∞-Categories in Topology

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§0. Introduction

At the QED Academy 2023 in Sonthofen, I gave a lecture course about *∞-categories in topology*. The goal of this course was ambitious and many omissions had to be made, but in the end we managed to have the Adams spectral sequence on the board. After the course I decided, for future use, to translate my handwritten German notes into English and into LaTeX. The result is this document. What began as a simple transscription quickly led to many additions, as it presented me with the tempting opportunity to provide many missing details. While I fell for this temptation many times, I hope that at their heart, these notes still display a faithful (albeit not essentially surjective) representation of my original course.

0.1. ∞-Categories in topology. — When I first learned ∞-category theory, I felt like I'd just been given a cheat code for homotopy theory. Before, I'd felt lost in all the technicalities and struggled to develop intuition. With ∞-categories, everything suddenly made sense and I finally started to see the elegance and the clarity that I'd been looking for so long.

So what are ∞-categories and what makes them so useful? Very roughly, an ∞-category not only contains 0-*morphisms* (objects) and 1-*morphism*, as an ordinary category, but also higher *n-morphisms* for all $n \geqslant 2$. It turns out that all the usual results and constructions from category theory can be carried over to ∞-categories—however, getting ∞-category theory off the ground is *much* more difficult than ordinary category theory: More than 30 years lie between the first definition of ∞-categories [BV73] and the first proof of the Yoneda lemma [L-HTT]! But it's worth the effort. Among many other applications, which we'll not attempt to survey here, ∞-categories provide an incredibly powerful framework to do homotopy theory in. Already the vague idea explained above has a topological flavour: *n*-morphisms in an ∞-category, which run between morphisms of lower order, are reminiscent of *n*-cells in a CW-complexes, whose boundary is made up of cells of lower dimension. And indeed, every CW-complex (and then by CW-approximation every topological space) *is* an example of an ∞-category! This is a key advantage of ∞-category theory over ordinary category theory:

> *Ordinary category theory can be used to do homotopy theory—but homotopy theory is (!) ∞-category theory.*

In particular, many formal ∞-categorical constructions, like presheaves or colimits, have a concrete topological meaning. It's surprising how many classical topological results can be reproved in a completely formal way just from abstract ∞-category theory! That number only increases if you bring algebra into the game. The ∞-category of *spectra* is still closely related to the homotopy theory of topological spaces, but at the same time it behaves a lot like the category of abelian groups. In particular, spectra admit a *tensor product* and so it makes sense to talk about rings and modules in spectra and to do algebra with them. As it turns out, algebra in the ∞-category of spectra is not a *tool*, like homology or homotopy groups in classical topology, but instead we'll be able to *reinterpret* many classical theorems as algebraic statements in the ∞-category of spectra.

For someone like me, how came from an algebra background originally, it's amazing to be able to do topology with just the tools I feel confident with: category theory and algebra. But ∞-category theory also does wonderful things in algebra. For example, the theory of *derived categories*—another notoriously technical subject—becomes *so much clearer* once you know spectra and the associated notion of a *stable ∞-category*. In general, the future of algebra and algebraic geometry is derived, and all things derived become much clearer if you approach them using ∞-category theory.

All of this is to say: You should learn ∞-category theory! It will be painful, but once the pain eases, you'll never look at mathematics the same way again.

0.2. Aim and scope of these notes. — The goal of these notes is to introduce ∞-categories and to explain many of their applications to topology. As prerequisites, you should feel comfortable with ordinary category theory; in addition, it would be beneficial to have a solid background in commutative algebra and topology (at least, you should have heard of simplicial sets, homology, and homotopy groups). None of these prerequisites are strictly necessary—we'll recall the necessary ordinary category theory in §1 and we'll reintroduce many classical topological constructions in a way that's convenient for us—but a solid background would certainly help you not to get overwhelmed by the material. Also, the more topology you already know, the more you'll be able to appreciate how the ∞-categorical viewpoint simplifies and clarifies classical topology.

These notes roughly consist of four parts: In §§2–5 we'll introduce ∞-categories as well as the technical ingredients that go into Lurie's proof of Yoneda's lemma. Unfortunately, this part contains several minor black boxes and a major one: I won't be able to prove Lurie's *straightening/unstraightening* equivalence. The second part is §6, in which we'll redevelop classical category theory in the setting of ∞-categories. In the third part, spanning §§7–8, we'll introduce spectra and their tensor product. Finally, the last part is §9, in which we'll apply our theory to topology (although many more applications are scattered throughout the text up to this point). One highlight will be the construction of the Adams spectral sequence.

0.3. Model independence and notation. — The model for ∞-categories we use in these notes will be *quasi-categories*. But there are many other approaches to ∞-categories, like *topologically* or *Kan-enriched categories*, *complete Segal spaces*, *1-complicial sets*, . . . Of course, all these approaches should be equivalent, but it's usually a non-trivial task to tranfer a result proven in one model into another model. A general theory of model independence that allows for such non-trivial transfers has been developed by Emily Riehl and Dominic Verity [RV22]. Another very interesting approach is being developed by Cisinksi–Cnossen–Nguyen–Walde [CCNT]: Instead of defining what an ∞-category is supposed to be, they axiomatise what constructions one is allowed to do with them, and then develop ∞-category theory abstractly from these axioms. In a sense, this is a much more consequent implementation of our steps (*b*) and (*c*) below.

Our approach towards a model-independent theory will, or at least we would, in an ideal version of these notes, proceed in the following steps:

(*a*)   First, we'll set up the framework of quasi-categories, by any means necessary.

(*b*)   After that, we'll prove (or black box) a few key statements in the model of quasi-categories. The statements themselves are model-independent, even though their proofs are not.

(*c*)   Finally, all further proofs will be done in a model-independent fashion.

If you prefer a different model of quasi-categories, you'll probably know how to do steps (*a*) and (*b*) in your model, and then, at least in an ideal world, everything from step (*c*) will work in your model too.

In reality, these notes don't quite live up to that ideal, but I dare say we come somewhat close. In §§2–5, we'll sketch how to get the theory of quasi-categories off the ground. This corresponds to steps (*a*) and (*b*) above, with most of (*b*) happening in §4 and §5. Everything from §6 onward mostly falls within step (*c*). I write "mostly" because, unfortunately, there are

still a few non-model-independent arguments scattered throughout the text, the worst offender probably being our treatment of cardinality bounds and filteredness in §6.8. I could, of course, tautologically claim that every non-model-independent proof still belongs to step (*b*), but that would be a poor excuse for my inability to come up with better arguments.

The transition to (mostly) model-independent arguments will be reflected in a change of terminology: Throughout §§2–5, we'll use the term *quasi-category*, we'll write $F(\mathcal{C}, \mathcal{D})$ for the mapping object in simplicial sets and we'll always write $N(\mathcal{E})$ for the quasi-category obtained as the nerve of an ordinary category $\mathcal{E}$. Starting from §6, we'll simply say ∞-*category*, write $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$, and consider every ordinary category $\mathcal{E}$ implicitly as an ∞-category, suppressing N everywhere. Only when we're using non-model-independent arguments, we'll switch back to the old terminology, to emphasise that what we're doing is morally questionable.

Also, since I'm doing my PhD in Bonn, I'm legally obliged to use the term *anima* for what most normal people would call *space* or ∞-*groupoid* or (in non-model-independent language) *Kan complex*.

§1. Category theory

We assume you are familiar with categories, functors, natural transformations and the Yoneda lemma. In fact, you will probably be familiar with most of the stuff in this section, so we'll leave out many proofs (but give them later in the ∞-categorical context).

§1.1. Adjunctions

1.1. Definition. — Let $L\colon \mathcal{C} \to \mathcal{D}$ be a functor.

(a) Let $y \in \mathcal{D}$. An object $x \in \mathcal{C}$ is a *right adjoint object to y under L* if there exists an equivalence
$$\operatorname{Hom}_{\mathcal{C}}(-, x) \simeq \operatorname{Hom}_{\mathcal{D}}\big(L(-), y\big)$$
in the functor category $\operatorname{Fun}(\mathcal{C}^{\mathrm{op}}, \operatorname{Set})$.

(b) A functor $R\colon \mathcal{D} \to \mathcal{C}$ is a *right adjoint of L* if there exists an equivalence
$$\operatorname{Hom}_{\mathcal{C}}\big(-, R(-)\big) \simeq \operatorname{Hom}_{\mathcal{D}}\big(L(-), -\big)$$
in the functor category $\operatorname{Fun}(\mathcal{C}^{\mathrm{op}} \times \mathcal{D}, \operatorname{Set})$. In this case we write $L \dashv R$.

There are obvious dual notions of *left adjoint objects/functors.*

1.2. Lemma ("Adjoints can be constructed pointwise"). — *A functor $L\colon \mathcal{C} \to \mathcal{D}$ has a right adjoint if and only if every $y \in \mathcal{D}$ has a right adjoint object $x \in \mathcal{C}$.*

Proof. One implication is trivial: If $R\colon \mathcal{D} \to \mathcal{C}$ is a right adjoint of L, then $R(y)$ is a right adjoint object of y for every $y \in \mathcal{D}$. The other implication is left as an exercise. We'll prove an ∞-categorical variant in Lemma 6.2. $\qquad\square$

1.3. Unit and counit. — Let $L\colon \mathcal{C} \rightleftarrows \mathcal{D} :R$ be an adjunction. For every $x \in \mathcal{C}$, the identity $\operatorname{id}_{L(x)}\colon L(x) \to L(x)$ is adjoint to a morphism $u_x\colon x \to RL(x)$. One can show that these morphisms assemble into a natural transformation $u\colon \operatorname{id}_{\mathcal{C}} \Rightarrow RL$, called the *unit* of the adjunction. Dually, there is a *counit* $c\colon LR \Rightarrow \operatorname{id}_{\mathcal{D}}$.

1.4. Lemma (Triangle identities). — *Let $L\colon \mathcal{C} \rightleftarrows \mathcal{D} :R$ be an adjunction. Then the diagrams*

$$
L \xrightarrow{\;Lu\;} LRL \qquad R \xrightarrow{\;uR\;} RLR
$$

$$
\begin{array}{ccc}
L \xrightarrow{\;Lu\;} LRL & & R \xrightarrow{\;uR\;} RLR \\
{\scriptstyle\operatorname{id}_L}\searrow \;\Vert{\scriptstyle cL} & and & {\scriptstyle\operatorname{id}_R}\searrow \;\Vert{\scriptstyle Rc} \\
L & & R
\end{array}
$$

commute. Conversely, if L, R are functors and $u : \operatorname{id}_{\mathcal{C}} \Rightarrow RL$, $c\colon LR \Rightarrow \operatorname{id}_{\mathcal{D}}$ are natural transformations such that the diagrams above commute, then L and R determine an adjunction.

Proof. Exercise. We'll prove an ∞-categorical variant in Lemma 6.5. $\qquad\square$

1.5. Corollary. — *Let $L\colon \mathcal{C} \rightleftarrows \mathcal{D} :R$ be an adjunction and let $\mathcal{I}$ be another category. Then the pre- and postcomposition functors determine adjunctions*

$$L \circ -\colon \operatorname{Fun}(\mathcal{I}, \mathcal{C}) \rightleftarrows \operatorname{Fun}(\mathcal{I}, \mathcal{D}) :R \circ -\,,$$
$$- \circ R\colon \operatorname{Fun}(\mathcal{C}, \mathcal{I}) \rightleftarrows \operatorname{Fun}(\mathcal{D}, \mathcal{I}) :- \circ L\,.$$

Proof. By Lemma 1.4, we only need to construct unit and a counit transformations satisfying the triangle identities. These are immediately inherited from the adjunction $L \dashv R$. $\qquad\square$

§1.2. Limits and colimits

1.6. Definition. — Let $F\colon \mathcal{I} \to \mathcal{C}$ be a functor. A *limit of F*, denoted $\lim F$ (or sometimes $\lim_{i \in \mathcal{I}} F(i)$), is a right adjoint object of F under the functor $\mathrm{const}\colon \mathcal{C} \to \mathrm{Fun}(\mathcal{I}, \mathcal{C})$ that sends $i \in \mathcal{I}$ to the constant functor with value i. Dually, a *colimit of F*, denoted $\mathrm{colim}\, F$ (or sometimes $\mathrm{colim}_{i \in \mathcal{I}} F(i)$), is a left adjoint object of F under const.

Concretely, Definition 1.6 means that we have the following natural bijections for all $x, y \in \mathcal{C}$:

$$\mathrm{Hom}_{\mathcal{C}}(x, \lim F) \cong \mathrm{Hom}_{\mathrm{Fun}(\mathcal{I}, \mathcal{C})}(\mathrm{const}\, x, F)\,,$$
$$\mathrm{Hom}_{\mathcal{C}}(\mathrm{colim}\, F, y) \cong \mathrm{Hom}_{\mathrm{Fun}(\mathcal{I}, \mathcal{C})}(F, \mathrm{const}\, y)\,.$$

1.7. Lemma. — *Left adjoint functors preserve colimits and right adjoint functors preserve limits.*

Proof. Let $L\colon \mathcal{C} \rightleftarrows \mathcal{D} \colon R$ be an adjunction and let $\mathcal{I}$ be another category. By Corollary 1.5, the postcomposition functors $L_* := L \circ -$ and $R_* := R \circ -$ determine an adjunction

$$L_*\colon \mathrm{Fun}(\mathcal{I}, \mathcal{C}) \rightleftarrows \mathrm{Fun}(\mathcal{I}, \mathcal{D}) \colon R_*$$

Now let $F\colon \mathcal{I} \to \mathcal{C}$ be a functor admitting a colimit $\mathrm{colim}\, F$. Since left adjoint functors clearly preserve left adjoint objects, we see that $L(\mathrm{colim}\, F)$ is a left adjoint object of F under $\mathrm{const}\, R(-)$. But $\mathrm{const}\, R(-) \simeq R_* \mathrm{const}\colon \mathcal{D} \to \mathrm{Fun}(\mathcal{I}, \mathcal{C})$. A left adjoint object of F under $R_* \mathrm{const}$ is also a left adjoint object of $L_* F$ under $\mathrm{const}\colon \mathcal{D} \to \mathrm{Fun}(\mathcal{I}, \mathcal{D})$ by the adjunction above. In summary, this proves that $L(\mathrm{colim}\, F)$ is a left adjoint object of $L_* F$ under const, which is precisely what we want. The case of limits is analogous. $\square$

1.8. Lemma ("Colimits in functor categories are computed pointwise."). — *Let $\mathcal{C}$, $\mathcal{D}$, and $\mathcal{I}$ be categories such that $\mathcal{D}$ has all $\mathcal{I}$-shaped colimits; that is, all functors $\mathcal{I} \to \mathcal{D}$ admit colimits. Then $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ has again all $\mathcal{I}$-shaped colimits and the evaluation functor*

$$\mathrm{ev}_x\colon \mathrm{Fun}(\mathcal{C}, \mathcal{D}) \longrightarrow \mathrm{Fun}(\{x\}, \mathcal{D}) \simeq \mathcal{D}$$

preserves $\mathcal{I}$-shaped colimits for all $x \in \mathcal{C}$. A dual assertion holds for limits.

Proof. By Lemma 1.2, the condition that $\mathcal{D}$ has all $\mathcal{I}$-shaped colimits implies that the functor $\mathrm{const}\colon \mathcal{D} \to \mathrm{Fun}(\mathcal{I}, \mathcal{D})$ has a left adjoint $\mathrm{colim}\colon \mathrm{Fun}(\mathcal{I}, \mathcal{D}) \to \mathcal{D}$. Under the "currying" equivalence

$$\mathrm{Fun}\big(\mathcal{I}, \mathrm{Fun}(\mathcal{C}, \mathcal{D})\big) \simeq \mathrm{Fun}\big(\mathcal{C}, \mathrm{Fun}(\mathcal{I}, \mathcal{D})\big)\,,$$

the functor $\mathrm{const}\colon \mathrm{Fun}(\mathcal{C}, \mathcal{D}) \to \mathrm{Fun}(\mathcal{I}, \mathrm{Fun}(\mathcal{C}, \mathcal{D}))$ corresponds to the postcomposition functor $\mathrm{const}_*\colon \mathrm{Fun}(\mathcal{C}, \mathcal{D}) \to \mathrm{Fun}(\mathcal{C}, \mathrm{Fun}(\mathcal{I}, \mathcal{D}))$. By Corollary 1.5, we have an adjunction

$$\mathrm{colim}_*\colon \mathrm{Fun}\big(\mathcal{C}, \mathrm{Fun}(\mathcal{I}, \mathcal{D})\big) \rightleftarrows \mathrm{Fun}\big(\mathcal{C}, \mathcal{D}\big) \colon \mathrm{const}_*\,.$$

Hence $\mathrm{const}\colon \mathrm{Fun}(\mathcal{C}, \mathcal{D}) \to \mathrm{Fun}(\mathcal{I}, \mathrm{Fun}(\mathcal{C}, \mathcal{D}))$ has a left adjoint too, which proves that $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ has $\mathcal{I}$-shaped colimits. The additional assertion that ev_x preserves $\mathcal{I}$-shaped colimits follows by unravelling how colim is constructed from colim_*. $\square$

§1.3. Kan extensions

1.9. Setup — Suppose we are given functors f and F as follows:

$$
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{\ F\ } & \mathcal{D} \\
{\scriptstyle f}\big\downarrow & \nearrow & \\
\mathcal{C}' & &
\end{array}
$$

Often one would like to extend F to a functor $F'\colon \mathcal{C}' \to \mathcal{D}$. Of course, in general there's no such functor making the diagram above commute, and if there is, it might not be unique.

1.10. Definition. — In the situation of Setup 1.9, a *left Kan extension of F along f*, denoted $\mathrm{Lan}_f F\colon \mathcal{C}' \to \mathcal{D}$, is a left adjoint object of F under $f^* = - \circ f\colon \mathrm{Fun}(\mathcal{C}',\mathcal{D}) \to \mathrm{Fun}(\mathcal{C},\mathcal{D})$. Dually, a *right Kan extension of F along f*, denoted $\mathrm{Ran}_f F\colon \mathcal{C}' \to \mathcal{D}$, is a right adjoint object of F under f^*.

1.11. Warning. — In general, even if the respective Kan extensions exist, the diagrams

$$
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{\ F\ } & \mathcal{D} \\
{\scriptstyle f}\big\downarrow & \nearrow & \\
\mathcal{C}' & {\scriptstyle \mathrm{Lan}_f F} &
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{\ F\ } & \mathcal{D} \\
{\scriptstyle f}\big\downarrow & \nearrow & \\
\mathcal{C}' & {\scriptstyle \mathrm{Ran}_f F} &
\end{array}
$$

only commute up to the indicated natural transformations. Indeed, the defining property from Definition 1.10 says that there are natural bijections

$$
\mathrm{Hom}_{\mathrm{Fun}(\mathcal{C}',\mathcal{D})}(\mathrm{Lan}_f F, F') \cong \mathrm{Hom}_{\mathrm{Fun}(\mathcal{C},\mathcal{D})}(F, F' \circ f)\,,
$$
$$
\mathrm{Hom}_{\mathrm{Fun}(\mathcal{C}',\mathcal{D})}(F', \mathrm{Ran}_f F) \cong \mathrm{Hom}_{\mathrm{Fun}(\mathcal{C},\mathcal{D})}(F' \circ f, F)\,,
$$

for all $F'\colon \mathcal{C}' \to \mathcal{D}$. Plugging in $F' = \mathrm{Lan}_f F$, then taking the image of $\mathrm{id}_{\mathrm{Lan}_f F}$ produces the indicated natural transformation; and likewise for $\mathrm{Ran}_f F$ (so in other words, we're considering the unit of the adjunction $\mathrm{Lan}_f \dashv f^*$ and the counit of $f^* \dashv \mathrm{Ran}_f$, respectively).

1.12. Example. — Let $\mathcal{C}' = *$, then $\mathrm{Fun}(\mathcal{C}',\mathcal{D}) \simeq \mathcal{D}$ and we see immediately that $\mathrm{Lan}_f F$ corresponds to $\mathrm{colim}\, F$ (if either exists). Likewise, $\mathrm{Ran}_f F$ corresponds to $\lim F$.

Next we set out to answer the question when Kan extensions exist. To this end, we need to introduce two constructions that will feature prominently throughout the text.

1.13. Construction. — Let $\mathcal{C}$ be a category and let $[1] := \{\bullet \to \bullet\}$ be the category with two objects and one non-identity morphism. The *arrow category of $\mathcal{C}$* is the category

$$
\mathrm{Ar}(\mathcal{C}) := \mathrm{Fun}\big([1],\mathcal{C}\big)\,.
$$

Concretely, objects in $\mathrm{Ar}(\mathcal{C})$ are morphisms $\alpha\colon x \to y$ in $\mathcal{C}$, and morphisms in $\mathrm{Ar}(\mathcal{C})$ are commutative diagrams

$$
\begin{array}{ccc}
x & \longrightarrow & x' \\
{\scriptstyle \alpha}\big\downarrow & & \big\downarrow{\scriptstyle \alpha'} \\
y & \longrightarrow & y'
\end{array}
$$

in $\mathcal{C}$. There are functors $s, t\colon \mathrm{Ar}(\mathcal{C}) \to \mathcal{C}$ ("source" and "target" projection) sending an arrow $(\alpha\colon x \to y)$ to x and y, respectively.

1.14. Construction. — Let $f\colon \mathcal{C} \to \mathcal{C}'$ be a functor and $x' \in \mathcal{C}'$. The *slice category of $\mathcal{C}$ over x'* is the pullback[(1.1)]

$$
\begin{array}{ccc}
\mathcal{C}_{/x'} & \longrightarrow & \mathrm{Ar}(\mathcal{C}') \\
\downarrow & \lrcorner & \downarrow {\scriptstyle (s,t)} \\
\mathcal{C} \times \{x'\} & \xrightarrow{f \times x'} & \mathcal{C}' \times \mathcal{C}'
\end{array}
$$

Concretely, objects in the slice category $\mathcal{C}_{/x'}$ are pairs $(x, f(x) \to x')$, where $x \in \mathcal{C}$ and $f(x) \to x'$ is a morphism in $\mathcal{C}'$. Morphisms in $\mathcal{C}_{/x'}$ are given by morphisms $\alpha\colon x \to y$ such that

$$
\begin{array}{ccc}
f(x) & \xrightarrow{f(\alpha)} & f(y) \\
\downarrow & /\!/\!/ & \downarrow \\
x' & =\!=\!=\!= & x'
\end{array}
$$

commutes. Dually, there's also $\mathcal{C}_{x'/}$, the *slice category of $\mathcal{C}$ under x'*.

1.15. Lemma (Kan extension formula). — *In the situation of Setup 1.9, assume that for all $x' \in \mathcal{C}'$ the following colimits exist in $\mathcal{D}$:*

$$
\operatorname*{colim}_{(x, f(x) \to x') \in \mathcal{C}_{/x'}} F(x) := \operatorname{colim}\left(\mathcal{C}_{/x'} \longrightarrow \mathcal{C} \xrightarrow{F} \mathcal{D} \right).
$$

Then $\mathrm{Lan}_f F$ exists and $\mathrm{Lan}_f F(x')$ is given by that colimit.

Proof. Exercise. We'll prove an ∞-categorical variant in Lemma 6.27. $\qquad\square$

1.16. Corollary. — *In the situation of Setup 1.9, assume that $f\colon \mathcal{C} \to \mathcal{C}'$ is fully faithful and that the colimits from Lemma 1.15 exist. Then the natural transformation $u_F\colon F \Rightarrow \mathrm{Lan}_f F \circ f$ from Warning 1.11 is an equivalence.*

Proof. If $x' = f(y)$ for some $y \in \mathcal{C}$, then f being fully faithful implies that the slice category $\mathcal{C}_{/f(y)}$ is equivalent to $\mathcal{C}_{/y}$ (that is, the slice category formed with respect to $\mathrm{id}_{\mathcal{C}}\colon \mathcal{C} \to \mathcal{C}$). The latter has a terminal object, namely $\{\mathrm{id}_y\}$. Hence

$$
\mathrm{Lan}_f\big(F f(y)\big) \cong \operatorname*{colim}_{(x, f(x) \to y) \in \mathcal{C}_{/f(y)}} F(x) \cong \operatorname*{colim}_{(x \to y) \in \mathcal{C}_{/y}} F(x) \cong F(y). \qquad\square
$$

To finish this subsection, we prove a result about the *category $\mathrm{PSh}(\mathcal{C}) := \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Set})$ of presheaves on $\mathcal{C}$*. This will seem rather technical at first, but, together with its ∞-categorical version, it will be invaluable throughout the text.

1.17. Theorem ("$\mathrm{PSh}(\mathcal{C})$ arises by freely adding colimits to $\mathcal{C}$."). — *Let $\mathcal{C}$ and $\mathcal{D}$ be categories, where $\mathcal{D}$ has all colimits. Let $\sharp_{\mathcal{C}}\colon \mathcal{C} \to \mathrm{PSh}(\mathcal{C})$ denote the Yoneda embedding, sending $x \in \mathcal{C}$ to $\mathrm{Hom}_{\mathcal{C}}(-, x)\colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$. Then restriction along $\sharp_{\mathcal{C}}$ induces an equivalence*

$$
\sharp_{\mathcal{C}}^*\colon \mathrm{Fun}^{\mathrm{colim}}\big(\mathrm{PSh}(\mathcal{C}), \mathcal{D}\big) \xrightarrow{\;\simeq\;} \mathrm{Fun}(\mathcal{C}, \mathcal{D}).
$$

Here $\mathrm{Fun}^{\mathrm{colim}}(\mathrm{PSh}(\mathcal{C}), \mathcal{D}) \subseteq \mathrm{Fun}(\mathrm{PSh}(\mathcal{C}), \mathcal{D})$ is the full subcategory spanned by the colimit-preserving functors. Furthermore, every colimit-preserving functor $\mathrm{PSh}(\mathcal{C}) \to \mathcal{D}$ admits a right adjoint.

[(1.1)]The pullback ist taken in the category of *small categories*, that is, those categories whose class of objects is a set. But the explicit description works with the weaker assumption that $\mathcal{C}$ and $\mathcal{C}'$ are *locally small*, meaning that $\mathrm{Hom}_{\mathcal{C}}(x, y)$ and $\mathrm{Hom}_{\mathcal{C}'}(x', y')$ are sets for all $x, y \in \mathcal{C}$, $x', y' \in \mathcal{C}'$.

To prove Theorem 1.17, we send two lemmas in advance.

1.18. Lemma ("Every presheaf is a colimit of representables."). — *Let $\mathcal{C}$ be a category. For every $E \in \mathrm{PSh}(\mathcal{C})$, the natural morphism*

$$\operatorname*{colim}_{(y, \mathrm{Hom}_{\mathcal{C}}(-,y) \to E) \in \mathcal{C}_{/E}} \mathrm{Hom}_{\mathcal{C}}(-, y) \xrightarrow{\cong} E$$

is an isomorphism.

Proof. Exercise (use Yoneda's lemma). We'll prove an ∞-categorical version in Lemma 6.31. $\square$

1.19. Lemma. — *Let $\mathcal{C}$ and $\mathcal{D}$ be categories, where $\mathcal{D}$ has all colimits. For every $F \colon \mathcal{C} \to \mathcal{D}$, the left Kan extension $\mathrm{Lan}_{\sharp_{\mathcal{C}}} F \colon \mathrm{PSh}(\mathcal{C}) \to \mathcal{D}$ (which exists due to Lemma 1.15) admits a right adjoint. The right adjoint sends $y \in \mathcal{D}$ to $\mathrm{Hom}_{\mathcal{D}}(F(-), y) \colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$.*

Proof. Exercise. We'll prove an ∞-categorical version in Lemma 6.32. $\square$

Furthermore, we need the following general lemma (which will occasionally be useful in the future too).

1.20. Lemma. — *Let $\mathcal{C}$ and $\mathcal{D}$ be categories and let $L \colon \mathcal{C} \rightleftarrows \mathcal{D} \colon R$ be an adjunction.*

(a) *The left adjoint L is fully faithful if and only if the unit transformation $u \colon \mathrm{id}_{\mathcal{C}} \Rightarrow RL$ is an equivalence.*

(b) *Suppose L is fully faithful and R is conservative (that is, if $\alpha \colon x \to y$ is a morphism in $\mathcal{D}$ such that $R(\alpha)$ is an isomorphism, then α is an isomorphism too). Then L and R are inverse equivalences of categories.*

Proof. To prove (a), first observe that for all elements $x, y \in \mathcal{C}$, the postcomposition map $(u_y)_* \colon \mathrm{Hom}_{\mathcal{C}}(x, y) \to \mathrm{Hom}_{\mathcal{C}}(x, RL(y))$ is given by

$$(u_y)_* \colon \mathrm{Hom}_{\mathcal{C}}(x, y) \xrightarrow{L} \mathrm{Hom}_{\mathcal{D}}\big(L(x), L(y)\big) \xrightarrow{\cong} \mathrm{Hom}_{\mathcal{C}}\big(x, RL(y)\big),$$

where the second map is the adjunction bijection. By Yoneda's lemma, $u_y \colon y \to RL(y)$ is an equivalence if and only if $(u_y)_* \colon \mathrm{Hom}_{\mathcal{C}}(x, y) \to \mathrm{Hom}_{\mathcal{C}}(x, RL(y))$ is a bijection for all x. By the above, this happens if and only if $L \colon \mathrm{Hom}_{\mathcal{C}}(x, y) \to \mathrm{Hom}_{\mathcal{D}}(L(x), L(y))$ is a bijection for all $x \in \mathcal{C}$. This proves (a).

For (b), the second of the triangle identities from Lemma 1.4 shows that $Rc \colon RLR \Rightarrow R$ is a natural equivalence. Since R is conservative, $c \colon LR \Rightarrow \mathrm{id}_{\mathcal{D}}$ must be an equivalence too. Since $u \colon \mathrm{id}_{\mathcal{C}} \Rightarrow RL$ is an equivalence by assumption, we are done. $\square$

Proof of Theorem 1.17. By Lemma 1.19 and Lemma 1.7, the adjunction $\mathrm{Lan}_{\sharp_{\mathcal{C}}} \dashv \sharp_{\mathcal{C}}^*$ restricts to an adjunction

$$\mathrm{Lan}_{\sharp_{\mathcal{C}}} \colon \mathrm{Fun}(\mathcal{C}, \mathcal{D}) \rightleftarrows \mathrm{Fun}^{\mathrm{colim}}\big(\mathrm{PSh}(\mathcal{C}), \mathcal{D}\big) \colon \sharp_{\mathcal{C}}^*.$$

Since $\sharp_{\mathcal{C}}$ is fully faithful, Corollary 1.16 implies that the unit $u \colon \mathrm{id}_{\mathrm{Fun}(\mathcal{C}, \mathcal{D})} \Rightarrow \sharp_{\mathcal{C}}^* \circ \mathrm{Lan}_{\sharp_{\mathcal{C}}}$ is an equivalence. Furthermore, it's clear that $\sharp_{\mathcal{C}}^*$ is conservative: If a natural transformation $\eta \colon F \Rightarrow G$ between colimit-preserving functors $F, G \colon \mathrm{PSh}(\mathcal{C}) \to \mathcal{D}$ is an equivalence when restricted to representable presheaves, then it is an equivalence everywhere, because every presheaf can be written as a colimit of representables (Lemma 1.18). Then Lemma 1.20(b) finishes the proof. $\square$

§2. The simplicial model

In this section, we'll introduce our model for ∞-categories and the main object of interest in §2, §4, and §5: quasi-categories! We'll see some first signs that quasi-categories behave a lot like ordinary categories and we'll define the quasi-category of quasi-categories Cat_∞.

§2.1. Recollections on simplicial sets

2.1. Definition. — (*a*) The *simplex category* Δ is the category whose objects are finite non-empty totally ordered sets $[n] = \{0 < 1 < \cdots < n\}$ for all $n \geqslant 0$ and whose morphisms are order-preserving maps, that is, maps $\alpha\colon [m] \to [n]$ such that $\alpha(0) \leqslant \alpha(1) \leqslant \cdots \leqslant \alpha(m)$.

(*b*) A *simplicial set* is a presheaf on Δ, that is, a functor $X\colon \Delta^{\mathrm{op}} \to \mathrm{Set}$. The *category of simplicial sets* is the category $\mathrm{sSet} := \mathrm{PSh}(\Delta) \simeq \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{Set})$ of presheaves on Δ.

2.2. Construction. — For all $i = 0, \dots, n$ and all $j = 0, \dots, n-1$ let $d_i\colon [n-1] \to [n]$ be the unique injective morphism in Δ that doesn't hit i and let $s_j\colon [n] \to [n-1]$ be the unique surjective morphism in Δ that hits j twice. It's straightforward to see that every morphism $\alpha\colon [m] \to [n]$ in Δ can be written as a composition of some s_j and some d_i. Therefore, a simplicial set can be described by the following data:

(*a*) Sets $X_n := X([n])$ for all $n \geqslant 0$.

(*b*) *Face maps* $d_i^*\colon X_n \to X_{n-1}$ for all $i = 0, \dots, n$.

(*c*) *Degeneracy maps* $s_j^*\colon X_{n-1} \to X_n$ for all $j = 0, \dots, n-1$.

The face and degeneracy maps satisfy $d_j^* \circ d_i^* = d_{i-1}^* \circ d_j^*$ and $s_j^* \circ s_i^* = s_{i-1}^* \circ s_j^*$ for all $i > j$ as well as

$$d_j^* \circ s_i^* = \begin{cases} s_i^* \circ d_{j-1}^* & \text{if } i < j-1 \\ \mathrm{id}_{X_{n-1}} & \text{if } i = j-1 \text{ or } i = j \\ s_{i-1}^* \circ d_j^* & \text{if } i > j \end{cases}.$$

It's customary to call elements of X_n *n-simplices of X*. An n-simplex is called *degenerate* if it is in the image of $s_j\colon X_{n-1} \to X_n$ for some j.

Let's give some first examples of simplicial sets and explain some basic constructions.

2.3. Boundaries and horns. — For all $n \geqslant 0$, the functor $\Delta^n := \mathrm{Hom}_\Delta(-, [n])\colon \Delta^{\mathrm{op}} \to \mathrm{Set}$ is a simplicial set, called the *n-simplex*. Yoneda's lemma implies that $\mathrm{Hom}_{\mathrm{sSet}}(\Delta^n, X) \cong X_n$ for all simplicial sets X. The maps $d_i\colon [n-1] \to [n]$ and $s_j\colon [n] \to [n-1]$ from Construction 2.2 induce maps $d_i\colon \Delta^{n-1} \to \Delta^n$ and $s_j\colon \Delta^n \to \Delta^{n-1}$ in sSet.[(2.1)] Using these maps, we can define the following sub-simplicial sets of Δ^n:

$$\partial \Delta^n := \bigcup_{i=0}^{n} \mathrm{im}\big(d_i\colon \Delta^{n-1} \to \Delta^n\big) \subseteq \Delta^n, \text{ the } \textit{boundary of } \Delta^n,$$

$$\Lambda_j^n := \bigcup_{\substack{i=0 \\ i \neq j}}^{n} \mathrm{im}\big(d_i\colon \Delta^{n-1} \to \Delta^n\big) \subseteq \Delta^n, \text{ the } \textit{j-horn in } \Delta^n.$$

[(2.1)]This may be confusing at first but the maps $d_i\colon \Delta^{n-1} \to \Delta^n$ and $s_j\colon \Delta^n \to \Delta^{n-1}$ really run in the indicated directions. The point is that while $\mathrm{Hom}_\Delta(-, [n])\colon \Delta^{\mathrm{op}} \to \mathrm{Set}$ is contravariant, the functor that assigns $[n] \mapsto \mathrm{Hom}_\Delta(-, [n])$, that is, the Yoneda embedding $よ_\Delta\colon \Delta \to \mathrm{PSh}(\Delta) \simeq \mathrm{sSet}$, is covariant.

Here the unions are taken degree-wise.[(2.2)] It's customary to call horns Λ_j^n *inner horns* if $0 < j < n$ and *outer horns* if $j = 0$ or $j = n$. Concretely, for all $m \geqslant 0$, the m-simplices of the boundary $\partial\Delta^n$ and the j-horn Λ_j^n are given by the following formulae:

$$(\partial\Delta^n)_m = \left\{\alpha\colon [m] \to [n] \mid [n] \neq \operatorname{im}(\alpha)\right\},$$
$$(\Lambda_j^n)_m = \left\{\alpha\colon [m] \to [n] \mid [n] \neq \operatorname{im}(\alpha) \cup \{j\}\right\}.$$

Here are some pictures in the case $n = 2$ (in the bottom line, the dotted lines mark the faces that are missing in the respective horns):

$$\Delta^2 = \begin{array}{c} 2 \\ \diagup\ {}_{/\!/\!/}\ \diagdown \\ 0 \longrightarrow 1 \end{array} \quad,\quad \partial\Delta^2 = \begin{array}{c} 2 \\ \diagup\ \ \diagdown \\ 0 \longrightarrow 1 \end{array}$$

$$\Lambda_0^2 = \begin{array}{c} 2 \\ \diagup\ \ \diagdown \\ 0 \longrightarrow 1 \end{array} \quad,\quad \Lambda_1^2 = \begin{array}{c} 2 \\ \diagup\ \ \diagdown \\ 0 \longrightarrow 1 \end{array} \quad,\quad \Lambda_2^2 = \begin{array}{c} 2 \\ \diagup\ \ \diagdown \\ 0 \cdots\!\cdots 1 \end{array}$$

2.4. Geometric Realisation. — These pictures suggest a geometric way to think about simplices, boundaries of simplices, and horns. In fact, we can associate to every simplicial set X a topological space (in fact, a CW-complex) $|X|$, called the *geometric realisation of X*. To describe this construction, we first define $|\Delta^n|$ to be the *topological n-simplex*, that is, the space $\{(t_0, \ldots, t_n) \in \mathbb{R}^n \mid 0 \leqslant t_i \leqslant 1, \sum_{i=1}^n t_i = 1\} \subseteq \mathbb{R}^n$. For all $i = 0, \ldots, n$ and all $j = 0, \ldots, n-1$ we define maps $|d_i|\colon |\Delta^{n-1}| \to |\Delta^n|$ and $|s_j|\colon |\Delta^n| \to |\Delta^{n-1}|$ via

$$|d_i|(t_0, \ldots, t_{n-1}) := (t_0, \ldots, t_{i-1}, 0, t_i, \ldots, t_n)$$
$$|s_j|(t_0, \ldots, t_n) := (t_0, \ldots, t_{i-1}, t_i + t_{i+1}, t_{i+2}, \ldots, t_n)$$

For general simplicial sets X, we can now construct $|X|$ by taking a topological n-simplex $|\Delta^n|$ for every $\sigma \in X_n$ and gluing them together according to the face and degeneracy maps above. More precisely, we take
$$|X| := \operatorname*{colim}_{(n, \Delta^n \to X)} |\Delta^n| \in \operatorname{Top}.$$

This agrees with the Kan extension formula from Lemma 1.15! So $|\cdot|\colon \mathrm{sSet} \to \mathrm{Top}$ must be the unique colimit-preserving extension, guaranteed by Theorem 1.17, of the functor $\Delta \to \mathrm{Top}$ that sends $[n] \mapsto |\Delta^n|$.

Furthermore, Theorem 1.17 guarantees that $|\cdot|\colon \mathrm{sSet} \to \mathrm{Top}$ admits a right adjoint, which we denote $\mathrm{Sing}\colon \mathrm{Top} \to \mathrm{sSet}$. By Lemma 1.19, it is given by $(\mathrm{Sing}\,Y)_n \cong \operatorname{Hom}_{\mathrm{Top}}(|\Delta^n|, Y)$. So $\mathrm{Sing}\,Y$ is indeed the construction you know from the definition of singular homology.

2.5. Nerve and homotopy category. — Every partially ordered set defines a category. In particular, we can regard the totally ordered sets $[n]$ as categories. Accordingly, we obtain a functor $U\colon \Delta \to \mathrm{Cat}$ into the category of small categories; U simply sends $[n] \mapsto [n]$. For every small category $\mathcal{C}$, this allows us to define a simplicial set $\mathrm{N}(\mathcal{C})$, called the *nerve of $\mathcal{C}$*, as the composition

$$\mathrm{N}(\mathcal{C})\colon \Delta^{\mathrm{op}} \xrightarrow{U^{\mathrm{op}}} \mathrm{Cat}^{\mathrm{op}} \xrightarrow{\operatorname{Hom}_{\mathrm{Cat}}(-, \mathcal{C})} \mathrm{Set}\,.$$

[(2.2)]Therefore, they're colimits in sSet, as limits and colimits in functor categories are computed pointwise by Lemma 1.8.

Concretely, $N(\mathcal{C})_n = \mathrm{Hom}_{\mathrm{Cat}}([n], \mathcal{C}) \cong \{x_0 \to \cdots \to x_n \text{ in } \mathcal{C}\}$ is the set of all chains of n morphisms in $\mathcal{C}$. The face maps $d_i^* \colon N(\mathcal{C})_n \to N(\mathcal{C})_{n-1}$ compose the $(i-1)^{\mathrm{st}}$ and i^{th} morphism in the chain (in the cases $i = 0$ or $i = n$, the face map d_0^* just discards x_0 and d_n^* just discards x_n). The degeneracy maps $s_j^* \colon N(\mathcal{C})_{n-1} \to N(\mathcal{C})_n$ insert an identity at the jth position.

Observe that the formula $N(\mathcal{C})_n \cong \mathrm{Hom}_{\mathrm{Cat}}([n], \mathcal{C})$ is exactly of the form of a right-adjoint as in Lemma 1.19! So what's the corresponding left adjoint? According to Theorem 1.17, it has to be the unique colimit-preserving extension of the functor $U \colon \Delta \to \mathrm{Cat}$ above.[(2.3)] We'll denote this extension by $h \colon \mathrm{sSet} \to \mathrm{Cat}$ and for a simplicial set X, we call $h(X)$ the *homotopy category of X*. The objects of $h(X)$ are the set of 0-simplices X_0. However, the morphisms of $h(X)$ are a little more difficult to describe. For example, $h(\Lambda_1^2)$ contains a morphism $\alpha \colon 0 \to 1$ and a morphism $\beta \colon 1 \to 2$; α and β are induced by the functors $[1] \cong h(\Delta^{\{0,1\}}) \to h(\Lambda_1^2)$ and $[1] \cong h(\Delta^{\{1,2\}}) \to h(\Lambda_1^2)$. Hence $h(\Lambda_1^2)$ must also contain a morphism $\beta \circ \alpha \colon 0 \to 2$, even though there's no 1-simplex from 0 to 2 in Λ_1^2. So in general, not all morphisms in $h(X)$ come from 1-simplices of X. Instead, we have to take *chains* of 1-simplices and quotient out a suitable equivalence relation. This is not too hard to make precise, but quite technical and we won't pursue it here. We'll see an explicit description in the case of quasi-categories in 2.13 below; the general description can be found in [F-HC$_{\mathrm{I}}$, Construction/Proposition II.24].

2.6. Mapping objects in simplicial sets. — For every simplicial set X the functor $- \times X \colon \mathrm{sSet} \to \mathrm{sSet}$ commutes with colimits.[(2.4)] Hence, by Theorem 1.17, it must be the unique colimit-preserving extension of the functor $\Delta \to \mathrm{sSet}$ sending $[n] \mapsto \Delta^n \times X$. But more importantly, $- \times X$ must have a right adjoint, which we denote $F(X, -) \colon \mathrm{sSet} \to \mathrm{sSet}$. By the formula in Lemma 1.19, the right adjoint is given by $F(X, Y)_n \cong F(\Delta^n \times X, Y)$.

At this point, let's take a moment to appreciate the power of Theorem 1.17: It gave us adjunctions

$$|\cdot| \colon \mathrm{sSet} \rightleftarrows \mathrm{Top} \colon \mathrm{Sing}, \quad h \colon \mathrm{sSet} \rightleftarrows \mathrm{Cat} \colon N, \quad \text{and} \quad - \times X \colon \mathrm{sSet} \rightleftarrows \mathrm{sSet} \colon F(X, -)$$

essentially for free!

§2.2. Quasi-categories and Kan complexes

In this subsection, we'll introduce *quasi-categories*, a class of simplicial sets that behaves very similarly to ordinary categories. To motivate the definition, we start with a lemma.

2.7. Lemma. — (*a*) *Let Y be a topological space and let $\mathrm{Sing}\, Y$ be the singular simplicial set of Y as in 2.4. For all $n \geqslant 1$ and all $0 \leqslant i \leqslant n$, every horn filling problem*

has a solution.

[(2.3)] Here we use implicitly that Cat has all colimits (which is easy to check, but not completely trivial, due to the same composition issues as in the description of $h(\Lambda_1^2)$).

[(2.4)] Using that limits and colimits in sSet are computed degree-wise by Lemma 1.8, this can be reduced to the fact that products in Set commute with colimits, which is straightforward to check.

(b)   *Let $\mathcal{C}$ be a small category and let $\mathrm{N}(\mathcal{C})$ be the nerve of $\mathcal{C}$ as in 2.5. For all $n \geqslant 2$ and all $0 < i < n$, every inner horn filling problem*

$$
\begin{array}{ccc}
\Lambda_i^n & \longrightarrow & \mathrm{N}(\mathcal{C}) \\
\downarrow & \nearrow & \\
\Delta^n & &
\end{array}
$$

has a unique solution. Furthermore, if X is a simplicial set with this horn filling property, then $X \cong \mathrm{N}(\mathcal{C})$ for some category $\mathcal{C}$ (which is necessarily the homotopy category $h(X)$).

Proof sketch. By the adjunction $|\cdot| \colon \mathrm{sSet} \rightleftarrows \mathrm{Top} \colon \mathrm{Sing}$ from 2.4, a horn filling problem as in (a) is equivalent to

$$
\begin{array}{ccc}
|\Lambda_i^n| & \longrightarrow & Y \\
\downarrow & \nearrow & \\
|\Delta^n| & &
\end{array}
$$

This one always has a solution since the topological space $|\Lambda_i^n|$ is a retract of $|\Delta^n|$. This proves (a). For (b), recall from 2.5 that a morphism $\Delta^n \to \mathrm{N}(\mathcal{C})$ corresponds to a chain $x_0 \to \cdots \to x_n$ in $\mathcal{C}$. But the morphisms $x_j \to x_{j+1}$ are already given by $\Lambda_i^n \to \mathrm{N}(\mathcal{C})$. This shows the unique horn filling assertion from (b). The additional assertion is more or less straightforward if you use the description of the homotopy category from 2.13 below. For a complete proof, see [F-HC$_\mathrm{I}$, Theorem II.25]. □

Recall that by Grothendieck's *homotopy hypothesis*, topological spaces should be the same as ∞-groupoids, so in particular, they should provide examples of ∞-categories. Furthermore, every ordinary category should give rise to an ∞-category too. So if we try to model ∞-categories by a specific class of simplicial sets, that class should contain $\mathrm{Sing}\,Y$ for every topological space Y and $\mathrm{N}(\mathcal{C})$ for every small category $\mathcal{C}$. It then feels reasonable to look for a common generalisation of the horn filling conditions from Lemma 2.7(a) and (b), which is precisely what the definition of quasi-categories does:

2.8. Definition (Boardman–Vogt, [BV73]). — A *quasi-category* (or ∞-*category*) is a simplicial set $\mathcal{C}$ such that for all $n \geqslant 2$ and all $0 < i < n$, every inner horn filling problem

$$
\begin{array}{ccc}
\Lambda_i^n & \longrightarrow & \mathcal{C} \\
\downarrow & \nearrow & \\
\Delta^n & &
\end{array}
$$

has a solution. If, moreover, all horn filling problems for $n \geqslant 1$ and $0 \leqslant i \leqslant n$ have solutions, then $\mathcal{C}$ is called a *Kan complex*. We let $\mathrm{Kan} \subseteq \mathrm{QCat} \subseteq \mathrm{sSet}$ denote the full subcategories spanned by Kan complexes and quasi-categories.

The rest of §2 as well as the entirety of §4 and §5 will be spent convincing you that Definition 2.8 is a sensible definition and that quasi-categories really behave like ordinary categories. Let's begin by giving a dictionary of the most basic categorical notions and their counterparts in the world of quasi-categories.

2.9. Objects and Morphisms. — Let $\mathcal{C}$ be a quasi-category. We'll use the following suggestive terminology. If x is a 0-simplex in $\mathcal{C}$, we'll say that *x is an object in $\mathcal{C}$* and write $x \in \mathcal{C}$ instead of $x \in \mathcal{C}_0$. We also write $\{x\} \to \mathcal{C}$ for the map $\Delta^0 \to \mathcal{C}$ induced by x. If α is a 1-simplex in $\mathcal{C}$ and $x = d_1^*(\alpha)$, $y = d_0^*(\alpha)$, we'll say that *$\alpha\colon x \to y$ is a morphism in $\mathcal{C}$*. For an object $x \in \mathcal{C}$, we'll call the degenerate 1-simplex $s_0^*(x) \in \mathcal{C}_1$ the *identity on x* and we'll write $\mathrm{id}_x\colon x \to x$.

2.10. Functors and natural transformations. — A *functor of quasi-categories* is simply a map of simplicial sets. If $\mathcal{C}$ and $\mathcal{D}$ are quasi-categories, then the construction $\mathrm{F}(\mathcal{C}, \mathcal{D})$ from 2.6 plays the role of the category of functors from $\mathcal{C}$ to $\mathcal{D}$. We'll show in Corollary 3.11 that $\mathrm{F}(\mathcal{C}, \mathcal{D})$ is indeed a quasi-category again. Furthermore, if $F, G\colon \mathcal{C} \to \mathcal{D}$ are functors of quasi-categories, then a *natural transformation $\eta\colon F \Rightarrow G$* is a functor $\eta\colon \Delta^1 \times \mathcal{C} \to \mathcal{D}$ such that the following diagram commutes:

$$
\begin{array}{ccc}
\{0\} \times \mathcal{C} & & \\
\downarrow & {\scriptstyle /\!/\!/} & \searrow^{F} \\
\Delta^1 \times \mathcal{C} & \xrightarrow{\ \eta\ } & \mathcal{D} \\
\uparrow & {\scriptstyle /\!/\!/} & \nearrow_{G} \\
\{1\} \times \mathcal{C} & &
\end{array}
$$

By 2.6, we may equivalently view η as a 1-simplex $\Delta^1 \to \mathrm{F}(\mathcal{C}, \mathcal{D})$ from F to G. That is, natural transformations are morphisms in the functor quasi-category, as they should be (except that we don't know yet that $\mathrm{F}(\mathcal{C}, \mathcal{D})$ is a quasi-category again). Further evidence that $\mathrm{F}(\mathcal{C}, \mathcal{D})$ is the right construction will be given in Lemma 2.14 below.

