathfrak{p} \in \operatorname{Spec} A$ are disjoint unions of spectra of separable field extensions of $\kappa(\mathfrak{p})$.*

(*e*) *Henselization and strict henselization commute with filtered colimits of local rings along local ring morphisms.*

(*f*) *If A is noetherian and universally catenary, then the same is true for A^{h} and A^{sh}.*

Sketch of a proof. Part (*a*) is easy: both A^{h} and A^{sh} are filtered colimits of flat A-algebras, hence flat themselves. Moreover, $A \to A^{\mathrm{h}}$ and $A \to A^{\mathrm{sh}}$ are local morphisms, hence even faithfully flat. We omit the proofs of (*b*) to (*e*), but refer to [$\mathrm{EGA}_{\mathrm{IV}/4}$, (18.6) and (18.8)].

For (*f*), we verify the criterion from Theorem 1.6.19(*c*) below. In the case of A^{h} and $\mathfrak{p} = 0$ we can use the fact that $\widehat{A}$ and $(A^{\mathrm{h}})^\wedge$ are isomorphic (see (1.6.3)). For general $\mathfrak{p}$, one uses the fact that $\mathfrak{p}$ already "comes from" some étale A-algebra B_λ in the colimit $A^{\mathrm{h}} = \operatorname{colim}_{\lambda \in \Lambda_0} B_\lambda$, i.e., there is some $\mathfrak{p}_\lambda \in \operatorname{Spec} B_\lambda$ such that $\mathfrak{p} = \mathfrak{p}_\lambda A^{\mathrm{h}}$. Indeed, $\mathfrak{p}$ is finitely generated as A^{h} is noetherian (Proposition 1.6.15(*e*)), so all generators are already contained in some B_λ. The generated ideal $\mathfrak{p}_\lambda$ is necessarily prime. Indeed, $A^{\mathrm{h}}/\mathfrak{p}_\lambda A^{\mathrm{h}} \cong B_\lambda^h/\mathfrak{p}_\lambda B_\lambda^{\mathrm{h}} \cong (B_\lambda/\mathfrak{p}_\lambda)^{\mathrm{h}}$ (using Proposition 1.6.15(*c*)) is a domain and faithfully flat over $B_\lambda/\mathfrak{p}_\lambda$ by (*a*). Now if $b \in B_\lambda/\mathfrak{p}_\lambda$ is a zero divisor and I is the kernel of the multiplication map $b\colon B_\lambda/\mathfrak{p}_\lambda \to B_\lambda/\mathfrak{p}_\lambda$, then $I \otimes_{B_\lambda/\mathfrak{p}_\lambda} (B_\lambda/\mathfrak{p}_\lambda)^{\mathrm{h}}$ is the kernel of $b\colon (B_\lambda/\mathfrak{p}_\lambda)^{\mathrm{h}} \to (B_\lambda/\mathfrak{p}_\lambda)^{\mathrm{h}}$, which is nonzero as $(B_\lambda/\mathfrak{p}_\lambda)^{\mathrm{h}}$ is faithfully flat. Thus $b = 0$ in $(B_\lambda/\mathfrak{p}_\lambda)^{\mathrm{h}}$. But then $(b) \otimes_{B_\lambda/\mathfrak{p}_\lambda} (B_\lambda/\mathfrak{p}_\lambda)^{\mathrm{h}}$ vanishes as well, proving $b = 0$ since $(B_\lambda/\mathfrak{p}_\lambda)^{\mathrm{h}}$ is faithfully flat. This reduces the general case to the special case $\mathfrak{p} = 0$ above.

For the strict henselization, one can again reduce to the case $\mathfrak{p} = 0$. Moreover, we use $(A^{\mathrm{sh}})^\wedge \cong (\widehat{A}^{\mathrm{sh}})^\wedge$ (by (1.6.4)) to reduce to the case where A is a complete domain. By Cohen's structure theorem (see [Stacks, Tag 032A]), A is a quotient of a complete regular local ring C, hence A^{sh} is a quotient of C^{sh}. Now C^{sh} is regular by (*b*), hence universally catenary, thus A^{sh} is universally catenary too. Applying Theorem 1.6.19 backwards, we see that all irreducible components of $(A^{\mathrm{sh}})^\wedge$ have the same dimension, as required. □

1.6.19. Theorem (Ratliff). — *For a noetherian local ring A, the following are equivalent:*

(*a*) *A is universally catenary, i.e., every A-algebra of finite type is catenary.*

(*b*) *$A[T]$ is catenary.*

(*c*) *For every prime ideal* $\mathfrak{p}$*, the irreducible components of* $\operatorname{Spec}\big((A/\mathfrak{p})^\wedge\big)$ *are all of the same dimension.*

Proof. Omitted. See [Mat89, Theorem 31.7] or [Stacks, Tag 0AW1]. □

1.6.20. Definition. — Recall the following notions from commutative algebra.

(*a*) A noetherian local ring A is a *G-ring* if the geometric fibres $\widehat{A}\otimes_A \overline{\kappa(\mathfrak{p})}$ of the ring map $A\rightarrow \widehat{A}$ are regular for every $\mathfrak{p}\in\operatorname{Spec} A$.

(*b*) A ring A is called *excellent* if it is noetherian and universally catenary, all its local rings are G-rings, and for every A-algebra B of finite type the regular locus $\{\mathfrak{p}\in\operatorname{Spec} B \mid B_\mathfrak{p} \text{ is regular}\}$ is open in $\operatorname{Spec} B$ (rings with this property are called "J-2 rings").

(*c*) A noetherian A is *universally Japanese* if for every prime ideal $\mathfrak{p}\in\operatorname{Spec} A$ and any finite field extension ℓ of $\kappa(\mathfrak{p})$ the normalization of $A/\mathfrak{p}$ in ℓ is a finitely generated A-module.

If A is a local ring, then the last condition in Definition 1.6.20(*b*), i.e. A being J-2, is an automatic consequence if A is a G-ring (see the appendix, Proposition* A.3.5 for a proof). Hence if A is local, one only has to verify that A is a G-ring and universally catenary.

In Definition 1.6.20(*c*), a hard theorem of Nagata shows that every A-algebra of finite type is universally Japanese too. A proof can be found in [Stacks, Tag 032E] (note that The Stacks Project calls rings with the property from Definition 1.6.20(*c*) *Nagata rings*, so what they show is that "every Nagata ring is universally Japanese"). Moreover, if A is excellent, then A is universally Japanese (we prove this in Proposition* A.3.7) and every A-algebra of finite type is excellent too.

1.6.21. Example. — If A is a DVR with quotient field K, then A is regular, hence universally Japanese. As remarked above, for the local ring A to be excellent it is necessary and sufficient that A is a G-ring. The fibre over the special point $\mathfrak{m}\in\operatorname{Spec} A$ is trivial, so $\widehat{A}\otimes_A\overline{K}$ is regular, hence only the generic fibre matters and we conclude that A is excellent iff $\widehat{A}\otimes_A\overline{K}$ is regular!

If $\widehat{K}$ denotes the quotient field of $\widehat{A}$, then $\widehat{A}\otimes_A\overline{K}\cong\widehat{K}\otimes_K\overline{K}$ (because tensoring with K is the same as localizing at a uniformizer $\pi\in A$). The latter is regular iff $\widehat{K}$ contains no inseparable field extension of K, which is trivial in characteristic 0. Being universally Japanese in this case is also equivalent to the same condition. Indeed, if you really dive into the proof of Proposition* A.3.7, you find out that being universally Japanese is equivalent to $\widehat{A}\otimes_A K\cong\widehat{K}$ being geometrically reduced over K, which is pretty much the above condition. But honestly, I'm too lazy to work this out.

The following fact has actually been given in the 13^{th} lecture, but I decided to relocate it since it seemed quite out of place (and Professor Franke likely just forgot to mention this).

1.6.22. Fact. — *Let A be a noetherian local ring.*

(*a*) *A is universally Japanese iff A^{h} is universally Japanese.*

(*b*) *A is a G-ring iff A^{h} is a G-ring. In this case A^{sh} is a G-ring as well.*

(*c*) *If A is excellent, then so is A^{h}.*

Proof. All but the second assertion of (*b*) are in [$\mathrm{EGA}_{\mathrm{IV}/4}$, (18.7)]. Said second assertion is proved in [Stacks, Tag 07QR] or [FK88, end of I.1]. □

1.6.5. The Artin Approximation Property

1.6.23. Definition/Lemma. — A noetherian local ring has the *Artin approximation property* (AAP for short) if it satisfies the following equivalent conditions:

(a) If $f_1, \dots, f_n \in A[X_1, \dots, X_m]$ and $\alpha \in \widehat{A}^m$ satisfy $f_i(\alpha) = 0$ for all $i = 1, \dots, n$, then for every $s \in \mathbb{N}$ there is an $a_s \in A^m$ such that $f_i(a_s) = 0$ for all $i = 1, \dots, n$ and such that the images of a_s and α in $A/\mathfrak{m}^s \cong \widehat{A}/\mathfrak{m}^s\widehat{A}$ coincide.

(b) If $F\colon \mathrm{Alg}_A \rightarrow \mathrm{Set}$ is a functor that comits with filtered colimits, and if $\varphi \in F(\widehat{A})$, then for every $s \in \mathbb{N}$ there is an $f_s \in F(A)$ such that the images of f and φ in $F(A/\mathfrak{m}^s) \cong F(\widehat{A}/\mathfrak{m}^s\widehat{A})$ coincide.

Proof of equivalence. We start with (a) $\Rightarrow$ (b). Since $\widehat{A}$ is a filtered colimit of its subalgebras B of finite type over A, there is such a subalgebra $B \cong A[X_1, \dots, X_m]/(f_1, \dots, f_n)$ and an element $\varphi' \in F(B)$ whose image in $F(\widehat{A})$ equals φ. The ring morphism $B \rightarrow A$ is given by an element $\alpha \in \widehat{A}^m$ satisfying $f_i(\alpha) = 0$ for $i = 1, \dots, n$. By (a) there is an element $a_s \in A^m$ such that $f_i(a_s) = 0$ and $a_s \equiv \alpha \mod \mathfrak{m}^s$. Let $\beta_s\colon B \rightarrow A$ be the morphism defined by a. Then $f = F(\beta_s)(\varphi') \in F(A)$ satisfies the desired conditions.

For (b) $\Rightarrow$ (a), all we need to do is to consider the functor $F\colon \mathrm{Alg}_A \rightarrow \mathrm{Set}$ given by $F(B) = \{b \in B \mid f_i(b) = 0 \text{ for all } i = 1, \dots, n\}$. It's easy to check that this indeed commutes with filtered colimits. $\square$

There is also a more general notion of having the AAP with respect to an ideal $I \subseteq A$. For more information, check out Guillaume Rond's paper [Ron18]. In the next few remarks we will derive some properties of rings having the AAP, construct some counterexamples, and finally link the AAP to the property of being excellent.

1.6.24. Remark. — Let A be a noetherian local ring with the AAP.

(a) A is henselian. To see this, use the separatedness of the $\mathfrak{m}$-adic topology on A to deduce Proposition 1.6.7(f) from Definition/Lemma 1.6.23(a) and the fact that $\widehat{A}$ is already henselian.

(b) If A is reduced, then so is $\widehat{A}$ (this follows immediately from Definition/Lemma 1.6.23(a)). Note that for A^{h} and A^{sh} this doesn't need the AAP, since it follows from Fact 1.6.18(b) and Serre's criterion that a ring is reduced iff it is R_0 and S_1.

(c) If $\widehat{A}$ is a domain, then A is algebraically closed in $\widehat{A}$ (this is just straightforward from Definition/Lemma 1.6.23(a)). *Yes, algebraically* and not just integrally! The polynomials in question need not be monic.

1.6.25. Example. — Here are some examples of very well-behaved rings that yet do not have the AAP.

(a) This counterexample is due to Nagata. Or maybe F.K. Schmidt. Whatever the case, you should have a look at [BGR84]. Let k be a field of characteristic $p > 0$ such that $[k : k^p]$ is infinite. Consider the ring

$$A = \left\{ \sum_{i=0}^{\infty} a_i T^i \;\middle|\; \begin{array}{c} a_i \in k, \text{ and the subfield of } k \text{ generated by } k^p \\ \text{and the } a_i \text{ is a finite field extension of } k^p \end{array} \right\}.$$

Then $\widehat{A} = k[\![T]\!]$ and $\widehat{A}^p \subseteq A$, so Remark 1.6.24($c$) fails. This shows that A is a DVR without the AAP. In this case the normalization of A in $K(f)$ fails to be a finitely generated A module for all $f \in k[\![T]\!] \setminus A$, where K denotes the quotient field of A (note that f is integral over K as $f^q \in K$).

(b) Let $x \in \mathbb{Q}_p$ be transcendental over $\mathbb{Q}$ and consider the ring
$$A = \{f \in \mathbb{Q}(T) \mid f(x) \in \mathbb{Z}_p \text{ and } f'(x) \in \mathbb{Z}_p\} \ .$$
Then A is a noetherian local ring and one has an isomorphism $\widehat{A} \cong \mathbb{Z}_p[T]/(T^2)$ sending $f \in \mathbb{Q}[T]$ to the image of $f(x) + f'(x)T$. In this case, Remark 1.6.24(b) fails, and the same holds after passing to A^{h} as $(A^{\mathrm{h}})^\wedge \cong \widehat{A}$. Note that the normalization of A in its field of quotients $\mathbb{Q}(T)$ is $\{f \in \mathbb{Q}(T) \mid f(x) \in \mathbb{Z}_p\}$, which is no finitely generated A-algebra.

Lecture 13
2nd Dec, 2019

1.6.26. Remark. — *Popescu's theorem* is a celebrated result, which implies that every henselian noetherian local G-ring has the AAP (see [Ron18] or [Stacks, Tag 07BW] for proofs). Conversely, if a noetherian local ring has the AAP, then it is excellent, as proved by Rotthaus in [Rot90]. In particular, a henselian G-ring is excellent, hence universally catenary (in fact, the latter is not that hard to show directly).

Artin's original paper shows that if A is the henselization (at an arbitrary prime ideal) of an algebra of finite type over a field or an excellent DVR, then A has the AAP (see [Art69, Theorem (1.10)]). This result is known as *Artin's approximation theorem.*

1.6.27. Corollary. — *Let A be a local G-ring.*

(a) *A^{h} is a domain iff $\widehat{A}$ is a domain.*

(b) *When $\widehat{A}$ is a domain, A^{h} is the algebraic closure of A in $\widehat{A}$ (no, we are not talking about the integral closure here).*

Proof. Recall that $\widehat{A} \cong (A^{\mathrm{h}})^\wedge$ by (1.6.3). Thus if $\widehat{A}$ is a domain, then so is A^{h} by faithful flatness. Conversely, assume A^{h} is a domain. By Fact 1.6.22(b), A^{h} is a G-ring again, hence it has the AAP by Popescu's theorem. Consider the functor $F\colon \mathrm{Alg}_{A^{\mathrm{h}}} \to \mathrm{Set}$ given by
$$F(B) = \left\{(b, b') \in B^2 \;\middle|\; bb' = 0\right\} \ .$$
It's clear that F commutes with filtered colimits: if $(b, b') \in F(B)$, where $B = \operatorname{colim}_{\lambda \in \Lambda} B_\lambda$ is a filtered colimit, then b and b are already contained in some B_λ, and their product must vanish in some B_μ for $\lambda \leqslant \mu$. So the AAP is applicable. Now if $F(\widehat{A})$ contains a non-trivial element (α, α') with $\alpha, \alpha' \neq 0$, then by the AAP there are elements $(a_s, a_s') \in F(A^{\mathrm{h}})$ satisfying $a_s a_s' = 0$. Moreover, for sufficiently large s we have $a_s, a_s' \neq 0$, contradicting the assumption that A^{h} is a domain. This proves (a).

For (b), note that étale A-algebras are quasi-finite, hence algebraic over A (albeit not necessarily integral). Hence A^{h} is algebraic over A. Since A^{h} has the AAP by Popescu's theorem, Remark 1.6.24(c) shows that it is algebraically closed in $\widehat{A}$. This proves (b). $\square$

The most important application of Artin's approximation theorem to étale cohomology is the following corollary.

1.6.28. Corollary. — *Let A be a henselian noetherian local ring with residue field k and let X be a proper scheme over $S = \operatorname{Spec} A$. Further let $S_0 = \operatorname{Spec} k$ and $X_0 = X \times_S S_0$. Then the functor*
$$\begin{aligned} \{\textit{finite étale } X\textit{-schemes}\} &\xrightarrow{\sim} \{\textit{finite étale } X_0\textit{-schemes}\} \\ Y &\longmapsto Y_0 = Y \times_X X_0 \end{aligned}$$

is an equivalence of categories, and $\pi_0(X_0) \to \pi_0(X)$ is a bijection. Consequently, for all geometric points $\overline{x}$ of X_0 we have an isomorphism $\pi_1^{\text{ét}}(X_0, \overline{x}) \xrightarrow{\sim} \pi_1^{\text{ét}}(X, \overline{x})$.

1.6.29. Remark. — (*a*) A heuristic reason why the proof of Corollary 1.6.28 is relatively complicated (in that it uses heavy machinery like Artin's approximation theorem) is the following. Assuming there is a suitable "étale homotopy theory", we would look at the exact sequence

$$\pi_2^{\text{ét}}(S, \overline{x}) \longrightarrow \pi_1^{\text{ét}}(X_0, \overline{x}) \longrightarrow \pi_1^{\text{ét}}(X, \overline{x}) \longrightarrow \pi_1^{\text{ét}}(S, \overline{x}) \,,$$

so we would need to investigate $\pi_2^{\text{ét}}(S, \overline{x})$, whatever that may be. At least so much can be said: $\pi_2^{\text{ét}}$ is even worse than in Algebraic Topology.

(*b*) A full proof of Corollary 1.6.28 can be found in [$\text{SGA}_{4\frac{1}{2}}$, Arcata IV Prop. 4.1] or [Stacks, Tag 0A48]. In the case where the dimensions of the fibres of $X \to S$ are $\leqslant 1$, the use of the AAP may be bypassed; see [$\text{SGA}_{4/3}$, Exposé XIII Prop. 2.1]. Professor Franke does not know whether the AAP can be avoided in general.

(*c*) Corollary 1.6.28 is a vast generalization of Proposition 1.6.10, which is in fact the special case $X = S$.

Proof of Corollary 1.6.28. After filling in some details that Professor Franke (intentionally) left out, the proof has become quite long, so I decided to split it into five steps.

Step 1. We establish the assertion about connected components first. Recall that for a noetherian scheme Y, the set of connected components $\pi_0(Y)$ are encoded in $\Gamma(Y, \mathcal{O}_Y)$ as the "minimal" idempotents, i.e., those that do not divide any other idempotent. Let $X_n = X \times_S \operatorname{Spec}(A/\mathfrak{m}^{n+1})$ be the n^{th} infinitesimal thickening of X_0. Using the baby case of Hensel's lemma for nilpotent ideals, we see that the idempotents of $\Gamma(X_0, \mathcal{O}_{X_0})$ are in canonical bijection with the idempotents of $\Gamma(X_n, \mathcal{O}_{X_n})$. By the theorem about formal functions, they are also in canonical bijection with the idempotents of

$$\Gamma(X, \mathcal{O}_X) \otimes_A \widehat{A} \cong \Gamma(X, \mathcal{O}_X)^\wedge \cong \lim_{n \in \mathbb{N}} \Gamma(X_n, \mathcal{O}_{X_n}) \,. \tag{$*$}$$

Since X is proper over S, the ring of global sections $B = \Gamma(X, \mathcal{O}_X)$ is a finite A-algebra, hence a finite product of finite local A-algebras by Proposition 1.6.7(*c*), and its idempotents are in canonical bijection with the idempotents of $B/\mathfrak{m}B$. Since $\widehat{A}$ is still henselian, ring $\widehat{B} \cong B \otimes_A \widehat{A}$ is likewise a product of local $\widehat{A}$-algebras, and its idempotents of $\widehat{B}$ are in canonical bijection with those of $\widehat{B}/\mathfrak{m}\widehat{B} \cong B/\mathfrak{m}B$. Now ($*$) and the arguments before conclude the proof that $\pi_0(X_0) \xrightarrow{\sim} \pi_0(X)$ is an isomorphism.

Step 2. To show that the functor in question is fully faithful we would like to invoke Lemma 1.6.11. Thus we only need to check that for finite étale X-schemes Y the map $\pi_0(Y_0) \to \pi_0(Y)$ is bijective. But such a Y is again proper over S, so the above argument can be applied again.

Step 3. For essential surjectivity, let $Y_0 \to X_0$ be finite étale. By Proposition 1.4.20, there are unique (up to unique isomorphism) finite étale X_n-schemes Y_n such that $Y_n \times_{X_n} X_0 \cong Y_0$. Uniqueness moreover tells us $Y_n \times_{X_n} X_{n-1} \cong Y_{n-1}$. Thus, $Y_n \cong \underline{\operatorname{Spec}}\, \mathcal{A}_n$, where $\mathcal{A}_n$ is a flat coherent $\mathcal{O}_{X_n}$-algebra, and $\mathcal{A}_n|_{X_{n-1}} \cong \mathcal{A}_{n-1}$. By Grothendieck's famous *existence theorem* (see [$\text{EGA}_{\text{III}/1}$, Théorème (5.1.4)] for the concrete statement) the sequence $(\mathcal{A}_n)_{n \in \mathbb{N}}$ gives rise to a unique coherent algebra $\widehat{\mathcal{A}}$ on $\widehat{X} = X \times_S \operatorname{Spec} \widehat{A}$ satisfying $\mathcal{A}_n \cong \widehat{\mathcal{A}}|_{X_n}$. We claim:

(⊠) *In our case, the coherent algebra $\widehat{\mathcal{A}}$ is flat (hence locally free) and $\widehat{Y} = \underline{\operatorname{Spec}}\,\widehat{\mathcal{A}}$ is a finite étale $\widehat{X}$-scheme.*

For the proof of flatness, it suffices to see that tensoring with $\widehat{\mathcal{A}}$ preserves monomorphisms of coherent modules on $\widehat{X}$. So let $\mathcal{F} \hookrightarrow \mathcal{G}$ be a monomorphism of coherent modules over $\widehat{X}$ and consider $\varphi\colon \mathcal{F} \otimes \widehat{\mathcal{A}} \to \mathcal{G} \otimes \widehat{\mathcal{A}}$. Since the $\mathcal{A}_n$ are flat over X_n by assumption, pulling back to X_n shows

$$\ker\varphi \subseteq \bigcap_{n\geqslant 1} \mathfrak{m}^n(\mathcal{F}\otimes\widehat{\mathcal{A}})$$

By Krull's intersection theorem, this implies that the support Z of $\ker\varphi$ doesn't intersect the closed subset of $\widehat{X}$ defined by X_0. But $\widehat{X} \to \operatorname{Spec}\widehat{A}$ is proper, so the image of Z is closed because Z is closed, hence the image contains the unique closed point $\mathfrak{m} \in \operatorname{Spec}\widehat{A}$ unless Z is empty. As Z doesn't intersect X_0, this shows that Z must be indeed empty. To get that $\widehat{Y}$ is étale, we must show (Proposition 1.4.10) that the trace induces a perfect pairing

$$\operatorname{Tr}_{\widehat{\mathcal{A}}/\widehat{X}}\colon \widehat{\mathcal{A}} \times \widehat{\mathcal{A}} \longrightarrow \mathcal{O}_{\widehat{X}}\,.$$

Knowing that $\widehat{\mathcal{A}}$ is a vector bundle on $\widehat{X}$, we may apply a similar argument as above to the kernel and cokernel of the induced map $\widehat{\mathcal{A}} \to \mathcal{H}om_{\widehat{X}}(\widehat{\mathcal{A}}, \mathcal{O}_{\widehat{X}})$ to see that this is indeed an isomorphism. This shows (⊠).

Step 4. In the case where A is complete, $X = \widehat{X}$ and the proof of essential surjectivity is finished. Otherwise we must find a way to "descend" $\widehat{Y}$ to a finite étale X-scheme Y. If A already has the AAP (i.e., if A is a G-ring by Popescu's theorem, see Remark 1.6.26), this is easily done: consider the functor $F\colon \mathrm{Alg}_A \to \mathrm{Set}$ defined by

$$F(B) = \{\text{isomorphism classes of étale coverings of } X_B = X \times_S \operatorname{Spec} B\}\,.$$

If F would commute with filtered colimits, then we could apply the AAP in the form of Definition/Lemma 1.6.23(b), which straight up provides the desired étale covering of X. Proving that F indeed commutes with filtered colimits is relatively easy but quite technical, so we give only a sketch: suppose we have $B = \operatorname{colim}_{\lambda\in\Lambda} B_\lambda$ and $\mathcal{S}$ is a flat coherent $\mathcal{O}_{X_B}$-algebra such that $\underline{\operatorname{Spec}}\,\mathcal{S} \to X_B$ is an étale covering. Then $\mathcal{S}$ is a vector bundle, so by A.1.2(e) it is the pullback of some vector bundle $\mathcal{S}_\lambda$ on X_λ. Choose finitely many local generators. Taking $\mu \geqslant \lambda$ large enough, we can achieve that the finitely many products (taken in $\mathcal{S}$) of these generators are already contained in the pullback $\mathcal{S}_\mu$ to X_μ and that $\mathcal{S}_\mu$ contains a global section that acts as a unit on all the chosen local generators. Then $\mathcal{S}_\mu$ has already an $\mathcal{O}_{X_\mu}$-algebra structure compatible with that of $\mathcal{S}$. Applying similar considerations to the kernel and cokernel of $\mathcal{S} \to \mathcal{H}om_{X_B}(\mathcal{S}, \mathcal{O}_{X_B})$ induced by the trace pairing, we see that these already vanish for $\mathcal{S}_\mu$ if μ is chosen large enough. So every étale covering of X_B comes from an étale covering of some X_μ. By similar arguments, every isomorphism between étale coverings of X_B already exists on some "finite stage" (i.e., comes from some X_μ). This more or less proves commutativity with filtered colimits.

Step 5. To remove the assumption that A has the AAP, one applies a similar reduction as in our (sketched) proof that F commutes with filtered colimits. Write $A = \operatorname{colim} A_\alpha$ as a filtered colimit over its subalgebras $A_\alpha \subseteq A$ which are of finite type over $\mathbb{Z}$. Using A.1.3(i), both $X \to S$ and $Y_0 \to X_0$ may be written as limits over $X_\alpha \to S_\alpha$ and $(Y_\alpha)_0 \to (X_\alpha)_0$ if α is large enough. By A.1.3(j) and (k), we may chose α even larger in order to achieve that both $X_\alpha \to S_\alpha$ and $(Y_\alpha)_0 \to (X_\alpha)_0$ are proper resp. finite étale. Put $X' = X_\alpha$, $A' = A_\alpha$ and so one. Now have A' of finite type over $\mathbb{Z}$, together with a proper morphism $X' \to S' = \operatorname{Spec} A'$ and a finite étale morphism $Y_0' \to X_0'$ such that our original situation is the base change of our new situation along $S \to S'$. Let $\mathfrak{p} = \mathfrak{m} \cap A'$ and let $A'' = (A'_{\mathfrak{p}})^{\mathrm{h}}$ be the henselization of A' with respect to its prime ideal $\mathfrak{p}$. Define X'', S'' etc. accordingly. Since $A'_{\mathfrak{p}}$ is an algebra of essentially finite type over $\mathbb{Q}$ or $\mathbb{Z}_{(p)}$ (depending on whether $\mathfrak{m} \cap \mathbb{Z} = (0)$ or $\mathfrak{m} \cap \mathbb{Z} = (p)$) and A'' is its henselization, we see that A'' has the AAP by Artin's approximation theorem [Art69, Theorem (1.10)]. So the argument from Step 4 is applicable and shows that $Y_0'' \to X_0''$ comes from some finite étale morphism $Y'' \to X''$. Since $A' \to A$ factors through A'' by naturality of henselization, we may base change Y'' along $S \to S''$ back to get $Y = Y'' \times_{S''} S \to X$, which is an étale covering satisfying $Y \times_X X_0 \cong Y_0$, as required. □

Lecture 14
6th Dec, 2019

1.6.30. Corollary. — *Let A be a henselian noetherian local ring with residue field k. Let $S = \operatorname{Spec} A$, $S_0 = \operatorname{Spec} k$, and $X \to S$ a proper morphism such that the dimension of $X_0 = X \times_S S_0$ is $\leqslant 1$. Then the canonical morphism*

$$\operatorname{Pic}(X) \twoheadrightarrow \operatorname{Pic}(X_0)$$

(given by pullback) is surjective.

1.6.31. Remark. — (*a*) Here, $\operatorname{Pic}(X)$ denotes the set of isomorphism classes of line bundles on X, equipped with its canonical group structure given by the tensor product.

(*b*) Without the restriction that $\dim X_0 \leqslant 1$, the assertion is wrong. For instance, $X \to S$ could be derived from $\xi\colon \mathfrak{X} \to \mathfrak{S}$, a universal surface over some moduli space of algebraic surfaces. Then the Hodge structure on $(R^2\xi_*\mathbb{Z})_s$ vanishes on complex points $s \in \mathfrak{S}(\mathbb{C})$, and if $S = \operatorname{Spec} \mathcal{O}_{\mathfrak{S},s_0}^{\mathrm{h}}$ is the spectrum of the henselization of the local ring at some $s_0 \in \mathfrak{S}$, one may be able to choose $c = c_1(\mathcal{L}_0) \in (R^2\xi_*\mathbb{Z})_{s_0}$ in such a way that it is a Hodge cycle at s_0 but there is no neighbourhood U of s_0 such that c is a Hodge cycle in $(R^2\xi_*\mathbb{Z})_s$ for $s \in U$. In that case, $\mathcal{L}_0$ is not in the image of $\operatorname{Pic}(X) \to \operatorname{Pic}(X_0)$.

(*c*) See [SGA$_{4\frac{1}{2}}$, Arcata IV Prop. 4.1] for a proof that doesn't use the AAP and works already for morphisms $X \to S$ that are only separated and satisfy $\dim X_0 \leqslant 1$.

Proof of Corollary 1.6.30. Let $\widehat{S} = \operatorname{Spec} \widehat{A}$ and $\widehat{X} = X \times_S \widehat{S}$. Recall that we may identify $\operatorname{Pic}(Y) \cong H^1(Y_{\mathrm{Zar}}, \mathcal{O}_Y^\times)$ for all schemes Y.[14] Let $S_n = \operatorname{Spec}(A/\mathfrak{m}^{n+1})$ and $X_n = X \times_S S_n$ be the n^{th} infinitesimal thickening of X_0. Topologically the X_n all coincide; algebraically, for all $n \geqslant 1$ there is a coherent ideal $\mathcal{J}_n \subseteq \mathcal{O}_{X_n}$ defining X_{n-1} as a closed subscheme of X_n. We have a short exact sequence

$$1 \longrightarrow 1 + \mathcal{J}_n \longrightarrow \mathcal{O}_{X_n}^\times \longrightarrow \mathcal{O}_{X_{n-1}}^\times \longrightarrow 1\,.$$

As $\mathcal{J}_n^2 = 0$, the sheaf $1 + \mathcal{J}_n$ (as a sheaf of abelian groups under multiplication) is isomorphic to $\mathcal{J}_n$ (as a sheaf of abelian groups via addition). Since sheaf cohomology only cares for the underlying topological space and the isomorphism class of the sheaf, we thus get an exact sequence

$$\operatorname{Pic}(X_n) \longrightarrow \operatorname{Pic}(X_{n-1}) \longrightarrow H^2(X_0, \mathcal{J}_n)$$

[14] Professor Franke required Y to be quasi-compact and separated, so that Čech cohomology and sheaf cohomology coincide. This restriction doesn't make sense for two reasons: (1) even on quasi-compact separated schemes, $H^\bullet$ and $\check{H}^\bullet$ coincide only for quasi-coherent sheaves, which $\mathcal{O}_Y^\times$ is not; (2) however, H^1 and $\check{H}^1$ always coincide, on arbitrary spaces and for arbitrary sheaves!

as part of the long exact sheaf cohomology sequence. Since $\dim X_0 \leqslant 1$, Grothendieck's theorem on cohomological dimension shows $H^2(X_0, \mathcal{J}_n) = 0$. Hence $\operatorname{Pic}(X_n) \to \operatorname{Pic}(X_{n-1})$ is surjective. Therefore, for every line bundle $\mathcal{L}_0$ on X_0 there is a sequence $(\mathcal{L}_n)_{n \in \mathbb{N}}$ of line bundles $\mathcal{L}_n$ on X_n satisfying $\mathcal{L}_n|_{X_{n-1}} \cong \mathcal{L}_{n-1}$. Using Grothendieck's existence theorem [$\mathrm{EGA}_{\mathrm{III}/1}$, Théorème (5.1.4)] we see that there is a line bundle $\widehat{\mathcal{L}}$ on $\widehat{X}$ with compatible isomorphisms $\widehat{\mathcal{L}}|_{X_n} \cong \mathcal{L}_n$ (a priori $\widehat{\mathcal{L}}$ is only coherent, but an argument as in Step 3 of the proof of Corollary 1.6.28 shows that $\widehat{\mathcal{L}}$ is automatically a line bundle).

This immediately settles the case where A is already complete. The case where A has the AAP is only slightly harder: consider the functor $F\colon \mathrm{Alg}_A \to \mathrm{Set}$ given by

$$F(B) = \operatorname{Pic}(X \times_S \operatorname{Spec} B)\,.$$

By a similar technical argument as in the proof of Corollary 1.6.28, F commutes with filtered colimits. Therefore the AAP is applicable and we are done. For the general case we construct A' and A'' as above, where $A' \subseteq A$ is a finite type $\mathbb{Z}$-algebra and A'' its henselization at $\mathfrak{p} = \mathfrak{m} \cap A'$. Then A'' has the AAP by Artin's approximation theorem, so the previous argument applies, and to finish the proof we just base change back to A. $\square$

1.7. Direct and Inverse Images of Étale Sheaves

1.7.1. Construction. — Let $f\colon X \to Y$ be a morphism of schemes. For any presheaf $\mathcal{F}$ on $X_{\text{ét}}$, let $f_*\mathcal{F}$ be the presheaf on $Y_{\text{ét}}$ defined by

$$\Gamma(V, f_*\mathcal{F}) = \Gamma(X \times_Y V, \mathcal{F})$$

for étale Y-schemes V. This $f_*\mathcal{F}$ is called the *direct image* or *pushforward* of $\mathcal{F}$ under f, and it's easy to check that $f_*\colon \mathrm{PSh}(X_{\text{ét}}) \to \mathrm{PSh}(Y_{\text{ét}})$ defines a functor between the presheaf categories.

Note that f_* restricts to a functor $f_*\colon \mathrm{Sh}(X_{\text{ét}}) \to \mathrm{Sh}(Y_{\text{ét}})$. Indeed, let $\mathcal{F}$ be a sheaf, $V \to Y$ an étale morphism and $\{V_i \to V\}_{i \in I}$ an étale cover. We must show that

$$\Gamma(V, f_*\mathcal{F}) \longrightarrow \prod_{i \in I} \Gamma(V_i, f_*\mathcal{F}) \overset{\mathrm{pr}_1^*}{\underset{\mathrm{pr}_2^*}{\rightrightarrows}} \prod_{i,j \in I} \Gamma(V_i \times_V V_j, f_*\mathcal{F})$$

is an equalizer diagram (I know, Definition 1.3.13(b) has a somewhat different condition, but we've seen—more or less—in the proof of Proposition 1.3.15 that the sheaf axiom in the covering sieves formalism is equivalent to the above; also this is where covering families really become easier). Observe that $X \times_Y V \to X$ is étale and $\{X \times_Y V_i \to X \times_Y V\}_{i \in I}$ is an étale cover again, because being étale and being jointly surjective is preserved under base change. Now plugging in the definition, the above diagram becomes a similar diagram for $\mathcal{F}$ and the chosen étale cover of $X \times_Y V$, hence it is indeed an equalizer diagram by the sheaf axiom for $\mathcal{F}$.

1.7.2. Remark. — Pushforward of (pre)sheaves on larger étale sites might be studied in the same way. However, if you insist on working in a noetherian setting (as we do in the lecture), you should take some care to make sure that $X \times_Y V$ stays (locally) noetherian.

1.7.3. Example. — If $\overline{x}\colon \operatorname{Spec} \kappa(x) \to X$ is a geometric point of X and M is any set, there is a constant sheaf M (this is some abuse of notation) on $\operatorname{Spec} \kappa(\overline{x})_{\text{ét}}$ given by the sheafification of the constant M-valued presheaf. But M can is also explicitly describable: since $\kappa(\overline{x})$ is separably closed, any étale $\kappa(\overline{x})$-scheme U is just a disjoint union of copies of $\operatorname{Spec}\kappa(\overline{x})$. Thus,

$$\Gamma(U, M) = M^{\# U}\,.$$

Now $\overline{x}_* M$ is a "skyscraper sheaf" on $X_{\text{ét}}$. That is, for any étale X-scheme U we have

$$\Gamma(U, \overline{x}_* M) = \prod_{\overline{u}} M\,,$$

where the product is taken over all geometric points $\overline{u}\colon \operatorname{Spec}\kappa(\overline{x}) \to U$ lifting $\overline{x}$. One easily checks that $\mathcal{F} \mapsto \mathcal{F}_{\overline{x}}$ as a functor $\operatorname{Sh}(X_{\text{ét}}) \to \mathrm{Set}$ is left-adjoint to $M \mapsto \overline{x}_* M$ as a functor $\mathrm{Set} \to \operatorname{Sh}(X_{\text{ét}})$.

1.7.4. Construction. — In the situation of Construction 1.7.1, let $\mathcal{G}$ be a presheaf on $Y_{\text{ét}}$. Define a presheaf $f^\sharp\mathcal{F}$ on $X_{\text{ét}}$ by

$$\Gamma(U, f^\sharp\mathcal{G}) = \operatorname*{colim}_{V \in \mathcal{C}_U} \Gamma(V, \mathcal{F})\,,$$

where the colimit is taken over the following category $\mathcal{C}_U$: the objects of $\mathcal{C}_U$ are commutative diagrams of the form

$$\begin{tikzcd} U \arrow[r, dotted] \arrow[d] & V \arrow[d, dotted] \\ X \arrow[r, "f"] & Y \end{tikzcd}\,,$$

where $V \to Y$ is étale. The morphisms of $\mathcal{C}_U$ are morphisms $V \to V'$ of étale Y-schemes such that

$$\begin{tikzcd} & V \arrow[d] \\ U \arrow[ur] \arrow[r] & V' \end{tikzcd}$$

commutes. If $U \to U'$ is a morphism of étale X-schemes, we have a functor $\mathcal{C}_{U'} \to \mathcal{C}_U$ sending any $(U' \to V \to Y) \in \mathcal{C}_{U'}$ to its composition with $U \to U'$. By the universal property of colimits, this gives a canonical morphism $\Gamma(U', f^\sharp\mathcal{G}) \to \Gamma(U, f^\sharp\mathcal{F})$, turning $f^\sharp\mathcal{G}$ indeed into a presheaf on $X_{\text{ét}}$.

In case $\mathcal{G}$ is already a sheaf, let $f^*\mathcal{G} = (f^\sharp\mathcal{G})^{\mathrm{Sh}}$ be the sheafification of $f^\sharp\mathcal{G}$. This sheaf is called the *inverse image* or *pullback* of $\mathcal{G}$ under f.

1.7.5. Fact. — *The category $\mathcal{C}_U$ is cofiltered. Therefore, the colimit defining $\Gamma(U, f^\sharp\mathcal{G})$ is indeed a filtered colimit.*

Proof. The arguments from the proof of Fact 1.6.2 can be copied verbatim. □

1.7.6. Remark. — (*a*) Our construction of stalks is a special case of Construction 1.7.4: if $\overline{x}\colon \operatorname{Spec}\kappa(\overline{x}) \to X$ is a geometric point of X, then there is a canonical isomorphism $\Gamma(\operatorname{Spec}\kappa(\overline{x}), \overline{x}^\sharp\mathcal{F}) \cong \mathcal{F}_{\overline{x}}$.

(*b*) From the universal property of colimits, one derives a functorial bijection

$$\operatorname{Hom}_{\operatorname{PSh}(X_{\text{ét}})}(f^\sharp\mathcal{G}, \mathcal{F}) \cong \operatorname{Hom}_{\operatorname{PSh}(Y_{\text{ét}})}(\mathcal{G}, f_*\mathcal{F})$$

for presheaves $\mathcal{F}$ and $\mathcal{G}$. In other words, $f^\sharp\colon \operatorname{PSh}(Y_{\text{ét}}) \rightleftarrows \operatorname{PSh}(X_{\text{ét}}) : f_*$ is an adjoint pair of functors. In particular, there is a canonical isomorphism $g^\sharp f^\sharp \cong (fg)^\sharp$. Indeed, this is a formal consequence of the fact that $f_* g_* \cong (fg)_*$, which is easily verified.

(*c*) Since f_* restricts to a functor $\mathrm{Sh}(X_{\text{ét}}) \to \mathrm{Sh}(Y_{\text{ét}})$ and sheafification is left-adjoint to the forgetful functor from sheaves to presheaves, it's a formal consequence that

$$\mathrm{Hom}_{\mathrm{Sh}(X_{\text{ét}})}(f^*\mathcal{G}, \mathcal{F}) \cong \mathrm{Hom}_{\mathrm{Sh}(Y_{\text{ét}})}(\mathcal{G}, f_*\mathcal{F})$$

for sheaves $\mathcal{F}$ and $\mathcal{G}$, i.e., $f^*\colon \mathrm{Sh}(Y_{\text{ét}}) \rightleftarrows \mathrm{Sh}(X_{\text{ét}}) : f_*$ are adjoint functors as well. As in (*b*), this formally implies that $g^* f^* \cong (fg)^*$ canonically. Thus, we have isomorphisms

$$(f^\sharp \mathcal{G})_{\overline{x}} \cong \mathcal{G}_{f(\overline{x})} \cong (f^*\mathcal{G})_{\overline{x}}$$

for all geometric points $\overline{x}$ of X. Here $f(\overline{x})$ denotes the composition $f \circ \overline{x}\colon \operatorname{Spec} \kappa(\overline{x}) \to Y$.

(*d*) For those of you who get off on coherence conditions, here is an explicit description of $f^*\mathcal{G}$: if $U \to X$ is étale, we have

$$\Gamma(U, f^*\mathcal{G}) = \left\{ (g_{\overline{u}}) \in \prod_{\overline{u}} \mathcal{G}_{f(\overline{u})} \;\middle|\; (g_{\overline{u}}) \text{ fulfills the coherence condition}^{\mathrm{TM}} \right\}.$$

Herein, the coherence condition$^{\mathrm{TM}}$ is the condition that the sieve of all étale morphisms $j\colon V \to U$ for which there exists an étale morphism $W \to Y$ fitting into a commutative diagram

$$\begin{tikzcd} V \arrow[rr, "\varphi"] \arrow[d, "j"'] & & W \arrow[d] \\ U \arrow[r] & X \arrow[r, "f"] & Y \end{tikzcd},$$

together with an element $g_V \in \Gamma(W, \mathcal{G})$ with the property that $g_{j(\overline{v})}$ equals the image of g_V in $\mathcal{G}_{\varphi(\overline{v})}$ for all geometric points $\overline{v}$ of V, is a covering sieve. In particular, for all étale $U \to X$, the canonical morphism

$$\Gamma(U, f^*\mathcal{G}) \hookrightarrow \prod_{\overline{u}} \mathcal{G}_{f(\overline{u})}$$

is injective, with $\overline{u}$ running over all geometric points of U.

(*e*) If $g \in \Gamma(Y, \mathcal{G})$, then the element $(\text{image of } g \text{ in } \mathcal{G}_{f(\overline{u})})_{\overline{u}} \in \prod_{\overline{u}} \mathcal{G}_{f(\overline{u})}$ satisfies the coherence condition from (*e*). This particular element will be denoted $f^*(g) \in \Gamma(X, f^*\mathcal{G})$. Another way to think of $f^*(g)$ is as the image of g in the colimit defining $\Gamma(X, f^\sharp\mathcal{G})$ (see Construction 1.7.4), which is then mapped to an element of $\Gamma(X, f^*\mathcal{G})$ via the sheafification map $f^\sharp\mathcal{G} \to f^*\mathcal{G}$.

(*f*) In case $f\colon X \to Y$ is étale itself, $f^*\mathcal{G}$ and $f^*(g)$ are just the restrictions to X.

Lecture 15
9th Dec, 2019

1.7.7. Proposition. — *Let $(\pi_{\beta,\alpha}\colon X_\beta \to X_\alpha)_{\beta \geqslant \alpha}$ be an inverse system of affine morphisms between quasi-compact and quasi-separated schemes. Put $X = \lim X_\alpha$, with structure morphisms $\pi_\alpha\colon X \to X_\alpha$. For some fixed α, let $\mathcal{F}_\alpha$ be a sheaf on $X_{\alpha,\text{ét}}$ and let $\mathcal{F} = \pi_\alpha^*\mathcal{F}_\alpha$ resp. $\mathcal{F}_\beta = \pi_{\beta,\alpha}^*\mathcal{F}_\alpha$ for $\beta \geqslant \alpha$ be its inverse images on X resp. X_β. Then*

$$\begin{aligned} \operatorname*{colim}_{\beta \geqslant \alpha} \Gamma(X_\beta, \mathcal{F}_\beta) &\xrightarrow{\sim} \Gamma(X, \mathcal{F}) \\ \big(\textit{image of } f_\beta \in \Gamma(X_\beta, \mathcal{F}_\beta)\big) &\longmapsto \xi_\beta^*(f_\beta) \end{aligned}$$

is a bijection. Here we use notation from using notation from Remark 1.7.6(e).

Proof. Step 1. We show injectivity. Let $f \in \Gamma(X_\beta, \mathcal{F}_\beta)$ and $f \in \Gamma(X_{\beta'}, \mathcal{F}_{\beta'})$ have the same image in $\Gamma(X, \mathcal{F})$ (note that we can't just check for an element with image 0 since we want to prove the assertion for sheaves of sets actually). We must show that there is some $\gamma \geqslant \beta, \beta'$ such that the images of f and f' in $\Gamma(X_\gamma, \mathcal{F}_\gamma)$ already coincide. Replacing f and f' by their images in $\Gamma(X_{\beta''}, \mathcal{F}_{\beta''})$ for some $\beta'' \geqslant \beta, \beta'$, we may assume $\beta = \beta'$. Let $\overline{x}$ be a geometric point of X_β whose underlying point x is in the image of $|X| \to |X_\beta|$. Then there is a geometric point $\overline{y}$ of X such that

$$\begin{array}{ccc} \operatorname{Spec}\kappa(y) & \xrightarrow{\overline{y}} & X \\ \big\downarrow & & \big\downarrow{\scriptstyle \xi_\beta} \\ \operatorname{Spec}\kappa(x) & \xrightarrow{\overline{x}} & X_\beta \end{array}$$

commutes. Hence the images of f and f' in $(\mathcal{F}_\beta)_{\overline{x}} \cong \mathcal{F}_{\pi_\beta(\overline{y})} \cong \mathcal{F}_{\overline{y}}$ coincide, as they are identified with the image of $\pi_\beta^*(f) = \pi_\beta^*(f')$ in $\mathcal{F}_{\overline{y}}$. Thus, by definition of the stalk $(\mathcal{F}_\beta)_{\overline{x}}$ as a filtered colimit, there is an étale neighbourhood $V \to X_\beta$ of $\overline{x}$ such that $f|_V = f'|_V$. The image U_x of $V \to X_\beta$ is open in X_β by Proposition 1.2.14, and satisfies $f|_{U_x} = f'|_{U_x}$ because $V \to U_x$ is surjective, hence generates an étale covering sieve of U_x. Let $U \subseteq X_\beta$ be the union of all U_x constructed in that fashion. Then the (Zariski-)sheaf axiom shows $f|_U = f'|_U$. Moreover $\pi_\beta \colon X \to X_\beta$ factors over U. Let $Z = X_\beta \setminus U$ equipped with any closed subscheme structure. Combining (a) and (b) from A.1.1 we see that $Z \times_{X_\beta} X \cong \lim_{\gamma \geqslant \beta}(Z \times_{X_\beta} X_\gamma)$ is empty, hence some $Z \times_{X_\beta} X_\gamma$ must be empty, so $X_\gamma \to X_\beta$ already factors through U. Then f and f' have the same image in $\Gamma(X_\gamma, \mathcal{F}_\gamma)$, as required.

Step 2. We show surjectivity. Let $f \in \Gamma(X, \mathcal{F})$ be given. By construction of $\mathcal{F}$ as a pullback of $\mathcal{F}_\alpha$, there is

(1) an étale cover $\{U^{(i)} \to X\}_{i \in I}$, in which I may be assumed to be finite since X is quasi-compact by A.1.1(b),

(2) together with étale morphisms $W^{(i)} \to X_\alpha$ that fit into a commutative diagram

$$\begin{array}{ccc} U^{(i)} & \longrightarrow & X \\ \big\downarrow & & \big\downarrow{\scriptstyle \xi_\alpha} \\ W^{(i)} & \longrightarrow & X_\alpha \end{array},$$

(3) together with elements $f^{(i)} \in \Gamma(W^{(i)}, \mathcal{F}_\alpha)$,

such that $f|_{U^{(i)}}$ is the inverse image of $f^{(i)}$ under the map explained in Remark 1.7.6(e). By A.1.3(k), all $U^{(i)} \to X$ are base changes of étale morphisms $U^{(i)}_{\beta_i} \to X_{\beta_i}$. Since there are only finitely many i, we may replace the β_i by some $\beta \geqslant \beta_i$ for all $i \in I$. We are also free to increase α. Thus we may assume all $U^{(i)} \to X$ are base changes of étale morphisms $U^{(i)}_\alpha \to X_\alpha$ with respect to $\xi_\alpha \colon X \to X_\alpha$. Applying a similar argument to the morphisms $U^{(i)} \to W^{(i)} \times_{X_\alpha} X$ obtained from the above diagram (observe that these morphisms are étale by Fact 1.4.6(b)), we may assume that these already come from étale morphisms $U^{(i)}_\alpha \to W^{(i)} \times_{X_\alpha} X_\alpha \cong W^{(i)}$. Then $W^{(i)}$ may be replaced by $U^{(i)}_\alpha$ and the $f^{(i)}$ with their restrictions $f^{(i)}|_{U^{(i)}_\alpha}$ accordingly, so that we may finally assume

$$U^{(i)} = W^{(i)} \times_{X_\alpha} X\,,$$

and the morphism $U^{(i)} \to W^{(i)}$ from the above diagram is just the projection to the first factor.