2.11. Arrows, slices, and Hom. — We let $\mathrm{Ar}(\mathcal{C}) \coloneqq \mathrm{F}(\Delta^1, \mathcal{C})$ denote the *arrow quasi-category* of $\mathcal{C}$. The inclusions $\{0\} \to \Delta^1$ and $\{1\} \to \Delta^1$ induce a source and a target projection $s, t\colon \mathrm{Ar}(\mathcal{C}) \to \mathcal{C}$. Furthermore, for $x, y \in \mathcal{C}$, we define the *Hom anima* $\mathrm{Hom}_{\mathcal{C}}(x, y)$ and the *slice quasi-category $\mathcal{C}_{x/}$* via the pullbacks

$$
\begin{array}{ccccc}
\mathrm{Hom}_{\mathcal{C}}(x, y) & \longrightarrow & \mathcal{C}_{x/} & \longrightarrow & \mathrm{Ar}(\mathcal{C}) \\
\downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow^{(s,t)} \\
\{x\} \times \{y\} & \longrightarrow & \{x\} \times \mathcal{C} & \longrightarrow & \mathcal{C} \times \mathcal{C}
\end{array}
$$

We'll prove in Corollary 4.4 that $\mathrm{Hom}_{\mathcal{C}}(x, y)$ is always an anima in the sense of Definition 2.18 below, and we'll prove in Corollary 3.11 that $\mathrm{Ar}(\mathcal{C})$ and $\mathcal{C}_{x/}$ are quasi-categories. So these constructions live up to their names. Furthermore, it follows from Lemma 2.14 below and $\Delta^1 \cong \mathrm{N}([1])$ that we have an isomorphism of simplicial sets $\mathrm{Ar}(\mathrm{N}(\mathcal{D})) \cong \mathrm{N}(\mathrm{Ar}(\mathcal{D}))$ for every ordinary category $\mathcal{D}$, so it makes sense to use the same notation as in Construction 1.13. Furthermore, since $\mathrm{N}\colon \mathrm{Cat} \to \mathrm{sSet}$ preserves pullbacks (being a right adjoint), it follows that $\mathrm{N}(\mathcal{D})_{y/} \cong \mathrm{N}(\mathcal{D}_{y/})$ for all $y \in \mathcal{D}$. Finally, it follows that $\mathrm{Hom}_{\mathrm{N}(\mathcal{D})}(x, y)$ is a *discrete simplicial set*, that is, a disjoint union of copies of Δ^0, with the indexing set being $\mathrm{Hom}_{\mathcal{D}}(x, y)$. So our construction of Hom recovers the usual notion for ordinary categories.

Be aware that $\mathcal{C}_{x/}$ is *not* the slice construction from [L-HTT, Proposition 1.2.9.2] or [Lan21, Definition 1.4.13]; instead, it corresponds to their *fat slice $\mathcal{C}^{x/}$*. I like our definition better because it is in line with Construction 1.14 and we'll avoid using the other slice construction

(or rather hide its unavoidable usages in black boxes). It can be shown that while the two slice constructions are not isomorphic, they are equivalent as quasi-categories (see [L-HTT, Proposition 4.2.1.5] or [Lan21, Proposition 2.5.27]), so once we're out of the simplicial swamp (that is, starting from §6), the distinction won't matter.

2.12. Compositions. — Morphisms in a quasi-category can be composed, albeit not uniquely. To explain how this works, let's first describe an equivalence relation on morphisms. For morphisms $\alpha, \alpha' \colon x \to y$ we say α *and* α' *are equivalent*, $\alpha \simeq \alpha'$, if the map $\sigma \colon \partial\Delta^2 \to \mathcal{C}$ represented by the hollow triangle

$$\sigma = \quad \begin{array}{c} y \\ {}^{\alpha'}\nearrow \quad \nwarrow {}^{\mathrm{id}_y} \\ x \xrightarrow[\alpha]{} y \end{array}$$

can be extended to a map $\overline{\sigma} \colon \Delta^2 \to \mathcal{C}$ satisfying $\overline{\sigma}|_{\partial\Delta^2} = \sigma$ (thus "filling" the triangle above). Even though the definition is asymmetric in α and α', it turns out that "$\simeq$" is an equivalence relation on $\mathcal{C}_1$. For reflexivity, we can fill the triangle by taking $\overline{\sigma}_{\mathrm{ref}} := s_1^*(\alpha)$ to be a degenerate simplex. For symmetry and transitivity, consider the maps $\vartheta_{\mathrm{sym}} \colon \Lambda_1^3 \to \mathcal{C}$ and $\vartheta_{\mathrm{trans}} \colon \Lambda_2^3 \to \mathcal{C}$ represented as the following hollow tetrahedra (each tetrahedron is missing its interior as well as one face; the missing faces have been highlighted):

$$\vartheta_{\mathrm{sym}} = \begin{array}{c} y \xleftarrow{\mathrm{id}_y} y \\ {}_{\alpha}\diagdown \; {}_{\alpha'}\diagdown \; {}^{\mathrm{id}_y}\diagdown {}^{\mathrm{id}_y} \\ x \xrightarrow[\alpha]{} y \end{array} \quad \text{and} \quad \vartheta_{\mathrm{trans}} = \begin{array}{c} y \xleftarrow{\mathrm{id}_y} y \\ {}_{\alpha''}\diagdown \; {}_{\alpha'}\diagdown \; {}^{\mathrm{id}_y}\diagdown {}^{\mathrm{id}_y} \\ x \xrightarrow[\alpha]{} y \end{array}$$

More precisely, the face $\vartheta_{\mathrm{sym}}|_{\Delta^{\{0,1,2\}}}$ is a 2-simplex witnessing $\alpha \simeq \alpha'$, whereas the faces $\vartheta_{\mathrm{sym}}|_{\Delta^{\{0,1,3\}}} = s_1^*(\alpha)$ and $\vartheta_{\mathrm{sym}}|_{\Delta^{\{1,2,3\}}} = s_1^*(\mathrm{id}_y) = s_1^* s_0^*(y)$ are degenerate simplices. Likewise, the faces $\vartheta_{\mathrm{trans}}|_{\Delta^{\{0,1,2\}}}$ and $\vartheta_{\mathrm{trans}}|_{\Delta^{\{0,2,3\}}}$ are 2-simplices witnessing $\alpha \simeq \alpha'$ and $\alpha' \simeq \alpha''$, respectively, whereas the face $\vartheta_{\mathrm{trans}}|_{\Delta^{\{1,2,3\}}} = s_1^*(\mathrm{id}_y) = s_1^* s_0^*(y)$ is a degenerate simplex. By Definition 2.8, the horns ϑ_{sym} and $\vartheta_{\mathrm{trans}}$ can be extended to 3-simplices $\overline{\vartheta}_{\mathrm{sym}}, \overline{\vartheta}_{\mathrm{trans}} \colon \Delta^3 \to \mathcal{C}$ satisfying $\overline{\vartheta}_{\mathrm{sym}}|_{\Lambda_1^3} = \vartheta_{\mathrm{sym}}$ and $\overline{\vartheta}_{\mathrm{trans}}|_{\Lambda_2^3} = \vartheta_{\mathrm{trans}}$ (in other words, the hollow tetrahedra can be "filled"). Now the face $\overline{\sigma}_{\mathrm{sym}} := \overline{\vartheta}_{\mathrm{sym}}|_{\Delta^{\{0,2,3\}}}$ is a 2-simplex witnessing $\alpha' \simeq \alpha$ and $\overline{\sigma}_{\mathrm{trans}} := \overline{\vartheta}_{\mathrm{trans}}|_{\Delta^{\{0,1,3\}}}$ is a 2-simplex witnessing $\alpha \simeq \alpha''$, which proves symmetry and transitivity.

Now let's define compositions. For morphisms $\alpha \colon x \to y$ and $\beta \colon y \to z$ in $\mathcal{C}$, consider the map $\sigma \colon \Lambda_1^2 \to \mathcal{C}$ represented by

$$\sigma = \quad \begin{array}{c} z \\ {}^{}\nearrow \quad \nwarrow {}^{\beta} \\ x \xrightarrow[\alpha]{} y \end{array}$$

By Definition 2.8, this horn admits a filler, that is, a morphism $\overline{\sigma} \colon \Delta^2 \to \mathcal{C}$ such that $\overline{\sigma}|_{\Lambda_1^2} = \sigma$. If $\gamma \colon x \to z$ is the morphism in $\mathcal{C}$ represented by $\overline{\sigma}|_{\Delta^{\{0,2\}}} \colon \Delta^{\{0,2\}} \to \mathcal{C}$, then γ is called *a composition of α and β* and we write $\gamma \simeq \beta \circ \alpha$. In particular, composition of morphisms is not unique in a general quasi-category, as filling inner horns is not unique.[(2.5)] However, if $\gamma, \gamma' \colon x \to z$ are any two compositions, then γ and γ' are equivalent in the sense of 2.12 above.

[(2.5)]Conversely, uniqueness of composition in ordinary categories accounts for uniqueness of filling inner horns in nerves of an ordinary categories, see Lemma 2.7(*b*).

Indeed, then we can consider the morphism $\vartheta\colon \Lambda^3_1 \to \mathcal{C}$ represented as follows (the missing face has been highlighted again):

$$\vartheta = \begin{array}{c}\end{array}$$

Concretely, $\vartheta|_{\Delta^{\{0,1,2\}}}$ and $\vartheta|_{\Delta^{\{0,1,3\}}}$ are 2-simplices that witness γ and γ' being compositions of α and β, whereas $\vartheta|_{\Delta^{\{1,2,3\}}} = s_1^*(\beta)\colon \Delta^{\{1,2,3\}} \to \mathcal{C}$ is a degenerate simplex. By Definition 2.8, we can extend ϑ to a map $\overline{\vartheta}\colon \Delta^3 \to \mathcal{C}$ and then the face $\overline{\sigma} := \overline{\vartheta}|_{\Delta^{\{0,2,3\}}}$ is a 2-simplex witnessing an equivalence $\gamma \simeq \gamma'$.

2.13. The homotopy category. — We can now describe the homotopy category $h(\mathcal{C})$ from 2.5 in more explicit terms. As already explained there, the objects of $h(\mathcal{C})$ are the 0-simplices $\mathcal{C}_0$, that is, the objects of $\mathcal{C}$. We've seen in 2.5 that the morphism may cause some problems since we might need to add compositions. However, by 2.12 above, compositions already exist in $\mathcal{C}$, they just might not be unique. So we find that the set of morphisms of $h(\mathcal{C})$ is given by $\mathcal{C}_1/\simeq$, the set of 1-simplices modulo the equivalence condition from 2.12.

To make this argument precise, one would have to check that $h(\mathcal{C})$ as described above satisfies the universal property of the colimit $\mathrm{colim}_{(n,\Delta^n\to\mathcal{C})}[n]$ in Cat. This is technical, but straightforward, and we leave the details to you.

As a consequence, we can prove that the construction $\mathrm{F}(-,-)$ from 2.6 is compatible with the functor category construction for ordinary categories.

2.14. Lemma. — *If $\mathcal{C}$ is a quasi-category and $\mathcal{D}$ is an ordinary category, then there is an isomorphism of simplicial sets*

$$\mathrm{F}\big(\mathcal{C}, \mathrm{N}(\mathcal{D})\big) \cong \mathrm{N}\big(\mathrm{Fun}(h(\mathcal{C}), \mathcal{D})\big)$$

In particular, if $\mathcal{C} \cong \mathrm{N}(\mathcal{C}')$ is the nerve of an ordinary category $\mathcal{C}'$, we get an isomorphism $\mathrm{F}(\mathrm{N}(\mathcal{C}'), \mathrm{N}(\mathcal{D})) \cong \mathrm{N}(\mathrm{Fun}(\mathcal{C}', \mathcal{D}))$.

Proof sketch. For all $n \geqslant 0$, we obtain the following chain of bijections, all of which are compatible with the simplicial structure maps:

$$\begin{aligned}
\mathrm{F}\big(\mathcal{C}, \mathrm{N}(\mathcal{D})\big)_n \cong \mathrm{Hom}_{\mathrm{sSet}}\big(\Delta^n \times \mathcal{C}, \mathrm{N}(\mathcal{D})\big) &\cong \mathrm{Hom}_{\mathrm{Cat}}\big(h(\Delta^n \times \mathcal{C}), \mathcal{D}\big) \\
&\cong \mathrm{Hom}_{\mathrm{Cat}}\big(h(\mathcal{C}) \times [n], \mathcal{D}\big) \\
&\cong \mathrm{Hom}_{\mathrm{Cat}}\big([n], \mathrm{Fun}(h(\mathcal{C}), \mathcal{D})\big) \\
&\cong \mathrm{N}\big(\mathrm{Fun}(h(\mathcal{C}), \mathcal{D})\big)_n.
\end{aligned}$$

In the first step, we use the definition of $\mathrm{F}(-,-)$ from 2.6. In the second step, we use the adjunction $h\colon \mathrm{sSet} \rightleftarrows \mathrm{Cat} \colon \mathrm{N}$ from 2.5. In the third step, we use that h commutes with products of quasi-categories, which follows from the description in 2.13. In the fourth step, we use "currying" for ordinary categories. Finally, in the fifth step we plug in the definition of $\mathrm{N}(\mathrm{Fun}(h(\mathcal{C}), \mathcal{D}))$.

To prove the "in particular", it suffices to see that the unit $u_{\mathcal{C}'}\colon \mathcal{C}' \to h\,\mathrm{N}(\mathcal{C}')$ of the adjunction $h \dashv \mathrm{N}$ from 2.5 is an isomorphism of categories (and we really need an isomorphism, not just an equivalence of categories). This is easy to check using Lemma 2.7(*b*) and the explicit description of $h\,\mathrm{N}(\mathcal{C})$ from 2.13. $\qquad\square$

2.15. Equivalences in quasi-categories. — We say that a morphism $\alpha\colon x \to y$ in $\mathcal{C}$ is an *equivalence* if it becomes an isomorphism in the homotopy category $h(\mathcal{C})$. Equivalently, α is an equivalence if and only if the horns $\sigma_{\text{left}}\colon \Lambda_0^2 \to \mathcal{C}$ and $\sigma_{\text{right}}\colon \Lambda_2^2 \to \mathcal{C}$ represented by

$$
\sigma_{\text{left}} = \quad
\begin{array}{c}
 x \\
{}^{\text{id}_y}\nearrow \quad \searrow \\
x \xrightarrow[\alpha]{} y
\end{array}
\qquad \text{and} \qquad
\sigma_{\text{right}} = \quad
\begin{array}{c}
 y \\
{}^{\text{id}_y}\nearrow \quad \nwarrow{}^{\alpha} \\
y \dashrightarrow x
\end{array}
$$

can be filled, that is, if and only if there are 2-simplices $\overline{\sigma}_{\text{left}}, \overline{\sigma}_{\text{right}}\colon \Delta^2 \to \mathcal{C}$ such that $\overline{\sigma}_{\text{left}}|_{\Lambda_0^2} = \sigma_{\text{left}}$ and $\overline{\sigma}_{\text{right}}|_{\Lambda_2^2} = \sigma_{\text{right}}$. Indeed, by 2.12 above, $\overline{\sigma}_{\text{left}}$ corresponds to a left inverse of α and $\overline{\sigma}_{\text{right}}$ corresponds to a right inverse. We say *x and y are equivalent* and write $x \simeq y$ if there exists an equivalence $\alpha\colon x \to y$.

2.16. Sub-quasi-categories. — If $\mathcal{C}$ is a quasi-category and $S_0 \subseteq \mathcal{C}_0$ is a set of 0-simplices, we can define a sub-simplicial set $\mathcal{C}[S_0] \subseteq \mathcal{C}$ by declaring that a simplex $\Delta^n \to \mathcal{C}$ belongs to $\mathcal{C}[S_0]$ if and only if all its vertices $\{i\} \to \Delta^n \to \mathcal{C}$ for $0 \leqslant i \leqslant n$ belong to S_0. It's straightforward to check that $\mathcal{C}[S_0]$ is a quasi-category again: If $\Lambda_i^n \to \mathcal{C}[S_0]$ is an inner horn, any filler $\Delta^n \to \mathcal{C}$ will automatically belong to $\mathcal{C}[S_0]$, because $\Lambda_i^n \to \Delta^n$ is a bijection on vertices whenever $n \geqslant 2$. We call $\mathcal{C}[S_0]$ the *full sub-quasi-category spanned by S_0*.

Similarly, assume $S_1 \subseteq \mathcal{C}_1$ is a set of 1-simplices which contains all identities and is closed under the equivalence relation from 2.12 as well as under compositions. We can define a sub-simplicial set $\mathcal{C}[S_1] \subseteq \mathcal{C}$ by declaring that a simplex $\Delta^n \to \mathcal{C}$ belongs to $\mathcal{C}[S_1]$ if and only if all its edges $\Delta^{\{i,j\}} \to \Delta^n \to \mathcal{C}$ for $0 \leqslant i, j \leqslant n$ belong to S_1. Once again, if $\Lambda_i^n \to \mathcal{C}[S_1]$ is an inner horn, any filler $\Delta^n \to \mathcal{C}$ will automatically belong to $\mathcal{C}[S_1]$, because any "missing" edge in $\Delta^n \smallsetminus \Lambda_i^n$ is a composition of edges in Λ_i^n. Hence $\mathcal{C}[S_1]$ is a quasi-category again, and we call it the *sub-quasi-category spanned by S_1* (and usually we'll emphasise that $\mathcal{C}[S_1]$ is not full).

2.17. The opposite quasi-category. — Every quasi-category $\mathcal{C}$ admits an *opposite quasi-category* $\mathcal{C}^{\text{op}}$. In fact, this construction works for arbitrary simplicial sets. Let $(-)^{\text{op}}\colon \text{Cat} \to \text{Cat}$ be the functor that sends a category to its opposite. Consider the composition

$$
\nabla\colon \Delta \xrightarrow{U} \text{Cat} \xrightarrow{(-)^{\text{op}}} \text{Cat} \xrightarrow{\text{N}} \text{sSet},
$$

where U and N are the functors from 2.5. This composition sends $[n] \mapsto \text{N}([n]^{\text{op}}) \cong \Delta^n$, since there is an isomorphism of categories $[n]^{\text{op}} \cong [n]$ given by sending $i \mapsto n - i$. Nevertheless, ∇ does *not* coincide with the Yoneda embedding $よ_\Delta\colon \Delta \to \text{sSet}$, which also sends $[n] \mapsto \Delta^n$, since the effect on morphisms is different (∇ "reverses the order" of face and degeneracy maps). According to Theorem 1.17, ∇ admits a unique colimit-preserving extension, which we denote $(-)^{\text{op}}\colon \text{sSet} \to \text{sSet}$. Intuitively, if X is a simplicial set, then X^{op} is given by inverting the direction of every 1-simplex and by reversing the order of all face and degeneracy maps. It's straightforward to check that $(-)^{\text{op}} \circ (-)^{\text{op}} \simeq \text{id}_{\text{sSet}}$ (so the right adjoint from Theorem 1.17 is just $(-)^{\text{op}}$ again) and that $\text{N}(\mathcal{D})^{\text{op}} \cong \text{N}(\mathcal{D}^{\text{op}})$ holds for every ordinary category $\mathcal{D}$. Furthermore, if $\mathcal{C}$ is a quasi-category, then so is $\mathcal{C}^{\text{op}}$, because $(-)^{\text{op}}$ transforms an inner horn inclusion $\Lambda_i^n \to \Delta^n$, where $n \geqslant 2$ and $0 < i < n$, into $\Lambda_{n-i}^n \to \Delta^n$, which is again an inner horn inclusion.

This finishes our preliminary ordinary-to-quasi-categories dictionary. Next, we'll introduce another notion that will play a central role in these notes.

2.18. Definition. — A quasi-category $\mathcal{C}$ is called an *anima* (plural *animae*) if all its morphisms are equivalences in the sense of 2.15. For an arbitrary quasi-category, we let $\mathrm{core}(\mathcal{C}) \subseteq \mathcal{C}$ be the (non-full) sub-quasi-category spanned by the equivalences, as defined in 2.16.

It follows immediately that $\mathrm{core}(\mathcal{C})$ is the largest anima contained in $\mathcal{C}$.

2.19. Theorem (Joyal, [Joy02, Corollary 1.4]). — *A quasi-category $\mathcal{C}$ is a Kan complex if and only if it is an anima.*

Proof. If $\mathcal{C}$ is a Kan complex, then the horns from 2.15 can be filled, so $\mathcal{C}$ is an anima. The converse is much harder to prove and we'll postpone it to Corollary 4.2. □

So on one hand, by Definition 2.18, animae are the analogues of groupoids in quasi-category theory. In fact, people used (and continue to use) the term ∞-*groupoid*, before Beilinson, Clausen, and Scholze decided to invent a new term. On the other hand, Theorem 2.19 says that animae are the same as *Kan complexes*. We'll see in §3 that for the purposes of homotopy theory, Kan complexes and topological spaces can be used interchangeably. This fits perfectly with Grothendieck's *homotopy hypothesis*, which predicts that the theory of ∞-groupoids/animae should essentially be the homotopy theory of topological spaces.

We'll keep the terms *anima* and *Kan complex* distinct until we've finished the proof that they coincide (Corollary 4.2). After that, we'll use the terms interchangeably. Starting from §6, we try to keep our arguments as model-independent as possible. Accordingly, we'll settle on *anima*, only using *Kan complex* to emphasise that a certain (non-model-independent) argument takes place in thequasi-categorical model.

§2.3. Simplicially enriched categories

Until now, we know a good supply of Kan complexes, given by $\mathrm{Sing}\,Y$ for every topological space Y (see Lemma 2.7(a)). We'll see in Theorem 3.26 that these exhaust essentially all Kan complexes. Besides that, our only other examples of quasi-categories are nerves of ordinary categories (see Lemma 2.7(b)). These can't possibly be all! The goal of this subsection is to provide a rich source of non-trivial examples of quasi-categories, using a fancier version of the nerve construction.

2.20. "Definition". — A *simplicially enriched category* $\mathcal{C}$ is the same as a category, except that the morphisms sets $\mathrm{Hom}_{\mathcal{C}}(x, y)$ for $x, y \in \mathcal{C}$ are replaced by simplicial sets $\mathrm{F}_{\mathcal{C}}(x, y)$. Composition of morphisms is now a map of simplicial sets $\circ\colon \mathrm{F}_{\mathcal{C}}(x, y) \times \mathrm{F}_{\mathcal{C}}(y, z) \to \mathrm{F}_{\mathcal{C}}(x, z)$ and the identity on any object $x \in \mathcal{C}$ is a 0-simplex $\mathrm{id}_x \in \mathrm{F}_{\mathcal{C}}(x, x)_0$. Composition and identities are supposed to satisfy some straightforward compatibilities that we won't spell out. Furthermore, if $\mathcal{C}$ and $\mathcal{D}$ are simplicially enriched categories, there is an obvious notion of a *simplicially enriched functor $F\colon \mathcal{C} \to \mathcal{D}$*. We let Cat_Δ denote the category of (small) simplicially enriched categories and simplicially enriched functors between them.

If you would like to see a formal definition of these notions, have a look at [Lan21, Definitions 1.2.34 and 1.2.35].

2.21. Construction. — We'll construct a "simplicially thickened" versions of the ordinary categories $[n]$ and use them to define a simplicial nerve functor $\mathrm{N}^\Delta\colon \mathrm{Cat}_\Delta \to \mathrm{sSet}$. This is originally due to Cordier and Porter [CP86].

To start with, the simplicially enriched category $\mathfrak{C}[\Delta^n]$ is given as follows: It's objects are $0, 1, \ldots, n$ and it's morphisms are given by

$$F_{\mathfrak{C}[\Delta^n]}(i,j) := \begin{cases} \emptyset & \text{if } i > j \\ \Delta^0 & \text{if } i = j \\ \square^{j-i-1} & \text{if } i < j \end{cases} .$$

Here $\square^n := (\Delta^1)^n$ is the *n-cube*. Note the shift by -1 in the definition! In particular, $F_{\mathfrak{C}[\Delta^n]}(i,i) = \Delta^0$ (and that 0-simplex is necessarily id_i), but also $F_{\mathfrak{C}[\Delta^n]}(i, i+1) = \square^0 \cong \Delta^0$. The composition map $\circ\colon F_{\mathfrak{C}[\Delta^n]}(i,j) \times F_{\mathfrak{C}[\Delta^n]}(j,k) \to F_{\mathfrak{C}[\Delta^n]}(i,k)$ is given by

$$\square^{j-i-1} \times \square^{k-j-1} \xrightarrow{\cong} \square^{j-i-1} \times \{1\} \times \square^{k-j-1} \subseteq \square^{k-i-1}$$

if $i < j < k$; in the other cases, there's only one possible composition map. The simplicially enriched categories $\mathfrak{C}[\Delta^n]$ can be assembled into a functor $\mathfrak{C}[-]\colon \Delta \to \mathrm{Cat}_\Delta$. A conceptual construction of this functor is given in [L-HTT, Definition 1.1.5.3] or [Lan21, Lemma 1.2.62]. Since it's quite annoying to unravel said conceptual construction, let us describe the simplicially enriched functors $\mathfrak{C}[d_i]\colon \mathfrak{C}[\Delta^{n-1}] \to \mathfrak{C}[\Delta^n]$ and $\mathfrak{C}[s_j]\colon \mathfrak{C}[\Delta^n] \to \mathfrak{C}[\Delta^{n-1}]$ explicitly: On objects, $\mathfrak{C}[d_i]$ and $\mathfrak{C}[s_j]$ are just given by d_i and s_j, repectively. For the effect on morphisms, let's first describe $\mathfrak{C}[d_i]\colon F_{\mathfrak{C}[\Delta^{n-1}]}(k, \ell) \to F_{\mathfrak{C}[\Delta^n]}(d_i(k), d_i(\ell))$ in the case $k < i \leqslant \ell$ (in all other cases, we simply get the identity). Then $d_i(k) = k$ and $d_i(\ell) = \ell + 1$ and the desired morphism is

$$\square^{\ell-k-1} \xrightarrow{\cong} \square^{i-k-1} \times \{0\} \times \square^{(\ell+1)-i-1} \subseteq \square^{(\ell+1)-k-1} .$$

Similarly, $\mathfrak{C}[s_j]\colon F_{\mathfrak{C}[\Delta^n]}(k, \ell) \to F_{\mathfrak{C}[\Delta^{n-1}]}(s_j(k), s_j(\ell))$ is only interesting for $k \leqslant j < \ell$. If $k = j$ or $j + 1 = \ell$, then the desired morphism $\square^{\ell-k-1} \cong (\Delta^1)^{\ell-k-1} \to (\Delta^1)^{\ell-k-2} \cong \square^{\ell-k-2}$ is given by forgetting the first or the last factor, respectively. If $k < j$ and $j + 1 < \ell$, then the desired morphism is

$$\square^{\ell-k-1} \cong \square^{j-k-1} \times \square^2 \times \square^{\ell-(j+1)-1} \longrightarrow \square^{j-k-1} \times \Delta^1 \times \square^{\ell-(j+1)-1} \cong \square^{\ell-k-2} ,$$

induced by the map $\square^2 \to \Delta^1$ that sends $(0,0) \in \square^2$ to $0 \in \Delta^1$ and the other three 0-simplices of $\square^2$ to $1 \in \Delta^1$.

It can be shown that the category Cat_Δ has all colimits (see [Lan21, Corollary 1.2.45]). Consequently, by Theorem 1.17, the functor above admits a unique colimit-preserving extension $\mathfrak{C}[-]\colon \mathrm{sSet} \to \mathrm{Cat}_\Delta$, which in turn has a right-adjoint $N^\Delta\colon \mathrm{Cat}_\Delta \to \mathrm{sSet}$, called the *simplicial nerve* or *coherent nerve*. By the formula from Lemma 1.19, the simplicial nerve is given by

$$N^\Delta(\mathcal{C})_n \cong \mathrm{Hom}_{\mathrm{Cat}_\Delta}\big(\mathfrak{C}[\Delta^n], \mathcal{C}\big) .$$

2.22. Lemma (Cordier–Porter, [CP86, Theorem 2.1]). — *Let $\mathcal{C}$ be a small simplicially enriched category. If $\mathcal{C}$ is even Kan-enriched, that is, if $F_\mathcal{C}(x,y)$ is a Kan complex for all $x, y \in \mathcal{C}$, then $N^\Delta(\mathcal{C})$ is a quasi-category.*

Proof sketch. By the adjunction $\mathfrak{C}[-]\colon \mathrm{sSet} \rightleftarrows \mathrm{Cat}_\Delta : N^\Delta$ from Construction 2.21, an inner horn filling problem for $N^\Delta(\mathcal{C})$ as in Definition 2.8 is equivalent to an extension problem

$$\begin{array}{ccc} \mathfrak{C}[\Lambda_i^n] & \xrightarrow{\ f\ } & \mathcal{C} \\ \downarrow & \nearrow & \\ \mathfrak{C}[\Delta^n] & & \end{array}$$

of simplicially enriched categories. The functor $\mathfrak{C}[\Lambda_i^n] \to \mathfrak{C}[\Delta^n]$ is a bijection on objects and an isomorphism on all but one simplicial sets of morphisms. The only difference between these two simplicially enriched categories is that $\mathrm{F}_{\mathfrak{C}[\Lambda_i^n]}(0,n) \to \mathrm{F}_{\mathfrak{C}[\Delta^n]}(0,n) \cong \square^{n-1}$ is not an isomorphism. Instead, $\mathrm{F}_{\mathfrak{C}[\Lambda_i^n]}(0,n)$ is given by deleting the interior and the bottom i-face of the $(n-1)$-cube $\square^{n-1}$. More precisely, if $\partial\square^{n-1} := \bigcup_{j=1}^{n-1}(\square^{j-1} \times (\{0\} \sqcup \{1\}) \times \square^{n-j-1})$ denotes the *boundary of the $(n-1)$-cube*, then $\mathrm{F}_{\mathfrak{C}[\Lambda_i^n]}(0,n) \to \mathrm{F}_{\mathfrak{C}[\Delta^n]}(0,n)$ can be identified with the inclusion of simplicial sets

$$\partial\square^{n-1} \smallsetminus \left(\square^{i-1} \times \{0\} \times \square^{n-i-1}\right) \subseteq \square^{n-1}.$$

To make this precise, one has to show that the description of $\mathfrak{C}[\Lambda_i^n]$ given above satisfies the universal property of $\operatorname{colim}_{(m,\Delta^m \to \Lambda_i^n)} \mathfrak{C}[\Delta^m]$ in Cat_Δ. This is not hard, but technical. A full argument is in [Lan21, Lemma 1.2.69].

So to solve the extension problem of simplicially enriched categories above, it's enough to solve the extension problem

$$
\begin{array}{ccc}
\partial\square^{n-1} \smallsetminus \left(\square^{i-1} \times \{0\} \times \square^{n-i-1}\right) & \overset{f}{\longrightarrow} & \mathrm{F}_{\mathcal{C}}\big(f(0), f(n)\big) \\
\downarrow & \nearrow & \\
\square^{n-1} & &
\end{array}
$$

of simplicial sets. This can be done by successive horn filling (or by applying the upcoming Lemma 3.9, which is also proved by successive horn filling), using the fact that $\mathrm{F}_{\mathcal{C}}(f(0), f(n))$ is a Kan complex, as $\mathcal{C}$ is supposed to be Kan-enriched. A complete argument is in [Lan21, Lemma 1.2.70]. $\qquad\square$

2.23. Example. — The category of simplicial sets can be turned into a simplicially enriched category sSet^Δ by putting $\mathrm{F}_{\mathrm{sSet}^\Delta}(X,Y) := \mathrm{F}(X,Y)$. This can be used to construct some interesting quasi-categories as follows:

(a)   Restricting to the full subcategory $\mathrm{Kan} \subseteq \mathrm{sSet}$ yields a simplicial enrichment Kan^Δ. Note that Kan^Δ is actually a Kan-enriched category, since $\mathrm{F}(X,Y)$ is a Kan complex whenever Y is a Kan complex, as we'll see in Corollary 3.11. Up to set-theoretic difficulties that we'll not address here, Lemma 2.22 shows that

$$\mathrm{An} := \mathrm{N}^\Delta(\mathrm{Kan}^\Delta)$$

is a quasi-category; we call it the *quasi-category of animae.*

(b)   For quasi-categories $\mathcal{C}$ and $\mathcal{D}$, the simplicial set $\mathrm{F}(\mathcal{C},\mathcal{D})$ is a quasi-category; once again, this will be shown in Corollary 3.11. Then $\operatorname{core}\mathrm{F}(\mathcal{C},\mathcal{D})$ from Definition 2.18 is an anima, hence a Kan complex by Theorem 2.19. So we can turn the category of quasi-categories QCat into a Kan-enriched category QCat^Δ by putting $\mathrm{F}_{\mathrm{QCat}^\Delta}(\mathcal{C},\mathcal{D}) := \operatorname{core}\mathrm{F}(\mathcal{C},\mathcal{D})$. By Lemma 2.22 (and up to set-theoretic difficulties),

$$\mathrm{Cat}_\infty := \mathrm{N}^\Delta(\mathrm{QCat}^\Delta)$$

is a quasi-category; we call it the *quasi-category of (small) quasi-categories.*

Let's unravel how the notions from 2.12, 2.13, and 2.15 look like in the cases of An and Cat_∞. A 1-simplex $\alpha\colon \Delta^1 \to \mathrm{Cat}_\infty$ is equivalently a simplicially enriched functor $\overline{\alpha}\colon \mathfrak{C}[\Delta^1] \to \mathrm{QCat}^\Delta$. Let $\mathcal{C} \coloneqq \overline{\alpha}(0)$ and $\mathcal{D} \coloneqq \overline{\alpha}(1)$. As we've seen in Construction 2.21, $\mathrm{F}_{\mathfrak{C}[\Delta^1]}(0,1) \cong \Delta^0$. Hence α is given by a morphism $\Delta^0 \to \mathrm{core}\,\mathrm{F}(\mathcal{C},\mathcal{D})$ of simplicial sets. In other words, a morphism in Cat_∞ is given by a functor $\mathcal{C} \to \mathcal{D}$ of quasi-categories, as we would expect.

Next, let's consider a 2-simplex $\sigma\colon \Delta^2 \to \mathrm{Cat}_\infty$, or equivalently, a simplicially enriched functor $\overline{\sigma}\colon \mathfrak{C}[\Delta^2] \to \mathrm{QCat}^\Delta$. Let $\mathcal{C} \coloneqq \overline{\sigma}(0)$, $\mathcal{D} \coloneqq \overline{\sigma}(1)$, and $\mathcal{E} \coloneqq \overline{\sigma}(2)$. Furthermore, let $F\colon \mathcal{C} \to \mathcal{D}$ and $G\colon \mathcal{D} \to \mathcal{E}$ be the functors of quasi-categories corresponding to the 1-simplices $\sigma|_{\Delta^{\{0,1\}}} = d_2^*(\sigma)$ and $\sigma|_{\Delta^{\{1,2\}}} = d_0^*(\sigma)$. Now $\mathrm{F}_{\mathfrak{C}[\Delta^2]}(0,2) \cong \Delta^1$ by Construction 2.21, so $\overline{\sigma}$ induces a map $\Delta^1 \to \mathrm{core}\,\mathrm{F}(\mathcal{C},\mathcal{E})$. By definition of the composition in $\mathfrak{C}[\Delta^2]$, we find that $\{0\} \to \Delta^1 \to \mathrm{core}\,\mathrm{F}(\mathcal{C},\mathcal{E})$ is $G \circ F\colon \mathcal{C} \to \mathcal{E}$, whereas $\{1\} \to \Delta^1 \to \mathrm{core}\,\mathrm{F}(\mathcal{C},\mathcal{E})$ is another functor $H\colon \mathcal{C} \to \mathcal{E}$. The morphism $\Delta^1 \to \mathrm{core}\,\mathrm{F}(\mathcal{C},\mathcal{E})$ is an equivalence $G \circ F \simeq H$ in $\mathrm{F}(\mathcal{C},\mathcal{E})$.

Therefore, if $F\colon \mathcal{C} \to \mathcal{D}$ and $G\colon \mathcal{D} \to \mathcal{E}$ are functors of quasi-categories, hence morphisms in Cat_∞, then a composition of F and G in the quasi-category Cat_∞, as defined in 2.12, is a functor $H\colon \mathcal{C} \to \mathcal{E}$ of quasi-categories together with an equivalence $G \circ F \simeq H$ in $\mathrm{F}(\mathcal{C},\mathcal{E})$. The same analysis can be done for An. So if $f\colon X \to Y$ and $g\colon Y \to Z$ are maps of Kan complexes, corresponding to morphisms in the quasi-category An, then a composition of f and g in the quasi-category An is a morphism $h\colon X \to Z$ together with a 1-simplex $\Delta^1 \to \mathrm{F}(X,Z)$ from $g \circ f$ to h. By 2.6, such a 1-simplex $\Delta^1 \to \mathrm{F}(X,Z)$ is equivalently a map $\eta\colon \Delta^1 \times X \to Z$ such that

$$
\begin{array}{c}
\{0\} \times X \\
\downarrow \qquad \text{///} \\
\Delta^1 \times X \xrightarrow{\ \eta\ } Z \\
\uparrow \qquad \text{///} \\
\{1\} \times X
\end{array}
\quad
\begin{array}{l}
\\ \overset{g \circ f}{\searrow} \\ \\ \\ \underset{h}{\nearrow}
\end{array}
$$

commutes. In other words, η is a *homotopy from $g \circ f$ to h*. In summary, we obtain the following slogans:

> *"Compositions in Cat_∞ are compositions in sSet up to equivalence of functors."*

> *"Compositions in An are compositions in sSet up to homotopy."*

Furthermore, this analysis shows that two functors of quasi-categories $F, G\colon \mathcal{C} \to \mathcal{D}$ are equivalent as morphisms in Cat_∞ in the sense of 2.12 if and only if they are equivalent as objects in $\mathrm{F}(\mathcal{C},\mathcal{D})$. Similarly, two morphisms of animae $f, g\colon X \to Y$ are equivalent as morphisms in An in the sense of 2.12 if and only they are homotopic. This somewhat explains the term *homotopy category*.

Finally, we see that an equivalence $\mathcal{C} \simeq \mathcal{D}$ in the quasi-category Cat_∞, as defined in 2.15, is given by functors of quasi-categories $F\colon \mathcal{C} \to \mathcal{D}$ and $G\colon \mathcal{D} \to \mathcal{C}$ together with equivalences $G \circ F \simeq \mathrm{id}_\mathcal{C}$ and $F \circ G \simeq \mathrm{id}_\mathcal{D}$, exactly as an equivalence of ordinary categories. Analogously, an equivalence $X \simeq Y$ in An is given by maps of Kan complexes $f\colon X \to Y$ and $g\colon Y \to X$, together with homotopies $g \circ f \simeq \mathrm{id}_X$ and $f \circ g \simeq \mathrm{id}_Y$. In other words, equivalences in An are simply *homotopy equivalences*. We'll explore this in much more detail in §3.

If X, Y are Kan complexes, then $\mathrm{F}(X,Y)$ is a Kan complex too, as we'll see in Corollary 3.11. Hence $\mathrm{F}(X,Y) = \mathrm{core}\,\mathrm{F}(X,Y)$ and therefore the Kan-enriched category Kan^Δ is a full sub-simplicially enriched category of QCat^Δ. Using the explicit formula for the simplicial nerve

from Construction 2.21, it's straightforward to see that $N^\Delta(-)$ sends full sub-simplicially enriched categories to full sub-quasi-categories in the sense of 2.16. Thus $An \subseteq Cat_\infty$ is a full sub-quasi-category of Cat_∞. In particular, if X and Y are Kan complexes, then

$$\mathrm{Hom}_{An}(X,Y) \xrightarrow{\;\cong\;} \mathrm{Hom}_{Cat_\infty}(X,Y)$$

is an isomorphism of simplicial sets. In general, Theorem 2.24 below describes the Hom anima from 2.11 in a simplicial nerve. A relatively short proof of that theorem was given by Fabian Hebestreit and Achim Krause in [FK20].

2.24. Theorem. — *Let $\mathcal{C}$ be a Kan-enriched category. Then there is a homotopy equivalence of Kan complexes*

$$\mathrm{Hom}_{N^\Delta(\mathcal{C})}(x,y) \simeq F_{\mathcal{C}}(x,y)\,.$$

In particular, $\mathrm{Hom}_{An}(X,Y) \simeq F(X,Y)$ *for all* $X,Y \in An$ *and* $\mathrm{Hom}_{Cat_\infty}(\mathcal{C},\mathcal{D}) \simeq \mathrm{core}\,F(\mathcal{C},\mathcal{D})$ *for all* $\mathcal{C},\mathcal{D} \in Cat_\infty$. ∎

2.25. Example. — We can also turn the category of ordinary categories Cat into a Kan enriched category[(2.6)] Cat^Δ via $F_{Cat^\Delta}(\mathcal{C},\mathcal{D}) := \mathrm{core}\,N(\mathrm{Fun}(\mathcal{C},\mathcal{D}))$. We let

$$Cat^{(2)} := N^\Delta(Cat^\Delta)$$

denote its simplicial nerve. According to Theorem 2.24, $\mathrm{Hom}_{Cat^{(2)}}(\mathcal{C},\mathcal{D}) \simeq \mathrm{core}\,N(\mathrm{Fun}(\mathcal{C},\mathcal{D}))$. In particular, we see that $Cat^{(2)}$ is different from $N(Cat)$, the nerve of the ordinary category of categories. Indeed, we've seen in 2.11 that $\mathrm{Hom}_{N(Cat)}(\mathcal{C},\mathcal{D})$ would be a discrete: a disjoint union of copies of Δ^0, where the indexing set is precisely the set of functors from $\mathcal{C}$ to $\mathcal{D}$. In contrast to that, $\mathrm{core}\,N(\mathrm{Fun}(\mathcal{C},\mathcal{D})) \simeq N(\mathrm{core}\,\mathrm{Fun}(\mathcal{C},\mathcal{D}))$, where $\mathrm{core}\,\mathrm{Fun}(\mathcal{C},\mathcal{D}) \subseteq \mathrm{Fun}(\mathcal{C},\mathcal{D})$ denotes the maximal groupoid contained in $\mathrm{Fun}(\mathcal{C},\mathcal{D})$. So $\mathrm{Hom}_{Cat^{(2)}}(\mathcal{C},\mathcal{D})$ is the nerve of a groupoid and usually not a discrete simplicial set.[(2.7)]

[(2.6)]Don't confuse Cat^Δ, the simplicially enriched category of categories, with Cat_Δ, the category of simplicially enriched categories.

[(2.7)]One says that $Cat^{(2)}$ is the 2-*category of categories*, and we've just seen why: $Cat^{(2)}$ not only knows about categories and functors, but through $\mathrm{Hom}_{Cat^{(2)}}(\mathcal{C},\mathcal{D}) \simeq N(\mathrm{core}\,\mathrm{Fun}(\mathcal{C},\mathcal{D}))$ it also contains information about natural equivalences between functors. In general, a quasi-category $\mathcal{E}$ is said to be an *n-category* if for all objects $x,y \in \mathcal{E}$ and all morphisms $f \in \mathrm{Hom}_{\mathcal{E}}(x,y)$ one has $\pi_i(\mathrm{Hom}_{\mathcal{E}}(x,y),f) \cong 0$ whenever $i \geqslant n$. Here π_i refers to the homotopy groups introduced in Construction 3.15 and we've used implicitly that $\mathrm{Hom}_{\mathcal{E}}(x,y)$ is a Kan complex, as will be shown in Corollaries 4.2 and 4.4. It is not hard to check that $Cat^{(2)}$ is indeed a 2-category. Indeed, we've seen that $\mathrm{Hom}_{Cat^{(2)}}(\mathcal{C},\mathcal{D}) \simeq N(\mathrm{core}\,\mathrm{Fun}(\mathcal{C},\mathcal{D}))$. By the observation in the proof of Lemma 4.8, we get

$$\pi_i\big(N(\mathrm{core}\,\mathrm{Fun}(\mathcal{C},\mathcal{D})),F\big) \cong \pi_{i-1}\big(\mathrm{Hom}_{N(\mathrm{core}\,\mathrm{Fun}(\mathcal{C},\mathcal{D}))}(F,F),\mathrm{id}_F\big)$$

for all $F \in N(\mathrm{core}\,\mathrm{Fun}(\mathcal{C},\mathcal{D}))$. But now $\mathrm{Hom}_{N(\mathrm{core}\,\mathrm{Fun}(\mathcal{C},\mathcal{D}))}(F,F)$ is a discrete simplicial set, because it is the Hom anima in the nerve of an ordinary category. So the right-hand side vanishes for $i-1 \geqslant 1$, as desired.

You might have expected the 2-category of categories to encompass all natural transformations, not only the natural equivalences. The reason for this confusion is an unfortunate oversimplification of language on our part: What we call ∞-*categories* (or *n-categories*) should more accurately be called $(\infty,1)$-categories (or $(n,1)$-*categories*). The first entry of the pair "$(\infty,1)$" signifies that such an object contains "d-morphisms" for every dimension $0 \leqslant d < \infty$, whereas the second entry refers to the fact that all d-morphisms for $d > 1$ are invertible. This is evidenced by the fact that $\mathrm{Hom}_{\mathcal{E}}(x,y)$ is a Kan complex for any quasi-category $\mathcal{E}$ and all $x,y \in \mathcal{E}$. Thanks to the effort of many mathematicians, we now have well-studied notions of (∞,k)-*categories* (with $(\infty,0)$-categories corresponding to animae and $(\infty,1)$-categories corresponding to what we call ∞-*categories* in these notes), in which only d-morphisms for $d > k$ need to be invertible. These have become important tools in modern mathematics—for example, it's sometimes necessary to use the fact that Cat_∞ can be enhanced to an $(\infty,2)$-category—but this goes beyond the scope of these notes.

Thanks to Lemma 2.14, the nerve functor $N\colon \mathrm{Cat} \to \mathrm{QCat}$ defines a fully faithful functor of simplicially enriched categories $N\colon \mathrm{Cat}^\Delta \to \mathrm{QCat}^\Delta$. Accordingly, we can regard $\mathrm{Cat}^{(2)}$ as the full sub-quasi-category of Cat_∞ spanned by those quasi-categories that are nerves of ordinary categories.

In a similar way, one can define equip the category of groupoids Grpd with a Kan enrichment Grpd^Δ (simply given by restriction from Cat^Δ) and we let

$$\mathrm{Grpd}^{(2)} \coloneqq N^\Delta(\mathrm{Grpd}^\Delta)$$

denotes its simplicial nerve. As above, $\mathrm{Grpd}^{(2)}$ is the full sub-quasi-category of Cat_∞ spanned by the nerves of groupoids. Since every nerve of a groupoid is a Kan complex (which follows from Corollary 4.2, but can also be checked by hand), we see that $\mathrm{Grpd}^{(2)}$ is also a full sub-quasi-category of An.

§3. Simplicial homotopy theory

The goal of this section is to describe how to do homotopy theory with simplicial sets instead of topological spaces. This doesn't quite work on the nose, since simplicial sets are much more rigid than topological spaces. For example, consider the naive definition of homotopies: Two maps $f, g \colon X \to Y$ are said to be *homotopic*, $f \simeq g$, if there exists a map $\eta \colon \Delta^1 \times X \to Y$ such that the diagram

$$
\begin{array}{ccc}
\{0\} \times X & & \\
\downarrow & \!\!\!/\!/\!/ & \overset{f}{\searrow} \\
\Delta^1 \times X & \overset{\eta}{\longrightarrow} & Z \\
\uparrow & \!\!\!/\!/\!/ & \nearrow \\
\{1\} \times X & & g
\end{array}
$$

commutes. This relation is *not* an equivalence relation! For example, if $d_1 \colon \Delta^0 \simeq \{0\} \to \Delta^1$ and $d_0 \colon \Delta^0 \simeq \{1\} \to \Delta^1$ are the two maps from the 0-simplex to the 1-simplex, then $d_1 \simeq d_0$, but $d_0 \not\simeq d_1$. So the relation is not symmetric (nor transitive). However, as we will see, everything works fine as long as we work with Kan complexes!

So the upshot of this section will be that instead of replacing topological spaces by arbitrary simplicial sets as a habitat for homotopy theory, we should replace them with Kan complexes. In view of Theorem 2.19, this fits perfectly with Grothendieck's *homotopy hypothesis* that ∞-groupoids should essentially be topological spaces.

§3.1. Fibrations and lifting properties

We start with several definitions that generalise the horn filling properties from Definition 2.8.

3.1. Definition. — We say that a map $f \colon X \to Y$ of simplicial sets *has lifting against* $i \colon A \to B$ if every lifting problem

$$
\begin{array}{ccc}
A & \longrightarrow & X \\
\scriptstyle i \downarrow & \nearrow & \downarrow \scriptstyle f \\
B & \longrightarrow & Y
\end{array}
$$

has a solution.

3.2. Definition. — Let $f \colon X \to Y$ be a map of simplicial sets.

(a)   We call f a *Kan fibration* if it has lifting agains all horn inclusions $\Lambda_i^n \to \Delta^n$ for $n \geqslant 1$ and $0 \leqslant i \leqslant n$. We call f a *left*, *right*, or *inner fibration*, if it has lifting against all horn inclusions for $0 \leqslant i < n$, all $0 < i \leqslant n$, or $0 < i < n$, respectively.

(b)   We call f a *trivial fibration* if it has lifting against all boundary inclusions $\partial \Delta^n \to \Delta^n$ for all $n \geqslant 0$.

3.3. Example. — A simplicial set X is a Kan complex if and only if $X \to *$ is a Kan fibration, and a quasi-category if and only if $X \to *$ is an inner fibration. Here and in the following we put $* := \Delta^0$ for convenience. Furthermore, if $f \colon X \to Y$ is a Kan fibration and Y is a Kan complex, then X is a Kan complex too. Similarly, if f is an inner fibration and Y is a quasi-category, then X is a quasi-category too.

To analyse lifting properties, we need to introduce yet another technical notion.

3.4. Definition. — A class Σ of morphisms of simplicial sets is called *saturated* if the following conditions are satisfied:

(a) Σ is *closed under pushouts*: If $(A \to B) \in \Sigma$ and $A \to C$ is an arbitrary map of simplicial sets, then $(C \to B \sqcup_A C) \in \Sigma$.

(b) Σ is *closed under retracts*: If we're given a commutative diagram

$$
\begin{array}{ccccc}
 & & \mathrm{id}_{A'} & & \\
 & \nearrow & & \searrow & \\
A' & \longrightarrow & A & \longrightarrow & A' \\
\downarrow{\scriptstyle i'} & & \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle i'} \\
B' & \longrightarrow & B & \longrightarrow & B' \\
 & \searrow & & \nearrow & \\
 & & \mathrm{id}_{B'} & &
\end{array}
$$

such that $(i\colon A \to B) \in \Sigma$, then also $(i'\colon A' \to B') \in \Sigma$.

(c) Σ is *closed under coproducts*: If $(A_i \to B_i) \in \Sigma$, then also $(\coprod A_i \to \coprod B_i) \in \Sigma$.

(d) Σ is *closed under* (*countable*) *infinite compositions*: If $A_0 \to A_1 \to A_2 \to \cdots$ are all in Σ, then also $(A_0 \to \operatorname{colim}_{n \geqslant 0} A_n) \in \Sigma$.

For an arbitrary class Σ of morphisms in sSet, the *saturation of* Σ, $\mathrm{sat}(\Sigma)$, is the smallest saturated class containing Σ.

3.5. Lemma. — *A morphism $f\colon X \to Y$ of simplicial sets has lifting against all $(A \to B) \in \Sigma$ if and only f has lifting against all $(A \to B) \in \mathrm{sat}(\Sigma)$.*

Proof sketch. It's straightforward to check that the class of morphisms that f has lifting against is saturated as in Definition 3.4. □

3.6. Definition. — (a)   A morphism of simplicial sets is called *anodyne* if it is contained in $\mathrm{sat}\{\Lambda_i^n \to \Delta^n \mid n \geqslant 1,\, 0 \leqslant i \leqslant n\}$, the saturation of all horn inclusions. Similarly, a morphism is called *left*, *right*, or *inner anodyne* if it is contained in the saturation of those horn inclusions where $0 \leqslant i < n$, $0 < i \leqslant n$, or $0 < i < n$, respectively.

(b)   A morphism of simplicial set is a *cofibration* if it is contained in $\mathrm{sat}\{\partial\Delta^n \to \Delta^n \mid n \geqslant 0\}$, the saturation of all boundary inclusions.

3.7. Example. — Using Lemma 3.5, we see that Kan fibrations have lifting against all anodyne morphisms and left/right/inner fibrations have lifting against all left/right/inner anodyne morphisms. Furthermore, trivial fibrations have lifting against all cofibrations.

3.8. Lemma. — *A map $i\colon A \to B$ is simplicial sets is a cofibration if and only if i is injective in every degree.*

Proof sketch. It's straightforward to check that degree-wise injectivity is closed under pushouts, retracts, coproducts, and infinite compositions, whence all cofibrations are degree-wise injective. Conversely, a degree-wise injective map can be built from boundary inclusions by successively adding simplices. This successive procedures needs pushouts (to add new simplices), coproducts (to add arbitrarily many simplices at once), and infinite compositions. □

3.9. Lemma. — *If $A \to B$ is anodyne and $A' \to B'$ is a cofibration, then*

$$A \times B' \sqcup_{A \times A'} B \times A' \longrightarrow B \times B'$$

is anodyne again. Analogous assertions are true for left/right/inner anodyne maps.

Proof sketch. Fix $A' \to B'$ and consider the class Σ of all morphisms $A \to B$ for which $A \times B' \sqcup_{A \times A'} B \times A' \to B \times B'$ is anodyne. Then Σ is easily checked to be saturated. Hence it suffices to consider the case where $A \to B$ is a horn inclusion $\Lambda_i^n \to \Delta^n$. By the same argument, we can reduce to the case where $A' \to B'$ is a boundary inclusion $\partial \Delta^m \to \Delta^m$. So it suffices to check that $\Lambda_i^n \times \Delta^m \sqcup_{\Lambda_i^n \times \partial \Delta^m} \Delta^n \times \partial \Delta^m \to \Delta^n \times \Delta^m$ is anodyne. This can be done by hand, explicitly writing said map as a sequence of horn inclusions. For a complete proof in all its gory details, see [Lan21, Lemma 1.3.31]. □

3.10. Corollary. — *If $i \colon A \to B$ is a cofibration and $f \colon X \to Y$ is a Kan fibration, then*

$$\mathrm{F}(B, X) \longrightarrow \mathrm{F}(B, Y) \times_{\mathrm{F}(A, Y)} \mathrm{F}(A, X)$$

is a Kan fibration. If $i \colon A \to B$ is anodyne, then the map above is even a trivial fibration. Analogous conclusions are true for left/right/inner fibrations and left/right/inner anodyne cofibrations.

Proof sketch. By playing around with the universal properties of pushouts and pullbacks as well as the adjunction from 2.6, we find that the following lifting problems are equivalent:

$$
\begin{array}{ccc}
\Lambda_i^n & \longrightarrow & \mathrm{F}(B, X) \\
\downarrow & \nearrow & \downarrow \\
\Delta^n & \longrightarrow & \mathrm{F}(B, Y) \times_{\mathrm{F}(A, Y)} \mathrm{F}(A, X)
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
\Lambda_i^n \times B \sqcup_{\Lambda_i^n \times A} \Delta^n \times A & \longrightarrow & X \\
\downarrow & \nearrow & \downarrow f \\
\Delta^n \times B & \longrightarrow & Y
\end{array}
$$

Since $\Lambda_i^n \times B \sqcup_{\Lambda_i^n \times A} \Delta^n \times A \to \Delta^n \times B$ is anodyne by Lemma 3.9 and $f \colon X \to Y$ has lifting against all anodyne maps by Lemma 3.5, the lifting problem on the right can be solved, proving that $\mathrm{F}(B, X) \to \mathrm{F}(B, Y) \times_{\mathrm{F}(A, Y)} \mathrm{F}(A, X)$ indeed has lifting against all horn inclusions. If $A \to B$ is anodyne, then the same argument shows that we even get lifting against all boundary inclusions. The other assertions are entirely analogous. □

3.11. Corollary. — *Let X be a Kan complex, $\mathcal{C}$ a quasi-category, and B an arbitrary simplicial set. Then $\mathrm{F}(B, X)$ is a Kan complex and $\mathrm{F}(B, \mathcal{C})$ is a quasi-category. In particular, $\mathrm{Ar}(\mathcal{C})$ is a quasi-category again, and if $x \in \mathcal{C}$ is an object, then the slice $\mathcal{C}_{x/}$ from 2.11 is a quasi-category too.*

Proof. For the first two assertions, apply Corollary 3.10 to the cofibration $\emptyset \to B$ and the Kan fibration $X \to *$ or the inner fibration $\mathcal{C} \to *$, respectively. The assertion about $\mathrm{Ar}(\mathcal{C})$ is just the case $B = \Delta^1$. Finally, for $\mathcal{C}_{x/}$ we use that $(s, t) \colon \mathrm{Ar}(\mathcal{C}) \to \mathcal{C} \times \mathcal{C}$ is an inner fibration by Corollary 3.10 applied to the cofibration $\partial \Delta^1 \to \Delta^1$. Hence its pullback $\mathcal{C}_{x/} \to \{x\} \times \mathcal{C}$ must be an inner fibration too and so $\mathcal{C}_{x/}$ is a quasi-category by Example 3.3. □

We conclude this subsection with an immensely useful lemma.

3.12. Lemma ("Quillen's small object argument"). — *Every morphism of simplicial sets $f\colon X \to Y$ can be factored as*

$$f\colon X \xrightarrow{\ i\ } \overline{X} \xrightarrow{\ \overline{f}\ } Y\,,$$

where i is anodyne and $\overline{f}$ is a Kan fibration. Similarly, every morphism of simplicial sets can be factored into a left/right/inner anodyne map followed by a left/right/inner fibration, and also into a cofibration followed by a trivial fibration.

Proof. We only prove the first assertion; the others are completely analogous. Let

$$\Sigma(f) := \left\{ \sigma = \begin{array}{ccc} \Lambda_i^n & \longrightarrow & X \\ \downarrow & /\!/\!/ & \downarrow f \\ \Delta^n & \longrightarrow & Y \end{array} \,\middle|\, n \geqslant 1,\, 0 \leqslant i \leqslant n \right\}$$

and consider the simplicial set $S(f)$ defined as the pushout

$$
\begin{array}{ccc}
\coprod\limits_{\sigma \in \Sigma(f)} \Lambda_i^n & \longrightarrow & X \\
\downarrow & \ulcorner. & \downarrow \\
\coprod\limits_{\sigma \in \Sigma(f)} \Delta^n & \longrightarrow & S(f)
\end{array}
$$

Then $X \to S(f)$ is anodyne, because it is a pushout of a coproduct of horn inclusions, and f factors as $f\colon X \to S(f) \to Y$. Let $X_0 := X$ and $f_0 := f$. Inductively putting $X_{n+1} := S(f_n)$, we get factorisations

$$f\colon X \longrightarrow X_n \xrightarrow{\ f_n\ } Y$$

for all $n \geqslant 0$, where $X \to X_n$ is anodyne. Now let $\overline{X} := \operatorname{colim}_{n \geqslant 0} X_n$ and let $\overline{f}\colon \overline{X} \to Y$ be the induced map. Since anodyne maps are closed under infinite compositions, $X \to \overline{X}$ is anodyne. So it suffices to show that $\overline{f}$ is a Kan fibration. Note that Λ_i^n is built from finitely many simplices, which are in turn glued along finitely many subsimplices. Hence, for every map $\sigma\colon \Lambda_i^n \to \overline{X}$, each of these finitely many simplices must occur at some finite stage of the colimit $\overline{X} := \operatorname{colim}_{n \geqslant 0} X_n$, and each gluing condition must be satisfied at some finite stage. Consequently, every $\sigma\colon \Lambda_i^n \to \overline{X}$ must factor through $X_m \to \overline{X}$ for $m \gg 0$. Consequently, by construction of X_{m+1}, every lifting problem involving σ can be solved as

$$
\begin{array}{ccccc}
\Lambda_i^n & \longrightarrow & X_m \longrightarrow X_{m+1} & \longrightarrow & \overline{X} \\
\downarrow & \nearrow & & & \downarrow \overline{f} \\
\Delta^n & \longrightarrow & & \longrightarrow & Y
\end{array}
$$

which proves that $\overline{f}\colon \overline{X} \to Y$ is a Kan fibration, as desired. $\qquad\square$

§3.2. Homotopy groups

The goal of this subsection is to introduce homotopy groups of Kan complexes (Construction 3.15) and to prove an analogue of Whitehead's theorem (Theorem 3.18). We start noting that the naive definition of homotopies from the beginning of §3 works fine if X is Kan.

3.13. Definition. — Let X be a Kan complex.

(a) We say that $x, y \in X$ *belong to the same connected component* and write $x \simeq y$ if there is a 1-simplex $\Delta^1 \to X$ from x to y. By Theorem 2.19, this is an equivalence relation and the notation is compatible with 2.15. We let $\pi_0(X) := X_0 / \simeq$ denote the set of *connected components of X*.

(b) Let A be an arbitrary simplicial set. We say that $f, g \colon A \to X$ are *homotopic* and write $f \simeq g$ if and only if they belong to the same connected component of $\mathrm{F}(A, X)$, which is a Kan complex by Corollary 3.11. A *homotopy* $\eta \colon f \Rightarrow g$ is a 1-simplex $\Delta^1 \to \mathrm{F}(A, X)$ from f to g.

3.14. Construction. — Let $A \subseteq B$ be an inclusion of arbitrary simplicial sets and $X \subseteq Y$ be an inclusion of Kan complexes. Consider the following pullback (taken in sSet):

$$
\begin{array}{ccc}
\mathrm{F}\big((B, A), (Y, X)\big) & \longrightarrow & \mathrm{F}(B, Y) \\
\downarrow & \lrcorner & \downarrow \\
\mathrm{F}(A, X) & \longrightarrow & \mathrm{F}(A, Y)
\end{array}
$$

Note that $\mathrm{F}(B, Y) \to \mathrm{F}(A, Y)$ is a Kan fibration by Corollary 3.10 and $\mathrm{F}(A, X)$ is a Kan complex by Corollary 3.11. Therefore $\mathrm{F}((B, A), (Y, X))$ is a Kan complex too.

3.15. Construction. — Let X be a Kan complex, $x \in X$ a point, and $n \geqslant 1$. Furthermore, recall from Construction 2.21 that we use $\square^n$ and $\partial\square^n$ to denote the n-cube $(\Delta^1)^n$ and its boundary $\bigcup_{i=1}^n \square^{i-1} \times (\{0\} \sqcup \{1\}) \times \square^{n-i}$. We define the nth *homotopy group of X with basepoint x* as

$$
\pi_n(X, x) := \pi_0 \, \mathrm{F}\big((\square^n, \partial\square^n), (X, x)\big).
$$

As the name suggests, $\pi_n(X, x)$ should be a group, so let's construct a group operation! Given elements $[\alpha], [\beta] \in \pi_n(X, x)$, represented by maps of pairs $\alpha, \beta \colon (\square^n, \partial\square^n) \to (X, x)$, we can define a map $(\alpha, \beta) \colon \Lambda_1^2 \times \square^{n-1} \to X$ by $(\alpha, \beta)|_{\Delta^{\{0,1\}} \times \square^{n-1}} := \alpha$ and $(\alpha, \beta)|_{\Delta^{\{1,2\}} \times \square^{n-1}} := \beta$; this is possible since α and β agree on the "overlap" $\{1\} \times \square^{n-1}$, as they're both equal to $\mathrm{const}\, x$ there. Now consider the extension problem

$$
\begin{array}{ccc}
\Lambda_1^2 \times \square^{n-1} \sqcup_{\Lambda_1^2 \times \partial\square^{n-1}} \Delta^2 \times \partial\square^{n-1} & \xrightarrow{(\alpha, \beta) \cup \mathrm{const}\, x} & X \\
\downarrow & {\scriptstyle \vartheta} \nearrow & \\
\Delta^2 \times \square^{n-1} & &
\end{array}
$$

Since the vertical arrow is anodyne by Lemma 3.9, this extension problem has a solution ϑ. By construction, $\vartheta|_{\partial(\Delta^{\{0,2\}} \times \square^{n-1})} = \mathrm{const}\, x$. We then define $[\alpha] \cdot [\beta] := [\vartheta|_{\Delta^{\{0,2\}} \times \square^{n-1}}]$.

3.16. Remark. — Let us explain how Construction 3.15 is related to the usual construction of the group structure on $\pi_n(X, x)$ from topology, as this nicely illustrates the "rigidity" of simplicial sets and how said rigidity is overcome by the Kan condition. First, observe that $\Lambda_1^2 \times \square^{n-1} \cong \Delta^{\{0,1\}} \times \square^{n-1} \sqcup_{\{1\} \times \square^{n-1}} \Delta^{\{1,2\}} \times \square^{n-1}$ is simply given by "stacking one cube on top of another". In topological spaces, we can identify two stacked cubes with another cube, which immediately yields the group operation. In simplicial sets, this identification no longer works; in fact, there isn't even a suitable map $\Delta^1 \times \square^{n-1} \to \Lambda_1^2 \times \square^{n-1}$. But instead we can use the zigzag $\Delta^{\{0,2\}} \times \square^{n-1} \to \Delta^2 \times \square^{n-1} \leftarrow \Lambda_1^2 \times \square^{n-1}$, thanks to the Kan condition.

3.17. Lemma. — *Let X be a Kan complex, let $x \in X$ be a point, and let $n \geqslant 1$.*

(a) *The operation $\cdot$ from Construction 3.15 is well-defined (that is, independent of the choices of α, β, and ϑ) and defines a group structure on $\pi_n(X, x)$.*

(b) *Let $n \geqslant 2$. By "permuting the coordinates of the cube $\square^n$" we obtain operations $\cdot_1, \cdot_2, \ldots, \cdot_n$, where $\cdot_1 = \cdot$. Then these operations all coincide and are commutative.*

Proof sketch. All assertions in (a) can be proved by solving extension problems of the form

$$(*) \qquad \begin{array}{ccc} A \times \square^{n-1} \sqcup_{A \times \partial\square^{n-1}} B \times \partial\square^{n-1} & \longrightarrow & X \\ \downarrow & \nearrow & \\ B \times \square^{n-1} & & \end{array}$$

where $A \to B$ is anodyne (so that a solution always exists by Lemma 3.9).[(3.1)]

Let's start with independence of the choice of ϑ. So let ϑ' be another choice. Using the same idea as in 2.12, we can pose an extension problem $(*)$, with $(A \to B) = (\Lambda_1^3 \to \Delta^3)$. Restricting any solution to $\Delta^{\{0,2,3\}} \times \square^{n-1}$ yields a homotopy of pairs $\vartheta|_{\Delta^{\{0,2\}} \times \square^{n-1}} \simeq \vartheta'|_{\Delta^{\{0,2\}} \times \square^{n-1}}$. To show that the choices of α and β don't matter, suppose we're given homotopies of pairs $\alpha \simeq \alpha'$ and $\beta \simeq \beta'$ and let $[\alpha'] \cdot [\beta'] = [\vartheta'|_{\Delta^{\{0,2\}} \times \square^{n-1}}]$. Using the given homotopies, we can write down an extension problem $(*)$, with $(A \to B) = (\Delta^1 \times \Lambda_1^2 \to \Delta^1 \times \Delta^2)$. Restricting to $\Delta^1 \times \Delta^{\{0,2\}} \times \square^{n-1}$ yields a homotopy of pairs $\vartheta|_{\Delta^{\{0,2\}} \times \square^{n-1}} \simeq \vartheta'|_{\Delta^{\{0,2\}} \times \square^{n-1}}$. This shows well-definedness. To show associativity, choose $(A \to B) = (\Delta^{\{0,1\}} \cup \Delta^{\{1,2\}} \cup \Delta^{\{2,3\}} \to \Delta^3)$. A neutral element is $\mathrm{const}\, x \colon (\square^n, \partial\square^n) \to (X, x)$; to show $[\alpha] \cdot [\mathrm{const}\, x] = [\alpha] = [\mathrm{const}\, x] \cdot [\alpha]$ for all α, simply solve the corresponding lifting problem via degenerate simplices. Finally, to construct inverses, we take inspiration from 2.15 and write down extension problems $(*)$ with $(A \to B) = (\Lambda_0^2 \to \Delta^2)$ and $(A \to B) = (\Lambda_2^2 \to \Delta^2)$ to construct a left and a right inverse. This finishes the proof sketch of (a).

For (b), we use the *Eckmann–Hilton trick*: We can show simultaneously that $\cdot_1 = \cdot_2$ and that both operations are commutative by verifying the single identity

$$\big([\alpha] \cdot_1 [\beta]\big) \cdot_2 \big([\alpha'] \cdot_1 [\beta']\big) = \big([\alpha] \cdot_2 [\alpha']\big) \cdot_1 \big([\beta] \cdot_2 [\beta']\big)$$

for all α, α', β, and β'![(3.2)] To show the Eckmann–Hilton identity, consider the extension problem

$$\begin{array}{ccc} \Lambda_1^2 \times \Lambda_1^2 \times \square^{n-2} \sqcup_{\Lambda_1^2 \times \Lambda_1^2 \times \partial\square^{n-2}} \Delta^2 \times \Delta^2 \times \partial\square^{n-2} & \xrightarrow{((\alpha,\beta),(\alpha',\beta')) \cup \mathrm{const}\, x} & X \\ \downarrow & \nearrow & \\ \Delta^2 \times \Delta^2 \times \square^{n-1} & \quad \rho & \end{array}$$

which has a solution by Lemma 3.9. Then observe that for any solution ρ, both sides of the Eckmann–Hilton identity are given by $[\rho|_{\Delta^{\{0,2\}} \times \Delta^{\{0,2\}} \times \square^{n-2}}]$. $\qquad\square$

3.18. Theorem ("Whitehead's theorem for Kan complexes"). — *Let $f \colon X \to Y$ be a morphism of Kan complexes. Then f is a homotopy equivalence if and only if it induces a bijection $\pi_0(X) \cong \pi_0(Y)$ and isomorphisms $\pi_n(X, x) \cong \pi_n(Y, f(x))$ for all $x \in X$ and all $n \geqslant 1$.*

[(3.1)] Also note that the diagram from Construction 3.15 is of this form too, with $(A \to B) = (\Lambda_1^2 \to \Delta^2)$

[(3.2)] If you haven't seen this trick before, it will probably blow your mind.

The proof of Theorem 3.18 will occupy the rest of this subsection. The first step is an analogue of the long exact sequence of a Serre fibration.

3.19. Lemma ("Long exact sequence of a fibration"). — *Let $f \colon X \to Y$ be a Kan fibration between Kan complexes. Let $x \in X$, let $y := f(x)$ be its image, and let $F := f^{-1}\{y\} = \{y\} \times_Y X$ be the fibre over y. Then there exists a long exact sequence of groups/pointed sets*

$$\cdots \longrightarrow \pi_n(F,x) \longrightarrow \pi_n(X,x) \longrightarrow \pi_n(Y,y) \overset{\partial}{\longrightarrow} \pi_{n-1}(F,x) \longrightarrow \cdots$$

$$\cdots \longrightarrow \pi_1(X,x) \longrightarrow \pi_1(Y,y) \overset{\partial}{\longrightarrow} \pi_0(F) \longrightarrow \pi_0(X) \longrightarrow \pi_0(Y) \, .$$

In low degrees, exactness means the following:

(a) *There is an action $\pi_1(Y,y) \times \pi_0(F) \to \pi_0(F)$ in such a way that the boundary map $\partial \colon \pi_1(Y,y) \to \pi_0(F)$ is given by acting on $[x] \in \pi_0(F)$, the stabiliser of $[x]$ is precisely the image of $\pi_1(X,x) \to \pi_1(Y,y)$, and two elements of $\pi_0(F)$ map to the same element in $\pi_0(X)$ if and only if they lie in the same orbit of the $\pi_1(Y,y)$-action.*

(b) *An element in $\pi_0(X)$ maps to the class $[y] \in \pi_0(Y)$ if and only if it lies in the image of $\pi_0(F) \to \pi_0(X)$.*

Proof sketch. You can take any proof of the long exact sequence of a Serre fibration, like [Hat02, Theorem 4.41] and adapt the arguments to the simplicial setting. To illustrate how this can be done, we'll explain how to construct the boundary map ∂. So let $[\alpha] \in \pi_{n+1}(Y,y)$, where $\alpha \colon (\square^{n+1}, \partial\square^{n+1}) \to (Y,y)$ is a map of pairs as usual. Consider the lifting problem

$$
\begin{array}{ccc}
\{0\} \times \square^n \sqcup_{\{0\} \times \square^n} \Delta^1 \times \partial\square^n & \xrightarrow{\text{const } x} & X \\
\Big\downarrow & \overset{\vartheta}{\nearrow} & \Big\downarrow f \\
\Delta^1 \times \square^n & \xrightarrow{\ \alpha\ } & Y
\end{array}
$$

which has a solution ϑ by Lemma 3.9. Then $\vartheta|_{\{1\} \times \square^n} \colon \{1\} \times \square^n \to X$ factors through $F \to X$ and it maps $\{1\} \times \partial\square^n$ to x. Thus we can define $\partial[\alpha] := [\vartheta|_{\{1\} \times \square^n}] \in \pi_n(F,x)$. $\qquad\square$

3.20. Remark. — We will often use Lemma 3.19 in conjunction with the five lemma to deduce that a map of Kan complexes induces a bijection on π_0 and isomorphisms on π_n for all basepoints and all $n \geqslant 1$ (and is thus a homotopy equivalence by Theorem 3.18). But the five lemma only applies for exact sequences of groups, not pointed sets. However, these arguments can be saved using the group action from Lemma 3.19(a). We will usually skip the verification in low degrees and just cite the five lemma.

We also need an alternative description of homotopy groups. This is how Goerss and Jardine define them in [GJ99, §I.7]; we chose the cubical approach since it makes the group multiplication easier to visualise.

3.21. Lemma. — *Let X be a Kan complex, let $x \in X$, and let $n \geqslant 0$. Then there is a bijection*

$$\pi_n(X,x) \cong \pi_0 \, \mathrm{F}\big((\Delta^n, \partial\Delta^n), (X,x)\big) \, .$$

Proof sketch. By cutting out a single n-simplex from $\square^n$, we can obtain a sub-simplicial set $C^n \subseteq \square^n$ such that $\partial\square^n \subseteq C^n$ is anodyne (in fact, it can be obtained by successively filling

horns) and $\square^n/C^n \cong \Delta^n/\partial\Delta^n$. For example, in the case $n = 2$, we can choose C^n as in the following picture:

$$\partial\square^2 \qquad C^2 \qquad \square^2$$
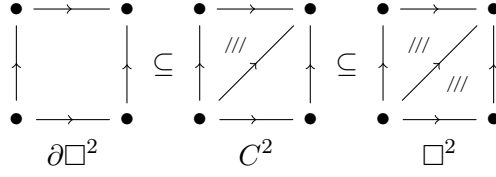
In general, the n-simplex $\Delta^n \to \square^n$ that we cut out to obtain C^n sends the vertex $\{i\}$ to the vertex $\{1\}^i \times \{0\}^{n-i}$.

Now observe that since x is just a point, we have $\mathrm{F}((B, A), (X, x)) \cong \mathrm{F}((B/A, *), (X, x))$ for every inclusion $A \subseteq B$ of simplicial sets. Since $\square^n/C^n \cong \Delta^n/\partial\Delta^n$, we only need to prove $\pi_0 \mathrm{F}((\square^n, \partial\square^n), (X, x)) \cong \pi_0 \mathrm{F}((\square^n, C^n), (X, x))$. This follows from a more general claim:

($\boxtimes$) *Let $A' \subseteq A$ be anodyne and let $A \subseteq B$ be any inclusion of simplicial sets. Then we have a bijection $\pi_0 \mathrm{F}((B, A), (X, x)) \cong \pi_0 \mathrm{F}((B, A'), (X, x))$.*

To prove ($\boxtimes$), put $F \coloneqq \mathrm{F}((B, A), (X, x))$ and $F' \coloneqq \mathrm{F}((B, A'), (X, x))$ for short. Consider the pullback $P \coloneqq \mathrm{F}(A', \{x\}) \times_{\mathrm{F}(A', X)} \mathrm{F}(A, X)$. Then $F' \to P$ is a Kan fibration, since it is a pullback of $\mathrm{F}(B, X) \to \mathrm{F}(A, X)$, which is Kan by Corollary 3.10. Manipulating pullbacks, we find $F \cong \mathrm{F}(A, \{x\}) \times_P F'$. Now $\mathrm{F}(A, \{x\}) \cong *$ is just a point, so F is a fibre of the Kan fibration $F' \to P$. Furthermore, $P \to \mathrm{F}(A', \{x\}) \cong *$ is a trivial fibration, since it is a pullback of $\mathrm{F}(A, X) \to \mathrm{F}(A', X)$, which is a trivial fibration by Corollary 3.10. By Lemma 3.22 below, this means that $P \to *$ is a homotopy equivalence and so all homotopy groups of P vanish. Using the long exact sequence from Lemma 3.19 (for every basepoint in F'; a single basepoint won't suffice), we can conclude $\pi_0(F) \cong \pi_0(F')$, as claimed. Note that this works even though we only have an exact sequence of pointed sets on π_0. This finishes the proof. Another proof of (a more general version of) ($\boxtimes$) is in [F-HC$_\mathrm{I}$, Lemma V.3.13]. $\qquad\square$

3.22. Lemma. — *If $f\colon X \to Y$ is a trivial fibration between Kan complexes*[(3.3)], *then f is a homotopy equivalence. Similarly, if $F\colon \mathcal{C} \to \mathcal{D}$ is a trivial fibration between quasi-categories, then F is an equivalence as in Example 2.23.*

Proof. Since trivial fibrations have lifting against all cofibrations, we can use the lifting problems

to first construct a map $g\colon Y \to X$ such that $f \circ g = \mathrm{id}_Y$ and then to construct a homotopy $\eta\colon g \circ f \Rightarrow \mathrm{id}_X$. Similarly, if $F\colon \mathcal{C} \to \mathcal{D}$ is a trivial fibration between quasi-categories, we get a functor $G\colon \mathcal{D} \to \mathcal{C}$ such that $F \circ G = \mathrm{id}_\mathcal{D}$ and a natural transformation $\eta\colon G \circ F \Rightarrow \mathrm{id}_\mathcal{C}$. To show that η is an equivalence in $\mathrm{F}(\mathcal{C}, \mathcal{C})$ (and thus prove that F and G are mutually inverse equivalences of quasi-categories), we lift furthermore against $\Delta^1 \times \mathcal{C} \to \mathrm{N}(J) \times \mathcal{C}$, where $J \coloneqq \{\bullet \rightleftarrows \bullet\}$ is the "free-living isomorphism", the category with two objects and a pair of mutually inverse isomorphisms between them. $\qquad\square$

[(3.3)] The only reason why we restrict ourselves to Kan complexes (or quasi-categories) is that we haven't defined what a homotopy equivalence of arbitrary simplicial sets would be. The lifting problems in the proof can be solved for arbitrary trivial fibrations $f\colon X \to Y$, with no assumptions on X or Y.

The crucial step in the proof of Theorem 3.18 is to show a "compression lemma", as in the proof of Whitehead's theorem in topology (compare to [Hat02, Lemma 4.6]).

3.23. Lemma ("Compression lemma"). — *Let X be a connected*[(3.4)] *Kan complex such that $\pi_n(X, x) \cong 0$ for all $x \in X$ and all $n \geqslant 1$. Let $A \subseteq B$ be an inclusion of simplicial sets, $f \colon B \to X$ a map and $\eta \colon \Delta^1 \times A \to X$ a homotopy from $f|_A$ to $\mathrm{const}\, x$. Then η can be extended to a homotopy $\bar\eta \colon \Delta^1 \times B \to X$ from f to $\mathrm{const}\, x$.*

Proof. We can construct $\bar\eta$ simplex by simplex, so it suffices to treat the case $A = \partial\Delta^n$ and $B = \Delta^n$ for some $n \geqslant 0$. Consider the extension problem

$$\{0\} \times \Delta^n \sqcup_{\{0\}\times\partial\Delta^n} \Delta^1 \times \partial\Delta^n \xrightarrow{(f,\,\eta)} X$$
$$\downarrow \qquad\qquad {}^{\vartheta}$$
$$\Delta^1 \times \Delta^n$$

which has a solution ϑ by Lemma 3.9. Then $\vartheta|_{\{1\}\times\partial\Delta^n} = \mathrm{const}\, x$, hence $\vartheta|_{\{1\}\times\Delta^n}$ defines an element in $\pi_0\,\mathrm{F}((\Delta^n, \partial\Delta^n), (X, x)) \cong \pi_n(X, x) \cong 0$ using Lemma 3.21 and our assumption on X. Hence there is a homotopy $\vartheta' \colon \Delta^1 \times \Delta^n \to X$ such that $\vartheta'|_{\Delta^1\times\partial\Delta^n} = \mathrm{const}\, x$ as well as $\vartheta'|_{\{0\}\times\Delta^n} = \vartheta|_{\{1\}\times\Delta^n}$ and $\vartheta'|_{\{1\}\times\Delta^n} = \mathrm{const}\, x$ (in other words, ϑ' is a homotopy $\vartheta|_{\{1\}\times\Delta^n} \Rightarrow \mathrm{const}\, x$ relative to the boundary $\partial\Delta^n$). To construct $\bar\eta$, we can now simply compose the homotopies ϑ and ϑ' (which we do similarly to Construction 3.15 and Remark 3.16, by solving an extension problem along the anodyne map $\Lambda^2_1 \times \Delta^n \to \Delta^2 \times \Delta^n$ and then restricting to $\Delta^{\{0,2\}} \times \Delta^n$). $\qquad\square$

3.24. Lemma. — *If X is a Kan complex as in Lemma 3.23, then $X \to *$ is a trivial fibration.*

Proof. We have to show that every extension problem of the following form is solvable:

$$\partial\Delta^n \xrightarrow{\sigma} X$$
$$\downarrow \qquad \nearrow$$
$$\Delta^n$$

Using Lemma 3.23 (applied to $A = \emptyset$ and $B = \partial\Delta^n$), there is a homotopy $\eta \colon \Delta^1 \times \partial\Delta^n \to X$ from σ to $\mathrm{const}\, x$. Now consider the extension problem

$$\Delta^1 \times \partial\Delta^n \sqcup_{\{1\}\times\partial\Delta^n} \{1\} \times \Delta^n \xrightarrow{(\eta,\,\mathrm{const}\,x)} X$$
$$\downarrow \qquad\qquad {}^{\vartheta}$$
$$\Delta^1 \times \Delta^n$$

which has a solution ϑ by Lemma 3.9 as usual. Then $\vartheta|_{\{0\}\times\Delta^n}$ provides a solution of the original extension problem. $\qquad\square$

3.25. Lemma. — *Let $f \colon X \to Y$ be a Kan fibration between Kan complexes and assume f satisfies the condition from Theorem 3.18. Then f is a trivial fibration.*

[(3.4)]That is, $\pi_0(X) = *$.

Proof. We have to show that every extension problem of the following form is solvable:

$$\begin{array}{ccc} \partial\Delta^n & \xrightarrow{\;\bar\sigma\;} & X \\ \downarrow & \nearrow & \downarrow{\scriptstyle f} \\ \Delta^n & \xrightarrow{\;\sigma\;} & Y \end{array}$$

Let $\eta\colon \Delta^1 \times \Delta^n \to \Delta^n$ be a homotopy from id_{Δ^n} to the constant map $\mathrm{const}\,n$ (such a homotopy can easily be constructed by hand). Then $\sigma \circ \eta\colon \Delta^1 \times \Delta^n \to Y$ is a homotopy from σ to $\mathrm{const}\,y$, where $y = \sigma(n)$. Now let $F := f^{-1}\{y\} = \{y\} \times_Y X$ be the fibre over y and consider the lifting problem

$$\begin{array}{ccc} \{0\} \times \partial\Delta^n & \xrightarrow{\quad\bar\sigma\quad} & X \\ \downarrow & \nearrow^{\vartheta} & \downarrow{\scriptstyle f} \\ \Delta^1 \times \partial\Delta^n & \xrightarrow{\;\sigma\circ\eta|_{\Delta^1 \times \partial\Delta^n}\;} & Y \end{array}$$

which can be solved by Lemma 3.9. Then $\vartheta|_{\{1\}\times\partial\Delta^n}\colon \{1\} \times \partial\Delta^n \to X$ factors through $F \to X$. Using the long exact sequence from Lemma 3.19 and the assumption on f, we see $\pi_n(F, x) \cong 0$ for all $x \in F$ and all $n \geqslant 0$. Hence $F \to *$ is a trivial fibration by Lemma 3.24 and so $\vartheta|_{\{1\}\times\partial\Delta^n}\colon \{1\} \times \partial\Delta^n \to F$ can be extended to a map $\bar\vartheta\colon \{1\} \times \Delta^n \to F$. Finally, consider the lifting problem

$$\begin{array}{ccc} \Delta^1 \times \partial\Delta^n \sqcup_{\{1\}\times\partial\Delta^n} \{1\} \times \Delta^n & \xrightarrow{\;(\vartheta,\bar\vartheta)\;} & X \\ \downarrow & \nearrow^{\rho} & \downarrow{\scriptstyle f} \\ \Delta^1 \times \Delta^n & \xrightarrow{\hspace{3cm}} & Y \end{array}$$

which can be solved by Lemma 3.9. Then $\rho|_{\{0\}\times\Delta^n}$ provides a solution for the original lifting problem and we're done. $\qquad\square$

Proof of Theorem 3.18. Let's first assume that $f\colon X \to Y$ is a homotopy equivalence. Then f clearly induces a bijection $\pi_0(X) \cong \pi_0(Y)$. But to get isomorphisms $\pi_n(X, x) \cong \pi_n(Y, y)$ for all $x \in X$, $y = f(x)$, and all $n \geqslant 1$, we have to show that $f\colon (X, x) \to (Y, y)$ is also a pointed homotopy equivalence, which is not entirely trivial.

It suffices to show that $\pi_0 \mathrm{F}((Y, y), (Z, z)) \to \pi_0 \mathrm{F}((X, x), (Z, z))$ is surjective for every pointed Kan complex (Z, z). Indeed, if this is true, then plugging in $(Z, z) = (X, x)$ yields a pointed map $g\colon (Y, y) \to (X, x)$ together with a pointed homotopy $g \circ f \simeq \mathrm{id}_{(X,x)}$. In particular, g is a homotopy equivalence too. Repeating the argument with g, we obtain $h\colon (X, x) \to (Y, y)$ together with $h \circ g \simeq \mathrm{id}_{(Y,y)}$. Then g is a pointed homotopy equivalence and thus f must be a pointed homotopy equivalence too. To show that $\pi_0 \mathrm{F}((Y, y), (Z, z)) \to \pi_0 \mathrm{F}((X, x), (Z, z))$ is surjective, first note that we have Kan fibrations

$$\mathrm{ev}_x\colon \mathrm{F}(X, Z) \longrightarrow \mathrm{F}(\{x\}, Z) \cong Z \quad\text{and}\quad \mathrm{ev}_y\colon \mathrm{F}(Y, Z) \longrightarrow \mathrm{F}(\{y\}, Z) \cong Z$$

by Corollary 3.10. Using Construction 3.14, we see that the fibres of these fibrations are given by $\mathrm{ev}_x^{-1}\{z\} \cong \mathrm{F}((X, x), (Z, z))$ and $\mathrm{ev}_y^{-1}\{z\} \cong \mathrm{F}((Y, y), (Z, z))$. Using Lemma 3.19, we obtain a diagram of exact sequences

$$\begin{array}{ccccccccc} \cdots \longrightarrow & \pi_1(Z, z) & \longrightarrow & \pi_0 \mathrm{F}((Y, y), (Z, z)) & \longrightarrow & \pi_0 \mathrm{F}(Y, Z) & \longrightarrow & \pi_0(Z) & \longrightarrow \cdots \\ & \| & & \downarrow & & \downarrow{\scriptstyle\cong} & & \| & \\ \cdots \longrightarrow & \pi_1(Z, z) & \longrightarrow & \pi_0 \mathrm{F}((X, x), (Z, z)) & \longrightarrow & \pi_0 \mathrm{F}(X, Z) & \longrightarrow & \pi_0(Z) & \longrightarrow \cdots \end{array}$$

Since we assume $f\colon X \to Y$ to be an unpointed homotopy equivalence, it induces a bijection $\pi_0 \mathrm{F}(Y, Z) \cong \pi_0 \mathrm{F}(X, Z)$, as indicated above. Then a diagram chase involving Lemma 3.19(a) shows that $\pi_0 \mathrm{F}((Y, y), (Z, z)) \to \pi_0 \mathrm{F}((X, x), (Z, z))$ is surjective. As argued above, this is what we need.

Conversely, assume that $f\colon X \to Y$ induces a bijection $\pi_0(X) \cong \pi_0(Y)$ and isomorphisms $\pi_n(X, x) \cong \pi_n(Y, y)$ for all $x \in X$, $y = f(x)$, and all $n \geqslant 1$. Lemma 3.12 allows us to choose a factorisation

$$f\colon X \xrightarrow{\ i\ } \overline{X} \xrightarrow{\ \overline{f}\ } Y$$

where i is anodyne and $\overline{f}$ is a Kan fibration. Then i and $\overline{f}$ are homotopy equivalences. Indeed, Corollary 3.10 shows that $\mathrm{F}(\overline{X}, Z) \to \mathrm{F}(X, Z)$ is a trivial fibration for every Kan complex Z, hence a bijection on π_0 by Lemma 3.22. Plugging in $Z = X$ and $Z = \overline{X}$ shows that i is a homotopy equivalence, as claimed. The Kan fibration $\overline{f}\colon \overline{X} \to Y$ is a trivial fibration by Lemma 3.25, hence a homotopy equivalence by Lemma 3.22. We are done! $\qquad\square$

§3.3. Simplicial approximation and model categories

3.26. Theorem (Simplicial approximation). — *For every Kan complex X we have a bijection $\pi_0(X) \cong \pi_0(|X|)$ and isomorphisms $\pi_n(X, x) \cong \pi_n(|X|, x)$ for all $x \in X$ and all $n \geqslant 1$. Similarly, for every topological space Y we have a bijection $\pi_0(Y) \cong \pi_0(\mathrm{Sing}\, Y)$ and isomorphisms $\pi_n(Y, y) \cong \pi_n(\mathrm{Sing}\, Y, y)$ for all $y \in Y$ and all $n \geqslant 1$. In particular, the adjunction*

$$|\cdot|\colon \mathrm{Kan} \rightleftarrows \mathrm{Top} \,{:}\, \mathrm{Sing}$$

from 2.4 induces homotopy equivalences $u_X\colon X \xrightarrow{\ \simeq\ } \mathrm{Sing}\, |X|$ for all Kan complexes X and weak equivalences $c_Y\colon |\mathrm{Sing}\, Y| \to Y$ for every topological space Y. $\qquad\blacksquare$

The proof of Theorem 3.26 is a technical headache. For a full proof, have a look at Fabian's and Christoph Winges' lecture notes [F-HC$_\mathrm{I}$, §V.5]; several versions of this theorem can also be found in [Hat02, §2.C].

Theorem 3.26 is an incarnation of Grothendieck's *homotopy hypothesis*. It tells us, essentially, that as long as we're only interested in topological spaces up to weak equivalence, or CW-complexes up to homotopy equivalence, we can safely pass to the category of Kan complexes, or better yet, to the quasi-category An from Example 2.23(a). In particular, everything we would ever like to know about homotopy groups (or homology groups etc.) will be captured by An! We'll see through many examples how this point of view leads to clean, abstract, and conceptually satisfying proofs of many classical topological results and ultimately to a deeper understanding of homotopy theory.

At this point it seems natural to leave a few words about *model categories*. Historically, these have played a dominating role in the development of ∞-category theory and to this day they are an indispensible tool in the foundations of the topic (especially in the proof of Lurie's straightening/unstraightening equivalence, Theorem 5.4) as well as in many other areas of topology. So dismissing them as a tool of the past would be blatantly ignorant and outright disrespectful. Still, model categories run contrary to the modern point of view that I'm trying to get across in these notes, and so I'll try to avoid them entirely—which is, of course, only achievable by conveniently hiding their unavoidable uses in black boxes. But at the very least, I should tell you the definition.

3.27. Definition. — Let $\mathcal{A}$ be a category with finite limits and colimits. A *model structure on $\mathcal{A}$* consists of 3 classes of morphisms C, F, and W (*cofibrations*, *fibrations*, and *weak equivalences*) satisfying the following properties:

(a) All isomorphisms of $\mathcal{A}$ are contained in each of the classes C, F, W, and these classes are all closed under retracts.

(b) W is closed under 2-out-of-3. That is, if two of f, g, and $g \circ f$ are weak equivalences, then so is the third.

(c) A lifting problem

$$
\begin{array}{ccc}
a & \longrightarrow & x \\
{\scriptstyle i}\downarrow & \nearrow & \downarrow{\scriptstyle f} \\
b & \longrightarrow & y
\end{array}
$$

with $i \in C$ a cofibration and $f \in F$ a fibration always has a solution provided that i is a *trivial cofibration* (a cofibration that is also a weak equivalence) or f is a *trivial fibration* (a fibration that is also a weak equivalence).

(d) Every morphism in $\mathcal{A}$ can be factored into a cofibration followed by a trivial fibration and into a trivial cofibration followed by a fibration. That is, if $a \to y$ is a morphism in $\mathcal{A}$, then there exist factorisations

$$ a \longrightarrow x \longrightarrow y \quad \text{and} \quad a \longrightarrow b \longrightarrow y \,, $$

where $(a \to x) \in C$ and $(x \to y) \in F \cap W$ as well as $(a \to b) \in C \cap W$ and $(b \to y) \in F$. Sometimes these factorisations are required to be functorial (which is satisfied in virtually all examples).

A category $\mathcal{A}$ equipped with a model structure is called a *model category*. If $\mathcal{A}$ is a model category, then $x \in \mathcal{A}$ is called *cofibrant* if the map from the initial object to x is a cofibration, and *fibrant* if the map from x to the terminal object is a fibration. We call x *bifibrant* if it is both fibrant and cofibrant.

3.28. Example. — Basically, the entirety of §3.1 can be summarised by saying that sSet carries a model structure in which cofibrations are exactly that, fibrations are Kan fibrations, and weak equivalences are morphisms that can be factored into an anodyne map followed by a trivial fibration. This model structure is called the *Kan–Quillen model structure.*

3.29. Example. — As the Kan–Quillen model structure "models" the quasi-category An from Example 2.23, it seems natural to ask whether there is another model structure on sSet that "models" Cat_∞. The naive attempt would be to ask that fibrations be inner fibrations and that trivial cofibrations be inner anodyne maps. But there are examples of cofibrations between quasi-categories that are equivalences in Cat_∞, but not inner anodyne; for example, the functor $\{0\} \to \mathrm{N}(J)$, where $J = \{\bullet \underset{\frown}{\overset{\frown}{}} \bullet\}$ is the "free-living isomorphism". It was an insight of Joyal how this can be fixed: There is a model structure on sSet, called the *Joyal model structure* such that cofibrations are just that and weak equivalences are those maps $A \to B$ such that $\pi_0 \, \mathrm{core} \, \mathrm{F}(B, \mathcal{C}) \to \pi_0 \, \mathrm{core} \, \mathrm{F}(A, \mathcal{C})$ is bijective for every quasi-category $\mathcal{C}$. Here $\pi_0 \, \mathrm{core}$ means the set of equivalence classes of objects; once we've proved the hard part of Theorem 2.19, this notation will be consistent with Definition 3.13(a). Weak equivalences in the Joyal model structure are called *Joyal equivalences*. We'll give another characterisation in Lemma 3.30 below.

Fibrant objects in the Joyal model structure are precisely quasi-categories. General fibrations in the Joyal model structure are harder to pin down. However, fibrations between quasi-categories are characterised by a lifting property: They are those inner fibrations that also have lifing against $\{0\} \to \mathrm{N}(J)$. We'll call these *isofibrations* and we'll meet them again in "Definition" 5.11(b). For proofs see [Joy08, Theorem 6.12] or [F-HC$_{\mathrm{II}}$, Theorem VIII.23].

3.30. Lemma. — *A map $A \to B$ of simplicial sets is a Joyal equivalence if and only if $\mathrm{F}(B,\mathcal{C}) \to \mathrm{F}(A,\mathcal{C})$ is an equivalence in Cat_∞ for every quasi-category $\mathcal{C}$.*

Proof. The "if" part is trivial, so assume $A \to B$ is a Joyal equivalence as in Example 3.29. By Lemma 3.12, we may choose an inner anodyne map $B \to \mathcal{B}$ into a quasi-category and a factorisation $A \to \mathcal{A} \to \mathcal{B}$ into an inner anodyne map followed by an inner fibration. Then $\mathcal{A}$ is a quasi-category too. Note that $\mathrm{F}(\mathcal{A},\mathcal{C}) \to \mathrm{F}(A,\mathcal{C})$ is a trivial fibration by Corollary 3.10 and thus an equivalence of quasi-categories by Lemma 3.22; the same is true for $\mathrm{F}(\mathcal{B},\mathcal{C}) \to \mathrm{F}(B,\mathcal{C})$. So it's enough to show that $\mathrm{F}(\mathcal{B},\mathcal{C}) \to \mathrm{F}(\mathcal{A},\mathcal{C})$ is an equivalence of quasi-categories, and for this, it's enough to show that our functor $F\colon \mathcal{A} \to \mathcal{B}$ is an equivalence of quasi-categories.

We know that $F^*\colon \pi_0 \operatorname{core} \mathrm{F}(\mathcal{B},\mathcal{C}) \to \pi_0 \operatorname{core} \mathrm{F}(\mathcal{A},\mathcal{C})$ is bijective for every quasi-category $\mathcal{C}$. Plugging in $\mathcal{C} = \mathcal{A}$ and choosing a preimage of $\mathrm{id}_\mathcal{A}$ yields a functor $G\colon \mathcal{B} \to \mathcal{A}$ together with an equivalence $G \circ F \simeq \mathrm{id}_\mathcal{A}$. Since F^* and $(G \circ F)^*$ are bijective, it follows that $G^*\colon \pi_0 \operatorname{core} \mathrm{F}(\mathcal{A},\mathcal{C}) \to \pi_0 \operatorname{core} \mathrm{F}(\mathcal{B},\mathcal{C})$ must too be bijective for every quasi-category $\mathcal{C}$. By the same argument, we obtain $H\colon \mathcal{A} \to \mathcal{B}$ together with an equivalence $H \circ G \simeq \mathrm{id}_\mathcal{B}$. Then G must be an isomorphism in $h(\mathrm{Cat}_\infty)$ and so F must be too. This shows that F is an equivalence in Cat_∞, as desired. □

3.31. Example. — There are also several model structures on Top. For example, there is the *Serre–Quillen model structure*, in which cofibrations are retracts of relative CW-inclusions, weak equivalences are just that, and fibrations are Serre fibrations.