For $i, j \in I$, let pr_1 and pr_2 be the projections from $W^{(i)} \times_{X_\alpha} W^{(j)}$ to its two factors. Then the preimages of $\mathrm{pr}_1^*(f^{(i)}), \mathrm{pr}_2^*(f^{(j)}) \in \Gamma(W^{(i)} \times_{X_\alpha} W^{(j)}, \mathcal{F}_\alpha)$ in the set $\Gamma(U^{(i)} \times_X U^{(j)}, \mathcal{F})$ coincide as they are both equal to the restriction of $f \in \Gamma(X, \mathcal{F})$ along the étale morphism $U^{(i)} \times_X U^{(j)} \to X$. Applying the injectivity assertion that was proved in Step 1 with X_α replaced by $W^{(i)} \times_{X_\alpha} W^{(j)}$, we get some $\beta_{i,j} \geqslant \alpha$ with the property that $\mathrm{pr}_1^*(f^{(i)})$ and $\mathrm{pr}_2^*(f^{(j)})$ already coincide in $\Gamma(X_{\beta_{i,j}} \times_{X_\alpha} W^{(i)} \times_{X_\alpha} W^{(j)}, \mathcal{F}_{\beta_{i,j}})$. Again, increasing α sufficiently much, we may assume without restriction that $\alpha = \beta_{i,j}$ for all $i, j \in I$. Thus $\mathrm{pr}_1^*(f^{(i)}) = \mathrm{pr}_2^*(f^{(j)})$ for all $i, j \in I$, i.e., the elements $f^{(i)} \in \Gamma(W^{(i)}, \mathcal{F}_\alpha)$ and $f^{(j)} \in \Gamma(W^{(j)}, \mathcal{F}_\alpha)$ become equal upon restriction to $W^{(i)} \times_{X_\alpha} W^{(j)}$. Thus, by the sheaf axiom the elements $f^{(i)}$ may be pasted together to an element $f_\alpha \in \Gamma(U, \mathcal{F}_\alpha)$, where U is the union over the images of $W^{(i)} \to X_\alpha$ (so U is open by Proposition 1.2.14). Clearly $\xi_\alpha \colon X \to X_\alpha$ factors over U, hence so does $\xi_{\beta,\alpha} \colon X_\beta \to X_\alpha$ for some $\beta \geqslant \alpha$ by the argument from Step 1. Now the preimage $f_\beta = \xi_{\beta,\alpha}^*(f_\alpha)$ is an element of $\Gamma(X_\beta, \mathcal{F}_\beta)$, and it maps to f by construction. This shows surjectivity. $\square$

Lecture 16
13th Dec, 2019

1.7.8. Remark. — For the sake of simplicity, we have formulated Proposition 1.7.7 for a partially ordered indexing set, but it remains valid for arbitrary cofiltered indexing categories $\mathcal{I}$ (in fact, we remarked in Appendix A.1 that these two notions are more or less interchangeable). In this case, the set $\{\beta \mid \beta \geqslant \alpha\}$ needs to be replaced by the *comma category* $\mathcal{I}/\alpha$ over α.

1.7.9. Corollary. — *Let $f \colon X \to Y$ be a morphism of schemes and $\overline{y}$ be a geometric point of Y. Put $X_{\overline{y}} = X \times_Y \operatorname{Spec} \mathcal{O}_{Y_{\text{ét}}, \overline{y}}$. Then there is a natural isomorphism*

$$(f_*\mathcal{F})_{\overline{y}} \cong \Gamma(X_{\overline{y}}, \mathrm{pr}_1^* \mathcal{F}),$$

where $\mathrm{pr}_1 \colon X_{\overline{y}} \to X$ denotes the projection to the first factor.

Proof. By Proposition 1.6.15(g) and the construction (1.6.2) in its proof, we can write the stalk at $\overline{y}$ as $\mathcal{O}_{Y_{\text{ét}}, \overline{y}} \cong \mathcal{O}_{Y,y}^{\mathrm{sh}} \cong \operatorname{colim}_{(V,\overline{v})} \Gamma(V, \mathcal{O}_V)$, where $(V, \overline{v})$ ranges through the affine étale neighbourhoods of the geometric point $\overline{y}$. Thus $X_{\overline{y}} \cong \lim_{(V,\overline{v})} X \times_Y V$. Applying Proposition 1.7.7 hence gives

$$\Gamma(X_{\overline{y}}, \mathrm{pr}_1^* \mathcal{F}) \cong \operatorname*{colim}_{(V,\overline{v})} \Gamma(X \times_Y V, \mathcal{F}) \cong \operatorname*{colim}_{(V,\overline{v})} \Gamma(V, f_*\mathcal{F}).$$

The colimit on the right-hand side is precisely $(f_*\mathcal{F})_{\overline{y}}$ (up to an easy cofinality argument to amend the fact that we only consider affine étale neighbourhoods here). $\square$

The following Fact 1.7.10 should have been given earlier, but Professor Franke forgot about it. It becomes particularly interesting by the fact that the functor $(-)_x$ of taking stalks at a geometric point is exact, whereas taking global sections $\Gamma(X, -)$ is not. So Fact 1.7.10 essentially says that strictly henselian rings have trivial higher étale cohomology!

1.7.10. Fact. — *Let $X = \operatorname{Spec} A$ where A is a strictly henselian ring (some people call these "strictly local rings") and $\mathcal{F}$ a sheaf on $X_{\text{ét}}$. Then*

$$\Gamma(X, \mathcal{F}) \xrightarrow{\sim} \mathcal{F}_{\overline{x}}$$

is an isomorphism for any geometric point $\overline{x}$ whose underlying point x is the unique closed point of A.

Proof. As every étale morphism with x in its image has a section by Proposition 1.6.7(e), we see that $(X, \overline{x})$ is cofinal in the category of étale neighbourhoods of $\overline{x}$. $\square$

1.7.11. Corollary. — *If $f\colon X \to Y$ is finite and $\overline{y}$ a geometric point of Y, then there is a canonical isomorphism*

$$(f_*\mathcal{F})_{\overline{y}} \xrightarrow{\sim} \prod_{\overline{x}} \mathcal{F}_{\overline{x}}\,,$$

where the product is taken over "all" geometric points $\overline{x}\colon \operatorname{Spec}\kappa(\overline{x}) \to X$ "lying over" $\overline{y}$.

1.7.12. Remark*. — In the lecture we were talking about "lifts" of $\overline{y}$ to X, but this is definitely not what we want. For example, $Y = \operatorname{Spec} k$ could be the spectrum of a separably closed field and $X = \operatorname{Spec} \ell$ for ℓ/k a non-trivial purely inseparable extension. Then the identity on Y is a geometric point, but admits no lift to X. So instead I wrote $\overline{x}$ "lying over" $\overline{y}$, but this too needs some explanation: obviously, $\overline{x}$ lying over $\overline{y}$ should include the condition that a diagram

$$\begin{tikzcd}
\operatorname{Spec}\kappa(\overline{x}) \arrow[r, "\overline{x}"] \arrow[d] & X \arrow[d, "f"] \\
\operatorname{Spec}\kappa(\overline{y}) \arrow[r, "\overline{y}"] & Y
\end{tikzcd}$$

commutes. Still we get into set-theoretic trouble since there is a whole proper class of such $\overline{x}$, and we would rather have a (finite) set of them! There are essentially three ways out of this:

(1) We could demand that $\kappa(\overline{y})$ and $\kappa(\overline{x})$ both equal some chosen separable closures $\kappa(y)^{\mathrm{sep}}$ and $\kappa(x)^{\mathrm{sep}}$ of their underlying ordinary points. We have seen in Remark 1.6.6 that this loses no information.

(2) We could define a preorder on the class of all $\overline{x}$ as above, with $\overline{x}' \leqslant x$ iff $\overline{x}\colon \operatorname{Spec}\kappa(x) \to X$ can be factored over $\overline{x}'\colon \operatorname{Spec}\kappa(x') \to X$. Note that $\overline{x}' \leqslant \overline{x}$ and $\overline{x} \leqslant \overline{x}'$ implies that the fields $\kappa(\overline{x})$ and $\kappa(\overline{x}')$ are isomorphic. Then we can take the finite set of isomorphism classes of $\overline{x}$ that are minimal with respect to "$\leqslant$" (it will become clear from the proof below that such a finite set indeed exists).

(3) We could demand that $\kappa(\overline{y})$ is algebraically closed rather than just separably closed. In this case the original approach works: a geometric point $\overline{x}$ "lies over" $\overline{y}$ iff it is a lift $\overline{x}\colon \operatorname{Spec}\kappa(\overline{y}) \to X$ of $\overline{y}\colon \operatorname{Spec}\kappa(\overline{y}) \to Y$.

Proof of Corollary 1.7.11. The question is local on Y, whence we may assume $Y = \operatorname{Spec} A$, $X = \operatorname{Spec} B$, such that B is finite over A. Observe that $\mathcal{O}_{Y_{\text{ét}},\overline{y}}$ is strictly henselian by Proposition 1.6.15(g). By Proposition 1.6.7(c) thus, the finite $\mathcal{O}_{Y_{\text{ét}},\overline{y}}$-algebra $S = B \otimes_A \mathcal{O}_{Y_{\text{ét}},\overline{y}}$ may be decomposed as

$$S = \prod_{i=1}^{n} S_i\,,$$

where the S_i are local and finite over $\mathcal{O}_{Y_{\text{ét}},\overline{y}}$. Moreover, if $\mathfrak{m}$ and k denote the maximal ideal and the residue field of $\mathcal{O}_{Y_{\text{ét}},\overline{y}}$, then the $S_i/\mathfrak{m}S_i$ are finite over k, hence artinian local rings. If $\mathfrak{m}_i$ denotes the unique prime ideal of S_i, then $\kappa(\mathfrak{m}_i)$ is finite over the searably closed field k, hence $\kappa(\mathfrak{m}_i)$ is separably closed itself. If the geometric point $\overline{y}$ has algebraically closed residue field $\kappa(\overline{y})$, then then it factors uniquely over each of the $\overline{x}_i$, because $\kappa(\mathfrak{m}_i)$ is a purely inseparable extension of k, hence $k \hookrightarrow \kappa(\overline{y})$ extends uniquely to a morphism $\kappa(\mathfrak{m}_i) \hookrightarrow \kappa(\overline{y})$. Thus, every $\overline{x}_i\colon \operatorname{Spec}\kappa(\mathfrak{m}_i) \to X$ is a geometric point lying over y in the sense of any of (1), (2), (3). Moreover, for any $\overline{x}$ as in Remark* 1.7.12, the morphism $B \to \kappa(\overline{x})$ factors over $B \otimes_A k \cong \prod_{i=1}^n S_i/\mathfrak{m}S_i$, which is artinian with maximal ideals $\mathfrak{m}_1, \dots, \mathfrak{m}_n$. Thus, the $\overline{x}_i$ are indeed "the" geometric points lying over $\overline{y}$, in the sense of any of (1), (2), or (3). Now we compute

$$(f_*\mathcal{F})_{\overline{x}} \cong \Gamma(\operatorname{Spec} S, \mathcal{F}) \cong \prod_{i=1}^n \Gamma(\operatorname{Spec} S_i, \mathcal{F}) \cong \prod_{i=1}^n \mathcal{F}_{\overline{x}_i}\,.$$

Here the first isomorphism follows from Corollary 1.7.9 as $X_{\overline{y}} = \operatorname{Spec} S$ in this case, the second follows from $\operatorname{Spec} S = \coprod_{i=1}^n \operatorname{Spec} S_i$ and the sheaf axiom, and the third isomorphism follows from Fact 1.7.10, using that the S_i are finite over $\mathcal{O}_{Y_{\text{ét}},\overline{y}}$, hence strictly henselian too by Proposition 1.6.15(b). □

1.7.13. Corollary. — *If $f\colon X \to Y$ is a finite, radiciel, and surjective morphism (thus a universal homeomorphism, see Remark 1.4.21), then f_* and f^* are mutually inverse equivalences of categories between $\operatorname{Sh}(X_{\text{ét}})$ and $\operatorname{Sh}(Y_{\text{ét}})$.*

Proof. This can be derived from Proposition 1.4.22, or using the previous Corollary 1.7.11 and verifying that the isomorphisms $(f^*\mathcal{G})_{\overline{x}} \cong \mathcal{G}_{f(\overline{x})}$ and $(f_*\mathcal{F})_{\overline{y}} \cong \mathcal{F}_{f^{-1}(\overline{y})}$ become inverse to each other when applied with $\mathcal{F} = f^*\mathcal{G}$ or $\mathcal{G} = f_*\mathcal{F}$. □

In the rest of this section Professor Franke gives some hints about how pushforward and pullback work in the pro-étale topology. Recall that $U \to X$ is weakly étale if it is flat and the diagonal $U \to U \times_X U$ is flat as well (Definition 1.4.27). The latter condition is automatic when $U \to X$ is a monomorphism, as in this case $U \cong U \times_X U$. Combining this observation with an argument analogous to the proof of Proposition 1.4.22, we see that for any pair $f, f'\colon U \to U'$ of morphisms of weakly étale X-schemes, the equalizer $\operatorname{Eq}(f, f') \to U$ is weakly étale. Clearly being weakly étale is preserved under base change, so all in all the proofs of Fact 1.6.2 and Fact 1.7.5 still work, and stalks at geometric points as well as inverse and direct images may be defined as we did in this lecture.

The *big* difference, however, is that while the category of weakly étale sheaves of sets as sufficiently many *topos points* (see [Stacks, Tag 00Y3]), the topos points given by stalks at geometric points are *not* sufficiently many! For instance, if $X = \operatorname{Spec} A$ is the spectrum of a Dedekind domain, then the morphism

$$U \coloneqq \coprod_{\mathfrak{m} \neq 0} \operatorname{Spec} A_{\mathfrak{m}} \longrightarrow X$$

is weakly étale, where the disjoint union is taken over all non-zero (and thus maximal) prime ideals of A. Moreover, every geometric point of X lifts to U. However, the sieve generated by U is no covering sieve, as faithfully flat descent *fails* for $U \to X$.

For the pro-étale topology, what mostly replaces stalks at geometric points are evaluations (i.e., taking sections, not stalks!) at $\operatorname{Spec} A$, where $\operatorname{Spec} A \to X$ is weakly étale and A is *strictly w-local*. For instance, if $\overline{x}$ is a geometric point of X, then $\operatorname{Spec} \mathcal{O}_{X_{\text{ét}},\overline{x}} \to X$ is weakly étale (being a filtered limit over étale morphisms to X) and one has $\mathcal{F}_{\overline{x}} = \Gamma(\operatorname{Spec} \mathcal{O}_{X_{\text{ét}},\overline{x}}, \mathcal{F})$ for weakly étale sheaves $\mathcal{F}$.

We now sketch the definition of what a w-local ring is supposed to be. Recall that a topological space X is *spectral* if it is sober and the quasi-compact open subsets form a topology base closed under arbitrary finite intersections (allowing the empty intersection, so X itself is quasi-compact). A theorem of Hochster says that spectral spaces are precisely the underlying spaces of spectra of rings. Bhatt/Scholze now call a spectral space X *w-local* if the closed points form a closed subset and the following equivalent conditions ([BS15, Lemma 2.1.4]) are satisfied:

(a) Every open cover of X splits, i.e., if $X = \bigcup_{i\in I}$ is an open cover, then $\coprod_{i\in I} U_i \to X$ has a section. Note that this is *not* the original condition from the lecture; Professor Franke explained to me afterwards that he made a mistake and will probably correct this in the next lecture.

(b) The closed points of X map homeomorphically to the set $\pi_0(X)$ of connected components (equipped with a suitable Zariski topology).

Also note that (a) implies that X has no higher cohomology, i.e., $H^i(X, \mathcal{F}) = 0$ for $i > 0$ and all (abelian) sheaves $\mathcal{F}$ on X.

A ring A is called *w-local* if $\operatorname{Spec} A$ is w-local, and *strictly w-local* if it is w-local and satisfies the following equivalent conditions ([BS15, Definition 2.2.1 and Lemma 2.2.9]):

(1) Every weakly étale faithfully flat $U \to \operatorname{Spec} A$ has a section.

(2) $A_{\mathfrak{m}}$ is strictly henselian for every maximal ideal $\mathfrak{m}$ of A.

One then shows that the inclusion of the (non-full) subcategory $\{w\text{-local rings}\} \subseteq \mathrm{Rings}$ has a left-adjoint $A \mapsto A^Z$ (see [BS15, Lemma 2.2.4]). This construction shall be sketched now. Recall that the *constructive topology* X_{const} on a spectral space X is the coarsest topology for which the quasi-compact open subsets of X are open-closed in X_{const}. Then X_{const} is spectral and compact Hausdorff. When X is noetherian, it has the description that $U \subseteq X_{\mathrm{const}}$ is open iff for all $x \in U$ the intersection $U \cap \overline{\{x\}}$ contains an open dense subset; the closure being taken in X. Then

$$A^Z = \left\{ (a_{\mathfrak{p}}) \in \prod_{\mathfrak{p}\in\operatorname{Spec} A} A_{\mathfrak{p}} \;\middle|\; \begin{array}{c} \text{there are a decomposition } (\operatorname{Spec} A)_{\mathrm{const}} = \coprod_{i=1}^n U_i \\ \text{into disjoint open subsets and } f_i \in A,\ a_i \in A_{f_i} \text{ s.th.} \\ V(f_i) \cap U_i = \emptyset \text{ and } a_{\mathfrak{p}} \text{ is the image of } a_i \text{ for all } x \in U_i \end{array} \right\}$$

In particular, $\mathbb{Z}^Z$ is the set of all families $(a_p) \in \prod_p \mathbb{Z}_{(p)}$, p running over the prime numbers, such that there is an $r \in \mathbb{Q}$ such that $a_p = r$ in $\mathbb{Q}$ for almost all p.

Chapter 2.

Cohomology

2

2.1. Definitions and Basic Facts

Lecture 17
16th Dec, 2019

Étale cohomology of schemes and morphisms between them will be constructed using the machinery of *right-derived functors*. We assume familiarity with the construction of the functors R^iF for any left-exact functor F between abelian categories. Since the R^iF are usually computed via injective resolutions (provided such resolutions exist), it is useful to have the following proposition.

2.1.1. Proposition. — *Let X be an arbitrary scheme.*

(*a*) *The category $\mathrm{Ab}(X_{\text{ét}})$ of sheaves of abelian groups on $X_{\text{ét}}$ has sufficiently many injectives.*

(*b*) *If $j\colon U \to X$ is étale and $\mathcal{I} \in \mathrm{Ab}(X_{\text{ét}})$ is injective, then $j^*\mathcal{I} \in \mathrm{Ab}(U_{\text{ét}})$ is injective too.*

Proof. The general way to do this is to use Grothendieck's [Tôh, Théorème 1.10.1], but for this lecture Professor Franke prefers a down-to-the-earth approach.

We begin with (*a*). For a geometric point $\overline{x}$ of X, the functor $\overline{x}_*\colon \mathrm{Ab} \to \mathrm{Ab}(X_{\text{ét}})$ of forming a "skyscraper sheaf" has the exact left-adjoint $(-)_{\overline{x}}\colon \mathrm{Ab}(X_{\text{ét}}) \to \mathrm{Ab}$ of forming the stalk at $\overline{x}$. Thus $\overline{x}_*$ preserves injective objects, i.e., if I is an injective abelian group, then $\overline{x}_*I$ is injective in $\mathrm{Ab}(X_{\text{ét}})$. Also recall $(\overline{x}_*I)_{\overline{x}} \cong I$. Now let $\mathcal{F}$ be an arbitrary abelian sheaf on $X_{\text{ét}}$. For every geometric point $\overline{x}$ choose an embedding $\mathcal{F}_{\overline{x}} \hookrightarrow I_{\overline{x}}$ into an injective abelian group, using that Ab has enough injectives. By the stalk-skyscraper adjunction, we get a morphism $\mathcal{F} \to \overline{x}_*I_{\overline{x}}$. Now consider

$$\mathcal{F} \longrightarrow \prod_{\overline{x}} \overline{x}_*I_{\overline{x}}\,,$$

where the product is taken over "all" (in the sense if Remark 1.6.6) geometric points of X. The product on the right-hand side is a product of injective objects, hence injective itself. It remains to show that the morphism is injective. So let $\mathcal{K}$ be its kernel. Composing with the projection to the x^{th} factor, we see that $\mathcal{K}$ lies in the kernel of $\mathcal{F} \to \overline{x}_*I_{\overline{x}}$. This kernel, however, vanishes at $\overline{x}$, hence $\mathcal{K}_{\overline{x}} = 0$ for all chosen $\overline{x}$, proving that $\mathcal{K} = 0$.

For (*b*), there are two approaches, which are both important on their own. One can use the fact that $j^*\overline{x}_*I \cong \bigoplus_{\overline{u}} \overline{u}_*I$, where the sum is taken over the preimages (in the sense of Remark* 1.7.12(1)) of $\overline{x}$ in U. In particular, this is a finite direct sum as j is quasi-finite, so the right-hand side stays injective if I is an injective abelian group. Then one may apply Lemma 2.1.2 below, using $\mathfrak{X} = \{\mathcal{I} \in \mathrm{Ab}(X_{\text{ét}}) \mid j^*\mathcal{I} \text{ is injective in } \mathrm{Ab}(U_{\text{ét}})\}$. Another proof uses Proposition 2.1.3 and the fact that every functor having an exact left-adjoint preserves injectivity. □

2.1.2. Lemma. — *Let $\mathcal{A}$ be an abelian category and $\mathfrak{X}$ a class of objects of $\mathcal{A}$ such that the following two conditions are satisfied.*

(a) *If $x \in \mathfrak{X}$ and $y \to x$ is a split monomorphism, then also $y \in \mathfrak{X}$.*

(b) *For every object $a \in \mathcal{A}$ there is a monomorphism $a \hookrightarrow x$ with $x \in \mathfrak{X}$.*

Then $\mathfrak{X}$ contains all injective objects of $\mathcal{A}$.

Proof. If i is injective, then the monomorphism $i \to x$ from (b) splits, hence $i \in \mathfrak{X}$ by (a). □

2.1.3. Proposition. — *If $j\colon U \to X$ is étale, then $j^*\colon \mathrm{Ab}(X_{\text{ét}}) \to \mathrm{Ab}(U_{\text{ét}})$ has an exact left-adjoint*

$$j_!\colon \mathrm{Ab}(U_{\text{ét}}) \longrightarrow \mathrm{Ab}(X_{\text{ét}})\,.$$

Concretely, if $\overline{x}$ is a geometric point of X and $\mathcal{F} \in \mathrm{Ab}(U_{\text{ét}})$, then $(j_!\mathcal{F})_{\overline{x}} \cong \bigoplus_{\overline{u}} \mathcal{F}_{\overline{u}}$, the sum being taken over all preimages of $\overline{x}$ in U.

Sketch of a proof. We first construct a functor $j_\sharp$ between the presheaf categories, which is left-adjoint to $j^\sharp$ from Construction 1.7.4. If $V \to X$ is étale, we put

$$\Gamma(V, j_\sharp\mathcal{F}) = \bigoplus_{\varphi\colon V \to U} \Gamma(V, \mathcal{F})\,,$$

where the sum is taken over all factorizations $\varphi\colon V \to U$ of $V \to X$ over j. The left-adjointness to $j^\sharp$ (which is a straightforward restriction functor in our case) as well as the formula for the stalks are verified by an easy calculation. Now define $j_!\mathcal{F} = (j_\sharp\mathcal{F})^{\mathrm{Sh}}$. If follows that $j_!$ is indeed a left-adjoint of j^*. Exactness can be seen from the calculation of stalks (but it holds even if we don't have a suitable notion of "stalks", see [Stacks, Tag 03DJ]). □

2.1.4. Remark. — The formula for stalks in Proposition 2.1.3 is left-adjoint to $j^*\overline{x}_*I \cong \bigoplus_{\overline{u}} \overline{u}_*I$, which was used earlier in the proof of Proposition 2.1.1.

2.1.5. Remark*. — In the case of an open immersion $j\colon U \hookrightarrow X$, the formula for stalks shows that $j_!\mathcal{F}$ is the usual "extension by zero". Moreover, in this case $\Gamma(V, j_\sharp\mathcal{F})$ equals $\Gamma(V, \mathcal{F})$ iff the image of the étale morphism $V \to X$ is contained in U, and is 0 else. In particular, there is a canonical monomorphism $j_\sharp\mathcal{F} \hookrightarrow j_*\mathcal{F}$. Sheafifying thus turns $j_!\mathcal{F}$ into a subsheaf of $j_*\mathcal{F}$. However, beware that this *doesn't work* for arbitrary étale morphisms j.

2.1.6. Definition (Finally!). — Let X be an arbitrary scheme.

(a) We denote by $H^i(X_{\text{ét}}, -)$ the i^{th} right-derived functor of the global sections functor $\Gamma(X, -)\colon \mathrm{Ab}(X_{\text{ét}}) \to \mathrm{Ab}$.

(b) For a morphism $f\colon X \to Y$ of schemes, let R^if_* denote the i^{th} right-derived functor of $f_*\colon \mathrm{Ab}(X_{\text{ét}}) \to \mathrm{Ab}(Y_{\text{ét}})$. In case of ambiguity, we use subscripts $R^if_{\text{ét},*}$ and $R^if_{\mathrm{Zar},*}$ to distinguish between derived pushforward on étale and Zariski sites.

(c) Let $\zeta_{X,*}\colon \mathrm{Ab}(X_{\text{ét}}) \to \mathrm{Ab}(X_{\mathrm{Zar}})$ denote the restriction to the Zariski site, i.e., for $\mathcal{F} \in \mathrm{Ab}(X_{\text{ét}})$, the sheaf $\zeta_{X,*}\mathcal{F}$ is obtained by restricting $\mathcal{F}$ to Zariski opens of X. We let $R^i\zeta_{X,*}$ denote its right-derived functors.

2.1.7. Remark. — By Proposition 2.1.1(b), the functors $H^i(U_{\text{ét}}, j^*(-))\colon \mathrm{Ab}(X_{\text{ét}}) \to \mathrm{Ab}$ are the derived functors of $\Gamma(U, -)\colon \mathrm{Ab}(X_{\text{ét}}) \to \mathrm{Ab}$, the functor of taking sections over U.

2.1.8. Remark. — If $\mathcal{O}$ is a sheaf of rings on $X_{\text{ét}}$, then $H^i(X_{\text{ét}}, -)$ are also the (underlying abelian groups of the) derived functors of $\Gamma(X,-)\colon \operatorname{Mod}_{\mathcal{O}} \to \operatorname{Mod}_{\Gamma(X,\mathcal{O})}$. In the case of sheaves on topological spaces this is usually proved via *flabby sheaves*, i.e., those $\mathcal{F}$ for which $\Gamma(U,\mathcal{F}) \twoheadrightarrow \Gamma(V,\mathcal{F})$ is surjective whenever $V \subseteq U$ are open subsets. One shows that cohomology can be computed via flabby resolutions and then everything is clear since being flabby is preserved under the forgetful functor $\operatorname{Mod}_{\mathcal{O}} \to \operatorname{Ab}(X)$.

However, this argument no longer works. For example, injective sheaves $\mathcal{I}$ on $X_{\text{ét}}$ are no longer "flabby" (in the naive sense that $\Gamma(U,\mathcal{I}) \twoheadrightarrow \Gamma(V,\mathcal{I})$ is surjective when $V \to U$ is étale), the problem being that $j_!\mathbb{Z}_U \to \mathbb{Z}_X$ is no longer a monomorphism (which is the essential ingredient in [Stacks, Tag 01EA]) because there can be more than one geometric point of U over a given geometric point of X.

Instead, the proof in [Stacks, Tag 03FA] uses Čech cohomology arguments. As Robin pointed out, there is better notion of "flabby sheaves" that works for arbitrary sites, called *limp sheaves* by The Stacks Project and *flasque sheaves* by [$\text{SGA}_{4/2}$, Exposé V.4].

2.1.9. Remark. — For $\mathcal{F} \in \operatorname{Ab}(X_{\text{ét}})$, the sheaf $R^i f_* \mathcal{F}$ is the sheafification of the presheaf $V \mapsto H^i((X \times_Y V)_{\text{ét}}, \mathcal{F})$ for $V \to Y$ étale. Here $\mathcal{F}$ should actually be replaced by the pullback of $\mathcal{F}$ to $X \times_V Y$, but such abuse of notation is very convenient and we will use it frequently.

To see why the assertion is true, let Φ^i denote the functors described above. Then the sequence $(\Phi^i)_{i \geqslant 0}$ forms a cohomological functor (i.e., takes short exact sequences to a long exact sequence), Φ^0 agrees with $R^0 f_*$, and Φ^i kills injective objects of $\operatorname{Ab}(X_{\text{ét}})$ for all $i > 0$ by Proposition 2.1.1(b). It is a well-known fact that in such a situation the Φ^i are indeed the derived functors of f_*. For some reason, Professor Franke decided to write down a very general version of said fact in Proposition 2.1.10 below.

2.1.10. Proposition. — *Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories.*

(a) *Let $\Phi^\bullet = (\Phi^i)_{i \geqslant 0}\colon \mathcal{A} \to \mathcal{B}$ be a cohomological functor. Then the Φ^i are the derived functors of $F = \Phi^0$ if one of the following "effaceability conditions" holds:*

(1) *If $\mathcal{B} = \operatorname{Mod}_R$ is the category of modules over a ring, then it is sufficient that for every $i > 0$, all $a \in \mathcal{A}$, and all $f \in \Phi^i(a)$, there is a monomorphism $j\colon a \hookrightarrow b$ in $\mathcal{A}$ such that $\Phi^i(j)(f) = 0$.*

(2) *If $\mathcal{B} = \operatorname{Mod}_{\mathcal{O}}$ is the category of modules over a sheaf of rings $\mathcal{O}$ on a topological space X or on $X_{\text{ét}}$, then it suffices that for every (geometric) point x of X, all $i > 0$, every $a \in \mathcal{A}$, and every $f \in \Phi^i(a)_x$, there is a monomorphism $j\colon a \hookrightarrow b$ in $\mathcal{A}$ such that $\Phi^i(j)(f) = 0$.*

(3) *If $\mathcal{B}$ is arbitrary, we need that for every $i > 0$ and all $a \in \mathcal{A}$ there is a monomorphism $j\colon a \hookrightarrow b$ such that $\Phi^i(j) = 0$.*

(4) *If $\mathcal{B}$ is arbitrary and $\mathcal{A}$ has enough injectives, it suffices to have $\Phi^i(a) = 0$ whenever $a \in \mathcal{A}$ is injective.*

(b) *Assume $\mathcal{A}$ has sufficiently many injectives. Let $F\colon \mathcal{A} \to \mathcal{B}$ be a left-exact functor and $\mathfrak{X} \subseteq \mathcal{A}$ a class of objects satisfying the conditions from Lemma 2.1.2 and in addition*

($*$) *If $0 \to x' \to x \to x'' \to 0$ is exact in $\mathcal{A}$ and $x', x \in \mathfrak{X}$, then also $x'' \in \mathfrak{X}$ and $Fx \to Fx''$ is an epimorphism.*

Then every injective object of $\mathcal{A}$ is in $\mathfrak{X}$ and $R^i F(x) = 0$ for $i > 0$ and all $x \in \mathfrak{X}$.

Sketch of a proof.* Conditions (3) and (4) of (a) are well-known effaceability criteria from [Tôh, Chapitre II]. To see that the weaker criteria (1) and (2) hold, go through the proof of (3) in this particular special case and generalize where necessary.

For (b), we already know that $\mathfrak{X}$ contains all injective objects since we proved this in Lemma 2.1.2. Now let $x \in \mathfrak{X}$. We can choose a short exact sequence $0 \to x \to t \to c \to 0$ with t injective. By $(*)$ we have $c \in \mathfrak{X}$ as well, and since t is injective, we get $R^i F(c) \cong R^{i+1} F(x)$ for all $i > 0$ from the long exact derived functor sequence. Thus it suffices to prove $R^1 F(x) = 0$. But $F(t) \to F(c)$ is an epimorphism by $(*)$, hence the long exact sequence together with $R^1 F(t) = 0$ show $R^1 F(x) = 0$, as required. □

2.1.11. Proposition. — *If $f\colon X \to Y$ and $g\colon Y \to Z$ are morphisms of schemes, then we have three Leray spectral sequences*

$$E_2^{p,q} = H^p(Y_{\text{ét}}, R^q f_* \mathcal{F}) \Longrightarrow H^{p+q}(X_{\text{ét}}, \mathcal{F}) , \quad E_2^{p,q} = R^p g_* R^q f_* \mathcal{F} \Longrightarrow R^{p+q}(g \circ f)_* \mathcal{F} ,$$
$$E_2^{p,q} = H^p(X_{\text{Zar}}, R^q \zeta_{X,*} \mathcal{F}) \Longrightarrow H^{p+q}(X_{\text{ét}}, \mathcal{F}) .$$

Moreover, if $F\colon \mathrm{Ab}(X_{\text{ét}}) \to \mathrm{Ab}(Y_{\text{Zar}})$ denotes the "forgetful pushforward" functor, then there are two spectral sequences[1]

$$E_2^{p,q} = R^p f_{\text{Zar},*} R^q \zeta_{X,*} \mathcal{F} \Longrightarrow R^{p+q} F(\mathcal{F}) , \quad E_2^{p,q} = R^p \zeta_{Y,*} R^q f_{\text{ét},*} \mathcal{F} \Longrightarrow R^{p+q} F(\mathcal{F}) .$$

Lecture 18
20th Dec, 2019

Proof. All of these become instances of the Grothendieck spectral sequence once we show that f_* and $\zeta_{X,*}$ map injective objects to acyclic ones (with respect to the respective second functor). In fact, both f_* and $\zeta_{X,*}$ even preserve injective objects! For f_* the reason is that it has the exact left-adjoint $f^*\colon \mathrm{Ab}(Y_{\text{ét}}) \to \mathrm{Ab}(X_{\text{ét}})$. For $\zeta_{X,*}$, there are at least two possible approaches:

(1) Construct an exact left-adjoint $\zeta_X^*\colon \mathrm{Ab}(X_{\text{Zar}}) \to \mathrm{Ab}(X_{\text{ét}})$. To do so, we first define an adjoint $\zeta_X^\sharp\colon \mathrm{PAb}(X_{\text{Zar}}) \to \mathrm{PAb}(X_{\text{ét}})$ of the forgetful functor $\mathrm{PAb}(X_{\text{ét}}) \to \mathrm{PAb}(X_{\text{Zar}})$. Similar to Construction 1.7.4 we define it via

$$\Gamma(U, \zeta_X^\sharp \mathcal{G}) = \operatorname*{colim}_V \Gamma(V, \mathcal{G}) = \Gamma(\text{image of } U \to X, \mathcal{G}) .$$

The colimit in the middle is taken over all Zariski-open subsets $V \subseteq X$ containing the image of $U \to X$. But since every étale morphism $U \to X$ has open image by Proposition 1.2.14, we get the equality on the right. It's easy to see that $\zeta_X^\sharp$ has the required property. Then $\zeta_X^* = (\zeta_X^\sharp)^{\mathrm{Sh}}$ is a left-adjoint of $\zeta_{X,*}$. If $\overline{x}$ is a geometric point of X with underlying point x, then $(\zeta_X^* \mathcal{G})_{\overline{x}} \cong (\zeta_X^\sharp \mathcal{G})_{\overline{x}} \cong \mathcal{G}_x$, so ζ_X^* is indeed exact.

(2) If $\overline{x}$ and x are as above, show that $\zeta_{X,*}\overline{x}_* I \cong x_* I$ for injective abelian groups I. Thus $\zeta_{X,*}\mathcal{I}$ stays injective if $\mathcal{I}$ is one of the injective objects from the proof of Proposition 2.1.1. Since there are sufficiently many of them, this suffices to provide us with the required spectral sequence.

Whichever approach you prefer, this finishes the proof. □

2.1.12. Proposition. — *If $f\colon X \to Y$ is a finite morphism of schemes, then $R^i f_* \mathcal{F} = 0$ for all $i > 0$ and all sheaves $\mathcal{F}$ on $X_{\text{ét}}$.*

[1] In the lecture we had a single spectral sequence $E_2^{p,q} = R^p f_{\text{Zar},*} R^q \zeta_{X,*} \mathcal{F} \Rightarrow \zeta_{Y,*} R^{p+q} f_{\text{ét},*} \mathcal{F}$, suggesting that the second of the above spectral sequences collapses. Unfortunately, this is not true (I asked Professor Franke).

Proof. For geometric points $\overline{y}$ of Y we have $(f_*\mathcal{F})_{\overline{y}} \cong \prod_{\overline{x}} \mathcal{F}_{\overline{x}}$ by Corollary 1.7.9 (with notation as explained there). Since exactness can be checked on stalks at geometric points, this shows that f_* is an exact functor, whence its higher derived functors vanish. $\square$

2.2. The Relation with Galois Cohomology

2.2.1. Remark. — Let X be an arbitrary scheme and $\overline{x}\colon \operatorname{Spec}\kappa(\overline{x}) \to X$ a geometric point with underlying point $x \in X$. We may identify $\kappa(x)$ with its image in $\kappa(\overline{x})$. Moreover, we have seen in Remark 1.6.6 that replacing $\kappa(\overline{x})$ by the separable closure $\kappa(x)^{\mathrm{sep}}$ of $\kappa(x)$ in it neither changes stalks nor étale neighbourhoods, so we may assume $\kappa(\overline{x}) = \kappa(x)^{\mathrm{sep}}$. If $\sigma \in G_x = \operatorname{Gal}(\kappa(\overline{x})/\kappa(x))$ and $(U,\overline{u})$ is an étale neighbourhood of $\overline{x}$, then also $(U, \overline{u}\circ\operatorname{Spec}(\sigma))$ is also a preimage of $\overline{x}$ in U. We thus get an action of G_x on the category of étale neighbourhoods of $\overline{x}$ via

$$\sigma\colon (U,\overline{u}) \longmapsto \big(U, \overline{u}\circ\operatorname{Spec}(\sigma)\big)\,!$$

Now if $\mathcal{F}$ is a sheaf on $X_{\text{ét}}$, then the action of G_x on the étale neighbourhoods induces an action on the stalk $\mathcal{F}_{\overline{x}}$, sending the image of $\varphi \in \Gamma(U,\mathcal{F})$ via $\overline{u}$ to the image of φ via $\overline{u}\circ\operatorname{Spec}(\sigma)$. It follows that the image of $\Gamma(X,\mathcal{F})$ is contained in the subset of G_x-invariants $\mathcal{F}_{\overline{x}}^{G_x} \subseteq \mathcal{F}_{\overline{x}}$.

This action of G_x on the set of lifts $\overline{u}$ of $\overline{x}$ to an étale X-scheme U is compatible with morphisms $f\colon U \to U'$ in $X_{\text{ét}}$ in the sense that $\sigma f(\overline{u}) = f(\sigma\overline{u})$, and if $j\colon U \to X$ is étale, then the image of $\Gamma(U,\mathcal{F})$ in $\prod_{j(\overline{u})=\overline{x}} \mathcal{F}_{\overline{x}}$ (the product is taken over all lifts of $\overline{x}$ to U) is contained in the subset $\big\{(\varphi_{\overline{u}}) \in \prod_{\overline{u}} \mathcal{F}_{\overline{x}} \bigm| \varphi_{\sigma\overline{u}} = \sigma\varphi_{\overline{u}}\big\}$ (here σ denotes both the action on geometric points and on $\mathcal{F}_{\overline{x}}$ by abuse of notation).

2.2.2. Proposition. — *Let $X = \operatorname{Spec} k$ be the spectrum of a field and consider the geometric point $\overline{x}\colon \operatorname{Spec} k^{\mathrm{sep}} \to X$. Let $G = \operatorname{Gal}(k^{\mathrm{sep}}/k)$ be the absolute Galois group of k and let G-*Mod *be the category of discrete abelian groups with a continuous G-action.*

(*a*) *We have an equivalence of categories*

$$\begin{aligned} \operatorname{Ab}(X_{\text{ét}}) &\xrightarrow{\sim} G\text{-Mod} \\ \mathcal{F} &\longmapsto \mathcal{F}_{\overline{x}}\,. \end{aligned}$$

(*b*) *Similarly, there are equivalences of categories* $\operatorname{Sh}(X_{\text{ét}}) \cong G$-Set *and* $\operatorname{Grp}(X_{\text{ét}}) \cong G$-Grp *between the categories of sheaves of sets/groups on $X_{\text{ét}}$ and the categories of discrete sets/groups with a continuous G-action.*

(*c*) *There is a canonical isomorphism $H^i(X_{\text{ét}},\mathcal{F}) \cong H^i(G,\mathcal{F}_{\overline{x}})$ of cohomological functors on* $\operatorname{Ab}(X_{\text{ét}})$. *Here $H^i(G,-)$ denotes the right-derived functor of $(-)^G\colon G$-*Mod $\to G$-Mod *(also known as "group cohomology").*

Sketch of a proof. We prove (*a*) and (*b*) by constructing a quasi-inverse functor in each of these cases. If F is a discrete set or (abelian) group with a continuous G-action, let $\mathcal{F}_F$ be the sheaf given by

$$\Gamma(U,\mathcal{F}_F) = \left\{(\varphi_{\overline{u}}) \in \prod_{\overline{u}} F \;\middle|\; \varphi_{\sigma\overline{u}} = \sigma\varphi_{\overline{u}}\right\}\,,$$

where $\overline{u}$ ranges over the lifts of $\overline{x}$ to U. If $j\colon V \to U$ is a morphism in $X_{\text{ét}}$ then the restriction of $\varphi \in \Gamma(U,\mathcal{F})$ to V is given by $\varphi|_V = (\varphi_{j(\overline{v})})$ where $\overline{v}$ ranges over the lifts of $\overline{x}$ to V.

This sheaf satisfies $(\mathcal{F})_{\overline{x}}$. Indeed, the category of étale neighbourhoods of $\overline{x}$ has a cofinal subsystem of objects $(U, \overline{u})$, where $U = \operatorname{Spec} \ell$. In this case G acts transitively on the lifts of $\overline{x}$ to U. Thus, projecting to the $\overline{u}^{\text{th}}$ factor provides an isomorphism $\Gamma(U, \mathcal{F}_F) \cong F$ in this case, and then the same follows for the stalk $\mathcal{F}_{\overline{x}}$ after taking colimits.

Conversely, if $F = \mathcal{F}_{\overline{x}}$, then Remark 2.2.1 gives a canonical morphism $\mathcal{F} \rightarrow \mathcal{F}_F$. As we have just seen, this induces an isomorphism $\mathcal{F}_{\overline{x}} = F \cong (\mathcal{F}_F)_{\overline{x}}$ on stalks at $\overline{x}$. But $\overline{x}$ is the only geometric point of $X = \operatorname{Spec} k$, hence $\mathcal{F} \xrightarrow{\sim} \mathcal{F}_F$ must be an isomorphism by Proposition 1.6.3(e). This proves (a) and (b).

For (c), it suffices to show that the equivalence of categories $\mathbf{Ab}(X_{\text{ét}}) \cong G\text{-}\mathbf{Mod}$ from (a) identifies the functors $\Gamma(X, -)$ and $(-)^G$. Indeed, if $F \in G\text{-}\mathbf{Mod}$, then

$$\Gamma(X, \mathcal{F}_F) = \{\varphi \in F \mid \varphi = \sigma\varphi\} = F^G\,,$$

because $G = \operatorname{Gal}(k^{\text{sep}}/k)$ acts trivially on the lifts of $\overline{x}$ to X (because, well, there's only one). This proves (c). $\square$

Proposition 2.2.2 shows that the étale cohomology of a point can be computed by Galois cohomology. Thus, in the rest of this section we compute some Galois cohomology groups.

2.2.3. Proposition. — *Let L/K be a Galois extension.*

(a) *We have $H^1(\operatorname{Gal}(L/K), L^\times) = 0$ (this is famously known as "Hilbert's theorem 90").*

(b) *We have $H^2(\operatorname{Gal}(L/K), L^\times) = \{[A] \in \operatorname{Br}(K) \mid A \text{ splits over } L\}$.*

Proof. Part (a) is proved in Corollary 2.3.13 assuming the special case that $L = L^{\text{sep}}$ is separably closed (and thus a separable closure of K). The general case can be deduced as follows: putting $G = \operatorname{Gal}(L^{\text{sep}}/K)$, we see that $H = \operatorname{Gal}(L^{\text{sep}}/L)$ is a closed subgroup of the pro-finite group G and $G/H \cong \operatorname{Gal}(L/K)$. Consider the Hochschild–Serre spectral sequence

$$E_2^{p,q} = H^p\big(G/H, H^q(H, L^{\text{sep},\times})\big) \Longrightarrow H^{p+q}\big(G, L^{\text{sep},\times}\big)\,.$$

From Corollary 2.3.13 we know $H^1(G, L^{\text{sep},\times}) = 0 = H^1(H, L^{\text{sep},\times})$. Thus $E_2^{0,1} = 0$. Hence the above spectral sequence shows $H^1(\operatorname{Gal}(L/K), L^\times) = E_2^{1,0} \cong H^1(G, L^{\text{sep},\times}) = 0$, as claimed. We omit the proof of part (b). $\square$

2.2.4. Definition. — We say a field k *has property C_n* if any homogeneous polynomial $f \in k[X_1, \dots, X_m]$ of degree $0 < d < \sqrt[n]{m}$ has a non-trivial zero in k^m.

2.2.5. Remark. — We are only interested in C_1 since the properties C_n for $n > 1$ seem to be quite useless.

To finish the section, we state two classical results on C_1 fields without proofs.

2.2.6. Proposition. — *Let K be a field having property C_1.*

(a) *Every algebraic extension of K is C_1 again.*

(b) *For all Galois extensions L/K and all $i > 0$ we have $H^i(\operatorname{Gal}(L/K), L^\times) = 0$.*

(c) *More generally, the absolute Galois group $G = \operatorname{Gal}(K^{\text{sep}}/K)$ has cohomological dimension $\leqslant 1$. That is, for any discrete continuous G-module T and all $i > 1$ we have $H^i(G, T) = 0$.*

2.2.7. Proposition (Tsen). — *If K is a field of transcendence degree 1 over a separably closed field, then K is C_1.*

2.3. The Relation between H^1 and Torsors

Lecture 19
10^{th} Jan, 2020

2.3.1. Definition. — Let $\mathcal{C}$ be a site and $\mathcal{G}$ a sheaf of groups on $\mathcal{C}$. A *$\mathcal{G}$-torsor* is a sheaf of sets $\mathcal{T}$ together with a morphism $\alpha\colon \mathcal{G}\times\mathcal{T}\to\mathcal{T}$ of sheaves of sets satisfying the usual axioms for a left $\mathcal{G}$-action as well as the following conditions:

(a) For every object x of $\mathcal{C}$, the class $\{y\to x \mid \mathcal{T}(x)\neq\emptyset\}$ is a covering sieve of x.

(b) With pr_2 denoting the projection to the second factor, the morphism
$$(\alpha,\mathrm{id}_{\mathcal{T}}\circ\mathrm{pr}_2)\colon \mathcal{G}\times\mathcal{T}\longrightarrow\mathcal{T}\times\mathcal{T}$$
is an isomorphism.

A *morphism of $\mathcal{G}$-torsors* is a morphism $\tau\colon\mathcal{T}\to\mathcal{T}'$ of sheaves of sets compatible with the $\mathcal{G}$-action, i.e., such that the diagram
$$\begin{tikzcd}
\mathcal{G}\times\mathcal{T} \arrow[r, "(\mathrm{id}_{\mathcal{G}},\tau)"] \arrow[d, "\alpha"'] & \mathcal{G}\times\mathcal{T} \arrow[d, "\alpha'"] \\
\mathcal{T} \arrow[r, "\tau"] & \mathcal{T}'
\end{tikzcd}$$
commutes. A torsor is called *trivial* if $\lim_{x\in\mathcal{C}}\mathcal{T}(x)\neq\emptyset$, or equivalently, if there is an isomorphism $\mathcal{G}\xrightarrow{\sim}\mathcal{T}$ of $\mathcal{G}$-torsors.

2.3.2. Remark. — The equivalence at the end of Definition 2.3.1 can be seen as follows: it's clear that $\mathcal{G}$ and thus every torsor isomorphic to it are trivial. Conversely, if we find compatible elements $t_x\in\mathcal{T}(x)$ defining an element of $\lim_{\mathcal{C}}\mathcal{T}$, then we have an isomorphism $\mathcal{G}\xrightarrow{\sim}\mathcal{T}$ sending $g\in\mathcal{G}(x)$ to gt_x (this is abuse of notation for $\alpha(g,t_x)$, of course). This is an isomorphism, because an inverse is given by sending $t\in\mathcal{T}(x)$ to the unique section $g\in\mathcal{G}(x)$ satisfying $t=gt_x$.

2.3.3. Lemma. — (a) *The category of $\mathcal{G}$-torsors on $\mathcal{C}$ is a groupoid.*

(b) *Let $\mathcal{I}$ be an injective object of the category of* $\mathrm{Ab}(\mathcal{C})$ *of sheaves of abelian groups on $\mathcal{C}$, then every $\mathcal{I}$-torsor is trivial.*

Proof. Part (a). Let $\tau\colon\mathcal{T}\to\mathcal{T}'$ be an arbitrary morphism of torsors. We need to show that $\mathcal{T}(x)\to\mathcal{T}'(x)$ is an isomorphism. First suppose $\mathcal{T}(x)$, and choose an element $t\in\mathcal{T}(x)$. By $\mathcal{G}$-equivariance, $\mathcal{T}(x)\to\mathcal{T}'(x)$ is given by $\tau(gt)=g\tau(t)$, hence an isomorphism since the $\mathcal{G}(x)$-action is simply transitive on both sides. In general, x can be covered by objects $y\in\mathcal{C}$ such that $\mathcal{T}(y)\neq\emptyset$, and the sheaf axiom shows that $\mathcal{T}(x)\to\mathcal{T}'(x)$ is an isomorphism in the general case as well.

Part (b). Since $\mathcal{I}$ is abelian, we will use additive notation. Define the "internal Hom" presheaf $\mathcal{J}=\mathcal{H}om_{\mathcal{C}}(\mathcal{T},\mathcal{I})$ on $\mathcal{C}$ by $\mathcal{J}(x)=\mathrm{Hom}_{\mathrm{PSh}(\mathcal{C}/x)}(\mathcal{T},\mathcal{I})$. It's easy to check that $\mathcal{J}$ is in fact a sheaf of abelian groups, the group structure given by addition on $\mathcal{I}$. Moreover, there is a canonical morphism $\iota\colon\mathcal{I}\to\mathcal{J}$ given by sending a section $i\in\mathcal{I}(x)$ to the constant i-valued morphism in $\mathcal{J}(x)=\mathrm{Hom}_{\mathrm{PSh}(\mathcal{C}/x)}(\mathcal{T},\mathcal{I})$, i.e., the morphism sending any $t\in\mathcal{T}(y)$ to the image of i in $\mathcal{I}(y)$. This ι is clearly a monomorphism. Thus, since $\mathcal{I}$ is injective, it must have a split $\pi\colon\mathcal{J}\to\mathcal{I}$.

We also have a morphism $\kappa\colon\mathcal{T}\to\mathcal{J}$ sending $t\in\mathcal{T}(x)$ to $(-)-t\in\mathrm{Hom}_{\mathrm{PSh}(\mathcal{C}/x)}(\mathcal{T},\mathcal{I})$. Here the morphism of sheaves $(-)-t$ sends any $t'\in\mathcal{T}(y)$ to the unique $i\in\mathcal{I}(y)$ such that $t'-i$ equals the image of t in $\mathcal{T}(y)$. Now its easy to see that
$$\mathcal{T}\xrightarrow{\kappa}\mathcal{J}\xrightarrow{\pi}\mathcal{I}$$

is not only a morphism of sheaves, but also a morphism of $\mathcal{I}$-torsors. Hence $\mathcal{T} \cong \mathcal{I}$ by part (a). $\square$

2.3.4. Definition. — Let $i\colon \mathcal{G} \to \mathcal{H}$ be a morphism of sheaves of groups on $\mathcal{C}$ and $\mathcal{T}$ a $\mathcal{G}$-torsor. An *i-splitting* of $\mathcal{T}$ is a morphism $\sigma\colon \mathcal{T} \to \mathcal{H}$ such that $\sigma(gt) = i(g)\sigma(t)$ for all $g \in \mathcal{G}(x)$, $t \in \mathcal{T}(x)$.

2.3.5. Remark. — An i-splitting of $\mathcal{T}$ as defined in Definition 2.3.4 is obviously equivalent to giving a trivialization of the $\mathcal{H}$-torsor $i_*\mathcal{T} = (\mathcal{T} \times \mathcal{H})/\mathcal{G}$. However, Professor Franke does not intend to define pushforwards of torsors in general.