An adjunction $L\colon \mathcal{A} \rightleftarrows \mathcal{B} \colon R$ between model categories is called a *Quillen adjunction* if the left adjoint L preserves cofibrations and trivial cofibrations, or equivalently, if the right adjoint R preserves fibrations and trivial fibrations. It is called a *Quillen equivalence* if, additionally, the following conditions hold:

(a) For every cofibrant object $x \in \mathcal{A}$ and every trivial cofibration $i\colon L(x) \to y$ into a fibrant object in $\mathcal{B}$, the composition $R(i) \circ u_x\colon x \to RL(x) \to R(y)$ is a weak equivalence in $\mathcal{A}$.

(b) For every fibrant object $y \in \mathcal{B}$ and every trivial fibration $f\colon x \to R(y)$ from a cofibrant object in $\mathcal{A}$, the composition $c_y \circ L(f)\colon L(x) \to LR(y) \to y$ is a weak equivalence in $\mathcal{B}$.

Then one way to understand Theorem 3.26 is that the adjunction $|\cdot|\colon \mathrm{sSet} \rightleftarrows \mathrm{Top} : \mathrm{Sing}$ from 2.4 is a Quillen equivalence.

§4. Joyal's lifting theorem

Let's begin by stating the theorem that this section owes its name to. We won't give a proof; the proof is not too difficult, at least compared to our later black box Theorem 5.4, but it uses some constructions (joins and thin slices) that we've avoided so far and will continue to avoid. If you're interested, Joyal's original proof [Joy02, Theorem 2.2] as well as the accounts in [Lan21, Theorem 2.1.8] or [L-Ker, Tag 01H0] are all very readable.

4.1. Theorem (Joyal's lifting theorem). — *Let $p\colon \mathcal{C} \to \mathcal{D}$ be an inner fibration of quasi-categories. Then for all $n \geqslant 2$, every lifting problem of the form*

$$
\begin{array}{ccc}
\Lambda_0^n \longrightarrow \mathcal{C} \\
\downarrow \quad \nearrow \quad \downarrow p \\
\Delta^n \longrightarrow \mathcal{D}
\end{array}
\quad or \quad
\begin{array}{ccc}
\Lambda_n^n \longrightarrow \mathcal{C} \\
\downarrow \quad \nearrow \quad \downarrow p \\
\Delta^n \longrightarrow \mathcal{D}
\end{array}
$$

in which the 1-simplex $\Delta^{\{0,1\}} \subseteq \Lambda_0^n \to \mathcal{C}$ or $\Delta^{\{n-1,n\}} \subseteq \Lambda_n^n \to \mathcal{C}$ is sent to an equivalence in $\mathcal{C}$, admits a solution. ∎

§4.1. Consequences of Joyal's lifting theorem

This subsection is devoted to convincing you what a ridiculously strong result Theorem 4.1 actually is. We begin with some simple corollaries and work our way up to two highly non-trivial theorems.

4.2. Corollary ("Animae and Kan complexes are the same"). — *A quasi-category $\mathcal{C}$ is a Kan complex if and only if it is an anima, that is, if and only if all its morphisms are equivalences.*

Proof. We've seen in Theorem 2.19 that Kan complexes are animae. So let's assume $\mathcal{C}$ is an anima. Since $\mathcal{C}$ is a quasi-category, it suffices to show that all outer horns $\Lambda_0^n \to \mathcal{C}$ and $\Lambda_n^n \to \mathcal{C}$ have fillers. For $n = 1$, this is clear, since we can extend $\{0\} \to \mathcal{C}$ or $\{1\} \to \mathcal{C}$ to a degenerate simplex $\Delta^1 \to \mathcal{C}$. For $n \geqslant 2$, we can apply Theorem 4.1 to the inner fibration $p\colon \mathcal{C} \to *$. □

4.3. Corollary ("Left fibrations over animae are Kan fibrations"). — *Let Y be a Kan complex and let $f\colon X \to Y$ be a left fibration. Then f is a Kan fibration and thus X is a Kan complex. A dual assertion holds for right fibrations.*

Proof sketch. We must show that for all $n \geqslant 1$ every lifting problem

$$
\begin{array}{ccc}
\Lambda_n^n \longrightarrow X \\
\downarrow \quad \nearrow \quad \downarrow f \\
\Delta^n \longrightarrow Y
\end{array}
$$

has a solution. Let's first consider the case $n = 1$. Since every morphism of Y is an equivalence by Corollary 4.2, the map $\Delta^1 \to Y$ extends to a map $\mathrm{N}(J) \to Y$, where $J := \{\bullet \rightleftarrows \bullet\}$ is the "free-living isomorphism", the category with two objects and a pair of mutually inverse isomorphisms between them. It can be shown via explicit horn filling that $\{1\} \to \mathrm{N}(J)$ is both left and right anodyne. Since f is a left fibration, we get a lift $\mathrm{N}(J) \to X$, which upon restriction along $\Delta^1 \to \mathrm{N}(J)$ yields a solution of our original lifting problem.

Now let $n \geqslant 2$. It suffices to show that every morphism in X is an equivalence, because then Theorem 4.1 will solve our lifting problem. So let $\alpha \colon x \to y$ be a morphism in X and consider the map $\sigma \colon \Lambda_0^2 \to X$ represented by

$$\sigma = \quad \begin{array}{c} x \\ {\scriptstyle \mathrm{id}_x} \nearrow \quad \raisebox{0.5ex}{$\scriptstyle ?$} \\ x \xrightarrow[\alpha]{} y \end{array}$$

Since Y is an anima, $f(\alpha)$ is an equivalence and so $\vartheta := f \circ \sigma \colon \Lambda_0^2 \to Y$ can be extended to a map $\overline{\vartheta} \colon \Delta^2 \to Y$. Since $f \colon X \to Y$ is a left fibration, we can lift $\overline{\vartheta}$ to a map $\overline{\sigma} \colon \Delta^2 \to X$ such that $\overline{\sigma}|_{\Lambda_0^2} = \sigma$ and $f \circ \overline{\sigma} = \overline{\vartheta}$. The 2-simplex $\overline{\sigma}$ shows that α has a left inverse β. Repeating the argument with β, we see that β itself has a left inverse. Then β must be an equivalence. Hence its right inverse α must be an equivalence too. $\qquad\square$

4.4. Corollary ("$\mathrm{Hom}_{\mathcal{C}}$ takes values in animae"). — *Let $\mathcal{C}$ be a quasi-category. Then for all $x, y \in \mathcal{C}$, the slice category projection $t \colon \mathcal{C}_{x/} \to \mathcal{C}$ from 2.11 is a left fibration and $\mathrm{Hom}_{\mathcal{C}}(x, y)$ is an anima.*

Proof sketch. By 2.11, $t \colon \mathcal{C}_{x/} \to \mathcal{C}$ is a pullback of $(s, t) \colon \mathrm{Ar}(\mathcal{C}) \to \mathcal{C} \times \mathcal{C}$, which is an inner fibration by Corollary 3.10. Hence $t \colon \mathcal{C}_{x/} \to \mathcal{C}$ is an inner fibration too and we only need to solve outer horn lifting problems

$$\begin{array}{ccc} \Lambda_0^n & \longrightarrow & \mathcal{C}_{x/} \\ \downarrow & \nearrow & \downarrow {\scriptstyle t} \\ \Delta^n & \longrightarrow & \mathcal{C} \end{array}$$

for all $n \geqslant 1$. Write $\mathcal{C}_{x/} \cong \{x\} \times_{\mathcal{C}, s} \mathrm{Ar}(\mathcal{C})$ as in 2.11. By the usual adjunction tricks, a horn lifting problem as above is equivalent to an extension problem

$$\begin{array}{ccc} \Lambda_0^n \times \Delta^1 \sqcup_{\Lambda_0^n \times \{1\}} \Delta^n \times \{1\} & \xrightarrow{f} & \mathcal{C} \\ \downarrow & \raisebox{-1ex}{$\scriptstyle \overline{f}$} \nearrow & \\ \Delta^n \times \Delta^1 & & \end{array}$$

with the additional condition that f satisfies $f|_{\Lambda_0^n \times \{0\}} = \mathrm{const}\, x$ and the extension must satisfy $\overline{f}|_{\Delta^n \times \{0\}} = \mathrm{const}\, x$. Such an extension problem can be written as a sequence of horn filling problems. Each horn is either an inner horn, which can be filled by Definition 2.8, or a horn whose first edge is sent to $\mathrm{const}\, x$, which can be filled by Theorem 4.1, or a horn that can be filled with a degenerate simplex. Up to the horn filling combinatorics, which we skip as usual, this proves that $t \colon \mathcal{C}_{x/} \to \mathcal{C}$ is a left fibration.

To prove that $\mathrm{Hom}_{\mathcal{C}}(x, y)$ is an anima, recall the pullback diagram

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{C}}(x, y) & \longrightarrow & \mathcal{C}_{x/} \\ \downarrow & \raisebox{0ex}{$\scriptstyle \lrcorner$} & \downarrow {\scriptstyle t} \\ \{y\} & \longrightarrow & \mathcal{C} \end{array}$$

from 2.11. It follows that $\mathrm{Hom}_{\mathcal{C}}(x, y) \to \{y\}$ is a left fibration. Hence $\mathrm{Hom}_{\mathcal{C}}(x, y)$ is a Kan complex by Corollary 4.3. $\qquad\square$

4.5. Theorem ("Equivalences of functors can be checked pointwise"). — *Let $F, G\colon \mathcal{C} \to \mathcal{D}$ be functors of quasi-categories and let $\eta\colon F \Rightarrow G$ be a natural transformation (that is, a 1-simplex $\Delta^1 \to \mathrm{F}(\mathcal{C}, \mathcal{D})$ from F to G). Then η is an equivalence of functors if and only if $\eta_x\colon F(x) \to G(x)$ is an equivalence in $\mathcal{D}$ for all $x \in \mathcal{C}$.*

If all $\eta_x\colon F(x) \to G(x)$ are equivalences, we can choose inverses $\vartheta_x\colon G(x) \to F(x)$, but already that step is non-canonical, since inverses are no longer unique in quasi-categories. To assemble the ϑ_x into a natural transformation $\vartheta\colon G \Rightarrow F$ involves infinitely more non-canonical choices, and we have to make them all in a coherent way. This is an impossible task to do by hand, but incredibly, Theorem 4.5 does it for us!

Proof sketch of Theorem 4.5. The "only if" part is clear. To prove the "if" part, we start with some general observations. Let $i\colon A \to B$ be a cofibration of simplicial sets and consider lifting problems of the form

$$
(*) \quad
\begin{array}{ccc}
\Lambda_0^2 & \longrightarrow & \mathrm{F}(B, \mathcal{D}) \\
\downarrow & \nearrow & \downarrow \\
\Delta^2 & \longrightarrow & \mathrm{F}(A, \mathcal{D})
\end{array}
\quad \text{or equivalently} \quad
(**) \quad
\begin{array}{c}
\Lambda_0^2 \times B \sqcup_{\Lambda_0^2 \times A} \Delta^2 \times A \longrightarrow \mathcal{D} \\
\downarrow \qquad\qquad\qquad \nearrow \\
\Delta^2 \times B
\end{array}
$$

Consider those $(**)$ for which the 1-simplex $\Delta^{\{0,1\}} \times \{b\} \to \Lambda_0^2 \times B \to \mathcal{D}$ is an equivalence in $\mathcal{D}$ for all $b \in B_0$. We claim:

($\boxtimes$) *Let Σ be the class of all cofibrations $i\colon A \to B$ such that every extension problem $(**)$, for which $\Delta^{\{0,1\}} \times \{b\} \to \Lambda_0^2 \times B \to \mathcal{D}$ is an equivalence in $\mathcal{D}$ for all $b \in B_0$, can be solved. Then Σ is saturated and contains $\partial\Delta^n \to \Delta^n$ for all $n \geqslant 0$.*

Saturatedness of Σ is straightforward to check. To see that Σ contains $\partial\Delta^n \to \Delta^n$, one uses Theorem 4.1; as usual, we skip the horn filling combinatorics. A full argument is in Fabian's notes [F-HC$_\mathrm{I}$, Lemma VII.2].

By ($\boxtimes$) and Lemma 3.8, Σ contains all cofibrations of simplicial sets. In particular, Σ contains $i\colon \coprod_{x \in \mathcal{C}} \{x\} \to \mathcal{C}$. Now let $\eta\colon F \Rightarrow G$ be a natural transformation such that $\eta_x\colon F(x) \to G(x)$ is an equivalence for all $x \in \mathcal{C}$. To construct a left inverse of η, consider the map $\sigma\colon \Lambda_0^2 \to \mathrm{F}(\mathcal{C}, \mathcal{D})$ represented by

$$
\sigma = \quad
\begin{array}{c}
F \\
{}^{\mathrm{id}_F}\nearrow \quad \nwarrow \\
F \underset{\eta}{\Longrightarrow} G
\end{array}
$$

Our assumption on η means that its image under $i^*\colon \mathrm{F}(\mathcal{C}, \mathcal{D}) \to \mathrm{F}(\coprod_{x \in \mathcal{C}} \{x\}, \mathcal{D}) \cong \prod_{x \in \mathcal{C}} \mathcal{D}$ is an equivalence. Hence $i^* \circ \sigma$ can be extended to a 2-simplex $\Delta^2 \to \mathrm{F}(\coprod_{x \in \mathcal{C}} \{x\}, \mathcal{D})$ and we obtain a lifting diagram

$$
\begin{array}{ccc}
\Lambda_0^2 & \longrightarrow & \mathrm{F}(\mathcal{C}, \mathcal{D}) \\
\downarrow & \nearrow & \downarrow \\
\Delta^2 & \longrightarrow & \mathrm{F}\left(\coprod_{x \in \mathcal{C}} \{x\}, \mathcal{D}\right)
\end{array}
$$

which has a solution by what the above arguments. Hence η has a left inverse $\vartheta\colon G \Rightarrow F$. Again, $\vartheta_x\colon G(x) \to F(x)$ must be equivalences for all $x \in \mathcal{C}$. Repeating the argument with

ϑ shows that ϑ must have a left inverse too. Then ϑ must be an equivalence and so its right inverse η must be an equivalence too. □

A similar miracle as Theorem 4.5 is the following theorem.

4.6. Theorem ("Fully faithful & essentially surjective implies equivalence"). — *A functor $F\colon \mathcal{C} \to \mathcal{D}$ of ∞-categories is an equivalence if and only if the following conditions are satisfied:*

(a) *F is fully faithful. That is, F induces homotopy equivalences of animae*

$$\operatorname{Hom}_{\mathcal{C}}(x, y) \xrightarrow{\simeq} \operatorname{Hom}_{\mathcal{D}}\big(F(x), F(y)\big)$$

for all $x, y \in \mathcal{C}$.

(b) *F is essentially surjective. That is, F induces a surjection $\pi_0 \operatorname{core}(\mathcal{C}) \to \pi_0 \operatorname{core}(\mathcal{D})$.*

4.7. Remark. — The "only if" part of Theorem 4.6 is easy. For later use, we remark that $F\colon \mathcal{C} \to \mathcal{D}$ being fully faithful implies that $\pi_0 \operatorname{core}(\mathcal{C}) \to \pi_0 \operatorname{core}(\mathcal{D})$ is injective. Indeed, this is purely an assertion about the homotopy categories of $\mathcal{C}$ and $\mathcal{D}$ and it follows from the fact that if $F\colon \mathcal{C} \to \mathcal{D}$ is a fully faithful functor of quasi-categories, then $h(F)\colon h(\mathcal{C}) \to h(\mathcal{D})$ is a fully faithful functor of ordinary categories. This in turn follows from the fact that $\operatorname{Hom}_{h(\mathcal{C})}(x, y) \cong \pi_0 \operatorname{Hom}_{\mathcal{C}}(x, y)$, which is straightforward to check from 2.13.

To prove the "if" part of Theorem 4.6, let's first consider the case where $\mathcal{C}$ and $\mathcal{D}$ are animae.

4.8. Lemma. — *Let $F\colon \mathcal{C} \to \mathcal{D}$ be a fully faithful and essentially surjective functor of animae. Then F is a homotopy equivalence.*

Proof. Since $\mathcal{C}$ and $\mathcal{D}$ are animae, we have $\mathcal{C} = \operatorname{core}(\mathcal{C})$ and $\mathcal{D} = \operatorname{core}(\mathcal{D})$. By Remark 4.7 and the fact that F is essentially surjective, we see that $\mathcal{C} \to \mathcal{D}$ is a bijection on path components. Hence we may assume without loss of generality that $\mathcal{C}$ and $\mathcal{D}$ are connected. Now choose $x \in \mathcal{C}$ and observe that

$$\pi_{n+1}(\mathcal{C}, x) \cong \pi_n\big(\operatorname{Hom}_{\mathcal{C}}(x, x), \operatorname{id}_x\big)$$

for all $n \geqslant 0$. Indeed, by the pullback square from 2.11, a map $(\square^n, \partial\square^n) \to (\operatorname{Hom}_{\mathcal{C}}(x, x), \operatorname{id}_x)$ is equivalently a morphism $\square^n \times \Delta^1 \to \mathcal{C}$ such that $\partial\square^n \times \Delta^1 \cup \square^n \times \{0, 1\} \to \mathcal{C}$ is constant on x. But that's just a map $(\square^{n+1}, \partial\square^{n+1}) \to (\mathcal{C}, x)$, as claimed.

Hence F being fully faithful implies that $\pi_{n+1}(\mathcal{C}, x) \cong \pi_{n+1}(\mathcal{D}, F(x))$ is an isomorphism for all $n \geqslant 0$. But then F is a homotopy equivalence by Theorem 3.18. □

Furthermore we need:

4.9. Lemma. — *Let $\mathcal{C}$ be a quasi-category and let $\mathcal{C}[S_1] \subseteq \mathcal{C}$ be a (not necessarily full) sub-quasi-category spanned by a collection $S_1 \subseteq \mathcal{C}_1$ of morphisms as in 2.16. Then for all $x, y \in \mathcal{C}$*

$$\operatorname{Hom}_{\mathcal{C}[S_1]}(x, y) \longrightarrow \operatorname{Hom}_{\mathcal{C}}(x, y)$$

is an equivalence onto the set of path components of morphisms from S_1.

Proof sketch. By unravelling 2.11, an n-simplex $\Delta^n \to \operatorname{Hom}_{\mathcal{C}}(x, y)$ is the same as a map $\sigma\colon \Delta^1 \times \Delta^n \to \mathcal{C}$ such that $\sigma|_{\{0\} \times \Delta^n} = \sigma|_{\{1\} \times \Delta^n} = \operatorname{const} x$. Then σ defines an n-simplex $\Delta^n \to \operatorname{Hom}_{\mathcal{C}[S_1]}(x, y)$ if and only if σ maps all morphisms in $\Delta^1 \times \Delta^n$ to S_1. Note that all morphisms in $\{0\} \times \Delta^n$ and $\{1\} \times \Delta^n$ are mapped to id_x, which is contained S_1 because we

assume that S_1 contains all identities. So it suffices to check that $\sigma|_{\Delta^1 \times \{i\}} \colon \Delta^1 \times \{i\} \to \mathcal{C}$ is contained in S_1 for all $i = 0, \ldots, n$, because all other morphisms in $\Delta^1 \times \Delta^n$ are generated under compositions by these as well as the morphisms in $\{0\} \times \Delta^n$ and $\{1\} \times \Delta^n$. This means that an n-simplex $\Delta^n \to \operatorname{Hom}_{\mathcal{C}}(x, y)$ belongs to $\operatorname{Hom}_{\mathcal{C}[S_1]}(x, y)$ if and only if all its vertices correspond to morphisms in S_1. In other words, $\operatorname{Hom}_{\mathcal{C}[S_1]}(x, y) \subseteq \operatorname{Hom}_{\mathcal{C}}(x, y)$ is the collection of path components of morphisms from S_1, as desired. $\qquad\square$

Proof sketch of Theorem 4.6. Assume F is fully faithful and essentially surjective. We'll show that $\operatorname{core} \mathrm{F}(K, \mathcal{C}) \to \operatorname{core} \mathrm{F}(K, \mathcal{D})$ is a homotopy equivalence of animae for all simplicial sets K (note that both $\mathrm{F}(K, \mathcal{C})$ and $\mathrm{F}(K, \mathcal{D})$ are indeed quasi-categories by Corollary 3.11). Once we have this, plugging in $K = \mathcal{D}$ yields a functor $G \colon \mathcal{D} \to \mathcal{C}$ with an equivalence $G \circ F \simeq \mathrm{id}_{\mathcal{C}}$. It's straightforward to see that G is again fully faithful and essentially surjective, so repeating the argument with G shows that G has a left inverse too. Then G must be an equivalence and so its right inverse F must be an equivalence too.

Case $K = *$. Since $\operatorname{core} \mathrm{F}(*, \mathcal{C}) \cong \operatorname{core}(\mathcal{C})$, we must show that $\operatorname{core}(\mathcal{C}) \to \operatorname{core}(\mathcal{D})$ is a homotopy equivalence of animae. By Lemma 4.9, $\operatorname{Hom}_{\operatorname{core}(\mathcal{C})}(x, y) \to \operatorname{Hom}_{\mathcal{C}}(x, y)$ is an equivalence onto those path components that correspond to equivalences from x to y. The same is true for $\mathcal{D}$, whence $\operatorname{core}(\mathcal{C}) \to \operatorname{core}(\mathcal{D})$ is fully faithful again. Clearly, it is essentially surjective too, so Lemma 4.8 shows that $\operatorname{core}(\mathcal{C}) \to \operatorname{core}(\mathcal{D})$ must be a homotopy equivalence.

Case $K = \Delta^n$, $n \geqslant 1$. Let $I^n := \bigcup_{i=1}^n \Delta^{\{i-1, i\}} \subseteq \Delta^n$ be the *n-spine*. It's straightforward to check that $I^n \to \Delta^n$ is inner anodyne, so $\mathrm{F}(\Delta^n, \mathcal{C}) \to \mathrm{F}(I^n, \mathcal{C})$ is a trivial fibration by Corollary 3.10. The same is true for $\mathcal{D}$. We may thus replace $K = \Delta^n$ by $K = I^n$. Now we claim:

($\boxtimes$) *If $i \colon A \to B$ is a cofibration of simplicial sets, then $i^* \colon \operatorname{core} \mathrm{F}(B, \mathcal{C}) \to \operatorname{core} \mathrm{F}(A, \mathcal{C})$ is a Kan fibration. Furthermore, for all $x_0, x_1, \ldots, x_n \in \mathcal{C}$, the following diagram is a pullback diagram of Kan complexes and its vertical arrows are Kan fibrations:*

$$
\begin{array}{ccc}
\operatorname{Hom}_{\mathcal{C}}(x_0, x_1) \times \cdots \times \operatorname{Hom}_{\mathcal{C}}(x_{n-1}, x_n) & \longrightarrow & \operatorname{core} \mathrm{F}(I^n, \mathcal{C}) \\
\downarrow & \ulcorner & \downarrow \\
\{x_0\} \times \cdots \times \{x_n\} & \longrightarrow & \operatorname{core}(\mathcal{C}) \times \cdots \times \operatorname{core}(\mathcal{C})
\end{array}
$$

We know that F induces a homotopy equivalence $(\operatorname{core}(\mathcal{C}))^{n+1} \simeq (\operatorname{core}(\mathcal{D}))^{n+1}$ by the case $K = *$. Furthermore, since we assume F to be fully faithful, we know that F induces homotopy equivalences $\prod_{j=1}^n \operatorname{Hom}_{\mathcal{C}}(x_{j-1}, x_j) \simeq \prod_{j=1}^n \operatorname{Hom}_{\mathcal{D}}(F(x_{j-1}), F(x_j))$. So if we believe ($\boxtimes$) (and its analogue for $\mathcal{D}$), then Lemma 3.19 plus the five lemma (plus Remark 3.20) show that $\operatorname{core} \mathrm{F}(I^n, \mathcal{C}) \to \operatorname{core} \mathrm{F}(I^n, \mathcal{D})$ induces a bijection on π_0 and isomorphisms on π_n for all basepoints and all $n \geqslant 1$. Hence $\operatorname{core} \mathrm{F}(I^n, \mathcal{C}) \to \operatorname{core} \mathrm{F}(I^n, \mathcal{D})$ must be a homotopy equivalence by Theorem 3.18.

To prove ($\boxtimes$), first note that $\mathrm{F}(B, \mathcal{C}) \to \mathrm{F}(A, \mathcal{C})$ is an inner fibration by Corollary 3.10. Furthermore, if $m \geqslant 2$ and $\sigma \colon \Lambda_j^m \to \operatorname{core} \mathrm{F}(B, \mathcal{C})$ is any m-dimensional horn (we allow $j = 0$ or $j = m$), then any m-simplex $\bar{\sigma} \colon \Delta^m \to \mathrm{F}(B, \mathcal{C})$ with $\bar{\sigma}|_{\Lambda_j^m} = \sigma$ is already contained in $\operatorname{core} \mathrm{F}(B, \mathcal{C})$. Indeed, equivalences in $\mathcal{C}$ are closed under 2-out-of-3, hence the edges of $\Delta^m \smallsetminus \Lambda_j^m$ will automatically be mapped to equivalences too. This observation immediately shows that $i^* \colon \operatorname{core} \mathrm{F}(B, \mathcal{C}) \to \operatorname{core} \mathrm{F}(A, \mathcal{C})$ is an inner fibration again. Furthermore, Theorem 4.1 shows that i^* has lifting against $\Lambda_0^m \to \Delta^m$ and $\Lambda_m^m \to \Delta^m$ for all $m \geqslant 2$. It remains to deal with the case $m = 1$, that is, to show lifting agains $\{0\} \to \Delta^1$ and $\{1\} \to \Delta^1$. Let's sketch how to prove

the former; the latter is analogous. Building B from A by successively attaching simplices, we can reduce to the case where $i \colon \partial\Delta^k \to \Delta^k$ is a simplex boundary inclusion. The case $k = 0$ is trivial. For $k \geqslant 1$, the map $\partial\Delta^k \to \Delta^k$ is bijective on 0-simplices and so it suffices to show that any extension problem

$$\partial\Delta^k \times \Delta^1 \sqcup_{\partial\Delta^k \times \{0\}} \Delta^k \times \{0\} \xrightarrow{\ \sigma\ } \mathcal{C}$$

$$\downarrow \qquad\qquad \overset{\overline{\sigma}}{\dashrightarrow}$$

$$\Delta^k \times \Delta^1$$

in which $\sigma|_{\{j\}\times\Delta^1} \colon \{j\} \times \Delta^1 \to \mathcal{C}$ is an equivalence in $\mathcal{C}$ for every 0-simplex $j \in (\partial\Delta^k)_0$, admits a solution. Indeed, it follows from Theorem 4.5 that any extension $\overline{\sigma} \colon \Delta^k \times \Delta^1 \to \mathcal{C}$ will automatically define a map $\Delta^1 \to \operatorname{core} \operatorname{F}(\Delta^k, \mathcal{C})$. To construct the desired extension, write it as a sequence of horn filling problems; each inner horn can be filled by Definition 2.8 and each outer horn by Theorem 4.1. As usual, we skip the horn filling combinatorics. This finishes the proof that $i^* \colon \operatorname{core} \operatorname{F}(B, \mathcal{C}) \to \operatorname{F}(A, \mathcal{C})$ is a Kan fibration.

Choosing i to be the cofibration $\{0\} \sqcup \cdots \sqcup \{n\} \to I^n$, we see that $\operatorname{core} \operatorname{F}(I^n, \mathcal{C}) \to (\operatorname{core}(\mathcal{C}))^{n+1}$ is indeed a Kan fibration. It remains to show that we get a pullback diagram. We can write I^n as an iterated pushout $I^n \cong \Delta^{\{0,1\}} \sqcup_{\{1\}} \cdots \sqcup_{\{n-1\}} \Delta^{\{n-1,n\}}$ and thus $\operatorname{F}(I^n, \mathcal{C})$ as an iterated pullback $\operatorname{F}(I^n, \mathcal{C}) \cong \operatorname{Ar}(\mathcal{C}) \times_{t,\mathcal{C},s} \cdots \times_{t,\mathcal{C},s} \operatorname{Ar}(\mathcal{C})$. Plugging in the definition of $\operatorname{Hom}_{\mathcal{C}}(x_{i-1}, x_i)$ from 2.11 yields the desired pullback diagram—except for one problem: The left vertical arrow reads $\operatorname{F}(I^n, \mathcal{C}) \to \mathcal{C}^{n+1}$ instead of $\operatorname{core} \operatorname{F}(I^n, \mathcal{C}) \to (\operatorname{core}(\mathcal{C}))^{n+1}$. To get core into the picture, observe that as a consequence of Corollary 4.2, $\operatorname{core} \colon \operatorname{QCat} \to \operatorname{Kan}$ is a right adjoint to the inclusion $\operatorname{Kan} \subseteq \operatorname{QCat}$. Hence core turns pullbacks in QCat into pullbacks in Kan. However, the pullbacks at hand are supposed to be taken in sSet, and in general it's not true that pullbacks in QCat or Kan coincide with those in sSet.[(4.1)] But if a pullback of quasi-categories, taken in sSet, happens to be a quasi-category again, then it's automatically a pullback in QCat too, and likewise for a pullback of Kan complexes that happens to be Kan again. Since we've seen that $\operatorname{F}(I^n, \mathcal{C}) \to \mathcal{C}^{n+1}$ is an inner fibration and $\operatorname{core} \operatorname{F}(I^n, \mathcal{C}) \to (\operatorname{core}(\mathcal{C}))^{n+1}$ is a Kan fibration, this is is true in our situation. So core preserves the pullback at hand and we conclude that

$$\operatorname{core}\big(\operatorname{Hom}_{\mathcal{C}}(x_0, x_1) \times \cdots \times \operatorname{Hom}_{\mathcal{C}}(x_{n-1}, x_n)\big) \longrightarrow \operatorname{core} \operatorname{F}(I^n, \mathcal{C})$$

$$\downarrow \qquad\qquad \ulcorner \qquad\qquad \downarrow$$

$$\operatorname{core}\big(\{x_0\} \times \cdots \times \{x_n\}\big) \longrightarrow \operatorname{core}(\mathcal{C}) \times \cdots \times \operatorname{core}(\mathcal{C})$$

is a pullback of simplicial sets. But $\{x_0\} \times \cdots \times \{x_n\}$ and $\operatorname{Hom}_{\mathcal{C}}(x_0, x_1) \times \cdots \times \operatorname{Hom}_{\mathcal{C}}(x_{n-1}, x_n)$ are Kan complexes (the latter by Corollary 4.4), hence coincide with their cores. This shows that we get a pullback as desired, thus finishing the proof of ($\boxtimes$) and the case $K = \Delta^n$.

Case K is finite-dimensional. A simplicial set K is called *finite-dimensional* if it has non-degenerate simplices in only finitely many degrees. We use induction on maximal dimension d of a non-degenerate simplex. The case $d = 0$ follows from the case $K = \Delta^0$ above. For the inductive step, we can write a $(d + 1)$-dimensional simplicial set K as a pushout of some d-dimensional simplicial set along a disjoint union $\coprod \partial\Delta^{d+1} \to \coprod \Delta^{d+1}$ of simplex boundary

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[(4.1)] It's not even true that pullbacks always exist in QCat and Kan.

inclusions. Accordingly, $\mathrm{F}(K,\mathcal{C})$ and $\mathrm{F}(K,\mathcal{D})$ can be written as pullbacks. By arguments as in ($\boxtimes$), we still get pullbacks after applying core and the legs $\operatorname{core}\mathrm{F}\left(\coprod\Delta^{d+1},\mathcal{C}\right) \to \mathrm{F}\left(\coprod\partial\Delta^{d+1},\mathcal{C}\right)$ and $\operatorname{core}\mathrm{F}\left(\coprod\Delta^{d+1},\mathcal{D}\right) \to \mathrm{F}\left(\coprod\partial\Delta^{d+1},\mathcal{D}\right)$ are Kan fibrations. Using the inductive hypothesis and the case $K = \Delta^{d+1}$ together with Lemma 3.19 and the five lemma (plus Remark 3.20), we see that $\operatorname{core}\mathrm{F}(K,\mathcal{C}) \to \operatorname{core}\mathrm{F}(K,\mathcal{D})$ induces a bijection on π_0 and isomorphisms on π_n for all basepoints and all $n \geqslant 1$. Hence $\operatorname{core}\mathrm{F}(K,\mathcal{C}) \to \operatorname{core}\mathrm{F}(K,\mathcal{D})$ must be a homotopy equivalence by Theorem 3.18.

General case. Write $K \cong \operatorname{colim}_{d\geqslant 0}\operatorname{sk}_d K$, where $\operatorname{sk}_d K$ is the *d-skeleton of K*. It is defined as the left Kan extension

$$\operatorname{sk}_d K := \operatorname{Lan}_{\Delta^{\mathrm{op}}_{\leqslant d}\to\Delta^{\mathrm{op}}}\left(K|_{\Delta^{\mathrm{op}}_{\leqslant d}}\right),$$

where $\Delta^{\mathrm{op}}_{\leqslant d} \subseteq \Delta^{\mathrm{op}}$ is the full subcategory spanned by $[0],\dots,[d]$. It's straightforward to see, using the Kan extension formula from Lemma 1.15, that $\operatorname{sk}_d K$ is d-dimensional and the transition maps $\operatorname{sk}_d K \to \operatorname{sk}_{d+1} K$ are cofibrations. By the finite-dimensional case, F induces equivalences $\operatorname{core}\mathrm{F}(\operatorname{sk}_d K,\mathcal{C}) \xrightarrow{\simeq} \operatorname{core}\mathrm{F}(\operatorname{sk}_d K,\mathcal{D})$ for all $d \geqslant 0$.

By the colimit above, $\mathrm{F}(K,\mathcal{C}) \cong \lim_{d\geqslant 0}\mathrm{F}(\operatorname{sk}_d K,\mathcal{C})$. This limit is preserved by core. Indeed, ($\boxtimes$) shows that $\operatorname{core}\mathrm{F}(\operatorname{sk}_{d+1} K,\mathcal{C}) \to \operatorname{core}\mathrm{F}(\operatorname{sk}_d K,\mathcal{C})$ is a Kan fibration and we can apply an argument as above. The same applies to $\mathcal{D}$ instead of $\mathcal{C}$. So it remains to see that equivalences of Kan complexes are preserved under limits along Kan fibrations. This can be shown using a Milnor sequence for homotopy groups, for example, or by hand, using a straightforward, but technical argument. See [F-HC$_\mathrm{I}$, Lemma VII.12] for example. $\qquad\square$

§4.2. Localisations of ∞-categories

4.10. Construction. — Let $\mathcal{C}$ be a quasi-category and $W \subseteq \mathcal{C}_1$ a subset of morphisms. We wish to construct the localisation $\mathcal{C} \to \mathcal{C}[W^{-1}]$, that is, the universal functor of quasi-categories that sends the morphisms from W to equivalences. To do so, consider the the pushout

$$
\begin{array}{ccc}
\coprod_W \Delta^1 & \longrightarrow & \mathcal{C} \\
\downarrow & \ulcorner. & \downarrow \\
\coprod_W \mathrm{N}(J) & \longrightarrow & \overline{\mathcal{C}}
\end{array}
$$

in simplicial sets, where $J := \{\bullet \rightleftarrows \bullet\}$ is the "free-living isomorphism", the category with two objects and a pair of mutually inverse isomorphisms between them. By Lemma 3.12, we can choose an inner anodyne map $\overline{\mathcal{C}} \to \mathcal{C}[W^{-1}]$ into a quasi-category. We call the composition $p\colon \mathcal{C} \to \mathcal{C}[W^{-1}]$ the *localisation of $\mathcal{C}$ at W*. We'll check in a moment that p is independent of the choices (up to equivalence), so the definite article is justified.

4.11. Lemma. — *For every quasi-category $\mathcal{D}$, the functor $p\colon \mathcal{C} \to \mathcal{C}[W^{-1}]$ from Construction 4.10 above induces an equivalence*

$$p^*\colon \operatorname{Hom}_{\mathrm{Cat}_\infty}\left(\mathcal{C}[W^{-1}],\mathcal{D}\right) \xrightarrow{\simeq} \operatorname{Hom}^W_{\mathrm{Cat}_\infty}(\mathcal{C},\mathcal{D}) \subseteq \operatorname{Hom}_{\mathrm{Cat}_\infty}(\mathcal{C},\mathcal{D}),$$

where $\operatorname{Hom}^W_{\mathrm{Cat}_\infty}(\mathcal{C},\mathcal{D}) \subseteq \operatorname{Hom}_{\mathrm{Cat}_\infty}(\mathcal{C},\mathcal{D})$ is the collection of path components of those functors $F\colon \mathcal{C} \to \mathcal{D}$ that send W to equivalences in $\mathcal{D}$.

Proof. Let $F^W(\mathcal{C}, \mathcal{D}) \subseteq F(\mathcal{C}, \mathcal{D})$ be the full sub-quasi-category (as in 2.16) spanned by those functors $F \colon \mathcal{C} \to \mathcal{D}$ that send W to equivalences in $\mathcal{D}$. We know from Theorem 2.24 that $\mathrm{Hom}_{\mathrm{Cat}_\infty}(\mathcal{C}[W^{-1}], \mathcal{D}) \simeq \mathrm{core}\, F(\mathcal{C}[W^{-1}], \mathcal{D})$ and $\mathrm{Hom}_{\mathrm{Cat}_\infty}(\mathcal{C}, \mathcal{D}) \simeq \mathrm{core}\, F(\mathcal{C}, \mathcal{D})$; it's then straightforward to check that

$$\mathrm{Hom}_{\mathrm{Cat}_\infty}^W(\mathcal{C}, \mathcal{D}) \simeq \mathrm{core}\, F^W(\mathcal{C}, \mathcal{D}) \,.$$

Since $\overline{\mathcal{C}} \to \mathcal{C}[W^{-1}]$ is inner anodyne by Construction 4.10, $F(\mathcal{C}[W^{-1}], \mathcal{D}) \to F(\overline{\mathcal{C}}, \mathcal{D})$ is a trivial fibration by Corollary 3.10.

We'll show that $\mathrm{core}\, F(\overline{\mathcal{C}}, \mathcal{D}) \to \mathrm{core}\, F^W(\mathcal{C}, \mathcal{D})$ is a trivial fibration too to finish the proof. This is straightforward, but a little annoying thanks to technicalities. The pushout from Construction 4.10 shows that

$$\begin{array}{ccc}
F(\overline{\mathcal{C}}, \mathcal{D}) & \longrightarrow & F(\mathcal{C}, \mathcal{D}) \\
\downarrow & \lrcorner & \downarrow \\
\prod_W F(\mathrm{N}(J), \mathcal{D}) & \longrightarrow & \prod_W F(\Delta^1, \mathcal{D})
\end{array}$$

is a pullback of simplicial sets. Note that $F(\overline{\mathcal{C}}, \mathcal{D}) \to F(\mathcal{C}, \mathcal{D})$ factors through the full sub-quasi-category $F^W(\mathcal{C}, \mathcal{D}) \subseteq F(\mathcal{C}, \mathcal{D})$ and $F(\mathrm{N}(J), \mathcal{D}) \to F(\Delta^1, \mathcal{D})$ factors through the full sub-quasi-category $F^{\{0 \to 1\}}(\Delta^1, \mathcal{D}) \subseteq F(\Delta^1, \mathcal{D})$. Since pullbacks behave well under passing to sub-simplicial sets, the following diagram is a pullback too:

$$\begin{array}{ccc}
F(\overline{\mathcal{C}}, \mathcal{D}) & \longrightarrow & F^W(\mathcal{C}, \mathcal{D}) \\
\downarrow & \lrcorner & \downarrow \\
\prod_W F(\mathrm{N}(J), \mathcal{D}) & \longrightarrow & \prod_W F^{\{0 \to 1\}}(\Delta^1, \mathcal{D})
\end{array}$$

To finish the proof, it's enough to show the following two claims:

($\boxtimes_1$) *The pullback above stays a pullback after applying* core *everywhere.*

($\boxtimes_2$) *The map* $\mathrm{core}\, F(\mathrm{N}(J), \mathcal{D}) \to \mathrm{core}\, F^{\{0 \to 1\}}(\Delta^1, \mathcal{D})$ *is a trivial fibration.*

To prove ($\boxtimes_1$), observe that $F(\mathrm{N}(J), \mathcal{D}) \to F(\Delta^1, \mathcal{D})$ is an inner fibration by Corollary 3.10 and $\mathrm{core}\, F(\mathrm{N}(J), \mathcal{D}) \to \mathrm{core}\, F(\Delta^1, \mathcal{D})$ is a Kan fibration by claim ($\boxtimes$) in the proof of Theorem 4.6. By an argument similar to Lemma 4.9, the fact that $F^{\{0 \to 1\}}(\Delta^1, \mathcal{D}) \subseteq F(\Delta^1, \mathcal{D})$ is a full sub-quasi-category implies that $\mathrm{core}\, F^{\{0 \to 1\}}(\Delta^1, \mathcal{D}) \subseteq \mathrm{core}\, F(\Delta^1, \mathcal{D})$ is a collection of path components. By inspection, this means that $F(\mathrm{N}(J), \mathcal{D}) \to F^{\{0 \to 1\}}(\Delta^1, \mathcal{D})$ must be an inner fibration too and $\mathrm{core}\, F(\mathrm{N}(J), \mathcal{D}) \to \mathrm{core}\, F^{\{0 \to 1\}}(\Delta^1, \mathcal{D})$ must be a Kan fibration too. By the same argument as in the proof of Theorem 4.6 it follows that core preserves the pullback, as required.

To prove ($\boxtimes_2$), we claim $\mathrm{core}\, F(\mathrm{N}(J), \mathcal{D}) \cong \mathrm{core}\, F(\mathrm{N}(J), \mathrm{core}(\mathcal{D}))$. Indeed, an n-simplex $\Delta^n \to \mathrm{core}\, F(\mathrm{N}(J), \mathcal{D})$ is the same as an n-simplex $\Delta^n \to F(\mathrm{N}(J), \mathcal{D})$ all of whose edges are mapped to equivalences; by 2.6 and Theorem 4.5, that's the same as a map $\sigma \colon \mathrm{N}(J) \times \Delta^n \to \mathcal{D}$ such that $\sigma|_{\{x\} \times \Delta^{\{i,j\}}} \colon \{x\} \times \Delta^{\{i,j\}} \to \mathcal{D}$ maps to an equivalence for all $x \in \mathrm{N}(J)$ and all edges $\Delta^{\{i,j\}} \subseteq \Delta^n$. But then all morphisms in $\mathrm{N}(J) \times \Delta^n$ must be mapped to equivalences, because every morphism in $\mathrm{N}(J)$ is already an equivalence. So σ necessarily factors through $\mathrm{core}(\mathcal{D}) \subseteq \mathcal{D}$, as desired.

Since $\mathrm{core}(\mathcal{D})$ is a Kan complex by Corollary 4.2, $\mathrm{F}(\mathrm{N}(J), \mathrm{core}(\mathcal{D}))$ is a Kan complex by Corollary 3.11 and so $\mathrm{core}\,\mathrm{F}(\mathrm{N}(J), \mathrm{core}(\mathcal{D})) \cong \mathrm{F}(\mathrm{N}(J), \mathrm{core}(\mathcal{D}))$. By analogous arguments, we get isomorphisms $\mathrm{core}\,\mathrm{F}^{\{0\to1\}}(\Delta^1, \mathcal{D}) \cong \mathrm{core}\,\mathrm{F}^{\{0\to1\}}(\Delta^1, \mathrm{core}(\mathcal{D})) \cong \mathrm{F}(\Delta^1, \mathrm{core}(\mathcal{D}))$. Now $\Delta^1 \to \mathrm{N}(J)$ is anodyne (in fact, both left and right anodyne) by an explicit horn filling argument. Hence $\mathrm{F}(\mathrm{N}(J), \mathrm{core}(\mathcal{D})) \to \mathrm{F}(\Delta^1, \mathrm{core}(\mathcal{D}))$ is a trivial fibration by Corollary 3.10. This finishes the proof of ($\boxtimes_2$) and we are done. $\qquad\square$

4.12. Corollary/Warning. — *If $\mathcal{C}$ is a (small) ordinary category and W a collection of morphisms in $\mathcal{C}$, then the localisation $\mathrm{N}(\mathcal{C})[W^{-1}]$ from Construction 4.10 is not necessarily the nerve of an ordinary category. But the homotopy category $h(\mathrm{N}(\mathcal{C})[W^{-1}])$ is equivalent to the localisation of $\mathcal{C}$ at W in the world of ordinary categories.*

Proof sketch. For counterexamples see Theorem 4.13 or the discussion in 6.34 below. The assertion about $h(\mathrm{N}(\mathcal{C})[W^{-1}])$ follows easily from a combination of Lemma 4.11 and Lemma 2.14 as well as the universal property of localisations in ordinary category theory. $\qquad\square$

So localisations provide another way to construct non-trivial examples of quasi-categories. In fact, both An and Cat_∞ can be constructed in this way:

4.13. Theorem. — *If Kan^Δ and QCat^Δ are the Kan-enriched categories from Example 2.23, then there are canonical equivalences of quasi-categories*

$$\mathrm{N}(\mathrm{Kan})\big[\{\textit{homotopy equivalences}\}^{-1}\big] \overset{\simeq}{\longrightarrow} \mathrm{N}^\Delta(\mathrm{Kan}^\Delta) = \mathrm{An}\,,$$

$$\mathrm{N}(\mathrm{QCat})\big[\{\textit{equivalences of quasi-categories}\}^{-1}\big] \overset{\simeq}{\longrightarrow} \mathrm{N}^\Delta(\mathrm{QCat}^\Delta) = \mathrm{Cat}_\infty\,. \qquad\blacksquare$$

4.14. Remark. — It's not hard to construct the functors in Theorem 4.13: By a direct inspection of their constructions, we can build a map $\mathrm{N}(\mathrm{Kan}) \to \mathrm{N}^\Delta(\mathrm{Kan}^\Delta)$ of simplicial sets (or rather simplicial *classes*, but we'll ignore the set-theoretic difficulties). Using Lemma 4.11, we only need to check that this map sends homotopy equivalences in Kan to equivalences in $\mathrm{N}^\Delta(\mathrm{Kan}^\Delta)$, which is clear from the unravelling in Example 2.23. An analogous argument works of course for $\mathrm{N}^\Delta(\mathrm{QCat}^\Delta)$.

However, proving that these functors are equivalences is not easy. There is a general notion of *simplicial model categories*: These are model categories $\mathcal{A}$ (Definition 3.27) together with a simplicial enrichment $\mathcal{A}^\Delta$ that interacts with the model structure in a certain way. One can show that the model structures on sSet from Examples 3.28 and 3.29 can be made into simplicial model structures. In general, if $\mathcal{A}$ is a simplicial model category and $\mathcal{A}^{\mathrm{cf}} \subseteq \mathcal{A}$, $(\mathcal{A}^\Delta)^{\mathrm{cf}} \subseteq \mathcal{A}^\Delta$ is the full subcategory respectively the full sub-simplicially enriched category spanned by the bifibrant objects, there is an equivalence of quasi-categories

$$\mathrm{N}(\mathcal{A}^{\mathrm{cf}})\big[\{\text{weak equivalences}\}^{-1}\big] \overset{\simeq}{\longrightarrow} \mathrm{N}^\Delta\big((\mathcal{A}^\Delta)^{\mathrm{cf}}\big)\,.$$

A proof can be found in [L-HA, Theorem 1.3.4.20].

4.15. Remark. — For a general model category $\mathcal{A}$, it's customary to call

$$\mathcal{A}_\infty := \mathrm{N}(\mathcal{A}^{\mathrm{cf}})\big[\{\text{weak equivalences}\}^{-1}\big]$$

the *underlying quasi-category of $\mathcal{A}$*. Its homotopy category $h(\mathcal{A}_\infty)$ is called the *homotopy category of $\mathcal{A}$*. By Corollary/Warning 4.12, this agrees with the ordinary localisation of $\mathcal{A}^{\mathrm{cf}}$ at the weak equivalences. Furthermore, if the factorisations from Definition 3.27(*d*) can be chosen

functorially (which is always the case in practice), then $\mathcal{A}_\infty$ could be obtained equally well by inverting the weak equivalences in either of $\mathcal{A}^c$, $\mathcal{A}^f$, or $\mathcal{A}$ itself, where $\mathcal{A}^c, \mathcal{A}^f \subseteq \mathcal{A}$ denote the full subcategories spanned by the cofibrant or fibrant objects, respectively. So in practice, all possible alternative definitions of $\mathcal{A}_\infty$ agree.

We'll sketch the argument why $\mathrm{N}(\mathcal{A}^c) \to \mathrm{N}(\mathcal{A})$ becomes an equivalences after localisation at all weak equivalences; the other cases are entirely analogous. Let's assume that $\mathcal{A}$ has functorial cofibrant replacements. That is, for $x \in \mathcal{A}$ the map from the initial object to x factors functorially through a trivial fibration $\eta_x \colon c(x) \to x$, where $c(x)$ is cofibrant. Then we get a natural transformation $\eta \colon c(-) \Rightarrow \mathrm{id}_{\mathrm{N}(\mathcal{A})}$ of endofunctors of $\mathrm{N}(\mathcal{A})$. Using Lemma 4.11, we can show that this natural transformation passes to the localisation at all weak equivalences.[(4.2)] After the localisation, η becomes an equivalence of endofunctors by Theorem 4.5. Then $\mathrm{N}(\mathcal{A}^c) \to \mathrm{N}(\mathcal{A})$ and $c \colon \mathrm{N}(\mathcal{A}) \to \mathrm{N}(\mathcal{A}^c)$ become equivalences of quasi-categories after localisation at all weak equivalences, as desired.

In general, it's hard to describe morphisms in any localisation. However, if $\mathcal{A}$ is a simplicial model category, then Remark 4.14 and Theorem 2.24 provide convenient access to the Hom animae in $\mathcal{A}_\infty$.

[(4.2)]Here's the full argument: Put $W \coloneqq \{\text{weak equivalences}\}$ for short; we wish to construct a natural transformation $\mathrm{N}(\mathcal{A})[W^{-1}] \times \Delta^1 \to \mathrm{N}(\mathcal{A})[W^{-1}]$. It's clear from the construction that $\mathrm{N}(\mathcal{A})[W^{-1}] \times \Delta^1$ can also be described as the localisation of $\mathrm{N}(\mathcal{A}) \times \Delta^1$ at $W \times \{\mathrm{id}_0\} \cup W \times \{\mathrm{id}_1\}$. Thus, by Lemma 4.11, it's enough to provide a natural transformation $\mathrm{N}(\mathcal{A}) \times \Delta^1 \to \mathrm{N}(\mathcal{A})[W^{-1}]$ that sends these morphisms to equivalences. Now the composition of $\eta \colon \mathrm{N}(\mathcal{A}) \times \Delta^1 \to \mathrm{N}(\mathcal{A})$ with the localisation functor $\mathrm{N}(\mathcal{A}) \to \mathrm{N}(\mathcal{A})[W^{-1}]$ does just that.

§5. Lurie's straightening equivalence

We've seen in 2.11 how to construct the Hom animae $\mathrm{Hom}_\mathcal{C}(x, y)$ in a quasi-category $\mathcal{C}$. But we never explained how to assemble these values into a functor $\mathrm{Hom}_\mathcal{C} \colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathrm{An}$ (where $\mathcal{C}^{\mathrm{op}}$ is as in 2.17). In this section, we give such a construction and prove the Yoneda lemma. To do this, we'll use Lurie's *straightening/unstraightening equivalence*, which deals with the problem of constructing functors $F \colon \mathcal{C} \to \mathrm{An}$ and $F \colon \mathcal{C} \to \mathrm{Cat}_\infty$. In the end, straightening/unstraightening will not only allow us to prove Yoneda's lemma, but the statement itself will be indispensable for developing quasi-category theory as a higher analogue of ordinary category theory.

§5.1. Cocartesian fibrations and the straightening equivalence

5.1. Some informal motivation. — Let's think about what a functor $F \colon \mathcal{C} \to \mathrm{Cat}_\infty$ looks like. Suppose $x, y \in \mathcal{C}$ are objects and $\alpha \colon x \to y$ is a morphism. Then $F(x)$, $F(y)$ will be quasi-categories and $F(\alpha) \colon F(x) \to F(y)$ will be a functor between them. So for every $u \in F(x)$, there will be an associated object $v \simeq F(\alpha)(u) \in F(y)$. In a picture, this could look as follows:
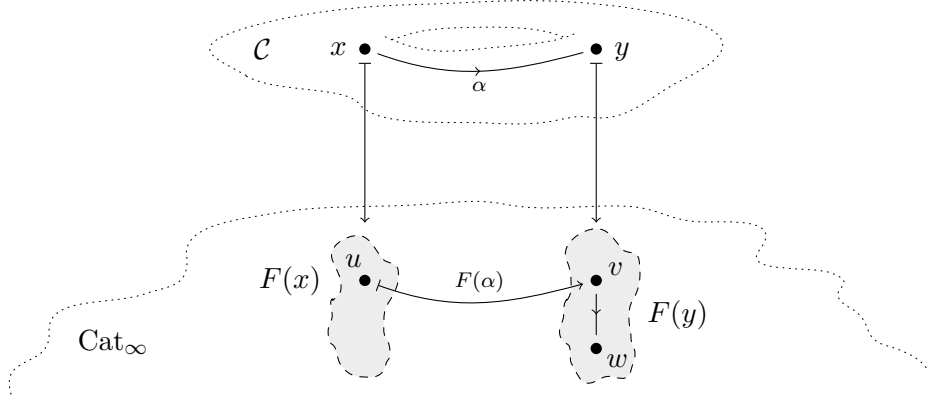
Now let's turn this picture upside down! Take $F(x)$ and $F(y)$ and place them above x and y. We would like to think of them as the fibres over x and y in some kind of fibration $p \colon \mathcal{U} \to \mathcal{C}$:
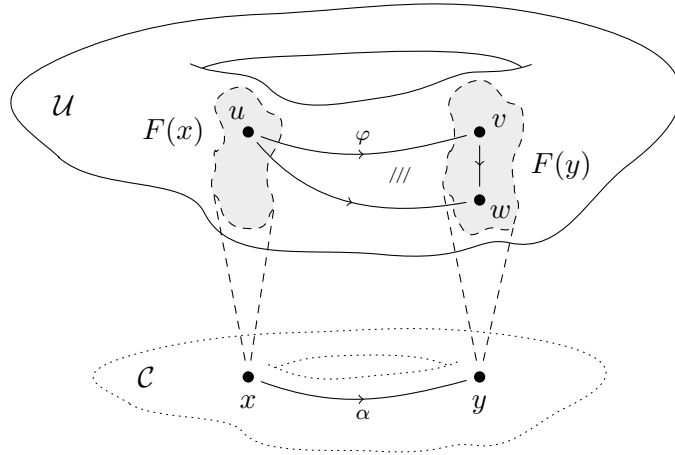
But, of course, $\mathcal{U}$ shouldn't just be a disjoint union of some fibres. Instead, we need to capture somehow that the values $F(x)$ of the functor F "vary functorially in x". To do this, we connect

every $u \in F(x)$ to its image $v \simeq F(\alpha)(u)$ by a new 1-simplex $\varphi \colon u \to v$. Then we keep adding further simplices to make sure that the object we end up with is a quasi-category. For example, in our picture we have a morphism $v \to w$ in $F(y)$, so we have to add a 2-simplex $\sigma \colon \Delta^2 \to \mathcal{U}$ in such a way that $\sigma|_{\Delta^{\{0,2\}}} \colon u \to w$ is a composition of φ and $v \to w$.

Here's a hopefully enlightening example. For any (not necessarily commutative ring) R let LMod_R denote the category of left R-modules. Intuitively, we would expect that there exists a functor $\mathrm{LMod}_{(-)}$ which sends a ring R to LMod_R and a morphism of rings $R \to R'$ to the base change functor $R' \otimes_R - \colon \mathrm{LMod}_R \to \mathrm{LMod}_{R'}$. However, $\mathrm{LMod}_{(-)}$ cannot take values in the category of categories Cat. The problem is that $R'' \otimes_{R'} (R' \otimes_R -) \simeq R'' \otimes_R -$ only holds as an *equivalence* of functors, but not as an *equality*. In other words, $\mathrm{LMod}_{(-)}$ can only be a functor

$$\mathrm{LMod}_{(-)} \colon \mathrm{N}(\mathrm{Ring}) \longrightarrow \mathrm{Cat}^{(2)} \subseteq \mathrm{Cat}_\infty$$

into the 2-*category of categories* from Example 2.25, which is a full sub-quasi-category of Cat_∞.

It's not clear how one would construct the functor $\mathrm{LMod}_{(-)} \colon \mathrm{N}(\mathrm{Ring}) \to \mathrm{Cat}_\infty$ by hand. So instead, let's guess the fibration $p \colon \mathcal{U} \to \mathrm{N}(\mathrm{Ring})$ that corresponds to it. To this end, let LMod be the category of pairs (R, M), where R is a ring and M is a left R-module. Morphisms $(R, M) \to (R', M')$ in LMod consist of a morphism of rings $R \to R'$ and a morphism of left R'-modules $R' \otimes_R M \to M'$. We claim that p should precisely be the induced functor of quasi-categories

$$p \colon \mathrm{N}(\mathrm{LMod}) \longrightarrow \mathrm{N}(\mathrm{Ring}) \,.$$

Note that the fibres of p are $p^{-1}\{R\} \simeq \mathrm{N}(\{R\} \times_{\mathrm{Ring}} \mathrm{LMod}) \simeq \mathrm{N}(\mathrm{LMod}_R)$ since N commutes with pullbacks. So the fibres indeed match up with the values of our desired functor. Moreover, the special 1-simplices φ from above occur naturally: Namely, for any $(R, M) \in \mathrm{LMod}$ and any $R \to R'$ there's a canonical morphism $\varphi \colon (R, M) \to (R', R' \otimes_R M)$ in LMod. Any other morphism $(R, M) \to (R', M')$ can be written uniquely as a composition of φ and a morphism $(R', R' \otimes_R M) \to (R', M')$ in $\mathrm{LMod}_{R'}$, so the corresponding 1-simplex in $\mathrm{N}(\mathrm{LMod})$ is indeed given by adding a suitable 2-simplex $\sigma \colon \Delta^2 \to \mathrm{N}(\mathrm{LMod})$ as above.

To summarise, we've given some motivation and an example that functors $F \colon \mathcal{C} \to \mathrm{Cat}_\infty$ should correspond to certain fibrations $p \colon \mathcal{U} \to \mathcal{C}$, in such a way that the values $F(x)$ correspond to the fibres $p^{-1}\{x\}$. Furthermore, we've seen that for every $\alpha \colon x \to y$ in $\mathcal{C}$ and every $u \in p^{-1}\{x\} \simeq F(x)$, there should be a special 1-simplex $\varphi \colon u \to v$ that connects u to its image under $F(\alpha)$. Every other 1-simplex from u to an object in $p^{-1}\{y\}$ should arise as a composition of φ with some morphism $v \to w$ in $p^{-1}\{y\} \simeq F(y)$.

These vague ideas are captured in a precise sense by the following definition:

5.2. Definition. — Let $p \colon \mathcal{U} \to \mathcal{C}$ be an inner fibration of quasi-categories.

(a) A morphism $\varphi \colon u \to v$ in $\mathcal{U}$ is called *p-cocartesian* if, for every $n \geqslant 2$, every lifting problem

$$
\begin{array}{ccc}
\Lambda_0^n & \longrightarrow & \mathcal{U} \\
\downarrow & \nearrow & \downarrow p \\
\Delta^n & \longrightarrow & \mathcal{C}
\end{array}
$$

in which $\Delta^{\{0,1\}} \subseteq \Lambda_0^n$ is sent to φ, has a solution.

(b)   We call p a *cocartesian fibration* if every lifting problem

$$
\begin{array}{ccc}
\{0\} & \longrightarrow & \mathcal{U} \\
\downarrow & \nearrow & \downarrow p \\
\Delta^1 & \longrightarrow & \mathcal{C}
\end{array}
$$

has a solution in which Δ^1 is sent to a p-cocartesian morphism.

There are dual notions of *p-cartesian morphisms* and *cartesian fibrations*, in which we use $\Delta^{\{n-1,n\}} \subseteq \Lambda_n^n \to \Delta^n$ and $\{1\} \to \Delta^1$ instead.

It's easy to identify those cocartesian fibrations that correspond to functors $F\colon \mathcal{C} \to \mathrm{An}$.

5.3. Lemma ("Left fibrations are cocartesian fibrations whose fibres are animae"). — *For a cocartesian fibration $p\colon \mathcal{U} \to \mathcal{C}$, the following conditions are equivalent:*

(a)   *Every morphism in $\mathcal{U}$ is p-cocartesian.*

(b)   *$p\colon \mathcal{U} \to \mathcal{C}$ is a left fibration.*

(c)   *All fibres $p^{-1}\{x\}$ for $x \in \mathcal{C}$ are animae.*

Proof sketch. The equivalence (a) $\Leftrightarrow$ (b) is clear and the implication (b) $\Rightarrow$ (c) follows from Corollary 4.3. To prove (c) $\Rightarrow$ (a), let $\varphi\colon u \to v$ be a morphism in $\mathcal{U}$. We wish to show that φ is p-cocartesian. Suppose we're given a lifting problem of the sort we're interested in: a diagram

$$
\begin{array}{ccc}
\Lambda_0^n & \xrightarrow{\sigma} & \mathcal{U} \\
\downarrow & \nearrow & \downarrow p \\
\Delta^n & \longrightarrow & \mathcal{C}
\end{array}
$$

such that $\sigma|_{\Delta^{\{0,1\}}}$ is $\varphi\colon u \to v$. We will construct a new lifting problem

$$
\begin{array}{ccccc}
\Lambda_0^n & \xrightarrow{d_1} & \Lambda_1^{n+1} & \xrightarrow{\overline{\sigma}} & \mathcal{U} \\
\downarrow & /\!/\!/ & \downarrow & \nearrow & \downarrow p \\
\Delta^n & \xrightarrow{d_1} & \Delta^{n+1} & \longrightarrow & \mathcal{C}
\end{array}
$$

such that $\sigma = \overline{\sigma} \circ d_1$ (note that d_1 maps $\Delta^{\{0,1\}}$ to $\Delta^{\{0,2\}}$, so φ is now the image of $\Delta^{\{0,2\}}$ under $\overline{\sigma}$). Since p is an inner fibration, we'll be able to solve the new lifting problem and get a solution for our original one.

Let's construct $\overline{\sigma}$! By Definition 5.2(b), applied to $p(\varphi)\colon \Delta^1 \to \mathcal{C}$, we can choose a p-cocartesian morphism $\varphi'\colon u \to v'$ such that $v' \in p^{-1}\{p(v)\}$. By Definition 5.2(a), applied to a suitable $\Lambda_0^2 \to \mathcal{U}$, we can find a morphism $\psi\colon v' \to v$ such that $\varphi \simeq \psi \circ \varphi'$. Note that ψ is an equivalence since $p^{-1}\{p(v)\}$ is an anima by assumption. Now we construct $\overline{\sigma}$ piece by piece. We put $\overline{\sigma}|_{d_1(\Lambda_0^n)} := \sigma$ and we send $\{1\}$ to v' as well as $\Delta^{\{0,1\}}$ to φ'. Furthermore, we send $\Delta^{\{0,1,2\}}$ to a 2-simplex witnessing $\varphi \simeq \psi \circ \varphi'$. The rest of $\Lambda_1^{n+1} \smallsetminus (d_1(\Lambda_0^n) \cup \Delta^{\{0,1,2\}})$ can be filled by a sequence of horn filling problems in which either the first edge is φ', so a filler exists by Definition 5.2(a), or the first edge is $\psi\colon v \to v'$, so a filler exists by Joyal's lifting theorem (Theorem 4.1) since ψ is an equivalence. As usual, we skip the horn filling combinatorics. $\square$

We can now state the straightening/unstraightening equivalence. As was, unfortunately, clear from the beginning, we won't give a proof here. The most readable proof available is probably due to Gijs Heuts [FHR21] with contributions by Fabian Hebestreit and Jaco Ruit, building on previous work by Cisinski and Nguyen. Lurie's original proof can be found in [L-HTT, §3.2]. There's also another approach by Cisinski [Cis19] in which straightening/unstraightening is much easier to obtain, but much more work is needed to identify $\mathrm{An} \subseteq \mathrm{Cat}_\infty$ with the full sub-quasi-category spanned by the Kan complexes.

5.4. Theorem (Straightening/unstraightening). — *Let $\mathcal{C}$ be a quasi-category.*

(a) *Let $\mathrm{Cocart}(\mathcal{C}) \subseteq \mathrm{Cat}_{\infty/\mathcal{C}}$ be the (non-full!) sub-quasi-category spanned by cocartesian fibrations over $\mathcal{C}$ and those maps that preserve cocartesian morphisms (see 2.16). Then there are inverse equivalences of quasi-categories*

$$\mathrm{St}^{\mathrm{cocart}} \colon \mathrm{Cocart}(\mathcal{C}) \underset{\simeq}{\overset{\simeq}{\rightleftarrows}} \mathrm{F}(\mathcal{C}, \mathrm{Cat}_\infty) \colon \mathrm{Un}^{\mathrm{cocart}}$$

called "straightening" and "unstraightening". For a cocartesian fibration $p \colon \mathcal{U} \to \mathcal{C}$, the value of $\mathrm{St}^{\mathrm{cocart}}(p) \colon \mathcal{C} \to \mathrm{Cat}_\infty$ at $x \in \mathcal{C}$ is given by the fibre $p^{-1}\{x\}$. Furthermore, $F \colon \mathcal{C} \to \mathcal{D}$ is a functor, then the unstraightening equivalence sends the precomposition functor $- \circ F \colon \mathrm{F}(\mathcal{D}, \mathrm{Cat}_\infty) \to \mathrm{F}(\mathcal{C}, \mathrm{Cat}_\infty)$ to the pullback $F^* \colon \mathrm{Cocart}(\mathcal{D}) \to \mathrm{Cocart}(\mathcal{C})$.

(b) *Let $\mathrm{Left}(\mathcal{C}) \subseteq \mathrm{Cat}_{\infty/\mathcal{C}}$ be the full sub-quasi-category spanned by the left fibrations over $\mathcal{C}$ (see 2.16). Then the equivalences from (a) restrict to equivalences*

$$\mathrm{St}^{\mathrm{left}} \colon \mathrm{Left}(\mathcal{C}) \underset{\simeq}{\overset{\simeq}{\rightleftarrows}} \mathrm{F}(\mathcal{C}, \mathrm{An}) \colon \mathrm{Un}^{\mathrm{left}} .$$

Dually, there are equivalences $\mathrm{Cart}(\mathcal{C}) \simeq \mathrm{F}(\mathcal{C}^{\mathrm{op}}, \mathrm{Cat}_\infty)$ and $\mathrm{Right}(\mathcal{C}) \simeq \mathrm{F}(\mathcal{C}^{\mathrm{op}}, \mathrm{An})$. Here $\mathcal{C}^{\mathrm{op}}$ is the opposite quasi-category from 2.17. ∎

Most of the time, one can treat Theorem 5.4 as a black box and work purely with the statement, without knowing how exactly and we don't need to know how exactly the functors $\mathrm{St}^{\mathrm{cocart}}$ and $\mathrm{Un}^{\mathrm{cocart}}$ are constructed!

Still, we'll spend the rest of §5.1 to give you some idea how the construction works. We'll start with a few examples that will become very relevant in 5.5. After that, we'll explain the effect of straightening on morphisms and discuss two classical examples in 5.7 and 5.8.

5.5. Example. — Let $\mathcal{C}$ be a quasi-category. The following are examples of cocartesian fibrations and their unstraightenings.

(a) For every quasi-category $\mathcal{D}$, the unique functor $\mathcal{D} \to *$ is a cocartesian fibrations, with the cocartesian morphisms given by the equivalences in $\mathcal{D}$. This is an easy application of Joyal's lifting theorem (Theorem 4.1). Furthermore, since cocartesian fibrations are clearly preserved under pullbacks, we see that $\mathrm{pr}_2 \colon \mathcal{D} \times \mathcal{C} \to \mathcal{C}$ is a cocartesian fibration for every quasi-category $\mathcal{C}$. The pr_2-cocartesian morphisms in $\mathcal{D} \times \mathcal{C}$ are precisely those that are equivalences in the $\mathcal{D}$-component. The straightening of $\mathrm{pr}_2 \colon \mathcal{D} \times \mathcal{C} \to \mathcal{C}$ is the constant functor $\mathrm{const}\,\mathcal{D} \colon \mathcal{C} \to \mathrm{Cat}_\infty$; this follows from the pullback statement in Theorem 5.4(a), but it's probably also pretty clear intuitively.

(b) For every $x \in \mathcal{C}$, we've seen in Corollary 4.4 that $t \colon \mathcal{C}_{x/} \to \mathcal{C}$ is even a left fibration. The fibre of t over $y \in \mathcal{C}$ is $\mathrm{Hom}_\mathcal{C}(x, y)$ by 2.11, so we can use the straightening of t as our definition of the Hom functor

$$\mathrm{Hom}_\mathcal{C}(x, -) \colon \mathcal{C} \longrightarrow \mathrm{An} .$$

Analogously, the dual construction $s\colon \mathcal{C}_{/y} \to \mathcal{C}$ is a right fibration and its cartesian straightening is, by definition, the contravariant Hom functor $\mathrm{Hom}_{\mathcal{C}}(-, y)\colon \mathcal{C}^{\mathrm{op}} \to \mathrm{An}$. This still leaves the question how to construct the two-variable Hom functor $\mathrm{Hom}_{\mathcal{C}}\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathrm{An}$, which we'll discuss in Constructions 5.21 and 5.22 below.

(c) The target projection $t\colon \mathrm{Ar}(\mathcal{C}) \to \mathcal{C}$ from 2.11 is a cocartesian fibration, and its is our definition of the functor
$$\mathcal{C}_{/-}\colon \mathcal{C} \longrightarrow \mathrm{Cat}_{\infty}$$
that sends $x \in \mathcal{C}$ to the slice quasi-category $\mathcal{C}_{/x}$. A morphism $\varphi\colon (\alpha\colon u \to u') \to (\beta\colon v \to v')$ in $\mathrm{Ar}(\mathcal{C})$, that is, a commutative diagram

$$
\begin{array}{ccc}
u & \longrightarrow & v \\
{\scriptstyle \alpha}\big\downarrow & {\scriptstyle /\!/\!/} & \big\downarrow {\scriptstyle \beta} \\
u' & \longrightarrow & v'
\end{array}
$$

in $\mathcal{C}$, is t-cocartesian if and only if $u \to v$ is an equivalence in $\mathcal{C}$. Proving this is a somewhat subtle and will lead us on a detour in §5.2. One way to see the "if"-part (which is the difficult part) would be to reformulate a lifting problem for t against $\Lambda_0^n \to \Delta^n$ into a lifting problem for $\mathcal{C} \to *$ against $\Lambda_0^n \times \Delta^1 \sqcup_{\Lambda_0^n \times \{1\}} \Delta^n \times \{1\} \to \Delta^n \times \Delta^1$, as in the proof of Corollary 3.10. Then one proves, using Joyal's lifting problem (Theorem 4.1), that a lifting problem of the latter kind is always solvable if the original lifting problem maps $\Delta^{\{0,1\}} \subseteq \Lambda_0^n$ to a morphism φ as above.

However, a much nicer proof of the "if"-part is provided by Lemma 5.13 and Lemma 5.16 below (except that there are some black boxes involved . . .). For the "only if"-part, it's enough to write down the correct lifting diagrams; we leave this to you.

(d) In the example from 5.1, it's not hard to show that $p\colon \mathrm{N}(\mathrm{LMod}) \longrightarrow \mathrm{N}(\mathrm{Ring})$ is a cocartesian fibration (compare the arguments in 5.7 below). Cocartesian lifts of a morphism $R \to R'$ have the form $(R, M) \to (R', R' \otimes_R M)$. In particular, we can define a functor
$$\mathrm{LMod}_{(-)}\colon \mathrm{Ring} \longrightarrow \mathrm{Cat}_{\infty}$$
as the straightening of the cocartesian fibration $\mathrm{N}(\mathrm{LMod}) \to \mathrm{N}(\mathrm{Ring})$. As we've already seen in 5.1, the fibres are given by $\mathrm{N}(\mathrm{LMod}_R)$. These are nerves of 1-categories and so $\mathrm{LMod}_{(-)}$ factors through $\mathrm{Cat}^{(2)} \subseteq \mathrm{Cat}_{\infty}$, as desired.

5.6. Straightening/unstraightening on morphisms — Suppose we're given a cocartesian fibration $p\colon \mathcal{U} \to \mathcal{C}$ and let $F\colon \mathcal{C} \to \mathrm{Cat}_{\infty}$ be its straightening. We know what F does on objects: It sends $x \in \mathcal{C}$ to the fibre $p^{-1}\{x\}$. We'll now explain what F does on morphisms. To this end, let $\mathrm{Ar}^{\mathrm{cocart}}(\mathcal{U}) \subseteq \mathrm{Ar}(\mathcal{U})$ denote the full sub-quasi-category spanned by the p-cocartertesian morphisms. Then
$$\mathrm{Ar}^{\mathrm{cocart}}(\mathcal{U}) \longrightarrow \mathrm{Ar}(\mathcal{C}) \times_{s,\mathcal{C}} \mathcal{U}$$
is a trivial fibration.[(5.1)] To see this, first consider the case where p is a left fibration. Then $\mathrm{Ar}^{\mathrm{cocart}}(\mathcal{U}) = \mathrm{Ar}(\mathcal{U})$ by Lemma 5.3(a). Furthermore, $\{0\} \to \Delta^1$ is left anodyne. Hence the map

---

[(5.1)]Intuitively, this says that given a morphism $\alpha\colon x \to y$ in $\mathcal{C}$ and an object $u \in p^{-1}\{x\}$, then lifting α to a p-cocartesian morphism $\varphi\colon u \to v$ can not only be done, but even in a unique way (up to contractible ambiguity). This fits perfectly into the picture from 5.1: Such a p-cocartesian lift φ connects $u \in p^{-1}\{x\} \simeq F(x)$ to its image under $F(\alpha)\colon F(x) \to F(y)$. So φ should be unique.

above is a trivial fibration by Corollary 3.10. In general, we can adapt the proof of Corollary 3.10 to show that $\mathrm{Ar}^{\mathrm{cocart}}(\mathcal{U}) \to \mathrm{Ar}(\mathcal{C}) \times_{s,\mathcal{C}} \mathcal{U}$ has lifting against $\partial\Delta^n \to \Delta^n$: Rewrite such a lifting problem as a lifting problem for $p\colon \mathcal{U} \to \mathcal{C}$ against $\partial\Delta^n \times \Delta^1 \sqcup_{\partial\Delta^n \times \{0\}} \Delta^n \times \{0\} \to \Delta^n \times \Delta^1$. The latter is a sequence of horn lifting problems, each of which can be solved either because p is an inner fibration or by employing Definition 5.2(a).

Given a morphism $\alpha\colon x \to y$ in $\mathcal{C}$, we can now give the desired description of the functor $F(\alpha)\colon p^{-1}\{x\} \to p^{-1}\{y\}$ as follows: Pull back $\mathrm{Ar}^{\mathrm{cocart}}(\mathcal{U}) \to \mathrm{Ar}(\mathcal{C}) \times_{s,\mathcal{C}} \mathcal{U}$ along $\{\alpha\} \to \mathrm{Ar}(\mathcal{C})$ to obtain a trivial fibration $\{\alpha\} \times_{\mathrm{Ar}(\mathcal{C})} \mathrm{Ar}(\mathcal{U}) \to \{\alpha\} \times_{s,\mathcal{C}} \mathcal{U} \cong \{x\} \times_{\mathcal{C}} \mathcal{U} \cong p^{-1}\{x\}$. Every trivial fibration admits a section. By choosing such a section and composing with the target projections $t\colon \mathrm{Ar}(\mathcal{U}) \to \mathcal{U}$ and $t\colon \mathrm{Ar}(\mathcal{C}) \to \mathcal{C}$, we obtain (up to natural equivalence) the desired functor

$$F(\alpha)\colon p^{-1}\{x\} \longrightarrow \{\alpha\} \times_{\mathrm{Ar}(\mathcal{C})} \mathrm{Ar}(\mathcal{U}) \xrightarrow{\ t\ } \{y\} \times_{\mathcal{C}} \mathcal{U} \cong p^{-1}\{y\}\,.$$

With (a lot) more care, one can continue these considerations to give a complete description of the functor $F\colon \mathcal{C} \to \mathrm{Cat}_\infty$. This was first done by Haugseng and is described in [Lan21, §3.3]. The proofs of straightening/unstraightening in [FHR21] or [L-HTT, §3.2] proceed instead by constructing a simplicially enriched functor $\mathfrak{C}[\mathcal{C}] \to \mathrm{sSet}^\Delta$, as they deduce Theorem 5.4 from a suitable Quillen equivalence of model categories.

5.7. Straightening/unstraightening and stacks — We'll briefly explain the relation between Theorem 5.4 and the language of stacks from algebraic geometry. If you already know stacks, this will hopefully make Theorem 5.4 less mysterious. If you'd like to learn about stacks, this remark will hopefully make the literature on stacks less mysterious (a modern treatment of stacks using ∞-categories can be found for example in [Kha23]). If you don't care about stacks at all, you can safely skip this remark.

In algebraic geometry, one is naturally lead to functors whose values should be groupoids. For example, given a scheme S and a group scheme G acting on S, one would like to study the functor $[S/G]\colon (\mathrm{Sch}_{/S})^{\mathrm{op}} \to \mathrm{Grpd}$ that sends any scheme X over S to the groupoid of G_X-torsors, where $G_X := G \times_S X$ is the base change of G to X. A morphism $f\colon X \to Y$ in $\mathrm{Sch}_{/S}$ should be sent to the pullback functor $f^*\colon \{G_Y\text{-torsors}\} \to \{G_X\text{-torsors}\}$. Here we run into a similar problem as in 5.1: If $g\colon Y \to Z$ is another morphism in $\mathrm{Sch}_{/S}$, then the associated pullback functors come with a natural *equivalence* $f^* \circ g^* \simeq (g \circ f)^*$, but that equivalence is not an *equality*. So $[S/G]$ can't exist as a functor $[S/G]\colon (\mathrm{Sch}_{/S})^{\mathrm{op}} \to \mathrm{Grpd}$ into the category of groupoids; instead, it's a functor

$$[S/G]\colon \mathrm{N}\big(\mathrm{Sch}_{/S}\big)^{\mathrm{op}} \longrightarrow \mathrm{Grpd}^{(2)}$$

into the 2-*category of groupoids* as introduced in Example 2.25.

The ancient algebraic geometers didn't have the language to deal with functors into a 2-category, but they made do with the tools of their time: They instead constructed a functor $p\colon \mathcal{U} \to \mathrm{Sch}_{/S}$ in such a way that $\mathrm{N}(p)\colon \mathrm{N}(\mathcal{U}) \to \mathrm{N}(\mathrm{Sch}_{/S})$ is a right fibration whose straightening $\mathrm{St}^{\mathrm{right}}(\mathrm{N}(p)) \simeq [S/G]$ is the functor above.[(5.2)] Explicitly, $\mathcal{U}$ is the category of pairs $(X, \mathcal{P})$, where $X \in \mathrm{Sch}_{/S}$ and $\mathcal{P}$ is a G_X-torsor. Morphisms $(X, \mathcal{P}) \to (Y, \mathcal{Q})$ in $\mathcal{U}$ are

---

[(5.2)]It's not a coincidence that the unstraightening of $[S/G]$ is the nerve of an ordinary category. Let $\mathcal{C}$ be any ordinary category, let $F\colon \mathrm{N}(\mathcal{C})^{\mathrm{op}} \to \mathrm{Cat}_\infty$ be any functor that lands in the full sub-quasi-category $\mathrm{Cat}^{(2)} \subseteq \mathrm{Cat}_\infty$ from Example 2.25, and let $p\colon \mathcal{U} \to \mathrm{N}(\mathcal{C})$ be the cartesian unstraightening of F. Then $\mathcal{U}$ is equivalent to the nerve of an ordinary category. We'll give a sketch of the proof, which uses the notion of *homotopy pullbacks* from §5.2. First, it's enough to show that $\mathrm{Hom}_{\mathcal{U}}(u, v)$ is a discrete anima for all $u, v \in \mathcal{U}$, because then the essentialy surjective functor $u_{\mathcal{U}}\colon \mathcal{U} \to \mathrm{N}(h(\mathcal{U}))$ (given by the unit of the adjunction $h \dashv \mathrm{N}$) is also fully faithful,

pairs (f, α) where $f \colon X \to Y$ is a morphism in $\mathrm{Sch}_{/S}$ and $\alpha \colon f^*\mathcal{Q} \xrightarrow{\cong} \mathcal{P}$ is an isomorphism of G_X-torsors. See [Ols16, Example 8.1.10] or [Stacks, Tag 036Z].

   More generally, a *fibred category* is a functor $p \colon \mathcal{U} \to \mathcal{C}$ of ordinary categories such that $\mathrm{N}(p) \colon \mathrm{N}(\mathcal{U}) \to \mathrm{N}(\mathcal{C})$ is a cartesian fibration.[(5.3)] The classical definition of cartesian morphisms, see [Ols16, Definition 3.1.1] or [Stacks, Tag 02XK], differs from Definition 5.2(a), but it's still equivalent, as we'll see in Lemma 5.16. By Theorem 5.4, the data of a fibred category defines a functor $\mathrm{St}^{\mathrm{cart}} \colon \mathrm{N}(\mathcal{C})^{\mathrm{op}} \to \mathrm{Cat}_\infty$, which necessarily factors through the full sub-quasi-category $\mathrm{Cat}^{(2)} \subseteq \mathrm{Cat}_\infty$ from Example 2.25, because the fibres of $\mathrm{N}(p) \colon \mathrm{N}(\mathcal{U}) \to \mathrm{N}(\mathcal{C})$ must be nerves of ordinary categories again, Conversely, we show in footnote (5.2) below that the unstraightening of such a functor is necessarily the nerve of an ordinary category.

   A *category fibred in groupoids* is a fibred category $p \colon \mathcal{U} \to \mathcal{C}$ such that all fibres are groupoids; equivalently, the associated functor $\mathrm{St}^{\mathrm{cart}} \colon \mathrm{N}(\mathcal{C})^{\mathrm{op}} \to \mathrm{Cat}^{(2)}$ factors through $\mathrm{Grpd}^{(2)} \subseteq \mathrm{Cat}^{(2)}$. Finally, if $\mathcal{C}$ is equipped with a Grothendieck topology, we call $p \colon \mathcal{U} \to \mathcal{C}$ a *stack* if the functor $\mathrm{St}^{\mathrm{right}}(\mathrm{N}(p)) \colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Grpd}^{(2)}$ is a sheaf. To formulate the sheaf condition, one needs an appropriate notion of limits in $\mathrm{Grpd}^{(2)}$ (or in An), which we'll see in Definition 6.9. Fortunately, these limits can be pinned down in explicit terms; for example, it's not too hard to unravel Lemma 6.14 to arrive at the description from [Ols16, §4.2] or [Stacks, Tag 026B]. In particular, the result is (the nerve of) a groupoid again.

   Thus, by exclusively working on the fibration side of the cartesian straightening equivalence, the theory of stacks can be and has been developed within the framework of ordinary category theory. But with today's tools, its actually possible to talk about stacks in the intended way: as functors into $\mathrm{Grpd}^{(2)}$. I find the latter much easier.

   We've seen in 5.7 that Theorem 5.4 is already interesting for functors into $\mathrm{Grpd}^{(2)} \subseteq \mathrm{An}$. But there's an even simpler case: functors into *sets!* As it turns out, even this simplest possible special case is interesting and recovers classical theory.

5.8. Straightening/unstraightening and covering theory. — An anima is called *discrete* if it is homotopy equivalent to a disjoint union of copies of the point $*$. Equivalently, all path

---

hence an equivalence by Theorem 4.6. To prove that $\mathrm{Hom}_{\mathcal{U}}(u,v)$ is discrete, observe that we have a map $\mathrm{Hom}_{\mathcal{U}}(u,v) \to \mathrm{Hom}_{\mathrm{N}(\mathcal{C})}(p(u),p(v))$. Since the target is a discrete anima, the source $\mathrm{Hom}_{\mathcal{U}}(u,v)$ is a disjoint union of the fibres of this map. So it's enough to show that each individual fibre is a discrete anima. Restricting to the fibre over $\alpha \in \mathrm{Hom}_{\mathrm{N}(\mathcal{C})}(p(u),p(v))$ amounts to base changing along the map $\alpha \colon \Delta^1 \to \mathrm{N}(\mathcal{C})$. So we may assume $\mathrm{N}(\mathcal{C}) \cong \Delta^1$.

   Now $\{0\} \to \Delta^1$ is fully faithful. Hence, if $\mathcal{U}_0 \coloneqq \{0\} \times_{\Delta^1} \mathcal{U}$ denotes the fibre over 0, then $\mathcal{U}_0 \to \mathcal{U}$ is fully faithful too. But $\mathcal{U}_0$ is also equivalent to the nerve of an ordinary category, because we assume our original functor F takes values in $\mathrm{Cat}^{(2)} \subseteq \mathrm{Cat}_\infty$. So if $u,v \in \mathcal{U}_0$, then $\mathrm{Hom}_{\mathcal{U}}(u,v) \simeq \mathrm{Hom}_{\mathcal{U}_0}(u,v)$ is a discrete anima. The same reasoning applies if u and v both belong to the fibre over 1. If $p(u) = 1$ and $p(v) = 0$, then the map $\mathrm{Hom}_{\mathcal{U}}(u,v) \to \mathrm{Hom}_{\Delta^1}(1,0) \simeq \emptyset$ forces $\mathrm{Hom}_{\mathcal{U}}(u,v) \simeq \emptyset$, which is discrete too. It remains to deal with the case $p(u) = 0$ and $p(v) = 1$. After choosing a p-cartesian lift $\varphi \colon u' \to v$ of $0 \to 1$, the dual of Lemma 5.16 below provides a homotopy pullback

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{U}}(u,u') & \xrightarrow{\varphi_*} & \mathrm{Hom}_{\mathcal{U}}(u,v) \\ {\scriptstyle p}\downarrow & \lrcorner_h & \downarrow{\scriptstyle p} \\ \mathrm{Hom}_{\Delta^1}(0,0) & \xrightarrow{p(\varphi)_*} & \mathrm{Hom}_{\Delta^1}(0,1) \end{array}$$

The bottom horizontal arrow is clearly a homotopy equivalence, so the top arrow must be one as well, whence $\mathrm{Hom}_{\mathcal{U}}(u,u') \simeq \mathrm{Hom}_{\mathcal{U}}(u,v)$. As we already know $\mathrm{Hom}_{\mathcal{U}}(u,u')$ to be discrete, we're done.

[(5.3)] It follows from the uniqueness statement in Lemma 2.7(b) that any map between nerves of ordinary categories is automatically an inner fibration. So to check whether such a map is a cartesian fibration, it's enough to show the existence of cartesian lifts.

components are contractible. Considering sets as discrete animae, it's easy to construct a functor $\mathrm{N}(\mathrm{Set}) \to \mathrm{An}$; this functor is fully faithful and an equivalence onto the full sub-quasi-category spanned by the discrete animae.[(5.4)]

Let X be an anima. A *covering of X* is a Kan fibration (or equivalently a left fibration, see Corollary 4.3) $p\colon X' \to X$ such that for all $x \in X$, the fibres $p^{-1}\{x\}$ are discrete animae. This recovers the usual notion of coverings from topology. More precisely, let's call a covering $p\colon X' \to X$ *strict* if the fibres $p^{-1}\{x\}$ are not only equivalent to but *isomorphic* to disjoint unions of copies of $*$. Then every covering is equivalent to a strict covering and the adjunction $|\cdot|\colon \mathrm{Kan} \rightleftarrows \mathrm{Top} : \mathrm{Sing}$ from 2.4 transform strict coverings of animae into usual coverings of topological spaces and vice versa.[(5.5)]

We let $\mathrm{Cov}(X) \subseteq \mathrm{Left}(X)$ denote the full sub-quasi-category spanned by the coverings of X. Under the straightening equivalence from Theorem 5.4(*b*), coverings $p\colon X' \to X$ correspond to those functors $F\colon X \to \mathrm{An}$ that land in discrete animae. Thus, we get an equivalence of quasi-categories

$$\mathrm{Cov}(X) \xrightarrow{\simeq} \mathrm{F}\big(X, \mathrm{N}(\mathrm{Set})\big).$$

Now recall $\mathrm{F}(X, \mathrm{N}(\mathrm{Set})) \cong \mathrm{N}(\mathrm{Fun}(h(X), \mathrm{Set}))$ from Lemma 2.14. But what is $h(X)$? We know $X \simeq \mathrm{Sing}\,|X|$ from the simplicial approximation theorem, hence $h(X) \simeq h(\mathrm{Sing}\,|X|)$. By the description in 2.13, the objects of $h(\mathrm{Sing}\,|X|)$ are given by $\mathrm{Sing}_0\,|X|$, the points of $|X|$. The morphisms of $h(\mathrm{Sing}\,|X|)$ are equivalence classes of $\mathrm{Sing}_1\,|X|$, that is, equivalence classes of paths in X. A quick unravelling of definitions shows that the equivalence relation is precisely for two paths to be homotopic. Hence $h(\mathrm{Sing}\,|X|)$ is precisely the *fundamental groupoid* $\Pi_1|X|$ of the topological space $|X|$, and therefore $h(X) \simeq \Pi_1|X|$. We have thus proved a classical classification result from topology:

5.9. Theorem (Classification of covering animae). — *Let X be an anima. Then $\mathrm{Cov}(X)$ is equivalent to the nerve of the ordinary category $\mathrm{Fun}(\Pi_1|X|, \mathrm{Set})$. In particular, there's an equivalence of ordinary categories*

$$h\big(\mathrm{Cov}(X)\big) \simeq \mathrm{Fun}\big(\Pi_1|X|, \mathrm{Set}\big).$$

[(5.4)]Let's sketch how to do this: One can equip Set with a trivial Kan enrichment Set^{Δ} in which $\mathrm{F}_{\mathrm{Set}^{\Delta}}(S, T)$ is just a disjoint union of $\mathrm{Hom}_{\mathrm{Set}}(S, T)$ many points. Sending $S \mapsto \coprod_{s \in S} *$ then defines a fully faithful simplicially enriched functor $\mathrm{Set}^{\Delta} \to \mathrm{Kan}^{\Delta}$. Applying $\mathrm{N}^{\Delta}(-)$, we obtain a fully faithful functor of quasi-categories $\mathrm{N}(\mathrm{Set}) \cong \mathrm{N}^{\Delta}(\mathrm{Set}^{\Delta}) \to \mathrm{N}^{\Delta}(\mathrm{Kan}^{\Delta})$, whose essential image are precisely the discrete animae.

[(5.5)]To see that every covering $p\colon X' \to X$ is equivalent to a strict one, we need to use some details of the construction of the equivalence from Theorem 5.4(*b*): Since the functor $F\colon X \to \mathrm{An}$ associated to X lands in discrete animae, we can factor it, up to equivalence, through a functor $F_0\colon X \to \mathrm{N}(\mathrm{Set})$. By Lemma 2.14, F_0 is induced by a functor of ordinary categories $\overline{F}_0\colon h(X) \to \mathrm{Set}$. If $\overline{p}\colon \mathcal{U} \to h(X)$ is the *Grothendieck construction* of $\overline{F}_0$, then $\mathrm{N}(\overline{p})\colon \mathrm{N}(\mathcal{U}) \to \mathrm{N}(h(X))$ can be shown to be a strict covering. Hence the pullback $\mathrm{N}(\mathcal{U}) \times_{\mathrm{N}(h(X))} X \to X$ is a strict covering too and equivalent to our original covering p.

Now suppose $p\colon X' \to X$ is a strict covering. Using that the fibres of p are disjoint unions of copies of $*$ together with the lifting properties of Kan fibrations, it's easy to see that for every $\Delta^n \to X$, the pullback $\Delta^n \times_X X'$ consists of a disjoint union of copies of Δ^n. This means that the preimage of any cell in the CW-complex $|X|$ under $|p|\colon |X'| \to |X|$ is a disjoint union of copies of that cell. Via some technical arguments that we omit, this shows that $|p|$ is a covering in the usual sense. Conversely, if $q\colon Y' \to Y$ is a covering of topological spaces, q is Serre fibration. Sing turns Serre fibrations into Kan fibrations because $|\cdot|\colon \mathrm{Kan} \rightleftarrows \mathrm{Top} : \mathrm{Sing}$ is a Quillen adjunction (even a Quillen equivalence) by Example 3.31. Hence $\mathrm{Sing}\,q\colon \mathrm{Sing}\,Y' \to \mathrm{Sing}\,Y$ is a Kan fibration. This could also be shown by an easy direct argument (observe that the pair $(|\Delta^n|, |\Lambda_i^n|)$ is homeomorphic to $([0,1]^{n-1} \times [0,1], [0,1]^{n-1} \times \{0\})$ and use the homotopy lifting property of covering spaces, see [Hat02, Proposition 1.30] for example). Since Sing preserves pullbacks and sends discrete topological spaces to disjoint unions of copies of $*$, we see that $\mathrm{Sing}\,q$ is indeed a strict covering.

It might not be immediately obvious, but Theorem 5.9 comprises all you would ever want to know about covering theory. Since it fits the theme of these notes, let us spell this out:

5.10. Corollary. — *Suppose X is connected and let a basepoint $x \in X$ be chosen.*

(a) *There's a Galois correspondence (that is, a bijection) between connected coverings of X and subgroups $H \subseteq \pi_1(X, x)$.*

(b) *X is simply connected (that is, $\pi_1(X, x) \cong 0$) if and only if every covering $p \colon X' \to X$ admits a section $s \colon X \to X'$ if and only if every covering of X splits into a disjoint union of copies of X.*

(c) *$\mathrm{Cov}(X)$ contains a unique object $\widetilde{p} \colon \widetilde{X} \to X$ (up to equivalence) with the property that $\widetilde{X}$ is simply connected. This covering $\widetilde{p} \colon \widetilde{X} \to X$ is called the universal covering of X. If $\mathrm{Aut}_X(\widetilde{X})$ denotes the group of deck transformations of $\widetilde{X}$, that is, the automorphism group of $p \colon \widetilde{X} \to X$ in $h(\mathrm{Cov}(X))$, then $\mathrm{Aut}_X(\widetilde{X}) \cong \pi_1(X, x)$.*

(d) *Suppose $p' \colon X' \to X$ is a covering and $x' \in X'$ is a point such that $p(x') = x$. Let $f \colon (Z, z) \to (X, x)$ be a morphism of pointed animae, where Z is connected too. Then the pointed lifting problem*

$$
\begin{array}{ccc}
 & & (X', x') \\
 & \overset{f'}{\nearrow} & \downarrow p \\
(Z, z) & \xrightarrow{\;f\;} & (X, x)
\end{array}
$$

has a solution if and only if the image of $f_* \colon \pi_1(Z, z) \to \pi_1(X, x)$ is contained in the image of $p_* \colon \pi_1(X', x') \to \pi_1(X, x)$. In this case, the lift f' is necessarily unique.

Proof sketch. Let's denote $G := \pi_1(X, x)$ for short. The crucial observation is that $\Pi_1|X|$ is equivalent to its full sub-groupoid spanned by $\{x\}$. This full sub-groupoid consists of one element x with $\mathrm{Hom}_{\Pi_1|X|}(x, x) \cong G$ many automorphisms. Therefore, the functor category $\mathrm{Fun}(\Pi_1|X|, \mathrm{Set})$ is equivalent to the category $G\text{-Set}$ of sets together with a left action of G and we obtain an equivalence of quasi-categories

$$
\mathrm{Cov}(X) \simeq \mathrm{N}(G\text{-Set})\,.
$$

With this observation, (a) is immediate: We just have to note that a G-set S is *connected*—that is, S can't be written as a disjoint union of two non-empty G-sets—if and only if S consists of a single G-orbit. This in turn happens if and only if $S \cong G/H$ is the set of left cosets for some subgroup $H \subseteq G$. Part (b) is just as trivial: We have $G \cong 0$ if and only if every G-set S has a fixed point (or in other words, the map $S \to *$ admits a G-equivariant section). Furthermore, $G \cong 0$ if and only if every G-set is a disjoint union of fixed points.

For (c), consider G with the natural action of itself as a G-set. We let $\widetilde{p} \colon \widetilde{X} \to X$ be the associated covering. Then $\mathrm{Aut}_X(\widetilde{X}) \cong \mathrm{Aut}_{G\text{-Set}}(G, G) \cong G$. Furthermore, it's easy to see that $\widetilde{X}$ is simply connected. Indeed, if $q \colon Y \to \widetilde{X}$ is a covering of $\widetilde{X}$, then $\widetilde{p} \circ q \colon Y \to X$ is a covering of X and so q determines a morphism in $\mathrm{Cov}(X)$. This morphism corresponds to a morphism $S \to G$ of G-sets. Every such morphism has a section, which shows that every covering of $\widetilde{X}$ has a section and so $\widetilde{X}$ is simply connected by (b). Conversely, suppose $p \colon X' \to X$ is a covering of X such that X' is simply connected. Let S be the associated G-set; by (a), we must have $S \cong G/H$ for some subgroup $H \subseteq G$. Let $\pi \colon G \to G/H$ be the canonical projection. By abuse of notation, $\pi \colon \widetilde{X} \to X'$ also denotes the associated morphism in $\mathrm{Cov}(X)$. It's easy

to see that π is a covering of X'.[5.6] Since X' is simply connected, π must admit a section $s\colon X' \to \widetilde{X}$. But then $\pi\colon G \to G/H$ also admits a section, which is only possible if H is the trivial subgroup.

For (d), the "only if"-part is trivial. For the "if"-part, we may assume that X' is connected; otherwise just replace X' by the connected component of x'. Then X' corresponds to a G-set of the form G/H. Furthermore, we must have $\pi_1(X', x') \cong H$. To see this, construct a map $\pi\colon \widetilde{X} \to X'$ as in the proof of (c). This is necessarily the universal covering of X'. Then $\mathrm{Aut}_{X'}(\widetilde{X}) \subseteq \mathrm{Aut}_X(\widetilde{X}) \cong G$ is the subgroup of those automorphisms $\tau\colon G \to G$ that satisfy $\pi \circ \tau = \pi$. Hence indeed $\mathrm{Aut}_{X'}(\widetilde{X})$. To solve our lifting problem, note that a lift of f is equivalent to a section of the pullback covering $p_Z\colon Z \times_X X' \to Z$. Since straightening/unstraightening transforms pullbacks into precompositions (see Theorem 5.4(a)), p_Z corresponds to the set G/H with $\pi_1(Z, z)$-action induced by $f_*\colon \pi_1(Z, z) \to \pi_1(X, x) = G$. By assumption, the image of f_* is contained in H, so the action is trivial. Hence G/H is a disjoint union of fixed points; each fixed point determines a section of p_Z. Together with the requirement $f'(z) = x'$, we then get a unique solution. $\qquad\square$

§5.2. Digression: Homotopy pullbacks

After getting acquainted with straightening/unstraightening, our next goal is to prove Yoneda's lemma. But before we can do that, we need to go on a brief detour about *homotopy pullbacks*. These guys will allow us to compute $\mathrm{Hom}_{\mathrm{Ar}(\mathcal{C})}$ and $\mathrm{Hom}_{\mathcal{C}/y}$ in terms of $\mathrm{Hom}_{\mathcal{C}}$ for any quasi-category $\mathcal{C}$, which will be used countless times throughout the rest of this text.