2.3.6. — From now on, all sheaves of groups will be abelian and we will use additive notation for convenience (except for the sheaf $\mathcal{O}_{X_{\text{ét}}}^\times$ considered in Fact 2.3.11 below). Consider a short exact sequence

$$0 \longrightarrow \mathcal{G} \xrightarrow{\;i\;} \mathcal{H} \xrightarrow{\;\pi\;} \mathcal{Q} \longrightarrow 0$$

in $\mathrm{Ab}(\mathcal{C})$ and let $(\mathcal{T}, \sigma)$ be a $\mathcal{G}$-torsor equipped with an i-splitting σ. If $\mathcal{T}(x) \neq \emptyset$, then $q_x = \pi(\sigma(t))$ does not depend on the choice of $t \in \mathcal{T}(x)$, as

$$\pi\bigl(\sigma(g+t)\bigr) = \pi\bigl(i(g) + \sigma(t)\bigr) = 0 + \pi\bigl(\sigma(t)\bigr)\,.$$

If x is arbitrary, then the sheaf axiom provides us with a unique element $q_x \in \mathcal{Q}(x)$ such that $v^*q_x = q_y$ whenever $v\colon y \to x$ is an x-object satisfying $\mathcal{T}(y) \neq \emptyset$. By compatibility, the q_x define an element $q \in \lim_{\mathcal{C}} \mathcal{Q}$.

It is easy to see that this q only depends on the isomorphism class of $(\mathcal{T}, \sigma)$; we denote it by $q(\mathcal{T}, \sigma)$ from now on. Moreover, if $h \in \lim_{\mathcal{C}} \mathcal{H}$ and $h + \sigma\colon \mathcal{T} \to \mathcal{H}$ is defined in the obvious way as $(h+\sigma)(t) = h + \sigma(t)$, then $q(\mathcal{T}, h+\sigma) = \pi(h) + q(\mathcal{T}, \sigma)$.

Conversely, suppose $q \in \lim_{\mathcal{C}} \mathcal{Q}$ is given. We put $\mathcal{T}_q(x) = \{h \in \mathcal{H}(x) \mid \pi(h) = q\}$ and let $\sigma_q\colon \mathcal{T}_q \to \mathcal{H}$ be the obvious embedding. Then $\mathcal{T}_q$ is a $\mathcal{G}$-torsor and σ_q an i-splitting. One easily checks $q(\mathcal{T}_q, \sigma_q) = q$ and that $\sigma\colon \mathcal{T} \to \mathcal{H}$ restricts to an isomorphism $\mathcal{T} \xrightarrow{\sim} \mathcal{T}_{q(\mathcal{T},\sigma)}$. Thus we obtain:

2.3.7. Lemma. — *The association $(\mathcal{T}, \sigma) \mapsto q(\mathcal{T}, \sigma)$ defines a bijection*

$$\{\mathcal{G}\textit{-torsors with an } i\textit{-splitting}\} \xrightarrow{\;\sim\;} \lim_{\mathcal{C}} \mathcal{Q}\,,$$

compatible with the action of $\lim_{\mathcal{C}} \mathcal{H}$ on both sides.

2.3.8. Remark. — If $\mathcal{H} = \mathcal{I}$ is injective, then i-splittings always exist as $i_*\mathcal{T}$ from Remark 2.3.5 is trivial by Lemma 2.3.3(b). Since we didn't and won't define pushforward of torsors, Professor Franke suggest alternatively to copy the proof of said result *mutatis mutandis.*

2.3.9. Proposition. — *Let $\mathcal{C}$ be a site. We denote by $H^i(-)$ the i^{th} right-derived functor of $\Gamma(-)\colon \mathrm{Ab}(\mathcal{C}) \to \mathrm{Ab}$ defined by $\Gamma(\mathcal{G}) = \lim_{\mathcal{C}} \mathcal{G}$. Then for any $\mathcal{G}$ one has a bijection*

$$\begin{aligned} \{\textit{iso. classes of } \mathcal{G}\textit{-torsors}\} &\xrightarrow{\;\sim\;} H^1(\mathcal{G}) \\ \mathcal{T} &\longmapsto \bigl(q(T,\sigma) \bmod \pi\,\Gamma(\mathcal{I})\bigr) \\ \mathcal{T}_q &\longleftarrow \bigl(q \bmod \pi\,\Gamma(\mathcal{I})\bigr)\,. \end{aligned}$$

Here we have chosen an arbitrary resolution $0 \to \mathcal{G} \xrightarrow{\;i\;} \mathcal{I} \xrightarrow{\;\pi\;} \mathcal{Q} \to 0$ with $\mathcal{I}$ injective, so that $H^1(\mathcal{F}) \cong \Gamma(\mathcal{Q})/\Gamma(\mathcal{I})$, and σ is any $\mathcal{I}$-splitting (which exists by Remark 2.3.8).

Sketch of a proof. This follows from the above and a few calculations. Also note that $\mathrm{Ab}(\mathcal{C})$ has enough injectives: this follows from [Tôh, Théorème 1.10.1], but we are only going to use Proposition 2.3.9 in cases where we already know the existence of sufficiently many injectives. $\square$

2.3.10. Remark*. — The identification with $H^1(\mathcal{G})$ suggests that there should be a canonical group structure on the set of isomorphism classes of $\mathcal{G}$-torsors. It can be explicitly described as follows: for torsors $\mathcal{T}$ and $\mathcal{T}'$, consider the sheaf

$$\mathcal{T} \times_{\mathcal{G}} \mathcal{T} = (\mathcal{T} \times \mathcal{T}')/\mathcal{G}\,.$$

Here "modding out $\mathcal{G}$" is abuse of notation for modding out the equivalence relation given by $(g+t,t') \sim (t, g+t')$ for all $g \in \mathcal{G}(x)$, $t \in \mathcal{T}(x)$, $t' \in \mathcal{T}'(x)$. Then $\mathcal{T} \times_{\mathcal{G}} \mathcal{T}'$ becomes a $\mathcal{G}$-torsor again in a canonical way.

We claim that $-\times_{\mathcal{G}}-$ corresponds to the addition in the group $H^1(\mathcal{G})$. By Proposition 2.3.9 it suffices to show that $\mathcal{T}_0 \cong \mathcal{G}$ is the neutral element (which is straightforward) and that $\mathcal{T}_{q+q'} \cong \mathcal{T}_q \times_G \mathcal{T}_{q'}$. Upon inspection, sections of $\mathcal{T}_q \times_G \mathcal{T}_{q'}$ are equivalence classes of pairs (h,h') such that $\pi(h) = q$ and $\pi(h') = q'$. Then its easy to check that $[h,h'] \mapsto h+h'$ is well-defined and induces the required isomorphism $\mathcal{T}_q \times_G \mathcal{T}_{q'} \xrightarrow{\sim} \mathcal{T}_{q+q'}$.

The main application of the theory of torsors to étale cohomology comes through $\mathcal{O}_{X_{\text{ét}}}^\times$-torsors, which happen to be just line bundles over X.

2.3.11. Fact. — *Let X be a scheme. There is an equivalence of groupoids*

$$\left\{\begin{array}{c}\textit{line bundles } \mathcal{L} \textit{ on } X, \\ \textit{isomorphisms of line bundles}\end{array}\right\} \xrightarrow{\sim} \left\{\mathcal{O}_{X_{\text{ét}}}^\times\textit{-torsors}\right\}\,.$$

Sketch of a proof. If $\mathcal{L}$ is a line bundle on X, we can construct an $\mathcal{O}_{X_{\text{ét}}}^\times$-torsor $\mathcal{T}_{\mathcal{L}}$ as follows: if $j\colon U \to X$ is an étale X-scheme, we put

$$\Gamma(U,\mathcal{T}_{\mathcal{L}}) = \left\{\lambda \in \Gamma(U, j^*\mathcal{L}) \;\middle|\; \lambda\colon \mathcal{O}_U \xrightarrow{\sim} \mathcal{L} \text{ is an isomorphism}\right\},$$

which has a natural $\Gamma(U, \mathcal{O}_{X_{\text{ét}}}^\times)$-action given by multiplication. This defines a functor $\mathcal{L} \mapsto \mathcal{T}_{\mathcal{L}}$ from the left-hand side to the right-hand side.

Conversely let $\mathcal{T}$ be an $\mathcal{O}_{X_{\text{ét}}}^\times$-torsor. We wish to construct a corresponding line bundle $\mathcal{L}_{\mathcal{T}}$ via faithfully flat descent (in the form of étale descent of course). Let's first assume $\mathcal{T}$ is trivial over X, i.e., $\Gamma(X,\mathcal{T}) \neq \emptyset$. We define $\mathcal{L}_{\mathcal{T}}$ as a sheaf of sets first: for open subsets $U \subseteq X$, put

$$\Gamma(U,\mathcal{L}_{\mathcal{T}}) = \left\{(t,f) \;\middle|\; t \in \Gamma(U,\mathcal{T}) \text{ and } f \in \Gamma(U,\mathcal{O}_X^\times)\right\}/\!\sim,$$

where $\sim$ is the equivalence relation defined as $(t,f) \sim (t',f')$ iff there is a $\lambda \in \Gamma(U, \mathcal{O}_X^\times)$ such that $t' = \lambda t$ and $f' = \lambda f$. Thus, fixing t, every equivalence class has a unique representative of the form (t,f). It's easy to check that addition and scalar multiplication on equivalence classes defined by $[t,f] + [t,f'] = [t, f+f']$ and $\lambda[t,f] = [t,\lambda f]$ are independent of the choice of t and turn $\mathcal{L}_{\mathcal{T}}$ into a line bundle over $\mathcal{O}_X$ (in fact, even a trivial one).

For arbitrary X, we always find an étale cover $\{U_i \to X\}_{i \in I}$ such that $\mathcal{T}$ is trivial over each U_i. Then the above construction gives line bundles $\mathcal{L}_i$ over U_i corresponding to $\mathcal{T}|_{U_i}$. It can be checked immediately that the $\mathcal{L}_i$ form a descent datum, hence define a line bundle $\mathcal{L}_{\mathcal{T}}$ on X via faithfully flat descent. We leave it to the reader to verify that the functors $\mathcal{L} \mapsto \mathcal{T}_{\mathcal{L}}$ and $\mathcal{T} \mapsto \mathcal{L}_{\mathcal{T}}$ are indeed quasi-inverse to each other. $\square$

2.3.12. Corollary. — *For any scheme X, we have $H^1(X_{\text{ét}}, \mathcal{O}^\times_{X_{\text{ét}}}) \cong \operatorname{Pic}(X)$.*

Sketch of a proof.* We get from Proposition 2.3.9 and Fact 2.3.11 that both sides are in canonical bijection as sets. It remains to check that this is an isomorphism of abelian groups as well. To this end, we check that for line bundles $\mathcal{L}$ and $\mathcal{L}'$ the canonical map $\Gamma(U, \mathcal{T}_{\mathcal{L}}) \times \Gamma(U, \mathcal{T}_{\mathcal{L}'}) \to \Gamma(U, \mathcal{T}_{\mathcal{L}\otimes\mathcal{L}'})$ sending (λ, λ') to $\lambda \otimes \lambda'$ induces an isomorphism

$$\mathcal{T}_{\mathcal{L}} \times_{\mathcal{O}_{X_{\text{ét}}}} \mathcal{T}_{\mathcal{L}'} \xrightarrow{\sim} \mathcal{T}_{\mathcal{L}\otimes\mathcal{L}'} .$$

Then Remark* 2.3.10 shows that the group structures on both sides coincide. □

2.3.13. Corollary (Hilbert's theorem 90)**.** — *If k is any field, then*

$$H^1\big(\operatorname{Gal}(k^{\text{sep}}/k), k^{\text{sep},\times}\big) = 0 .$$

Proof.* Let $X = \operatorname{Spec} k$ and $\overline{x}\colon \operatorname{Spec} k^{\text{sep}} \to X$. Then $k^{\text{sep},\times} \cong \mathcal{O}^\times_{X_{\text{ét}},\overline{x}}$. Hence Proposition 2.2.2(c) and Corollary 2.3.12 show

$$H^1\big(\operatorname{Gal}(k^{\text{sep}}/k), k^{\text{sep},\times}\big) \cong H^1\big(X_{\text{ét}}, \mathcal{O}^\times_{X_{\text{ét}}}\big) \cong \operatorname{Pic}(X) .$$

But the right-hand side is trivial because X is just a point. □

2.4. Applications of Čech Cohomology

Even though Čech cohomology usually does not compute sheaf cohomology in general, it is still a powerful tool, especially for comparing cohomology of sheaves on different sites. For example, we will show that the étale cohomology of quasi-coherent sheaves coincides with their ordinary Zariski cohomology.

2.4.1. Construction. — Let $\mathcal{F} \in \operatorname{PAb}(X_{\text{ét}})$ be a presheaf of abelian groups on the étale site over X (in fact, the construction that follows would work for arbitrary categories $\mathcal{C}$ admitting fibre products, but we will stick with $X_{\text{ét}}$ for convenience). For any $U \to X$ in $X_{\text{ét}}$ put

$$\check{C}^n\big(\{U \to X\}, \mathcal{F}\big) = \Gamma(\underbrace{U \times_X \cdots \times_X U}_{n+1 \text{ factors}}, \mathcal{F}) .$$

For $0 \leqslant i \leqslant n+1$ let $d_i\colon \check{C}^n(\{U \to X\}, \mathcal{F}) \to \check{C}^{n+1}(\{U \to X\}, \mathcal{F})$ be the morphism induced by

$$\operatorname{pr}_{0,\dots,\hat{\imath},\dots,n+1}\colon \underbrace{U \times_X \cdots \times_X U}_{n+2 \text{ factors}} \longrightarrow \underbrace{U \times_X \cdots \times_x U}_{n+1 \text{ factors}}$$

omitting the i^{th} factor (and the numbering of factors starts at 0). This gives a cochain complex, called the *Čech complex* of $\{U \to X\}$ with coefficients in $\mathcal{F}$,

$$\check{C}^\bullet\big(\{U \to X\}, \mathcal{F}\big) = \Big(\check{C}^0\big(\{U \to X\}, \mathcal{F}\big) \xrightarrow{\check{d}} \check{C}^1\big(\{U \to X\}, \mathcal{F}\big) \xrightarrow{\check{d}} \dots\Big) .$$

Its differential is $\check{d} = \sum_{i=0}^{n+1} (-1)^i d_j$ in degree n. The cohomology groups

$$\check{H}^i\big(\{U \to X\}, \mathcal{F}\big) = H^i \check{C}^\bullet\big(\{U \to X\}, \mathcal{F}\big)$$

are called the *Čech cohomology groups* of $\mathcal{F}$.

Lecture 20
13th Jan, 2020

2.4.2. Remark. — If $f\colon V \to U$ is a morphism of étale X-schemes, then we have an induced morphism

$$\check{f}^\bullet\colon \check{C}^\bullet\big(\{U \to X\}, \mathcal{F}\big) \longrightarrow \check{C}^\bullet\big(\{V \to X\}, \mathcal{F}\big)$$

of Čech complexes. The map $\check{f}^n\colon \check{C}^n(\{U \to X\}, \mathcal{F}) \to \check{C}^n(\{V \to X\}, \mathcal{F})$ in degree n is the one induced by $(f\,\mathrm{pr}_0, \dots, f\,\mathrm{pr}_n)\colon V \times_X \dots \times_X V \to U \times_X \dots \times_X U$ (there are $n+1$ factors everywhere).

If $g\colon V \to U$ is another morphism in $X_{\text{ét}}$, then $\check{f}^\bullet$ and $\check{g}^\bullet$ are cochain homotopic. In particular, they induce the same maps on Čech cohomology. A cochain homotopy can be constructed as follows: for all $0 \leqslant l \leqslant n-1$, let $h_l\colon \check{C}^n(\{U \to X\}, \mathcal{F}) \to \check{C}^{n-1}(\{V \to X\}, \mathcal{F})$ be the map induced by

$$(f\,\mathrm{pr}_0, \dots, f\,\mathrm{pr}_l, g\,\mathrm{pr}_l, \dots, g\,\mathrm{pr}_{n-1})\colon \underbrace{V \times_X \dots \times_X V}_{n \text{ factors}} \longrightarrow \underbrace{U \times_X \dots \times_X U}_{n+1 \text{ factors}}$$

(observe that pr_l occurs twice). One can verify that $\check{h}^\bullet$ given by $\check{h}^n = \sum_{l=0}^{n-1}(-1)^l h_l$ in degree n is a cochain homotopy. See [AG₂, Lemma 1.2.1] for more details; this also contains Professor Franke's computation of $h_l d_k$ via a sweet six-fold case distinction, which I am certainly not going to include in these notes.

2.4.3. Lemma. — *Suppose $j\colon U \to X$ is étale and admits a section $s\colon X \to U$. Then the Čech complex $\check{C}^\bullet(\{U \to X\}, \mathcal{F})$ is acyclic in positive degrees for every $\mathcal{F} \in \mathrm{PAb}(X_{\text{ét}})$.*

Proof. Let's first consider the special case $j = \mathrm{id}_X\colon X \to X$. In this case we easily compute

$$\check{C}^\bullet\big(\{X = X\}, \mathcal{F}\big) = \Big(\Gamma(X, \mathcal{F}) \xrightarrow{0} \Gamma(X, \mathcal{F}) \xrightarrow{\mathrm{id}} \Gamma(X, \mathcal{F}) \xrightarrow{0} \Gamma(X, \mathcal{F}) \xrightarrow{\mathrm{id}} \dots\Big),$$

and the assertion is clear. In general, the endomorphisms of $\check{C}^\bullet(\{U \to X\}, \mathcal{F})$ induced by id_U and $s \circ j$ are cochain homotopic by Remark 2.4.2. But the map induced by $s \circ j$ factors over $\check{C}^\bullet(\{X = X\}, \mathcal{F})$, which has vanishing cohomology in positive degrees, hence the same must be true for $\check{C}^n(\{U \to X\}, \mathcal{F})$. □

2.4.1. Étale Cohomology of Quasi-Coherent Sheaves

2.4.4. Proposition. — *Let X be a scheme. If $\mathcal{U}\colon U = \bigcup_{j \in J} U_j$ is a Zariski-open cover of some X-scheme U, then $\check{H}^i(\mathcal{U}, -)$ denotes the ordinary Čech cohomology.*

(a) *Let $\mathfrak{X}$ be the class of all $\mathcal{F} \in \mathrm{Ab}(X_{\text{ét}})$ such that $\check{H}^i(\{V' \to V\}, \mathcal{F}) = 0$ for all $i > 0$ whenever $V' \to V$ is a surjective morphism of affine étale X-schemes. Then $\mathfrak{X}$ satisfies the assumptions of Proposition 2.1.10(b) for each of the functors $\zeta_{U,*}$, $U \in X_{\text{ét}}$.*

(b) *Let $\mathfrak{Y}$ be the class of all $\mathcal{F} \in \mathfrak{X}$ that also satisfy $\check{H}^i(\mathcal{V}, \mathcal{F}) = 0$ for all $i > 0$, whenever $\mathcal{V}\colon V = \bigcup_{j=1}^n V_j$ is a finite cover of a quasi-compact and quasi-separated $V \in X_{\text{ét}}$ by quasi-compact Zariski-open subsets V_j. Then $\mathfrak{Y}$ satisfies the assumptions of Proposition 2.1.10(b) for each of the functors $\Gamma(U, -)$, where $U \in X_{\text{ét}}$ is quasi-compact and quasi-separated.*

(c) *Let $\mathfrak{Y}'$ be the class of all $\mathcal{F} \in \mathfrak{X}$ that also satisfy $\check{H}^i(\mathcal{V}, \mathcal{F}) = 0$ for all $i > 0$ whenever $\mathcal{V}\colon V = \bigcup_{j \in J} V_i$ is any Zariski-open cover of an arbitrary $V \in X_{\text{ét}}$. Then $\mathfrak{Y}$ satisfies the assumptions of Proposition 2.1.10(b) for each of the functors $\Gamma(U, -)$, $U \in X_{\text{ét}}$.*

2.4.5. Remark. — (a) All assertions of Proposition 2.4.4 remain true if we replace $X_{\text{ét}}$ by X_{fppf} or X_{fpqc} (and "surjective étale" in (a) by "fppf" or "fpqc" respectively). This will become apparent during the proof.

(b) Professor Franke assumed X to be locally noetherian in Proposition 2.4.4(b) (or at least stressed that X may be arbitrary in (b)), but I don't quite see why that restriction should be necessary. Please correct me if I'm wrong!

(c) If every finite subset points of X is contained in an affine open subset, then [Mil80] shows that the canonical map

$$\check{H}^i(X_{\text{ét}},\mathcal{F}) = \operatorname*{colim}_{\mathcal{U}} \check{H}^i(\mathcal{U},\mathcal{F}) \xrightarrow{\sim} H^i(X_{\text{ét}},\mathcal{F})$$

(the colimit is taken over all étale covers $\mathcal{U} = \{U_j \to X\}_{j\in J}$ of X) is an isomorphism.

Proof of Proposition 2.4.4. We begin with (a). We first show that every $\mathcal{F}$ has a monomorphism $\mathcal{F} \hookrightarrow \mathcal{G}$ for some $\mathcal{G} \in \mathfrak{X}$. There are two possible approaches. One could either show directly that all injective objects of $\mathrm{Ab}(X_{\text{ét}})$ are in $\mathfrak{X}$. This is what [Stacks, Tag 03AW] does, and their proof works for arbitrary sites (in particular, for X_{fppf} and X_{fpqc} as well).

Alternatively, we could take $\mathcal{G} = \prod_{\overline{x}} \overline{x}_* I_{\overline{x}}$ as in the proof of Proposition 2.1.1(a). This is easier and more down-to-the-earth, but it only works for $X_{\text{ét}}$. To see that $\mathcal{G} \in \mathfrak{X}$, it suffices to consider the special case $\mathcal{G} = \overline{x}_* I$ because the Čech complex and thus also Čech cohomology commute with products. Now its easy to check that

$$\check{C}^\bullet\big(\{V' \to V\}, \overline{x}_* I\big) \cong \check{C}^\bullet\big(\{V' \times_X \operatorname{Spec}\kappa(\overline{x}) \to V \times_X \operatorname{Spec}\kappa(\overline{x})\}, I\big)\,.$$

But $V' \times_X \operatorname{Spec}\kappa(\overline{x})$ and $V \times_X \operatorname{Spec}\kappa(\overline{x})$ are both finite disjoint unions of copies of $\operatorname{Spec}\kappa(\overline{x})$ by Lemma 1.4.9(b), hence this morphism has a section. Thus both Čech complexes above must be acyclic in positive degrees by Lemma 2.4.3, proving $\overline{x}_* I \in \mathfrak{X}$.

It is clear that direct summands of objects in $\mathfrak{X}$ are in $\mathfrak{X}$ again, because direct summands of acyclic complexes stay acyclic. It remains to show that for any short exact sequence $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ with $\mathcal{F}, \mathcal{F}' \in \mathfrak{X}$, we also have $\mathcal{F}'' \in \mathfrak{X}$. We claim:

$(*)$ *For any surjective morphism $V' \to V$ of affine étale X-schemes, we get a short exact sequence of Čech complexes*

$$0 \longrightarrow \check{C}^\bullet\big(\{V' \to V\}, \mathcal{F}'\big) \longrightarrow \check{C}^\bullet\big(\{V' \to V\}, \mathcal{F}\big) \longrightarrow \check{C}^\bullet\big(\{V' \to V\}, \mathcal{F}''\big) \longrightarrow 0$$

Let's first see how $(*)$ implies (a). Taking the long exact cohomology sequence of the above short exact sequence of complexes, we get $\check{H}^i(\{V' \to V\}, \mathcal{F}'') = 0$ for all $i > 0$, hence $\mathcal{F}'' \in \mathfrak{X}$. Moreover, we obtain that $\Gamma(V,\mathcal{F}) \twoheadrightarrow \Gamma(V,\mathcal{F}'')$ is surjective (because $\{V' \to V\}$ is an étale cover, we have $\check{H}^0(\{V' \to V\}, -) = \Gamma(V,-)$ by the sheaf axiom). Now let $U \in X_{\text{ét}}$ be arbitrary. As seen above $\Gamma(V,\mathcal{F}) \twoheadrightarrow \Gamma(V,\mathcal{F}'')$ is surjective for all affine open subsets $V \subseteq U$, hence $\zeta_{U,*}\mathcal{F} \to \zeta_{U,*}\mathcal{F}''$ is indeed an epimorphism of sheaves. This shows $(*) \Rightarrow (a)$.

To prove $(*)$, observe that it suffices to show that $\Gamma(V,\mathcal{F}) \twoheadrightarrow \Gamma(V,\mathcal{F}'')$ is surjective for all affine étale X-schemes V, because the $V' \times_V \cdots \times_V V'$ occuring in the Čech complex above are affine étale X-schemes again. So let $f'' \in \Gamma(V,\mathcal{F}'')$. We need to construct a preimage in $\Gamma(V,\mathcal{F})$. Since $\mathcal{F} \to \mathcal{F}''$ is an epimorphism of sheaves, we find an étale cover $\{V_j \to V\}_{j\in J}$ such that $f''|_{V_j}$ has a preimage in $\Gamma(V_j,\mathcal{F})$ for all $j \in J$. Without restriction, all V_j are affine. Moreover, since V is affine, it is quasi-compact, hence we may assume that the indexing set J is finite. Then $\pi\colon V' \to V$, where $V' = \coprod_{j\in J} V_j$, is an affine étale cover of V with the same properties.

In particular, $\pi^* f'' = f''|_{V'}$ has a preimage $f \in \Gamma(V', \mathcal{F})$. Now consider the Čech complexes of $\{V' \to V\}$ with coefficients in $\mathcal{F}'$, $\mathcal{F}$, and $\mathcal{F}''$ respectively. We abbreviate them as $'\check{C}^\bullet$, $\check{C}^\bullet$, and $''\check{C}^\bullet$ for convenience. Consider the element the image of the element $\check{d}(f) = (\mathrm{pr}_1^* - \mathrm{pr}_2^*)f \in \check{C}^1 = \Gamma(V' \times_V V', \mathcal{F})$ under $\check{C}^1 \to {}''\check{C}^1$. This vanishes, because $\mathrm{pr}_1^* \pi^* f'' = f''|_{V' \times_V V'} = \mathrm{pr}_2^* \pi^* f''$. Hence $\check{d}(f)$ must be the image of some $f' \in {}'\check{C}^1$. Note that $\check{d}(f') = \check{d}^2(f) = 0$. But $''\check{C}^\bullet$ is acyclic in positive degrees because $\mathcal{F}' \in \mathfrak{X}$, so we can write $f' = \check{d}(\varphi')$ for some $\varphi' \in {}'\check{C}^0$. By construction, $\check{d}(f - \varphi) = 0$. Hence $f - \varphi'$ defines a global section $\varphi \in \Gamma(V, \mathcal{F})$, whose image is f''. This proves surjectivity and thus $(*)$.

The proof of (b) and (c) is pretty much the same: we first show that every $\mathcal{F}$ has a monomorphism $\mathcal{F} \hookrightarrow \mathcal{G}$ for some $\mathcal{G} \in \mathfrak{Y}$ or $\mathcal{G} \in \mathfrak{Y}'$ respectively and $\mathfrak{Y}$, $\mathfrak{Y}'$ are closed under taking direct summands. This can be done just as in (a). Now let $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ be a short exact sequence with $\mathcal{F}, \mathcal{F}' \in \mathfrak{Y}$ resp. $\mathcal{F}, \mathcal{F}' \in \mathfrak{Y}'$. We claim

$(*')$ *Let $\mathcal{V}\colon V = \bigcup_{j \in J} V_j$ be a finite/arbitrary cover of a quasi-compact quasi-separated/arbitrary $V \in X_{\text{ét}}$ by quasi-compact/arbitrary Zariski-open subsets V_j. Then we get a short exact sequence of Čech complexes*

$$0 \longrightarrow \check{C}^\bullet(\mathcal{V}, \mathcal{F}') \longrightarrow \check{C}^\bullet(\mathcal{V}, \mathcal{F}) \longrightarrow \check{C}^\bullet(\mathcal{V}, \mathcal{F}'') \longrightarrow 0\,.$$

As in (a), $(*')$ immediately implies $\mathcal{F}'' \in \mathfrak{Y}$ resp. $\mathcal{F}'' \in \mathfrak{Y}'$ via the long exact cohomology sequence. Moreover, it is clear that $\Gamma(U, \mathcal{F}) \to \Gamma(U, \mathcal{F}'')$ is surjective for all quasi-compact quasi-separated/arbitrary $U \in X_{\text{ét}}$, because we can just take $U = V$ in $(*')$. Thus $(*')$ implies (b) and (c).

To prove $(*')$, it suffices to show that $\Gamma(V, \mathcal{F}) \to \Gamma(V, \mathcal{F}'')$ is surjective, because the intersections $\bigcap_{l=0}^n V_{j_l}$ occuring in the Čech complex are again quasi-compact quasi-separated/arbitrary. Also note that $0 \to \zeta_{V,*}\mathcal{F}' \to \zeta_{V,*}\mathcal{F} \to \zeta_{V,*}\mathcal{F}'' \to 0$ is a short exact sequence of Zariski sheaves, since $R^1\zeta_{V,*}\mathcal{F}' = 0$ by (a) and Proposition 2.1.10(b). Thus, if $f'' \in \Gamma(V, \mathcal{F}'')$ is given, then we can find a Zariski-open cover $\mathcal{V}\colon V = \bigcup_{j \in J} V_j$ such that each $f''|_{V_j}$ has a preimage in $\Gamma(V_j, \mathcal{F})$ (a priori, we would only have found an étale cover $\{V_j \to V\}_{j \in J}$ with that property). Without restriction all V_j are affine. In the case of (b), we may moreover assume that J is finite, as V is quasi-compact. Now the rest of the proof of (a) can be copied. $\square$

2.4.6. Definition. — For the purpose of this lecture, let an $\mathcal{O}_{X_{\text{ét}}}$-module $\mathcal{F}$ be called *quasi-coherent* if all $\zeta_{U,*}\mathcal{F}$ for $U \in X_{\text{ét}}$ are quasi-coherent, and for every morphism $f\colon V \to U$ in $X_{\text{ét}}$ the canonical morphism $f^*\zeta_{U,*}\mathcal{F} \xrightarrow{\sim} \zeta_{V,*}\mathcal{F}$ is an isomorphism.

2.4.7. Proposition. — *Let X be any scheme. If $\mathcal{F}$ is a quasi-coherent $\mathcal{O}_{X_{\text{ét}}}$-module, then $R^i\zeta_{U,*}\mathcal{F} = 0$ for all $i > 0$, $U \in X_{\text{ét}}$. Moreover, $\zeta_{X,*}$ induces an equivalence of categories*

$$\zeta_{X,*}\colon \{\textit{quasi-coherent } \mathcal{O}_{X_{\text{ét}}}\textit{-modules}\} \xrightarrow{\sim} \{\textit{quasi-coherent } \mathcal{O}_{X_{\text{Zar}}}\textit{-modules}\}\,.$$

2.4.8. Remark. — As in Remark 2.4.5, all assertions of Proposition 2.4.7 remain true if $X_{\text{ét}}$ is replaced by X_{fpqc} or X_{fppf}. The proof remains the same.

Proof of Proposition 2.4.7. The first assertion follows from Proposition 2.4.4(a) and Proposition 2.1.10(b), once we show that all quasi-coherent modules are contained in $\mathfrak{X}$. Thus, let $\mathcal{F}$ be quasi-coherent and let $V' \to V$ be a surjective morphism of affine étale X-schemes. We need to show that $\check{C}^\bullet(\{V' \to V\}, \mathcal{F})$ is acyclic in positive degrees. This can be checked after faithfully flat base change. Write $V = \operatorname{Spec} A$ and $V' = \operatorname{Spec} A'$, then A' happens to be faithfully flat over A. Thus, it suffices to show that

$$\check{C}^\bullet\big(\{V' \to V\}, \mathcal{F}\big) \otimes_A A' \cong \check{C}^\bullet\big(\{\mathrm{pr}_1 \colon V' \times_V V' \to V'\}, \mathcal{F}\big)$$

is acyclic in positive degrees. But $\mathrm{pr}_1 \colon V' \times_V V' \to V'$ has a section, namely the diagonal $\Delta_{V'/V} \colon V' \to V' \times_V V'$, hence the complex on the right-hand side is acyclic by Lemma 2.4.3. This proves the first assertion.

The second assertion is basically just a consequence of faithfully flat descent. More precisely, we can construct a quasi-inverse functor ι as follows: if $\mathcal{G}$ is a quasi-coherent $\mathcal{O}_{X_{\mathrm{Zar}}}$-module and $j \colon U \to X$ is étale, put $\Gamma(U, \iota(\mathcal{G})) = \Gamma(U, j^*\mathcal{G})$. Here $j^* \colon \mathrm{QCoh}_X \to \mathrm{QCoh}_U$ indicates the usual pullback of quasi-coherent modules in the Zariski-topology. Using Proposition 1.2.11, it's easy to check that $\zeta_{X,*}$ and ι are indeed quasi-inverse functors. $\square$

We thus arrive at the promised comparison result for étale and Zariski cohomology of quasi-coherent sheaves.

2.4.9. Corollary. — *Let X be an arbitrary scheme.*

(*a*) *If $\mathcal{F}$ is a quasi-coherent $\mathcal{O}_{X_{\text{ét}}}$-module, then $H^i(X_{\text{ét}}, \mathcal{F}) \cong H^i(X_{\mathrm{Zar}}, \zeta_{X,*}\mathcal{F})$ for all $i \geqslant 0$.*

(*b*) *If X is a noetherian scheme over $\mathbb{F}_p$, then $H^i(X_{\text{ét}}, \mathbb{Z}/p\mathbb{Z}) = 0$ for all $i > \dim X + 1$.*

Proof. Part (*a*) follows from Proposition 2.4.7 and Proposition 2.1.11: the spectral sequence $E_2^{i,j} = H^i(X_{\mathrm{Zar}}, R^j\zeta_{X,*}\mathcal{F}) \Rightarrow H^{i+j}(X_{\text{ét}}, \mathcal{F})$ collapses as $R^j\zeta_{X,*}\mathcal{F} = 0$ for $j > 0$. For (*b*), we use the short exact sequence

$$0 \longrightarrow \mathbb{Z}/p\mathbb{Z} \longrightarrow \mathcal{O}_{X_{\text{ét}}} \xrightarrow{\varphi^* - \mathrm{id}} \mathcal{O}_{X_{\text{ét}}} \longrightarrow 0$$

of étale sheaves, where $\varphi \colon X \to X$ denotes the absolute Frobenius. This sequence is exact as there are Artin–Schreier coverings (see Proposition 1.5.12). In the corresponding long exact sequence we see that $H^i(X_{\text{ét}}, \mathcal{O}_{X_{\text{ét}}}) = H^i(X, \mathcal{O}_X) = 0$ for $i > \dim X$ by (*a*) and Grothendieck's dimension theorem [Tôh, Théorème 3.6.5]. Hence $H^i(X_{\text{ét}}, \mathbb{Z}/p\mathbb{Z}) = 0$ for $i > \dim X + 1$. $\square$

2.4.10. Remark*. — If X is separated and of finite type over an algebraically closed extension $k/\mathbb{F}_p$ (for example, X could be a variety over $\overline{\mathbb{F}}_p$), then even

$$H^{\dim X + 1}(X_{\text{ét}}, \mathbb{Z}/p\mathbb{Z}) = 0$$

(thanks to Robin for pointing this out). A proof can be found in [Stacks, Tag 0A3N].

2.4.2. Étale Cohomology of Inverse Limits

We finish this section with a comparison theorem for étale cohomology of inverse limits of schemes. This will be a generalization of Proposition 1.7.7.

2.4.11. Situation. — Let $\mathcal{I}$ be a (small) cofiltered index category. Suppose we are given the following bunch of data:

(1) A system $(X_\alpha)_{\alpha \in \mathcal{I}}$ of quasi-compact and quasi-separated schemes with affine transition maps $\pi_\mu \colon X_\beta \to X_\alpha$ for all $\mu \in \mathrm{Hom}_{\mathcal{I}}(\beta, \alpha)$. We denote $X = \lim_{\mathcal{I}} X_\alpha$ (this exists by A.1.1) with structure maps $\pi_\alpha \colon X \to X_\alpha$.

(2) Let $(\mathcal{F}_\alpha, \varphi_\mu)_\alpha \in \mathcal{I}$ be a system of sheaves $\mathcal{F}_\alpha \in \mathrm{Ab}(X_{\alpha,\text{ét}})$ together with transition morphisms $\varphi_\mu\colon \pi_\mu^*\mathcal{F}_\alpha \to \mathcal{F}_\beta$ such that the diagram

$$\begin{tikzcd} \pi_{\mu\nu}^*\mathcal{F}_\alpha \arrow[r, "\sim"] \arrow[dr, "\varphi_{\mu\nu}"'] & \pi_\nu^*\pi_\mu^*\mathcal{F}_\alpha \arrow[r, "\pi_\nu^*(\varphi_\mu)"] & \pi_\nu^*\mathcal{F}_\beta \arrow[dl, "\varphi_\nu"] \\ & \mathcal{F}_\gamma & \end{tikzcd}$$

commutes for all composable morphisms $\mu \in \mathrm{Hom}_\mathcal{I}(\beta,\alpha)$ and $\nu \in \mathrm{Hom}_\mathcal{I}(\gamma,\beta)$. We denote the category of all such systems by $\mathcal{A}$.

For $\alpha \in \mathcal{I}$ we denote by $\mathcal{I}/\alpha$ the slice category of α-objects. For every object $\beta \in \mathcal{I}/\alpha$ (which is actually a morphism $\mu\colon \beta \to \alpha$), the set of homomorphisms $\mathrm{Hom}_{\mathcal{I}/\alpha}(\beta,\alpha)$ has precisely one element (corresponding to μ). We write $\pi_{\beta,\alpha}\colon X_\beta \to X_\alpha$ for the corresponding morphism.

2.4.12. Proposition. — *Suppose we are in 2.4.11. Then for all $i \geqslant 0$ there is a canonical isomorphism*

$$\operatorname*{colim}_{\mathcal{I}} H^i\left(X_{\alpha,\text{ét}}, \mathcal{F}_\alpha\right) \xrightarrow{\sim} H^i\left(X_{\text{ét}}, \operatorname*{colim}_{\mathcal{I}} \pi_\alpha^*\mathcal{F}_\alpha\right).$$

For $i = 0$ this morphism fits (as the dashed arrow) into a commutative diagram

$$\begin{tikzcd} \operatorname*{colim}_{\mathcal{I}/\alpha} \Gamma\left(X_\beta, \pi_{\beta,\alpha}^*\mathcal{F}_\alpha\right) \arrow[rr, "\sim", "\text{Proposition 1.7.7}"'] \arrow[d] & & \Gamma\left(X, \pi_\alpha^*\mathcal{F}_\alpha\right) \arrow[d] \\ \operatorname*{colim}_{\mathcal{I}/\alpha} \Gamma(X_\beta, \mathcal{F}_\beta) \arrow[r, "\sim"] & \operatorname*{colim}_{\mathcal{I}} \Gamma(X_\beta, \mathcal{F}_\beta) \arrow[r, dashed, "\sim"] & \Gamma\left(X, \operatorname*{colim}_{\mathcal{I}} \pi_\beta^*\mathcal{F}_\beta\right) \end{tikzcd}.$$

In this diagram, the top arrow is the isomorphism from Proposition 1.7.7.

Lecture 21
17th Jan, 2020

2.4.13. Remark. — In general, if X is an arbitrary scheme and $(\mathcal{G}_\alpha)_{\alpha\in\mathcal{I}}$ is a filtered system in $\mathrm{Ab}(X_{\text{ét}})$, one can construct $\operatorname{colim} \mathcal{G}_\alpha$ in $\mathrm{Ab}(X_{\text{ét}})$ as the sheafification of the presheaf $U \mapsto \operatorname{colim} \Gamma(U, \mathcal{G}_\alpha)$. If V is a quasi-compact open subset of X, then the canonical morphism

$$\operatorname*{colim}_{\mathcal{I}} \Gamma(V, \mathcal{G}_\alpha) \xrightarrow{\sim} \Gamma\left(V, \operatorname*{colim}_{\mathcal{I}} \mathcal{G}_\alpha\right)$$

is an isomorphism. Indeed, let $\mathcal{G}$ denote the presheaf $U \mapsto \operatorname{colim} \Gamma(U, \mathcal{G}_\alpha)$. We first claim that $\mathcal{G}$ satisfies the sheaf axiom with respect to étale covers $\{U_i \to U\}_{i\in I}$ in which the indexing set I is finite. So what we want to show is that

$$\Gamma(U, \mathcal{G}) \longrightarrow \prod_{i\in I} \Gamma(U_i, \mathcal{G}) \underset{\mathrm{pr}_2^*}{\overset{\mathrm{pr}_1^*}{\rightrightarrows}} \prod_{i,j\in I} \Gamma(U_i \times_U U_j, \mathcal{G})$$

is an equalizer diagram. Since the $\mathcal{G}_\alpha$ are sheaves, this is true if $\mathcal{G}$ is replaced by $\mathcal{G}_\alpha$. But since I is finite, the equalizer in question is a finite limit, and finite limits commute with filtered colimits, hence the same must be true for $\mathcal{G}$.

Now if V is quasi-compact, then every étale cover $\{V_i \to V\}_{i\in I}$ admits a finite subcover, i.e., a finite subset $J \subseteq I$ such that $\{V_i \to V\}_{i\in I}$ is already an étale cover. Going through the explicit description of $\Gamma(V, \mathcal{G}^{\mathrm{Sh}})$ in Proposition 1.6.3(c), we thus obtain that the canonical morphism $\Gamma(V, \mathcal{G}) \xrightarrow{\sim} \Gamma(V, \mathcal{G}^{\mathrm{Sh}})$ is an isomorphism, as claimed. This will be used in the proof of Proposition 2.4.12.

Proof of Proposition 2.4.12. Observe that the category $\mathcal{A}$ from Situation 2.4.11 is abelian. In fact, all required categorical constructions—finite biproducts, kernels, cokernels, equality of image and coimage—can be carried out component-wise (here we use that the pullback functor $\pi_\mu^*\colon \mathrm{Ab}(X_{\alpha,\text{ét}}) \to \mathrm{Ab}(X_{\beta,\text{ét}})$ is exact), so $\mathcal{A}$ being abelian follows from the fact that all $\mathrm{Ab}(X_{\alpha,\text{ét}})$ are abelian. For $\mathcal{F} = (\mathcal{F}_\alpha, \varphi_\mu) \in \mathcal{A}$, let's denote

$$H^i_{\text{left}}(\mathcal{F}) = \operatorname*{colim}_{\mathcal{I}} H^i\big(X_{\alpha,\text{ét}}, \mathcal{F}_\alpha\big) \quad \text{and} \quad H^i_{\text{right}}(\mathcal{F}) = H^i\Big(X_{\text{ét}}, \operatorname*{colim}_{\mathcal{I}} \pi_\alpha^* \mathcal{F}_\alpha\Big)\,.$$

Our strategy is to show that $H^i_{\text{left}}(-)\colon \mathcal{A} \to \mathrm{Ab}$ and $H^i_{\text{right}}(-)\colon \mathcal{A} \to \mathrm{Ab}$ are the respective right-derived functors of $H^0_{\text{left}}(-)$ and $H^0_{\text{right}}(-)$, and that $H^0_{\text{left}}(-) \cong H^0_{\text{right}}(-)$. By the universal property of right-derived functors, this will immediately settle the proof.

By exactness of filtered colimits we see that $H^\bullet_{\text{left}}(-)$ and $H^\bullet_{\text{right}}(-)$ are cohomological δ-functors. Moreover, we calculate

$$\operatorname*{colim}_{\mathcal{I}} \Gamma\left(X_\alpha, \mathcal{F}_\alpha\right) \cong \operatorname*{colim}_{\mathcal{I}} \operatorname*{colim}_{\mathcal{I}/\alpha} \Gamma\left(X_\beta, \mathcal{F}_\beta\right) \cong \operatorname*{colim}_{\mathcal{I}} \Gamma\left(X, \pi_\alpha^* \mathcal{F}_\alpha\right) \cong \Gamma\Big(X, \operatorname*{colim}_{\mathcal{I}} \pi_\alpha^* \mathcal{F}_\alpha\Big)\,.$$

The left isomorphism comes from the fact that $\operatorname{colim}_{\mathcal{I}/\alpha} \Gamma(X_\beta, \mathcal{F}_\beta) \cong \Gamma(X_\beta, \mathcal{F}_\beta)$ since $\mathcal{I}$ is filtered. For the middle isomorphism we apply Proposition 1.7.7 to the filtered category $\mathcal{I}/\alpha$ and take $\operatorname{colim}_{\mathcal{I}}$ afterwards. The isomorphism on the right follows from Remark 2.4.13. This calculation shows $H^0_{\text{left}}(-) \cong H^0_{\text{right}}(-)$.

It remains to show that both cohomological functors $H^\bullet_{\text{left}}(-)$ and $H^\bullet_{\text{right}}(-)$ are effaceable in the sense of (3) of Proposition 2.1.10(a). For arbitrary schemes S, let $\mathfrak{Y}_S$ denote the class of objects in $\mathrm{Ab}(S_{\text{ét}})$ specified in Proposition 2.4.4(b). Furthermore, let $\mathfrak{Y}$ be the class of all $\mathcal{H} = (\mathcal{H}_\alpha, \psi_\mu) \in \mathcal{A}$ satisfying $\mathcal{H}_\alpha \in \mathfrak{Y}_{X_\alpha}$ for all α. We must show that

(1) *for all* $\mathcal{F} \in \mathcal{A}$ *there exists a monomorphism* $\mathcal{F} \hookrightarrow \mathcal{H}$ *into some* $\mathcal{H} \in \mathfrak{Y}$*, and*

(2) *for all* $\mathcal{H} \in \mathfrak{Y}$ *we have* $H^i_{\text{left}}(\mathcal{H}) = 0 = H^i_{\text{right}}(\mathcal{H})$ *for all* $i > 0$.

We start with (2). If $\mathcal{H} \in \mathfrak{Y}$, then $H^i(X_{\alpha,\text{ét}}, \mathcal{H}_\alpha) = 0$ for all α and all $i > 0$ by Proposition 2.4.4(b), using that all X_α are quasi-compact and quasi-separated. Hence $H^i_{\text{left}}(\mathcal{H}) = 0$ for $i > 0$. The fact that $H^i_{\text{right}}(\mathcal{H}) = 0$ for all $i > 0$ will follows at once from the following claim:

$(*)$ *We have* $\operatorname{colim}_{\mathcal{I}} \pi_\alpha^* \mathcal{H}_\alpha \in \mathfrak{Y}_X$.

To prove $(*)$, we must check the two conditions. So let $V' \to V$ be a surjective morphism of affine étale X-schemes. Combining (g), (i), and (k) from Appendix A.1, we see that it can be written as a base change of a morphism $V'_\alpha \to V_\alpha$ of affine étale X_α-schemes for some α. Moreover, there is a $\beta \in \mathcal{I}/\alpha$ such that $V'_\beta = V'_\alpha \times_{X_\alpha} X_\beta \to V_\alpha \times_{X_\alpha} X_\beta = V_\beta$ is surjective. Indeed, $V'_\alpha \to V_\alpha$ is étale, hence its image is open. Let Z_α be the complement of its image. Then $Z_\alpha \times_{X_\alpha} X = \emptyset$ since $V' \to V$ is surjective, hence already $Z_\alpha \times_{X_\alpha} X_\beta = \emptyset$ for some $\beta \in \mathcal{I}/\alpha$ by A.1.1(b). This shows that $V'_\beta \to V_\beta$ is indeed surjective.

We have $\operatorname{colim}_{\mathcal{I}/\beta} \pi_\gamma^* \mathcal{H}_\gamma = \operatorname{colim}_{\mathcal{I}} \pi_\alpha^* \mathcal{H}_\alpha$ since the category $\mathcal{I}$ is filtered. Now observe that

$$\operatorname*{colim}_{\mathcal{I}/\beta} \check{C}^\bullet\big(\{V'_\gamma \to V_\gamma\}, \mathcal{H}_\gamma\big) \cong \check{C}^\bullet\Big(\{V' \to V\}, \operatorname*{colim}_{\mathcal{I}/\beta} \pi_\gamma^* \mathcal{H}_\gamma\Big)\,.$$

Indeed, every term in the Čech complex on the right-hand side is a finite product of terms of the form $\Gamma(U, \operatorname{colim}_{\mathcal{I}/\beta} \pi_\gamma^* \mathcal{H}_\gamma)$, where U is some fibre product of V' over V, hence quasi-compact and quasi-separated. Thus we can pull out the colimit since we have already proved

Proposition 2.4.12 in the case $i = 0$. Now $\check{C}^\bullet(\{V'_\gamma \to V_\gamma\}, \mathcal{H}_\gamma)$ is acyclic in positive degrees for all $\gamma \in \mathcal{I}/\beta$, because $\mathcal{H}_\gamma \in \mathfrak{Y}_{X_\gamma}$. Since filtered colimits are exact, the above equality shows that $\check{C}^\bullet(\{V' \to V\}, \operatorname{colim}_{\mathcal{I}/\beta} \pi_\gamma^* \mathcal{H}_\gamma)$ is exact in positive degrees, which is what we wanted to show.

In a similar manner, using A.1.2(h), one shows that for every cover $\mathcal{V}\colon V = \bigcup_{j=1}^n V_i$ of a quasi-compact quasi-separated $V \in X_{\text{ét}}$ by quasi-compact Zariski-open V_j the Čech complex $\check{C}^\bullet(\mathcal{V}, \operatorname{colim}_{\mathcal{I}} \pi_\alpha^* \mathcal{H}_\alpha)$ is acyclic in positive degrees. This ultimately shows $\operatorname{colim}_{\mathcal{I}} \pi_\alpha^* \mathcal{H}_\alpha \in \mathfrak{Y}_X$, hence the proof of $(*)$ is complete.

It remains to prove (1). Fix some $\gamma \in \mathcal{I}$. For every geometric point $\overline{x}$ of X_γ and every abelian group Φ let $\mathcal{H}(\overline{x}, \gamma, \Phi)$ be the system $\big(\prod_{\nu\colon \gamma \to \alpha} \pi_\nu(\overline{x})_* \Phi\big)_\alpha$. For any $\mu\colon \beta \to \alpha$ in $\mathcal{I}$ and all $\vartheta\colon \gamma \to \beta$, there is a canonical morphism $\pi_\mu^* \pi_{\mu\vartheta}(\overline{x})_* \Phi \to \pi_\vartheta(\overline{x})_* \Phi$, which is adjoint to the identity on $\pi_{\mu\vartheta}(\overline{x})_* \Phi$ via the pullback-pushforward adjunction. Composing with $\pi_\mu^*(\mathrm{pr}_{\mu\vartheta})\colon \pi_\mu^*\big(\prod_{\nu\colon \alpha \to \gamma} \pi_\nu(\overline{x})_* \Phi\big) \to \pi_\mu^* \pi_{\mu\vartheta}(\overline{x})_* \Phi$ yields a canonical morphism

$$\eta_\mu\colon \pi_\mu^* \left(\prod_{\nu\colon \gamma \to \alpha} \pi_\nu(\overline{x})_* \Phi \right) \longrightarrow \prod_{\vartheta\colon \gamma \to \beta} \pi_\vartheta(\overline{x})_* \Phi\,,$$

and its straightforward to check that these η_μ turn $\mathcal{H}(\overline{x}, \gamma, \Phi)$ into an object of $\mathcal{A}$. Moreover, $\mathcal{H}(\overline{x}, \gamma, \Phi)$ is an element of $\mathfrak{Y}$. Indeed, the skycraper sheaves $\pi_\nu(\overline{x})_* \Phi$ are elements of $\mathfrak{Y}_{X_\alpha}$, as seen in the proof of Proposition 2.4.4, and $\mathfrak{Y}_{X_\alpha}$ is stable under taking products (because the Čech complex commutes with products and products are exact in Ab).