5.11. "Definition". — Suppose we're given a diagram of Kan complexes or quasi-categories

$$
\begin{array}{ccc}
X \longrightarrow X' \\
\downarrow \quad \swArrow \quad \downarrow \\
Y \longrightarrow Y'
\end{array}
\quad \text{or} \quad
\begin{array}{ccc}
\mathcal{C} \longrightarrow \mathcal{C}' \\
\downarrow \quad \swArrow \quad \downarrow \\
\mathcal{D} \longrightarrow \mathcal{D}'
\end{array}
$$

that commutes up to homotopy or up to natural equivalence, respectively (so that the corresponding diagram in An or Cat_∞-commutes; see the discussion in Example 2.23). We say that the diagram is a *homotopy pullback* if its image in An or Cat_∞ is a pullback in the ∞-categorical sense (which we will only define in Definition 6.9(a) below).

As stated, this "definition" doesn't lead to vicious circles, but once you try to prove anything with it, it surely does. So let us give a workable definition, which is equivalent (albeit not obviously so):

(a) A diagram of Kan complexes is a *homotopy pullback* if it is homotopy equivalent to an honest pullback of Kan complexes in which one leg is a Kan fibration.

(b) A diagram of quasi-categories is a *homotopy pullback* if it is equivalent to an honest pullback of quasi-categories legs in which one leg is an *isofibration* (or *categorical fibration* in Lurie's terminology). That is, it is an inner fibration and has the lifting property against

---

[5.6]We only need to show that π is a Kan fibration, because the fact that the fibres of π are discrete follows easily from the fact that the fibres of $p\colon X' \to X$ and $p \circ \pi = \widetilde{p}\colon \widetilde{X} \to X$ are both discrete. To show that π is a Kan fibration, consider any simplex $\sigma\colon \Delta^n \to X'$ and its image $p \circ \sigma\colon \Delta^n \to X$ in X. To solve any horn lifting problem involving σ, we may as well base change $\widetilde{X}$ and X along $p \circ \sigma$. But the pullbacks $\Delta^n \times_{p\circ\sigma, X, p} X'$ and $\Delta^n \times_{p\circ\sigma, X, \widetilde{p}} \widetilde{X}$ are both disjoint unions of copies of Δ^n because we assume $p\colon X' \to X$ and $\widetilde{p}\colon \widetilde{X} \to X$ to be coverings. So the new horn lifting problem has a solution for trivial reasons.

$\{0\} \to \mathrm{N}(J)$. Here $J := \{\bullet \rightleftarrows \bullet\}$ is the category of two objects and a pair of mutually inverse isomorphisms between them, so lifting against $\{0\} \to \mathrm{N}(J)$ means that we can lift equivalences.

So homotopy pullbacks, or equivalently, pullbacks in An or Cat_∞ can be computed as follows: First write down the diagram as a diagram of simplicial sets. Then replace one of its legs by an equivalence followed by a Kan fibration or an isofibration; this can be done by Lemma 3.12.[(5.7)] Finally, take the usual pullback along that Kan or isofibration.

As a consequence, with some care, homotopy pullbacks can usually be manipulated in the same way as ordinary pullbacks. We'll use this freely throughout the rest of this section.

5.12. Model category fact. — There's a notion of *homotopy pullbacks* in any model category. This is done in any sensible treatment of model categories; see for example [Cis19, Definition 2.3.22] or [F-HC$_{\mathrm{II}}$, Definition VIII.49(vi)]. An honest pullback diagram in a model category is automatically a homotopy pullback as well if all objects are fibrant and at least one of the legs is a fibration; see e.g. [Cis19, Proposition 2.3.27] for a proof. Thus, for the Kan–Quillen model structure and the Joyal model structure on sSet (see Examples 3.28 and 3.29) the above "definition" is recovered. ∎

5.13. Lemma. — *Let $\mathcal{C}$ be a quasi-category and let $\alpha\colon x \to y$, $\alpha'\colon x' \to y'$ be morphisms in $\mathcal{C}$. Then there exists a homotopy pullback diagram of animae*

$$
\begin{array}{ccc}
\mathrm{Hom}_{\mathrm{Ar}(\mathcal{C})}\big((\alpha\colon x \to y),(\alpha'\colon x' \to y')\big) & \longrightarrow & \mathrm{Hom}_{\mathcal{C}}(y,y') \\
\downarrow & \ulcorner_h & \downarrow{\alpha^*} \\
\mathrm{Hom}_{\mathcal{C}}(x,x') & \xrightarrow{\ \ \alpha'_*\ \ } & \mathrm{Hom}_{\mathcal{C}}(x,y')
\end{array}
$$

Here the pre- and postcomposition maps α^* and α'_* are defined by means of the functors $\mathrm{Hom}_{\mathcal{C}}(x,-)\colon \mathcal{C} \to \mathrm{An}$ *and* $\mathrm{Hom}_{\mathcal{C}}(-,y')\colon \mathcal{C}^{\mathrm{op}} \to \mathrm{An}$ *from Example 5.5(b).* ∎

5.14. Remark. — The only proof I know is in Fabian's handwritten notes [F-HC$_{\mathrm{II}}$, Proposition VIII.5]. It's not particularly difficult: You work directly with the definition of $\mathrm{Hom}_{\mathrm{Ar}(\mathcal{C})}$ to write it as an honest pullback in which both legs are Kan fibrations. Then you check that the corners of the pullback are homotopy equivalent to $\mathrm{Hom}_{\mathcal{C}}(x,x')$, $\mathrm{Hom}_{\mathcal{C}}(y,y')$, and $\mathrm{Hom}_{\mathcal{C}}(x,y')$, respectively. Along the way, you should also check (but this will be quite apparent from the description in 5.6) that the maps you obtain are really the pre- and postcomposition maps α^* and α'_* as defined above.

Also note that in the case where $\mathcal{C}$ is an ordinary category we recover our original description of morphisms in an arrow category from Construction 1.13. Indeed, in this case all Hom animae are discrete (that is, sets), hence the ambiguity of α^* and α'_* up to homotopy goes away. Furthermore, any map of discrete Kan complexes is automatically a Kan fibration, so the homotopy pullback agrees with the pullback on the nose by "Definition" 5.11(a). If you think about this pullback, that's exactly how morphisms in $\mathrm{Ar}(\mathcal{C})$ are described.

[(5.7)]To replace a functor of quasi-categories by an equivalence followed by an isofibration, a small variation of the argument from Lemma 3.12 is needed. The problem is that $\mathrm{N}(J)$ has countably many non-degenerate simplices, whereas Λ^n_i had only finitely many. This has the effect that it's no longer sufficient to iterate the construction of $S(f)$ countably many times. To fix this, we simply do $\aleph_1$ many iterations instead of $\aleph_0$ many.

5.15. Corollary. — *Let $\mathcal{C}$ be a quasi-category, let $y \in \mathcal{C}$ be an object, and let $\alpha \colon x \to y$ and $\alpha' \colon x' \to y$ be morphisms in $\mathcal{C}$. Then there exists a homotopy pullback diagram of animae*

$$
\begin{array}{ccc}
\mathrm{Hom}_{\mathcal{C}_{/y}}\big((\alpha \colon x \to y),(\alpha' \colon x' \to y)\big) & \longrightarrow & \{\alpha\} \\
\downarrow & \lrcorner_h & \downarrow \\
\mathrm{Hom}_{\mathcal{C}}(x, x') & \xrightarrow{\ \alpha'_*\ } & \mathrm{Hom}_{\mathcal{C}}(x, y)
\end{array}
$$

Here the postcomposition map α'_* is again defined as in Lemma 5.13.

Proof sketch. We use the following pullback square from 2.11:

$$
\begin{array}{ccc}
\mathcal{C}_{/y} & \longrightarrow & \mathrm{Ar}(\mathcal{C}) \\
\downarrow & \lrcorner & \downarrow \\
\{y\} & \longrightarrow & \mathcal{C}
\end{array}
$$

In general, its straightforward to check that Hom in a pullback of quasi-categories is the pullback of Hom in each component. Then we plug in Lemma 5.13 and check that everything works out with homotopy pullbacks too. For a complete proof, see [F-HC$_{\mathrm{II}}$, Corollary VIII.6], where Fabian deduces the result from Lemma 5.13 as we do here, or [L-HTT, Lemma 5.5.5.12], in which Lurie gives a direct argument. $\qquad\square$

Homotopy pullbacks can be used to give an equivalent characterisation of cocartesian edges. In fact, the terminology *(co)cartesian morphism* was originally introduced in the classical theory of stacks (see 5.7), where it was defined using the criterion from Lemma 5.16 below.[(5.8)] The equivalence with Definition 5.2(*a*) is due to Lurie; see [L-HTT, Proposition 2.4.4.3] or [Lan21, Corollary 3.1.16].

5.16. Lemma. — *Let $p \colon \mathcal{U} \to \mathcal{C}$ be an inner fibration of quasi-categories. Then a morphism $\varphi \colon u \to v$ is p-cocartesian if and only if the following diagram is a homotopy pullback of animae for every $w \in \mathcal{U}$:*

$$
\begin{array}{ccc}
\mathrm{Hom}_{\mathcal{U}}(v, w) & \xrightarrow{\ \varphi^*\ } & \mathrm{Hom}_{\mathcal{U}}(u, w) \\
p\downarrow & \lrcorner_h & \downarrow p \\
\mathrm{Hom}_{\mathcal{C}}\big(p(v), p(w)\big) & \xrightarrow{p(\varphi)^*} & \mathrm{Hom}_{\mathcal{C}}\big(p(u), p(w)\big)
\end{array}
$$

Here the precomposition maps φ^* and $p(\varphi)^*$ are once again defined as in Lemma 5.13. $\qquad\blacksquare$

To finish our excursion into homotopy pullbacks, we introduce a variant of cocartesian fibrations that is occasionally quite useful, but will only play a very minor role in these notes.

5.17. Definition. — Let $p \colon \mathcal{U} \to \mathcal{C}$ be an inner fibration of quasi-categories.

(*a*) Let $\varphi \colon u \to v$ be a morphism in $\mathcal{U}$, corresponding to a map $\varphi \colon \Delta^1 \to \mathcal{U}$. We call φ a *locally p-cocartesian morphism* if it is $p_{p\circ\varphi}$-cocartesian, where $p_{p\circ\varphi} \colon \Delta^1 \times_{p\circ\varphi,\mathcal{C}} \mathcal{U} \to \Delta^1$ denotes the pullback of p along $p \circ \varphi \colon \Delta^1 \to \mathcal{C}$.

[(5.8)]Of course, in the classical theory the homotopy pullback of animae was replaced by an ordinary pullback of sets. But observe that a homotopy pullback of simplicial sets, in which all participating objects are disjoint unions of copies of $*$, must automatically be an ordinary pullback. The reason is that any map between two such simplicial sets is automatically a Kan fibration and any homotopy equivalence is automatically an isomorphism.

(*b*)   We call p a *locally cocartesian fibration* if the pullback $p_\alpha\colon \Delta^1 \times_{\alpha,\mathcal{C}} \mathcal{U} \to \Delta^1$ is a cocartesian fibration for every $\alpha\colon \Delta^1 \to \mathcal{C}$.

There are dual notions of *locally p-cartesian morphisms* and *locally cartesian fibrations*.

5.18. Corollary. — *Let $p\colon \mathcal{U} \to \mathcal{C}$ be a locally cocartesian fibration. Then p is a cocartesian fibration if and only if the set of locally p-cartesian morphisms is closed under composition.*

Proof sketch. First assume that p is a cocartesian fibration. Then cocartesian lifts are unique up to equivalence, as we've seen in 5.6. A morphism being cocartesian is preserved under pullbacks. Hence every p-cocartesian morphism φ is also $p_{p \circ \varphi}$-cocartesian. The above-mentioned uniqueness then implies that every locally p-cocartesian morphism must also be p-cocartesian. So locally p-cocartesian morphisms being closed under composition reduces to the same assertion about p-cocartesian morphisms, which is easy to check (for example, using Lemma 5.16).[(5.9)]

Conversely, assume that locally p-cocartesian morphisms are closed under compositions. Let $\varphi\colon u \to v$ be locally p-cocartesian. We wish to show that φ is also p-cocartesian. To this end, we'll verify that the diagram from Lemma 5.16 is a homotopy pullback for all $w \in \mathcal{U}$. Using Theorem 3.18, Lemma 3.19, and the five lemma (plus Remark 3.20), it's enough to show that for every $\alpha \in \mathrm{Hom}_{\mathcal{C}}(p(v), p(w))$, the induced map on homotopy fibres over α is a homotopy equivalence. So fix $\alpha\colon p(v) \to p(w)$ in $\mathcal{C}$. Let $\mathcal{U}_\alpha := \Delta^1 \times_{\alpha,\mathcal{C}} \mathcal{U}$ be the fibre over α. Furthermore, let $\psi\colon v \to v'$ be a locally p-cocartesian lift of α (so that $p(v') = p(w)$). We claim that the homotopy-commutative diagram

(∗)
$$
\begin{array}{ccccc}
\mathrm{Hom}_{\mathcal{U}_\alpha}(v',w) & \xrightarrow{\ \psi^*\ } & \mathrm{Hom}_{\mathcal{U}}(v,w) & \xrightarrow{\ \varphi^*\ } & \mathrm{Hom}_{\mathcal{U}}(u,w) \\
\big\downarrow & {{/\!/\!/}_h} & \big\downarrow{\scriptstyle p} & {{/\!/\!/}_h} & \big\downarrow{\scriptstyle p} \\
\{\mathrm{id}_{p(w)}\} & \xrightarrow{\ \alpha^*\ } & \mathrm{Hom}_{\mathcal{C}}\big(p(v), p(w)\big) & \xrightarrow{\ p(\varphi)^*\ } & \mathrm{Hom}_{\mathcal{C}}\big(p(u), p(w)\big)
\end{array}
$$

exhibits $\mathrm{Hom}_{\mathcal{U}_{p(w)}}(v',w)$ both as the homotopy fibre of $\mathrm{Hom}_{\mathcal{U}}(v,w) \to \mathrm{Hom}_{\mathcal{C}}(p(v), p(w))$ over $\{\alpha\}$ and the homotopy fibre of $\mathrm{Hom}_{\mathcal{U}}(u,w) \to \mathrm{Hom}_{\mathcal{C}}(p(u), p(w))$ over $\{\alpha \circ p(\varphi)\}$. As explained above, if we could show this, we would be done.

To see this, observe that Hom animae in pullbacks are given as pullbacks of Hom animae in the respective factors (which is straightforward to see from 2.11 and we'll see a more general assertion in Lemma 6.76(*a*)). Combining this with the assumption that ψ is locally p-cocartesian and Lemma 5.16, we see that the following diagram consists of a homotopy pullback square and a pullback square on the nose:

$$
\begin{array}{ccccc}
\mathrm{Hom}_{\mathcal{U}_\alpha}(v',w) & \xrightarrow{\ \psi^*\ } & \mathrm{Hom}_{\mathcal{U}_\alpha}(v,w) & \longrightarrow & \mathrm{Hom}_{\mathcal{U}}(v,w) \\
\big\downarrow & {\llcorner\!\lrcorner}_h & \big\downarrow{\scriptstyle p} & {\llcorner\!\lrcorner} & \big\downarrow{\scriptstyle p} \\
\mathrm{Hom}_{\Delta^1}(0,0) & \longrightarrow & \mathrm{Hom}_{\Delta^1}(0,1) & \xrightarrow{\ \alpha\ } & \mathrm{Hom}_{\mathcal{C}}\big(p(v), p(w)\big)
\end{array}
$$

Hence the outer rectangle must be a homotopy pullback too. Since $\mathrm{Hom}_{\Delta^1}(0,0) \simeq * \simeq \{\mathrm{id}_{p(w)}\}$, it follows that the left square in (∗) is a homotopy pullback. Since, by assumption, any choice of composition $\psi \circ \varphi$ is locally p-cocartesian, the same argument can be used to show that the outer rectangle in (∗) is a homotopy pullback too. This proves that $\mathrm{Hom}_{\mathcal{U}_{p(w)}}(v',w)$ agrees with both homotopy fibres in question and we're done. $\qquad\square$

[(5.9)]Closedness under composition is also clear intuitively: If $\beta\colon y \to z$ is another morphism in $\mathcal{C}$ and we compose a p-cocartesian lift of α with a p-cocartesian lift of β, then we have connected an element of $F(x)$ with its image under $F(\beta) \circ F(\alpha) \simeq F(\beta \circ \alpha)\colon F(x) \to F(z)$. And that's a p-cocartesian lift of $\beta \circ \alpha$.

§5.3. The Yoneda lemma

5.19. Theorem (Quasi-categorical Yoneda lemma). — *Let $\mathcal{C}$ be a quasi-category, $x \in \mathcal{C}$ an object, and $E \colon \mathcal{C} \to \mathrm{An}$ a functor. Then evaluation at id_x induces an equivalence of animae*

$$\mathrm{ev}_{\mathrm{id}_x} \colon \ \mathrm{Hom}_{\mathrm{F}(\mathcal{C}, \mathrm{An})}\big(\mathrm{Hom}_{\mathcal{C}}(x, -), E\big) \xrightarrow{\simeq} E(x) \,.$$

Here $\mathrm{Hom}_{\mathcal{C}}(x, -) \colon \mathcal{C} \to \mathrm{An}$ is the functor from Example 5.5(b). A dual statement holds for the contravariant Hom functor $\mathrm{Hom}_{\mathcal{C}}(-, x)$ and $\mathrm{F}(\mathcal{C}^{\mathrm{op}}, \mathrm{An})$.

For the proof we need, more or less, the fact that $* \simeq \{\mathrm{id}_x\} \to \mathcal{C}_{x/}$ is a left anodyne map. This is proved in [Lan21, Lemma 4.1.4] or [F-HC$_{\mathrm{II}}$, Corollary D.7]. Their proofs use some constructions we haven't mentioned yet, but we can circumvent these at the cost of showing a slightly weaker statement, which will still be sufficient for our purposes.

5.20. Lemma. — *Let $\mathcal{C}$ be a quasi-category and $x \in \mathcal{C}$ an object. For every left fibration $X \to \mathcal{C}$, the natural map*

$$\mathrm{F}\big(\mathcal{C}_{x/}, X\big) \xrightarrow{\simeq} \mathrm{F}(*, X) \times_{\mathrm{F}(*, \mathcal{C})} \mathrm{F}\big(\mathcal{C}_{x/}, \mathcal{C}\big)$$

is an equivalence of quasi-categories.

Proof sketch. We call a cofibration $A \to B$ of simplicial sets *weakly left anodyne* if the natural map $\mathrm{F}(B, X) \to \mathrm{F}(A, X) \times_{\mathrm{F}(A, \mathcal{C})} \mathrm{F}(B, \mathcal{C})$ is an equivalence of quasi-categories for all left fibrations $X \to \mathcal{C}$. Every left anodyne map is weakly left anodyne by Corollary 3.10. Our goal is to show that $* \to \mathcal{C}_{x/}$ is weakly left anodyne.

The idea to show this is as follows: Intuitively, it's clear that $\mathrm{id}_x \in \mathcal{C}_{x/}$ is an initial object. Therefore, there should be a natural transformation $\eta \colon \mathrm{const}\{\mathrm{id}_x\} \Rightarrow \mathrm{id}_{\mathcal{C}_{x/}}$. So η witnesses the fact that $* \to \mathcal{C}_{x/}$ is a homotopy equivalence—except that $\mathcal{C}_{x/}$ is not an anima. Still, as we'll see, η can then be leveraged to show the desired statement.

To construct η, we can proceed as follows: The identity on $\mathrm{Ar}(\mathcal{C})$ is adjoint to a map $\mathrm{Ar}(\mathcal{C}) \times \Delta^1 \to \mathcal{C}$. Combining this with the map $\mathcal{C} \simeq \mathrm{F}(*, \mathcal{C}) \to \mathrm{F}(\Delta^1, \mathcal{C}) \simeq \mathrm{Ar}(\mathcal{C})$ induced by $\Delta^1 \to *$ provides a map $\mathrm{Ar}(\mathcal{C}) \times \Delta^1 \to \mathrm{Ar}(\mathcal{C})$; restricting this to $\mathcal{C}_{x/}$ yields the desired map $\eta \colon \mathcal{C}_{x/} \times \Delta^1 \to \mathcal{C}_{x/}$. Putting $\mathcal{C}_{x/}^{\triangleleft} := (\mathcal{C}_{x/} \times \Delta^1)/(\mathcal{C}_{x/} \times \{0\})$, it's easy to check that η factors through a map $\overline{\eta} \colon \mathcal{C}_{x/}^{\triangleleft} \to \mathcal{C}_{x/}$. The map $* \to \mathcal{C}_{x/}$ induces a map $\Delta^1 \simeq *^{\triangleleft} \to \mathcal{C}_{x/}^{\triangleleft}$. This fits into a diagram

$$
\begin{array}{ccccc}
* & \longrightarrow & \Delta^1 & \longrightarrow & * \\
\downarrow & /\!/\!/ & \downarrow & /\!/\!/ & \downarrow \\
\mathcal{C}_{x/} & \longrightarrow & \mathcal{C}_{x/}^{\triangleleft} & \xrightarrow{\ \overline{\eta}\ } & \mathcal{C}_{x/}
\end{array}
$$

which exhibits $* \to \mathcal{C}_{x/}$ as a retract of $\Delta^1 \to \mathcal{C}_{x/}^{\triangleleft}$. Hence to show that $* \to \mathcal{C}_{x/}$ is weakly left anodyne, it's enough to show the same for $\Delta^1 \to \mathcal{C}_{x/}^{\triangleleft}$. For this, note that $\{0\} \to \Delta^1$ is left anodyne, and so is the composition $\{0\} \to \Delta^1 \to \mathcal{C}_{x/}^{\triangleleft}$ since it is a pushout of $\mathcal{C}_{x/} \times \{0\} \to \mathcal{C}_{x/} \times \Delta^1$. It's easy to check that being weakly left anodyne is closed under 2-out-of-3, and so $\Delta^1 \to \mathcal{C}_{x/}^{\triangleleft}$ must be weakly left anodyne too. $\qquad\square$

Proof sketch of Theorem 5.19. Let $p \colon \mathcal{U} \to \mathcal{C}$ be the unstraightening of $E \colon \mathcal{C} \to \mathrm{An}$. Then

$$
\begin{aligned}
\mathrm{Hom}_{\mathrm{F}(\mathcal{C}, \mathrm{An})}\big(\mathrm{Hom}_{\mathcal{C}}(x, -), E\big) &\simeq \mathrm{Hom}_{\mathrm{Left}(\mathcal{C})}\big(t \colon \mathcal{C}_{x/} \to \mathcal{C}, p \colon \mathcal{U} \to \mathcal{C}\big) \\
&\simeq \mathrm{Hom}_{\mathrm{Cat}_{\infty/\mathcal{C}}}\big(t \colon \mathcal{C}_{x/} \to \mathcal{C}, p \colon \mathcal{U} \to \mathcal{C}\big)
\end{aligned}
$$

using the straightening equivalence (Theorem 5.4(b)) and the fact that $\mathrm{Left}(\mathcal{C}) \to \mathrm{Cat}_{\infty/\mathcal{C}}$ is fully faithful. By Corollary 5.15, the diagram

$$
\begin{array}{ccc}
\mathrm{Hom}_{\mathrm{Cat}_{\infty/\mathcal{C}}}\big((t\colon \mathcal{C}_{x/} \to \mathcal{C}),(p\colon \mathcal{U} \to \mathcal{C})\big) & \longrightarrow & * \\
\downarrow & \lrcorner_h & \downarrow \\
\mathrm{Hom}_{\mathrm{Cat}_{\infty}}\big(\mathcal{C}_{x/},\mathcal{U}\big) & \longrightarrow & \mathrm{Hom}_{\mathrm{Cat}_{\infty}}\big(\mathcal{C}_{x/},\mathcal{C}\big)
\end{array}
$$

is a homotopy pullback, where $*$ is sent to $t\colon \mathcal{C}_{x/} \to \mathcal{C}$. Now recall from Theorem 2.24 that $\mathrm{Hom}_{\mathrm{Cat}_{\infty}}(-,-) \simeq \mathrm{core}\,\mathrm{F}(-,-)$. Furthermore, we claim that the following diagrams are homotopy pullbacks:

$$
\begin{array}{ccc}
\mathrm{F}\big(\mathcal{C}_{x/},\mathcal{U}\big) & \longrightarrow & \mathrm{F}\big(\mathcal{C}_{x/},\mathcal{C}\big) \\
\downarrow & \lrcorner_h & \downarrow \\
\mathcal{U} & \longrightarrow & \mathcal{C}
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
\mathrm{core}\,\mathrm{F}\big(\mathcal{C}_{x/},\mathcal{U}\big) & \longrightarrow & \mathrm{core}\,\mathrm{F}\big(\mathcal{C}_{x/},\mathcal{C}\big) \\
\downarrow & \lrcorner_h & \downarrow \\
\mathrm{core}(\mathcal{U}) & \longrightarrow & \mathrm{core}(\mathcal{C})
\end{array}
$$

For the left one, we use Lemma 5.20 and "Definition" 5.11(b): We only need to check that $p\colon \mathcal{U} \to \mathcal{C}$ is an isofibration. But any left fibration has lifting against $\{0\} \to \mathrm{N}(J)$, as this map is left anodyne (by an explicit horn filling argument). To see that the right square is a homotopy pullback too, we need to check that $\mathrm{core}\colon \mathrm{QCat} \to \mathrm{Kan}$ preserves homotopy pullbacks. The deeper reason for this is of course that $\mathrm{core}\colon \mathrm{Cat}_{\infty} \to \mathrm{An}$ is right adjoint to the inclusion $\mathrm{An} \subseteq \mathrm{Cat}_{\infty}$ (see Example 6.3(a)). For a direct argument, we can use 5.12: By an easy application of Joyal's lifting theorem (Theorem 4.1), core transforms isofibrations into Kan fibrations and then by arguments as in the proof of Theorem 4.6 we can show that core preserves pullbacks of quasi-categories in which at least one leg is an isofibration.

Combining the homotopy pullbacks so far (this kind of manipulation is fine as we've argued in "Definition" 5.11), we find that

$$
\begin{array}{ccc}
\mathrm{Hom}_{\mathrm{Cat}_{\infty/\mathcal{C}}}\big((t\colon \mathcal{C}_{x/} \to \mathcal{C}),(p\colon \mathcal{U} \to \mathcal{C})\big) & \longrightarrow & * \\
\downarrow & \lrcorner_h & \downarrow \\
\mathrm{core}(\mathcal{U}) & \longrightarrow & \mathrm{core}(\mathcal{C})
\end{array}
$$

is a homotopy pullback, where $*$ is sent to $x \in \mathrm{core}(\mathcal{C})$. As observed above, $\mathcal{U} \to \mathcal{C}$ is an isofibration and so $\mathrm{core}(\mathcal{U}) \to \mathrm{core}(\mathcal{C})$ is a Kan fibration. Thus, the homotopy pullback agrees with the ordinary pullback. Now $\mathrm{core}(\mathcal{U}) \times_{\mathrm{core}(\mathcal{C})} \{x\} \cong \mathrm{core}(\mathcal{U} \times_{\mathcal{C}} \{x\}) \cong \mathcal{U} \times_{\mathcal{C}} \{x\} \simeq F(x)$ using that the fibres $\mathcal{U} \times_{\mathcal{C}} \{x\} =: p^{-1}\{x\}$ of p are animae and compute the values of F. This is what we wanted to prove. $\qquad\square$

Finally, we would like to construct the functor $\mathrm{Hom}_{\mathcal{C}}\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathrm{An}$ and then use it to prove the embedding version of Yoneda's lemma. There are several constructions for $\mathrm{Hom}_{\mathcal{C}}$ and we'll outline two possibilities in Constructions 5.21 and 5.22 below. We won't prove that they are equivalent (they are), but we won't ever need that either. So you can just choose whichever is your favourite.

5.21. Construction. — Consider the functor $\mathcal{C}_{/-}\colon \mathcal{C} \to \mathrm{Cat}_{\infty}$ from Example 5.5(c). For every $x \in \mathcal{C}$ there is a natural functor $\mathcal{C}_{/x} \to \mathcal{C}$, so we would expect that $\mathcal{C}_{/-}$ lifts to a functor

$\mathcal{C}_{/-}\colon \mathcal{C} \to \mathrm{Cat}_{\infty/\mathcal{C}}$. To construct such a lift, first note that for all quasi-categories $\mathcal{D}$ and all $y \in \mathcal{D}$, we have an equivalence $\mathrm{F}(\mathcal{C}, \mathcal{D}_{/y}) \simeq \mathrm{F}(\mathcal{C}, \mathcal{D})_{/\operatorname{const} y}$ (in fact, even an isomorphism of simplicial sets), as can be checked by a simple calculation. Furthermore, Theorem 5.4(*a*) and Example 5.5(*a*) imply that $\mathrm{F}(\mathcal{C}, \mathrm{Cat}_\infty)_{/\operatorname{const} \mathcal{C}} \simeq \mathrm{Cocart}(\mathcal{C})_{/(\mathrm{pr}_2 \colon \mathcal{C}\times\mathcal{C}\to\mathcal{C})}$ holds. So to lift our functor $\mathcal{C}_{/-}$ to $\mathrm{Cat}_{\infty/\mathcal{C}}$, it suffices to observe that the following diagram is a morphism of cocartesian fibrations over $\mathcal{C}$:

$$
\begin{array}{ccc}
\mathrm{Ar}(\mathcal{C}) & \xrightarrow{\ (s,t)\ } & \mathcal{C} \times \mathcal{C} \\
{\scriptstyle t}\downarrow & {\scriptstyle /\!/\!/} \;\;\; \swarrow {\scriptstyle \mathrm{pr}_2} & \\
\mathcal{C} & &
\end{array}
$$

By the dual of Corollary 4.4, $\mathcal{C}_{/x} \to \mathcal{C}$ is a right fibration for all $x \in \mathcal{C}$, hence $\mathcal{C}_{/-}\colon \mathcal{C} \to \mathrm{Cat}_{\infty/\mathcal{C}}$ takes values in the full sub-quasi-category $\mathrm{Right}(\mathcal{C}) \subseteq \mathrm{Cat}_{\infty/\mathcal{C}}$. Since $\mathrm{Right}(\mathcal{C}) \simeq \mathrm{F}(\mathcal{C}^{\mathrm{op}}, \mathrm{An})$ by the dual of Theorem 5.4(*b*), we obtain a functor

$$
\yoneda_{\mathcal{C}}\colon \mathcal{C} \xrightarrow{\ \mathcal{C}_{/-}\ } \mathrm{Right}(\mathcal{C}) \xrightarrow{\ \mathrm{St}^{\mathrm{right}}\ } \mathrm{F}(\mathcal{C}^{\mathrm{op}}, \mathrm{An}) \,,
$$

which we take as our definition of the Yoneda embedding (we'll see in Corollary 5.25 below that it is indeed fully faithful). Finally, we let $\mathrm{Hom}_{\mathcal{C}}\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathrm{An}$ be the image of $\yoneda_{\mathcal{C}}$ under the "currying" equivalence $\mathrm{F}(\mathcal{C}, \mathrm{F}(\mathcal{C}^{\mathrm{op}}, \mathrm{An})) \simeq \mathrm{F}(\mathcal{C}^{\mathrm{op}} \times \mathcal{C}, \mathrm{An})$. We define the *twisted arrow quasi-category* $(s,t)\colon \mathrm{TwAr}(\mathcal{C}) \to \mathcal{C}^{\mathrm{op}} \times \mathcal{C}$ to be the unstraightening of $\mathrm{Hom}_{\mathcal{C}}\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathrm{An}$ via Theorem 5.4(*b*).

Construction 5.21 has the advantage that it allows for a straightforward proof of Corollary 5.25 below. On the downside, however, the unstraightening $\mathrm{TwAr}(\mathcal{C})$ is very inexplicit in this description. So alternatively, one can write down an explicit simplicial model for $(s,t)\colon \mathrm{TwAr}(\mathcal{C}) \to \mathcal{C}^{\mathrm{op}} \times \mathcal{C}$ and define $\mathrm{Hom}_{\mathcal{C}}$ to be its straightening.

5.22. Construction. — For an ordinary category $\mathcal{C}$, we define the *twisted arrow category* $\mathrm{TwAr}(\mathcal{C})$ to be the ordinary category whose objects are arrows $\alpha\colon x \to y$ in $\mathcal{C}$ and whose morphisms $(\alpha\colon x \to y) \to (\alpha'\colon x' \to y')$ are "twisted" commutative squares

$$
\begin{array}{ccc}
x & \longleftarrow & x' \\
{\scriptstyle \alpha}\downarrow & {\scriptstyle /\!/\!/} & \downarrow{\scriptstyle \alpha'} \\
y & \longrightarrow & y'
\end{array}
$$

There are functors $s\colon \mathrm{TwAr}(\mathcal{C}) \to \mathcal{C}^{\mathrm{op}}$ and $t\colon \mathrm{TwAr}(\mathcal{C}) \to \mathcal{C}$ that send $\alpha\colon x \to y$ to x and y, respectively. For a quasi-category $\mathcal{C}$, we re-define $\mathrm{TwAr}(\mathcal{C})$ to be the simplicial set given by

$$
\mathrm{TwAr}(\mathcal{C})_n \vcentcolon= \mathrm{Hom}_{\mathrm{sSet}}\big(\mathrm{N}([n]^{\mathrm{op}} \star [n]), \mathcal{C}\big) \,.
$$

Here $[n]^{\mathrm{op}} \star [n]$ is the *join* of the totally ordered sets $[n]^{\mathrm{op}}$ and $[n]$. In general, if $\mathcal{I}$ and $\mathcal{J}$ are ordinary categories, we let $\mathcal{I} \star \mathcal{J}$ be the category obtained from the disjoint union $\mathcal{I} \sqcup \mathcal{J}$ by adding precisely one morphism $i \to j$ for all $i \in \mathcal{I}$, $j \in \mathcal{J}$.

The natural maps $(\Delta^n)^{\mathrm{op}} \cong \mathrm{N}([n]^{\mathrm{op}}) \to \mathrm{N}([n]^{\mathrm{op}} \star [n])$ and $\Delta^n \cong \mathrm{N}([n]) \to \mathrm{N}([n]^{\mathrm{op}} \star [n])$ induce maps of simplicial sets $s\colon \mathrm{TwAr}(\mathcal{C}) \to \mathcal{C}^{\mathrm{op}}$ and $t\colon \mathrm{TwAr}(\mathcal{C}) \to \mathcal{C}$. It turns out that $(s,t)\colon \mathrm{TwAr}(\mathcal{C}) \to \mathcal{C}^{\mathrm{op}} \times \mathcal{C}$ is always a left fibration; in particular, $\mathrm{TwAr}(\mathcal{C})$ is a quasi-category. See [L-HA, Proposition 5.2.1.3] or [Lan21, Proposition 4.2.4] for proofs. We can then define $\mathrm{Hom}_{\mathcal{C}}\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathrm{An}$ to be the straightening of $(s,t)\colon \mathrm{TwAr}(\mathcal{C}) \to \mathcal{C}^{\mathrm{op}} \times \mathcal{C}$.

5.23. Remark. — If $\mathcal{C}$ is an ordinary category, then $\mathrm{N}(\mathrm{TwAr}(\mathcal{C})) \simeq \mathrm{TwAr}(\mathrm{N}(\mathcal{C}))$, no matter how you define the right-hand side. So the notational overload checks out. If you use Construction 5.22, this equivalence is even an isomorphism of simplicial sets and straightforward to verify. If you use Construction 5.21 instead, the proof is still not too hard, but it requires you to know how straightening/unstraightening works under the hood, at least for ordinary categories (in which case straightening/unstraightening is known as the *Grothendieck construction*).

Let's do a quick reality checks for our newly constructed functor $\mathrm{Hom}_{\mathcal{C}}$:

5.24. Lemma. — *Let* $\mathrm{Hom}_{\mathcal{C}} \colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathrm{An}$ *be the functor from Construction* 5.21 *or from Construction* 5.22. *For all* $x, y \in \mathcal{C}$, *the restrictions*

$$\mathrm{Hom}_{\mathcal{C}} \,|_{\{x\} \times \mathcal{C}} \colon \mathcal{C} \longrightarrow \mathrm{An} \quad and \quad \mathrm{Hom}_{\mathcal{C}} \,|_{\mathcal{C}^{\mathrm{op}} \times \{y\}} \colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathrm{An}$$

agree with the functors $\mathrm{Hom}_{\mathcal{C}}(x, -)$ *and* $\mathrm{Hom}_{\mathcal{C}}(-, y)$ *constructed in Example* 5.5(*b*).

Proof sketch, assuming Construction 5.21. It's straightforward to see from the construction that $\mathrm{Hom}_{\mathcal{C}} \,|_{\mathcal{C}^{\mathrm{op}} \times \{y\}} \colon \mathcal{C}^{\mathrm{op}} \to \mathrm{An}$ is the straightening of the right fibration $\mathcal{C}_{/y} \to \mathcal{C}$, which is also the definition of $\mathrm{Hom}_{\mathcal{C}}(-, y)$ in Example 5.5(*b*). Now let $\mathcal{U} \to \mathcal{C}$ be the unstraightening of $\mathrm{Hom}_{\mathcal{C}} \,|_{\{x\} \times \mathcal{C}}$. Note that evaluating a functor $T \colon \mathcal{C}^{\mathrm{op}} \to \mathrm{An}$ at $x \in \mathcal{C}^{\mathrm{op}}$ is the same as restriction along $\mathrm{F}(\mathcal{C}^{\mathrm{op}}, \mathrm{An}) \to \mathrm{F}(\{x\}, \mathrm{An}) \simeq \mathrm{An}$. By the dual of Theorem 5.4(*b*), this corresponds to the pullback functor $x^* \colon \mathrm{Right}(\mathcal{C}) \to \mathrm{Right}(\{x\}) \simeq \mathrm{An}$ under the right straightening equivalence. So $\mathrm{Hom}_{\mathcal{C}} \,|_{\{x\} \times \mathcal{C}}$ can be described as the composition

$$\mathrm{Hom}_{\mathcal{C}} \,|_{\{x\} \times \mathcal{C}} \colon \mathcal{C} \xrightarrow{\;\mathcal{C}_{/-}\;} \mathrm{Right}(\mathcal{C}) \xrightarrow{\;x^*\;} \mathrm{Right}\big(\{x\}\big) \simeq \mathrm{An} \,.$$

By "inspection"[(5.10)], this means that the following diagram is a homotopy pullback:

$$
\begin{array}{ccc}
\mathcal{U} & \longrightarrow & \{x\} \times \mathcal{C} \\
\downarrow & \llcorner_h & \downarrow \\
\mathrm{Ar}(\mathcal{C}) & \xrightarrow{(s,t)} & \mathcal{C} \times \mathcal{C}
\end{array}
$$

To compute this homotopy pullback, we use "Definition" 5.11(*b*): We claim that $(s, t) \colon \mathrm{Ar}(\mathcal{C}) \to \mathcal{C} \times \mathcal{C}$ is already an isofibration. Indeed, it's an inner fibration by Corollary 3.10 and lifting of equivalences follows easily from Theorem 4.5. So we can just take the pullback on the nose, which is $\mathcal{C}_{x/}$ by 2.11. But $\mathrm{Hom}_{\mathcal{C}}(x, -) \colon \mathcal{C} \to \mathrm{An}$ was defined to be the straightening of $t \colon \mathcal{C}_{x/} \to \mathcal{C}$. This shows $\mathrm{Hom}_{\mathcal{C}} \,|_{\{x\} \times \mathcal{C}} \simeq \mathrm{Hom}_{\mathcal{C}}(x, -)$. $\qquad\square$

[(5.10)]Unfortunately, verifying that the above diagram is a homotopy pullback requires us to know a little more about how the straightening/unstraightening equivalence is constructed. The idea is to rewrite the equivalence $\mathrm{Hom}_{\mathcal{C}} \,|_{\{x\} \times \mathcal{C}} \simeq x^* \circ (\mathcal{C}_{/-})$ as a pullback $\mathrm{Hom}_{\mathcal{C}} \,|_{\{x\} \times \mathcal{C}} \simeq (\mathcal{C}_{/-}) \times_{\mathrm{const}\,\mathcal{C}} \mathrm{const}\{x\}$ in the functor quasi-category $\mathrm{F}(\mathcal{C}, \mathrm{Cat}_\infty)$ and then to transform this into a pullback in the quasi-category $\mathrm{Cocart}(\mathcal{C})$ via Theorem 5.4(*a*). This immediately yields that the diagram is a pullback in $\mathrm{Cocart}(\mathcal{C})$, hence a homotopy pullback of quasi-categories. However, to make this argument work as stated, we would need Lemma 6.12 below, which would lead to circular reasoning. So instead, one has to show that $x^* \circ (\mathcal{C}_{/-})$ can be written as a homotopy pullback in a suitable simplicial model category whose underlying quasi-category (in the sense of Remarks 4.14 and 4.15) is $\mathrm{F}(\mathcal{C}, \mathrm{Cat}_\infty)$. In fact, the proof of Theorem 5.4 works by deducing it from a Quillen equivalence between simplicial model categories . . .

Proof sketch, assuming Construction 5.22. In this case, we need to show that the pullbacks of $(s,t)\colon \mathrm{TwAr}(\mathcal{C}) \to \mathcal{C}^{\mathrm{op}} \times \mathcal{C}$ along $\{x\} \times \mathcal{C} \to \mathcal{C}^{\mathrm{op}} \times \mathcal{C}$ and $\mathcal{C}^{\mathrm{op}} \times \{y\} \to \mathcal{C}^{\mathrm{op}} \times \mathcal{C}$ are equivalent to $\mathcal{C}_{x/} \to \mathcal{C}$ and $(\mathcal{C}_{/y})^{\mathrm{op}} \to \mathcal{C}^{\mathrm{op}}$, respectively. This is not quite trivial; see [L-HA, Proposition 5.2.1.10] or [Lan21, Lemma 4.2.7]. □

The embedding version of Yoneda's lemma is now a simple consequence.

5.25. Corollary. — *For every quasi-category $\mathcal{C}$, the Yoneda embedding $よ_{\mathcal{C}}\colon \mathcal{C} \to \mathrm{F}(\mathcal{C}^{\mathrm{op}}, \mathrm{An})$ is fully faithful.*

Proof sketch. We must show that $よ_{\mathcal{C}}$ induces equivalences

$$\mathrm{Hom}_{\mathcal{C}}(x,y) \xrightarrow{\simeq} \mathrm{Hom}_{\mathrm{F}(\mathcal{C}^{\mathrm{op}}, \mathrm{An})}\big(\mathrm{Hom}_{\mathcal{C}}(-,x), \mathrm{Hom}_{\mathcal{C}}(-,y)\big)$$

for all $x, y \in \mathcal{C}$. It's clear from Theorem 5.19 that both sides are equivalent via evaluation at id_x. If you go with Construction 5.21, it's straightforward to see that the morphism induced by $よ_{\mathcal{C}}$ is an inverse to evaluation at id_x, so it is an equivalence too. If you prefer Construction 5.22, this needs a little more work, which we omit. □

§5.4. Remarks on model independence

We explained in 0.3 that, at least in an ideal version of these notes, we would proceed via the following steps:

(a) First, throughout §§2–5, we would set up the framework of quasi-categories.

(b) After that, we would identify a few key model-independent statements and prove (or black box) them in the model of quasi-categories.

(c) Finally, starting from §6, all further proofs would be done in a model-independent fashion.

Step (a) is done by now, whereas step (c) lays ahead, so in this subsection, we'll concern ourselves with step (b).

What are these key model-independent statements supposed to be? Of course, Theorems 4.5 and 4.6 are among them, as is the ∞-categorical Yoneda lemma Theorem 5.19. But we'll need a few more. Let's start with three more reality checks for the Hom functor.

5.26. Lemma. — *Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor of quasi-categories. Then the natural maps $\mathrm{Hom}_{\mathcal{C}}(x,y) \to \mathrm{Hom}_{\mathcal{D}}(F(x), F(y))$ assemble into a natural transformation*

$$\mathrm{Hom}_{\mathcal{C}}(-,-) \Longrightarrow \mathrm{Hom}_{\mathcal{D}}\big(F(-), F(-)\big)$$

in $\mathrm{F}(\mathcal{C}^{\mathrm{op}} \times \mathcal{C}, \mathrm{An})$. Here $\mathrm{Hom}_{\mathcal{C}}$ and $\mathrm{Hom}_{\mathcal{D}}$ are the functors from Constructions 5.21 or 5.22.

Proof sketch, assuming Construction 5.21. Consider the morphism $t\colon \mathcal{C} \times_{F,\mathcal{D},s} \mathrm{Ar}(\mathcal{D}) \to \mathcal{D}$. It is a cocartesian fibration, which can be shown using Lemmas 5.13 and 5.16 in the same way as Example 5.5(c). Hence the following diagram is a diagram of cocartesian fibrations over $\mathcal{C}$:

After unravelling Construction 5.21 and using the fact that precompositions correspond to pullbacks under straightening/unstraightening (see Theorem 5.4(*a*)), the diagram above will induce the desired natural transformation $\mathrm{Hom}_{\mathcal{C}}(-,-) \Rightarrow \mathrm{Hom}_{\mathcal{D}}(F(-), F(-))$, provided we can show the following claim:

(⊠) *The cocartesian straightening of the middle vertical arrow is the composite functor*

$$\mathcal{C} \xrightarrow{F} \mathcal{D} \xrightarrow{\mathcal{D}_{/-}} \mathrm{Cat}_{\infty/\mathcal{D}} \xrightarrow{F^*} \mathrm{Cat}_{\infty/\mathcal{C}}.$$

To prove (⊠) it's enough to show that the straightening of $t\colon \mathcal{C} \times_{F,\mathcal{D},s} \mathrm{Ar}(\mathcal{D}) \to \mathcal{D}$ is $F^* \circ \mathcal{D}_{/-}$, since once again, precompositions correspond to pullbacks. Now the following diagram is a homotopy pullback:

$$\begin{array}{ccc}
\mathcal{C} \times_{F,\mathcal{D},s} \mathrm{Ar}(\mathcal{D}) & \longrightarrow & \mathcal{C} \times \mathcal{D} \\
\downarrow & \lrcorner_h & \downarrow{\scriptstyle F \times \mathrm{id}_{\mathcal{D}}} \\
\mathrm{Ar}(\mathcal{D}) & \xrightarrow{(s,t)} & \mathcal{D} \times \mathcal{D}
\end{array}$$

(in fact, it's a pullback on the nose, and $(s,t)\colon \mathrm{Ar}(\mathcal{D}) \to \mathcal{D} \times \mathcal{D}$ is an isofibration; see the argument in the proof of Lemma 5.24). By a similar "inspection" as in the proof of Lemma 5.24, this observation shows that the straightening of $t\colon \mathcal{C} \times_{F,\mathcal{D},s} \mathrm{Ar}(\mathcal{D}) \to \mathcal{D}$ is indeed $F^* \circ \mathcal{D}_{/-}$, thus proving (⊠). $\square$

Proof sketch, assuming Construction 5.22. It's clear from the construction that F induces a map $\mathrm{TwAr}(\mathcal{C}) \to \mathrm{TwAr}(\mathcal{D})$. By the universal property of pullbacks, this factors over a map

$$\mathrm{TwAr}(\mathcal{C}) \longrightarrow (\mathcal{C}^{\mathrm{op}} \times \mathcal{C}) \times_{F^{\mathrm{op}} \times F, \mathcal{D}^{\mathrm{op}} \times \mathcal{D}, (s,t)} \mathrm{TwAr}(\mathcal{D}).$$

The latter is a morphism of left fibrations over $\mathcal{C}^{\mathrm{op}} \times \mathcal{C}$. Since precompositions correspond to pullbacks under straightening/unstraightening (see Theorem 5.4(*a*)), we get a natural transformation $\mathrm{Hom}_{\mathcal{C}}(-,-) \Rightarrow \mathrm{Hom}_{\mathcal{D}}(-,-) \circ (F^{\mathrm{op}} \times F) \simeq \mathrm{Hom}_{\mathcal{D}}(F(-), F(-))$, as desired. $\square$

5.27. Lemma. — *Let $F, G\colon \mathcal{C} \to \mathcal{D}$ be functors of quasi-categories and let $\eta\colon F \Rightarrow G$ be a natural transformation. Then the natural transformation from Lemma 5.26 as well as η_* (postcomposition with η) and η^* (precomposition with η) fit into a commutative diagram*

$$\begin{array}{ccc}
\mathrm{Hom}_{\mathcal{C}}(-,-) & \overset{G}{\Longrightarrow} & \mathrm{Hom}_{\mathcal{D}}\big(G(-), G(-)\big) \\
{\scriptstyle F}\Vert & /\!/\!/ & \Vert{\scriptstyle \eta^*} \\
\mathrm{Hom}_{\mathcal{D}}\big(F(-), F(-)\big) & \overset{\eta_*}{\Longrightarrow} & \mathrm{Hom}_{\mathcal{D}}\big(F(-), G(-)\big)
\end{array}$$

in the quasi-category $\mathrm{F}(\mathcal{C}^{\mathrm{op}} \times \mathcal{C}, \mathrm{An})$.

Proof sketch. First observe that if $\mathcal{C}'$ is another quasi-category, then $\mathrm{Hom}_{\mathcal{C} \times \mathcal{C}'} \simeq \mathrm{Hom}_{\mathcal{C}} \times \mathrm{Hom}_{\mathcal{C}'}$ holds in $\mathrm{F}((\mathcal{C} \times \mathcal{C}')^{\mathrm{op}} \times (\mathcal{C} \times \mathcal{C}'), \mathrm{An})$. Depending on whether you use Construction 5.21 or Construction 5.22, this basically reduces to the observations that $\mathrm{Ar}(\mathcal{C} \times \mathcal{C}') \cong \mathrm{Ar}(\mathcal{C}) \times \mathrm{Ar}(\mathcal{C}')$ and $\mathrm{TwAr}(\mathcal{C} \times \mathcal{C}') \cong \mathrm{TwAr}(\mathcal{C}) \times \mathrm{TwAr}(\mathcal{C}')$, respectively, but in each case you need some model-category arguments to do the reduction, similar to the "inspection" in the proof of Lemma 5.24. We'll skip these arguments.

Now we regard η as a functor $\eta\colon \Delta^1 \times \mathcal{C} \to \mathcal{D}$. Then Lemma 5.26 can be applied to η to obtain a natural transformation

$$\mathrm{Hom}_{\Delta^1 \times \mathcal{C}}(-,-) \Longrightarrow \mathrm{Hom}_{\mathcal{D}}\big(\eta(-),\eta(-)\big)$$

in the functor quasi-category $\mathrm{F}((\Delta^1 \times \mathcal{C})^{\mathrm{op}} \times (\Delta^1 \times \mathcal{C}), \mathrm{An})$. Applying the usual "currying" isomorphism, we obtain $\mathrm{F}((\Delta^1 \times \mathcal{C})^{\mathrm{op}} \times (\Delta^1 \times \mathcal{C}), \mathrm{An}) \cong \mathrm{F}((\Delta^1)^{\mathrm{op}} \times \Delta^1, \mathrm{F}(\mathcal{C}^{\mathrm{op}} \times \mathcal{C}, \mathrm{An}))$. Since $(\Delta^1)^{\mathrm{op}} \times \Delta^1 \cong \Delta^1 \times \Delta^1 \cong \square^2$, an object in $\mathrm{F}((\Delta^1)^{\mathrm{op}} \times \Delta^1, \mathrm{F}(\mathcal{C}^{\mathrm{op}} \times \mathcal{C}, \mathrm{An}))$ corresponds to a commutative square in $\mathrm{F}(\mathcal{C}^{\mathrm{op}} \times \mathcal{C}, \mathrm{An})$. By unravelling the constructions, $\mathrm{Hom}_{\Delta^1 \times \mathcal{C}}(-,-)$ corresponds to the following square:

$$
\begin{array}{ccc}
\mathrm{Hom}_{\mathcal{C}}(-,-) \times \mathrm{Hom}_{\Delta^1}(1,0) & \Longrightarrow & \mathrm{Hom}_{\mathcal{C}}(-,-) \times \mathrm{Hom}_{\Delta^1}(1,1) \\
\Big\Downarrow & /\!/\!/ & \Big\Downarrow \\
\mathrm{Hom}_{\mathcal{C}}(-,-) \times \mathrm{Hom}_{\Delta^1}(0,0) & \Longrightarrow & \mathrm{Hom}_{\mathcal{C}}(-,-) \times \mathrm{Hom}_{\Delta^1}(0,1)
\end{array}
$$

Indeed, this follows from the fact that $\mathrm{Hom}_{\Delta^1 \times \mathcal{C}} \simeq \mathrm{Hom}_{\Delta^1} \times \mathrm{Hom}_{\mathcal{C}}$, as we've checked above; also note that $\mathrm{Hom}_{\Delta^1}(1,0)$ sits in the top left corner rather than $\mathrm{Hom}_{\Delta^1}(0,0)$ because of the way in which we identified $(\Delta^1)^{\mathrm{op}} \times \Delta^1$ with $\square^2$. Now observe that $\mathrm{Hom}_{\Delta^1}(1,0) \cong \emptyset$, whereas $\mathrm{Hom}_{\Delta^1}(0,0) \cong \mathrm{Hom}_{\Delta^1}(0,1) \cong \mathrm{Hom}_{\Delta^1}(1,1) \cong *$. This implies that the top left corner of the diagram above is $\mathrm{const}\,\emptyset$, whereas the other corners are given by $\mathrm{Hom}_{\mathcal{C}}(-,-)$. Similarly, $\mathrm{Hom}_{\mathcal{D}}(\eta(-),\eta(-))$ corresponds to the following commutative square:

$$
\begin{array}{ccc}
\mathrm{Hom}_{\mathcal{D}}\big(G(-),F(-)\big) & \overset{\eta_*}{\Longrightarrow} & \mathrm{Hom}_{\mathcal{D}}\big(G(-),G(-)\big) \\
{\scriptstyle \eta^*}\Big\Downarrow & /\!/\!/ & \Big\Downarrow{\scriptstyle \eta^*} \\
\mathrm{Hom}_{\mathcal{D}}\big(F(-),F(-)\big) & \overset{\eta_*}{\Longrightarrow} & \mathrm{Hom}_{\mathcal{D}}\big(F(-),G(-)\big)
\end{array}
$$

The natural transformation $\mathrm{Hom}_{\Delta^1 \times \mathcal{C}}(-,-) \Rightarrow \mathrm{Hom}_{\mathcal{D}}(\eta(-),\eta(-))$ from Lemma 5.26 corresponds to a morphism between these commutative squares. By inspection, this yields the desired commutative square. $\qquad\square$

5.28. Lemma ("Precomposition commutes with postcomposition"). — *Let $\mathcal{C}, \mathcal{D}$ be quasi-categories and let $\eta\colon F \Rightarrow G$, $\eta'\colon F' \Rightarrow G'$ be natural transformations in $\mathrm{F}(\mathcal{C},\mathcal{D})$. Then the following diagram commutes in the quasi-category $\mathrm{F}(\mathcal{C}^{\mathrm{op}} \times \mathcal{C}, \mathrm{An})$:*

$$
\begin{array}{ccc}
\mathrm{Hom}_{\mathcal{C}}\big(G(-),F'(-)\big) & \overset{\eta'_*}{\Longrightarrow} & \mathrm{Hom}_{\mathcal{D}}\big(G(-),G'(-)\big) \\
{\scriptstyle \eta^*}\Big\Downarrow & /\!/\!/ & \Big\Downarrow{\scriptstyle \eta^*} \\
\mathrm{Hom}_{\mathcal{D}}\big(F(-),F'(-)\big) & \overset{\eta'_*}{\Longrightarrow} & \mathrm{Hom}_{\mathcal{D}}\big(F(-),G'(-)\big)
\end{array}
$$

Proof sketch. The composition $\mathrm{Hom}_{\mathcal{D}} \circ (\eta^{\mathrm{op}} \times \eta')\colon (\Delta^1 \times \mathcal{C})^{\mathrm{op}} \times (\Delta^1 \times \mathcal{C}) \to \mathrm{An}$ can be viewed as a commutative square $\square^2 \to \mathrm{F}(\mathcal{C}^{\mathrm{op}} \times \mathcal{C}, \mathrm{An})$, similar to the proof of Lemma 5.27. Upon unravelling of definitions, this is the square above. $\qquad\square$

Another model-independent statement, which will turn out to be very useful in developing ∞-category theory, is Lurie's straightening/unstraightening equivalence Theorem 5.4. This may come as a little surprise: After all, the way we defined cocartesian and left fibrations very much relies on the simplicial model. Here's a model-independent re-definition:

5.29. Cocartesian and left fibrations, model-independently. — If $p\colon \mathcal{U} \to \mathcal{C}$ is a functor of ∞-categories, Lemma 5.16 provides a model-independent definition of a morphism $\varphi\colon u \to v$ in $\mathcal{U}$ being *p-cocartesian*.[(5.11)] The condition that cocartesian lifts exist (Definition 5.2(b)) can be replaced by the condition that

$$
\begin{array}{ccc}
\mathrm{Ar}^{\mathrm{cocart}}(\mathcal{U}) & \longrightarrow & \mathrm{Ar}(\mathcal{C}) \\
\downarrow & \lrcorner & \downarrow {\scriptstyle s} \\
\mathcal{U} & \xrightarrow{\quad p \quad} & \mathcal{C}
\end{array}
$$

is a pullback in Cat_∞, where $\mathrm{Ar}^{\mathrm{cocart}}(\mathcal{U}) \subseteq \mathrm{Ar}(\mathcal{U})$ is the full sub-∞-category spanned by the p-cocartesian morphisms. This model-independent definition of cocartesian fibrations recovers Definition 5.2, as we'll see in Lemma 5.30 below. Finally, in light of Lemma 5.3, we can redefine p to be a *left fibration* if it is a cocartesian fibration and $\mathrm{Ar}^{\mathrm{cocart}}(\mathcal{U}) \subseteq \mathrm{Ar}(\mathcal{U})$ is an equivalence of ∞-categories.

In particular, Theorem 5.4 can be reinterpreted as a model-independent statement about equivalences of ∞-categories

$$
\mathrm{Cocart}(\mathcal{C}) \simeq \mathrm{Fun}(\mathcal{C}, \mathrm{Cat}_\infty) \quad \text{and} \quad \mathrm{Left}(\mathcal{C}) \simeq \mathrm{Fun}(\mathcal{C}, \mathrm{An})
$$

for any ∞-category $\mathcal{C}$. This statement absolutely belongs to step (b) and it will play a much more prominent role in our treatment of ∞-categories than it does in ordinary category theory.

5.30. Lemma. — *The functor $\mathrm{Ar}^{\mathrm{cocart}}(\mathcal{U}) \to \mathcal{U} \times_{\mathcal{C},s} \mathrm{Ar}(\mathcal{C})$ is always fully faithful (where the pullback is taken in Cat_∞). Furthermore, the following conditions are equivalent:*

(a) *$\mathrm{Ar}^{\mathrm{cocart}}(\mathcal{U}) \to \mathcal{U} \times_{\mathcal{C},s} \mathrm{Ar}(\mathcal{C})$ is also essentially surjective. In particular, it must be an equivalence (by Theorem 4.6), so that $p\colon \mathcal{U} \to \mathcal{C}$ is a cocartesian fibration in the model-independent sense.*

(b) *For every factorisation $p\colon \mathcal{U} \to \mathcal{U}' \to \mathcal{C}$ into an equivalence of quasi-categories followed by an isofibration, $\mathcal{U}' \to \mathcal{C}$ is a cocartesian fibration in the old sense.*

(c) *For some factorisation $p\colon \mathcal{U} \to \mathcal{U}' \to \mathcal{C}$ into an equivalence of quasi-categories followed by an isofibration, $\mathcal{U}' \to \mathcal{C}$ is a cocartesian fibration in the old sense.*

Proof sketch. To show that $\mathrm{Ar}^{\mathrm{cocart}}(\mathcal{U}) \to \mathcal{U} \times_{\mathcal{C},s} \mathrm{Ar}(\mathcal{C})$ is fully faithful, use Lemma 5.13 to compute Hom on either side and Lemma 5.16 to show that they coincide. This is fun to figure out yourself so we'll leave it to you.

The implication (b) $\Rightarrow$ (c) is trivial. For (c) $\Rightarrow$ (a), it's enough to show that the functor $\mathrm{Ar}^{\mathrm{cocart}}(\mathcal{U}') \to \mathcal{U}' \times_{\mathcal{C},s} \mathrm{Ar}(\mathcal{C})$ is essentially surjective, because $\mathcal{U} \to \mathcal{U}'$ is an equivalence of quasi-categories. Since $\mathcal{U}' \to \mathcal{C}$ is a cocartesian fibration, it's an isofibration too, hence the pullback $\mathcal{U}' \times_{\mathcal{C},s} \mathrm{Ar}(\mathcal{C})$ in Cat_∞ can also be taken in simplicial sets (see "Definition" 5.11(b)). Furthermore, the condition from Definition 5.2(b) translates into $\mathrm{Ar}^{\mathrm{cocart}}(\mathcal{U}') \to \mathcal{U}' \times_{\mathcal{C},s} \mathrm{Ar}(\mathcal{C})$ being surjective on 0-simplices. Hence it must be an essentially surjective functor of quasi-categories.

By the same reasoning, to show (a) $\Rightarrow$ (b), we must show that for all factorisations $p\colon \mathcal{U} \to \mathcal{U}' \to \mathcal{C}$ into an equivalence followed by an isofibration, the map of simplicial sets $\mathrm{Ar}^{\mathrm{cocart}}(\mathcal{U}') \to \mathcal{U}' \times_{\mathcal{C},s} \mathrm{Ar}(\mathcal{C})$ is surjective on 0-simplices. The condition from (a) tells us

---

[(5.11)]Of course, instead of "homotopy pullback", one has to say "pullback square in the ∞-category An", which is a model-independent notion as will hopefully become clear in Definition 6.9.

this map is an equivalence of quasi-categories. In particular, it hits every equivalence class of 0-simplices. To show that it really hits every 0-simplex, it then suffices to show that $\mathrm{Ar}^{\mathrm{cocart}}(\mathcal{U}') \to \mathcal{U}' \times_{\mathcal{C},s} \mathrm{Ar}(\mathcal{C})$ is an isofibration. Observe that $\mathrm{Ar}(\mathcal{U}') \to \mathcal{U}' \times_{\mathcal{C},s} \mathrm{Ar}(\mathcal{C})$ is an isofibration—it's an inner fibration by Corollary 3.10 and lifting of equivalences can be shown by a straightforward argument, using that the isofibration $\mathcal{U}' \to \mathcal{C}$ admits lifting of equivalences too. Since $\mathrm{Ar}^{\mathrm{cocart}}(\mathcal{U}') \subseteq \mathrm{Ar}(\mathcal{U}')$ is a full sub-quasi-category in the sense of 2.16 and furthermore closed under equivalences, the map $\mathrm{Ar}^{\mathrm{cocart}}(\mathcal{U}') \to \mathrm{Ar}(\mathcal{U}')$ must be an isofibration. It follows that $\mathrm{Ar}^{\mathrm{cocart}}(\mathcal{U}') \to \mathcal{U}' \times_{\mathcal{C},s} \mathrm{Ar}(\mathcal{C})$ is an isofibration too, as desired. □

The example of cocartesian and left fibrations is only the first of many instances where our constructions with quasi-categories can be retconned into model-independent constructions. We'll see more of that in model category fact 6.13.

To finish this subsection, we'll briefly remark how to make the Yoneda lemma (in the version from Theorem 5.19) functorial, since this will become relevant later.

5.31. Functoriality of the Yoneda lemma. — Let $\mathcal{C}$ be a quasi-category. Let's write $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{An}) =: \mathrm{PSh}(\mathcal{C})$ for the *∞-category of presheaves on $\mathcal{C}$.*[(5.12)] For an object $x \in \mathcal{C}$ and a presheaf $E \colon \mathcal{C}^{\mathrm{op}} \to \mathrm{An}$, the dual of Theorem 5.19 tells us that

$$\mathrm{ev}_{\mathrm{id}_x} \colon \mathrm{Hom}_{\mathrm{PSh}(\mathcal{C})}\big(\mathsf{よ}_{\mathcal{C}}(x), E\big) \xrightarrow{\ \simeq\ } E(x)$$

is an equivalence. It turns out that this equivalence is functorial both in x and in E. Of course, thanks to Theorem 4.5, the only difficulty lies in making the map $\mathrm{ev}_{\mathrm{id}_x}$ functorial. It's straightforward to make it functorial in E, but I couldn't find any easy argument for functoriality in x. So if you do, please tell me!

One way of producing this natural transformation is to construct it on the level of simplicially enriched functors. Implicitly, this requires that the construction of $\mathrm{Hom}_{\mathcal{C}} \colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathrm{An}$ via simplicially enriched functors agrees with our construction; at least in the case of Construction 5.22, this was done by Lurie [L-HA, Proposition 5.2.1.11]. A somewhat nicer argument, which is at least model-independent (but makes heavy use of §6), goes as follows: Let's instead construct an inverse of $\mathrm{ev}_{\mathrm{id}_x}$ in a functorial way. That is, we are looking for a natural transformation

$$\eta \colon \mathrm{id}_{\mathrm{PSh}(\mathcal{C})} \Longrightarrow \mathrm{Hom}_{\mathrm{PSh}(\mathcal{C})}\big(\mathsf{よ}_{\mathcal{C}}(-), -\big).$$

We'll see in Lemma 6.31 that $\mathrm{id}_{\mathrm{PSh}(\mathcal{C})}$ is the ∞-categorical left Kan extension of $\mathsf{よ}_{\mathcal{C}}$ along itself. So it's enough to construct a natural transformation $\mathsf{よ}_{\mathcal{C}} \Rightarrow \mathsf{よ}_{\mathcal{C}}^* \mathrm{Hom}_{\mathrm{PSh}(\mathcal{C})}(\mathsf{よ}_{\mathcal{C}}(-), -)$; this can be taken to be the image of the natural transformation $\mathrm{Hom}_{\mathcal{C}}(-, -) \Rightarrow \mathrm{Hom}_{\mathrm{PSh}(\mathcal{C})}(\mathsf{よ}_{\mathcal{C}}(-), \mathsf{よ}_{\mathcal{C}}(-))$ from Lemma 5.26 under the "currying" equivalence $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}} \times \mathcal{C}, \mathrm{An}) \simeq \mathrm{Fun}(\mathcal{C}, \mathrm{PSh}(\mathcal{C}))$.

Even though this argument looks super fishy, I think it doesn't run into vicious circles. As far as I can see, we only used material up to Lemma 6.31, whereas the first time we need functoriality of the Yoneda lemma will be, conveniently, in Lemma 6.32.

[(5.12)] In §1 we had defined $\mathrm{PSh}(\mathcal{C}) := \mathrm{Fun}(\mathcal{C}, \mathrm{Set})$ for ordinary categories. This is, of course, *not* compatible with the new definition. In the following, all presheaves will be presheaves of animae and we'll never talk about presheaves of sets again.

§6. ∞-Category theory

Armed with Lurie's straightening equivalence and the quasi-categorical Yoneda lemma, we will spend §§6.1–6.4 redeveloping the theory from §1 (and more) in the setting of quasi-categories. In §6.5 we will see a first major application to topology. After that, there will be a lengthy appendix (§§6.6–6.10) in which we discuss presentable ∞-categories and prove Lurie's adjoint functor theorem.

Even though, implicitly, we work with quasi-categories, our arguments in §§6.1–6.4 will be almost entirely model-independent; the same is true, at least in large parts, for §§6.5–6.10. So from now on, instead of *quasi-categories*, we'll simply write ∞-*categories*. We'll consider ordinary categories as ∞-categories via the nerve construction, but we'll always suppress N in our notation.[(6.1)] Furthermore, we'll write $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ instead of $\mathrm{F}(\mathcal{C}, \mathcal{D})$ for ∞-categories $\mathcal{C}$ and $\mathcal{D}$. We'll only switch back to the old terminology in the few instances where non-model-independent arguments are used. I believe these few exceptions could easily be treated in any other model of ∞-categories as well.

§6.1. Adjunctions

6.1. Definition. — Let $L\colon \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories.

(a) Let $y \in \mathcal{D}$. An object $x \in \mathcal{C}$ is a *right adjoint object to y under L* if there exists an equivalence
$$\mathrm{Hom}_{\mathcal{C}}(-, x) \simeq \mathrm{Hom}_{\mathcal{D}}\big(L(-), y\big)$$
in the functor category $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{An})$.

(b) A functor $R\colon \mathcal{D} \to \mathcal{C}$ is a *right adjoint of L* if there exists an equivalence
$$\mathrm{Hom}_{\mathcal{C}}\big(-, R(-)\big) \simeq \mathrm{Hom}_{\mathcal{D}}\big(L(-), -\big)$$
in the functor category $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}} \times \mathcal{D}, \mathrm{An})$. In this case we write $L \dashv R$.

There are obvious dual notions of *left adjoint objects/functors.*

6.2. Lemma ("Adjoints can be constructed pointwise")**.** — *A functor $L\colon \mathcal{C} \to \mathcal{D}$ has a right adjoint if and only if every $y \in \mathcal{D}$ has a right adjoint object $x \in \mathcal{C}$.*

Proof. One implication is trivial: If $R\colon \mathcal{D} \to \mathcal{C}$ is a right adjoint of L, then $R(y)$ is a right adjoint object of y for every $y \in \mathcal{D}$. For the other implication, consider $\mathrm{Hom}_{\mathcal{D}}(L(-), -)\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{D} \to \mathrm{An}$ as a functor $\overline{R}\colon \mathcal{D} \to \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{An})$. Our assumption implies that $\overline{R}$ takes values in the image of the Yoneda embedding $よ_{\mathcal{C}}\colon \mathcal{C} \to \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{An})$; namely, $\overline{R}(y) \simeq \mathrm{Hom}_{\mathcal{C}}(-, x)$ if $x \in \mathcal{C}$ is a right adjoint object of $y \in \mathcal{D}$ under L. Since $よ_{\mathcal{C}}$ is an equivalence onto its image, we obtain a functor $R\colon \mathcal{D} \to \mathcal{C}$ with the required properties. $\square$

6.3. Example. — It's clear that any adjunction of ordinary categories is also an adjunction of ∞-categories. Furthermore, we already know some non-trivial examples of adjunctions of ∞-categories:

[(6.1)]In particular, the partially ordered set $[n]$ is now identified with its nerve, the quasi-category Δ^n. However, we'll continue to write Δ^n. There are at least to reasons for this. First, I believe Δ^n is easier to parse for a human brain. Second, once we reach §7, we'll consider simplicial objects in ∞-categories, and it will become necessary to keep the ∞-category Δ^n and the object $[n] \in \Delta$ notationally separate.

(*a*)   The inclusion $\mathrm{An} \subseteq \mathrm{Cat}_\infty$ is fully faithful (by Theorem 2.24 and Corollary 3.11) and has both adjoints: A right adjoint $\mathrm{core} \colon \mathrm{Cat}_\infty \to \mathrm{An}$ and a left adjoint $|\cdot| \colon \mathrm{Cat}_\infty \to \mathrm{An}$ sending $\mathcal{C}$ to $|\mathcal{C}|$, the localisation of $\mathcal{C}$ at all its morphisms.

(*b*)   For every ∞-category $\mathcal{C}$, the functor $- \times \mathcal{C} \colon \mathrm{Cat}_\infty \to \mathrm{Cat}_\infty$ has a right adjoint, which sends an ∞-category $\mathcal{D}$ to $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$.

Both (*a*) and (*b*) can easily be seen using Lemma 6.2: For (*a*), it's enough to check that the functors $i \colon \mathrm{core}(\mathcal{D}) \to \mathcal{D}$ and $p \colon \mathcal{C} \to |\mathcal{C}|$ induce functorial equivalences

$$i_* \colon \mathrm{Hom}_{\mathrm{An}}\big(-, \mathrm{core}(\mathcal{D})\big) \overset{\simeq}{\Longrightarrow} \mathrm{Hom}_{\mathrm{Cat}_\infty}(-, \mathcal{D}), \quad p^* \colon \mathrm{Hom}_{\mathrm{An}}\big(|\mathcal{C}|, -\big) \overset{\simeq}{\Longrightarrow} \mathrm{Hom}_{\mathrm{Cat}_\infty}(\mathcal{C}, -)$$

via post- and precomposition, respectively. Indeed, equivalences can be checked pointwise by Theorem 4.5 and then we can apply Lemma 4.11 (and a similar assertion for core). For (*b*), observe that we have an *evaluation functor* $\mathrm{ev} \colon \mathrm{Fun}(\mathcal{C}, \mathcal{D}) \times \mathcal{C} \to \mathcal{D}$ for all ∞-categories $\mathcal{C}$ and $\mathcal{D}$. If we work with quasi-categories, this functor is simply given by the counit of the adjunction $- \times \mathcal{C} \colon \mathrm{sSet} \rightleftarrows \mathrm{sSet} \colon \mathrm{F}(\mathcal{C}, -)$. Using Lemma 6.2, it's enough to show that the composition

$$\mathrm{Hom}_{\mathrm{Cat}_\infty}\big(-, \mathrm{Fun}(\mathcal{C}, \mathcal{D})\big) \overset{- \times \mathcal{C}}{\Longrightarrow} \mathrm{Hom}_{\mathrm{Cat}_\infty}\big(- \times \mathcal{C}, \mathrm{Fun}(\mathcal{C}, \mathcal{D}) \times \mathcal{C}\big) \overset{\mathrm{ev}_*}{\Longrightarrow} \mathrm{Hom}_{\mathrm{Cat}_\infty}(- \times \mathcal{C}, \mathcal{D})$$

is an equivalence. Again, Theorem 4.5 allows us to check this pointwise, and then Theorem 2.24 reduces everything to the adjunction $- \times \mathcal{C} \colon \mathrm{sSet} \rightleftarrows \mathrm{sSet} \colon \mathrm{F}(\mathcal{C}, -)$ of ordinary categories.

In particular, (*b*) allows us to define a functor $\mathrm{Fun}(\mathcal{C}, -) \colon \mathrm{Cat}_\infty \to \mathrm{Cat}_\infty$. With a little more work[(6.2)], these functors can be assembled into a two-argument functor

$$\mathrm{Fun}(-, -) \colon \mathrm{Cat}_\infty^{\mathrm{op}} \times \mathrm{Cat}_\infty \longrightarrow \mathrm{Cat}_\infty \,.$$

6.4. Unit and counit. — Let $L \colon \mathcal{C} \rightleftarrows \mathcal{D} \colon R$ be an adjunction. We wish to construct a *unit* transformation $u \colon \mathrm{id}_\mathcal{C} \Rightarrow RL$ and a *counit* transformation $c \colon LR \Rightarrow \mathrm{id}_\mathcal{D}$ in such a way that the adjunction equivalence $\eta \colon \mathrm{Hom}_\mathcal{D}(L(-), -) \overset{\simeq}{\Longrightarrow} \mathrm{Hom}_\mathcal{C}(-, R(-))$ and its inverse are given by

$$\mathrm{Hom}_\mathcal{D}\big(L(-), -\big) \overset{R}{\Longrightarrow} \mathrm{Hom}_\mathcal{C}\big(RL(-), R(-)\big) \overset{u^*}{\Longrightarrow} \mathrm{Hom}_\mathcal{C}\big(-, R(-)\big),$$

$$\mathrm{Hom}_\mathcal{C}\big(-, R(-)\big) \overset{L}{\Longrightarrow} \mathrm{Hom}_\mathcal{D}\big(L(-), LR(-)\big) \overset{c_*}{\Longrightarrow} \mathrm{Hom}_\mathcal{D}\big(L(-), -\big).$$

To construct u, we start with $\mathrm{Hom}_\mathcal{C}(-, -) \Rightarrow \mathrm{Hom}_\mathcal{D}(L(-), L(-)) \simeq \mathrm{Hom}_\mathcal{C}(-, RL(-))$, where the first one is induced by functoriality of L and the second by the given adjunction. We can

---

[(6.2)] Here's the argument: First, let $* * $ be the discrete category on two objects. In Remark 6.10 below, we'll construct a functor $\lim \colon \mathrm{Fun}(* *, \mathrm{Cat}_\infty) \to \mathrm{Cat}_\infty$. Under the identification $\mathrm{Fun}(* *, \mathrm{Cat}_\infty) \simeq \mathrm{Cat}_\infty \times \mathrm{Cat}_\infty$, this functor sends a pair $(\mathcal{C}, \mathcal{D})$ of ∞-categories to the product $\mathcal{C} \times \mathcal{D}$. By "currying", this functor corresponds to a functor $P \colon \mathrm{Cat}_\infty \to \mathrm{Fun}(\mathrm{Cat}_\infty, \mathrm{Cat}_\infty)$ sending $\mathcal{C}$ to $P(\mathcal{C}) \simeq - \times \mathcal{C} \colon \mathrm{Cat}_\infty \to \mathrm{Cat}_\infty$. By the way, this is also how you construct the functor $- \times \mathcal{C}$ in (*b*). As we've seen above, $P(\mathcal{C})$ is a left adjoint, and so P factors through the full sub-∞-category $\mathrm{Fun}^{\mathrm{L}}(\mathrm{Cat}_\infty, \mathrm{Cat}_\infty) \subseteq \mathrm{Fun}(\mathrm{Cat}_\infty, \mathrm{Cat}_\infty)$ spanned by the left adjoint functors. By Corollary 6.8 below, extracting adjoints induces an equivalence of ∞-categories $\mathrm{Fun}^{\mathrm{L}}(\mathrm{Cat}_\infty, \mathrm{Cat}_\infty) \simeq \mathrm{Fun}^{\mathrm{R}}(\mathrm{Cat}_\infty, \mathrm{Cat}_\infty)^{\mathrm{op}}$. Thus, P^{op} can be regarded as a functor $P^{\mathrm{op}} \colon \mathrm{Cat}_\infty^{\mathrm{op}} \to \mathrm{Fun}^{\mathrm{R}}(\mathrm{Cat}_\infty, \mathrm{Cat}_\infty)$, sending $\mathcal{C}$ to $P^{\mathrm{op}}(\mathcal{C}) \simeq \mathrm{Fun}(\mathcal{C}, -)$. "Currying" back, we obtain the desired functor $\mathrm{Fun}(-, -) \colon \mathrm{Cat}_\infty^{\mathrm{op}} \times \mathrm{Cat}_\infty \to \mathrm{Cat}_\infty$.

It's true that the functor $\mathrm{core}\, \mathrm{Fun}(-, -)$ agrees with $\mathrm{Hom}_{\mathrm{Cat}_\infty}(-, -)$, but this is not so easy to see (and we won't need it). One way would be to turn $\mathrm{F}(-, -) \colon \mathrm{QCat} \times \mathrm{QCat} \to \mathrm{QCat}$ into a Kan-enriched functor and show that $\mathrm{N}^\Delta(\mathrm{F}(-, -))$ agrees with $\mathrm{Fun}(-, -)$. This is easy since N^Δ turns Kan-enriched adjunctions into adjunctions of ∞-categories. Then one has to check that $\mathrm{Hom}_{\mathrm{Cat}_\infty}(-, -)$ agrees with N^Δ applied to $\mathrm{core}\, \mathrm{F}(-, -) \colon \mathrm{QCat} \times \mathrm{QCat} \to \mathrm{Kan}$. At least for Construction 5.22, this is done in [L-HA, Proposition 5.2.1.11].

consider this composition as a natural transformation $よ_\mathcal{C} \Rightarrow よ_\mathcal{C} \circ RL$ in $\mathrm{Fun}(\mathcal{C}, \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{An}))$. Since $よ_\mathcal{C}$ is fully faithful (Corollary 5.25), we obtain a natural transformation $u\colon \mathrm{id}_\mathcal{C} \Rightarrow RL$.

With this definition, the natural transformations $u^* \circ R$ and η agree pointwise; more precisely, for every fixed $x \in \mathcal{C}$ they induce the same transformation $\mathrm{Hom}_\mathcal{D}(L(x), -) \Rightarrow \mathrm{Hom}_\mathcal{C}(x, R(-))$. Indeed, Yoneda's lemma (Theorem 5.19) shows that any such transformation is uniquely determined by the image of $\mathrm{id}_{L(x)}$ under $\mathrm{Hom}_\mathcal{D}(L(x), L(x)) \to \mathrm{Hom}_\mathcal{C}(x, RL(x))$. Now by construction, both $u^* \circ R$ and η map $\mathrm{id}_{L(x)}$ to $u_x\colon x \to RL(x)$.

In ordinary categories two natural transformations that agree pointwise must be equal, because natural transformations are just pointwise data subject to certain conditions. But in ∞-land the situation is more subtle and we need a slightly convoluted argument to conclude $u^* \circ R \simeq \eta$.

First observe that since $u^* \circ R$ is still an equivalence by Theorem 4.5, since it agrees pointwise with η. Consider

$$\eta \circ (u^* \circ R)^{-1}\colon \mathrm{Hom}_\mathcal{C}\big(-, R(-)\big) \overset{\simeq}{\Longrightarrow} \mathrm{Hom}_\mathcal{C}\big(-, R(-)\big).$$

Write this as a natural equivalence $よ_\mathcal{C} \circ R \overset{\simeq}{\Longrightarrow} よ_\mathcal{C} \circ R$. Since $よ_\mathcal{C}$ is fully faithful, $\eta \circ (u^* \circ R)^{-1}$ must be induced by a natural equivalence $\varepsilon\colon R \overset{\simeq}{\Longrightarrow} R$. Put $\overline{u} := \varepsilon L \circ u\colon \mathrm{id}_\mathcal{C} \Rightarrow RL$. Then $\eta \simeq \overline{u}^* \circ R$. Indeed, this follows from the diagram

$$
\begin{array}{ccccc}
\mathrm{Hom}_\mathcal{D}\big(L(-), -\big) & \overset{R}{\Longrightarrow} & \mathrm{Hom}_\mathcal{C}\big(RL(-), R(-)\big) & \overset{\overline{u}^*}{\Longrightarrow} & \mathrm{Hom}_\mathcal{C}\big(-, R(-)\big) \\
R \big\Vert & /\!/\!/ & \big\Vert \varepsilon_* & /\!/\!/ & \big\Vert \varepsilon_* \\
\mathrm{Hom}_\mathcal{C}\big(RL(-), R(-)\big) & \overset{\varepsilon^*}{\Longrightarrow} & \mathrm{Hom}_\mathcal{C}\big(RL(-), R(-)\big) & \overset{\overline{u}^*}{\Longrightarrow} & \mathrm{Hom}_\mathcal{C}\big(-, R(-)\big)
\end{array}
$$

in which the left square commutes by Lemma 5.27 and the right square by Lemma 5.28. To conclude $u \simeq \overline{u}$, we plug $\eta \simeq \overline{u} \circ R$ into the construction of u to obtain the following commutative diagram:

$$
\begin{array}{ccccc}
\mathrm{Hom}_\mathcal{C}(-, -) & \overset{L}{\Longrightarrow} & \mathrm{Hom}_\mathcal{D}\big(L(-), L(-)\big) & \overset{R}{\Longrightarrow} & \mathrm{Hom}_\mathcal{D}\big(RL(-), RL(-)\big) \\
\big\Vert & & /\!/\!/ & & \big\Vert \overline{u}^* \\
\mathrm{Hom}_\mathcal{C}(-, -) & & \overset{u_*}{=\!=\!=\!=\!=\!=\!=\!=} & & \mathrm{Hom}_\mathcal{D}\big(-, RL(-)\big)
\end{array}
$$

Using Lemma 5.27 we conclude $u_* \simeq \overline{u}_*$ as natural transformations $よ_\mathcal{C} \Rightarrow よ_\mathcal{C} \circ RL$, and thus indeed $u \simeq \overline{u}$, as desired.

6.5. Lemma (Triangle identities). — *Let $L\colon \mathcal{C} \rightleftarrows \mathcal{D} \colon R$ be an adjunction of ∞-categories. Then there are commutative diagrams*

$$
\begin{array}{ccc}
L & \overset{Lu}{\Longrightarrow} & LRL \\
& {\scriptstyle \mathrm{id}_L} \searrow {\scriptstyle /\!/\!/} & \big\Vert cL \\
& & L
\end{array}
\quad and \quad
\begin{array}{ccc}
R & \overset{uR}{\Longrightarrow} & RLR \\
& {\scriptstyle \mathrm{id}_R} \searrow {\scriptstyle /\!/\!/} & \big\Vert Rc \\
& & R
\end{array}
$$

Conversely, let $L\colon \mathcal{C} \to \mathcal{D}$ and $R\colon \mathcal{D} \to \mathcal{C}$ be functors. Suppose we're given a natural transformation $u\colon \mathrm{id}_\mathcal{C} \Rightarrow RL$ as well as pointwise morphisms $c_y\colon LR(y) \to y$ for all $y \in \mathcal{D}$ such that $c_{L(x)} \circ L(u_x)$ and $R(c_y) \circ u_{R(y)}$ are equivalences (but not necessarily the identity on $L(x)$ and $R(y)$, respectively) for all $x \in \mathcal{C}$, $y \in \mathcal{D}$. Then L and R determine an adjunction.

Proof. First suppose $L\colon \mathcal{C} \rightleftarrows \mathcal{D} \colon R$ is an adjunction. Consider the diagram

$$
\begin{array}{ccccc}
\mathrm{Hom}_{\mathcal{D}}\big(L(-),-\big) & \xRightarrow{\;R\;} & \mathrm{Hom}_{\mathcal{C}}\big(RL(-),R(-)\big) & \xRightarrow{\;u^*\;} & \mathrm{Hom}_{\mathcal{C}}\big(-,R(-)\big) \\
& & \Big\Vert L & /\!/\!/ & \Big\Vert L \\
(cL)^* & /\!/\!/ & \mathrm{Hom}_{\mathcal{D}}\big(LRL(-),LR(-)\big) & \xRightarrow{(Lu)^*} & \mathrm{Hom}_{\mathcal{D}}\big(L(-),LR(-)\big) \\
& & \Big\Vert c_* & /\!/\!/ & \Big\Vert c_* \\
& & \mathrm{Hom}_{\mathcal{D}}\big(LRL(-),-\big) & \xRightarrow{(Lu)^*} & \mathrm{Hom}_{\mathcal{D}}\big(L(-),-\big)
\end{array}
$$

The top right square commutes by functoriality of L, the bottom right square commutes since pre- and postcomposition commute by Lemma 5.28, and the left cell commutes by Lemma 5.27. Now walking around the perimeter shows that $(cL)^* \circ (Lu)^* \colon \mathrm{Hom}_{\mathcal{D}}(L(-),-) \Rightarrow \mathrm{Hom}_{\mathcal{D}}(L(-),-)$ agrees with $c_* \circ L \circ u^* \circ R$. This is the identity, since we've arranged in 6.4 that $c_* \circ L$ and $u^* \circ R$ are inverses. This establishes the first triangle identity; the second one is analogous.

Conversely, suppose $L\colon \mathcal{C} \to \mathcal{D}$, $R\colon \mathcal{D} \to \mathcal{C}$ are functors and $u\colon \mathrm{id}_{\mathcal{C}} \Rightarrow RL$, $c_y\colon LR(y) \to y$ satisfy the conditions above. For L and R to be adjoints, it suffices that the composition

$$
\mathrm{Hom}_{\mathcal{D}}\big(L(-),-\big) \xRightarrow{\;R\;} \mathrm{Hom}_{\mathcal{C}}\big(RL(-),R(-)\big) \xRightarrow{\;u^*\;} \mathrm{Hom}_{\mathcal{C}}\big(-,R(-)\big)
$$

is a natural equivalence. By Theorem 4.5, it's enough to check this pointwise. In the case where $c_{L(x)} \circ L(u_x) \simeq \mathrm{id}_{L(x)}$ and $R(c_y) \circ u_{R(y)} \simeq \mathrm{id}_{R(y)}$ are the identity on $L(x)$ and $R(y)$, respectively, plugging x and y into the diagram above (as well as its analogue for the second triangle identity) shows that

$$
\mathrm{Hom}_{\mathcal{C}}\big(x,R(y)\big) \xrightarrow{\;L\;} \mathrm{Hom}_{\mathcal{D}}\big(L(x),LR(y)\big) \xrightarrow{\;c_{y,*}\;} \mathrm{Hom}_{\mathcal{D}}\big(L(x),y\big)
$$

is a pointwise inverse. If $c_{L(x)} \circ L(u_x)$ and $R(c_y) \circ u_{R(y)}$ are merely equivalences, but not necessarily the identity on $L(x)$ and $R(y)$, this argument is still enough to show that the composition $u^* \circ R\colon \mathrm{Hom}_{\mathcal{D}}(L(-),-) \Rightarrow \mathrm{Hom}_{\mathcal{C}}(-,R(-))$ is a natural equivalence. $\qquad \square$

6.6. Corollary. — *Let $L\colon \mathcal{C} \rightleftarrows \mathcal{D} \colon R$ be an adjunction and let $\mathcal{I}$ be another category. Then the pre- and postcomposition functors determine adjunctions*

$$
\begin{aligned}
L \circ -\colon\ & \mathrm{Fun}(\mathcal{I},\mathcal{C}) \rightleftarrows \mathrm{Fun}(\mathcal{I},\mathcal{D}) \colon R \circ -\,, \\
- \circ R\colon\ & \mathrm{Fun}(\mathcal{C},\mathcal{I}) \rightleftarrows \mathrm{Fun}(\mathcal{D},\mathcal{I}) \colon - \circ L\,.
\end{aligned}
$$

Proof. The proof of Corollary 1.5 can be copied verbatim. $\qquad \square$

To finish this subsection about adjunctions, we connect adjunctions to the theory of straightening/unstraightening. This won't be needed in the main text, but it shows up in the appendices and is a standard fact in other treatments of ∞-categories.

6.7. Lemma. — *Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories, corresponding to a functor $\Delta^1 \to \mathrm{Cat}_\infty$ (see Example 2.23), which in turn corresponds to a cocartesian fibration $p\colon \mathcal{U} \to \Delta^1$ by Theorem 5.4(a). Then the following are equivalent:*

(a) *F admits a right adjoint $G\colon \mathcal{D} \to \mathcal{C}$.*

(b) *The cocartesian fibration $p\colon \mathcal{U} \to \Delta^1$ is also a cartesian fibration.*

Furthermore, in this case G agrees with the functor classified by the cartesian straightening $\operatorname{St}^{\mathrm{cart}}(p)\colon (\Delta^1)^{\mathrm{op}} \to \operatorname{Cat}_\infty$.

Proof. The crucial observation is the following claim:

(⊠) *The functor* $\operatorname{Hom}_{\mathcal{D}}(F(-), -)\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{D} \to \operatorname{An}$ *is equivalent to the composition*

$$\mathcal{C}^{\mathrm{op}} \times \mathcal{D} \xrightarrow{i_0^{\mathrm{op}} \times i_1} \mathcal{U}^{\mathrm{op}} \times \mathcal{U} \xrightarrow{\operatorname{Hom}_{\mathcal{U}}} \operatorname{An}.$$

Here the first arrow is given by $i_0\colon \mathcal{C} \simeq \{0\} \times_{\Delta^1} \mathcal{U} \to \mathcal{U}$ *and* $i_1\colon \mathcal{D} \simeq \{1\} \times_{\Delta^1} \mathcal{U} \to \mathcal{U}$.

To prove (⊠), first observe that $i_0\colon \mathcal{C} \to \mathcal{U}$ and $i_1\colon \mathcal{D} \to \mathcal{U}$ are fully faithful. Indeed, Hom animae in pullbacks are given as pullbacks of Hom animae in the respective factors (which is straightforward to see from 2.11 and we'll see a more general assertion in Lemma 6.76(*a*)). So pullbacks of fully faithful functors are still fully faithful and it remains to observe that $\{0\} \to \Delta^1$ and $\{1\} \to \Delta^1$ are both fully faithful, which is obvious. Now consider the following commutative square in $\operatorname{Cat}_\infty$:

$$\begin{array}{ccc} \mathcal{C} & = & \mathcal{C} \\ \| & /\!/\!/ & \downarrow F \\ \mathcal{C} & \xrightarrow{F} & \mathcal{D} \end{array}$$

It can be viewed as a natural transformation $\operatorname{const} \mathcal{C} \Rightarrow F$ in $\operatorname{Fun}(\Delta^1, \operatorname{Cat}_\infty)$. After cocartesian unstraightening, it thus induces a morphism $\varphi\colon \Delta^1 \times \mathcal{C} \to \mathcal{U}$ in $\operatorname{Cocart}(\Delta^1)$. Consider the composite

$$(\Delta^1 \times \mathcal{C})^{\mathrm{op}} \times \mathcal{D} \xrightarrow{\varphi^{\mathrm{op}} \times i_1} \mathcal{U}^{\mathrm{op}} \times \mathcal{U} \xrightarrow{\operatorname{Hom}_{\mathcal{U}}} \operatorname{An}.$$

By unravelling the definitions, this composite can be regarded as a natural transformation $\eta\colon \operatorname{Hom}_{\mathcal{U}}(i_1 F(-), i_1(-)) \Rightarrow \operatorname{Hom}_{\mathcal{U}}(i_0(-), i_1(-))$ in $\operatorname{Fun}(\mathcal{C}^{\mathrm{op}} \times \mathcal{D}, \operatorname{An})$. We wish to show that η is an equivalence of functors. By Theorem 4.5 this can be checked pointwise. So fix $x \in \mathcal{C}$, $y \in \mathcal{D}$. By unravelling the constructions $\eta_{(x,y)}\colon \operatorname{Hom}_{\mathcal{U}}(i_1 F(x), i_1(y)) \to \operatorname{Hom}_{\mathcal{U}}(i_0(x), i_1(y))$ is given by precomposition with the morphism $\varphi_x\colon i_0(x) \to i_1 F(x)$ given as the image of $\Delta^1 \times \{x\} \to \Delta^1 \times \mathcal{C}$ under φ. As φ preserves cocartesian lifts, φ_x must be a cocartesian morphism. Since $\operatorname{Hom}_{\Delta^1}(1,1) \simeq \operatorname{Hom}_{\Delta^1}(0,1)$, Lemma 5.16 implies that precomposition with φ_x must be an equivalence. Thus η is an equivalence of functors, as desired. To finish the proof of (⊠), it remains to observe $\operatorname{Hom}_{\mathcal{U}}(i_1 F(-), i_1(-)) \simeq \operatorname{Hom}_{\mathcal{D}}(F(-), -)$ as we've checked above that $i_1\colon \mathcal{D} \to \mathcal{U}$ is fully faithful.

Now assume that $p\colon \mathcal{U} \to \Delta^1$ is a cartesian fibration too and let $G\colon \mathcal{D} \to \mathcal{C}$ correspond to $\operatorname{St}^{\mathrm{cart}}(p)\colon (\Delta^1)^{\mathrm{op}} \to \operatorname{Cat}_\infty$. Then (⊠) and its dual provide an equivalences of functors $\operatorname{Hom}_{\mathcal{D}}(F(-), -) \simeq \operatorname{Hom}_{\mathcal{U}}(i_0(-), i_1(-)) \simeq \operatorname{Hom}_{\mathcal{C}}(-, G(-))$, so F and G are adjoints. This proves (*b*) $\Rightarrow$ (*a*).

Conversely, suppose $G\colon \mathcal{D} \to \mathcal{C}$ is a right adjoint of F. Fix $y \in \mathcal{D}$. Then (⊠) and the fact that i_0 is fully faithful shows

$$\operatorname{Hom}_{\mathcal{U}}\big(i_0(-), i_0 G(y)\big) \simeq \operatorname{Hom}_{\mathcal{C}}\big(-, G(y)\big) \simeq \operatorname{Hom}_{\mathcal{D}}\big(F(-), y\big) \simeq \operatorname{Hom}_{\mathcal{U}}\big(i_0(-), i_1(y)\big).$$

The image of $\operatorname{id}_{i_0 G(y)}$ defines a morphism $\psi_y\colon i_0 G(y) \to i_1(y)$ in $\mathcal{U}$. Furthermore, Yoneda's lemma (or more precisely, the dual of Theorem 5.19) shows that any natural transformation $\operatorname{Hom}_{\mathcal{C}}(-, G(y)) \simeq \operatorname{Hom}_{\mathcal{U}}(i_0(-), i_0 G(y)) \Rightarrow \operatorname{Hom}_{\mathcal{U}}(i_0(-), i_1(y))$ is uniquely determined by the image of $\operatorname{id}_{G(y)}$. That uniqueness ensures that the chain of equivalences above is must be given

by postcomposition with ψ_y. Hence the dual of Lemma 5.16 shows that ψ_y is a p-cartesian morphism and we have constructed a sufficient supply of p-cartesian lifts. This finishes the proof of $(a) \Rightarrow (b)$. □

6.8. Corollary ("Extracting adjoints is functorial"). — *Let* $\mathrm{Fun}^{\mathrm{L}}, \mathrm{Fun}^{\mathrm{R}} \subseteq \mathrm{Fun}$ *denote the full sub-∞-categories spanned by the left/right adjoint functors and let* $\mathrm{Cat}_{\infty}^{\mathrm{L}}, \mathrm{Cat}_{\infty}^{\mathrm{R}} \subseteq \mathrm{Cat}_{\infty}$ *be the non-full sub-∞-categories (in the sense of 2.16) spanned by all objects but only the left/right adjoint functors.*

(a) *For all ∞-categories $\mathcal{C}$ and $\mathcal{D}$, sending a left adjoint functor $L\colon \mathcal{C} \to \mathcal{D}$ to its right adjoint $R\colon \mathcal{D} \to \mathcal{C}$ can be turned into an equivalence of ∞-categories $\mathrm{Fun}^{\mathrm{L}}(\mathcal{C}, \mathcal{D})^{\mathrm{op}} \simeq \mathrm{Fun}^{\mathrm{R}}(\mathcal{D}, \mathcal{C})$.*

(b) *There exists an equivalence of ∞-categories $\mathrm{Cat}_{\infty}^{\mathrm{L}} \simeq (\mathrm{Cat}_{\infty}^{\mathrm{R}})^{\mathrm{op}}$ which is the identity on objects and sends morphisms in $\mathrm{Cat}_{\infty}^{\mathrm{L}}$, that is, left adjoint functors $L\colon \mathcal{C} \to \mathcal{D}$, to their right adjoints $R\colon \mathcal{D} \to \mathcal{C}$.*

Proof sketch. For the equivalence in (a), it suffices show that the essential images of $\mathrm{Fun}^{\mathrm{L}}(\mathcal{C}, \mathcal{D})$ and $\mathrm{Fun}^{\mathrm{R}}(\mathcal{D}, \mathcal{C})^{\mathrm{op}}$ under the fully faithful Yoneda embeddings

$$\mathrm{Fun}^{\mathrm{R}}(\mathcal{D}, \mathcal{C}) \xrightarrow{(\text{よ}_{\mathcal{C}})_*} \mathrm{Fun}\big(\mathcal{D}, \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{An})\big) \simeq \mathrm{Fun}(\mathcal{C}^{\mathrm{op}} \times \mathcal{D}, \mathrm{An}),$$

$$\mathrm{Fun}^{\mathrm{L}}(\mathcal{C}, \mathcal{D})^{\mathrm{op}} \simeq \mathrm{Fun}^{\mathrm{R}}(\mathcal{C}^{\mathrm{op}}, \mathcal{D}^{\mathrm{op}}) \xrightarrow{(\text{よ}_{\mathcal{D}^{\mathrm{op}}})_*} \mathrm{Fun}\big(\mathcal{C}^{\mathrm{op}}, \mathrm{Fun}(\mathcal{D}, \mathrm{An})\big) \simeq \mathrm{Fun}(\mathcal{C}^{\mathrm{op}} \times \mathcal{D}, \mathrm{An})$$

coincide. Using Lemma 6.2 and the definition of the Yoneda embedding, it's straightforward to check that both essential images consist of those functors $H\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{D} \to \mathrm{An}$ such that for every $x \in \mathcal{C}$ there exists a $y \in \mathcal{D}$ such that $H(x, -) \simeq \mathrm{Hom}_{\mathcal{D}}(y, -)$ and for every $y' \in \mathcal{D}$ there exists an $x' \in \mathcal{C}$ such that $H(-, y') \simeq \mathrm{Hom}_{\mathcal{C}}(-, x')$. This proves that there exists an equivalence $\mathrm{Fun}^{\mathrm{L}}(\mathcal{C}, \mathcal{D})^{\mathrm{op}} \simeq \mathrm{Fun}^{\mathrm{R}}(\mathcal{D}, \mathcal{C})$ as desired. Furthermore if $L\colon \mathcal{C} \rightleftarrows \mathcal{D} :R$ is an adjunction, then the Yoneda embeddings above send both R and L to the functor $\mathrm{Hom}_{\mathcal{C}}(-, R(-)) \simeq \mathrm{Hom}_{\mathcal{D}}(L(-), -)\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{D} \to \mathrm{An}$. So the equivalence we've constructed is really given by extracting adjoints.