By the same argument, $\mathfrak{Y}$ is closed under products (and products in $\mathcal{A}$ can be taken component-wise). Now let $\mathcal{F} \in \mathcal{A}$ be an arbitrary element and let $\mathcal{H}(\overline{x}, \gamma, (\mathcal{F}_\gamma)_{\overline{x}})$ be as above (in the special case where $\Phi = (\mathcal{F}_\gamma)_{\overline{x}}$ is the stalk of $\mathcal{F}_\gamma$ at $\overline{x}$). There is a canonical morphism

$$\mathcal{F} \longrightarrow \mathcal{H}\big(\overline{x}, \gamma, (\mathcal{F}_\gamma)_{\overline{x}}\big)$$

given as follows: for all α and all $\nu\colon \gamma \to \alpha$ in $\mathcal{I}$, the morphism $(\mathcal{F}_\alpha)_{\pi_\nu(\overline{x})} = (\pi_\mu^* \mathcal{F}_\alpha)_{\overline{x}} \to (\mathcal{F}_\gamma)_{\overline{x}}$ of stalks at $\overline{x}$ induces a morphism from the constant $(\mathcal{F}_\alpha)_{\pi_\nu(\overline{x})}$-valued sheaf on X_α to $\pi_\nu(\overline{x})_*((\mathcal{F}_\gamma)_{\overline{x}})$. Composing this with the canonical morphism from $\mathcal{F}_\alpha$ to the constant $(\mathcal{F}_\alpha)_{\pi_\nu(\overline{x})}$-valued sheaf on X_α yields a morphism $\mathcal{F}_\alpha \to \pi_\nu(\overline{x})_*((\mathcal{F}_\gamma)_{\overline{x}})$. By varying α and ν we obtain the required morphism $\mathcal{F} \to \mathcal{H}(\overline{x}, \gamma, (\mathcal{F}_\gamma)_{\overline{x}})$.

Similar to the proof of Proposition 2.1.1(a), we can show that the ensuing morphism $\mathcal{F} \to \prod_{\gamma, \overline{x}} \mathcal{H}(\overline{x}, \gamma, (\mathcal{F}_\gamma)_{\overline{x}})$ is a monomorphism, where the product is taken over all $\gamma \in \mathcal{I}$ and all geometric points $\overline{x}$ of X_γ. Since $\mathfrak{Y}$ is closed under products as seen above, this finally shows (1). Hence $H^\bullet_{\mathrm{right}}(-)$ is effaceable and we are done. ☐

2.4.14. Corollary. — *If X is quasi-compact and quasi-separated and $(\mathcal{F}_\alpha)_{\alpha \in \mathcal{I}}$ a filtered system in* $\mathrm{Ab}(X_{\text{ét}})$*, then for all $i \geqslant 0$ there is a canonical isomorphism*

$$\operatorname*{colim}_{\mathcal{I}} H^i(X_{\text{ét}}, \mathcal{F}_\alpha) \xrightarrow{\sim} H^i\Big(X_{\text{ét}}, \operatorname*{colim}_{\mathcal{I}} \mathcal{F}_\alpha\Big)\,.$$

Proof. Apply Proposition 2.4.12 with all X_α equal to X. ☐

2.4.15. Corollary. — *Let $f\colon X \to Y$ be a morphism of schemes, $\overline{y}$ a geometric point of Y, and $X_{\overline{y}} = X \times_Y \operatorname{Spec} \mathcal{O}_{Y_{\text{ét}}, \overline{y}}$ the “fibre over $\overline{y}$”, then for $\mathcal{F} \in \mathrm{Ab}(X_{\text{ét}})$ there is a natural isomorphism*

$$(R^i f_* \mathcal{F})_{\overline{y}} \xrightarrow{\sim} H^i\big(X_{\overline{y}, \text{ét}}, \mathrm{pr}_1^* \mathcal{F}\big)$$

for all $i \geqslant 0$. For $i = 0$, this becomes the isomorphism from Corollary 1.7.9.

Proof. To construct the morphism in question, observe that both sides are cohomological functors (because both pr_1^* and taking stalks at $\overline{y}$ is exact). In particular, the left-hand side are the derived functors of $f_*(-)_{\overline{y}}\colon \mathrm{Ab}(X_{\text{ét}}) \to \mathrm{Ab}$ because it clearly vanishes on injective objects. Thus, the morphism in question arises naturally from Corollary 1.7.9 and the universal property of derived functors.

The sheaf $R^if_*\mathcal{F}$ is the sheafification of the presheaf $V \mapsto H^i(X \times_Y V, \mathcal{F})$ on $Y_{\text{ét}}$. In particular, their stalks at $\overline{y}$ coincide. Thus we can write

$$(R^if_*\mathcal{F})_{\overline{y}} \cong \operatorname*{colim}_{(V,\overline{v})} H^i(X \times_Y V, \mathcal{F})\,,$$

where the colimit is taken over all affine étale neighbourhoods $(V,\overline{v})$ of $\overline{y}$. By Proposition 2.4.12 the right-hand side is isomorphic to $H^i(X_{\overline{y},\text{ét}}, \mathrm{pr}_1^*\mathcal{F})$, and we win. $\square$

2.4.16. Warning*. — Later we would like to apply Corollary 2.4.15 in the case where $\mathcal{F} = \mathcal{O}_{X_{\text{ét}}}$ or $\mathcal{F} = \mathcal{O}_{X_{\text{ét}}}^\times$. It seems obvious that $\mathrm{pr}_1^*\mathcal{O}_{X_{\text{ét}}} = \mathcal{O}_{X_{\overline{y},\text{ét}}}$ and same for $\mathcal{O}_{X_{\text{ét}}}^\times$, but in fact, it's *not*! If $f\colon X \to S$ is a general morphism of schemes, then $f^*\mathcal{O}_{S_{\text{ét}}} \to \mathcal{O}_{X_{\text{ét}}}$ is usually *not an isomorphism*, and neither is $f^*\mathcal{O}_{S_{\text{ét}}}^\times \to \mathcal{O}_{X_{\text{ét}}}^\times$, unless f is étale! Indeed, if it were, then $\mathcal{O}_{X,x}$ and $\mathcal{O}_{S,s}$ would have the same strict henselization whenever $s = f(x)$ (since f^* preserves stalks at geometric points), but this is clearly nonsense. That's why the following lemma* is not completely trivial.

2.4.17. Lemma*. — *Let $X = \lim_{\mathcal{I}} X_\alpha$ be a limit over a cofiltered system of schemes with affine transition maps, and let $\pi_\alpha\colon X \to X_\alpha$ be the structure morphisms. Then*

$$\operatorname*{colim}_{\mathcal{I}} \mathcal{O}_{X_{\alpha,\text{ét}}} \cong \mathcal{O}_{X_{\text{ét}}} \quad \textit{and} \quad \operatorname*{colim}_{\mathcal{I}} \mathcal{O}_{X_{\alpha,\text{ét}}}^\times \cong \mathcal{O}_{X_{\text{ét}}}^\times\,.$$

In particular, in the situation of Corollary 2.4.15, we get

$$\left(R^if_*\mathcal{O}_{X_{\text{ét}}}\right)_{\overline{y}} \cong H^i\left(X_{\overline{y},\text{ét}}, \mathcal{O}_{X_{\overline{y},\text{ét}}}\right) \quad \textit{and} \quad \left(R^if_*\mathcal{O}_{X_{\text{ét}}}^\times\right)_{\overline{y}} \cong H^i\left(X_{\overline{y},\text{ét}}, \mathcal{O}_{X_{\overline{y},\text{ét}}}^\times\right).$$

Proof.* Without restriction, $\mathcal{I}$ has a final object 0. Since the assertion is local with respect to X_0, we may assume that X_0, and thus all X_α and X, are affine. Let $V \in X_{\text{ét}}$ be an affine étale X-scheme. Combining (i), (g), and (k) from Appendix A.1, we can write $V = V_\alpha \times_{X_\alpha} X$ for some affine étale X_α-scheme $V_\alpha \in X_{\alpha,\text{ét}}$. Put $V_\beta = V_\alpha \times_{X_\alpha} X_\beta$ for $\beta \in \mathcal{I}/\alpha$. Since $V \cong \lim_{\mathcal{I}/\alpha} V_\beta$, and everything is affine, we get $V = \operatorname{Spec}(\operatorname{colim}_{\mathcal{I}/\alpha}\Gamma(V_\beta, \mathcal{O}_{V_\beta}))$. Thus, we calculate

$$\Gamma(V, \mathcal{O}_{X_{\text{ét}}}) = \Gamma(V, \mathcal{O}_V) = \operatorname*{colim}_{\mathcal{I}/\alpha}\Gamma\left(V_\beta, \mathcal{O}_{V_\beta}\right) \cong \Gamma\left(V, \operatorname*{colim}_{\mathcal{I}/\alpha} \upsilon_\beta^*\mathcal{O}_{V_\beta}\right),$$

where $\upsilon_\beta\colon V \to V_\beta$ denotes the base change of π_β, so the right-most isomorphism follows from Proposition 2.4.12. This is almost what we want, except for a small technical argument to exchange υ_β for π_β. Observe that $\Gamma(V, \pi_\beta^*\mathcal{O}_{X_\beta}) \cong \Gamma(V, \upsilon_\beta^*\mathcal{O}_{V_\beta})$, because pullbacks along the étale morphisms $V \to X$ and $V_\beta \to X_\beta$ are just restrictions. Using this together with Remark 2.4.13 on the quasi-compact quasi-separated scheme V shows

$$\Gamma\left(V, \operatorname*{colim}_{\mathcal{I}/\alpha}\upsilon_\beta^*\mathcal{O}_{V_\beta}\right) \cong \operatorname*{colim}_{\mathcal{I}/\alpha}\Gamma\left(V, \upsilon_\beta^*\mathcal{O}_{V_\beta}\right) \cong \operatorname*{colim}_{\mathcal{I}/\alpha}\Gamma\left(V, \pi_\beta^*\mathcal{O}_{X_\beta}\right) \cong \Gamma\left(V, \operatorname*{colim}_{\mathcal{I}/\alpha}\pi_\beta^*\mathcal{O}_{X_\beta}\right).$$

This shows $\operatorname{colim}_{\mathcal{I}}\mathcal{O}_{X_{\alpha,\text{ét}}} \cong \mathcal{O}_{X_{\text{ét}}}$. The second assertion $\operatorname{colim}_{\mathcal{I}}\mathcal{O}_{X_{\alpha,\text{ét}}}^\times \cong \mathcal{O}_{X_{\text{ét}}}^\times$ is analogous. Finally, the additional assertions about stalks can be deduced from Proposition 2.4.12 in the same way as Corollary 2.4.15. $\square$

2.5. Constructible Sheaves

Unfortunately, before we can get into the fun business of proving the proper base change theorem, we need to discuss yet another technical notion. The reason is as follows: to construct ℓ-adic cohomology eventually, we would like to consider étale cohomology with coefficients in a constant sheaf associated to a finite abelian group. However, the category of such sheaves is very badly behaved: it is far from being abelian, and being a constant sheaf is neither preserved under pushforward nor under extension by zero. So the goal for this section is to construct a category of sheaves with better properties.

2.5.1. Noetherian Sheaves

2.5.1. Definition/Lemma. — An object x of an arbitrary category $\mathcal{C}$ is called *noetherian* if the following equivalent conditions are satisfied:

(a) Any ascending sequence $x_0 \hookrightarrow x_1 \hookrightarrow \dots \hookrightarrow x_n \hookrightarrow \dots \hookrightarrow x$ of subobjects of x stabilizes.

(b) Any set $\mathfrak{S}$ of subobjects of x has a *maximal* element $s^* \in \mathfrak{S}$ in the following sense: any monomorphism $s^* \hookrightarrow s$ of subobjects of x into an object $s \in \mathfrak{S}$ is already an isomorphism.

*Proof of equivalence**. (b) $\Rightarrow$ (a) is trivial, and (a) $\Rightarrow$ (b) follows from Zorn's lemma. □

2.5.2. Fact. — (a) *Noetherianness of objects of an abelian category is preserved under taking subobjects, quotients, extensions, and finite direct sums.*

(b) *If $\{j_k \colon Z_k \rightarrow X\}$, $k = 1, \dots, m$, is a finite jointly surjective set of locally closed immersions of schemes and $\mathcal{F} \in \mathrm{Ab}(X_{\text{ét}})$ is a sheaf such that all $j_k^*\mathcal{F}$ are noetherian in $\mathrm{Ab}(X_{k,\text{ét}})$, then $\mathcal{F}$ is noetherian.*

(c) *If Φ is a finite abelian group and X is a noetherian scheme, then the constant sheaf Φ_X on $X_{\text{ét}}$ is noetherian in $\mathrm{Ab}(X_{\text{ét}})$.*

(d) *If X is quasi-compact, then for $\mathcal{F} \in \mathrm{Ab}(X_{\text{ét}})$ to be noetherian is local with respect to $X_{\text{ét}}$. That is, if $\{U_i \rightarrow X\}_{i \in I}$ is an étale cover such that each $\mathcal{F}|_{U_{i,\text{ét}}}$ is noetherian, then X is noetherian as well.*

Proof. Part (a) can be proved as in the special case where the abelian category under consideration is Mod_R for some ring R. For (b), let $(\mathcal{F}_n)_{n \in \mathbb{N}}$ be an ascending sequence of subsheaves of $\mathcal{F}$. Since $j_k^* \colon \mathrm{Ab}(X_{\text{ét}}) \rightarrow \mathrm{Ab}(Z_{k,\text{ét}})$ is exact, $(j_k^*\mathcal{F}_n)_{n \in \mathbb{N}}$ is an ascending sequence of subsheaves of $j_k^*\mathcal{F}$, hence stabilizes for $n \geqslant N_i$. Since the morphisms j_k are jointly surjective, every geometric point $\overline{x}$ of X factors over some Z_k. Since $(\mathcal{F}_n)_{\overline{x}} = (j_k^*\mathcal{F}_n)_{\overline{x}}$ (see Remark 1.7.6), the sequence $((\mathcal{F}_n)_{\overline{x}})_{n \in \mathbb{N}}$ stabilizes for $n \geqslant N_k$. Thus $(\mathcal{F}_n)_{n \in \mathbb{N}}$ stabilizes for $n \geqslant \max N_k$, proving that $\mathcal{F}$ is noetherian.

In (c), we may assume that $\Phi = \mathbb{Z}/p\mathbb{Z}$ for some prime p, using (a) and the fact that every abelian group Φ has a filtration $0 = \Phi_0 \subseteq \Phi_1 \subseteq \dots \subseteq \Phi_n = \Phi$ such that all subquotients Φ_i/Φ_{i-1} are isomorphic to $\mathbb{Z}/p\mathbb{Z}$ for some prime p. Now let $\mathcal{F}$ be a subsheaf of the constant sheaf $\Phi_X = \mathbb{Z}/p\mathbb{Z}_X$. We claim:

($*$) *There is an open subset $U \subseteq X$ such that $\mathcal{F}_{\overline{x}} \neq 0$ iff the geometric point $\overline{x}$ factors over U. Moreover, if $V \in X_{\text{ét}}$ is a connected étale X-scheme, then*

$$\Gamma(V, \mathcal{F}) = \begin{cases} \mathbb{Z}/p\mathbb{Z} & \textit{if } V \rightarrow X \textit{ factors over } U \\ 0 & \textit{else} \end{cases}.$$

Since X is noetherian, hence has finitely many connected components, so we may reduce $(*)$ to the case that X is connected. In particular, $\Gamma(X, \mathbb{Z}/p\mathbb{Z}_X) = \mathbb{Z}/p\mathbb{Z}$. Thus, if $\overline{x}$ is a geometric point of X, then $\mathcal{F}_{\overline{x}}$ is non-zero iff it contains the image of the global section $1 \in \Gamma(X, \mathbb{Z}/p\mathbb{Z}_X)$. In this case, 1 is also contained in $\Gamma(V, \mathcal{F})$ for some étale neighbourhood $(V, \overline{v})$ of $\overline{x}$. Let $U_{\overline{x}}$ be the image of $V \to X$. Then $U_{\overline{x}}$ is open by Proposition 1.2.14. Moreover, $\Gamma(U_{\overline{x}}, \mathcal{F})$ is the equalizer of $\mathrm{pr}_1^*, \mathrm{pr}_2^* \colon \Gamma(V, \mathcal{F}) \rightrightarrows \Gamma(V \times_{U_{\overline{x}}} V, \mathcal{F})$ by the sheaf axiom, hence contains the image of $1 \in \Gamma(X, \mathbb{Z}/p\mathbb{Z}_X)$ as well.

Now let U be the union of all $U_{\overline{x}}$ as above. $\Gamma(U, \mathcal{F})$ contains 1 too, and it's straightforward to check that U has the property from $(*)$. Moreover, if $V \to X$ is étale and factors over U, then also $1 \in \Gamma(V, \mathcal{F})$, hence $\Gamma(V, \mathcal{F}) = \mathbb{Z}/p\mathbb{Z}$ if V is connected. Conversely, if $V \to X$ doesn't factor over U, then there is some geometric point $\overline{x}$ of X that factors over V but not over U, so that $\mathcal{F}_{\overline{x}} = 0$. Then $\Gamma(V, \mathcal{F})$ cannot contain the image of $1 \in \Gamma(X, \mathbb{Z}/p\mathbb{Z}_X)$, or 1 would also be contained in $\mathcal{F}_{\overline{x}}$. Then $\Gamma(V, \mathcal{F}) = 0$ if V is connected. This shows $(*)$.

In particular, $\mathcal{F}$ is uniquely determined by the open subset U from $(*)$. Now let $(\mathcal{F}_n)_{n \in \mathbb{N}}$ be a ascending sequence of subsheaves of $\mathbb{Z}/p\mathbb{Z}_X$. Then the corresponding sequence $(U_n)_{n \in \mathbb{N}}$ of open subsets of X is descending, hence stabilizes for $n \geqslant N$ as X is noetherian. Thus $(\mathcal{F}_n)_{n \in \mathbb{N}}$ stabilizes for $n \geqslant N$ and $\mathbb{Z}/p\mathbb{Z}$ is indeed noetherian.

For (d), if $\{U_i \to X\}_{i \in I}$ is as above, then we may find a finite subset $J \subseteq I$ such that $\{U_i \to X\}_{i \in J}$ is already an étale cover, because X is quasi-compact. If $(\mathcal{F}_n)_{n \in \mathbb{N}}$ is an ascending sequence of subsheaves, then each sequence $(\mathcal{F}_n|_{U_i, \text{ét}})_{n \in \mathbb{N}}$ for $i \in J$ stabilizes for $n \geqslant N_i$ by assumption. Hence the original sequence stabilizes for $n \geqslant \max N_i$. $\square$

2.5.3. Lemma. — *Let X be an arbitrary scheme. Let $\mathcal{F} \in \mathrm{Ab}(X_{\text{ét}})$ be noetherian and $\overline{\eta}$ a geometric point of X such that the closure $\overline{\{\eta\}}$ of the underlying point η contains an open neighbourhood of η. If $\mathcal{F}_{\overline{\eta}} = 0$, then there is an open neighbourhood $U \subseteq X$ of η such that the restriction $\mathcal{F}|_{U_{\text{ét}}} = 0$.*

Proof. As Robin pointed out, the proof becomes a bit clearer if we use the characterization from Definition/Lemma 2.5.1(b) rather than (a). Let

$$\mathfrak{S} = \{\mathcal{F}' \subseteq \mathcal{F} \mid \mathcal{F}'|_{U_{\text{ét}}} = 0 \text{ for some open neighbourhood } U \text{ of } \eta\} .$$

Since $\mathcal{F}$ is noetherian, $\mathfrak{S}$ contains a $\subseteq$-maximal element $\mathcal{F}^*$. Let U^* be the corresponding open neighbourhood of η. Without restriction, we may assume that U^* is contained in $\overline{\{\eta\}}$. If $\mathcal{F}|_{U^*_{\text{ét}}} \neq 0$, then there exists an étale U^*-scheme $V \in U^*_{\text{ét}}$ and a non-zero section $0 \neq s \in \Gamma(V, \mathcal{F})$. The geometric point $\overline{\eta}$ can be lifted to V because the image of V is open and contained in $\overline{\{\eta\}}$, hence contains the generic point η. However, $\mathcal{F}_{\overline{\eta}} = 0$, hence there must be an étale V-scheme W such that s vanishes in $\Gamma(W, \mathcal{F})$. Let U denote the image of $j \colon W \to X$ and let $\mathcal{F}_0$ be the subsheaf generated by the section s. That is, $\mathcal{F}_0$ is the image of $s \colon j_! \mathbb{Z}_W \to \mathcal{F}$ (which is adjoint to $s \colon \mathbb{Z}_W \to \mathcal{F}|_{W_{\text{ét}}}$). Then $\Gamma(W, \mathcal{F}_0) = 0$, hence also $\mathcal{F}_0|_{U_{\text{ét}}} = 0$ by the sheaf axiom and the fact that $\mathcal{F}_0$ is generated by s. In particular, $\mathcal{F}^* + \mathcal{F}_0$ (the sum is taken as subsheaves of $\mathcal{F}$) is an element of $\mathfrak{S}$, since it vanishes on $(U^* \cap U)_{\text{ét}}$. But also $\mathcal{F}^* \subsetneq (\mathcal{F}^* + \mathcal{F}_0)$ $\mathcal{F}^*|_{U^*_{\text{ét}}} = 0$ but s is a non-zero section over $U^*_{\text{ét}}$. This contradicts maximality of $\mathcal{F}^*$. $\square$

2.5.4. Counterexample*. — The original statement of Lemma 2.5.3 came without the condition that $\overline{\{\eta\}}$ contains an open neighbourhood of η. This version is wrong though. For example, suppose X is a connected noetherian scheme of dimension $\dim X \geqslant 1$, $x \in X$ is a closed point, and $\overline{x}$ a geometric point lifting x. Let $U = X \setminus \{x\}$ and $j \colon U \hookrightarrow X$ its open embedding. Then $j_!(\mathbb{Z}/p\mathbb{Z}_U)$ is a subsheaf of $\mathbb{Z}/p\mathbb{Z}_X$ (beware that this is only true for open embeddings, not for arbitrary étale $j\colon V \to X$) and satisfies $j_!(\mathbb{Z}/p\mathbb{Z}_U)_{\overline{x}} = 0$. However, $\Gamma(V, j_!(\mathbb{Z}/p\mathbb{Z}_U)) \neq 0$ for all $V \in X_{\text{ét}}$ whose image in X does not contain x.

2.5.2. LCC Sheaves and Constructible Sheaves

Lecture 22
20[th] Jan, 2020

2.5.5. Definition/Lemma. — Let X be a noetherian scheme. An object $\mathcal{F} \in \mathrm{Ab}(X_{\text{ét}})$ is *locally constant constructible* ("lcc" for short) if it satisfies the following equivalent conditions:

(a) The sheaf $\mathcal{F}$ is representable by a finite étale (commutative) group scheme F over X. That is, $\mathcal{F}$ is isomorphic to $\mathrm{Hom}_{\mathrm{Sch}/X}(-, F)$, which is an étale sheaf (even an fpqc sheaf) by Example 1.3.16.

(b) For every connected component Y of X there is a finite surjective étale morphism $Y' \to Y$ such that $\mathcal{F}|_{Y'_{\text{ét}}}$ is a constant sheaf given by a finite abelian group.

(c) The sieve $\mathcal{S} = \{U \in X_{\text{ét}} \mid \mathcal{F}|_{U_{\text{ét}}}$ is a constant sheaf, given by a finite abelian group$\}$ is a covering sieve.

Proof of equivalence. Throughout the proof we may assume that X is connected. We start with (a) $\Rightarrow$ (b). Let $\overline{x}$ be a chosen geometric point of X. If $\mathcal{F}$ is represented by the finite étale X-group scheme F, then F is given by a finite group $\Phi = \mathrm{Fib}_{\overline{x}}(F) = \mathcal{F}_{\overline{x}}$ with a continuous $\pi_1^{\text{ét}}(X, \overline{x})$-action. Put $K = \ker(\pi_1^{\text{ét}}(X, \overline{x}) \to \mathrm{Aut}(\Phi))$ and let $X' \to X$ be the Galois covering with Galois group $G = \pi_1^{\text{ét}}(X, \overline{x})/K$ (if these arguments seem mysterious to you, have a look at Theorem 1.5.10(a) again). Then $\mathcal{F}|_{X'_{\text{ét}}}$ is constant, given by Φ. Indeed, for $\mathcal{F}|_{X'_{\text{ét}}}$ to be constant it suffices to check $\Phi \times_X X' \cong F \times_X X'$ (this is not hard to see). Since the fibre functor $\mathrm{Fib}_{\overline{x}}\colon \mathrm{F\acute{E}t}/X \xrightarrow{\sim} \pi_1^{\text{ét}}(X, \overline{x})\text{-FSet}$ is an equivalence, it suffices to construct a $\pi_1^{\text{ét}}(X, \overline{x})$-equivariant bijection

$$\coprod_{\varphi \in \Phi} G \xrightarrow{\sim} \Phi \times G$$

(both sides are equal to $\Phi \times G$ as sets, but the $\pi_1^{\text{ét}}(X, \overline{x})$-action is the diagonal action on the right-hand side and induced by G on the left-hand side). By construction, G comes with a morphism $G \to \mathrm{Aut}(\Phi)$, and then $(\varphi, g) \mapsto (g\varphi, g)$ gives the required $\pi_1^{\text{ét}}(X, \overline{x})$-equivariant bijection. This finishes the proof of (a) $\Rightarrow$ (b).

The implication (b) $\Rightarrow$ (c) is trivial. For (c) $\Rightarrow$ (a) we use faithfully flat descent (with some care) as follows: choose an étale cover $\{U_i \to X\}_{i \in I}$ such that $\mathcal{F}|_{U_{i,\text{ét}}}$ is constant with value Φ_i. Put $F_i = \Phi_i \times U_i$, equipped with the obvious group scheme structure. It's straightforward to check $\mathcal{F}|_{U_{i,\text{ét}}} \cong \mathrm{Hom}_{\mathrm{Sch}/U_i}(-, F_i)$. By Yoneda's lemma, the F_i form a descent datum for surjective finite étale $F \to X$. It is clear that F is a commutative group scheme. Indeed, being a group scheme over X can be completely described in terms of morphisms between X, F, $F \times_X F$, and $F \times_X F \times_X F$. Since faithfully flat descent is an assertion about an equivalence of categories, these morphisms can be obtained from the corresponding morphisms for the F_i. $\square$

2.5.6. Fact. — *The property of being lcc is preserved under pushforward along finite étale morphisms between noetherian schemes.*

Proof.* Let $f\colon X' \to X$ be a finite étale between noetherian schemes, and $\mathcal{F}$ an lcc sheaf on $X'_{\text{ét}}$. The property of being lcc is local with respect to the étale topology (this is easy to see from Definition/Lemma 2.5.5(c)). Since étale coverings are étale-locally split by

Lemma* 1.5.5(a), we may assume that X is connected, $X' = S \times X$ for some finite discrete S, and $f = \mathrm{pr}_2$ is the projection to X. Now let $\{U_i \to X'\}_{i \in I}$ be an étale cover of X' such that each $\mathcal{F}|_{U_{i,\text{ét}}}$ is a constant sheaf with value Φ_i. Since $X' \to X$ is surjective (or $X' = \emptyset$ as X is connected, but this case is trivial anyway), we see that $\{U_i \to X\}_{i \in I}$ is an étale cover of X too. Using $X' = S \times X$, it's straightforward to check that $f_*\mathcal{F}|_{U_{i,\text{ét}}}$ is a constant sheaf with value $\Phi_i^{\oplus S}$. This proves that $f_*\mathcal{F}$ is lcc by Definition/Lemma 2.5.5. $\square$

2.5.7. Lemma. — *Let X be noetherian. If $\mathcal{F} \in \mathrm{Ab}(X_{\text{ét}})$ is a noetherian torsion sheaf, then there is an non-empty open subset $\emptyset \neq U \subseteq X$ such that $\mathcal{F}|_U$ is lcc.*

2.5.8. Remark*. — An étale sheaf being *torsion* could reasonably mean one of the following three conditions:

(a) There is an integer $N \neq 0$ such that $N\mathcal{F} = 0$.

(b) For every section s of $\mathcal{F}$ there is an integer $N \neq 0$ such that $Ns = 0$.

(c) For every geometric point $\overline{x}$ of X and all $s \in \mathcal{F}_{\overline{x}}$ there is an integer $N \neq 0$ such that $Ns = 0$.

Clearly (c) is the weakest condition. Since X is noetherian, every $V \in X_{\text{ét}}$ is quasi-compact, which implies $(c) \Rightarrow (b)$ by a standard argument. Finally $(b) \Rightarrow (a)$ follows immediately from $\mathcal{F}$ being noetherian. So there's no ambiguity.

Proof of Lemma 2.5.7. First note that it suffices to find an étale X-scheme V such that $\mathcal{F}|_{V_{\text{ét}}}$ is lcc. Indeed, if U denotes the image of the étale morphism $V \to X$, then U is Zariski-open and by Definition/Lemma 2.5.5(c) we see immediately that $\mathcal{F}|_{U_{\text{ét}}}$ is lcc as well, as claimed. In particular, we are free to replace X by any $V \in X_{\text{ét}}$.

Let $\overline{\eta}$ be a geometric point of X whose underlying point η is the generic point of an irreducible component of X. Let $\Phi = \mathcal{F}_{\overline{\eta}}$. Our goal is to show that

(1) *Φ is a finite abelian group,*

(2) *and after replacing X by some "suitably small" étale neighbourhood of $\overline{\eta}$, there exists a morphism $\sigma\colon \Phi_X \to \mathcal{F}$ of sheaves inducing an isomorphism on stalks at $\overline{\eta}$.*

In (2), there's actually a technical details that is easy to miss: if we replace X by some étale neighbourhood $(X', \overline{\eta}')$ of $\overline{\eta}$, we must ensure that the new underlying point η' is still the generic point of some irreducible component of X' (otherwise we couldn't apply Lemma 2.5.3). This is true because $0 = \dim \mathcal{O}_{X,\eta} = \dim \mathcal{O}_{X',\eta'}$ by Remark* 1.4.15. So there's nothing to worry about.

Let's first see how (1) and (2) finish the proof. Note that Φ_X is noetherian by (1) and Fact 2.5.2(c), hence $\ker \sigma$ and $\operatorname{coker} \sigma$ are noetherian by Fact 2.5.2(a). Since both vanish at $\overline{\eta}$ by (2), Lemma 2.5.3 shows that $\ker \sigma$ and $\operatorname{coker} \sigma$ vanish on $U_{\text{ét}}$ for some Zariski-open neighbourhood U of η. Hence σ is an isomorphism on $U_{\text{ét}}$ and we are done.

To show (1), we claim that there exist finitely many étale neighbourhoods V_i, $i = 1, \dots, n$, of $\overline{\eta}$, and sections $s_i \in \Gamma(V_i, \mathcal{F})$, such that the images of s_i in $\mathcal{F}_{\overline{\eta}}$ generate this abelian group. Indeed, otherwise we could find an infinite sequence $(V_i, s_i)_{i \in \mathbb{N}}$ such that each s_j is not contained in the subgroup of $\mathcal{F}_{\overline{\eta}}$ generated by $s_1, \dots, s_{j-1}$. Let $\mathcal{F}_j$ be the subsheaf of $\mathcal{F}$ generated by the sections $s_1, \dots, s_j$, as in the proof of Lemma 2.5.3. Then $\mathcal{F}_1 \subsetneq \mathcal{F}_2 \subsetneq \cdots$ is an infinite properly ascending sequence of subobjects of $\mathcal{F}$ (as can be seen from the stalks at $\overline{\eta}$), contradicting the assumption that $\mathcal{F}$ be noetherian. This shows (1): indeed, since $\mathcal{F}$ is torsion, $\Phi = \mathcal{F}_{\overline{\eta}}$ is a finitely generated torsion abelian group, hence finite.

Choose N such that $N\mathcal{F} = 0$ and let K be the kernel of the map $(\mathbb{Z}/N\mathbb{Z})^{\oplus n} \to \Phi$ induced by $s_1, \dots, s_n$. Since there are only finitely many V_i and the category of étale neighbourhoods is filtered (Fact 1.6.2), we may assume $V_1 = \dots = V_n = V$. As argued above, we can put $V = X$ without restriction. Then $s_1, \dots, s_n \in \Gamma(X, \mathcal{F})$ are global sections and thus define a map $(\mathbb{Z}/N\mathbb{Z})_X^{\oplus n} \to \mathcal{F}$. To prove (2), it suffices to show that $K_X \hookrightarrow (\mathbb{Z}/N\mathbb{Z})_X^{\oplus n} \to \mathcal{F}$ vanishes after replacing X by a sufficiently small étale neighbourhood of $\overline{\eta}$. Note that the image of K_X in $\mathcal{F}$ is noetherian by Fact 2.5.2(a), (c), and vanishes at $\overline{\eta}$ by construction, hence it vanishes over $U_{\text{ét}}$ for some Zariski-neighbourhood U of η, using Lemma 2.5.3. Therefore replacing X by U does the trick. $\square$

2.5.9. Definition. — Let X be a noetherian scheme. We call a sheaf $\mathcal{F} \in \operatorname{Ab}(X_{\text{ét}})$ *constructible* if it satisfies the following equivalent (by Proposition 2.5.20 and the discussion below) conditions:

(c_1) There is an ascending sequence $\emptyset = U_0 \subseteq U_1 \subseteq \dots \subseteq U_n = X$ (a *stratification*) of open subsets such that the restriction $\mathcal{F}|_{(U_k \setminus U_{k-1})_{\text{ét}}}$ to the reduced closed subscheme $U_k \setminus U_{k-1}$ of U_k is lcc for all $k = 1, \dots, n$.

(c_1^-) There is a finite jointly surjective family $\{j_k \colon X_k \hookrightarrow X\}$, $k = 1, \dots, n$, of locally closed immersions such that all $j_k^*\mathcal{F}$ are lcc.

(c_2) There are finitely many finite morphisms $\{p_k \colon X_k \to X\}$, $k = 1, \dots, n$, together with finite abelian groups Φ_k, such that there exists a monomorphism

$$\mathcal{F} \hookrightarrow \bigoplus_{k=1}^{n} p_{k,*}\Phi_k \,.$$

(c_3) $\mathcal{F}$ is noetherian and torsion.

The proof of equivalence of the four conditions is by no means easy and covers the rest of the section. We will chop it up into a series of lemmas and facts.

2.5.10. Lemma. — *Conditions (c_1) and (c_3) are equivalent. In particular, a locally constant constructible sheaf is noetherian.*

Proof. We start with $(c_1) \Rightarrow (c_3)$. Let $\emptyset = U_0 \subseteq U_1 \subseteq \dots \subseteq U_n = X$ be as in (c_1). Put $Z_k = U_k \setminus U_{k-1}$, equipped with its reduced subscheme structure inherited from U_k, and let $j_k \colon Z_k \hookrightarrow X$ be the corresponding locally closed immersion. Then all $j_k^*\mathcal{F}$ are locally constant constructible, hence noetherian by Fact 2.5.2(c), (d). Thus Fact 2.5.2(b) shows that $\mathcal{F}$ is noetherian. To show that $\mathcal{F}$ is torsion, it suffices to check that every stalk is torsion (Remark* 2.5.8). But stalks are preserved under j_k^* and every $j_k^*\mathcal{F}$ is lcc, hence its stalks are torsion. This ultimately proves that $\mathcal{F}$ is torsion too.

It remains to show $(c_3) \Rightarrow (c_1)$. Let $\mathcal{F}$ be noetherian torsion sheaf. Put $U_0 = \emptyset$, choose U_1 as in Lemma 2.5.7, and let $Z = X \setminus U_1$ (equipped with its reduced closed subscheme structure). Let $i \colon Z \hookrightarrow X$ denote the corresponding closed immersion. Once we show that $i^*\mathcal{F}$ is a noetherian torsion sheaf again, the assertion will follow immediately by noetherian induction. If $\mathcal{G}$ is any sheaf on $Z_{\text{ét}}$ and $\overline{x}$ a geometric point of X, then

$$(i_*\mathcal{G})_{\overline{x}} = \begin{cases} \mathcal{G}_{\overline{x}} & \text{if } \overline{x} \text{ factors over } Z \\ 0 & \text{else} \end{cases} ,$$

as follows from Corollary 1.7.11. In particular, since i^* preserve stalks, we get that the canonical map $\mathcal{F} \to i_* i^* \mathcal{F}$ is an epimorphism. Thus $i_* i^* \mathcal{F}$ is noetherian again by Fact 2.5.2(a). Moreover, if $(\mathcal{G}_n)_{n \in \mathbb{N}}$ is a strictly ascending sequence of subsheaves of $i^* \mathcal{F}$, then our calculation of stalks shows that $(i_* \mathcal{G}_n)_{n \in \mathbb{N}}$ is a strictly ascending sequence of subsheaves of $i_* i^* \mathcal{F}$, contradicting the fact that the latter is noetherian. Thus $i^* \mathcal{F}$ must be noetherian too. It's torsion for trivial reasons, whence the proof is complete. □

2.5.11. Fact. — (a) *Condition (c_2) is stable under taking subsheaves, finite direct sums, pushforward p_* along finite morphisms p, and arbitrary pullbacks f^*. Moreover, every lcc sheaf satisfies (c_2).*

(b) *If $\{i_k \colon X_k \hookrightarrow X\}$, $k = 1, \dots, n$, are the irreducible components of X and all restrictions $i_k^* \mathcal{F}$ satisfy (c_2), then $\mathcal{F}$ satisfies (c_2).*

Proof. In (a), stability under taking subobjects is trivial. The same goes for finite direct sums. For the other two stability assertions, let X be noetherian, $\mathcal{F}$ a sheaf on $X_{\text{ét}}$ satisfying (c_2), and let $(p_k \colon X_i \to X, \Phi_k)$ be as in (c_2). If $p \colon X \to X'$ is a finite morphism of noetherian schemes, then the $p \circ p_k \colon X_k \to X'$ are still finite morphisms and $p_* \mathcal{F} \hookrightarrow \bigoplus_{i=1}^n (p \circ p_k)_* \Phi_k$ is still a monomorphism. Similarly, let $f \colon X' \to X$ be an arbitrary morphism between noetherian schemes. Put $X'_k = X' \times_X X_k$ and let $f_k \colon X'_k \to X_k$ and $p'_k \colon X'_k \to X'$ be the base changes of f and p_k. We claim:

($*$) *The canonical morphism $f^* p_{k,*} \Phi_k \xrightarrow{\sim} p'_{k,*} f_k^* \Phi_k$ is an isomorphism.*

In fact, ($*$) is a special case of a much more general assertion about finite base change of étale sheaves; see Fact* 2.5.12(a) below.

Now ($*$) immediately implies that $f^* \mathcal{F}$ satisfies (c_2) again: indeed, since f^* is exact, the morphism $f^* \mathcal{F} \hookrightarrow \bigoplus_{i=1}^n f^* p_{k,*} \Phi_k$ is still injective, and by ($*$) the right-hand side maps injectively into $\bigoplus_{k=1}^n p'_{k,*} \Phi_k$.

It remains to show that every lcc sheaf satisfies (c_2). Let X'_k be étale coverings of the connected components of X as in Definition/Lemma 2.5.5(b). Then $p_k \colon X'_k \to X$ is still finite and $p_k^* \mathcal{F}$ is a constant sheaf by assumption. Moreover, an inspection of stalks, using Corollary 1.7.11 as in the proof of ($*$), shows that the canonical morphism $\mathcal{F} \hookrightarrow \bigoplus_{k=1}^n p_{k,*} q_k^* \mathcal{F}$ is injective. This finishes the proof of (a). In the situation of (b), a similar argument shows that $\mathcal{F} \hookrightarrow \bigoplus_{k=1}^n i_{k,*} i_k^* \mathcal{F}$ is injective, and (b) follows at once. □

2.5.12. Fact*. — *Consider a pullback square*

$$\begin{tikzcd} Y' \arrow[r, "j'"] \arrow[d, "p'"'] \arrow[dr, phantom, "\lrcorner"] & Y \arrow[d, "p"] \\ X' \arrow[r, "j"] & X \end{tikzcd}$$

of arbitrary (in particular, not necessarily noetherian) schemes.

(a) *If p, and thus p', are finite, then the base change morphism $j^* p_* \mathcal{F} \xrightarrow{\sim} p'_* j'^* \mathcal{F}$ is a natural isomorphism for all sheaves $\mathcal{F} \in \mathrm{Ab}(Y_{\text{ét}})$.*

(b) *If j, and thus j', are étale, then the other base change morphism $j'_! p'^* \mathcal{F} \xrightarrow{\sim} p^* j_! \mathcal{F}$ is a natural isomorphism for all sheaves $\mathcal{F} \in \mathrm{Ab}(X'_{\text{ét}})$.*

(c) *If both cases occur simultaneously, i.e., if p, p' are finite and j, j' are étale, then there is a natural isomorphism $p_* j'_! \mathcal{F} \xrightarrow{\sim} j_! p'_* \mathcal{F}$ for all $\mathcal{F} \in \mathrm{Ab}(Y'_{\text{ét}})$.*

*Proof**. Part (a). It suffices to check this on stalks. Let $\overline{x}'$ be a geometric point of X' and $\overline{x} = j(\overline{x}')$. We may assume that $\kappa(\overline{x}') = \kappa(\overline{x})$ are algebraically closed. Using that stalks are preserved under pullback, together with Corollary 1.7.11 and Remark* 1.7.12(3), we get

$$(j^* p_* \mathcal{F})_{\overline{x}'} = \prod_{\overline{y}} \mathcal{F}_{\overline{y}} \quad \text{and} \quad (p'_* j'^* \mathcal{F})_{\overline{x}'} = \prod_{\overline{y}'} \mathcal{F}_{j'(\overline{y}')}\,,$$

where $\{\overline{y}\}$ is the set of lifts of $\overline{x} \colon \operatorname{Spec}\kappa(\overline{x}) \to X$ to Y, and $\{\overline{y}'\}$ is the set of lifts of $\overline{x}' \colon \operatorname{Spec}\kappa(\overline{x}') \to X'$ to Y'. Thus it suffices to show that $j' \colon \{\overline{y}'\} \xrightarrow{\sim} \{\overline{y}\}$ is a bijection. But this follows immediately from $Y' = X' \times_X Y$ and the universal property of fibre products. This shows (a).

Part (b) is similar. Let $\overline{y}$ be a geometric point of Y and $\overline{x} = p(\overline{y})$ (this time $\kappa(\overline{y})$ doesn't need to be algebraically closed). We can express $(j'_! p'^* \mathcal{F})_{\overline{y}}$ and $(p^* j_! \mathcal{F})_{\overline{y}}$ using Proposition 2.1.3, and again the assertion reduces to the fact that $p' \colon \{\overline{y}'\} \xrightarrow{\sim} \{\overline{x}'\}$ is bijective, where the left-hand side is the set of lifts of $\overline{y}$ to Y' and the right-hand side is the set of lifts of $\overline{x}$ to X'. This is clear again.

Part (c). There are two ways to construct this isomorphism. First, we can start with the unit $\mathrm{id} \to j'^* j_!$ of the $j'_!$-j'^*-adjunction. Applying p'_* and (a) gives a morphism $p'_* \mathcal{F} \to p'_* j'^* j'_! \mathcal{F} \cong j^* p_* p'_! \mathcal{F}$. Via the $j_!$-j^* adjunction we get $j_! p'_* \mathcal{F} \to p_* j'_! \mathcal{F}$ as required. Alternatively, we could start with the counit $p'^* p'_* \to \mathrm{id}$ and use (b). Investigating stalks as before, we see that both constructions are mutually inverse, and (c) follows. $\square$

2.5.13. Remark. — If we could generalize Fact 2.5.11(a) to show that condition (c_2) is preserved under pushforward q_* along *quasi-finite* morphisms q rather than just finite morphisms, then $(c_1) \Rightarrow (c_2)$ would easily follow. However, the straightforward strategy to prove this fails: if we factor a quasi-finite morphism $q \colon Y \to X$ as

$$\begin{tikzcd} & \overline{Y} \arrow[dr, "\overline{q}"] & \\ Y \arrow[ur, hook, "j"] \arrow[rr, "q"] & & X \end{tikzcd}$$

according to Zariski's main theorem, such that $\overline{q}$ is finite and j is an open embedding, then $j_* \Phi_Y$ may fail to be a constant sheaf again (see Counterexample 2.5.15 below). This is a major obstacle in the proof, and will eventually force us to do (slightly) messy reductions to universally Japanese schemes.

2.5.14. Lemma. — *Let X be a noetherian normal scheme and $j \colon U \hookrightarrow X$ an open embedding. If Φ is any group or even just a set, then $j_* \Phi_U \cong \Phi_X$ holds as sheaves of groups or sets on $X_{\text{ét}}$.*

Proof. Without loss of generality let X be connected, hence irreducible (as X is noetherian). Let π_0 denote the connected components of a scheme. Since $\Gamma(V, \Phi_X) = \Phi^{\oplus \pi_0(V)}$ holds for all $V \in X_{\text{ét}}$, what we need to show is $\pi_0(U \times_X V) \cong \pi_0(V)$ for all V. Without loss of generality let V by connected. Using Serre's normality criterion and Lemma* A.2.4, we see that V is normal again, hence irreducible if it is connected. In particular, all open subschemes of V are irreducible again. This shows that $U \times_X V$ and V are both connected and we are done. $\square$

2.5.15. Counterexample. — The assumption that X be normal may not be dropped in Lemma 2.5.14. For example, let X be the scheme from Example 1.5.21 and $X_2 \to X$

the étale covering of degree 2 constructed there. If $U = X \setminus \{[0] = [\infty]\}$, then $U \times_X X_2$ has two connected components, whereas X_2 is connected. In particular, $\Gamma(X_2, \Phi_{X_2}) \cong \Phi$, but $\Gamma(X_2, j_*\Phi_U) \cong \Gamma(U \times_X X_2, \Phi_U) \cong \Phi^{\oplus 2}$. This shows that $j_*\Phi_U$ is no constant sheaf, providing a counterexample as promised in Remark 2.5.13.

2.5.16. Lemma. — *Let $q\colon Y \to X$ be a quasi-finite quasi-compact morphism, where X is a noetherian and universally Japanese scheme. Then for any finite group Φ, the sheaf $q_*\Phi_Y$ on X satisfies (c_2).*

Proof. We never really defined what it means for a scheme to be universally Japanese, but it's really just the obvious thing, i.e., that every point $x \in X$ has an affine open universally Japanese neighbourhood in the sense of Definition 1.6.20(c) (also see [Stacks, Tag 033S]). The proof proceeds in several reduction steps, to eventually end up in a situation where Lemma 2.5.14 can be applied. The assumption that X is universally Japanese is needed to ensure that normalization behaves nicely.

Step 1. We reduce to the case where X and Y are integral. Let $X_1, \dots, X_n$ be the irreducible components of X (equipped with their reduced closed subscheme structure) and put $Y_k = Y \times_X X_k$. Let $q_k\colon Y_k \to X$ and $i_k\colon Y_k \hookrightarrow Y$ be the canonical morphisms, so that $q_k = q \circ i_k$. As argued in the proof of Fact 2.5.11, $\Phi_Y \hookrightarrow \bigoplus_{k=1}^n i_{k,*}\Phi_{Y_k}$ is injective. Hence so is $q_*\Phi_Y \hookrightarrow \bigoplus_{k=1}^n q_{k,*}\Phi_{Y_k}$. Hence it suffices to treat the case where $X = X_k$ is irreducible. We should also remark that X_k is still universally Japanese, since being universally Japanese is preserved under morphisms of finite type.

Similarly, let $Y_1, \dots, Y_m$ be the irreducible components of Y, and let $q_k\colon Y_k \to X$ be the restriction of q. As before, $q_*\Phi_Y \hookrightarrow \bigoplus_{k=1}^m q_{k,*}\Phi_{Y_k}$, so without restriction $Y = Y_k$ is irreducible. Thus, X and Y are irreducible. Moreover, if X^{red} denotes the reduction of X, then $X_{\text{ét}}$ and $X^{\mathrm{red}}_{\text{ét}}$ are equivalent as sites by Proposition 1.4.20 (here we use that the nilradical $\mathrm{nil}(\mathcal{O}_X)$ is nilpotent as X is noetherian). Thus replacing X by X^{red} does not affect the category $\mathrm{Ab}(X_{\text{ét}})$. The same is true for replacing Y by Y^{red}. Therefore we may assume that X and Y are integral.

Step 2. We reduce to the case where Y is normal. Since X is universally Japanese and Y is of finite type over X, Y is universally Japanese too. Hence the normalization $p\colon \widetilde{Y} \to Y$ is a finite surjective morphism. Then the canonical morphism $\Phi_Y \hookrightarrow p_*p^*\Phi_Y = p_*\Phi_{\widetilde{Y}}$ is injective, which can be seen by an investigation of stalks via Corollary 1.7.11. Thus we may replace Y by $\widetilde{Y}$ without restriction.

Step 3. Now we get to business and start with the actual proof. Our goal is to show that there exists a factorization

$$\begin{tikzcd} & \overline{Y} \arrow[dr, "\overline{q}"] & \\ Y \arrow[ur, hook, "j"] \arrow[rr, "q"] & & X \end{tikzcd}$$

such that j is an open embedding (up to some technical detail, see below), $\overline{q}$ is finite, and $\overline{Y}$ is a *normal* scheme, i.e., we want to invoke a somewhat stronger version of Zariski's main theorem (the upside is that this version is actually explicit). Let $\overline{Y}$ be the *normalization of X in Y*. This means the following: let L be the function field of Y, i.e., the stalk at the generic point of the integral scheme Y. Let $\mathcal{A}$ be the subsheaf of the constant sheaf L_X on X, consisting of those sections that are integral over $\mathcal{O}_X$. Then we put $\overline{Y} = \underline{\mathrm{Spec}}\,\mathcal{A}$. Clearly $\overline{Y}$ is a connected normal scheme. Moreover, $\overline{q}\colon \overline{Y} \to X$ is a finite morphism. This follows since X is universally Japanese, L is a finite extension of the function field K, because the base change $Y \times_X \operatorname{Spec} K \to \operatorname{Spec} K$ is quasi-finite with target a field, hence finite.[2]

Since Y is normal itself, it factors over $j\colon Y \to \overline{Y}$. Applying Zariski's main theorem in the version of [Stacks, Tag 00Q9] does *not quite* show that j is an open embedding, but at least we can find a finite affine Zariski-open cover $Y = \bigcup_{k=1^n} U_k$ such that each $j_k = j|_{U_k}\colon U_k \hookrightarrow \overline{Y}$ is an open embedding. Thus we obtain "almost" the desired factorization of q. But in fact, what we got is enough: $j_*\Phi_Y \hookrightarrow \bigoplus_{k=1}^n j_{k,*}\Phi_{U_k}$ is injective by the sheaf axiom, and $j_{k,*}\Phi_{U_k} = \Phi_{\overline{Y}}$ by Lemma 2.5.14, hence $q_*\Phi_Y \hookrightarrow \bigoplus_{k=1}^n \overline{q}_*\Phi_{\overline{Y}}$. The right-hand side satisfies (c_2) straight by definition and we win. $\square$

2.5.17. Lemma. — *Let $q\colon Y \to X$ be a quasi-finite quasi-compact morphisms, where X is a universally Japanese noetherian scheme. Then condition (c_2) is preserved under the pushforward q_*. In particular, (c_1) implies (c_2) in this case.*

Proof. Suppose $\mathcal{F}$ satisfies (c_2) and let $\mathcal{F} \hookrightarrow \bigoplus_{k=1}^n p_{k,*}(\Phi_k)_{X_s}$ be the associated monomorphism. Since pushforward is left-exact, $q_*\mathcal{F} \hookrightarrow (q \circ p_k)_*(\Phi_k)_{X_k}$ is still injective, and since $q \circ p_k$ is quasi-finite, the summands on the right-hand side all satisfy (c_2) by Lemma 2.5.16. Hence so does $q_*\mathcal{F}$ by Fact 2.5.11(a), proving the first assertion.

For the second one, let $\emptyset = U_0 \subseteq U_1 \subseteq \cdots \subseteq U_n = X$ be a stratification as in (c_1). Let $Z_k = U_k \setminus U_{k-1}$, carrying the the reduced closed subscheme structure inherited from U_k, and let $i_k\colon Z_k \hookrightarrow X$ denote the corresponding locally closed immersion. We claim:

$(*)$ *The canonical morphism $\mathcal{F} \hookrightarrow \bigoplus_{k=1}^n i_{k,*}i_k^*\mathcal{F}$ is injective.*

This was just claimed in the lecture, but is not quite trivial, so we give it a proper proof. As usual, injectivity can be checked on stalks. Let $\overline{x}$ be a geometric point of X and choose k such that $\overline{x}$ is contained in Z_k. Let Z be the connected component of Z_k containing $\overline{x}$, $Z' = Z_k \setminus Z$, and $i\colon Z \hookrightarrow X$, $i'\colon Z' \hookrightarrow X$ the respective restrictions of i_k. Then $i_{k,*}i_k^*\mathcal{F} \cong i_*i^*\mathcal{F} \oplus i'_*i'^*\mathcal{F}$, so it suffices to show that $\mathcal{F} \to i_*i^*\mathcal{F}$ is injective at $\overline{x}$.

Since $i^*\mathcal{F}$ is lcc and Z is connected, we find a surjective finite étale morphism $V \to Z$ such that $i^*\mathcal{F}|_{V_{\text{ét}}}$ is constant, given by a finite abelian group Φ (using Definition/Lemma 2.5.5(c)). In particular, $\mathcal{F}_{\overline{x}} \cong (i^*\mathcal{F})_{\overline{x}} \cong \Phi$. So we must show that $\Phi \to (i_*i^*\mathcal{F})_{\overline{x}}$ is injective. Using Corollary 1.7.9, we find

$$(i_*i^*\mathcal{F})_{\overline{x}} \cong \Gamma\left(Z_{\overline{x}}, \mathrm{pr}_1^*\, i^*\mathcal{F}\right),$$

where $Z_{\overline{x}} = Z \times_X \operatorname{Spec} \mathcal{O}_{X_{\text{ét}},\overline{x}}$. We wish to compute the right-hand side via the étale cover $\{V \times_Z Z_{\overline{x}} \to Z_{\overline{x}}\}$. Note that the pullback of $\mathrm{pr}_1^*\, i^*\mathcal{F}$ to $V \times_Z Z_{\overline{x}}$ is the constant Φ-valued sheaf, because it coincides with the pullback of $i^*\mathcal{F}$ along $V \times_Z Z_{\overline{x}} \to V \to Z$. Thus, the sheaf axiom yields

$$\Gamma\left(Z_{\overline{x}}, \mathrm{pr}_1^*\, i^*\mathcal{F}\right) \cong \mathrm{Eq}\left(\Gamma\left(V \times_Z Z_{\overline{x}}, \Phi\right) \underset{\mathrm{pr}_2^*}{\overset{\mathrm{pr}_1^*}{\rightrightarrows}} \Gamma\left(V \times_Z V \times_Z Z_{\overline{x}}, \Phi\right)\right).$$

Note that $V \times_Z Z_{\overline{x}}$ is non-empty, because $V \to Z$ is surjective and $Z_{\overline{x}} \neq \emptyset$ (because it contains $\overline{x}$). Hence $\Gamma(V \times_Z Z_{\overline{x}}, \Phi) \cong \Phi^{\oplus m}$ is a finite non-empty direct sum of copies of Φ, according to the number m of connected components of $V \times_Z Z_{\overline{x}}$. Clearly the diagonal morphism $\Delta\colon \Phi \hookrightarrow \Phi^{\oplus m}$ equalizes pr_1^* and pr_2^*. This shows that $\Phi \hookrightarrow \Gamma(Z_{\overline{x}}, \mathrm{pr}_1^*\, i^*\mathcal{F})$ is injective, as required. We thus proved $(*)$.

[2]It would have sufficed to have L/K a finitely generated field extension (which we get for free as Y has finite type over X), since in that case the algebraic closure of K in L is always finite over L (this fact is not so well-known and not entirely trivial either; see [Bou90, §14.7 Corollary 1]).

The rest is easy: each $i_k^*\mathcal{F}$ is lcc, hence satisfies (c_2) by Fact 2.5.11(a), hence $i_{k,*}i_k^*\mathcal{F}$ satisfies (c_2) because $i_{k,*}$ is quasi-finite. Thus $\mathcal{F}$ satisfies (c_2) as well. □

2.5.18. Remark*. — In the lecture it was claimed that q_* preserves (c_2) for arbitrary X, so Lemma 2.5.17 (and later Proposition 2.5.20(b)) would be true without the hypothesis that X is universally Japanese. I'm somewhat sceptical though. The crucial part in the proof presented in the lecture was to show that for every finite abelian group Φ, the pushforward $q_*\Phi_Y$ satisfies property (c_2) again. To deduce the general case, we wrote q as a base change

$$\begin{tikzcd} Y \arrow[r, "q"] \arrow[d, "\pi_Y"'] \arrow[dr, phantom, "\lrcorner"] & X \arrow[d, "\pi_X"] \\ Y' \arrow[r, "q'"] & X' \end{tikzcd},$$

in which $q'\colon Y' \to X'$ is a quasi-finite morphism between schemes of finite type over $\mathbb{Z}$. This is always possible, combining (d), (i), and (l) from Appendix A.1. Since schemes of finite type over $\mathbb{Z}$ are universally Japanese, we know that $q'_*\Phi_{Y'}$ satisfies (c_2). The problem, however, is that the base change morphism

$$\pi_X^* q'_*\Phi_{Y'} \longrightarrow q_*\pi_Y^*\Phi_{Y'} \cong q_*\Phi_Y$$

is not necessarily an isomorphism, unless q is finite (Fact* 2.5.12). And I don't see either why this should be true after choosing X' and Y' "large enough". In fact, I bet there's an open embedding $j\colon U \hookrightarrow X$ of noetherian schemes such that $j_*\Phi_U$ is not even a noetherian sheaf on X; but I'm no Nagata, so I didn't succeed in constructing one .

2.5.19. Lemma*. — *Despite Remark* 2.5.18, it's still true that (c_1) implies (c_2) for arbitrary noetherian schemes X. Moreover, if $j\colon V \hookrightarrow X$ is étale and Φ is a finite abelian group, then $j_!\Phi_V$ satisfies (c_1).*

Proof.* We start with the second assertion. We may cover V by finitely many Zariski-open subsets $V_1, \dots, V_n$, such that each $j_k = j|_{V_k}\colon V_k \to X$ is étale and separated. Using the universal property of extension by zero, we get a canonical morphism $\bigoplus_{k=1}^n j_{k,!}\Phi_{V_k} \to j_!\Phi_V$. Since the V_k cover V, an inspection of stalks (using Proposition 2.1.3) shows that this morphism is an epimorphism. In particular, it suffices to prove that each $j_{k,!}\Phi_{V_k}$ satisfies (c_1), because (c_1) is equivalent to (c_3) by Lemma 2.5.10, and quotients of noetherian torsion sheaves are clearly noetherian torsion again.

Replacing j by j_k, we may thus assume that j is separated. What we are actually going to show is that $j_!\Phi_V$ satisfies (c_2); the implication $(c_2) \Rightarrow (c_1)$ will be shown later in the proof of Proposition 2.5.20 (and there's no circular reasoning involved). Since j is quasi-finite and separated, Zariski's main theorem shows that j may be factored as $j = p \circ i$, where $p\colon \overline{V} \to X$ is finite and $i\colon V \hookrightarrow \overline{V}$ is an open embedding. Our next claim is that $j_!\mathcal{F} \cong p_*i_!\mathcal{F}$ for any sheaf $\mathcal{F} \in \mathrm{Ab}(V_{\text{ét}})$. Indeed, using the explicit construction in the proof of Proposition 2.1.3, one easily constructs a functorial morphism $j_\sharp\mathcal{F} \to p_*i_\sharp\mathcal{F}$ on the level of presheaves. Sheafifying gives the desired canonical morphism. To see that it is an isomorphism, we use the descriptions of stalks from Corollary 1.7.11 and Proposition 2.1.3 (we omit the details since this is just straightforward).

In particular, $j_!\Phi_V \cong p_*i_!\Phi_V$. Note that the counit of the $i_!$-i^* provides a canonical morphism $i_!\Phi_V \cong i_!i^*\Phi_{\overline{V}} \hookrightarrow \Phi_{\overline{V}}$, which is injective as one can see on stalks. Thus we get an injective morphism $j_!\Phi_V \cong p_* i_! \Phi_V \hookrightarrow p_* \Phi_{\overline{V}}$, which proves that $j_!\Phi_V$ satisfies (c_2). This shows the second assertion.

The first assertion can be proved by reduction to the universally Japanese case. Suppose we find a morphism $\pi\colon X \to X'$, such that X' is of finite type over $\mathbb{Z}$ and $\mathcal{F} = \pi^*\mathcal{F}'$ for some sheaf $\mathcal{F}' \in \mathrm{Ab}(X'_{\text{ét}})$ satisfying condition (c_1). By Lemma 2.5.17, we know that $\mathcal{F}'$ also satisfies (c_2). So we find pairs $(p'_k\colon X'_k \to X, \Phi_k)$ consisting of finite morphisms p'_k and finite abelian groups Φ_k, together with an injective morphism $\mathcal{F}' \hookrightarrow \bigoplus_{k=1}^n p'_{k,*}\Phi_k$. Since π^* is exact, we thus get $\mathcal{F} = \pi^*\mathcal{F}' \hookrightarrow \bigoplus_{k=1}^n \pi^* p'_{k,*}\Phi_k$. Now let $X_k = X'_k \times_{X'} X$ and let $p_k\colon X_k \to X$ be the base change of p'_k. Then Fact* 2.5.12(a) shows $\pi^* p'_{k,*}\Phi_k \cong p_{k,*}\Phi_k$, hence $\mathcal{F}$ satisfies (c_2) as well.

It remains to construct X' and $\mathcal{F}'$ with the required properties. By (d) in Appendix A.1, we may write $X = \lim X_\alpha$ as a cofiltered limit over schemes X_α of finite type over $\mathbb{Z}$, and let $\pi_\alpha\colon X \to X_\alpha$ be the structure morphisms. We would like to show that any $\mathcal{F}$ satisfying (c_1) can be written as $\pi_\alpha^*\mathcal{F}_\alpha$ for some $\mathcal{F}_\alpha \in \mathrm{Ab}(X_{\alpha,\text{ét}})$ satisfying (c_1). To this end we claim:

$(*)$ *Every $\mathcal{F}$ satisfying property (c_1) can be written as*

$$\mathcal{F} \cong \operatorname{coker}\left(\bigoplus_{l=1}^m j_{l,!}\left(\mathbb{Z}/M_l\mathbb{Z}_{V_l}\right) \xrightarrow{\varphi} \bigoplus_{k=1}^n i_{k,!}\left(\mathbb{Z}/N_k\mathbb{Z}_{U_k}\right)\right),$$

where $i_k\colon U_k \to X$ and $j_l\colon V_l \to X$ are étale morphisms, N_k and M_l are non-zero integers, and φ is any morphism of sheaves (this is stolen from [Stacks, Tag 095N] by the way).

Let's first describe how $(*)$ finishes the proof. By (l) from Appendix A.1, we may choose α large enough such that all i_k and j_l are base changes of étale morphisms $i_{k,\alpha}\colon U_{k,\alpha} \to X_\alpha$ and $j_{l,\alpha}\colon V_{l,\alpha} \to X_\alpha$. For all $\beta \geqslant \alpha$ let $i_{k,\beta}\colon U_{k,\beta} \to X_\beta$ and $j_{l,\beta}\colon V_{l,\beta} \to X_\beta$ be the base changes of $i_{k,\alpha}$ and $j_{l,\alpha}$. Using Fact* 2.5.12(b) we see $i_{k,!}(\mathbb{Z}/N_k\mathbb{Z}) \cong \pi_\beta^* i_{k,\beta,!}(\mathbb{Z}/N_k\mathbb{Z})$ and similar for $j_{l,!}(\mathbb{Z}/M_l\mathbb{Z})$. Thus it suffices to show that φ can be written as a pullback $\pi_\beta^*(\varphi_\beta)$ of some $\varphi_\beta\colon \bigoplus j_{l,\beta,!}(\mathbb{Z}/M_l\mathbb{Z}) \to \bigoplus i_{k,\beta,!}(\mathbb{Z}/N_k\mathbb{Z})$ for $\beta \geqslant \alpha$ sufficiently large.

Let $\mathcal{G} = \bigoplus i_{k,!}(\mathbb{Z}/N_k\mathbb{Z})$ and $\mathcal{G}_\beta = \bigoplus i_{k,\beta,!}(\mathbb{Z}/N_k\mathbb{Z})$ denote the sheaves on the right-hand side. Observe that

$$\operatorname{Hom}_{\mathrm{Ab}(X_{\text{ét}})}\left(j_{l,*}(\mathbb{Z}/M_l\mathbb{Z}), \mathcal{G}\right) \cong \operatorname{Hom}_{\mathrm{Ab}(V_{l,\text{ét}})}\left(\mathbb{Z}/M_l\mathbb{Z}, j_l^*\mathcal{G}\right) \cong \Gamma\left(V_l, j_l^*\mathcal{G}\right)[M_l]\,,$$

where $(-)[M_l]$ denotes the M_l-torsion part, and a similar isomorphism holds for $\mathcal{G}_\beta$. Using Proposition 1.7.7 together with the fact that taking M_l-torsion commutes with filtered colimits, we can write

$$\Gamma\left(V_l, j_l^*\mathcal{G}\right)[M_l] \cong \operatorname*{colim}_{\beta \geqslant \alpha} \Gamma\left(V_{l,\beta}, j_{l,\beta}^*\mathcal{G}\right)[M_l]\,.$$

In particular, any morphism $j_{l,!}(\mathbb{Z}/M_l\mathbb{Z}) \to \mathcal{G}$ comes from some "finite stage", i.e., is the base change of some $j_{l,\beta,!}(\mathbb{Z}/M_l\mathbb{Z}) \to \mathcal{G}_\beta$ for sufficiently large $\beta \geqslant \alpha$. This shows that φ can be written as $\pi_\beta^*(\varphi_\beta)$ for some large enough $\beta \geqslant \alpha$.

It remains to prove $(*)$. Note that every sheaf of the form $\bigoplus i_{k,\beta,!}(\mathbb{Z}/N_k\mathbb{Z})$ satisfies (c_1) by the first part. Since property (c_1) is inherited by subobjects (because it is equivalent to (c_3) by Lemma 2.5.10, and for (c_3) this is trivial), it suffices to show that every $\mathcal{F}$ admits an epimorphism $\bigoplus i_{k,\beta,!}(\mathbb{Z}/N_k\mathbb{Z}) \twoheadrightarrow \mathcal{F}$. In fact, allow arbitrary indexing sets K rather than just finite sets $K = \{1, \dots, n\}$, then such an epimorphism $\bigoplus_{k\in K} i_{k,\beta,!}(\mathbb{Z}/N_k\mathbb{Z}) \twoheadrightarrow \mathcal{F}$

exists for every sheaf $\mathcal{F}$ whose stalks are torsion abelian groups, without any finiteness conditions. Indeed, we can just take $K = \coprod_{\overline{x}} \mathcal{F}_{\overline{x}}$, where $\overline{x}$ ranges over all geometric points of X, and choose (V_k, N_k) accordingly. But in our situation $\mathcal{F}$ is noetherian in addition to being torsion, so an easy argument shows that there exists a finite subset $K' \subseteq K$ such that $\bigoplus_{k \in K'} i_{k,\beta,!}(\mathbb{Z}/N_k\mathbb{Z}) \twoheadrightarrow \mathcal{F}$ is already an isomorphism. This finishes the proof of $(*)$. □

2.5.20. Proposition. — *Let X be a noetherian scheme.*

(*a*) *The conditions (c_1), (c_1^-), (c_2), and (c_3) are equivalent. The full subcategory of constructible sheaves is stable under subobjects, quotients, extensions, and finite direct sums (in particular, it is abelian). Moreover, the image of a constructible sheaf under f^* for arbitrary morphisms f, p_* for finite morphisms p, and $j_!$ for étale morphisms j, stays constructible.*

(*b*) *If $q\colon Y \to X$ is quasi-finite morphism and X is universally Japanese, then the image of constructible sheaves under q_* stays constructible as well.*

(*c*) *Every torsion sheaf on $X_{\text{ét}}$ in the sense of Remark* 2.5.8(c) is the filtered colimit of its constructible subsheaves.*

We postpone the proof of Proposition 2.5.20 until after one corollary and two remaining preparatory lemmas.

2.5.21. Corollary. — *Let X be a noetherian scheme. Then the cohomological functor $H^\bullet(X_{\text{ét}}, -)$ is effaceable on the category of constructible sheaves in the sense of* (1) *from Proposition* 2.1.10(*a*).

Proof. Every sheaf has a monomorphism $\mathcal{F} \hookrightarrow \mathcal{G} = \prod_{\overline{x}} \overline{x}_* \mathcal{F}_{\overline{x}}$, where the sheaf on the right-hand side is acyclic, which can be seen from Proposition 2.4.4(*b*) and its proof. If $\mathcal{F}$ is constructible, then it is a noetherian torsion sheaf, hence annihilated by some integer $N \neq 0$. Then clearly also $N\mathcal{G} = 0$, so $\mathcal{G}$ is a torsion sheaf as well (albeit probably not constructible). By Proposition 2.5.20(*c*), we can write $\mathcal{G} = \operatorname{colim} \mathcal{H}_\alpha$, where the colimit is taken over the filtered system of all constructible subsheaves $\mathcal{F} \subseteq \mathcal{H}_\alpha \subseteq \mathcal{G}$. Thus, by Proposition 2.4.12 (together with the fact that noetherian schemes are always quasi-compact quasi-separated),

$$0 = H^i(X_{\text{ét}}, \mathcal{G}) = \operatorname*{colim}_{\alpha} H^i(X_{\text{ét}}, \mathcal{H}_\alpha).$$

for all $i > 0$. In particular, for every $\eta \in H^i(X_{\text{ét}}, \mathcal{F})$ there is an α such that η is mapped to 0 under $H^i(X_{\text{ét}}, \mathcal{F}) \to H^i(X_{\text{ét}}, \mathcal{H}_\alpha)$. This shows effaceability in the sense of (1) from Proposition 2.1.10(*a*). □

2.5.22. Lemma. — *Let $p\colon Y \to X$ be a finite morphism of noetherian schemes, and $\mathcal{F}$ an lcc sheaf in* $\operatorname{Ab}(Y_{\text{ét}})$. *There exists a dense open subset $Y \subseteq X$ such that $p_*\mathcal{F}|_{U_{\text{ét}}}$ is lcc again.*

Proof. We first reduce everything to a sufficiently simple situation. The assertion is local on X, so we may also assume that $X = \operatorname{Spec} A$ is affine. Then $Y = \operatorname{Spec} B$ is affine as well. Doing induction on the number of generators of B over A, we may assume $B = A[T]/I$ for some ideal I. We may further assume that X and Y are reduced, because replacing them by their reductions X^{red} and Y^{red} does not affect the categories $\operatorname{Ab}(X_{\text{ét}})$ and $\operatorname{Ab}(Y_{\text{ét}})$ by Proposition 1.4.20. Let $\eta_1, \dots, \eta_n$ denote the generic points of the irreducible components of X. An open subset $U \subseteq X$ is dense iff it contains all η_i. Thus we may replace $X = \operatorname{Spec} A$ by an affine open subset containing η_1 but none of $\eta_2, \dots, \eta_n$ to reduce to the case where

$\operatorname{Spec} A$ is irreducible and reduced, hence A is a domain. Moreover, we may assume that $I = (f)$ is a principal ideal generated by a monic polynomial $f \in A[T]$. Indeed, let K be the fraction field of A and $f_1, \dots, f_m \in A[T]$ generators of I. The ideal $IK[T] \subseteq K[T]$ is principal, hence generated by some $f \in K[T]$, which thus divides $f_1, \dots, f_m$. Then there is an $\alpha \in A$ such that f is already contained in the localization $A_\alpha[T]$ and divides $f_1, \dots, f_m$ in $A_\alpha[T]$. Then $IA_\alpha[T]$ is the principal ideal generated by f, as required.

If the derivative $f' \neq 0$, then $A \to A[T]/(f)$ is étale at the generic point $0 \in \operatorname{Spec} A$. Hence, using Proposition 1.4.16(e) there is an $\alpha \in A$ such that $A_\alpha \to A_\alpha[T]/(f)$ is étale. Thus we may assume that $p\colon Y \to X$ is finite étale. In this case $p_*\mathcal{F}$ is lcc by Fact 2.5.6.

In particular, this settles the case where K has characteristic 0. If K has positive characteristic, we may write $f(T) = g(T^q)$, where q is a power of the characteristic and $g' \neq 0$. Put $C = A[T]/(g)$, so that C can be obtained from $B = A[T]/(f)$ as $C \cong B[T^{1/q}]$. Thus, putting $Y' = \operatorname{Spec} C$, we obtain that $r\colon Y' \to Y$ is finite, bijective, and radiciel in the sense of Remark 1.4.21. Hence it is a universal homeomorphism and thus $r_*\colon \operatorname{Ab}(Y'_{\text{ét}}) \to \operatorname{Ab}(Y_{\text{ét}})$ is an equivalence of categories, using Proposition 1.4.22. Therefore we may replace $p\colon Y \to X$ by $p \circ r\colon Y' \to X$ and apply the previous argument. $\square$

2.5.23. Lemma. — *Let $p\colon Y \to X$ be a finite morphism of noetherian schemes and $\mathcal{F} \in \operatorname{Ab}(Y_{\text{ét}})$ be a sheaf satisfying property (c_1). Then $p_*\mathcal{F}$ is noetherian.*

Proof. In the lecture we just said "Lemma 2.5.22 and noetherian induction", but actually there's quite some technical stuff to do. Before we start, let's state for the record that property (c_1) is preserved under pullbacks along closed immersions. Indeed, (c_1) and (c_3) are equivalent by Lemma 2.5.10. Moreover, we have seen in the proof of that lemma that being noetherian is preserved under pullbacks along closed immersions. The same is true for being torsion for obvious reasons, thus proving the claim. We will use this property several times throughout the proof.

Step 1. We reduce to the case where X and Y are affine and integral. As usual, Proposition 1.4.20 allows us to replace X and Y by their reductions X^{red} and Y^{red}. Since being noetherian can be checked Zariski-locally by Fact 2.5.2(b), we may assume that $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$ are affine. Moreover, we may assume that X is irreducible. Indeed, let $X_1, \dots, X_n$ be the irreducible components of X and $i_k\colon X_k \hookrightarrow X$. Put $Y_k = Y \times_X X_k$, and let $p_k\colon Y_k \to X_k$ and $j_k\colon Y_k \hookrightarrow Y$ be the base changes of p and i_k. Then $j_k^*\mathcal{F}$ satisfy (c_1) again (as argued above) and $\mathcal{F} \hookrightarrow \bigoplus_{k=1}^n j_{k,*}j_k^*\mathcal{F}$ is a monomorphism (such an argument already occured in the proof of Fact 2.5.11(b)). Hence $p_*\mathcal{F} \hookrightarrow \bigoplus_{k=1}^n i_{k,*}p_{k,*}j_k^*\mathcal{F}$ and thus it suffices to show that $p_{k,*}j_k^*\mathcal{F}$ is noetherian. Therefore we may assume that X is irreducible, hence A is a domain.

Similarly, let $Y_1, \dots, Y_m$ be the irreducible components of Y, and $j_k\colon Y_k \hookrightarrow Y$. Then $\mathcal{F} \hookrightarrow \bigoplus_{k=1}^m j_{k,*}j_k^*\mathcal{F}$ is injective, hence so is $p_*\mathcal{F} \hookrightarrow \bigoplus_{k=1}^m (p \circ j_k)_*j_k^*\mathcal{F}$, hence it suffices to proof the assertion with $p\colon Y \to X$ replaced by $p \circ j_k\colon Y_k \to X$. Therefore we may also assume that B is a domain.

Step 2. We reduce to the case where $\mathcal{F}$ is lcc. Let $\emptyset = V_0 \subseteq V_1 \subseteq \dots \subseteq V_n = Y$ be a stratification according to (c_1) and put $Z_1 = Y \setminus V_1$. Since p is finite, $X' = p(Z_1)$ is a closed subset. Observe that since B is finite over A and both are domains, every $\beta \in B$ satisfies an equation of the form $\beta^n + a_{n-1}\beta^{n-1} + \dots + a_0 = 0$ for some $a_k \in A$ and $a_0 \neq 0$. Thus, every non-zero prime ideal $0 \neq \mathfrak{q} \in \operatorname{Spec} B$ has a non-zero preimage in A. This proves that X' does not contain the generic point $0 \in \operatorname{Spec} A$, hence $U = X \setminus X'$ is non-empty open. Equip X' with its reduced closed subscheme structure and let $i\colon X' \hookrightarrow X$.

Put $Y' = Y \times_X X'$ and let $p'\colon Y' \to X'$, $j\colon Y \hookrightarrow Y$ be the base changes of p and i. By the noetherian induction hypothesis and Fact* 2.5.12(a), we know that $p'_* f^*\mathcal{F} \cong i^* p_*\mathcal{F}$ is noetherian. Using Fact 2.5.2(b), it thus remains to check that $p_*\mathcal{F}|_{U_{\text{ét}}}$ is noetherian. Put $V = p^{-1}(U)$. Then $V \subseteq V_1$, hence $\mathcal{F}|_{V_{\text{ét}}}$ is lcc. Replacing $p\colon Y \to X$ by $p|_V\colon V \to U$ finishes the second reduction step.

Step 3. We finish the noetherian induction. By Step 2, we may assume that $\mathcal{F}$ is lcc. By Lemma 2.5.22 there is a dense open $U \subseteq X$ such that $p_*\mathcal{F}|_{U_{\text{ét}}}$ is noetherian. Put $X' = X \setminus U$ (equipped with its reduced closed subscheme structure) and $Y' = Y \times_X X'$ and let i, p', and j be as above. By the noetherian induction hypothesis and Fact* 2.5.12(a), we know that $p'_* j^*\mathcal{F} \cong i^* p_*\mathcal{F}$ is noetherian too. Then Fact 2.5.2(b) shows that $p_*\mathcal{F}$ is noetherian too, and the proof is finally complete. □

Proof of Proposition 2.5.20. We already know $(c_3) \Leftrightarrow (c_1) \Rightarrow (c_2)$ from Lemma 2.5.10 and Lemma* 2.5.19. For $(c_2) \Rightarrow (c_3)$, we must show that for finite abelian groups Φ and finite morphisms $p\colon Y \to X$, the sheaf $p_*\Phi_Y$ is noetherian and torsion. The latter is trivial, and noetherianness follows from Lemma 2.5.23. Finally, $(c_1) \Rightarrow (c_1^-)$ is obvious, and $(c_1^-) \Rightarrow (c_3)$ follows from the fact that lcc sheaves are noetherian and torsion by Lemma 2.5.10 together with Fact 2.5.2(b). This shows equivalence of the four conditions.

The stability assertions under f^* and p_* follow from immediately from Fact 2.5.11(a), and the stability assertion under q_* from part (b) is proved in Lemma 2.5.17. It remains to show stability under $j_!$, where $j\colon U \to X$ is étale. Using characterization (c_2) and exactness of $j_!$, we see that it suffices to show that for finite morphisms $p\colon V \to U$ and finite abelian groups Φ, the sheaf $j_! p_*\Phi_V$ satisfies the equivalent conditions again. By an argument as in the proof* of Lemma* 2.5.19, we may reduce to the case where j is separated. Then $j \circ p\colon V \to X$ is separated as well, hence Zariski's main theorem allows us to construct a factorization $V \hookrightarrow \overline{V} \to X$. Now consider the diagram

$$\begin{tikzcd}
V \arrow[dr, "p'"] \arrow[drr, hook, bend left, "i"] \arrow[ddr, bend right, "p"'] & & \\
& U \times_X \overline{V} \arrow[r, "j'"] \arrow[d, "\overline{p}'"'] \arrow[dr, phantom, "\lrcorner"] & \overline{V} \arrow[d, "\overline{p}"] \\
& U \arrow[r, "j"] & X
\end{tikzcd} .$$

The base changes j' and $\overline{p}'$ are étale resp. finite again. Hence p is finite too. Since $i = j' \circ p$ is an open embedding, hence étale, Fact 1.4.6(b) shows that p' is also étale. In particular, $p'_*\mathcal{F} \cong p'_!\mathcal{F}$ for any étale sheaf $\mathcal{F} \in \mathrm{Ab}(V_{\text{ét}})$. Indeed, this is obvious if the finite étale morphism is a split étale covering. But every finite étale morphism is étale-locally a split étale covering by Lemma* 1.5.5(a), and whether two étale sheaves are equal can be tested étale-locally. Thus $i_!\Phi_V \cong j'_! p'_! \Phi_V \cong j'_! p_*\Phi_V$. Therefore, we compute

$$j_! p_*\Phi_V \cong j_! \overline{p}'_* p'_* \Phi_V \cong \overline{p}_* j'_! p'_* \Phi_V \cong \overline{p}_* i_! \Phi_V \,,$$

where Fact* 2.5.12(c) was used for the middle isomorphism. Now Lemma* 2.5.19 shows that $i_!\Phi_V$ satisfies (c_1) again, hence so does $\overline{p}_* i_!\Phi_V$ by stability under pushforward along finite morphisms. This finishes the proof of (a) and (b).

Part (c) is basically trivial: every sheaf on $X_{\text{ét}}$ is the filtered colimit of its finitely generated subsheaves (i.e., those subsheaves generated by finitely many sections). If $\mathcal{G}$ is torsion, then every subsheaf generated by finitely many sections is the image of $\bigoplus_{k=1}^{n} j_{k,!}(\mathbb{Z}/N_k\mathbb{Z}_{V_k}) \to \mathcal{G}$ for some étale morphisms $j_k\colon V_k \to X$ and some integers $N_k \neq 0$. By (a), the left-hand side is constructible, hence so is its image in $\mathcal{G}$. □

2.5.24. Remark. — To finish this section on constructible sheaves, Professor Franke mentions two results from [FK88, Section I, Proposition 4.17 and 4.18].

(a) Let X be noetherian and write $X = \lim X_\alpha$ as a limit of noetherian schemes X_α along affine transition maps. Let $\pi_\alpha\colon X \to X_\alpha$ be the structure morphisms. If $\mathcal{F}$ is a constructible sheaf in $\mathrm{Ab}(X_{\text{ét}})$, then there exists an index α and a constructible sheaf $\mathcal{F}_\alpha \in \mathrm{Ab}(X_{\alpha,\text{ét}})$ such that $\mathcal{F} \cong \pi_\alpha^*\mathcal{F}_\alpha$. We proved this essentially in the proof of Lemma* 2.5.19.

(b) Let $\mathcal{F} = \pi_\alpha^*\mathcal{F}_\alpha$ and $\mathcal{G} = \pi_\alpha^*\mathcal{G}_\alpha$ be constructible sheaves which are N-torsion for some integer $N \neq 0$ (in other words, they are modules over the constant $\mathbb{Z}/N\mathbb{Z}$-valued sheaf). For all $\beta \geqslant \alpha$ let $\mathcal{F}_\beta$ and $\mathcal{G}_\beta$ denote the pullbacks of $\mathcal{F}_\alpha$ and $\mathcal{G}_\alpha$ to X_β. Then for all $i \geqslant 0$ there is an isomorphism

$$\operatorname*{colim}_{\beta \geqslant \alpha} \mathcal{E}xt^i_{\mathbb{Z}/N\mathbb{Z}_{X_\beta}}(\mathcal{F}_\beta, \mathcal{G}_\beta) \xrightarrow{\sim} \mathcal{E}xt^i_{\mathbb{Z}/N\mathbb{Z}_X}(\mathcal{F}, \mathcal{G})\,.$$

2.6. Cohomology of Curves

Lecture 23
24^{th} Jan, 2020

2.6.1. Disclaimer. — Throughout this section, k will be a separably closed field, and all results are valid in this case. However, for some of the proofs it will be convenient to assume that k is even algebraically closed. This is made possible by the following observation: let X be a quasi-compact quasi-separated scheme over k, which is separably closed. Let $\mathcal{F} \in \mathrm{Ab}(X_{\text{ét}})$. For all field extensions K/k let $X_K = X \times_k \operatorname{Spec} K$, let $\pi_{K/k}\colon X_K \to X$ be the canonical projection, and put $\mathcal{F}_K = \pi_{K/k}^*\mathcal{F}$. In the special case $K = \overline{k}$, we have $X_{\overline{k}} \cong \lim_{\ell/k} X_\ell$, where the limit is taken over all finite extensions ℓ/k. Thus, Proposition 2.4.12 implies

$$H^i\big(X_{\overline{k},\text{ét}}, \mathcal{F}_{\overline{k}}\big) \cong \operatorname*{colim}_{\ell/k} H^i\big(X_{\ell,\text{ét}}, \mathcal{F}_\ell\big)$$

for all $i \geqslant 0$. Since k is separably closed, every finite extension ℓ/k is purely inseparable. Thus, $\operatorname{Spec} \ell \to \operatorname{Spec} k$ is finite, radiciel, and surjective, thus a universal homeomorphism by Remark 1.4.21. Therefore, we can use Proposition 1.4.22 to show that $\pi_{\ell/k,*}$ and $\pi_{\ell/k}^*$ are mutually quasi-inverse equivalences of categories between $\mathrm{Ab}(X_{\text{ét}})$ and $\mathrm{Ab}(X_{\ell,\text{ét}})$. In particular, all $H^i(X_{\ell,\text{ét}}, \mathcal{F}_\ell)$ in the above colimit are isomorphic to $H^i(X_{\text{ét}}, \mathcal{F})$, and we obtain a single isomorphism

$$H^i\big(X_{\overline{k},\text{ét}}, \mathcal{F}_{\overline{k}}\big) \cong H^i(X_{\text{ét}}, \mathcal{F})$$

in this case. This usually allows us to reduce to the case where $\overline{k}$ is algebraically closed. Not always though; see Warning* 2.4.16.

2.6.2. Terminology. — In this lecture, a *curve* is a one-dimensional scheme C of finite type over k (note: no connectedness, smoothness, or properness assumptions). Whenever a point of C is called x, it is assumed to be a closed point unless otherwise specified. If x is a closed point, then $\kappa(x)$ is finite over k by Hilbert's Nullstellensatz, hence separably closed too. Thus, every closed point is a geometric point as well. By abuse of notation, we also denote $x\colon \operatorname{Spec} \kappa(x) \to X$.

2.6.3. Proposition. — *Let C be a smooth curve over the separably closed field k. Then*

$$H^i\big(C_{\text{ét}}, \mathcal{O}_{C_{\text{ét}}}^\times\big) \cong \begin{cases} \Gamma(C, \mathcal{O}_C)^\times & \textit{if } i = 0 \\ \operatorname{Pic}(C) & \textit{if } i = 1 \\ 0 & \textit{else} \end{cases}.$$

Proof. Without losing generality C is connected, hence integral (using smoothness). Let K be the function field of C and $\eta\colon V = \operatorname{Spec} K \to C$ the generic point of C. Using Proposition 2.2.7 together with Proposition 2.2.2(c) and Proposition 2.2.6(b) shows $H^i(V_{\text{ét}}, \mathcal{O}_{V_{\text{ét}}}^\times) = 0$ for $i > 0$, because K is a C_1 field. Moreover, we claim:

($*$) *The higher derived direct images $R^i\eta_*\mathcal{O}_{V_{\text{ét}}}^\times$ vanish for $i > 0$.*

Indeed, if x is any closed point geometric point of C, or equivalently, a closed point (see 2.6.2) then Lemma* 2.4.17 shows

$$\big(R^i\eta_*\mathcal{O}_{V_{\text{ét}}}^\times\big)_x \cong H^i\big(V_{x,\text{ét}}, \mathcal{O}_{V_{x,\text{ét}}}^\times\big)\,,$$

where $V_x = U \times_C \operatorname{Spec} \mathcal{O}_{C_{\text{ét}},x} \cong \operatorname{Spec}(K \otimes_{\mathcal{O}_{C,x}} \mathcal{O}_{C,x}^{\text{sh}})$. We claim that $K_x = K \otimes_{\mathcal{O}_{C,x}} \mathcal{O}_{C,x}^{\text{sh}}$ is a C_1 field. To see this, first note that $\mathcal{O}_{C,x}^{\text{sh}}$ is a filtered colimit of domains. In fact, because C is smooth, hence normal, every $\mathcal{O}_{U,u}$ occuring in (1.6.1) is a normal domain by Lemma* A.2.4 and Serre's normality criterion. Therefore $\mathcal{O}_{C,x}^{\text{sh}}$ is a domain, and thus its localization K_x is one as well. Moreover, by (1.6.1) we may also write $K_x \cong \operatorname{colim} K \otimes \Gamma(U, \mathcal{O}_U)$. Every $K \otimes \Gamma(U, \mathcal{O}_U)$ is étale over K, hence finite, proving that every element of K_x is integral over K. Hence K_x is an algebraic field extension of K (Professor Franke even claimed that K_x is the maximal extension of K that is unramified at x) and thus C_1 by Proposition 2.2.6(a). In particular, we have $H^i(V_{x,\text{ét}}, \mathcal{O}_{V_{x,\text{ét}}}^\times) = 0$ for $i > 0$. Thus $R^i\eta_*\mathcal{O}_{V_{\text{ét}}}^\times$ vanishes at all closed points x, which is enough to prove ($*$) because C is Jacobson (see Remark 1.6.4).

Using ($*$) and the Leray spectral sequence (Proposition 2.1.11) we obtain

$$H^i\big(C_{\text{ét}}, \eta_*\mathcal{O}_{V_{\text{ét}}}^\times\big) \cong H^i\big(V_{\text{ét}}, \mathcal{O}_{V_{\text{ét}}}^\times\big) \cong \begin{cases} K^\times & \text{if } i = 0 \\ 0 & \text{else} \end{cases}.$$

To compute the cohomology of $\mathcal{O}_{C_{\text{ét}}}^\times$, we consider the following sequence of étale sheaves

$$0 \longrightarrow \mathcal{O}_{C_{\text{ét}}}^\times \longrightarrow \eta_*\mathcal{O}_{V_{\text{ét}}}^\times \xrightarrow{\text{div}} \bigoplus_{x \in C} x_*\mathbb{Z} \longrightarrow 0\,.$$

We would like to show it is exact. Let $U \in C_{\text{ét}}$ be an affine connected étale C-scheme. Then U is a connected smooth curve and one easily checks that $\Gamma(U, \eta_*\mathcal{O}_{V_{\text{ét}}}^\times) \cong K_U^\times$, where K_U is the function field of U. Moreover, $\Gamma(U, x_*\mathbb{Z}) = \bigoplus_u \mathbb{Z}$, where the direct sum is taken over all closed points $u \in U$ lying over x. Since a direct sum can be written as a filtered colimit and U is quasi-compact and quasi-separated, Corollary 2.4.14 thus shows

$$\Gamma\left(U, \bigoplus_{x \in C} x_*\mathbb{Z}\right) \cong \bigoplus_{x \in C} \Gamma(U, x_*\mathbb{Z}) \cong \operatorname{Div} U\,,$$

where $\operatorname{Div} U$ denotes the group of divisors on U, as usual. Now exactness of the sequence in question follows from the fact that a similar sequence of sheaves is exact in the Zariski topology by some well-known facts about divisors.

This allows us to compute the required cohomology groups using the long exact cohomology sequence! We have $\Gamma(C, x_*\mathbb{Z}) \cong \mathbb{Z}$ and $H^i(C_{\text{ét}}, x_*\mathbb{Z}) = 0$ for $i > 0$. The latter assertion follows either from the fact that $x\colon \operatorname{Spec} \kappa(x) \to X$ is finite together with Proposition 2.1.12 and the Leray spectral sequence, or from Čech cohomology and Proposition 2.4.4(b). Since C is quasi-compact and quasi-separated, we can thus apply Corollary 2.4.14 to obtain

$$H^i\left(C_{\text{ét}}, \bigoplus_{x\in C} x_*\mathbb{Z}\right) \cong \bigoplus_{x\in X} H^i(C_{\text{ét}}, x_*\mathbb{Z}) = \begin{cases} \operatorname{Div} C & \text{if } i = 0 \\ 0 & \text{else} \end{cases} .$$

And the assertion follows from the long exact cohomology sequence and our calculation of $H^i(C_{\text{ét}}, \eta_*\mathcal{O}_{V_{\text{ét}}}^\times)$ above. In particular, we get a reproof of Corollary 2.3.12 in the special case where X is a smooth curve. ☐

2.6.4. — Recall that for any scheme S there's a sheaf $\mu_n = \mu_{n,S}$ of n^{th} *roots of unity* on $S_{\text{ét}}$, given as the kernel of the n^{th} power map $(-)^n\colon \mathcal{O}_{S_{\text{ét}}}^\times \to \mathcal{O}_{S_{\text{ét}}}^\times$. If n is invertible on S, then μ_n fits into a short exact sequence

$$0 \longrightarrow \mu_n \longrightarrow \mathcal{O}_{S_{\text{ét}}}^\times \xrightarrow{(-)^n} \mathcal{O}_{S_{\text{ét}}}^\times \longrightarrow 0$$

(exactness follows more or less from the existence of Kummer coverings, see Proposition 1.5.11). Also note that in this case $f^*\mu_{n,S} \cong \mu_{n,X}$ for any morphism $f\colon X \to S$, despite Warning* 2.4.16. The reason is that $\mu_{n,S}$ is representable by the scheme $\underline{\operatorname{Spec}}\, \mathcal{O}_S[T]/(T^n - 1)$, which is étale over S if n is invertible on S. Then an abstract nonsense argument (using the f^*-f_* adjunction and the Yoneda lemma) shows that $f^*\mu_{n,S}$ is representable by the base change $\underline{\operatorname{Spec}}\, \mathcal{O}_X[T]/(T^n - 1)$, i.e., is isomorphic to $\mu_{n,X}$, as claimed. In particular, the argument from 2.6.1 works and you may assume that k is algebraically closed in the proof of Corollary 2.6.5 below if that makes you feel better (it isn't needed though).

Also, in case you wonder: yes, $\mathcal{O}_{S_{\text{ét}}}$ and $\mathcal{O}_{S_{\text{ét}}}^\times$ are representable too; the representing objects are the *additive group* $\mathbb{G}_{a,S} = \mathbb{A}^1_S$ and the *multiplicative group* $\mathbb{G}_{m,S} = \underline{\operatorname{Spec}}\, \mathcal{O}_S[T, T^{-1}]$ respectively. No, this doesn't contradict Warning* 2.4.16, because $\mathbb{G}_{a,S}$ and $\mathbb{G}_{m,S}$ are no elements of $S_{\text{ét}}$, hence the Yoneda argument doesn't work any more. So good thing our proof of Proposition 2.6.3 works for arbitrary separably closed k.

2.6.5. Corollary. — *If C is a proper smooth connected curve of genus g over the separably closed field k and ℓ a prime number different from* $\operatorname{char} k$*, then*

$$H^i(C_{\text{ét}}, \mu_{\ell^n}) \cong \begin{cases} \mu_{\ell^n}(k) & \textit{if } i = 0 \\ \operatorname{Pic}^0(C)[\ell^n] & \textit{if } i = 1 \\ \mathbb{Z}/\ell^n\mathbb{Z} & \textit{if } i = 2 \\ 0 & \textit{else} \end{cases} .$$

The group $\operatorname{Pic}^0(C)[\ell^n]$ *of ℓ^n-torsion in* $\operatorname{Pic}^0(C)$ *is non-canonically isomorphic to* $(\mathbb{Z}/\ell^n\mathbb{Z})^{\oplus 2g}$.

2.6.6. Remark. — (a) The same result, but non-canonically, holds for $H^\bullet(C_{\text{ét}}, \mathbb{Z}/\ell^n\mathbb{Z}_C)$, as $\mu_{\ell^n} \cong \mathbb{Z}/\ell^n\mathbb{Z}_C$ after choosing a primitive $(\ell^n)^{\text{th}}$ root of unity in k (which exists, as k is separably closed).

(b) Professor Franke points out that Corollary 2.6.5 must have been a special moment for the guys inventing étale cohomology: for the first time, we see that étale cohomology with coefficients in $\mathbb{Z}/\ell^n\mathbb{Z}_C$ behaves like singular cohomology of surfaces of genus g (which we think of as curves over $\mathbb{C}$, that's why the $\mathbb{R}$-dimension is 2).

(*c*) If the characteristic of k is $p > 0$, then we have a short exact sequence

$$0 \longrightarrow \mathbb{Z}/p\mathbb{Z}_C \longrightarrow \mathcal{O}_{C_{\text{ét}}} \xrightarrow{\varphi^* - \mathrm{id}} \mathcal{O}_{C_{\text{ét}}} \longrightarrow 0\,,$$

where $\varphi^* = (-)^p \colon \mathcal{O}_{C_{\text{ét}}} \to \mathcal{O}_{C_{\text{ét}}}$ is the Frobenius. Exactness of this sequence follows more or less from the existence of Artin–Schreier coverings, see Proposition 1.5.12. In combination with Corollary 2.4.9(a) this can be used to compute

$$H^i\big(\mathbb{P}^1_k, \mathbb{Z}/p\mathbb{Z}\big) \cong \begin{cases} \mathbb{Z}/p\mathbb{Z} & \text{if } i = 0 \\ 0 & \text{else} \end{cases}\,.$$

In particular, this is not what algebraic topology predicts for $H^\bullet_{\text{sing}}(\mathbb{CP}^1, \mathbb{Z}/p\mathbb{Z})$. Thus étale cohomology with p-torsion coefficients does not give the "correct answer" (and thus *crystalline cohomology* was born).

Sketch of a proof of Corollary 2.6.5. Observe that ℓ^n is invertible in k. Of course we use the short exact sequence

$$0 \longrightarrow \mu_{\ell^n} \longrightarrow \mathcal{O}^\times_{C_{\text{ét}}} \xrightarrow{(-)^{\ell^n}} \mathcal{O}^\times_{C_{\text{ét}}} \longrightarrow 0$$

from 2.6.4 to compute the required cohomology groups. Since $H^i(C_{\text{ét}}, \mathcal{O}^\times_{C_{\text{ét}}}) = 0$ for $i > 1$ by Proposition 2.6.3, it's clear that $H^1(C_{\text{ét}}, \mu_{\ell^n}) = 0$ for $i > 2$. In degrees 0 and 1 we get an exact sequence

$$0 \longrightarrow \Gamma\big(C, \mu_{\ell^n}\big) \longrightarrow \Gamma(C, \mathcal{O}_C)^\times \xrightarrow{(-)^{\ell^n}} \Gamma(C, \mathcal{O}_C)^\times \longrightarrow H^1\big(C_{\text{ét}}, \mu_{\ell^n}\big)\,.$$

Because C is integral and proper, $\Gamma(C, \mathcal{O}_C)$ is a finite field extension of k. But $\mu_{\ell^n}(k) = \mu_{\ell^n}(\overline{k})$ since k is separably closed, hence $\Gamma(C, \mu_{\ell^n}) \cong \mu_{\ell^n}(k)$ as claimed. Moreover, our argument shows that $\Gamma(C, \mathcal{O}_C)$ is a separably closed field itself, hence the morphism in the middle is surjective. Thus, the remaining two cohomology groups fit into an exact sequence

$$0 \longrightarrow H^1\big(C_{\text{ét}}, \mu_{\ell^n}\big) \longrightarrow \mathrm{Pic}(C) \xrightarrow{(-)^{\otimes \ell^n}} \mathrm{Pic}(C) \longrightarrow H^2\big(C_{\text{ét}}, \mu_{\ell^n}\big) \longrightarrow 0\,.$$

It is a classical theorem that the Picard functor $\mathrm{Pic}_{C/k}$ is representable by a scheme $\underline{\mathrm{Pic}}_C$, the *Picard scheme* of C. It has a decomposition $\underline{\mathrm{Pic}}_C = \coprod_{d \in \mathbb{Z}} \underline{\mathrm{Pic}}_C^d$, where $\underline{\mathrm{Pic}}_C^d$ parametrizes line bundles of degree d. The 0^{th} component $\underline{\mathrm{Jac}}_C = \underline{\mathrm{Pic}}_C^0$ is called the *Jacobian of* C. It is an abelian variety of dimension g over k.

In particular, we may write $\underline{\mathrm{Pic}}(C) \cong \underline{\mathrm{Jac}}_C(k) \times \mathbb{Z}$, and the morphism $(-)^{\otimes \ell^n}$ in question is given by multiplication by ℓ^n. A classical theorem about abelian varieties A of dimension g over k states that for $N \neq 0$ the morphism $N \colon A \to A$ is finite flat of degree N^{2g}. In particular, $A(k)$ is divisible. See [Jac, Theorem 10(b)] for instance (the proof given there is a bit lengthy). This shows that $\ell^n \colon \underline{\mathrm{Jac}}_C(k) \to \underline{\mathrm{Jac}}_C(k)$ is surjective with kernel $\mathrm{Pic}^0(C)[\ell^n]$. Hence

$$\ell^n \colon \underline{\mathrm{Jac}}_C(k) \times \mathbb{Z} \longrightarrow \underline{\mathrm{Jac}}_C(k) \times \mathbb{Z}$$

has kernel $\mathrm{Pic}^0(C)[\ell^n]$ and cokernel $\mathbb{Z}/\ell^n\mathbb{Z}$, as claimed.

It remains to identify $\mathrm{Pic}^0(C)_{\ell^n} \cong (\mathbb{Z}/\ell^n\mathbb{Z})^{\oplus 2g}$. If N is invertible in k, then $N \colon A \to A$ is even étale (indeed, this can be checked after base change to $\overline{k}$; in this case the proof of [Jac, Corollary 3.2.4] shows that the sheaf of relative Kähler differentials associated to $N \colon A \to A$

vanishes at the origin $0 \in A(k)$, hence at all closed points, hence everywhere as A is Jacobson). In our concrete situation, we obtain that the kernel $\underline{\mathrm{Jac}}_C[\ell^n]$ of $\ell^n\colon \underline{\mathrm{Jac}}_C \to \underline{\mathrm{Jac}}_C$, i.e., the fibre over the zero section $0\colon \operatorname{Spec} k \to \underline{\mathrm{Jac}}_C(k)$, is a finite étale scheme of degree ℓ^{2gn} over k. But k is separably closed, hence such a scheme must be a disjoint union of ℓ^{2ng} copies of k. Thus $\underline{\mathrm{Jac}}_C[\ell^n](k)$ is a group of order ℓ^{2ng}. It is clearly annihilated by ℓ^n. Moreover, its subgroup $\underline{\mathrm{Jac}}_C[\ell^{n-1}](k)$ of ℓ^{n-1}-torsion elements has order $\ell^{2(n-1)g}$ by the same argument. By the classification of finite abelian groups, the only possibility is $\underline{\mathrm{Jac}}_C[\ell^n](k) \cong (\mathbb{Z}/\ell^n\mathbb{Z})^{\oplus 2g}$, as claimed $\square$

For the rest of the section we now work towards a very general vanishing result for étale cohomology of torsion sheaves on curves over k.

2.6.7. Fact. — *Let C be a curve over k and $\mathcal{F}$ a torsion sheaf on C. Moreover, let $\overline{\eta}_1, \dots, \overline{\eta}_n$ be geometric points whose underlying points $\eta_1, \dots, \eta_n$ are the generic point of the irreducible components of C.*

(*a*) *If $\mathcal{F}_{\overline{\eta}_j} = 0$ for all j, then $H^i(C_{\text{ét}}, \mathcal{F}) = 0$ for $i > 0$.*

(*b*) *If $\varphi\colon \mathcal{F} \to \mathcal{G}$ is a morphism of torsion sheaves which induces isomorphisms at all geometric points $\overline{\eta}_1, \dots, \overline{\eta}_n$, then $\varphi_*\colon H^i(C_{\text{ét}}, \mathcal{F}) \xrightarrow{\sim} H^i(C_{\text{ét}}, \mathcal{G})$ is an isomorphism for all $i > 1$. If φ is an epimorphism, then φ_* is an isomorphism for $i = 1$ too.*

Proof. Part (*a*). By Proposition 2.5.20(*c*) and Corollary 2.4.14 we may assume that $\mathcal{F}$ is constructible. Then Lemma 2.5.3 implies $\mathcal{F}|_{U_{\text{ét}}} = 0$ for some open dense subset $U \subseteq C$. The rest $C \setminus U$ consists of only finitely many points. Hence the canonical morphism $\mathcal{F} \to \bigoplus_{x \in C \setminus U} x_* \mathcal{F}_x$ is an isomorphism (as can be seen on stalks). Now Proposition 2.4.4(*b*) and the arguments in its proof show that $H^i(C_{\text{ét}}, \mathcal{F}) = 0$ for $i > 0$, as required.

For part (*b*), we split φ into two short exact sequences $0 \to \mathcal{K} \to \mathcal{F} \to \mathcal{B} \to 0$ and $0 \to \mathcal{B} \to \mathcal{G} \to \mathcal{Q} \to 0$ in which all occuring sheaves are torsion and $\mathcal{K}_{\overline{\eta}_j} = 0 = \mathcal{Q}_{\overline{\eta}_j}$ for all j. Taking long exact cohomology sequences and applying (*a*) twice proves the assertion. $\square$

2.6.8. Lemma. — *Let C be a smooth affine curve over k and Φ a finite abelian group. Then $H^i(C_{\text{ét}}, \Phi_C) = 0$ for $i > 1$.*

Proof.* In this proof we may assume that k is algebraically closed by 2.6.1. Every finite abelian group has a filtration $0 = \Phi_0 \subseteq \Phi_1 \subseteq \dots \subseteq \Phi_n = \Phi$ such that all subquotients Φ_j/Φ_{j-1} are isomorphic to $\mathbb{Z}/\ell\mathbb{Z}$ for some prime ℓ. By induction and the long exact cohomology sequence, it thus suffices to deal with the case $\Phi \cong \mathbb{Z}/\ell\mathbb{Z}$. If $\ell \neq \operatorname{char} k$, we can copy the proof of Corollary 2.6.5, with the following two modifications:

(1) The ℓ^{th} power map $(-)^\ell\colon \Gamma(C, \mathcal{O}_C)^\times \to \Gamma(C, \mathcal{O}_C)^\times$ may well fail to be surjective, producing additional $H^1(C_{\text{ét}}, \mathbb{Z}/\ell\mathbb{Z}_C)$, but we don't care.

(2) To get $H^2(C_{\text{ét}}, \mathbb{Z}/\ell\mathbb{Z}_C) = 0$, we must show that $(-)^{\otimes \ell}\colon \operatorname{Pic}(C) \to \operatorname{Pic}(C)$ is surjective. Since k is algebraically closed, C admits a unique *compactification*, i.e., an open embedding $C \hookrightarrow \overline{C}$ into a smooth proper curve (see [Har77, Section I.6] for instance). Note that the restriction morphism $\operatorname{Pic}^0(\overline{C}) \twoheadrightarrow \operatorname{Pic}(C)$ is surjective. Indeed, every line bundle on C is given by some (non-unique) divisor $D \in \operatorname{Div} C$, and we can always choose integer coefficients for the remaining points $x \in \overline{C} \setminus C$ to obtain a divisor $\overline{D} \in \operatorname{Div} \overline{C}$ satisfying $\overline{D}|_C = D$ and $\deg \overline{D} = 0$. It follows from the arguments in Corollary 2.6.5 that $(-)^\ell\colon \operatorname{Pic}^0(\overline{C}) \twoheadrightarrow \operatorname{Pic}^0(\overline{C})$ is surjective. Hence the same is true for $\operatorname{Pic}(C)$.

If $\ell = p = \operatorname{char} k$, we need a different argument. In this case we have a short exact sequence $0 \to \mathbb{Z}/p\mathbb{Z}_C \to \mathcal{O}_{C_{\text{ét}}} \to \mathcal{O}_{C_{\text{ét}}} \to 0$ as in Remark 2.6.6(c). By Corollary 2.4.9(a) we know $H^i(C_{\text{ét}}, \mathcal{O}_{C_{\text{ét}}}) \cong H^i(C_{\text{Zar}}, \mathcal{O}_C)$. But C is affine, so $H^i(C_{\text{Zar}}, \mathcal{O}_C) = 0$ for $i > 0$. Thus, by the long exact cohomology sequence, the assertion holds in this case as well. □

2.6.9. Situation. — From now on, until the end of proof of Lemma 2.6.13, we will usually assume that we are in the following situation:

(1) C is an affine (this condition was missing in the lecture) irreducible curve over the algebraically closed field k, and $j\colon U \hookrightarrow C$ is an open embedding such that U is smooth over k.

(2) $\mathcal{F}$ is a sheaf on $U_{\text{ét}}$.

(3) We fix a closed point $x \in U$. In the special case where $\mathcal{F}$ is an lcc sheaf, we denote $K = \ker(\pi_1^{\text{ét}}(U, x) \to \operatorname{Aut}(\mathcal{F}_x))$ (see the proof of Lemma 2.6.12 for some justification where this morphism comes from).

Our goal is eventually to show that $H^i(C_{\text{ét}}, j_*\mathcal{F}) = 0$ for all $i > 1$ if $\mathcal{F}$ is torsion and lcc. This is done by a beautiful trick, called *méthode de la trace* (which was a well-known technique in Galois cohomology even before people used it in étale cohomology). Here, "trace" refers to the counit morphism $p_!p^* \to \mathrm{id}$ appearing in the proof of Lemma 2.6.10 below. See [SGA$_{4/3}$, Exposé IX.5] for more information.

2.6.10. Lemma. — *Assume we are in Situation 2.6.9. Let $p\colon U' \to U$ be a finite étale morphism. Since $U' \to X$ is still étale, we may choose a diagram (according to Zariski's main theorem)*

$$\begin{tikzcd} U' \arrow[r, hook, "j'"] \arrow[d, "p"'] & C' \arrow[d, "p'"] \\ U \arrow[r, hook, "j"] & C \end{tikzcd}$$

such that p' is finite and j' an open embedding. If $H^i(C'_{\text{ét}}, j'_*p^*\mathcal{F}) = 0$ for some $i \geqslant 0$, then the degree $[U' : U]$ of U' over U annihilates $H^i(C_{\text{ét}}, j_*\mathcal{F})$.

Proof. Since p is finite étale, the functor p^* has both a left-adjoint $p_!$ and a right-adjoint p_*. In fact, one has a functor isomorphism $p_! \xrightarrow{\sim} p_*$. This is obvious if p is a split étale covering. But every finite étale morphism is étale-locally a split étale covering by Lemma* 1.5.5(a), so in general the isomorphism can be defined étale-locally. In particular, composing unit and counit morphisms, we obtain a canonical morphism $\mathcal{F} \to p_*p^*\mathcal{F} \cong p_!p^*\mathcal{F} \to \mathcal{F}$, which equals multiplication by $[U' : U]$ (this can be checked on stalks).

Thus, it suffices to show $H^i(C_{\text{ét}}, j_*p_*p^*\mathcal{F}) = 0$. But the above diagram shows $j_*p_* \cong p'_*j'_*$, hence

$$H^i(C_{\text{ét}}, j_*p_*p^*\mathcal{F}) \cong H^i(C_{\text{ét}}, p'_*j'_*p^*\mathcal{F}) \cong H^i(C_{\text{ét}}, j'_*p^*\mathcal{F}) = 0$$

by assumption. The second isomorphism uses the fact that the Leray spectral sequence (Proposition 2.1.11) collapses for the finite morphism p' by Proposition 2.1.12. □

2.6.11. Lemma. — *In Situation 2.6.9, if $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ is a short exact sequence of torsion sheaves in* $\operatorname{Ab}(U_{\text{ét}})$ *and $H^i(C_{\text{ét}}, j_*\mathcal{F}') = 0 = H^i(C_{\text{ét}}, j_*\mathcal{F}'')$ for all $i > 1$, then also*

$$H^i(C_{\text{ét}}, j_*\mathcal{F}) = 0 \quad \textit{for all } i > 1\,.$$

Proof. Since j_* is left-exact, we have a short exact sequence $0 \to j_*\mathcal{F}' \to j_*\mathcal{F} \to \mathcal{Q} \to 0$, where $\mathcal{Q} = \ker(j_*\mathcal{F}'' \to R^1 j_*\mathcal{F}')$. By the long exact cohomology sequence it suffices to show $H^i(C_{\text{ét}}, \mathcal{Q}) = 0$ for $i > 1$. But since U is dense in C, Fact 2.6.7(b) is clearly applicable to $\mathcal{Q} \hookrightarrow j_*\mathcal{F}''$, hence the claim. □

2.6.12. Lemma. — *In Situation 2.6.9, assume that $\mathcal{F}$ is an ℓ^m-torsion lcc sheaf for some prime ℓ, and that the open subgroup K has index ℓ^n in $\pi_1^{\text{ét}}(U, x)$. Then*

$$H^i(C_{\text{ét}}, j_*\mathcal{F}) = 0 \quad \textit{for all } i > 1\,.$$

Proof. We do induction on $\#\mathcal{F}_x$. The first task is to check that this stalk is indeed a finite set. Since $\mathcal{F}$ is lcc, there is a surjective étale covering $U' \to U$ such that $\mathcal{F}|_{U'_{\text{ét}}}$ is a constant sheaf given by some finite abelian group Φ (that's Definition/Lemma 2.5.5(b)). Then $\mathcal{F}_x = \Phi$ is finite. Moreover, if $\#\mathcal{F}_x < \ell$, then $\Phi = 0$ because Φ must be ℓ^m-torsion for some $m \geqslant 0$. Thus $\mathcal{F} = 0$ and the assertion is trivial.

In general, observe that if F denotes the finite étale U-group scheme representing $\mathcal{F}$ (that's Definition/Lemma 2.5.5(a)), then $\mathrm{Fib}_x(F) = \Phi = \mathcal{F}_x$. In particular, $\pi_1^{\text{ét}}(U, x)$ acts on $\mathcal{F}_x$ via Theorem 1.5.10(a), and for functoriality reasons, the action must be via group automorphisms on $\mathcal{F}_x$. All $\pi_1^{\text{ét}}(U, x)$-orbits in $\mathcal{F}_x$ have cardinality a divisor of $\#\mathcal{F}_x = \ell^m$, hence their cardinality is an integral power of ℓ. Since there is an orbit (the orbit of $0 \in \mathcal{F}_x$) of cardinality 1, but $\#\mathcal{F}_x$ is divisible by ℓ, we deduce that there must be more one-point orbits. In other words, the subgroup $G \subseteq \mathcal{F}_x$ of fixed points of $\pi_1^{\text{ét}}(U, x)$ must be non-zero.

If $G \subsetneq \mathcal{F}_x$ is a proper subgroup, consider the short exact sequence

$$0 \longrightarrow G_U \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}/G_U \longrightarrow 0$$

on $U_{\text{ét}}$. In this case the induction hypothesis is applicable to G_U and $\mathcal{F}/G_U$, and the conclusion follows from Lemma 2.6.11.

It remains to deal with the case where $G = \mathcal{F}_x$, so that $\mathcal{F}_x = \mathrm{Fib}_x(F)$ is a discrete $\pi_1^{\text{ét}}(U, x)$-module. Then Theorem 1.5.10(a) shows that $F \cong \Phi \times U$ is a split group scheme, thus $\mathcal{F} \cong \Phi_U$ is a constant sheaf. Applying Fact 2.6.7(b) to $\Phi_C \to j_*\Phi_U$ shows $H^i(C_{\text{ét}}, j_*\Phi_U) \cong H^i(C_{\text{ét}}, \Phi_C)$ for $i > 1$. Now it seems like Lemma 2.6.8 should do the trick, but we need yet another technical argument to circumvent the fact that C is not smooth. As always, Proposition 1.4.20 allows us to replace C by its reduction C^{red}, so without restriction the irreducible curve C is integral. Consider its normalization $p\colon \widetilde{C} \to C$. Then p is finite because schemes of finite type over a field are universally Japanese, and $\widetilde{C}$ is a normal scheme of dimension 1 over the algebraically closed field k, hence smooth. Using Lemma 2.6.8 and Proposition 2.1.12 we thus get $0 = H^i(\widetilde{C}_{\text{ét}}, \Phi_{\widetilde{C}}) \cong H^i(C_{\text{ét}}, p_*\Phi_{\widetilde{C}})$ for $i > 1$. But the restriction $p^{-1}(U) \xrightarrow{\sim} U$ is an isomorphism because U is already smooth, hence $\Phi_C \to p_*\Phi_{\widetilde{C}}$ is an isomorphism over $U_{\text{ét}}$, and Fact 2.6.7(b) finally seals the deal. □

2.6.13. Lemma. — *In Situation 2.6.9, assume that $\mathcal{F}$ is an ℓ^m-torsion lcc sheaf for some prime ℓ. Then we always have*

$$H^i(C_{\text{ét}}, j_*\mathcal{F}) = 0 \quad \textit{for all } i > 1\,.$$

Proof. If $\pi_1^{\text{ét}}(U, x)/K$ is an ℓ-group, then Lemma 2.6.12 does it. Otherwise let $K' \subseteq \pi_1^{\text{ét}}(U, x)$ be the inverse image of an ℓ-Sylow subgroup of $\pi_1^{\text{ét}}(U, x)/K$. Then the index $(\pi_1^{\text{ét}}(U, x) : K')$ is coprime to ℓ. The left cosets $\pi_1^{\text{ét}}(U, x)/K'$ form a finite set (but not a group in general) equipped with a natural continuous $\pi_1^{\text{ét}}(U, x)$-action. Hence it defines an étale covering

$p\colon U' \to U$ via Theorem 1.5.10(a). Also note that U' is connected because the action of $\pi_1^{\text{ét}}(U,x)$ on $\pi_1^{\text{ét}}(U,x)/K'$ is transitive, and if $x' \in U'$ is a lift of x, then $\pi_1^{\text{ét}}(U',x') = K'$.

Choose $j'\colon U' \hookrightarrow C'$ and $p'\colon C' \to C$ as in Lemma 2.6.10. Since being lcc is an étale-local property by Definition/Lemma 2.5.5(c), $\mathcal{F}' = p^*\mathcal{F}$ is lcc again, and satisfies $\mathcal{F}'_{x'} = \mathcal{F}_x$ because p^* preserves stalks. Thus, the kernel of $K' = \pi_1^{\text{ét}}(U',x') \to \operatorname{Aut}(\mathcal{F}'_{x'}) = \operatorname{Aut}(\mathcal{F}_x)$ is given by K again, and by construction of K', the index $(K' : K)$ is a power of ℓ. Thus Lemma 2.6.12 is applicable and shows $H^i(C'_{\text{ét}}, j'_*\mathcal{F}') = 0$ for $i > 1$. And now comes the trick! Lemma 2.6.10 shows that $H^i(C_{\text{ét}}, j_*\mathcal{F})$ is annihilated by $[U' : U]$. By construction, $[U' : U] = (\pi_1^{\text{ét}}(U,x) : K')$ is coprime to ℓ. But since $\mathcal{F}$ is an ℓ^m-torsion sheaf, $H^i(C_{\text{ét}}, j_*\mathcal{F})$ is also annihilated by ℓ^m. Thus $H^i(C_{\text{ét}}, j_*\mathcal{F})$ is annihilated by $\gcd(\ell^m, [U' : U]) = 1$ for $i > 1$, hence vanishes, as required. □

2.6.14. Lemma. — *If C is an affine irreducible curve over an algebraically closed field k, and $\mathcal{F}$ a constructible sheaf on C, then*

$$H^i(C_{\text{ét}}, \mathcal{F}) = 0 \quad \textit{for all } i > 1\,.$$

Proof. As usual, Proposition 1.4.20 allows us to replace C by C^{red}, hence we may assume that C is integral. Since constructible sheaves are noetherian and torsion, we find a filtration $0 = \mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \cdots \subseteq \mathcal{F}_n = \mathcal{F}$ such that all $\mathcal{F}_j/\mathcal{F}_{j-1}$ are ℓ^m-torsion for some prime ℓ. By induction and the long exact cohomology sequence, it thus suffices to consider the case where $\mathcal{F}$ is ℓ^m-torsion itself.

Since C is integral and k is a perfect field by assumption, Grothendieck's generic freeness theorem shows that C is *generically smooth*, i.e., there is a non-empty open subset $U \subseteq C$ such that U is smooth over k. Since $\mathcal{F}$ is constructible, we find another non-empty open subset $U_1 \subseteq C$ such that $\mathcal{F}|_{U_{1,\text{ét}}}$ is lcc. Replace U by $U \cap U_1$, which is non-empty as C is irreducible. Put $j\colon U \hookrightarrow C$. Then $\mathcal{F}|_{U_{\text{ét}}}$ is lcc and we are in a situation where Lemma 2.6.13 is applicable. Thus $H^i(C_{\text{ét}}, j_*\mathcal{F}|_{U_{\text{ét}}}) = 0$. But $\mathcal{F} \to j_*\mathcal{F}|_{U_{\text{ét}}}$ is an isomorphism over $U_{\text{ét}}$, hence Fact 2.6.7(b) proves $H^i(C_{\text{ét}}, \mathcal{F}) \cong H^i(C_{\text{ét}}, j_*\mathcal{F}|_{U_{\text{ét}}})$ for $i > 1$. This finishes the proof. □

After all these special cases, we can finally prove the desired result in full generality.

2.6.15. Proposition. — *Let C be a curve over a separably closed field k and $\mathcal{F}$ a torsion sheaf on $C_{\text{ét}}$.*

(a) *If C is affine, we have $H^i(C_{\text{ét}}, \mathcal{F}) = 0$ for $i > 1$.*

(b) *If C is arbitrary, we have $R^i\zeta_{C,*}\mathcal{F} = 0$ for $i > 1$, and $H^i(C_{\text{ét}}, \mathcal{F}) = 0$ for $i > 2$.*

Proof. In (a), we may assume that $\mathcal{F}$ is constructible, because of Proposition 2.5.20(c) and Corollary 2.4.14. Moreover, since pullbacks of constructible sheaves are constructible again by Proposition 2.5.20(a), it's safe to apply 2.6.1, whence we may assume that k is algebraically closed. Let $C_1, \dots, C_n$ be the irreducible components of C (equipped with their reduced closed subscheme structures) and $i_j\colon C_j \hookrightarrow C$ their closed embeddings. Then $\mathcal{F} \hookrightarrow \bigoplus_{j=1}^n i_{j,*}i_j^*\mathcal{F}$ is a monomorphism, and induces isomorphisms on stalks at geometric points lying over the generic points of $C_1, \dots, C_n$. Thus Fact 2.6.7(b) is applicable and we obtain

$$H^i(C_{\text{ét}}, \mathcal{F}) \cong \bigoplus_{j=1}^n H^i\big(C_{\text{ét}}, i_{j,*}i_j^*\mathcal{F}\big) \cong \bigoplus_{j=0}^n H^i\big(C_{j,\text{ét}}, i_j^*\mathcal{F}\big) = 0$$

for all $i > 0$. Here we used Proposition 2.1.12 for the second isomorphism and Lemma 2.6.14 to get 0 on the right-hand side.

The first assertion of (b) follows from (a) after sheafification. For the second assertion, consider the Leray-type spectral sequence from Proposition 2.1.11

$$E_2^{p,q} = H^p\big(C_{\mathrm{Zar}}, R^q\zeta_{C,*}\mathcal{F}\big) \Longrightarrow H^{p+q}(C_{\text{ét}}, \mathcal{F})\,.$$

Part (a) shows that $E_2^{p,q} = 0$ for $q > 1$. Also C_{Zar} is a one-dimensional noetherian topological space, hence $E_2^{p,q} = 0$ for $p > 1$ by Grothendieck's theorem on cohomological dimension. Thus $E_2^{p,q} = 0$ if $p + q > 2$, which proves that the limit $H^i(C_{\text{ét}}, \mathcal{F})$ vanishes for $i > 2$, as claimed. This finishes the proof, the lecture, and the section ... □

... well, at least on Professor Franke's part. However, to give a complete proof of the proper base change theorem (Theorem 3.1.3 in the next chapter), it seems to me that we need another result about étale cohomology of proper curves (see the discussion in Remark 3.1.7). We will later generalize it to arbitrary dimensions (see Corollary* 3.1.5), but the the proof of the general case depends on the one-dimensional case. So here we go.

2.6.16. Proposition*. — *Let K/k be a (not necessarily algebraic) extension of separably closed fields. Let C be a proper curve over k and $C_K = C \times_k \operatorname{Spec} K$. If $\mathcal{F}$ is a torsion sheaf on $C_{\text{ét}}$ and $\mathcal{F}_K$ its pullback to $C_{K,\text{ét}}$, then the canonical morphism*

$$H^i(C_{\text{ét}}, \mathcal{F}) \xrightarrow{\sim} H^i\big(C_{K,\text{ét}}, \mathcal{F}_K\big)$$

is an isomorphism for all $i \geqslant 0$.

Sketch of a proof.* Fortunately, the proof is virtually the same as the proof of Proposition 2.6.15. Basically all we need to do is to replace every assertion of the form "$H^i(C_{\text{ét}}, \mathcal{G}) = 0$ for some sheaf $\mathcal{G}$ and some $i > 1$" by "Proposition* 2.6.16 holds for $\mathcal{G}$". To make this a bit clearer, we go through each of the steps. Before we start, observe that 2.6.1 allows us to replace k and K by their algebraic closures. So in what follows, all fields are algebraically closed.

(1) *Suppose C is smooth and proper, and Φ is a finite abelian group. Then Proposition* 2.6.16 holds for the constant sheaf Φ_C, i.e., $H^i(C_{\text{ét}}, \Phi_C) \xrightarrow{\sim} H^i(C_{K,\text{ét}}, \Phi_{C_K})$ for all $i \geqslant 0$.*

Using induction, the long exact cohomology sequence, and the five lemma, we can reduce (1) to the case $\Phi = \mathbb{Z}/\ell\mathbb{Z}$ for some prime ℓ. If ℓ is invertible in k, then the assertion follows basically from Corollary 2.6.5 and the fact that the genus of a smooth curve doesn't change upon base change. The case where $\ell = p > 0$ is the characteristic of k uses the Artin–Schreier sequence of course, but there's a trick involved, whence we refer to [Stacks, Tag 0A3P].

(2) *Suppose C is proper and $\mathcal{F}$ is as in Fact 2.6.7(a). Then Proposition* 2.6.16 holds for $\mathcal{F}$. Moreover, if $\varphi\colon \mathcal{F} \to \mathcal{G}$ is a morphism as in Fact 2.6.7(b) and Proposition* 2.6.16 holds for either of $\mathcal{F}$ or $\mathcal{G}$, then it holds for the other one as well.*

To see the first assertion, we have $\mathcal{F} \cong \bigoplus_{x \in C \setminus U} x_*\mathcal{F}_x$, as observed in the proof of Fact 2.6.7(a). Thus, the assertion reduces to a rather trivial property of the étale cohomology of a point. The second assertion follows from the first, using the same argument as in the proof of Fact 2.6.7(b), plus the five lemma.

(3) *Claim* (1) *holds for arbitrary proper curves, not only for smooth ones.*

If C is integral, we can consider its normalization $p\colon \widetilde{C} \to C$. Using generic smoothness, we see that $\Phi_C \to p_*\Phi_{\widetilde{C}}$ is an isomorphism over some dense open subset $U \subseteq C$. Thus, by (2), it suffices to prove the assertion for $p_*\Phi_{\widetilde{C}}$ instead of Φ_C. Now $H^i(C_{\text{ét}}, p_*\Phi_{\widetilde{C}}) \cong H^i(\widetilde{C}_{\text{ét}}, \Phi_{\widetilde{C}})$ by

Proposition 2.1.12. Moreover, pushforward along the finite morphism p commutes with base change to $\widetilde{C}_K$ and C_K respectively by Fact* 2.5.12(a), hence (1) shows that the assertion holds for $p_*\Phi_{\widetilde{C}}$, as required.

The reduction from arbitrary proper curves to proper integral curves is done by a similar argument, as in the proof of Proposition 2.6.15(b) for example. This settles the case of constant sheaves, and thus by (2) also the case of sheaves which are only constant on a dense open subset. In general, however, constructible sheaves are only constant after restriction to an étale C-scheme rather than an open subscheme of C. Thus, the *méthode de la trace* has to be invoked once again.

(4) *Assume we are in the situation of Lemma* 2.6.10*, except that C, and thus C', are proper rather than affine. Denote by $\pi\colon C_K \to C$ and $\pi'\colon C'_K \to C'$ the canonical projections. If*

$$H^i\big(C'_{\text{ét}}, j'_*p^*\mathcal{F}\big) \xrightarrow{\sim} H^i\big(C'_{K,\text{ét}}, \pi'^*j'_*p^*\mathcal{F}\big)$$

*is an isomorphism (in other words, if Proposition** 2.6.16 *holds for $j'_*p^*\mathcal{F}$), then kernel and cokernel of*

$$H^i\big(C_{\text{ét}}, j_*\mathcal{F}\big) \longrightarrow H^i\big(C_{K,\text{ét}}, \pi^*j_*\mathcal{F}\big)$$

are annihilated by $[U' : U]$.

It suffices to prove that $H^i(C_{\text{ét}}, j_*p_*p^*\mathcal{F}) \xrightarrow{\sim} H^i(C_{K,\text{ét}}, \pi^*j_*p_*p^*\mathcal{F})$ is an isomorphism, because as in the proof of Lemma 2.6.10, we get that $\mathcal{F} \to p_*p^*\mathcal{F} \cong p_!p^*\mathcal{F} \to \mathcal{F}$ is multiplication by $[U' : U]$. As done there, we calculate $H^i(C_{\text{ét}}, j_*p_*p^*\mathcal{F}) \cong H^i(C'_{\text{ét}}, j'_*p^*\mathcal{F})$, and the assertion follows from the assumption about $j'_*p^*\mathcal{F}$ plus Fact* 2.5.12(a) to ensure that pushforward along the finite morphism p behaves well under base change.

(5) *Assume we are in Situation* 2.6.9*, except that C is proper rather than affine. Given a short exact sequence $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ such that Proposition** 2.6.16 *holds for $j_*\mathcal{F}'$ and $j_*\mathcal{F}''$, it also holds for $j_*\mathcal{F}$.*

To prove (5), use (2) and the arguments from the proof of Lemma 2.6.11, plus the five lemma.

(6) *Assume C is proper and integral, and $j\colon U \hookrightarrow C$ is a non-empty open subset which is smooth over k. Let $\mathcal{F}$ be an ℓ^m-torsion lcc sheaf on U for some prime ℓ. Then Proposition** 2.6.16 *holds for $j_*\mathcal{F}$.*

Choose a closed/geometric point $x \in U$ and let $K = \ker(\pi_1^{\text{ét}}(U, x) \to \operatorname{Aut}(\mathcal{F}_x))$. If K has index ℓ^n in $\pi_1^{\text{ét}}(U, x)$, we use (5) and the same inductive argument as in the proof of Lemma 2.6.12 to reduce the assertion to the case where $\mathcal{F}$ is constant (i.e. the case $G = \mathcal{F}_x$ in the proof of Lemma 2.6.12). Using (2), we then replace $j_*\mathcal{F}$ by a constant sheaf on C, and (3) does the job.

For general K, we construct an étale covering $p\colon U' \to U$ as in the proof of Lemma 2.6.13. Using (4), we thus see that kernel and cokernel of $H^i(C_{\text{ét}}, j_*\mathcal{F}) \to H^i(C_{K,\text{ét}}, \pi^*j_*\mathcal{F})$ are annihilated by $[U' : U]$. But since $\mathcal{F}$ is ℓ^m-torsion, the kernel and cokernel above are annihilated by ℓ^m too. Thus they must vanish, as $\gcd(\ell^m, [U' : U]) = 1$ by construction.

(7) *Proposition** 2.6.16 *is true.*

Let's first assume C is proper and integral and $\mathcal{F}$ is a constructible sheaf on C. As in the proof of Lemma 2.6.14, we may assume that $\mathcal{F}$ is ℓ^m-torsion for some prime ℓ. Moreover, there exists a dense open subscheme $j\colon U \hookrightarrow C$ such that U is smooth and $\mathcal{F}|_{U_{\text{ét}}}$ is lcc. By (2) we may replace $\mathcal{F}$ by $j_*\mathcal{F}|_{U_{\text{ét}}}$, which satisfies the assertion by (6). In general, we use Proposition 2.5.20 and Corollary 2.4.14 to replace torsion sheaves by constructible sheaves, and an argument as in (3) to reduce to the case where C is integral. ∎

CHAPTER 3.

Proper Base Change

3

3.1. Formulation and Proof of the Main Theorem

LECTURE 24
27th Jan, 2020

3.1.1. — Let $g\colon X' \to X$ be a morphism of schemes. If $\mathcal{F} \in \mathrm{Ab}(X_{\text{ét}})$ is an étale sheaf, we have a functorial morphism $g^*\colon \Gamma(X, \mathcal{F}) \to \Gamma(X', g^*\mathcal{F})$. Applying the universal property of derived functors gives a pullback morphism

$$g^*\colon H^\bullet(X_{\text{ét}}, \mathcal{F}) \longrightarrow H^\bullet(X'_{\text{ét}}, g^*\mathcal{F}) \tag{3.1.1}$$

of cohomological functors (note that the right-hand side is indeed a cohomological functor, because $g^*\colon \mathrm{Ab}(X_{\text{ét}}) \to \mathrm{Ab}(X'_{\text{ét}})$ is exact). Now suppose g sits in a pullback diagram

$$\begin{tikzcd} X' \arrow[r, "g"] \arrow[d, "p'"'] \arrow[dr, phantom, "\lrcorner"] & X \arrow[d, "p"] \\ S' \arrow[r, "f"] & S \end{tikzcd} .$$

For all $i \geqslant 0$ let $\mathcal{H}^i(\mathcal{F})$ be the presheaf on $S_{\text{ét}}$ given by $U \mapsto H^i((X \times_S U)_{\text{ét}}, \mathcal{F})$. Likewise, let $\mathcal{H}^i(g^*\mathcal{F})$ be the presheaf on $S'_{\text{ét}}$ given by $U' \mapsto H^i((X' \times_{S'} U')_{\text{ét}}, g^*\mathcal{F})$. Observe that the sheafification of $\mathcal{H}^i(\mathcal{F})$ is $R^ip_*\mathcal{F}$ and the sheafification of $\mathcal{H}^i(g^*\mathcal{F})$ is $R^ip'_*g^*\mathcal{F}$. Moreover, (3.1.1) provides a canonical morphism

$$\mathcal{H}^i(\mathcal{F}) \longrightarrow f_*\mathcal{H}^i(g^*\mathcal{F}) .$$

Indeed, after unraveling definitions we see that such a morphism of presheaves is given by a system of compatible morphism $H^\bullet((X \times_S U)_{\text{ét}}, \mathcal{F}) \to H^\bullet((X' \times_S U)_{\text{ét}}, g^*\mathcal{F})$ for $U \in S_{\text{ét}}$, which can be obtained by applying (3.1.1) to the base change $X' \times_S U \to X \times_S U$ of g instead to g itself (then also compatibility is clear from the universal property of derived functors). Thus, after sheafification, we get a canonical morphism $R^ip_*\mathcal{F} \to f_*R^ip'_*g^*\mathcal{F}$. Taking the adjoint and letting i vary finally provides a natural morphism of cohomological functors

$$f^*R^\bullet p_*\mathcal{F} \longrightarrow R^\bullet p'_*g^*\mathcal{F} , \tag{3.1.2}$$

which is called the *base change morphism.*

3.1.2. Remark. — We would like to work out some important special cases, and analyze the behaviour of (3.1.2) under composition.

(*a*) If g is finite, then $R^ig_*g^*\mathcal{F} = 0$ for all $i > 0$ by Proposition 2.1.12, hence an isomorphism $H^i(X'_{\text{ét}}, g^*\mathcal{F}) \cong H^i(X_{\text{ét}}, g_*g^*\mathcal{F})$ holds by the Leray spectral sequence (Proposition 2.1.11). The same is true more generally when g is only integral. The idea is to write $g\colon X' \to X$ as a cofiltered limit over finite morphisms $g_\alpha\colon X_\alpha \to X$ (which is, at least étale-locally, always possible). One then checks that Proposition 2.4.12 provides an isomorphism $\operatorname{colim} \mathcal{H}^i(g_\alpha^*\mathcal{F}) \cong \mathcal{H}^i(g^*\mathcal{F})$. Sheafification, being a left-adjoint functor, commutes with colimits, hence $0 = \operatorname{colim} R^i g_{\alpha,*} g_\alpha^* \mathcal{F} \cong R^i g_* g^* \mathcal{F}$ for all $i > 0$, as claimed.

In this case, (3.1.1) may be obtained by applying $H^\bullet(X_{\text{ét}}, -)$ to the unit $\mathcal{F} \to g_* g^* \mathcal{F}$ of the g^*-g_* adjunction (which, as usual, follows from the uniqueness part of the universal property of derived functors).

(b) A special case of (3.1.2) is the following: suppose $\overline{s}\colon \operatorname{Spec}\kappa(\overline{s}) \to S$ is a geometric point of S, and we have $S' = \operatorname{Spec}\kappa(s)$ and $f = \overline{s}\colon \operatorname{Spec}\kappa(\overline{s}) \to S$. Then the base change morphism takes the form

$$(R^\bullet p_*\mathcal{F})_{\overline{s}} \longrightarrow H^\bullet\big(X_{0,\text{ét}}, \operatorname{pr}_1^*\mathcal{F}\big)\,,$$

where $X_0 = X \times_S \operatorname{Spec}\kappa(\overline{s})$ and $\operatorname{pr}_1\colon X_0 \to X$ is the projection to the first factor.

(c) The base change morphism (3.1.2) satisfies the obvious compatibility relation: given a double pullback diagram

$$\begin{tikzcd} X'' \arrow[r, "g'"] \arrow[d, "p''"'] \arrow[dr, phantom, "\lrcorner"] & X' \arrow[r, "g'"] \arrow[d, "p'"] \arrow[dr, phantom, "\lrcorner"] & X \arrow[d, "p"] \\ S'' \arrow[r, "f'"] & S' \arrow[r, "f"] & S \end{tikzcd}\,,$$

the diagram of base change morphisms

$$\begin{tikzcd} f'^* f^* R^\bullet p_* \mathcal{F} \arrow[r, "(3.1.2)"] \arrow[d, "f'^*(3.1.2)"'] & R^\bullet p''_*(f \circ f')^* \mathcal{F} \arrow[d, equal] \\ f'^* R^\bullet p'_* g^* \mathcal{F} \arrow[r, "(3.1.2)"] & R^\bullet p''_* f'^* f^* \mathcal{F} \end{tikzcd}$$

is commutative. Like all assertions of these kind, this follows from the uniqueness part of the universal property of derived functors.

And now, after months of teasing and foreshadowing, now is the moment when we finally state and prove the proper base change theorem!

3.1.3. Theorem (Artin/Grothendieck)**.** — *Suppose we are in the situation of* 3.1.1 *(all schemes are not necessarily noetherian). If $p\colon X \to S$ is proper and $\mathcal{F}$ is a torsion sheaf on $X_{\text{ét}}$ in the sense of Remark** 2.5.8(c)*, then the base change morphism* (3.1.2) *is an isomorphism*

$$f^* R^i p_* \mathcal{F} \xrightarrow{\sim} R^i p'_* g^* \mathcal{F}$$

for all $i \geqslant 0$.

Before we get into the proof, we take the time to discuss some corollaries. The first one wasn't mentioned in the lecture, and is actually not a corollary, but a generalization to derived categories (stop whining already!). I decided to include it anyway, because—believe me—it will make the "dévissage" arguments in the proof of Theorem 3.1.3 *so much cleaner*! We denote by $D^+(X_{\text{ét}})$ the bounded below derived category of $\operatorname{Ab}(X_{\text{ét}})$.

3.1.4. Corollary*. — *Suppose we are in the situation of 3.1.1. If $p\colon X \to S$ is proper and $K^\bullet \in D^+(X_{\text{ét}})$ is a complex whose cohomology are torsion sheaves, then the canonical morphism*

$$f^* Rp_* K^\bullet \xrightarrow{\sim} Rp'_* g^* K^\bullet$$

is an isomorphism in $D^+(S'_{\text{ét}})$.

Proof.* This is a formal consequence of Theorem 3.1.3, but I haven't done such a thing before, so I try to be in-depth. We first reduce the assertion to bounded complexes $K^\bullet \in D^b(X_{\text{ét}})$. Fix a cohomological degree i in which to check the isomorphism. $Rp_* K^\bullet$ can be computed as follows: choose a quasi-isomorphism $K^\bullet \xrightarrow{\sim} \mathcal{I}^\bullet$ into a complex of injective sheaves, and take $Rp_* K^\bullet = p_* \mathcal{I}^\bullet$. The standard procedure to produce $\mathcal{I}^\bullet$ is to take a Cartan–Eilenberg resolution $K^\bullet \to \mathcal{J}^{\bullet,\bullet}$ and then take $\mathcal{I}^\bullet$ to be the total complex of $\mathcal{J}^{\bullet,\bullet}$. Let $\tau_{\leqslant n} K^\bullet$ be the soft truncation of $K^\bullet$, i.e., the complex obtained by replacing K_n by $\ker(K^n \to K^{n+1})$ and K^j for $j > n$ by 0. Also note that we can choose a Cartan–Eilenberg resolution of $\tau_{\leqslant n} K^\bullet$ in such a way that it is part of a Cartan–Eilenberg resolution of $K^\bullet$. In particular, the induced injective resolutions of both complexes will coincide up to degree $n-1$. Choosing n large enough, we get that $f^* Rp_* K^\bullet$ and $f^* Rp_* \tau_{\leqslant n} K^\bullet$ coincide in cohomological degree i. A similar argument applies to the right-hand side, whence we have reduced the assertion to finite complexes.

To deal with finite complexes, we do induction on the number of non-zero cohomological degrees. If that number is 1, we have $K^\bullet \cong \mathcal{F}[i]$ for some torsion sheaf $\mathcal{F}$ placed in some degree i, and the assertion follows from Theorem 3.1.3. Now assume that $K^\bullet$ is concentrated between degrees 0 and $n+1$. The mapping cone of $\tau_{\leqslant n} K^\bullet \to K^\bullet$ is quasi-isomorphic to a complex consisting of $\mathcal{F} = H^{n+1}(K^\bullet)$, which is a torsion sheaf, placed in degree $n+1$. Applying the induction hypothesis to $\tau_{\leqslant n} K\bullet$ and $\mathcal{F}[n+1]$ proves the assertion for $K^\bullet$ by means of the distinguished triangle $\tau_{\leqslant n} K^\bullet \to K^\bullet \to \mathcal{F}[n+1] \to \tau_{\leqslant n} K^\bullet[1]$ in $D^+(X_{\text{ét}})$. $\square$

We would like to discuss another two corollaries. The first one wasn't in the lecture, but it's closely related to a technical complication that Professor Franke just ignored—and so did Deligne in [$\text{SGA}_{4\frac{1}{2}}$, Arcata IV] (or I'm just too dumb to see a trivial argument). So it's definitely worth pointing out.

3.1.5. Corollary*. — *Let K/k be a (not necessarily algebraic) extension of separably closed fields. Let X be a proper scheme over k and $X_K = X \times_k \operatorname{Spec} K$. If $\mathcal{F}$ is a torsion sheaf on $X_{\text{ét}}$ and $\mathcal{F}_K$ its pullback to $X_{K,\text{ét}}$, then the canonical morphism*

$$H^i(X_{\text{ét}}, \mathcal{F}) \xrightarrow{\sim} H^i\big(X_{K,\text{ét}}, \mathcal{F}_K\big)$$

is an isomorphism for all $i \geqslant 0$.

3.1.6. Corollary. — *Let $p\colon X \to S$ be proper and $S = \operatorname{Spec} A$, where A is a strictly henselian local ring with residue field k. Let $S_0 = \operatorname{Spec} k$ and $\overline{s}\colon S_0 \hookrightarrow S$ denote the corresponding geometric point. Then for all torsion sheaves $\mathcal{F}$ on $X_{\text{ét}}$ the canonical morphism*

$$H^i(X_{\text{ét}}, \mathcal{F}) \xrightarrow{\sim} H^i\big(X_{0,\text{ét}}, \operatorname{pr}_1^* \mathcal{F}\big)$$

is an isomorphism for all $i \geqslant 0$, where $X_0 = X \times_S S_0$ and $\operatorname{pr}_1\colon X_0 \to X$ is the projection to the first factor.

3.1.7. Remark. — Corollary* 3.1.5 is the special case of Theorem 3.1.3 applied in the case where $S = \operatorname{Spec} k$ and $S' = \operatorname{Spec} K$. Similarly, Corollary 3.1.6 is the special case of Theorem 3.1.3 applied in the situation of Remark 3.1.2(b) (if that's not clear to you, it will be after the discussion that follows).

But conversely, Theorem 3.1.3 already follows from Corollary* 3.1.5 and Corollary 3.1.6! Indeed, to check whether the base change morphism is an isomorphism is a stalk-wise task. So let $\overline{s}'$ be a geometric point of S' with image $f(\overline{s}') = \overline{s}$ in S. Put $X_{\overline{s}} = X \times_S \operatorname{Spec} \mathcal{O}_{S_{\text{ét}},\overline{s}}$ and $X'_{\overline{s}'} = X' \times_{S'} \operatorname{Spec} \mathcal{O}_{S'_{\text{ét}},\overline{s}'}$. Then Corollary 2.4.15 shows

$$\left(f^* R^i p_* \mathcal{F}\right)_{\overline{s}'} \cong H^i\left(X_{\overline{s},\text{ét}}, \mathcal{F}\right) \quad \text{and} \quad \left(R^i p'_* g^* \mathcal{F}\right)_{\overline{s}'} \cong H^i\left(X'_{\overline{s}',\text{ét}}, g^* \mathcal{F}\right),$$

where we omit some pr_1^* to not overload the notation. Let X_0 and X'_0 be obtained from $X_{\overline{s}}$ and $X_{\overline{s}'}$ as in Corollary 3.1.6. Then

$$H^i\left(X_{\overline{s},\text{ét}}, \mathcal{F}\right) \cong H^i\left(X_{0,\text{ét}}, \mathrm{pr}_1^* \mathcal{F}\right) \cong H^i\left(X'_{0,\text{ét}}, \mathrm{pr}_1^* g^* \mathcal{F}\right) \cong H^i\left(X'_{\overline{s}',\text{ét}}, g^* \mathcal{F}\right),$$

where the middle isomorphism follows from Corollary* 3.1.5 and the outer ones follow from Corollary 3.1.6. This shows that Theorem 3.1.3 indeed follows from its to corollaries. However, I don't see how Corollary 3.1.6 alone would suffice, as was claimed in the lecture and by Deligne in [$\mathrm{SGA}_{4\frac{1}{2}}$, Arcata IV]. In general, the extension $\kappa(\mathcal{O}_{S'_{\text{ét}},\overline{s}'})/\kappa(\mathcal{O}_{S_{\text{ét}},\overline{s}})$ of residue fields can be arbitrarily bad, and there's no reason why we could reduce to the case where the extension is finite or even just algebraic. So we need to work around this f*ck-up here. For that reason, some of the proofs are a bit different from the lecture.

Apart from that annoyance, our considerations above can be adapted if we impose one of the following additional restrictions.

(1) To prove Theorem 3.1.3 in the special case where the fibres of $p\colon X \to S$ have dimension $\leqslant 1$, it suffices to prove Corollary* 3.1.5 and Corollary 3.1.6 in the corresponding special cases. This is particularly nice, because we already know Corollary* 3.1.5 in that situation (Proposition* 2.6.16 for dimension one; the zero-dimensional case is quite trivial).

(2) To prove Theorem 3.1.3 in the special case where $p\colon X \to S$ is not only proper, but locally (on S) projective, it suffices to prove Corollary* 3.1.5 and Corollary 3.1.6 in the corresponding special cases.

3.1.8. Remark*. — The upshot of Remark 3.1.7 is that instead of Corollary* 3.1.5 and Corollary 3.1.6, it also suffices to prove the following single assertion, which somehow combines both:

(⊠) *Assume we are in the situation of Corollary 3.1.6 and notation is as given there, with the only exception that $\overline{s}\colon \operatorname{Spec} \kappa(\overline{s}) \to S$ is an arbitrary geometric point over k and $S_0 = \operatorname{Spec} \kappa(\overline{s})$. Then the natural morphism*

$$H^i(X_{\text{ét}}, \mathcal{F}) \xrightarrow{\sim} H^i\left(X_{0,\text{ét}}, \mathrm{pr}_1^* \mathcal{F}\right).$$

is an isomorphism for all $i \geqslant 0$.

The first application of (⊠) is to reduce everything to the noetherian case. Once we proved Lemma* 3.1.9 below, we will always assume we are in a noetherian situation without further mention.

3.1.9. Lemma*. — *It suffices to prove Theorem 3.1.3 in the special case where S and X are noetherian and $\mathcal{F}$ is a constructible sheaf on $X_{\text{ét}}$.*

Proof.* We first observe that if $(\mathcal{F}_\alpha)$ is a filtered system of sheaves on $X_{\text{ét}}$ such that Theorem 3.1.3 holds for all $\mathcal{F}_\alpha$, then it also holds for $\mathcal{F} = \operatorname{colim} \mathcal{F}_\alpha$. Indeed, it suffices to check ($\boxtimes$) above. Using that pullbacks commute with colimits and Corollary 2.4.14, we see that we can pull out the colimit on both sides of $H^i(X_{\text{ét}}, \operatorname{colim} \mathcal{F}) \rightarrow H^i(X_{0,\text{ét}}, \operatorname{colim} \operatorname{pr}_1^* \mathcal{F}_\alpha)$, and the claim follows immediately.

In particular, we may assume that $\mathcal{F}$ is of the form

$$\mathcal{F} \cong \operatorname{coker}\left(\bigoplus_{l \in \mathfrak{L}} j_{l,!}\big(\mathbb{Z}/M_l\mathbb{Z}_{V_l}\big) \xrightarrow{\varphi} \bigoplus_{k \in \mathfrak{K}} i_{k,!}\big(\mathbb{Z}/N_k\mathbb{Z}_{U_k}\big)\right),$$

where $\mathfrak{K}$ and $\mathfrak{L}$ are finite indexing sets, $i_k \colon U_k \rightarrow X$ and $j_l \colon V_l \rightarrow X$ are étale morphisms, N_k and M_l are non-zero integers, and φ is any morphism of sheaves. Indeed, if we drop the condition that $\mathfrak{K}$ and $\mathfrak{L}$ be finite, every torsion sheaf can be written in that form. Thus, every torsion sheaf can be written as a cofiltered limit of sheaves of the above form, as required.

To see that the noetherian constructible case suffices, it's again enough to check ($\boxtimes$); moreover, we may assume that $\mathcal{F}$ is as above. Write $S = \lim S_\alpha$ as a cofiltered limit of spectra of noetherian strictly henselian rings. This is always possible: A can be written as $\operatorname{colim} T_\alpha$, where the colimit is taken over all finite type $\mathbb{Z}$-subalgebras of A. Now let $\mathfrak{m}_\alpha \in \operatorname{Spec} T_\alpha$ be the preimage of the maximal ideal $\mathfrak{m}$ of A, and put $A_\alpha = (T_\alpha)^{\text{sh}}_{\mathfrak{m}_\alpha}$. Then A_α is strictly henselian and noetherian by Proposition 1.6.15(e), and $S = \lim \operatorname{Spec} A_\alpha$, as required. Using ($i$) and ($j$) from Appendix A.1, we can write $p \colon X \rightarrow S$ as a simulatenous pullback and cofiltered limit of $p_\alpha \colon X_\alpha \rightarrow S_\alpha$ for sufficiently large α. Moreover, $\mathcal{F}$ can be written as the pullback of some sheaf $\mathcal{F}_\alpha \in \operatorname{Ab}(X_{\alpha,\text{ét}})$, which has the same form as $\mathcal{F}$. This can be seen from the arguments in the proof of Lemma* 2.5.19 (which is stated in a noetherian setting, but that's not used in the part of the proof we are interested in). In particular, $\mathcal{F}_\alpha$ is constructible by Proposition 2.5.20(a).

Let $\mathcal{F}_\beta$ for $\beta \geqslant \alpha$ be the pullbacks of $\mathcal{F}_\alpha$ to X_β. Then Proposition 2.4.12 shows

$$\operatorname*{colim}_{\beta \geqslant \alpha} H^i\big(X_{\beta,\text{ét}}, \mathcal{F}_\beta\big) \cong H^i(X_{\text{ét}}, \mathcal{F})\,.$$

Since $\bar{s}$ is a geometric point of ever S_β as well, we can express $H^i(X_{0,\text{ét}}, \operatorname{pr}_1^* \mathcal{F})$ as a similar colimit. Thus ($\boxtimes$) reduces to the noetherian constructible case, as claimed. $\square$

3.1.10. Lemma. — *Suppose we are given a commutative diagram*

$$\begin{tikzcd} Y' \arrow[r, "h"] \arrow[d, "q'"'] \arrow[dr, phantom, "\lrcorner"] & Y \arrow[d, "q"] \\ X' \arrow[r, "g"] \arrow[d, "p'"'] \arrow[dr, phantom, "\lrcorner"] & X \arrow[d, "p"] \\ S' \arrow[r, "f"] & S \end{tikzcd}$$

in which S, X, and Y are noetherian, and p, q are proper. If Theorem 3.1.3 holds for p and q, then it holds for $p \circ q$ as well.

Sketch of a proof.* Professor Franke's intent was to use Corollary 3.1.6 and Remark 3.1.2(a) to avoid a messy Leray spectral sequence argument, but his proof is flawed as pointed out in Remark 3.1.7. However, it turns out that the brute-force Leray approach is not too horrible either. By Lemma* 3.1.9, we may assume that $\mathcal{F}$ is a constructible sheaf on $Y_{\text{ét}}$. Using Proposition 2.1.11, we get two spectral sequences

$$E_2^{i,j} = f^*R^ip_*R^jq_*\mathcal{F} \Longrightarrow f^*R^{i+j}(p \circ q)_*\mathcal{F}\,, \quad E_2^{i,j} = R^ip'_*R^jq'_*h^*\mathcal{F} \Longrightarrow R^{i+j}(p' \circ q')_*h^*\mathcal{F}$$

(for the first one we use that f^* is an exact functor). Since $\mathcal{F}$ is constructible, it is annihilated by some non-zero integer $N \neq 0$. Then $R^jq_*\mathcal{F}$ is annihilated by N as well, and thus a torsion sheaf. Therefore, since Theorem 3.1.3 holds for p and q, we get

$$f^*R^ip_*R^jq_*\mathcal{F} \cong R^ip'_*g^*R^jq_*\mathcal{F} \cong R^ip'_*R^jq'_*h^*\mathcal{F}\,.$$

Thus the E_2-pages of both spectral sequences are isomorphic (to be honest, we would have to check that their differentials agree as well; to verify this, you need to investigate functoriality properties of Cartan–Eilenberg resolutions, which I'm certainly not going to do here), proving that their limits coincide as well. □

3.1.11. Lemma. — *Suppose $p\colon X \to S$ is a proper morphism of noetherian schemes such that $\dim X_0 \leqslant 1$. Let $\mathcal{F}$ be a constant sheaf on $X_{\text{ét}}$ given by the finite abelian group Φ. Then Corollary 3.1.6 holds in cohomological degrees 0 and 1, and in higher degrees one still gets a surjection*

$$H^i(X_{\text{ét}}, \mathcal{F}) \twoheadrightarrow H^i\big(X_{0,\text{ét}}, \operatorname{pr}_1^*\mathcal{F}\big)$$

for all $i > 1$.

Proof. Observe that the right-hand side vanishes for $i > 2$ by Proposition 2.6.15, so only the cases $i = 0, 1, 2$ are interesting. As usual, we may assume that $\Phi = \mathbb{Z}/\ell\mathbb{Z}$ for some prime ℓ, using induction and the long exact cohomology sequence. Recall that Corollary 1.6.28 gave an equivalence of categories

$$\{\text{finite étale } X\text{-schemes}\} \longrightarrow \{\text{finite étale } X_0\text{-schemes}\}\,.$$

In particular, $\pi_0(X_0) \xrightarrow{\sim} \pi_0(X)$ is a bijection, and $\Gamma(X, \mathbb{Z}/\ell\mathbb{Z}) \cong \Gamma(X_0, \mathbb{Z}/\ell\mathbb{Z})$ follows. Also we may assume that X and X_0 are both connected.

For an arbitrary scheme Y and any integer N, the cohomology group $H^1(Y_{\text{ét}}, \mathbb{Z}/N\mathbb{Z})$ classifies $\mathbb{Z}/N\mathbb{Z}$-torsors on $Y_{\text{ét}}$ up to isomorphism (Proposition 2.3.9). We claim that isomorphism classes of $\mathbb{Z}/N\mathbb{Z}$ torsors are in bijection with isomorphism classes of $\mathbb{Z}/N\mathbb{Z}$-principal bundles over Y (see Definition/Lemma 1.5.6). Indeed, for $\mathcal{T}$ an $\mathbb{Z}/N\mathbb{Z}$-torsor, we choose an étale cover $\{V_i \to Y\}_{i \in I}$ over which $\mathcal{T}$ trivializes. Gluing the finite étale V_i-schemes $\mathbb{Z}/N\mathbb{Z} \times V_i$ according to $\mathcal{T}$ we obtain a finite étale Y-scheme $\widetilde{Y}$ via faithfully flat descent. By construction, $\widetilde{Y}$ is a $\mathbb{Z}/N\mathbb{Z}$-principal bundle. Conversely, given $\widetilde{Y}$, we can define $\mathcal{T}$ as follows: if $V \in Y_{\text{ét}}$, then $\Gamma(V, \mathcal{T})$ is the set of sections of the morphism $\widetilde{Y} \times_Y V \to V$, equipped with the $\mathbb{Z}/N\mathbb{Z}$-action coming from $\widetilde{Y}$. It's straightforward to check that these constructions are mutually inverse. In particular, if Y is connected, then Theorem 1.5.10(a) together with some easy arguments as in the proof of Proposition 1.5.11 show that

$$H^1(Y_{\text{ét}}, \mathbb{Z}/N\mathbb{Z}) \cong \operatorname{Hom}_{\text{cont}}\big(\pi_1^{\text{ét}}(Y, \overline{y}), \mathbb{Z}/N\mathbb{Z}\big)$$

for an arbitrary base point $\overline{y}$. In our situation we have $\pi_1^{\text{ét}}(X_0, \overline{x}) \cong \pi_1^{\text{ét}}(X, \overline{x})$ for any base point $\overline{x}$. Therefore the above considerations show that $H^1(X_{\text{ét}}, \mathbb{Z}/\ell\mathbb{Z}) \cong H^1(X_{0,\text{ét}}, \mathbb{Z}/\ell\mathbb{Z})$, as required.

It remains to deal with the $i = 2$ case. If $\ell = p > 0$ is the characteristic of k, then we actually have $H^2(X_{0,\text{ét}}, \mathbb{Z}/p\mathbb{Z}) = 0$! Indeed, using 2.6.1 we may check this after base change to $\overline{k}$, where we can apply Remark* 2.4.10 (or rather the reference given there). So from now on, we assume that ℓ is invertible in k, and hence on all of S. The first step is to do a horribly messy reduction to the case where k is algebraically closed. This was not done in the lecture, but it's necessary; otherwise the generic smoothness arguments later in the proof go wrong.

Let $\mathcal{I}$ be the category of finite A-algebras B such that $\operatorname{Spec} B \rightarrow \operatorname{Spec} A$ is a universal homeomorphism. Morphisms in $\mathcal{I}$ are given by local morphisms of local A-algebras (observe that every such B is necessarily local because $\operatorname{Spec} B$ can have only one closed point). We claim:

(1) $\mathcal{I}$ is a filtered category.

(2) The colimit $\operatorname{colim}_{\mathcal{I}} B = \overline{A}$ exists and is a strictly henselian local ring with residue field $\overline{k}$.

For (1), observe that tensor products of objects in $\mathcal{I}$ are contained in $\mathcal{I}$ again for rather trivial reasons. Moreover, if $\alpha, \alpha' \colon B \rightarrow B'$ is a pair of parallel morphisms, then $\operatorname{Coeq}(\alpha, \alpha')$ is an element of $\mathcal{I}$ again. To see this, first observe that α and α' must induce the same map of underlying topological spaces between $\operatorname{Spec} B'$ and $\operatorname{Spec} B$. Moreover, let $\mathfrak{q}' \in \operatorname{Spec} B'$ with images $\mathfrak{q} \in \operatorname{Spec} B$ and $\mathfrak{p} \in \operatorname{Spec} A$. Then $\kappa(\mathfrak{q})$ and $\kappa(\mathfrak{q}')$ are purely inseparable extensions of $\kappa(\mathfrak{p})$, hence any morphism $\kappa(\mathfrak{q}) \rightarrow \kappa(\mathfrak{q}')$ is uniquely determined. This shows that $\operatorname{Spec} \operatorname{Coeq}(\alpha, \alpha')$ contains all points of $\operatorname{Spec} B'$, hence $\operatorname{Spec} \operatorname{Coeq}(\alpha, \alpha') \rightarrow \operatorname{Spec} A$ is a homeomorphism again. Moreover, $\operatorname{Coeq}(\alpha, \alpha')$ is clearly finite and radiciel over A because the same must be true for B' (see Remark 1.4.21), proving $\operatorname{Coeq}(\alpha, \alpha') \in \mathcal{I}$.

For (2), observe that $\mathcal{I}$ is essentially small, hence the colimit exists. Being a colimit over local morphisms between local rings, $\overline{A}$ is local again. Using Proposition 1.6.7(a), we easily get that $\overline{A}$ is henselian again. It remains to show $\overline{A}/\mathfrak{m} \cong \overline{k}$. It's clear that $\overline{A}/\mathfrak{m}$ is algebraic over k. Conversely, every element of $\overline{k}$ is a q^{th} root of some element $\overline{a} \in k$, where q is a power of the characteristic p. Let $a \in A$ be a lift of $\overline{a}$ and let B be obtained from A by adjoining a q^{th} root of a. Then $\operatorname{Spec} B \rightarrow \operatorname{Spec} A$ is finite, radiciel, and surjective on spectra, hence a universal homeomorphism by Remark 1.4.21. This proves that $\overline{A}/\mathfrak{m}$ contains all of $\overline{k}$.

Now let $\overline{S} = \operatorname{Spec} \overline{A}$, put $\overline{X} = X \times_S \overline{S}$, and let $\overline{X}_0 = \overline{X} \times_{\overline{S}} \operatorname{Spec} \overline{k}$ be the fibre over the closed point. Adapting the arguments from 2.6.1, we see that

$$H^i(X_{\text{ét}}, \mathbb{Z}/\ell\mathbb{Z}) \cong H^i\big(\overline{X}_{\text{ét}}, \mathbb{Z}/\ell\mathbb{Z}\big) \quad \text{and} \quad H^i\big(X_{0,\text{ét}}, \mathbb{Z}/\ell\mathbb{Z}\big) \cong H^i\big(\overline{X}_{0,\text{ét}}, \mathbb{Z}/\ell\mathbb{Z}\big)\,.$$

This reduces our original situation to the case where the residue field is algebraically closed. There's only one problem: $\overline{A}$ might not be noetherian any more. We will see below how to deal with this.

To compute cohomology with coefficients in $\mathbb{Z}/\ell\mathbb{Z}$, we do the usual trick and replace $\mathbb{Z}/\ell\mathbb{Z}$ by μ_ℓ, as done before e.g. in Corollary 2.6.5. Since ℓ is invertible in k, hence in $\overline{A}$, we can use the standard short exact sequence to get a diagram

$$\begin{tikzcd}
H^1\big(\overline{X}_{\text{ét}}, \mathcal{O}^\times_{\overline{X}_{\text{ét}}}\big) \arrow[r] \arrow[d] & H^2\big(\overline{X}_{\text{ét}}, \mu_\ell\big) \arrow[r] \arrow[d] & H^2\big(\overline{X}_{\text{ét}}, \mathcal{O}^\times_{\overline{X}_{\text{ét}}}\big) \arrow[d] \\
H^1\big(\overline{X}_{0,\text{ét}}, \mathcal{O}^\times_{\overline{X}_{0,\text{ét}}}\big) \arrow[r] & H^2\big(\overline{X}_{0,\text{ét}}, \mu_\ell\big) \arrow[r] & 0
\end{tikzcd}$$

with exact rows. Once we verify that the two implicit assertions in this diagram are true, it will follow immediately that $H^2(\overline{X}_{\text{ét}}, \mu_\ell) \rightarrow H^2(\overline{X}_{0,\text{ét}}, \mu_\ell)$ is surjective, as claimed.

To see that the left vertical arrow is surjective, we need to check that $\operatorname{Pic}(\overline{X}) \twoheadrightarrow \operatorname{Pic}(\overline{X}_0)$ is surjective (see Corollary 2.3.12). Unfortunately, this doesn't follow immediately from Corollary 1.6.30, since $\overline{A}$ might not be noetherian. As a workaround, we write $\overline{X} \cong \lim_{\mathcal{I}} X_\alpha$ and $\overline{X}_0 \cong \lim_{\mathcal{I}} X_{\alpha,0}$ according to the definition of $\overline{A}$. Using (e) from Appendix A.1, we get that $\operatorname{colim}_{\mathcal{I}} \operatorname{Pic}(X_\alpha) \twoheadrightarrow \operatorname{Pic}(\overline{X})$ is surjective (in fact, it is even an isomorphism). The same is true for $\operatorname{colim}_{\mathcal{I}} \operatorname{Pic}(X_{\alpha,0}) \twoheadrightarrow \operatorname{Pic}(\overline{X}_0)$. But we know that $\operatorname{Pic}(X_\alpha) \twoheadrightarrow \operatorname{Pic}(X_{\alpha,0})$ is surjective by Corollary 1.6.30, and surjectivity behaves well under colimits, hence we are done.

It remains to explain why there is a 0 in the bottom right corner. If $\dim \overline{X}_0 = 0$, this is rather trivial. Now assume $\dim \overline{X}_0 = 1$. Observe that we may replace $\overline{X}$ and $\overline{X}_0$ by their reductions $\overline{X}^{\mathrm{red}}$ and $\overline{X}_0^{\mathrm{red}}$ because this doesn't affect the cohomology with coefficients in $\mathbb{Z}/\ell\mathbb{Z}$ by Proposition 1.4.20, and neither does it affect surjectivity of $\operatorname{Pic}(\overline{X}) \twoheadrightarrow \operatorname{Pic}(\overline{X}_0)$, see the proof of Corollary 1.6.30. After reducing everything, we can thus apply the following general fact:

$(*)$ *Let C be a reduced curve over an algebraically closed field k. Then $H^i(C_{\text{ét}}, \mathcal{O}_C^\times) = 0$ for all $i > 1$.*

If C is smooth, this follows directly from Proposition 2.6.3. Now assume C is only integral. Let $p\colon \widetilde{C} \to C$ be its normalization. Since k is perfect, we can apply generic smoothness to see that p is an isomorphism over some dense open $U \subseteq C$. Then $\mathcal{O}_{C_{\text{ét}}}^\times \to p_*\mathcal{O}_{\widetilde{C}_{\text{ét}}}^\times$ is an isomorphism over $U_{\text{ét}}$. Thus

$$H^i\big(C_{\text{ét}}, \mathcal{O}_{C_{\text{ét}}}^\times\big) \cong H^i\big(C_{\text{ét}}, p_*\mathcal{O}_{\widetilde{C}_{\text{ét}}}^\times\big) \cong H^i\big(\widetilde{C}_{\text{ét}}, \mathcal{O}_{\widetilde{C},\text{ét}}^\times\big) = 0$$

The second isomorphism uses Proposition 2.1.12 and the first one follows from Fact 2.6.7(b). "Oh, but why are $\mathcal{O}_{C_{\text{ét}}}^\times$ and $p_*\mathcal{O}_{\widetilde{C}_{\text{ét}}}^\times$ torsion sheaves?", I hear you object, and you are totally right: they aren't, most likely. However, torsion was only needed in Fact 2.6.7(a), to get $\mathcal{F}|_{U_{\text{ét}}} = 0$ for some open dense subset $U \subseteq C$. Here we get this for free, so the argument works.

The reduction from reduced curves to integral curves is similar: let $C_1, \dots, C_n$ be the irreducible components of C, equipped with their reduced closed subscheme structures, and $i_j\colon C_j \hookrightarrow C$. Since C is reduced, $\mathcal{O}_{C,\text{ét}}^\times \to \bigoplus_{j=1}^n i_{j,*}\mathcal{O}_{C_{j,\text{ét}}}^\times$ is an isomorphism over $U_{\text{ét}}$ for some dense open subset $U \subseteq C$ again, and we can apply the same argument once again. $\square$

We would like to upgrade Lemma 3.1.11 to get a complete proof of Corollary 3.1.6 in the special case $\dim X_0 \leqslant 1$. It turns out that the only missing ingredient is a completely formal argument.

3.1.12. Lemma. — *Let $\mathcal{A}$ be an abelian category. Let $\mathfrak{X} \subseteq \mathcal{A}$ be a class of objects such that every $a \in \mathcal{A}$ admits a monomorphism $a \hookrightarrow x$ with $x \in \mathfrak{X}$. Let moreover $\Phi^\bullet, \Psi^\bullet\colon \mathcal{A} \to \mathrm{Ab}$ be cohomological functors such that $\Phi^\bullet$ is effaceable in the sense of* (1) *from Proposition* 2.1.10(a), *and let $\varphi^\bullet\colon \Phi^\bullet \to \Psi^\bullet$ be a natural transformation of cohomological functors. Then the following are equivalent:*

(a) *$\varphi^\bullet$ is an isomorphism.*

(b) *For all $x \in \mathfrak{X}$ and all $i \geqslant 0$, $\varphi^i\colon \Phi^i(x) \twoheadrightarrow \Psi^i(x)$ is surjective, and bijective if $i = 0$.*

Proof.* This is an "amusing exercise". See [SGA$_{4\frac{1}{2}}$, Arcata IV Lem. 3.6]—where they also say "amusing exercise", haha lol 😆. Nevermind then, here's a proof.

Suppose (b) holds. We prove that $\varphi^i\colon \Phi^i(a) \twoheadrightarrow \Psi^i(a)$ is bijective using induction on i. Let $i = 0$. By assumption on $\mathfrak{X}$, we find an exact sequence $0 \to a \to x \to x''$. Since Φ^0 and Ψ^0 are right-exact, we get a diagram

$$\begin{tikzcd} 0 \arrow[r] & \Phi^0(a) \arrow[r] \arrow[d, "\varphi^0"'] & \Phi^0(x) \arrow[r] \arrow[d, "\varphi^0"', "\wr"] & \Phi^0(x') \arrow[d, "\varphi^0"', "\wr"] \\ 0 \arrow[r] & \Psi^0(a) \arrow[r] & \Psi^0(x) \arrow[r] & \Psi^0(x') \end{tikzcd},$$

proving that $\varphi^0\colon \Phi^0(a) \to \Psi^0(a)$ is an isomorphism too.

Now let $i > 0$ and the assertion be true for $i - 1$. We first prove that φ^i is injective. Let $f \in \Phi^i(a)$ be in the kernel of φ^i. Since $\Phi^\bullet$ is effaceable and $\mathfrak{X}$ has enough objects, we find a monomorphism $a \hookrightarrow x$ such that the image of f in $\Phi^i(x)$ vanishes. Consider the diagram

$$\begin{tikzcd} \Phi^{i-1}(x) \arrow[r] \arrow[d, "\varphi^{i-1}"', "\wr"] & \Phi^{i-1}(x/a) \arrow[r] \arrow[d, "\varphi^{i-1}"', "\wr"] & \Phi^i(a) \arrow[r] \arrow[d, "\varphi^i"'] & \Phi^i(x) \arrow[r] \arrow[d, "\varphi^i"'] & \Phi^i(x/a) \arrow[d, "\varphi^i"'] \\ \Psi^{i-1}(x) \arrow[r] & \Psi^{i-1}(x/a) \arrow[r] & \Psi^i(a) \arrow[r] & \Psi^i(x) \arrow[r] & \Psi^i(x/a) \end{tikzcd}.$$

By exactness of the top row, f is the image of some $f' \in \Phi^{i-1}(x/a)$. Let $g' \in \Psi^{i-1}(x/a)$ be the image of f'. Then the image of g' in $\Psi^i(a)$ equals $\varphi^i(f) = 0$, hence g' is the image of some $g'' \in \Psi^{i-1}(x)$ by exactness of the bottom row. Using that the first and the second vertical arrows are isomorphisms by the induction hypothesis, we find that $g'' = \varphi^{i-1}(f'')$ for some $f'' \in \Phi^{i-1}(x)$, and that f is the image of f'' in $\Phi^i(a)$. But then $f = 0$ by exactness, as required.

To show bijectivity of φ^i, we use the above diagram again. Now we know that the fourth vertical arrow is not only surjective, but bijective, and the fifth vertical arrow is injective. Thus, by the five lemma, φ^i must be bijective too. This finishes the induction. $\square$

3.1.13. Lemma. — *Corollary 3.1.6 holds in the special case where* $\dim X_0 \leqslant 1$. *Thus, Theorem 3.1.3 holds when all fibres of* $p\colon X \to S$ *have dimension* $\leqslant 1$. *In particular, it holds when* p *is finite and for* $X = \mathbb{P}^1_S$.

Proof. Assume we are in the situation of Corollary 3.1.6, with X and S noetherian. Let $\mathcal{A}$ be the category of constructible sheaves on $X_{\text{ét}}$ and $\mathfrak{X} \subseteq \mathcal{A}$ the class of objects of the form $\bigoplus_{l=1}^n p_{l,*}\Phi_l$, where $p_l\colon X_l \to X$ are finite morphisms and Φ_l are finite abelian groups. By characterization (c_2) from Definition 2.5.9, every constructible sheaf admits a monomorphism into an object of $\mathfrak{X}$. Moreover, let $\Phi^\bullet = H^\bullet(X_{\text{ét}}, -)$ and $\Psi^\bullet = H^\bullet(X_{0,\text{ét}}, \operatorname{pr}_1^*(-))$. Then $\Phi^\bullet$ and $\Psi^\bullet$ are cohomological functors $\mathcal{A} \to \mathrm{Set}$, and $\Phi^\bullet$ is effaceable by Corollary 2.5.21. We claim that the natural transformation $H^\bullet(X_{\text{ét}}, -) \to H^\bullet(X_{0,\text{ét}}, \operatorname{pr}_1^*(-))$ satisfies the condition from Lemma 3.1.12(b) for the class $\mathfrak{X}$. Indeed, we have isomorphisms

$$H^i\left(X_{\text{ét}}, \bigoplus_{l=1}^n p_{l,*}\Phi_l\right) \cong \bigoplus_{l=1}^n H^i\big(X_{l,\text{ét}}, \Phi_l\big)\,.$$

These follow from Proposition 2.1.12 and the Leray spectral sequence (Proposition 2.1.11). Now put $X_{l,0} \coloneqq X_l \times_S S_0$. By the same argument as before, plus Fact* 2.5.12(a), we get

$$H^i\left(X_{0,\text{ét}}, \operatorname{pr}_1^* \bigoplus_{l=1}^n p_{l,*}\Phi_l\right) \cong \bigoplus_{l=1}^n H^i\big(X_{l,0,\text{ét}}, \operatorname{pr}_1^*\Phi_l\big)\,.$$

Using Lemma 3.1.11 on X_l and $X_{l,0}$, we deduce that the condition from Lemma 3.1.12(b) is indeed satisfied, and we get isomorphisms everywhere. In other words, Corollary 3.1.6 holds for constructible sheaves in the special case where S and X are noetherian and $\dim X_0 \leqslant 1$. Since Corollary* 3.1.5 holds in that special case too (Proposition* 2.6.16), we conclude using Remark 3.1.7 and Lemma* 3.1.9 that Theorem 3.1.3 holds in full generality for morphisms whose fibres have dimension $\leqslant 1$. □

3.1.14. Lemma. — *Theorem 3.1.3 is true for all morphisms $p\colon X \to S$ of the form $X \hookrightarrow \mathbb{P}^n_S \to S$, where the first arrow is a closed immersion.*

Sketch of a proof. By Lemma 3.1.13, Theorem 3.1.3 holds when p is a closed immersion, hence by Lemma 3.1.10 it suffices to prove the assertion for $X = \mathbb{P}^n_S$. We do induction on n. The case $n = 1$ was done above, so let's assume $n > 1$ and the assertion is true for $n - 1$.

Our construction follows [SGA$_{4\frac{1}{2}}$, Arcata III.4]. Let $u\colon Z \hookrightarrow \mathbb{P}^n_S = \underline{\operatorname{Proj}}\,\mathcal{O}_S[t_0, \dots, t_n]$ be the closed embedding defined by the ideal (t_0, t_1). Let $\pi\colon \widetilde{X} \to \mathbb{P}^n_S$ be the blow-up of Z. The fibres of π are isomorphic to $\mathbb{P}^1$ over Y and trivial everywhere else, hence Theorem 3.1.3 holds for π by Lemma 3.1.13. Moreover, there is a natural morphism $\mathbb{P}^n_S \setminus Z \to \mathbb{P}^1_S$ sending points $[x_0 : \dots : x_n]$ to $[x_0 : x_1]$ (yes, that's not really a definition; if you want to, work out how that looks as a morphism of schemes). It can be extended to a morphism $\widetilde{p}\colon \widetilde{X} \to \mathbb{P}^1_S$. We thus obtain a diagram

$$\begin{tikzcd} \widetilde{X} \arrow[r, "\pi"] \arrow[d, "\widetilde{p}"'] & \mathbb{P}^n_S \arrow[d, "p"] \\ \mathbb{P}^1_S \arrow[r] & S \end{tikzcd} .$$

It can be shown that locally on $\mathbb{P}^1_S$–more precisely, after restricting to the two copies of $\mathbb{A}^1_S$ inside—the morphism $\widetilde{p}$ is isomorphic to $\mathrm{pr}_2\colon \mathbb{P}^{n-1}_S \times_S \mathbb{A}^1_S \to \mathbb{A}^1_S$. Thus the induction hypothesis is applicable. Since Theorem 3.1.3 is local on the base, this shows that the assertion is true for $\widetilde{p}$. By the above diagram and Lemma 3.1.10 we conclude that Theorem 3.1.3 is true for $p \circ \pi\colon \widetilde{X} \to S$. That is, if $\widetilde{g}\colon \widetilde{X}' \to \widetilde{X}$ denotes the base change of $g\colon X' \to X$ and $\pi'\colon \widetilde{X}' \to X'$ the base change of π, then

$$f^* R^i(p \circ \pi)_* \widetilde{\mathcal{F}} \xrightarrow{\sim} R^i(p' \circ \pi')_* \widetilde{g}^* \widetilde{\mathcal{F}}$$

holds for all $i \geqslant 0$ and all torsion sheaves $\widetilde{\mathcal{F}}$ on $\widetilde{X}_{\text{ét}}$.

Now let $\mathcal{F}$ be a constructible sheaf on $\mathbb{P}^n_{S,\text{ét}}$. The classical approach to prove Theorem 3.1.3 for $\mathcal{F}$ would be to look at $\mathcal{F} \to \pi_*\pi^*\mathcal{F}$, whose kernel and cokernel have support in Z since π is an isomorphism away from Z. An easy argument (see below) shows that Theorem 3.1.3 thus holds for the kernel and the cokernel, so it suffices to prove it for $\pi_*\pi^*\mathcal{F}$ as well. This is where things get messy: one looks at the Leray spectral sequences converging to $f^*R^i(p \circ \pi)_*\pi^*\mathcal{F}$ and $R^i(p' \circ \pi')_*\widetilde{g}^*\pi^*\mathcal{F}$. Knowing that their limits coincide somehow provides information about their E_2-pages, which we are ultimately interested in.

These arguments become way more cleaner when formulated in the framework of derived categories—which is why we introduced Corollary* 3.1.4! We first claim:

($*$) *If $\widetilde{K}^\bullet \in D^+(\widetilde{X}_{\text{ét}})$ is a bounded below complex with torsion cohomology, then Corollary* 3.1.4 holds for $p\colon X \to S$ if we plug in the complex $R\pi_*\widetilde{K}^\bullet$. The same is true for $Ru_*K^\bullet$ if $K^\bullet \in D^+(Z_{\text{ét}})$ has torsion cohomology.*

To prove $(*)$, we calculate

$$f^*Rp_*R\pi_*\widetilde{K}^\bullet \cong Rp'_*R\pi'_*\widetilde{g}^*\widetilde{K}^\bullet \cong Rp'_*g^*R\pi_*\widetilde{K}^\bullet\,.$$

The left isomorphism uses that Corollary* 3.1.4 holds for $(p\circ\pi)\colon \widetilde{X}\to S$ together with the "Leray spectral sequence" isomorphisms $Rp_*R\pi_*\cong R(p\circ\pi)_*$ and $Rp'_*R\pi'_*\cong R(p'\circ\pi')_*$. The right isomorphism uses that Corollary* 3.1.4 holds for $\pi\colon\widetilde{X}\to\mathbb{P}^n_S$. This proves the claim about $R\pi_*\widetilde{K}^\bullet$. The claim about $Ru_*K^\bullet$ can be done in the exact same way, using that Corollary* 3.1.4 holds both for the closed embedding $u\colon Z\hookrightarrow\mathbb{P}^n_S$ by Lemma 3.1.13 and for $u\circ p\colon Z\to S$ by the induction hypothesis, as $Z\cong\mathbb{P}^{n-2}_S$. We thus proved $(*)$.

Given a constructible sheaf $\mathcal{F}$, consider the complex $C^\bullet\in D^+(X_{\text{ét}})$ given by $\mathcal{F}$ placed in degree -1 and (any representative of) $R\pi_*$ placed in non-negative degrees. In formulas, $C^\bullet=(\mathcal{F}[-1]\to R\pi_*\pi^*\mathcal{F})$, where the morphism between degrees -1 and 0 is given by the canonical morphism $\mathcal{F}\to\pi_*\mathcal{I}^0$ for $\pi^*\mathcal{F}\to\mathcal{I}^\bullet$ any injective resolution. Observe that the cohomology sheaves $H^i(C^\bullet)$ have support contained in Y. Indeed, in degrees $i>0$ this follows from the fact that $R^i\pi_*\pi^*\mathcal{F}$ vanishes outside of Z because π is an isomorphism away from Z. In degrees 0 and -1, we use the same argument for $\mathcal{F}\to\pi_*\pi^*\mathcal{F}$ which is an isomorphism away from Y. Therefore, $C^\bullet\cong Ru_*K^\bullet$ for some complex $K^\bullet\in D^+(Y_{\text{ét}})$ with torsion cohomology; in fact, the canonical morphism $C^\bullet\to Ru_*u^*C^\bullet$ is a quasi-isomorphism (here we also use that u_* is exact because of Proposition 2.1.12).

Finally, we have an obvious map of complexes $R\pi_*\pi^*\mathcal{F}\to C^\bullet$. Its mapping cone is quasi-isomorphic to $\mathcal{F}[-1]$ because, well, $R\pi_*\pi^*\mathcal{F}$ and $C^\bullet$ coincide except in degree -1. Thus, we obtain a distinguished triangle

$$R\pi_*\pi^*\mathcal{F}\longrightarrow C^\bullet\longrightarrow\mathcal{F}[-1]\longrightarrow R\pi_*\pi^*\mathcal{F}[1]$$

in $D^+(X_{\text{ét}})$. Since Corollary* 3.1.4 holds for $R\pi_*\pi^*\mathcal{F}$ and $C^\bullet\cong Ru_*K^\bullet$ by $(*)$, this triangle shows that it holds for $\mathcal{F}[-1]$ as well, and thus for $\mathcal{F}$ too. $\square$

Proof of Theorem 3.1.3. The rest of the proof is basically another "dévissage" as in the proof of Lemma 3.1.14. Using Lemma* 3.1.9, it suffices to deal with the case where S and X are noetherian. Moreover, the assertion is local with respect to S, whence we may assume that S is affine (depending on your version of Chow's lemma this isn't even needed). By Chow's lemma, [EGA_{II}, Théorème (5.6.1)], we find a diagram

$$\begin{tikzcd} X \arrow[dr, "p"'] & \widetilde{X} \arrow[l, "\pi"'] \arrow[r, hook] \arrow[d] & \mathbb{P}^n_S \arrow[dl] \\ & S & \end{tikzcd}$$

in which π is "projective" (in the EGA sense) and there exists a dense open subset $U\subseteq X$ such that $\pi^{-1}(U)\xrightarrow{\sim}U$ is an isomorphism. Locally on X, EGA-projective morphisms are of the form $\widetilde{X}\hookrightarrow\mathbb{P}^m_X\to X$; depending on which version of Chow's lemma you want to use, you may even assume that π has this form not only locally. In any case, Corollary* 3.1.4 holds for π.

Let $Z=X\setminus U$, equipped with any closed subscheme structure, and denote $u\colon Z\hookrightarrow X$. By the principle of noetherian induction, we may assume that Corollary* 3.1.4 holds for $u\circ p\colon Z\to S$. Now, starting from $(*)$, the proof of Lemma 3.1.14 can be copied verbatim. $\square$

3.2. Derived Direct Images with Compact Support

Lecture 25
31st Jan, 2020

To start off todays lecture, we recall the famous *Nagata compactification theorem.*

3.2.1. Theorem. — *Let $f\colon X \to S$ be a separated*[1] *morphism of finite type, and assume S is quasi-compact and quasi-separated. Then there exists a factorization*

$$X \overset{j}{\hookrightarrow} \overline{X} \xrightarrow{\overline{f}} S\,,$$

in which j is a quasi-compact[2] *open embedding and $\overline{f}$ is a proper morphism.*

While Nagata's original proof used the outdated language of varieties, the first scheme-theoretic proof was given by Lütkebohmert in [Lüt93], using a different approach than Nagata. A translation of his original proof into scheme-theoretic language was given by Deligne [Del10]. Vojta [Voj07] and Conrad [Con07] have written expositions of Deligne's proof. It should be mentioned that Conrad takes extra care to make the arguments work in the non-noetherian case as well. Finally, [Stacks, Tag 0F41] has another proof of the non-noetherian case.

3.2.2. — We would like to get a generalization of Theorem 3.1.3 that drops the condition that X be proper over S. This necessarily involves the higher derived direct images by some sort of "compactly supported cohomology". We give a tentative definition: suppose $f\colon X \to S$ is separated and of finite type, and S is quasi-compact and quasi-separated, so that f admits a *compactification*, i.e., a factorization as in Theorem 3.2.1. If $\mathcal{F}$ is a torsion sheaf in $\mathrm{Ab}(X_{\text{ét}})$, we put

$$R^i f_! \mathcal{F} = R^i \overline{f}_* j_! \mathcal{F}$$

for all $i \geqslant 0$. Of course, it is not at all clear why $R^i f_! \mathcal{F}$ should be independent of the choices of j and $\overline{f}$, so that's the first thing to show.

3.2.3. Definition. — Given $f\colon X \to S$ as above, we define a *category $\mathcal{C}$ of compactifications* of f as follows: its objects are factorizations of f as in Theorem 3.2.1. A *morphism of compactifications* is a commutative diagram

$$\begin{tikzcd} & \overline{X}' \arrow[d, "\pi"] \arrow[dr, "\overline{f}'"] & \\ X \arrow[ur, hook, "j'"] \arrow[r, hook, "j"] & \overline{X} \arrow[r, "\overline{f}"] & S \end{tikzcd}$$

such that $\pi^{-1}(\overline{X} \setminus X) = \overline{X}' \setminus X$.

3.2.4. Remark*. — Actually, the additional condition in Definition 3.2.3 poses no real restriction: the morphism $\pi^{-1}(X) \to X$ is proper because it is a base change of π, and $j'\colon X \hookrightarrow \pi^{-1}(X)$ defines a section. But sections of separated morphisms are closed immersions, hence the open embedding $j'\colon X \hookrightarrow \pi^{-1}(X)$ must be the inclusion of an open-closed subscheme. In particular, it is an isomorphism as long as X is scheme-theoretically dense in $\overline{X}'$. Of course this is not satisfied for all compactifications, but at least the class of compactifications with that property is cofinal in $\mathcal{C}$, because we can always replace $\overline{X}$ be the scheme theoretic image of X in it (see [Stacks, Tag 01R8]).

[1] This was not mentioned in the lecture, but since proper morphisms and open embeddings are separated, it is clearly a necessary condition

[2] In the lecture we omitted the quasi-compactness condition, since we only worked in a noetherian setting.

3.2.5. Remark*. — To show that $R^i f_! \mathcal{F}$ is independent of the choice of compactification (in the the best possible approximate sense of the word "independent") it suffices to prove the following assertion (and I'm afraid Professor Franke didn't really make that point in the lecture).

(⊠) *Let $f\colon X \to S$ be separated of finite type and let S be quasi-compact and quasi-separated. Then the category $\mathcal{C}$ from Definition 3.2.3 is cofiltered. Moreover, for any morphism π in $\mathcal{C}$ and any torsion sheaf $\mathcal{F} \in \operatorname{Ab}(X_{\text{ét}})$, there are isomorphisms*

$$\iota_\pi \colon R^i \overline{f}'_* j'_! \mathcal{F} \xrightarrow{\sim} R^i \overline{f}_* j_! \mathcal{F}$$

for all $i \geqslant 0$. These are natural in $\mathcal{F}$, assemble into an isomorphism of cohomological functors, and satisfy $\iota_{\pi \circ \pi'} = \iota_\pi \circ \iota_{\pi'}$.

Once (⊠) is proved, you could define $R^i f_! \mathcal{F} = \lim_{\mathcal{C}} R^i \overline{f}_* j_! \mathcal{F}$, with transition morphisms provided by ι_π, to obtain a definition that is truly independent of the choice of compactification (but we will usually omit that technical step).

The main part of proving (⊠) is done by Lemma 3.2.6 and Lemma* 3.2.7 below

3.2.6. Lemma. — *We obtain the following isomorphisms.*

(*a*) *Consider a pullback diagram (we only need $\overline{p}^{-1}(\overline{X} \setminus X) = \overline{Y} \setminus Y$, but then the other conditions below imply that the diagram is cartesian)*

$$\begin{tikzcd} Y \arrow[r, hook, "k"] \arrow[d, "p"'] \arrow[dr, phantom, "\lrcorner"] & \overline{Y} \arrow[d, "\overline{p}"] \\ X \arrow[r, hook, "j"] & \overline{X} \end{tikzcd}$$

in which j, k are open embeddings and $\overline{p}$, p are a proper morphism. For every torsion sheaf $\mathcal{F}$ on $Y_{\text{ét}}$ and all $i \geqslant 0$, there are unique isomorphisms

$$j_! R^i p_* \mathcal{F} \xrightarrow{\sim} R^i \overline{p}_* k_! \mathcal{F}\,,$$

which are natural in $\mathcal{F}$, assemble into an isomorphism of cohomological functors, and fit, for $i = 0$, into a commutative diagram

$$\begin{tikzcd} j_! p_* \mathcal{F} \arrow[r, "\sim"] \arrow[d, hook] & \overline{p}_* k_! \mathcal{F} \arrow[d, hook] \\ j_* p_* \mathcal{F} \arrow[r, "\sim"] & \overline{p}_* k_* \mathcal{F} \end{tikzcd}\,.$$

(*b*) *If π is a morphism of compactifications as in Definition 3.2.3, one has*

$$R^i \pi_* j'_! \mathcal{F} \cong \begin{cases} j_! \mathcal{F} & \text{if } i = 0 \\ 0 & \text{else} \end{cases}\,.$$

Proof. Part (*a*). First recall that $j_! p_* \mathcal{F}$ and $\overline{p}_* k_! \mathcal{F}$ are subsheaves of $j_* p_* \mathcal{F}$ and $\overline{p}_* k_* \mathcal{F}$ in a canonical way; see Remark* 2.1.5. Thus, to get the desired commutative diagram, we must ensure that the bottom isomorphism identifies the subsheaves under consideration. They clearly coincide over the image of X in $\overline{X}$. Thus it suffices to check whether they coincide on stalks at a geometric point $\overline{x}$ with image in $\overline{X} \setminus X$. Then clearly $(j_! p_* \mathcal{F})_{\overline{x}} = 0$ by the description of stalks in Proposition 2.1.3. To compute the stalk of $\overline{p}_* k_! \mathcal{F}$, we use Corollary 1.7.9 and Corollary 3.1.6 to get more generally

$$\left(R^i \overline{p}_* k_! \mathcal{F}\right)_{\overline{x}} \cong H^i\left(Y_{\overline{x},\text{ét}}, \operatorname{pr}_1^* k_! \mathcal{F}\right) \cong H^i\left(Y_{0,\text{ét}}, \operatorname{pr}_1^* k_! \mathcal{F}\right),$$

where $Y_{\overline{x}} = Y \times_X \operatorname{Spec} \mathcal{O}_{X_{\text{ét}},\overline{x}}$ and $Y_0 = Y \times_X \operatorname{Spec} \kappa(\overline{x})$. As $\overline{p}^{-1}(\overline{X} \setminus X) = \overline{Y} \setminus Y$, we see that the projection $\operatorname{pr}_1 \colon Y_0 \to Y$ has image inside $\overline{Y} \setminus Y$. Thus $\operatorname{pr}_1^* k_! \mathcal{F} = 0$, and it's clear that $(\overline{p}_* k_! \mathcal{F})_{\overline{x}} = 0$ as well.

To construct the required morphism $j_! R^i p_* \mathcal{F} \xrightarrow{\sim} R^i \overline{p}_* k_! \mathcal{F}$, first observe that both sides are cohomological functors by exactness of $j_!$ and $k_!$. In particular, it suffices to show that $j_! R^i p_*$ are the derived functors of $j_! p_*$. Clearly it will be enough to show that the functors $R^i p_*$ are effaceable for $i > 0$ in the sense of (3) from Proposition 2.1.10(a). This seems trivial on first glance, but there's a subtlety actually: we would like $R^i p_*$ to be effaceable *when restricted to the full subcategory of torsion sheaves*! There are two ways to fix this.

(1) The natural transformation $j_! p_* \to \overline{p}_* k_!$ exists not only for torsion sheaves (but it might only be injective in general). The reason is that $(j_! p_* \mathcal{F})_{\overline{x}} = 0$ for $\overline{x}$ with image in $\overline{X} \setminus X$, so nothing can go wrong. In this case there's no trap and $R^i p_*$ is just effaceable.

(2) It is, in fact, true that $R^i p_*$ is effaceable on the category of torsion sheaves. The argument is similar to Corollary 2.5.21. Using Proposition 2.4.4, it's quite easy to check that $R^i p_*$ vanishes on $\prod_{\overline{x}} \overline{x}_* \mathcal{F}_{\overline{x}}$ for $i > 0$. However, this sheaf might not be torsion since stalks of products are weird, unless $\mathcal{F}$ is annihilated by some integer $N \neq 0$. To fix this problem, write $\mathcal{F} = \operatorname{colim}_{N \in \mathbb{N}} \mathcal{F}[N]$ as a colimit over its N-torsion subsheaves, and consider

$$\mathcal{F} \hookrightarrow \operatorname*{colim}_{n \in \mathbb{N}} \prod_{\overline{x}} \overline{x}_* \mathcal{F}[N]_{\overline{x}}.$$

The sheaf on the right-hand side is torsion, and it can be shown that $R^i p_*$ vanishes on it, using Corollary 2.4.14.

Either way, we get the desired uniquely determined morphism of cohomological functors. To see that it's an isomorphism, first observe that everything is clear over the image of X in $\overline{X}$. Thus, it suffices to check that we get an isomorphism on stalks at $\overline{x}$ with image in $\overline{X} \setminus X$. In this case we have $(j_! R^i p_* \mathcal{F})_{\overline{x}} = 0 = (R^i \overline{p}_* k_! \mathcal{F})_{\overline{x}}$ by the above calculation.

This finishes the proof of (a). Part (b) is an immediate consequence, because it is just the special case

$$\begin{tikzcd} X \arrow[r, hook, "j'"] \arrow[d, equal] & \overline{X}' \arrow[d, "\pi"] \\ X \arrow[r, hook, "j"] & \overline{X} \end{tikzcd}.$$

We are done. $\square$

3.2.7. Lemma*. — *Let S be quasi-compact quasi-separated, and $f \colon X \to S$ a separated morphism of finite type. Then the category $\mathcal{C}$ of compactifications of f, introduced in Definition 3.2.3, is cofiltered.*

Proof. Although this lemma was not in the lecture, its proof was (at least partially). First observe that $\mathcal{C}$ is non-empty because of Theorem 3.2.1. There are two more conditions to check. For the reader's convenience, we first explain the constructions and prove their correctness afterwards, so you can skip these parts if you want.

(1) Let $\overline{f}\colon \overline{X} \to S$ and $\overline{f}'\colon \overline{X}' \to S$ be compactifications of f. We must construct a compactification $\overline{f}''\colon \overline{X}'' \to S$ that "dominates both", i.e., fits into a diagram

$$\begin{tikzcd} & \overline{X}' \arrow[dr, "\overline{f}'"] & \\ X \arrow[ur, hook, "j'"] \arrow[r, hook, "j''"] \arrow[dr, hook, "j"'] & \overline{X}'' \arrow[u, "\pi'"'] \arrow[r, "\overline{f}"] \arrow[d, "\pi"] & S \\ & \overline{X} \arrow[ur, "\overline{f}"'] & \end{tikzcd}$$

We can construct $\overline{X}''$ as the scheme-theoretic image of $(j, j')\colon X \to \overline{X} \times_S \overline{X}'$, i.e., the "smallest" closed subscheme over which (j, j') factors (see [Stacks, Tag 01R5]). Then π and π' are induced by pr_1 and pr_2 respectively.

(2) Let $\pi, \pi'\colon \overline{X}' \to X$ be a pair of parallel morphism of compactifications. We must construct a compactification $\overline{f}''\colon \overline{X}'' \to S$ together with a morphism $\pi''\colon \overline{X}'' \to \overline{X}'$ such that $\pi \circ \pi'' = \pi' \circ \pi''$. To get this, we simply take $\overline{X}'' = \mathrm{Eq}(\pi, \pi')$, and $\pi''\colon \overline{X}'' \to \overline{X}'$ is the canonical morphism.

We start proving that (1) works. The first step is to show that $j''\colon X \hookrightarrow \overline{X}''$ is a quasi-compact open embedding again. Since j and j' are quasi-compact and quasi-compactness is preserved under base change, $X \times_S X \hookrightarrow \overline{X} \times_S X$ and $\overline{X} \times_S X \hookrightarrow \overline{X}'$ are quasi-compact open immersions. Moreover, the diagonal $\Delta\colon X \to X \times_S X$ is a closed immersion because X is separated over S. Hence the composition $(j, j')\colon X \to \overline{X} \times_S \overline{X}'$ of these three morphisms is quasi-compact again, so j'' is quasi-compact as well. Now [Stacks, Tag 01R8] can be applied to see that $\overline{X}'' \cap (X \times_S X)$ is the scheme-theoretic image of Δ. Combining this with the fact that Δ is a closed immersion, hence an isomorphism onto its scheme-theoretic image, we get that X is mapped isomorphically to the open subscheme $\overline{X}'' \cap (X \times_S X)$ of $\overline{X}''$, so $j''\colon X \hookrightarrow \overline{X}''$ is indeed a quasi-compact open immersion. Since $\overline{X}''$ is a closed subscheme of the proper S-scheme $\overline{X} \times_S \overline{X}'$, we see that $\overline{f}''\colon \overline{X}'' \to S$ is proper again, as required.

For (2), first observe that $\overline{X}'' = \mathrm{Eq}(\pi, \pi')$ is a closed subscheme of $\overline{X}$ since $\overline{X}'$ is separated over S. Thus $\overline{f}''\colon \overline{X}'' \to S$ is proper again. Moreover, since $\pi|_X = \pi'|_X = \mathrm{id}_X$, we see that the quasi-compact open embedding $j'\colon X \hookrightarrow \overline{X}'$ factors over $\overline{X}''$, hence we get $j''\colon X \hookrightarrow \overline{X}''$ as required. □

We are now ready to prove (⊠) from Remark* 3.2.5, which settles that $R^i f_! \mathcal{F}$ is independent of the choice of compactification.

Sketch of proof of (⊠)*. Cofilteredness of $\mathcal{C}$ was proved in Lemma* 3.2.7 above. To get the isomorphisms

$$\iota_\pi\colon R^i\overline{f}'_* j'_! \mathcal{F} \xrightarrow{\sim} R^i\overline{f}_* j_! \mathcal{F}\,,$$

apply $R^i\overline{f}_*$ to Lemma 3.2.6(b) and use the Leray spectral sequence (Proposition 2.1.11), which conveniently collapses. We omit the verification that $\iota_{\pi\circ\pi'} = \iota_\pi \circ \iota_{\pi'}$. □

3.2.8. Definition. — Let $f\colon X \to S$ be a separated morphism of finite type between arbitrary schemes. Let further $\mathcal{F}$ be a torsion sheaf on $X_{\text{ét}}$.

(a) The i^{th} *higher direct image of* $\mathcal{F}$ *with compact support* is defined as follows: if S is quasi-compact quasi-separated, let $\overline{f}\colon \overline{X} \to S$ be a Nagata compactification as in Theorem 3.2.1. Then put $R^i f_! \mathcal{F} = R^i\overline{f}_* j_! \mathcal{F}$. In general, we cover S by quasi-compact quasi-separated open subschemes, on which the above construction may be used, and glue everything together by means of the canonical isomorphisms from (⊠).

(*b*) If S is the spectrum of a separably closed field, the i^{th} *cohomology of* $\mathcal{F}$ *with compact support* is defined as $H^i_c(X_{\text{ét}}, \mathcal{F}) = H^i(\overline{X}_{\text{ét}}, j_!\mathcal{F})$.

3.2.9. Proposition. — *Let S be an arbitrary scheme.*

(*a*) *Suppose $f\colon X \to Y$ and $g\colon Y \to S$ are separated morphisms of finite type, and $\mathcal{F}$ is a torsion sheaf in* $\mathrm{Ab}(X_{\text{ét}})$*. Then there exists a Leray spectral sequence*

$$E_2^{p,q} = R^p g_! R^q f_! \mathcal{F} \Longrightarrow R^{p+q}(g \circ f)_! \mathcal{F}$$

which is functorial in $\mathcal{F}$.

(*b*) *Let $f\colon X \to S$ be separated of finite type. Let $j\colon U \hookrightarrow X$ be a quasi-compact*[3] *open embedding and $i\colon Z \hookrightarrow X$ be a closed embedding with image $X \setminus U$. Let $u = f \circ j\colon U \to S$ and $v = f \circ i\colon Z \to S$ denote the structure morphisms. For every torsion sheaf $\mathcal{F}$ in* $\mathrm{Ab}(X_{\text{ét}})$*, there is a long exact "excision sequence"*

$$\ldots \longrightarrow R^p u_!(\mathcal{F}|_{U_{\text{ét}}}) \longrightarrow R^p f_! \mathcal{F} \longrightarrow R^p v_!(\mathcal{F}|_{Z_{\text{ét}}}) \longrightarrow R^{p+1} u_!(\mathcal{F}|_{U_{\text{ét}}}) \longrightarrow \ldots .$$

which is functorial in $\mathcal{F}$.

(*c*) *Suppose we are given a pullback diagram*

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] \arrow[dr, phantom, "\lrcorner"] & X \arrow[d, "f"] \\ S' \arrow[r, "g"] & S \end{tikzcd}$$

in which f, f' are separated morphisms of finite type, and g, g' are arbitrary. Then for all $i \geqslant 0$ and all torsion sheaves $\mathcal{F}$ in $\mathrm{Ab}(X_{\text{ét}})$*, we get isomorphisms*

$$g^* R^i f_! \mathcal{F} \xrightarrow{\sim} R^i f'_! g'^* \mathcal{F},$$

which are functorial in $\mathcal{F}$ and assemble into an isomorphism of cohomological functors. In particular, for any geometric point $\overline{s}$ of S we get an isomorphism

$$\big(R^i f_! \mathcal{F}\big)_{\overline{s}} \cong H^i_c\big((X \times_S \operatorname{Spec} \kappa(\overline{s}))_{\text{ét}}, \operatorname{pr}_1^* \mathcal{F}\big).$$

Proof. All assertions are local on S, whence we may assume that S is quasi-compact and quasi-separated throughout the proof. We start with (*a*). Choose a compactification $\overline{g}\colon \overline{Y} \to S$ of g according to Theorem 3.2.1. Our first goal is to construct a diagram

$$\begin{tikzcd} X \arrow[r, hook, "j"] \arrow[dr, "f"'] & \overline{X} \arrow[r, hook, "j'"] \arrow[d, "\overline{f}"] \arrow[dr, phantom, "\lrcorner"] & \overline{X}' \arrow[d, "\overline{f}'"] \\ & Y \arrow[r, hook, "k"] & \overline{Y} \end{tikzcd}$$

in which the horizontal arrows are quasi-compact open embeddings as indicated, and $\overline{f}$, $\overline{f}'$ are proper. Applying Theorem 3.2.1 to $g \circ f\colon X \to S$, we get a quasi-compact open

[3]This condition was not in the lecture, since we only worked in a noetherian setting.

embedding $j_0\colon X \hookrightarrow \overline{X}_0$ such that $\overline{X}_0$ is proper over S. Let $\overline{X}'$ be the scheme-theoretic image of $(j_0, k\circ f)\colon X \to \overline{X}_0 \times_S \overline{Y}$. A slight generalization of our arguments in the proof of Lemma* 3.2.7 shows that we obtain a quasi-compact open embedding $j''\colon X \hookrightarrow \overline{X}'$. Moreover, the projection $\mathrm{pr}_2\colon \overline{X}_0 \times_S \overline{Y} \to \overline{Y}$ induces a proper morphism $\overline{f}'\colon \overline{X}' \to \overline{Y}$. Let $\overline{X} = \overline{f}'^{-1}(Y)$ and let $j'\colon \overline{X} \hookrightarrow \overline{X}'$ and $\overline{f}\colon \overline{X} \to Y$ be the base changes of k and $\overline{f}'$ respectively. We see that $\overline{f}$ is proper and j'' factors over j', thus $j'' = j'\circ j$ for some quasi-compact open embedding $j\colon X \hookrightarrow \overline{X}$. In particular, $\overline{f}$ is a compactification of $f\colon X \to Y$ and we obtain a diagram of the desired kind.

Given the above diagram, part (a) is a straightforward consequence of the Leray spectral sequence (Proposition 2.1.11). Consider

$$E_2^{p,q} = R^p\overline{g}_*R^q\overline{f}_*(j'\circ j)_!\mathcal{F} \Longrightarrow R^{p+q}(\overline{g}\circ\overline{f}')_*(j'\circ j)_!\mathcal{F}$$

Since $\overline{g}\circ\overline{f}'\colon \overline{X}' \to S$ is proper, the limit of the above spectral sequence is nothing else but $R^i(g\circ f)_!\mathcal{F}$ by Definition 3.2.8(a). To analyze its E_2-page, observe that $(j'\circ j)_! \cong j'_!\circ j_!$. Applying Lemma 3.2.6$(a)$ to the pullback square in the above diagram thus gives

$$E_2^{p,q} = R^p\overline{g}_*R^q\overline{f}_*(j'\circ j)_!\mathcal{F} \cong R^p\overline{g}_*k_!R^q\overline{f}_*j_!\mathcal{F} \cong R^pg_!R^qf_!\mathcal{F}\,,$$

where the isomorphism on the right-hand side follows straight from Definition 3.2.8(a) and the fact that $\overline{f}$, $\overline{g}$ are compactifications of f, g respectively. We thus obtain a spectral sequence of the desired kind.

Part (b). Consider the short exact sequence $0 \to j_!j^*\mathcal{F} \to \mathcal{F} \to i_*i^*\mathcal{F} \to 0$. Applying the cohomological functor $R^\bullet f_!$ to this sequence certainly produces some long exact sequence, and we only need to verify $R^pf_!j_!j^*\mathcal{F} \cong R^pu_!(\mathcal{F}|_{U_{\text{ét}}})$ and $R^pf_!i_*i^*\mathcal{F} \cong R^pv_!(\mathcal{F}|_{Z_{\text{ét}}})$. To see this, we use the spectral sequences from (a) applied to $u = f\circ j$ and $v = f\circ i$. In both cases it's easy to check that the spectral sequence collapses on its E_2-page, providing the desired isomorphisms (also see Lemma* 3.2.10(c) below).

Part (c). Since we may assume that S is quasi-compact and quasi-separated, we can choose a Nagata-compactification of f by Theorem 3.2.1. Base-changing everything to S' thus provides a diagram

$$\begin{tikzcd}
X' \arrow[r, "g'"] \arrow[d, hook, "j'"'] \arrow[dr, phantom, "\lrcorner"] & X \arrow[d, hook, "j"] \\
\overline{X}' \arrow[r, "\overline{g}'"] \arrow[d, "\overline{f}'"'] \arrow[dr, phantom, "\lrcorner"] & \overline{X} \arrow[d, "\overline{f}"] \\
S' \arrow[r, "g"] & S
\end{tikzcd}$$

Now the required base change isomorphism follows from the computation

$$g^*R^if_!\mathcal{F} = g^*R^i\overline{f}_*j_!\mathcal{F} \cong R^i\overline{f}'_*\overline{g}'^*j_!\mathcal{F} \cong R^i\overline{f}'_*j'_!g'^*\mathcal{F} = Rf'_!g'^*\mathcal{F}\,.$$

The left isomorphism is just Theorem 3.1.3, and the right isomorphism follows from Fact* 2.5.12(b). The additional assertion about $(R^if_!\mathcal{F})_{\overline{s}}$ is the special case $S = \operatorname{Spec}\kappa(\overline{s})$. $\square$

The following lemma didn't show up in the lecture (at least not in this form), but it should be mentioned once to be citeable later.

3.2.10. Lemma*. — *Let $f\colon X \to S$ be a separated morphism of finite type and let $p\colon Y \to X$ be another morphism. Let $\mathcal{F} \in \mathrm{Ab}(Y_{\text{ét}})$ be a torsion sheaf.*

(*a*) *If f is proper, then there's a functor isomorphism $R^i f_! \xrightarrow{\sim} R^i f_*$ for all $i \geqslant 0$.*

(*b*) *If p is finite, then $R^i f_!(p_* \mathcal{F}) \cong R^i(p \circ f)_!$ for all $i \geqslant 0$.*

(*c*) *If p is quasi-finite and separated, then $R^i p_! \mathcal{F} = 0$ for $i > 0$.*

*Proof**. Part (*a*) is trivial. For (*b*), we apply (*a*) to the proper morphism p and use Proposition 2.1.12 to see that the spectral sequence from Proposition 3.2.9(*a*) collapses. For (*c*), everything is local on S, whence we may assume that S is quasi-compact and quasi-separated. Applying Zariski's main theorem, we see that the Nagata compactification $\overline{p}\colon \overline{Y} \rightarrow X$ can be chosen such that $\overline{p}$ is finite. Then everything follows from (*a*) and Proposition 2.1.12. □

3.2.11. Proposition. — *Let $f\colon X \rightarrow S$ be a separated morphism of finite type and let $\mathcal{F}$ be a torsion sheaf in* $\mathrm{Ab}(X_{\text{ét}})$.

(*a*) *If $S = \operatorname{Spec} k$ is the spectrum of a separably closed field k, then $H_c^i(X_{\text{ét}}, \mathcal{F}) = 0$ for all $i > 2\dim X$.*

(*b*) *If $\overline{s}$ is a geometric point of S, then $(R^i f_! \mathcal{F})_{\overline{s}} = 0$ for all $i > 2\dim(X \times_S \operatorname{Spec} \kappa(\overline{s}))$.*

Proof. Clearly (*b*) follows from (*a*) and Proposition 3.2.9(*c*). In particular, (*b*) in the special case $X = \mathbb{A}_S^1$ follows from (*a*) in the special case where X is a curve over k. But we already know this special case because of Proposition 2.6.15(*b*)—up to the slight subtlety that we need to choose $\overline{X}$ in Definition 3.2.8(*b*) in such a way that it is again a curve over k! But that's no problem: in fact, it's always possible to choose $\overline{X}$ such that X is scheme-theoretically dense in it; see Remark* 3.2.4. The upshot is that (*b*) holds for $X = \mathbb{A}_S^1$.

In particular, if $\pi\colon \mathbb{A}_k^d \rightarrow \mathbb{A}_k^{d-1}$ denotes the canonical projection forgetting the last coordinate, then $R^q \pi_! \mathcal{F} = 0$ for all $q > 2$ and all torsion sheaves $\mathcal{F}$. We use this to prove that (*a*) holds in the special case $X = \mathbb{A}_k^d$ via induction on d. The cases $d = 0$ and $d = 1$ are trivial resp. treated above. Now assume $d > 1$ and the assertion is true for $d - 1$. Then the spectral sequence

$$E_2^{p,q} = H_c^p\big(\mathbb{A}_{k,\text{ét}}^{d-1}, R^q \pi_! \mathcal{F}\big) \Longrightarrow H_c^{p+q}\big(\mathbb{A}_{k,\text{ét}}^d, \mathcal{F}\big)$$

(this is a version of Proposition 3.2.9(*a*), using that étale sheaves over a separably closed field are uniquely determined by their stalk at the only geometric point) satisfies $E_2^{p,q} = 0$ whenever $p > 2(d-1)$ or $q > 2$, thus the limit $H^{p+q}(\mathbb{A}_k^d, \mathcal{F})$ vanishes whenever $p + q > 2d$, as required. We will reduce the general case to the case of $X = \mathbb{A}_k^d$.

Step 1. We reduce the assertion to the case where X is affine. So suppose the affine case is known. Then the general case can be done using induction on the number of affine opens needed to cover X. Suppose that number is n. Write $X = U \cup X'$, where U is affine and X' can be covered by $n - 1$ affine opens. Since intersections of affine opens are affine, using that X is separated, we see that the induction hypothesis is applicable to $U' = U \cap X'$. Now everything follows from the following claim:

($*$) *In the situation of (a), if $X = U \cup V$ is an open cover, then there exists a natural "Mayer–Vietoris sequence"*

$$\ldots \longrightarrow H_c^p\big((U \cap V)_{\text{ét}}, \mathcal{F}\big) \longrightarrow H^p(U_{\text{ét}}, \mathcal{F}) \oplus H^p(V_{\text{ét}}, \mathcal{F}) \longrightarrow H^p(X_{\text{ét}}, \mathcal{F}) \longrightarrow \ldots .$$

More generally, a similar sequence exists for arbitrary X, replacing cohomology with compact support $H_c^p(-, \mathcal{F})$ by higher direct images with compact support $R^p(-)_! \mathcal{F}$.

Claim $(*)$ is a formal consequence of the excision sequence from Proposition 3.2.9(b), applying it twice to $Z = X \setminus U = V \setminus (U \cap V)$. The argument works just as in Algebraic Topology and is perhaps an amusing exercise if you don't know this already.

Step 2. So now let X be affine. The next step is to reduce things to the case where X is integral. This is done by the method that was used over and over again in Section 2.5 and Section 2.6. As usual, Proposition 1.4.20 allows us to replace X by its reduction X^{red}. Now let $X_1, \dots, X_n$ be the irreducible components of X, equipped with their integral closed subscheme structures, and denote $i_j\colon X_j \hookrightarrow X$. Consider $\varphi\colon \mathcal{F} \to \bigoplus_{j=1}^n i_{j,*} i_j^* \mathcal{F}$. Lemma* 3.2.10($b$) shows

$$H_c^i\left(X_{\text{ét}}, \bigoplus_{j=1}^n i_{j,*} i_j^* \mathcal{F}\right) \cong \bigoplus_{j=1}^n H_c^i\left(X_{j,\text{ét}}, i_j^* \mathcal{F}\right).$$

In particular, if we assume the assertion is true in the affine integral case, then the right-hand side vanishes for $i > 2 \max \dim X_j = 2 \dim X$. Moreover, φ is an isomorphism over $U_{\text{ét}}$ for some dense open subset $U \subseteq X$. In particular, $\ker \varphi$ and $\operatorname{coker} \varphi$ are of the form $u_* \mathcal{K}$ and $u_* \mathcal{Q}$, where $u\colon Z \hookrightarrow X$ is any closed subscheme with image $X \setminus U$, and $\mathcal{K}, \mathcal{Q} \in \mathrm{Ab}(Z_{\text{ét}})$. Since $\dim Z < \dim X$, we may use induction to assume that the assertion is true for Z as well. Then Lemma* 3.2.10(b) again shows

$$H_c^i(X_{\text{ét}}, u_* \mathcal{K}) \cong H_c^i(Z_{\text{ét}}, \mathcal{K}) \quad \text{and} \quad H_c^i(X_{\text{ét}}, u_* \mathcal{Q}) \cong H_c^i(Z_{\text{ét}}, \mathcal{Q}),$$

and the right-hand sides vanish for $i > 2 \dim Z$ by assumption. Writing down some long exact sequences finally shows $H_c^i(X_{\text{ét}}, \mathcal{F}) = 0$ for $i > 2 \dim X$, as required.

Step 3. We deal with the case where X is affine and integral, thus finishing the proof. By Noether normalization we find a finite morphism $p\colon X \to \mathbb{A}_k^d$, where we necessarily have $d = \dim X$. Using Lemma* 3.2.10(b) we get $H_c^i(X_{\text{ét}}, \mathcal{F}) \cong H_c^i(\mathbb{A}_{k,\text{ét}}^d, p_* \mathcal{F}) = 0$ for $i > 2d$, as required. □

3.3. Finiteness Theorems

We finish the lecture with two finiteness results, which we don't have the time to prove. But who knows? Maybe we will come to that in Professor Franke's seminar next semester!

3.3.1. Theorem. — *Let $f\colon C \to S$ be a proper smooth relative curve and n a positive integer which is invertible on S. Moreover, let $\mu_{n,C}$ and $\mu_{n,S}$ denote the sheaves of n^{th} roots of unity on $C_{\text{ét}}$ and $S_{\text{ét}}$ respectively. Then we have isomorphisms*

$$R^i f_* \mu_{n,C} \cong \begin{cases} \mu_{n,S} & \text{if } i = 0 \\ \underline{\mathrm{Jac}}_{C/S}[n] & \text{if } i = 1 \\ \mathbb{Z}/n\mathbb{Z}_S & \text{if } i = 2 \\ 0 & \text{else} \end{cases}.$$

3.3.2. Theorem. — *If $f\colon X \to S$ is a separated morphism of finite type beween noetherian schemes, then for any constructible sheaf $\mathcal{F} \in \mathrm{Ab}(X_{\text{ét}})$, the higher direct images with compact support $R^i f_! \mathcal{F}$ are constructible again for all $i \geqslant 0$.*

3.3.3. Corollary. — *If X is a separated scheme of finite type over a separably closed field k and $\mathcal{F} \in \mathrm{Ab}(X_{\text{ét}})$ is constructible, then the cohomology groups $H_c^i(X, \mathcal{F})$ are finite abelian groups for all $i \geqslant 0$.*

Appendix A.

Some Supplementary Material A

A.1. On Inverse Limits of Schemes

In the following, all references refer to [Stacks] and colim always denotes a colimit over filtered partially ordered set. Note that colimits over filtered categories may, with some care, be reduced to colimits over filtered posets (Tag 0032).

A.1.1. Ascending properties. — Let (S_α) be a cofiltered system of schemes. We investigate whether the limit over this system exists, and list some properties:

(a) By Tag 01YX, the cofiltered limit $\lim S_\alpha$ exists if all transition morphisms are affine. Moreover, if 0 is an initial object in the (surpressed) index poset and $U_0 \subseteq S_0$ is an open subscheme (or a closed one, or actually any S_0-scheme), then

$$\lim_\alpha (U_0 \times_{S_0} S_\alpha) \cong U_0 \times_{S_0} \lim_\alpha S_\alpha$$

(since fibre products are limits too, this follows from abstract nonsense). In fact, we can describe $S = \lim S_\alpha$ explicitly. If 0 is initial as above, then $S_\alpha \cong \underline{\operatorname{Spec}}_{S_0} \mathcal{S}_\alpha$ for some quasi-coherent $\mathcal{O}_{S_0}$-algebras $\mathcal{S}_\alpha$. Then $\mathcal{S} = \operatorname{colim}_\alpha \mathcal{S}_\alpha$ is quasi-coherent again and $S \cong \underline{\operatorname{Spec}}_{S_0} \mathcal{S}$.

(b) In Tag 01Z2 it is proved that if all S_α are non-empty and quasi-compact, then S is non-empty. Actually S is also quasi-compact again, as follows from (c) and Tychonoff's theorem. Using this assertion together with (a), we see moreover that if all S_α are quasi-separated, then the same is true for S.

(c) By Tag 0CUF we have a homeomorphism $|S| \cong \lim |S_\alpha|$ on underlying topological spaces.

A.1.2. Descending properties of (sub-)objects. — It is often a useful fact to know that if $S = \lim S_\alpha$ has a certain property $\mathcal{P}$, then also all "sufficiently large" S_β (i.e., all $\beta \geqslant \alpha$ for some fixed α) have $\mathcal{P}$. In combination with (d), this is a standard procedure to reduce assertions about arbitrary schemes to the noetherian case.

(d) By Tag 01ZA, every quasi-compact and quasi-separated scheme S can be written as a limit $S = \lim S_\alpha$, where the S_α are of finite type over $\mathbb{Z}$, all transition maps are affine, and the index category can be chosen to be a directed poset.

(e) Suppose $S = \lim S_\alpha$, where all S_α (and thus S by (b)) are quasi-compact and quasi-separated. By Tag 01ZR, every finitely presented $\mathcal{O}_S$-module $\mathcal{F}$ (in particular, every coherent module if S is noetherian) can be written as the pullback of a finitely presented $\mathcal{F}_\alpha$ over S_α for some suitable α. If $\mathcal{F} = \mathcal{V}$ is a vector bundle, then by Tag 0B8W we may even assume that $\mathcal{V}_\alpha$ is already a vector bundle too by.

(f) Suppose $S = \lim S_\alpha$, where all S_α are quasi-compact and quasi-separated. If S is quasi-affine, then Tag 01Z6 shows that there exists an α such that all S_β are quasi-affine for $\beta \geqslant \alpha$.

(g) Suppose $S = \lim S_\alpha$, where all S_α are quasi-compact and quasi-separated. If S is affine, then Tag 01Z7 shows that there exists an α such that all S_β are affine for $\beta \geqslant \alpha$.

(h) Suppose $S = \lim S_\alpha$, where all S_α are quasi-compact and quasi-separated, and denote by $\pi_\alpha \colon S \rightarrow S_\alpha$ the canonical projections. Then every quasi-compact open subset $U \subseteq S$ is of the form $U = \pi_\alpha^{-1}(U_\alpha)$ for some α and some quasi-compact open subset $U_\alpha \subseteq S_\alpha$. Since Professor Franke couldn't find a Stacks Project reference (it would have been Tag 01Z4), the proof below is his own.

Proof of (h). Without restriction the index category has a final object 0 (we can always pass to the cofinal subcategory of all α-objects). Since $\pi_0 \colon S \rightarrow S_0$ is affine and S_0 is quasi-separated, S is quasi-separated as well (alternatively we could have used A.1.1(b)). Write $S_0 = \bigcup_{j=1}^n U_j$ as a finite union of affine open subschemes U_j, using that S_0 is quasi-compact. Then the $\pi_0^{-1}(U_j)$ are affine and the $U \cap \pi_0^{-1}(U_j)$ are still quasi-compact as S is quasi-separated.

It suffices to show that all $U \cap \pi_0^{-1}(U_j)$ are of the form $\pi_{\alpha_j}^{-1}(U_{\alpha_j})$ for some α_j and some quasi-compact open $U_{\alpha_j} \subseteq S_{\alpha_j} \times_{S_0} U_j$. Indeed, if we choose $\alpha \geqslant \alpha_1, \dots, \alpha_n$, then the open subscheme

$$U_\alpha \coloneqq \bigcup_{j=1}^n U_{\alpha_j} \times_{S_{\alpha_j}} S_\alpha \subseteq S_\alpha$$

will satisfy the desired condition.

Thus, we may assume that $S_0 = U_j$ and thus all S_α and S are affine. Say $S_\alpha = \operatorname{Spec} R_\alpha$ and $S = \operatorname{Spec} R = \operatorname{Spec}(\operatorname{colim}_\alpha R_\alpha)$. As U is quasi-compact, it has a finite open cover $U = \bigcup_{k=1}^m \operatorname{Spec} R \setminus V(f_k)$. Since $f_1, \dots, f_m$ are finitely many elements, they must already be contained in some R_α. Now $U_\alpha \coloneqq \operatorname{Spec} R_\alpha \setminus V(f_1, \dots, f_m)$ does the job. □

A.1.3. Descending properties of morphisms. — Same as for objects, we would like to know that if a morphism $f \colon X \rightarrow S$ has some property $\mathcal{P}$ and can be written as a limit over $f_\alpha \colon X_\alpha \rightarrow S_\alpha$, then already some f_α has property $\mathcal{P}$. Together with (i) below, this often allows to prove results about morphisms of finite type between noetherian schemes also for morphisms of finite presentation between arbitary schemes.

(i) Let $f \colon X \rightarrow S$ be a finitely presented morphism, and assume $S = \lim S_\alpha$, where all S_α are quasi-compact and quasi-separated and all transition morphisms are affine. By Tag 01ZM, there exists an index α for which there is a scheme X_α and a finitely presented morphism $f_\alpha \colon X_\alpha \rightarrow S_\alpha$ such that $X = X_\alpha \times_{S_\alpha} S$ and f is the base change of f_α. In particular, writing $X_\beta = X_\alpha \times_{S_\alpha} S_\beta$ for all $\beta \geqslant \alpha$, we see that f can be written as a cofiltered limit over finitely presented morphisms $f_\beta \colon X_\beta \rightarrow S_\beta$ for $\beta \geqslant \alpha$.

Moreover, if $g \colon Y \rightarrow S$ is another finitely presented morphism and written as a limit over $g_\beta \colon Y_\beta \rightarrow S_\beta$ for $\beta \geqslant \alpha$ as above, then

$$\operatorname*{colim}_{\beta \geqslant \alpha} \operatorname{Hom}_{\mathrm{Sch}/S_\beta}(X_\beta, Y_\beta) \cong \operatorname{Hom}_{\mathrm{Sch}/S}(X, Y)\,.$$

Combining this result with (d), we see that every finitely presented morphism between quasi-compact quasi-separated schemes can be written as a cofiltered limit of morphisms between schemes of finite type over $\mathbb{Z}$.

(j) By Tag 0204, if $f\colon X \to S$ is proper morphism and written as a cofiltered limit over finitely presented morphisms $f_\beta\colon X_\beta \to S_\beta$ as in (i), then already f_β is proper for some $\beta \geqslant \alpha$. The idea is to use Chow's lemma to reduce everything to a similar question about projective morphisms, which is easier to treat.

(k) By Tag 07RP, if $f\colon X \to S$ is étale and written as a cofiltered limit as in (i), then already f_β is étale for some $\beta \geqslant \alpha$.

(l) The same conclusion holds for a variety of properties. A comprehensive list can be found in [$\mathrm{EGA}_{\mathrm{IV}/3}$, Théorème (8.10.5)]; but let us mention that being quasi-finite is one of these properties since we have to apply Zariski's main theorem in a non-noetherian situation to prove Proposition 1.6.7.

A.2. The Conditions R_k and S_k

Serre's conditions R_k and S_k have shown up a few times: first in the proof of the Zariski–Nagata purity theorem (Theorem 1.5.18) in Lecture 7 and then again in the construction of henselizations in Lecture 11. So I decided to write down proofs of the facts we have used.

Naturally, depth of modules will occur, together with some local cohomology. The standard reference for this is of course [SGA_2], but I also find [Har67] quite readable (not least because it is in English).

A.2.1. Definition*. — Let X be a locally notherian scheme.

(a) We say X *satisfies* R_k at the point $x \in X$ if $\mathcal{O}_{X,x}$ is regular or $\dim \mathcal{O}_{X,x} > k$. In case X satisfies R_k at all $x \in X$, we just say X *satisfies* R_k.

(b) If $\mathcal{F}$ is a coherent $\mathcal{O}_X$-module, we say $\mathcal{F}$ *satisfies* S_k at the point $x \in X$ if we have $\operatorname{depth} \mathcal{F}_x \geqslant \min\{k, \dim(\operatorname{supp} \mathcal{F}_x)\}$. In case $\mathcal{O}_X$ satisfies S_k at all points $x \in X$, we often just say X *satisfies* S_k.

(c) If $\mathcal{F}$ is coherent and satisfies $\operatorname{depth} \mathcal{F}_x \geqslant \min\{k, \dim \mathcal{O}_{X,x}\}$ at some $x \in X$, then we say $\mathcal{F}$ *satisfies* S'_k *at* x. As usual, if this is true for all $x \in X$, we just say $\mathcal{F}$ *satisfies* S'_k.

A.2.2. Warning*. — Beware that Definition* A.2.1 is *non-standard terminology*! But we need it for Lemma* A.2.3 below. Note that this lemma wouldn't be true if S'_1, S'_2 were replaced by S_1, S_2 respectively. For example, part (a) is false when $U = \operatorname{Spec} A$ where A is a DVR with maximal ideal $\mathfrak{m}$, $V = U \setminus \{\mathfrak{m}\}$ (which is dense), and $\mathcal{F}$ is the coherent module associated to $A/\mathfrak{m}$.

However, at least in the case $\mathcal{F} = \mathcal{O}_X$, and more generally for vector bundles, the conditions S_k and S'_k coincide.

A.2.3. Lemma*. — *Let X be a locally noetherian scheme and $\mathcal{F}$ a coherent $\mathcal{O}_X$-module.*

(a) *$\mathcal{F}$ satisfies S'_1 iff $\Gamma(U, \mathcal{F}) \hookrightarrow \Gamma(V, \mathcal{F})$ is injective whenever V is a dense open subset of the open subset $U \subseteq X$.*

(b) *$\mathcal{F}$ satisfies S'_2 iff it satisfies S_1 and $\Gamma(U, \mathcal{F}) \xrightarrow{\sim} \Gamma(V, \mathcal{F})$ whenever $V \subseteq U$ are open subsets of X such that* $\operatorname{codim}(Z, U) \geqslant 2$ *for every connected component Z of $U \setminus V$.*

Proof.* Recall the following characterization of depth via local cohomology: let A be a local ring with maximal ideal $\mathfrak{m}$ and M a finite module over A. Then $\operatorname{depth} M$ is the smallest non-negative integer $d \geqslant 0$ such that

$$0 \neq H^d_{\mathfrak{m}}(M) = \operatorname*{colim}_{n \in \mathbb{N}} \operatorname{Ext}^d_A(A/\mathfrak{m}^n, M)\,.$$

See e.g. [Har67, Theorem 3.8] for a proof. Now let $V \subseteq U \subseteq X$ be open subsets, let $Z = U \setminus V$ and consider the local cohomology sequence

$$0 \longrightarrow H^0_Z(U,\mathcal{F}) \longrightarrow \Gamma(U,\mathcal{F}) \longrightarrow \Gamma(V,\mathcal{F}) \longrightarrow H^1_Z(U,\mathcal{F}) \longrightarrow H^1(U,\mathcal{F}) \longrightarrow \dots .$$

For every point $x \in Z$ we have $\mathcal{H}^p_Z(\mathcal{F})_x \cong H^p_{\mathfrak{m}_{X,x}}(\mathcal{F}_x)$. In particular, the above shows that $\operatorname{depth}\mathcal{F}_x \geqslant 1$ holds iff $\mathcal{H}^0_Z(\mathcal{F})_x = 0$ and $\operatorname{depth}\mathcal{F}_x \geqslant 2$ iff $\mathcal{H}^0_Z(\mathcal{F})_x = \mathcal{H}^1_Z(\mathcal{F}) = 0$. If $\mathcal{F}$ satisfies S_1', then $\operatorname{depth}\mathcal{F}_x = 0$ can only happen if $\dim\mathcal{O}_{X,x} = 0$. But if V is dense in U, then every $x \in Z$ has codimension at least 1, so we obtain $\mathcal{H}^0_Z(\mathcal{F}) = 0$ in this case. Then also $H^0_Z(U,\mathcal{F}) = 0$ and $\Gamma(U,\mathcal{F}) \hookrightarrow \Gamma(V,\mathcal{F})$ is indeed injective.

In a similar manner, $\mathcal{F}$ having S_2' implies that $H^1_Z(U,\mathcal{F}) = 0$ whenever $\operatorname{codim}(Z,U) \geqslant 2$, so the long exact local cohomology sequence shows that $\Gamma(U,\mathcal{F}) \xrightarrow{\sim} \Gamma(V,\mathcal{F})$ is an isomorphism, as claimed.

Conversely, assume that $\mathcal{F}$ satisfies the condition from (a). We need to show that $\mathcal{F}$ is S_1'. The only critical case is when $x \in X$ is a point such that $\operatorname{depth}\mathcal{F}_x = 0$. In this case, we have $H^0_{\mathfrak{m}_{X,x}}(\mathcal{F}_x) \neq 0$. Thus, if U is an affine open neighbourhood of x and $Z = \overline{\{x\}}$, then $H^0_Z(U,\mathcal{F}) \neq 0$ because $\mathcal{H}^0_Z(\mathcal{F})$ is the quasi-coherent sheaf associated to $H^0_Z(U,\mathcal{F})$ and its stalk at x is non-vanishing. Putting $V = U \setminus U \setminus \overline{\{x\}}$, we see that $\Gamma(U,\mathcal{F}) \to \Gamma(V,\mathcal{F})$ can't be injective. Thus V can't be dense in U, whence x must have codimension 0. So S_1' still holds at x.

In the same way we see that the condition from (b) implies that $\mathcal{F}$ is S_2'; the only additional ingredient is that $H^1(U,\mathcal{F}) = 0$ if U is affine. □

A.2.4. Lemma*. — *Let $f\colon X \to Y$ be a morphism of locally finite type between locally noetherian schemes. Let $x \in X$ and $y = f(x)$ such that f is étale at x. Then:*

(a) *X satisfies R_k at x iff Y satisfies R_k at y.*

(b) *X satisfies S_k at x iff Y satisfies S_k at y.*

Proof.* We have seen in the proof of Proposition 1.4.14 that $\dim\mathcal{O}_{X,x} = \dim\mathcal{O}_{Y,y}$ and that $\mathcal{O}_{X,x}$ is regular iff $\mathcal{O}_{Y,y}$ is. This immediately shows (a). For (b), what remains to check is $\operatorname{depth}\mathcal{O}_{X,x} = \operatorname{depth}\mathcal{O}_{Y,y}$. Suppose $a_1,\dots,a_n \in \mathfrak{m}_{Y,y}$ form a maximal regular sequence. Since $\mathcal{O}_{X,x}$ is flat over $\mathcal{O}_{Y,y}$, we immediately verify that the images of the a_i in $\mathfrak{m}_{X,x}$ form a regular sequence too. This shows $\operatorname{depth}\mathcal{O}_{X,x} \geqslant \operatorname{depth}\mathcal{O}_{Y,y}$.

For the converse, we may mod out the ideal $(a_1,\dots,a_n)$ to replace $\mathcal{O}_{X,x}$ and $\mathcal{O}_{Y,y}$ by $\mathcal{O}_{X,x}/(a_1,\dots,a_n)$ and $\mathcal{O}_{Y,y}/(a_1,\dots,a_n)$. Thus, we are still in a situation where $f\colon X \to Y$ is étale at x, but now additionally $\operatorname{depth}\mathcal{O}_{Y,y} = 0$. This happens iff $\mathfrak{m}_{Y,y}$ is an associated prime ideal of $\mathcal{O}_{Y,y}$ (see [Hom, Lemma 2.3.1] for example). So let $a \in \mathcal{O}_{Y,y}$ such that $\mathfrak{m}_{Y,y} = \operatorname{Ann}_{\mathcal{O}_{Y,y}}(a)$, i.e., $\mathfrak{m}_{Y,y}$ is the kernel of the multiplication map $a\colon \mathcal{O}_{Y,y} \to \mathcal{O}_{Y,y}$. Since $\mathcal{O}_{X,x}$ is flat over $\mathcal{O}_{Y,y}$, the kernel of $a\colon \mathcal{O}_{X,x} \to \mathcal{O}_{X,x}$ is given by $\mathfrak{m}_{Y,y} \otimes_{\mathcal{O}_{Y,y}} \mathcal{O}_{X,x} \cong \mathfrak{m}_{Y,y}\mathcal{O}_{X,x}$. But since $f\colon X \to Y$ is unramified at x, we get $\mathfrak{m}_{Y,y}\mathcal{O}_{X,x} = \mathfrak{m}_{X,x}$ (see Proposition 1.4.1(c)). Thus $\mathfrak{m}_{X,x}$ is an associated prime ideal of $\mathcal{O}_{X,x}$, which shows $\operatorname{depth}\mathcal{O}_{X,x} = 0$. We are done! □

A.3. More on G-Rings and Excellent Rings

In the 12^{th} lecture, when we introduced G-rings and excellent rings and all this stuff, Professor Franke casually dropped some facts of the form "if a noetherian ring is X and Y, then it is already Z". Since none of them were obvious to me, I decided to write up proper proofs. Some of them are my own, some are taken from [Stacks].

Before we begin with the proofs, we expand a bit upon the terminology that came up in Definition 1.6.20, and introduce a bit more. Unless otherwise specified, all references in this section are to [Stacks].

A.3.1. Notation*. — Throughout this section, if A is a noetherian ring and $\mathfrak{p} \in \operatorname{Spec} A$ a prime ideal, then $\widehat{A}_\mathfrak{p}$ denotes the completion of $A_\mathfrak{p}$ with respect to its maximal ideal $\mathfrak{p}A_\mathfrak{p}$. The localization of $\widehat{A}$ at $\mathfrak{p}$ (assuming A is local itself) will instead be denoted $\widehat{A} \otimes_A A_\mathfrak{p}$. If $\mathfrak{q} \in \operatorname{Spec} \widehat{A}$ is a prime ideal, the localization of $\widehat{A}$ at $\mathfrak{q}$ will also be denoted $\widehat{A}_\mathfrak{q}$, but this shouldn't stir any confusion as it will always be clear whether $\mathfrak{q}$ is a prime ideal in $\widehat{A}$ or A.

A.3.2. Definition*. — An arbitrary noetherian ring A is called a *G-ring* if for all primes $\mathfrak{p} \in \operatorname{Spec} A$ the localization $A_\mathfrak{p}$ is a local G-ring in the sense of Definition 1.6.20(a), i.e., for all primes $\mathfrak{q} \in \operatorname{Spec} A_\mathfrak{p}$ the geometric fibres $\widehat{A}_\mathfrak{p} \otimes_{A_\mathfrak{p}} \overline{\kappa(\mathfrak{q})}$ are regular.

A.3.3. Remark*. — A priori it is not clear that a local G-ring in the sense of Definition 1.6.20(a) is also a G-ring in the sense of Definition* A.3.2. Indeed, the latter is a condition on the geometric fibres of $A_\mathfrak{p} \rightarrow \widehat{A}_\mathfrak{p}$ (here $\widehat{A}_\mathfrak{p}$ is the completion of $A_\mathfrak{p}$ with respect to $\mathfrak{p}A_\mathfrak{p}$, and not the localization of $\widehat{A}$ at $\mathfrak{p}$) for *all* primes $\mathfrak{p} \in \operatorname{Spec} A$, whereas the former only concerns the case where $\mathfrak{p}$ is the maximal ideal of A. Nevertheless, these conditions turn out to be equivalent. See Tag 07PT for a proof (and note that The Stacks Project guys use Definition* A.3.2 as a definition of G-rings).

A.3.4. Definition*. — For a noetherian scheme X let $\operatorname{Reg}(X) = \{x \in X \mid \mathcal{O}_{X,x} \text{ regular}\}$ denote the *regular locus* of X. A noetherian ring A is called

(a) J-0 if $\operatorname{Reg}(\operatorname{Spec} A)$ contains a non-empty open subset.

(b) J-1 if $\operatorname{Reg}(\operatorname{Spec} A)$ is open.

(c) J-2 if any A-algebra of finite type is J-1.

With this terminology, an noetherian ring A is *excellent* iff it is a G-ring, J-2, and universally catenary. One of the assertions in the lecture was that if A is local, the J-2 condition isn't needed. In fact, something slightly more general is true.

A.3.5. Proposition*. — *Let A be a semi-local G-ring. Then A is J-2.*

Proof.* The proof consists of five steps. The first four are to reduce the assertion to the fact that complete noetherian local domains are J-0. This is proved in the fifth step, using the Cohen structure theorem.

Step 1. We first show that it suffices to prove that semi-local G-rings which are domains are J-0. By some general results on J-2 rings, it suffices to prove that every finite A-algebra B that happens to be a domain is J-0. In fact, by (4) of Tag 07PC it suffices to prove an even weaker condition. Clearly such B is semi-local again. Moreover, it is still a G-ring. Indeed, it can be shown more generally that any algebra of essentially finite type over a G-ring is a G-ring again. A full proof is in Tag 07PV, but in the case of a finite extension there is actually a simple argument: if $\mathfrak{q}_1, \dots, \mathfrak{q}_n$ are the primes over $\mathfrak{p} \in \operatorname{Spec} A$, then

$$\widehat{A}_\mathfrak{p} \otimes_A B \cong \prod_{i=1}^{n} \widehat{B}_{\mathfrak{q}_i}$$

(this follows basically from the Chinese remainder theorem; see Tag 07N9 for a full proof). Thus, $\widehat{B}_{\mathfrak{q}_i} \otimes_B \overline{\kappa(\mathfrak{q}_i)}$ is a factor of $\widehat{A}_{\mathfrak{p}} \otimes_A \overline{\kappa(\mathfrak{p})}$, hence regular. This shows that B is indeed a semi-local G-ring and a domain. Therefore it suffices to show that these guys are J-0, as claimed.

Step 2. Now let A be a semi-local G-ring which is a domain. We show that we can further reduce to the case where A is local. Indeed, let $\mathfrak{m}_1, \dots, \mathfrak{m}_n$ be the maximal ideals of A. Assuming the local case has been settled, we find non-zero $f_i \in A$ such that the localizations $A_{\mathfrak{m}_i}[f_i^{-1}]$ are regular. Then $A[(f_1 \cdots f_n)^{-1}]$ is regular too, proving that A is J-0.

Step 3. Now let A be local, in addition to the other assumptions. Suppose it has been shown that $\widehat{A}$ is J-2 (mind that $\widehat{A}$ might not be a domain any more). Suppose $\mathfrak{q} \in \operatorname{Spec} \widehat{A}$ is a prime such that the localization $\widehat{A}_{\mathfrak{q}}$ is regular, and let $\mathfrak{p} = \mathfrak{q} \cap A$. Then $A_{\mathfrak{p}}$ is regular too. Indeed, this follows from the characterization of regularity via finiteness of global dimension and the fact that $\widehat{A}_{\mathfrak{q}}$ is faithfully flat over $A_{\mathfrak{p}}$ (see Tag 00OF for more details). Conversely, if $A_{\mathfrak{p}}$ is regular and $\mathfrak{q} \in \operatorname{Spec} \widehat{A}$ is a prime over $\mathfrak{p}$, then the localization $\widehat{A}_{\mathfrak{q}}$ is regular too by Tag 031E, using that the fibres $\widehat{A} \otimes_A \kappa(\mathfrak{p})$ are regular (a priori we only know this for the geometric fibres, but then it holds for the ordinary fibres as well). This shows that if $f\colon \operatorname{Spec} \widehat{A} \rightarrow \operatorname{Spec} A$ denotes the induced morphism on schemes, then

$$\operatorname{Reg}(\operatorname{Spec} \widehat{A}) = f^{-1}\bigl(\operatorname{Reg}(\operatorname{Spec} A)\bigr)\,.$$

Since we assume $\widehat{A}$ is J-2, the left-hand side is open. Thus $\operatorname{Reg}(\operatorname{Spec} A)$ is open as well, because f is fpqc and thus a quotient map on underlying topological spaces by Proposition 1.2.13. Clearly $\operatorname{Reg}(\operatorname{Spec} A)$ is non-empty, because A is a domain, hence regular at (0). This shows that A is J-0, as required.

Step 4. By Step 3, it suffices to show that complete local rings (which are automatically G-rings) are J-2. Thus A may be assumed complete from now on. This came at the cost of losing the information that A is a domain though. However, this can be quickly regained: as in Step 1, let B be a finite A-algebra that is a domain. We are to show that B is J-0. Since A is henselian, B is a finite product of local A-algebras (Proposition 1.6.7(c)). But then B must already be local, or it wouldn't be a domain. Moreover, B is complete with respect to the maximal ideal of A, hence complete with respect to its own maximal ideal. This shows that it suffices to consider the case where A is a complete local domain.

Step 5. After all these reductions, let the actual proof begin! By the Cohen structure theorem in the form of Tag 032D we find a subring $A_0 \subseteq A$ such that A is finite over A_0 and $A_0 \cong \Lambda[\![X_1, \dots, X_d]\!]$, where Λ is a field or a Cohen ring (i.e., a complete DVR with uniformizer p a prime number). In particular, A_0 is regular. Let $K_0 \subseteq K$ denote the fraction fields of A_0 and A. This is a finite field extension. If it is separable, Lemma* A.3.6 below immediately shows that A is J-0.

Now assume $K_0 \subseteq K$ is not separable. This can only happen in positive characteristic, so in particular $A_0 \cong k[\![X_1, \dots, X_d]\!]$, where k is the residue field of A. Let N/K_0 be the normal closure of K/K_0 and let L/K_0 be fixed field under $\operatorname{Aut}(N/K_0)$. Then N/K is Galois and L/K_0 is purely inseparable. Moreover, $L \subseteq K$ since there a purely inseparable element has no other conjugates than itself. Let $B = A \cap L$. Then A is finite over B and the extension K/L on fraction fields is separable. Thus, by Lemma* A.3.6 it is enough to show that B is J-0. Hence we may assume $A = B$ and that the extension K/K_0 is purely inseparable.

In this case we may choose a power q of the characteristic p such that $K^q \subseteq K$. Let $k^{1/q}$ be the field obtained by adjoining all q^{th} roots of elements of k and consider the ring

$$A_0^{1/q} \coloneqq k^{1/q}[\![X_1^{1/q}, \dots, X_d^{1/q}]\!] .$$

By construction, $A \subseteq A_0^{1/q}$, and this is an integral ring extension (but probably not finite). Also note that $A_0^{1/q}$ is flat over A_0. Indeed, $k^{1/q}[X_0^{1/q}, \dots, X_d^{1/q}]$ is flat (and even free) over $k[X_1, \dots, X_d]$, and flatness is preserved under completion (this actually needs a small argument, but we omit that here). Moreover, using Grothendieck's generic freeness theorem (see Tag 051R or [Jac, Proposition A.2.1]), there is some non-zero $f \in A_0$ such that A_f is free over $(A_0)_f$. Then A_f is faithfully flat over $(A_0)_f$ as $A_0 \subseteq A$ is an inclusion of domains. We claim that this implies that $(A_0^{1/q})_f$ is flat over A. Indeed, since faithful flatness is preserved under base change, $(A_0^{1/q})_f \otimes_{(A_0)_f} A_f$ is faithfully flat over $(A_0^{1/q})_f$ and flat over A_f, so an easy argument shows that $(A_0^{1/q})_f$ is indeed flat over A_f. By going-up, $A_f \subseteq (A_0^{1/q})_f$ is even a faithfully flat ring extension, because it is integral. But now we are done, since $A_0^{1/q}$ is clearly regular, hence A_f is regular too by Tag 07NG. ☐

It remains to show the claimed lemma about extensions of domains which are separable on fraction fields.

A.3.6. Lemma*. — *Let $R \subseteq S$ be a finite extension of noetherian domains such that the induced extension L/K of fraction fields is separable. Then R is J-0 iff S is J-0.*

Proof.* The basic idea of the proof is that such a ring map is *generically smooth*. That is, we claim the following:

$(*)$ *There is an element $f \in R$ such that $R_f \to S_f$ is smooth (and actually even étale).*

Indeed, by Grothendieck's generic freeness theorem (see Tag 051R or [Jac, Proposition A.2.1]), there is a non-zero $f \in R$ such that S_f and $(\Omega_{S/R})_f$ are free over R. Then actually $(\Omega_{S/R})_f = 0$, because its stalk $\Omega_{L/K}$ at the generic point $(0) \in \operatorname{Spec} S$ vanishes as L/K is separable. This shows that $R_f \to S_f$ is flat and unramified, hence étale as claimed.

Now assume R is J-0. Then $f \in R$ from $(*)$ may be chosen in such a way that R_f is regular and $R_f \to S_f$ is smooth. Thus some general result shows that S_f is regular too (see Tag 07NF for example).

Conversely, let S be J-0 and choose some non-zero $g \in S$ such that S_g is regular. Since S is finite over R, the morphism $\operatorname{Spec} S \to \operatorname{Spec} R$ is proper. Hence the image of $V(g)$ in $\operatorname{Spec} R$ is closed. But it can't be all of $\operatorname{Spec} R$, as otherwise $\dim S/gS \geqslant \dim R$, contradicting the going-up theorem. Thus the image of $V(g)$ is contained in $V(f)$ for some non-zero $f \in R$; moreover, we may assume that f is as in $(*)$. Then $R_f \to S_f$ is smooth and S_f is regular. Since $R_f \subseteq S_f$ is a finite extension, S_f is actually faithfully flat over R_f. Then R_f must be regular too by Tag 00OF. ☐

The second not at all trivial remark from the 12^{th} lecture was that excellent rings are always universally Japanese in the sense of Definition 1.6.20. In fact, catenarity is not even needed.

A.3.7. Proposition*. — *Let A be a noetherian ring which is a G-ring and J-2 (such rings are called quasi-excellent). Then A is universally Japanese. In particular, this holds for all excellent rings.*

The first step is to prove the assertion for complete noetherian local domains, which is already not quite trivial.

A.3.8. Lemma*. — *Let A be a complete noetherian local domain (so A is automatically a G-ring, hence J-2 by Proposition* A.3.5, but we won't need that). If L/K is a finite extension of the fraction field of A, then the integral closure of A in L is finite over A.*

Proof.* The proof we give here is a concrete version of the proof in Tag 032W that uses more general results. By the Cohen structure theorem we find a subring $A_0 \subseteq A$ such that A is finite over A_0 and $A_0 \cong \Lambda[\![X_1, \dots, X_d]\!]$, where Λ is a field or a Cohen ring. So it suffices to show the theorem for $A = \Lambda[\![X_1, \dots, X_d]\!]$. In this case A is normal. Let K be its quotient field. If L/K is a separable extension, then finiteness of the integral closure of A in L is a well-known result, using that the trace form $\mathrm{Tr}_{L/K} \colon L \times L \to K$ is non-degenerate (see Tag 032L for details). In particular, this completely handles the case where K has characteristic 0.

So from now on we may assume that the characteristic is $p > 0$. Then $\Lambda = k$ is the residue field of A, so we need to show that the integral closure of $A = k[\![X_1, \dots, X_d]\!]$ in L is finite over A. We use induction on d. The case $d = 0$ is trivial, so let $d \geqslant 1$ and assume the assertion holds for $d - 1$. By an argument similar to the last step of the proof of Proposition* A.3.5, it suffices to consider the case where L/K is purely inseparable. Choose a power q of p such that $L^q \subseteq K$ and let B denote the integral closure of A in L. Let $Y_i = X_i^{1/q}$. Then surely B is contained in

$$A^{1/q} \coloneqq k^{1/q}[\![Y_1, \dots, Y_d]\!] \,.$$

Consider the ring morphism $\varepsilon \colon A^{1/q} \to A^{1/q}$ that fixes the variables $Y_1, \dots, Y_{d-1}$ and sends Y_d to 0. We may actually assume that $Y_d \in B$ and that ε restricts to a morphism $B \to B$. Indeed, L can be generated by elements $\beta_1, \dots, \beta_n \in A^{1/q}$ and we are free to enlarge L by adjoining Y_d and $\varepsilon(\beta_1), \dots, \varepsilon(\beta_n)$.

Now we claim that B/Y_dB is finite over $A/X_dA \cong k[\![X_1, \dots, X_{d-1}]\!]$. Indeed, these rings can be identified with the images of B and A under ε. Certainly $\varepsilon(B)$ is integral over $\varepsilon(A)$, so we only need to see that the integral closure of $\varepsilon(A) \cong k[\![X_1, \dots, X_{d-1}]\!]$ in L is finite over $\varepsilon(A)$. This follows from the induction hypothesis. Here we used the following fact: if K_{d-1} is the fraction field of $k[\![X_1, \dots, X_{d-1}]\!]$, then the algebraic closure of K_{d-1} in L is finite over K. We leave the proof to the reader.[1]

Since $Y_d^{i-1}B/Y_d^iB \cong B/Y_dB$ and $Y_d^q = X_d$, we see that B/X_dB too is finite over A/X_dX. Note that A is X_d-complete. So as soon as we show $\bigcap_{n \geqslant 1} X_d^n B = 0$, Tag 031D will imply that B is finite over A. Indeed, if some $b \in B$ is contained in the intersection, then $b^q \in \bigcap_{n \geqslant 1} X_d^n A = 0$, hence $b = 0$. □

Proof of Proposition A.3.5. Step 1.* We simplify the assertion. Let $\mathfrak{p} \in \operatorname{Spec} A$ and $\ell/\kappa(\mathfrak{p})$ a finite extension. We are to show that the integral closure of A in ℓ is finite over A. There is a ring $A/\mathfrak{p} \subseteq B \subseteq \ell$ such that ℓ is the fraction field of B and B is finite over A. Then it suffices to show that the integral closure of B in its fraction field ℓ is finite over B. Since algebras of finite type over quasi-excellent rings are quasi-excellent again (Tag 07QU), we may assume that $A = B$ is a domain and it suffices to show that the integral closure of A in its fraction field is finite over A (or in other words, "A is N-1").

Step 2. We reduce to the case where A is local. Since A is J-2 by assumption, the regular locus $\operatorname{Reg}(\operatorname{Spec} A)$ is open, and non-empty because A is a domain. Thus, we find

[1] *Hint:* $K \subseteq K_{d-1}(\!(X_d)\!)$. Now show that $K_{d-1}^{1/q}$ and $K_{d-1}(\!(X_d)\!)$ are linearly disjoint.

a non-zero $f \in A$ such that A_f is regular, hence normal. Thus we may apply Tag 0333 to see that it suffices to show that all localizations $A_\mathfrak{m}$ at maximal ideals are N-1. Since $A_\mathfrak{m}$ is quasi-excellent again, we may indeed assume that A is local.

Step 3. We prove that A is N-1 under the assumption that $\widehat{A}$ is reduced (which will be proved in Step 4). Let $\mathfrak{q}_1, \dots, \mathfrak{q}_n \in \operatorname{Spec} \widehat{A}$ be the minimal primes. Since $\widehat{A}$ is reduced, hence R_0 and S_1, these are all the associated primes of $\widehat{A}$. Since $\widehat{A}$ is reduced, the canonical map

$$\widehat{A} \longrightarrow \prod_{i=1}^{n} \widehat{A}/\mathfrak{q}_i$$

is injective. Since each $\widehat{A}/\mathfrak{q}_i$ is a complete noetherian local domain, Lemma* A.3.8 shows that its integral closure in $\kappa(\mathfrak{q}_i)$ is finite over it. Thus, the integral closure of $\widehat{A}$ in $\prod_{i=1}^n \kappa(\mathfrak{q}_i)$ is finite over $\widehat{A}$. Now let B be the integral closure of A in its fraction field K. Using that $\widehat{A}$ is flat over A, we see that $B \otimes_A \widehat{A} \to K \otimes_A \widehat{A}$ is injective. The right-hand side is the localization of $\widehat{A}$ at the multiplicative set $A \setminus \{0\}$, which only consists of non-zero divisors because A is a domain and $\widehat{A}$ is flat over A. Thus, $K \otimes_A \widehat{A}$ is contained in the localization of $\widehat{A}$ at all non-zero divisors. This localization is precisely $\prod_{i=1}^n \kappa(\mathfrak{q}_i)$, because the $\mathfrak{q}_i$ comprise all the associated primes of $\widehat{A}$. All in all, this shows that

$$B \otimes_A \widehat{A} \longrightarrow \prod_{i=1}^{n} \kappa(\mathfrak{q}_i)$$

is injective. But $B \otimes_A \widehat{A}$ is clearly contained in the integral closure of $\widehat{A}$ in $\prod_{i=1}^n \kappa(\mathfrak{q}_i)$ as noted above, this shows that $B \otimes_A \widehat{A}$ is finite over $\widehat{A}$. Then B is finite over A (if $\{\sum_i b_{i,j} \otimes a_i\}$ are generators of $B \otimes_A \widehat{A}$, then $\{b_{i,j}\}$ are generators of B, using that $\widehat{A}$ is faithfully flat over A). We are done.

Step 4. We show that $\widehat{A}$ is indeed reduced. Using the younger sibling of Serre's normality criterion, what we have to show is that $\widehat{A}$ is R_0 and S_1. Note that this is fulfilled for A since A is a domain, hence reduced. Moreover, the fibres of $A \to \widehat{A}$ are regular (even geometrically regular), hence the fibres are R_n and S_n for every $n \geqslant 0$. Now both R_0 and S_1 are "ascending" conditions in the sense that if they hold for the source and the fibres, then they hold for the target as well as long as the morphism in question is flat (see Tag 0336, but in these particular cases everything is easily checkable by hand). This shows that $\widehat{A}$ is indeed reduced and this rather lengthy proof has finally come to an end. $\square$

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