To prove (b), we grossly neglect set theory and regard both $\mathrm{Cat}_{\infty}^{\mathrm{L}}$ and $\mathrm{Cat}_{\infty}^{\mathrm{R}}$ as objects in Cat_{∞}. This can be repaired by considering universes or, with some care, by imposing cardinality bounds (similar to the argument in Lemma 7.18 below, where we do this in detail). We'll show that there exists a functorial bijection $\pi_0 \mathrm{Hom}_{\mathrm{Cat}_{\infty}}(\mathcal{C}, \mathrm{Cat}_{\infty}^{\mathrm{L}}) \cong \pi_0 \mathrm{Hom}_{\mathrm{Cat}_{\infty}}(\mathcal{C}, (\mathrm{Cat}_{\infty}^{\mathrm{R}})^{\mathrm{op}})$ for all ∞-categories $\mathcal{C}$; if we can do this, then the Yoneda lemma in the ordinary category $h(\mathrm{Cat}_{\infty})$ will show that $\mathrm{Cat}_{\infty}^{\mathrm{L}}$ and $\mathrm{Cat}_{\infty}^{\mathrm{R}}$ are isomorphic in the homotopy category, hence equivalent as ∞-categories. We know $\mathrm{Hom}_{\mathrm{Cat}_{\infty}}(\mathcal{C}, \mathrm{Cat}_{\infty}^{\mathrm{L}}) \simeq \mathrm{core}\,\mathrm{Fun}(\mathcal{C}, \mathrm{Cat}_{\infty}^{\mathrm{L}})$ by Theorem 2.24 and $\mathrm{Fun}(\mathcal{C}, \mathrm{Cat}_{\infty}^{\mathrm{L}}) \simeq \mathrm{Cocart}(\mathcal{C})$ by Theorem 5.4(a). Let $F\colon \mathcal{C} \to \mathrm{Cat}_{\infty}$ be a functor and $p\colon \mathcal{U} \to \mathcal{C}$ be its cocartesian unstraightening. By Lemma 6.7, F factors through $\mathrm{Cat}_{\infty}^{\mathrm{L}} \to \mathrm{Cat}_{\infty}$ if and only if for all $\alpha\colon \Delta^1 \to \mathcal{C}$, the pullback $p_{\alpha}\colon \Delta^1 \times_{\alpha, \mathcal{C}} \mathcal{U} \to \Delta^1$ is not only a cocartesian, but also a cartesian fibration. In other words, p is a *locally cartesian fibration* in the sense of Definition 5.17. Since right adjoints compose, it's clear that locally p-cartesian morphisms are closed under composition, and so p is automatically a cartesian fibration by Corollary 5.18. In summary, we obtain a bijection

$$\pi_0 \mathrm{Hom}_{\mathrm{Cat}_{\infty}}(\mathcal{C}, \mathrm{Cat}_{\infty}^{\mathrm{L}}) \cong \pi_0 \mathrm{core}\,\mathrm{Bicart}(\mathcal{C}),$$

where we define $\mathrm{Bicart}(\mathcal{C}) \subseteq \mathrm{Cat}_{\infty/\mathcal{C}}$ as the non-full sub-∞-category spanned by the *bicartesian fibrations*. That is, objects of $\mathrm{Bicart}(\mathcal{C})$ are those $p\colon \mathcal{U} \to \mathcal{C}$ that are both cocartesian and cartesian fibrations, and morphisms are those functors in $\mathrm{Cat}_{\infty/\mathcal{C}}$ that preserve both p-cocartesian

and p-cartesian morphisms. In the same way, we find bijections

$$\pi_0 \operatorname{Hom}_{\mathrm{Cat}_\infty}\big(\mathcal{C}, (\mathrm{Cat}_\infty^{\mathrm{R}})^{\mathrm{op}}\big) \cong \pi_0 \operatorname{Hom}_{\mathrm{Cat}_\infty}\big(\mathcal{C}^{\mathrm{op}}, \mathrm{Cat}_\infty^{\mathrm{R}}\big) \cong \pi_0 \operatorname{core} \operatorname{Bicart}(\mathcal{C}).$$

Hence $\pi_0 \operatorname{Hom}_{\mathrm{Cat}_\infty}(\mathcal{C}, \mathrm{Cat}_\infty^{\mathrm{L}}) \cong \pi_0 \operatorname{Hom}_{\mathrm{Cat}_\infty}(\mathcal{C}, (\mathrm{Cat}_\infty^{\mathrm{R}})^{\mathrm{op}})$ and so $\mathrm{Cat}_\infty^{\mathrm{L}} \simeq (\mathrm{Cat}_\infty^{\mathrm{R}})^{\mathrm{op}}$, as argued above. By unravelling the cases $\mathcal{C} \simeq *$ and $\mathcal{C} \simeq \Delta^1$ (the latter using Lemma 6.7), we find that this adjunction is really the identity on objects and given by extracting adjoints on morphisms. $\qquad\square$

§6.2. Limits and colimits

6.9. Definition. — Let $\mathcal{I}$ and $\mathcal{C}$ be ∞-categories.

(a) Let $F\colon \mathcal{I} \to \mathcal{C}$ be a functor of ∞-categories. A *colimit of F*, denoted $\operatorname{colim} F$ (or sometimes $\operatorname{colim}_{i \in \mathcal{I}} F(i)$), is a left adjoint object of F under $\operatorname{const}\colon \mathcal{C} \to \operatorname{Fun}(\mathcal{I}, \mathcal{C})$ that sends $x \in \mathcal{C}$ to the constant functor with value x. Dually, a *limit of F*, denoted $\lim F$ (or sometimes $\lim_{i \in \mathcal{I}} F(i)$), is a right adjoint object of F under const.

(b) We say that $\mathcal{C}$ *has all $\mathcal{I}$-shaped colimits* or *all $\mathcal{I}$-shaped limits* if all functors $\mathcal{I} \to \mathcal{C}$ admit colimits or limits, respectively.

6.10. Remark. — If $\mathcal{C}$ has all $\mathcal{I}$-shaped colimits, then Lemma 6.2 implies that forming colimits assembles into a functor $\operatorname{colim}\colon \operatorname{Fun}(\mathcal{I}, \mathcal{C}) \to \mathcal{C}$. The same is true for limits.

6.11. Lemma. — *Left adjoint functors between ∞-categories preserve colimits and right adjoint functors preserve limits.*

Proof. The proof of Lemma 1.7 can be copied verbatim. $\qquad\square$

6.12. Lemma ("Colimits in functor ∞-categories are computed pointwise."). — *Let $\mathcal{C}$, $\mathcal{D}$, and $\mathcal{I}$ be ∞-categories such that $\mathcal{D}$ has all $\mathcal{I}$-shaped colimits. Then $\operatorname{Fun}(\mathcal{C}, \mathcal{D})$ has again all $\mathcal{I}$-shaped colimits and the evaluation functor*

$$\operatorname{ev}_x\colon \operatorname{Fun}(\mathcal{C}, \mathcal{D}) \longrightarrow \operatorname{Fun}(\{x\}, \mathcal{D}) \simeq \mathcal{D}$$

preserves $\mathcal{I}$-shaped colimits for all $x \in \mathcal{C}$. A dual assertion holds for limits.

Proof. The proof of Lemma 1.8 can be copied verbatim. $\qquad\square$

Our next goal is to analyse limits and colimits in the ∞-categories An and Cat_∞. We start with a procedure for computing pullbacks and pushouts which is very useful in practice.

6.13. Pushouts and pullbacks in An and Cat_∞. — Pushouts and pullbacks in An or Cat_∞ can be computed using the following recipe:

(a) Write down the diagram on the level of Kan complexes or quasi-categories.

(b) For pushouts, use Lemma 3.12 to replace one leg by a cofibration. For pullbacks, use Lemma 3.12 to replace one leg by a Kan fibration/isofibration (depending on whether you take the pullback in An or Cat_∞, respectively).

(c) Take the pushout or pullback in sSet.

(d) For pushouts, the result of (*c*) will usually not be a Kan complex/quasi-category, so we need to use Lemma 3.12 once again to replace it by a Kan complex/quasi-category. For pullbacks, this step is unnecessary.

We've already seen the case of pullbacks in "Definition" 5.11. The procedure above is a consequence of the general model category fact that a pushout of cofibrant objects in a model category is automatically a homotopy pushout too if at least one leg is a cofibration, and a pullback of fibrant objects is a homotopy pullback if at least one leg is a fibration. See [Cis19, Corollary 2.3.28] for a proof of the general fact and [L-HTT, Theorem 4.2.4.1, Remark A.3.3.14] or [F-HC$_{II}$, Theorem X.21] for a proof that homotopy colimits/limits in a simplicial model category agree with colimits/limits in the underlying ∞-category.

The procedure above implies that many pullback constructions we've seen so far with simplicial sets are also pullbacks in An or Cat$_\infty$ and can thus be reinterpreted as model-independent constructions. For example, the diagram from 2.11 defining $\mathcal{C}_{x/}$ and $\mathrm{Hom}_\mathcal{C}(x,y)$ is also a pullback in Cat$_\infty$, because $(s,t)\colon \mathrm{Ar}(\mathcal{C}) \to \mathcal{C} \times \mathcal{C}$ is an isofibration (see the proof of Lemma 5.24). As another example, if $p\colon \mathcal{U} \to \mathcal{C}$ is a cocartesian fibration, then the fibre $p^{-1}\{x\}$, which computes the value of the associated functor $\mathrm{St}^{\mathrm{cocart}}\colon \mathcal{C} \to \mathrm{Cat}_\infty$ at x, can also be identified with the ∞-categorical pullback $\{x\} \times_\mathcal{C} \mathcal{U}$, because any cocartesian fibration p is automatically an isofibration. We'll often use these facts without mention. Let us also mention, and later use without mention, that An $\subseteq$ Cat$_\infty$ preserves both pushouts and pullbacks; in fact, it preserves all limits and colimits by Example 6.3(a) and Lemma 6.11. ∎

But there's also a description of limits and colimits that works in full generality and doesn't rely on the simplicial model.[(6.3)] To formulate this, we need to introduce some notation. Let $\mathcal{I}$ be an ∞-category and let $p\colon \mathcal{U} \to \mathcal{I}$ a cocartesian fibration. Furthermore, let $\mathrm{Fun}_\mathcal{I}(\mathcal{I},\mathcal{U}) := \mathrm{Fun}(\mathcal{I},\mathcal{U}) \times_{\mathrm{Fun}(\mathcal{I},\mathcal{I})} \{\mathrm{id}_\mathcal{I}\}$, the pullback being taken in Cat$_\infty$ (but we could take it in sSet as well by 6.13) and let

$$\mathrm{Fun}_\mathcal{I}^{\mathrm{cocart}}(\mathcal{I},\mathcal{U}) \subseteq \mathrm{Fun}_\mathcal{I}(\mathcal{I},\mathcal{U})$$

be the full sub-∞-category spanned by those $\mathcal{I} \to \mathcal{U}$ such that all morphisms in $\mathcal{I}$ are sent to p-cocartesian morphisms. Note that if p is a left fibration, then

$$\mathrm{Fun}_\mathcal{I}^{\mathrm{cocart}}(\mathcal{I},\mathcal{U}) \simeq \mathrm{Fun}_\mathbb{I}(\mathcal{I},\mathcal{U}) \simeq \mathrm{Hom}_{\mathrm{Cat}_{\infty/\mathcal{I}}}(\mathcal{I},\mathcal{U}) \,.$$

Indeed, the first equivalence is clear since in this case all morphisms in $\mathcal{U}$ are p-cocartesian by Lemma 5.3. The second equivalence follows from Corollary 5.15 combined with the facts that core: An $\to$ Cat$_\infty$ preserves pullbacks (because it is a right adjoint by Example 6.3(a)) and that $\mathrm{Fun}_\mathcal{I}(\mathcal{I},\mathcal{U})$ is already an anima (by Corollary 3.10 and Corollary 4.3).

6.14. Lemma. — *Let $F\colon \mathcal{I} \to \mathrm{Cat}_\infty$ be a functor and let $p\colon \mathcal{U} \to \mathcal{I}$ be its cocartesian unstraightening. Then the colimit and the limit of F in* Cat$_\infty$ *are given by*

$$\operatorname*{colim}_{i\in\mathcal{I}} F(i) \simeq \mathcal{U}\big[\{\text{cocartesian morphisms}\}^{-1}\big] \quad and \quad \lim_{i\in\mathcal{I}} F(i) \simeq \mathrm{Fun}_\mathcal{I}^{\mathrm{cocart}}(\mathcal{I},\mathcal{U}) \,.$$

In particular, if F takes values in An*, then the colimit and the limit of F in* An *are given by*

$$\operatorname*{colim}_{i\in\mathcal{I}} F(i) \simeq |\mathcal{U}| \quad and \quad \lim_{i\in\mathcal{I}} F(i) \simeq \mathrm{Hom}_{\mathrm{Cat}_{\infty/\mathcal{I}}}(\mathcal{I},\mathcal{U}) \,.$$

For the proof, we need the following lemma. In Corollary 6.17, a more general version of Lemma 6.15 is proved, but we need this special case as an input.

[(6.3)]I'd like to see a proof of model category fact 6.13 using only Lemma 6.14 below; I'm not sure if this works, so I'll leave it to you to figure out.

6.15. Lemma. — *For every ∞-category $\mathcal{C}$, the functor $\mathrm{Hom}_{\mathrm{Cat}_\infty}(\mathcal{C}, -)\colon \mathrm{Cat}_\infty \to \mathrm{An}$ preserves pullbacks.*

Proof sketch. As explained in footnote (6.2) in Example 6.3, Lurie constructs an equivalence of functors $\mathrm{Hom}_{\mathrm{Cat}_\infty}(-, -) \simeq \mathrm{core}\,\mathrm{Fun}(-, -)$ in [L-HA, Proposition 5.2.1.11], provided that you go with Construction 5.22. Thanks to Lemma 5.24, this proves $\mathrm{Hom}_{\mathrm{Cat}_\infty}(\mathcal{C}, -) \simeq \mathrm{core}\,\mathrm{Fun}(\mathcal{C}, -)$, no matter whether you use Construction 5.21 or 5.22. Now the claim is obvious, since both $\mathrm{core}\colon \mathrm{Cat}_\infty \to \mathrm{An}$ and $\mathrm{Fun}(\mathcal{C}, -)\colon \mathrm{Cat}_\infty \to \mathrm{Cat}_\infty$ are right adjoints by Example 6.3.

In fact, to show preservation of pullbacks in this way, we can get away with a little less than Lurie's result: We only need that $\mathrm{Hom}_{\mathrm{Cat}_\infty}(\mathcal{C}, -)$ and $\mathrm{core}\,\mathrm{Fun}(\mathcal{C}, -)$ agree on objects and morphisms. The former is clear by Theorem 2.24. Unfortunately, the latter still needs some care (and simplicial arguments): We know what $\mathrm{core}\,\mathrm{Fun}(-, -)$ does on morphisms, because it agrees with $\mathrm{N}^\Delta(\mathrm{core}\,\mathrm{F}(-, -))$ (the argument is in Example 6.3). For $\mathrm{Hom}_{\mathrm{Cat}_\infty}(\mathcal{C}, -)$, we need to unravel what straightening does on morphisms; this is quite nasty, but doable via 5.6. □

Proof sketch of Lemma 6.14. The idea in all of these statements is that the unstraightening of a constant functor $\mathrm{const}\,X$ is precisely the projection $\mathrm{pr}_2\colon X \times \mathcal{I} \to \mathcal{I}$. Let's first consider the case of colimits in An and see where this ideas takes us. To show that $|\mathcal{U}|$ is the desired colimit, we want an equivalence $\mathrm{Hom}_{\mathrm{An}}(|\mathcal{U}|, -) \simeq \mathrm{Hom}_{\mathrm{Fun}(\mathcal{I}, \mathrm{An})}(F, \mathrm{const}(-))$. Let's start manipulating the right-hand side. By Theorem 5.4(*b*), $\mathrm{Fun}(\mathcal{I}, \mathrm{An}) \simeq \mathrm{Left}(\mathcal{I})$, hence

$$\mathrm{Hom}_{\mathrm{Fun}(\mathcal{I}, \mathrm{An})}\big(F, \mathrm{const}(-)\big) \simeq \mathrm{Hom}_{\mathrm{Left}(\mathcal{I})}\Big(\mathcal{U}, \mathrm{Un}^{\mathrm{left}}\big(\mathrm{const}(-)\big)\Big).$$

The unstraightening of $\mathrm{const}\,X\colon \mathcal{I} \to \mathrm{An}$ is the projection $X \times \mathcal{I} \to \mathcal{I}$, functorially in $X \in \mathrm{An}$ (this is a consequence of the fact that precomposition corresponds to pullback in Theorem 5.4(*a*)). So we can continue our manipulations as follows:

$$
\begin{aligned}
\mathrm{Hom}_{\mathrm{Left}(\mathcal{I})}(\mathcal{U}, - \times \mathcal{I}) &\simeq \mathrm{Hom}_{\mathrm{Cat}_{\infty/\mathcal{I}}}(\mathcal{U}, - \times \mathcal{I}) \\
&\simeq \mathrm{Hom}_{\mathrm{Cat}_\infty}(\mathcal{U}, - \times \mathcal{I}) \times_{\mathrm{Hom}_{\mathrm{Cat}_\infty}(\mathcal{U}, \mathcal{I})} \{p\} \\
&\simeq \big(\mathrm{Hom}_{\mathrm{Cat}_\infty}(\mathcal{U}, -) \times \mathrm{Hom}_{\mathrm{Cat}_\infty}(\mathcal{U}, \mathcal{I})\big) \times_{\mathrm{Hom}_{\mathrm{Cat}_\infty}(\mathcal{U}, \mathcal{I})} \{p\} \\
&\simeq \mathrm{Hom}_{\mathrm{Cat}_\infty}(\mathcal{U}, -).
\end{aligned}
$$

In the first step we use that $\mathrm{Left}(\mathcal{I}) \to \mathrm{Cat}_{\infty/\mathcal{I}}$ is fully faithful. In the second step we use Corollary 5.15; by Lemma 6.12, the pullback is automatically functorial provided the square from Corollary 5.15 is functorial, which it clearly is by construction. In the third step, we use $\mathrm{Hom}_{\mathrm{Cat}_\infty}(\mathcal{U}, - \times \mathcal{I}) \simeq \mathrm{Hom}_{\mathrm{Cat}_\infty}(\mathcal{U}, -) \times \mathrm{Hom}_{\mathrm{Cat}_\infty}(\mathcal{U}, \mathcal{I})$ by Lemma 6.15. Finally, in the fourth step we use that $\mathrm{Hom}_{\mathrm{Cat}_\infty}(\mathcal{U}, \mathcal{I}) \times_{\mathrm{Hom}_{\mathrm{Cat}_\infty}(\mathcal{U}, \mathcal{I})} \{p\} \simeq \{p\}$ is just a point.

It remains to observe $\mathrm{Hom}_{\mathrm{An}}(|\mathcal{U}|, -) \simeq \mathrm{Hom}_{\mathrm{Cat}_\infty}(\mathcal{U}, -)$ because $|\cdot|\colon \mathrm{Cat}_\infty \to \mathrm{An}$ is left adjoint to the inclusion $\mathrm{An} \subseteq \mathrm{Cat}_\infty$. Thus, we have proved $\mathrm{colim}_{i \in \mathcal{I}} F(i) \simeq |\mathcal{U}|$ by verifying that $|\mathcal{U}|$ satisfies the desired universal property.

Let us now indicate the necessary changes to prove the other cases. For limits in An, we can use a similar calculation; the crucial step is $\mathrm{Hom}_{\mathrm{Cat}_\infty}(-, \mathrm{Fun}_{\mathcal{I}}(\mathcal{I}, \mathcal{U})) \simeq \mathrm{Hom}_{\mathrm{Cat}_{\infty/\mathcal{I}}}((-) \times \mathcal{I}, \mathcal{U})$, which uses Lemma 6.15, the adjunction from Example 6.3(*b*), and Corollary 5.15. We leave the details to you. When taking colimits or limits in Cat_∞, we can no longer argue that $\mathrm{Cocart}(\mathcal{I}) \to \mathrm{Cat}_{\infty/\mathcal{I}}$ is fully faithful. Instead, in the colimit case, Lemma 4.9 shows that $\mathrm{Hom}_{\mathrm{Cocart}(\mathcal{I})}(\mathcal{U}, - \times \mathcal{I}) \subseteq \mathrm{Hom}_{\mathrm{Cat}_{\infty/\mathcal{I}}}(\mathcal{U}, \mathcal{C} \times \mathcal{I})$ is a collection of path components; we have to check that it agrees with $\mathrm{Hom}_{\mathrm{Cat}_\infty}(\mathcal{U}[\{\text{cocartesian morphisms}\}^{-1}], -) \subseteq \mathrm{Hom}_{\mathrm{Cat}_\infty}(\mathcal{U}, -)$,

which is also a collection of path components by Lemma 4.11. This reduces us to checking a condition on the level of π_0-sets, which is straightforward. A similar argument applies in the limit case. $\qquad\square$

6.16. Corollary. — *Let $F\colon \mathcal{I} \to \mathcal{C}$ be a functor of ∞-categories. A natural transformation $c_F\colon \mathrm{const}\, y \Rightarrow F$ exhibits $y \in \mathcal{C}$ as a limit of F if and only if the natural map*

$$c_F^*\colon\ \mathrm{Hom}_{\mathcal{C}}(x, y) \xrightarrow{\ \simeq\ } \lim_{i \in \mathcal{I}} \mathrm{Hom}_{\mathcal{C}}\big(x, F(i)\big)$$

is an equivalence for all $x \in \mathcal{C}$. A dual assertion holds for colimits.

Proof. The unstraightening of $\mathrm{Hom}_{\mathcal{C}}(x, F(-))\colon \mathcal{I} \to \mathrm{An}$ is the left fibration $F^*(\mathcal{C}_{x/}) \to \mathcal{I}$, the pullback of the slice-∞-category projection $t\colon \mathcal{C}_{x/} \to \mathcal{C}$ along $F\colon \mathcal{I} \to \mathcal{C}$. Hence, according to Lemma 6.14, $\lim_{i \in \mathcal{I}} \mathrm{Hom}_{\mathcal{C}}(x, F(i)) \simeq \mathrm{Hom}_{\mathrm{Cat}_{\infty/\mathcal{I}}}(\mathcal{I}, F^*(\mathcal{C}_{x/}))$. Let us now manipulate the right-hand side as follows:

$$
\begin{aligned}
\mathrm{Hom}_{\mathrm{Cat}_{\infty/\mathcal{I}}}\big(\mathcal{I}, F^*(\mathcal{C}_{x/})\big) &\simeq \mathrm{Hom}_{\mathrm{Cat}_{\infty}}\big(\mathcal{I}, F^*(\mathcal{C}_{x/})\big) \times_{t, \mathrm{Hom}_{\mathrm{Cat}_{\infty}}(\mathcal{I}, \mathcal{I})} \{\mathrm{id}_{\mathcal{I}}\} \\
&\simeq \mathrm{Hom}_{\mathrm{Cat}_{\infty}}\big(\mathcal{I}, \{x\} \times_{\mathcal{C}, s} \mathrm{Ar}(\mathcal{C}) \times_{t, \mathcal{C}, F} \mathcal{I}\big) \times_{t, \mathrm{Hom}_{\mathrm{Cat}_{\infty}}(\mathcal{I}, \mathcal{I})} \{\mathrm{id}_{\mathcal{I}}\} \\
&\simeq \{\mathrm{const}\, x\} \times_{\mathrm{Hom}_{\mathrm{Cat}_{\infty}}(\mathcal{I}, \mathcal{C}), s} \mathrm{Hom}_{\mathrm{Cat}_{\infty}}\big(\mathcal{I}, \mathrm{Ar}(\mathcal{C})\big) \times_{t, \mathrm{Hom}_{\mathrm{Cat}_{\infty}}(\mathcal{I}, \mathcal{C})} \{F\} \\
&\simeq \mathrm{Hom}_{\mathrm{Fun}(\mathcal{I}, \mathcal{C})}(\mathrm{const}\, x, F)\,.
\end{aligned}
$$

In the first step we plug in Corollary 5.15 to write $\mathrm{Hom}_{\mathrm{Cat}_{\infty/\mathcal{I}}}$ as a pullback. In the second step, we plug in $F^*(\mathcal{C}_{x/}) \cong \{x\} \times_{\mathcal{C}, s} \mathrm{Ar}(\mathcal{C}) \times_{t, \mathcal{C}, F} \mathcal{I}$. In the third step we use Lemma 6.15 and simplify the pullback. Finally, in the fourth step we write $\mathrm{Hom}_{\mathrm{Cat}_{\infty}}(\mathcal{I}, \mathcal{C}) \simeq \mathrm{core}\,\mathrm{Fun}(\mathcal{I}, \mathcal{C})$ and $\mathrm{Hom}_{\mathrm{Cat}_{\infty}}(\mathcal{I}, \mathrm{Ar}(\mathcal{C})) \simeq \mathrm{Hom}(\Delta^1, \mathrm{Fun}(\mathcal{I}, \mathcal{C})) \simeq \mathrm{core}\,\mathrm{Ar}(\mathrm{Fun}(\mathcal{I}, \mathcal{C}))$ and use the definition of $\mathrm{Hom}_{\mathrm{Fun}(\mathcal{I}, \mathcal{C})}(\mathrm{const}\, x, F)$ from 2.11; as we've seen in model category fact 6.13, the pullbacks in sSet from 2.11 can be taken in Cat_{∞} as well, and since $\mathrm{Hom}_{\mathrm{Fun}(\mathcal{I}, \mathcal{C})}(\mathrm{const}\, x, F)$ is an anima anyway, it doesn't matter that we apply core everywhere.

Therefore, at least pointwise, c_F^* takes the form $c_F^*\colon \mathrm{Hom}_{\mathcal{C}}(x, y) \to \mathrm{Hom}_{\mathrm{Fun}(\mathcal{I}, \mathcal{C})}(\mathrm{const}\, x, F)$ for all $x \in \mathcal{C}$. Since a natural transformation is an equivalence if and only if it is a pointwise equivalence (Theorem 4.5), we are done. $\qquad\square$

6.17. Corollary. — *For every ∞-category $\mathcal{C}$, the functors $\mathrm{Hom}_{\mathcal{C}}(x, -)\colon \mathcal{C} \to \mathrm{An}$ and $\mathrm{Hom}_{\mathcal{C}}(-, y)\colon \mathcal{C}^{\mathrm{op}} \to \mathrm{An}$ preserve limits for all $x, y \in \mathcal{C}$ (note that limits in $\mathcal{C}^{\mathrm{op}}$ correspond to colimits in $\mathcal{C}$). Likewise, the Yoneda embedding $\natural_{\mathcal{C}}\colon \mathcal{C} \to \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{An})$ preserves limits.*

Proof. The first two assertions follow immediately from Corollary 6.16. The last one follows from the first plus the fact that limits and equivalences in functor categories are pointwise by Lemma 6.12 and Theorem 4.5. $\qquad\square$

§6.3. Coinitial and initial functors

Our next goal is to develop a general theory of diagrams that have the same limit or colimit. This is summarised by the following theorem due to Joyal, with a first written proof appearing in [L-HTT, Theorem 4.1.3.1].

6.18. Theorem (Joyal's version of Quillen's theorem A). — *For a functor $\alpha\colon \mathcal{I} \to \mathcal{J}$ of ∞-categories, the following are equivalent:*

(a)   *For every ∞-category $\mathcal{C}$ and every $F\colon \mathcal{J} \to \mathcal{C}$, the functor F has a colimit if and only if $F \circ \alpha$ has a colimit. Furthermore, in this case the following natural map is an equivalence:*

$$\operatorname*{colim}_{i\in\mathcal{I}} F\big(\alpha(i)\big) \xrightarrow{\simeq} \operatorname*{colim}_{j\in\mathcal{J}} F(j)\,.$$

(b)   *For every right fibration $f\colon X \to \mathcal{J}$, the following natural map is an equivalence:*

$$\operatorname{Hom}_{\operatorname{Cat}_{\infty/\mathcal{J}}}(\mathcal{I}, X) \xrightarrow{\simeq} \operatorname{Hom}_{\operatorname{Cat}_{\infty/\mathcal{J}}}(\mathcal{J}, X)\,.$$

(c)   *For every $j \in \mathcal{J}$, the slice-∞-category $\mathcal{I}_{j/} := \mathcal{I} \times_{\mathcal{J}} \mathcal{J}_{j/}$ is weakly contractible. That is, we have $|\mathcal{I}_{j/}| \simeq *$.*

A dual assertion holds for limits, left fibrations, and the slice-∞-categories $\mathcal{I}_{/j}$.

6.19. Definition. — If $\alpha\colon \mathcal{I} \to \mathcal{J}$ satisfies the equivalent conditions from Theorem 6.18, then α is called *coinitial*. Dually, α is called *initial* if it satisfies the dual equivalent conditions for limits.[(6.4)]

6.20. Example. — The following are examples of coinitial functors:

(a)   Right anodyne maps are coinitial. It's clear from Corollary 3.10 and Corollary 5.15 that the condition from Theorem 6.18(*b*) is satisfied.

(b)   Right adjoint functors $\alpha\colon \mathcal{I} \to \mathcal{J}$ are coinitial. Indeed, if β is a left adjoint, then $\alpha^*\colon \operatorname{Fun}(\mathcal{J},\mathcal{C}) \rightleftarrows \operatorname{Fun}(\mathcal{I},\mathcal{C}) :\beta^*$ is an adjunction by Corollary 6.6 and so to verify the condition $\operatorname{colim}_{\mathcal{I}} \circ \alpha^* \simeq \operatorname{colim}_{\mathcal{J}}$ from Theorem 6.18(*a*), it's enough to check $\beta^* \circ \operatorname{const} \simeq \operatorname{const}$, which is clear.

(c)   Localisations $p\colon \mathcal{I} \to \mathcal{I}[W^{-1}]$ are coinitial. One way to see this is that localisations are right anodyne, since by construction, p factors into $\mathcal{I} \to \overline{\mathcal{I}} \to \mathcal{I}[W^{-1}]$, where the second arrow is inner anodyne and the first arrow is right anodyne, because $\Delta^1 \to J$ is right anodyne. Then (*a*) does it.

But of course there's also a synthetic way to see this. Since the precomposition functor $p^*\colon \operatorname{Fun}(\mathcal{I}[W^{-1}],\mathcal{C}) \to \operatorname{Fun}(\mathcal{I},\mathcal{C})$ is fully faithful by Lemma 4.11, we have

$$\operatorname{Hom}_{\operatorname{Fun}(\mathcal{I},\mathcal{C})}(F \circ p, \operatorname{const} y) \simeq \operatorname{Hom}_{\operatorname{Fun}(\mathcal{I}[W^{-1}],\mathcal{C})}(F, \operatorname{const} y)\,,$$

functorially in $F\colon \mathcal{I}[W^{-1}] \to \mathcal{C}$ and all $y \in \mathcal{C}$, which proves that the condition from Theorem 6.18(*a*) is satified.

Proof of Theorem 6.18, (a) $\Leftrightarrow$ (b). Let $F\colon \mathcal{J} \to \operatorname{An}$ be a functor with unstraightening $\mathcal{U} \to \mathcal{J}$. Then the pullback $\alpha^*(\mathcal{U}) \to \mathcal{I}$ is the unstraightening of $F \circ \alpha\colon \mathcal{I} \to \operatorname{An}$. Lemma 6.14 shows $\lim_{j\in\mathcal{J}} F(j) \simeq \operatorname{Hom}_{\operatorname{Cat}_{\infty/\mathcal{J}}}(\mathcal{J},\mathcal{U})$. Similarly,

$$\lim_{i\in\mathcal{I}} F\big(\alpha(i)\big) \simeq \operatorname{Hom}_{\operatorname{Cat}_{\infty/\mathcal{I}}}(\mathcal{I}, \alpha^*(\mathcal{U})) \simeq \operatorname{Hom}_{\operatorname{Cat}_{\infty/\mathcal{I}}}(\mathcal{I},\mathcal{U})\,;$$

[(6.4)]Calling these functors (*co-*)*initial* is slightly nonstandard: People usually use the terms (*co-*)*final*, but there seems to be no universally agreed convention on which of these terms refers to limits and which refers to colimits. In any case, our naming convention is objectively the correct one. It is uniquely determined by the following two desiderata:

(a)   The concept for colimits should have the presyllable *co-*, whereas the concept for limits should have no presyllable at all.

(b)   The inclusion of an initial object should be *initial*, or maybe *cofinal*, but definitely not *final*. Likewise for terminal objects.

here the second equivalence is a quick calculation using Corollary 5.15 and Corollary 6.16. This shows that (*b*) holds if and only if (*a*) holds for functors $F\colon \mathcal{J} \to \mathrm{An}$. Now let $F\colon \mathcal{J} \to \mathcal{C}$ be an arbitrary functor. By Corollary 6.16, $\natural_{\mathcal{C}}\colon \mathcal{C} \to \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{An})$ preserves limits and it is fully faithful, so (*a*) holds for $F\colon \mathcal{J} \to \mathcal{C}$ if and only if it holds for $\natural_{\mathcal{C}} \circ F\colon \mathcal{J} \to \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{An})$. Finally, limits in $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{An})$ are computed pointwise by Lemma 6.12 and equivalences can be checked pointwise by Theorem 4.5, so (*a*) holds for functors into $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{An})$ if and only if it holds for functors into An. This finishes the proof of (*a*) $\Leftrightarrow$ (*b*). $\qquad\square$

Before we can prove (*a*) $\Rightarrow$ (*c*) $\Rightarrow$ (*b*), we need another lemma.

6.21. Lemma. — *A cartesian fibration $p\colon \mathcal{U} \to \mathcal{J}$ satisfies the conclusion of Theorem 6.18(*b*) if and only if the fibres $p^{-1}\{j\}$ of p are weakly contractible, that is, $|p^{-1}\{j\}| \simeq *$ for all $j \in \mathcal{J}$.*

Proof. Let $E\colon \mathcal{J}^{\mathrm{op}} \to \mathrm{Cat}_\infty$ be the straightening of $p\colon \mathcal{U} \to \mathcal{J}$ and let $f\colon X \to \mathcal{J}$ be a right fibration with straightening $F\colon \mathcal{J}^{\mathrm{op}} \to \mathrm{An}$. Then the cartesian straightening equivalence (the dual of Theorem 5.4(*a*)) shows

$$\mathrm{Hom}_{\mathrm{Cat}_{\infty/\mathcal{J}}}(\mathcal{U}, X) \simeq \mathrm{Hom}_{\mathrm{Cart}(\mathcal{J})}(\mathcal{U}, X) \simeq \mathrm{Hom}_{\mathrm{Fun}(\mathcal{J}^{\mathrm{op}}, \mathrm{Cat}_\infty)}(E, F)\,.$$

Note that the first equivalence holds even though $\mathrm{Cart}(\mathcal{J}) \to \mathrm{Cat}_{\infty/\mathcal{J}}$ is not fully faithful, since we're mapping into a right fibration where every morphism is cartesian (by the dual of Lemma 5.3). Now $|\cdot|\colon \mathrm{Cat}_\infty \to \mathrm{An}$ is left adjoint to the inclusion $\mathrm{An} \subseteq \mathrm{Cat}_\infty$ by Example 6.3(*a*) and that adjunction persists to functor-∞-categories by Corollary 6.6. Thus

$$\mathrm{Hom}_{\mathrm{Fun}(\mathcal{J}^{\mathrm{op}}, \mathrm{Cat}_\infty)}(E, F) \simeq \mathrm{Hom}_{\mathrm{Fun}(\mathcal{J}^{\mathrm{op}}, \mathrm{Cat}_\infty)}\bigl(|E|, F\bigr)\,.$$

The cartesian straightening of $\mathrm{id}_{\mathcal{J}}\colon \mathcal{J} \to \mathcal{J}$ is $\mathrm{const}\,*\colon \mathcal{J}^{\mathrm{op}} \to \mathrm{An}$. By the same arguments as above we then obtain

$$\mathrm{Hom}_{\mathrm{Cat}_{\infty/\mathcal{J}}}(\mathcal{J}, X) \simeq \mathrm{Hom}_{\mathrm{Fun}(\mathcal{J}^{\mathrm{op}}, \mathrm{Cat}_\infty)}(\mathrm{const}\,*, F)\,.$$

Putting everything together, we see that the condition from Theorem 6.18(*b*) is satisfied if and only if $p\colon \mathcal{U} \to \mathcal{J}$ induces an equivalence $|E| \Rightarrow \mathrm{const}\,*$ in $\mathrm{Fun}(\mathcal{J}^{\mathrm{op}}, \mathrm{An})$. Since equivalences can be checked pointwise (Theorem 4.5), this becomes precisely the condition that all fibres of p are weakly contractible. $\qquad\square$

Note that Lemma 6.21 and Theorem 6.18(*c*) look very much alike, but are a priori two different criteria for a cartesian fibration $p\colon \mathcal{U} \to \mathcal{J}$ to be coinitial. As a reality check, let's see that they are indeed equivalent. This isn't necessary to complete our proof of Theorem 6.18, but we'll need it later.

6.22. Lemma. — *Let $p\colon \mathcal{U} \to \mathcal{J}$ be a cartesian fibration. Then for every $j \in \mathcal{J}$, the natural functor $p^{-1}\{j\} \to \mathcal{U} \times_{\mathcal{J}} \mathcal{J}_{j/}$ admits a right adjoint. In particular, we obtain a homotopy equivalence of animae $|p^{-1}\{j\}| \simeq |\mathcal{U} \times_{\mathcal{J}} \mathcal{J}_{j/}|$.*

Proof sketch. By Lemma 6.2, right adjoints can be constructed pointwise. This can be done as follows: Fix an object $(u, \overline{\varphi}) \in \mathcal{U} \times_{\mathcal{J}} \mathcal{J}_{j/}$, given by an element $u \in \mathcal{U}$ and a morphism $\overline{\varphi}\colon j \to p(u)$ in $\mathcal{J}$. Let $\varphi\colon u' \to u$ be a p-cartesian lift of $\overline{\varphi}$. Then $u' \in p^{-1}\{j\}$ is a right adjoint object to $(u, \overline{\varphi})$ under $p^{-1}\{j\} \to \mathcal{U} \times_{\mathcal{J}} \mathcal{J}_{j/}$. To see this, note that, by construction, we have

a morphism $c\colon (u', \mathrm{id}_j) \to (u, \overline{\varphi})$ in $\mathcal{U} \times_{\mathcal{J}} \mathcal{J}_{j/}$ (which will play the role of the counit); using Theorem 4.5, we have to show that the composition

$$\mathrm{Hom}_{p^{-1}\{j\}}(u'', u') \longrightarrow \mathrm{Hom}_{\mathcal{U} \times_{\mathcal{J}} \mathcal{J}_{j/}}\big((u'', \mathrm{id}_j), (u', \mathrm{id}_j)\big) \xrightarrow{c_*} \mathrm{Hom}_{\mathcal{U} \times_{\mathcal{J}} \mathcal{J}_{j/}}\big((u'', \mathrm{id}_j), (u, \overline{\varphi})\big)$$

is an equivalence for all $u'' \in p^{-1}\{j\}$. Now use the characterisation of cartesian morphisms from the dual of Lemma 5.16 together with Corollary 5.15 and the fact that Hom animae in pullbacks of ∞-categories are pullbacks of the respective Hom animae (which is straightforward to see; we'll prove a more general statement in Lemma 6.76(*a*)) to show that both sides are equivalent to $\mathrm{Hom}_{\mathcal{U}}(u'', u) \times_{\mathrm{Hom}_{\mathcal{J}}(j, p(u))} \{\overline{\varphi}\}$ and that the morphism between them is equivalent to the identity. We'll leave the details to you. $\qquad\square$

Proof of Theorem 6.18, (*a*) $\Rightarrow$ (*c*) $\Rightarrow$ (*b*). Assume (*a*) holds true and consider the functor $\mathrm{Hom}_{\mathcal{J}}(j_0, -)\colon \mathcal{J} \to \mathrm{An}$ for some $j_0 \in \mathcal{J}$. Its unstraightening is the slice-∞-category projection $t\colon \mathcal{J}_{j_0/} \to \mathcal{J}$, hence $\mathrm{colim}_{j \in \mathcal{J}} \mathrm{Hom}_{\mathcal{J}}(j_0, j) \simeq |\mathcal{J}_{j_0/}|$ by Lemma 6.14. But $\mathcal{J}_{j_0/}$ has an initial element given by id_{j_0}, and so $\{\mathrm{id}_{j_0}\} \rightleftarrows \mathcal{J}_{j_0/}$ is an adjunction. Since adjunctions induce homotopy equivalences after $|\cdot|$, we conclude $|\mathcal{J}_{j_0/}| \simeq *$.

Now consider $\mathrm{Hom}_{\mathcal{J}}(j_0, \alpha(-))\colon \mathcal{I} \to \mathrm{An}$. Its unstraightening is $\mathcal{I}_{j_0/} \simeq \mathcal{I} \times_{\mathcal{J}} \mathcal{J}_{j_0/}$ (here we use that precomposition with α corresponds to pullback along α under the unstraightening equivalence, see Theorem 5.4). Combining Lemma 6.14 with condition (*a*), we obtain

$$|\mathcal{I}_{j_0/}| \simeq \operatorname*{colim}_{i \in \mathcal{I}} \mathrm{Hom}_{\mathcal{J}}\big(j_0, \alpha(i)\big) \simeq \operatorname*{colim}_{j \in \mathcal{J}} \mathrm{Hom}_{\mathcal{J}}(j_0, j) \simeq *,$$

as claimed. This finishes the proof of the implication (*a*) $\Rightarrow$ (*c*).

Now assume (*c*). We can factor $\alpha\colon \mathcal{I} \to \mathcal{J}$ into $\mathcal{I} \to \mathcal{I} \times_{\mathcal{J},s} \mathrm{Ar}(\mathcal{J}) \to \mathcal{J}$, where the first functor sends $i \in \mathcal{I}$ to the pair $(i, \mathrm{id}_{\alpha(i)}\colon \alpha(i) \to \alpha(i))$ and the second functor is induced by the target projection $t\colon \mathrm{Ar}(\mathcal{J}) \to \mathcal{J}$. It's straightforward to verify that $\mathcal{I} \to \mathcal{I} \times_{\mathcal{J},s} \mathrm{Ar}(\mathcal{J})$ is right adjoint to the projection $s\colon \mathcal{I} \times_{\mathcal{J},s} \mathrm{Ar}(\mathcal{J}) \to \mathcal{I}$.[(6.5)] By Example 6.20(*b*), we see that the functor $\mathcal{I} \to \mathcal{I} \times_{\mathcal{J},s} \mathrm{Ar}(\mathcal{J})$ satisfies (*a*), hence also (*b*). Furthermore, a slight generalisation of Example 5.5(*c*) (which can be proved by the same argument) shows that $t\colon \mathcal{I} \times_{\mathcal{J},s} \mathrm{Ar}(\mathcal{J}) \to \mathcal{J}$ is a cocartesian fibration. Its fibres $t^{-1}\{j\} \simeq \mathcal{I} \times_{\mathcal{J}} \mathcal{J}_{j/}$ are weakly contractible. Hence $t\colon \mathcal{I} \times_{\mathcal{J},s} \mathrm{Ar}(\mathcal{J}) \to \mathcal{J}$ satisfies (*b*) by Lemma 6.21. We conclude that $\alpha\colon \mathcal{I} \to \mathcal{J}$ must satisfy (*b*) as well. Indeed, if $f\colon X \to \mathcal{J}$ is a right fibration, then

$$\mathrm{Hom}_{\mathrm{Cat}_{\infty/\mathcal{J}}}(\mathcal{J}, X) \simeq \mathrm{Hom}_{\mathrm{Cat}_{\infty/\mathcal{J}}}\big(\mathcal{I} \times_{\mathcal{J},s} \mathrm{Ar}(\mathcal{J}), X\big)$$

$$\simeq \mathrm{Hom}_{\mathrm{Cat}_{\infty/\mathcal{I} \times_{\mathcal{J},s}\mathrm{Ar}(\mathcal{J})}}\big(\mathcal{I} \times_{\mathcal{J},s} \mathrm{Ar}(\mathcal{J}), t^*(X)\big).$$

In the first equivalence we use (*b*) for $t\colon \mathcal{I} \times_{\mathcal{J},s} \mathrm{Ar}(\mathcal{J}) \to \mathcal{J}$. In the second equivalence we let $t^*(X) \to \mathcal{I} \times_{\mathcal{J},s} \mathrm{Ar}(\mathcal{J})$ be the pullback of f along t and use Lemma 6.23(*a*) below. Now a pullback of a right fibration is again a right fibration, whence

$$\mathrm{Hom}_{\mathrm{Cat}_{\infty/\mathcal{I} \times_{\mathcal{J},s}\mathrm{Ar}(\mathcal{J})}}\big(\mathcal{I} \times_{\mathcal{J},s} \mathrm{Ar}(\mathcal{J}), t^*(X)\big) \simeq \mathrm{Hom}_{\mathrm{Cat}_{\infty/\mathcal{I} \times_{\mathcal{J},s}\mathrm{Ar}(\mathcal{J})}}\big(\mathcal{I}, t^*(X)\big)$$

$$\simeq \mathrm{Hom}_{\mathrm{Cat}_{\infty/\mathcal{J}}}(\mathcal{I}, X).$$

In the first equivalence we use (*b*) for $\mathcal{I} \to \mathcal{I} \times_{\mathcal{J},s} \mathrm{Ar}(\mathcal{J})$ and in the second we use Lemma 6.23(*a*) below again. This finishes the proof of the implication (*c*) $\Rightarrow$ (*b*). $\qquad\square$

[(6.5)]For example, one could use Lemma 5.13; alternatively, unit and counit as well as the triangle identities are easily constructed by hand.

§6.4. Kan extensions

We're now working towards an ∞-categorical analogue of Theorem 1.17. Our first goal is to construct left Kan extensions for presheaf categories. As it turns out, this is most easily done in the fibration picture.

6.23. Lemma. — *Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories.*

(a) *The pullback functor $F^*\colon \mathrm{Cat}_{\infty/\mathcal{D}} \to \mathrm{Cat}_{\infty/\mathcal{C}}$ has a left adjoint, namely the forgetful functor $\mathrm{Cat}_{\infty/\mathcal{C}} \to \mathrm{Cat}_{\infty/\mathcal{D}}$ that sends $f\colon \mathcal{C}' \to \mathcal{C}$ to $F \circ f\colon \mathcal{C}' \to \mathcal{D}$.*

(b) *The inclusion $\mathrm{Right}(\mathcal{D}) \subseteq \mathrm{Cat}_{\infty/\mathcal{D}}$ has a left adjoint that sends $g\colon \mathcal{D}' \to \mathcal{D}$ to $q\colon Y \to \mathcal{D}$, where*

$$\mathcal{D}' \longrightarrow Y \xrightarrow{\ q\ } \mathcal{D}$$

is any factorisation of g into a coinitial functor followed by a right fibration.

(c) *The functor $F^*\colon \mathrm{Right}(\mathcal{D}) \to \mathrm{Right}(\mathcal{C})$ has a left adjoint $F_!\colon \mathrm{Right}(\mathcal{C}) \to \mathrm{Right}(\mathcal{D})$. On objects, $F_!$ is given as follows: Let $p\colon X \to \mathcal{C}$ be a right fibration and let*

$$X \longrightarrow Y \xrightarrow{\ q\ } \mathcal{D}$$

be any factorisation of $F \circ p$ into a coinitial functor followed by a right fibration. Then we have $F_!(p\colon X \to \mathcal{C}) \simeq (q\colon Y \to \mathcal{D})$. In particular, all such factorisations are equivalent.

Proof. For (a), note that left adjoints can be constructed pointwise by Lemma 6.2, so its enough to show that $F \circ f\colon \mathcal{C}' \to \mathcal{C}$ is a left adjoint object to $f\colon \mathcal{C}' \to \mathcal{C}$ under F^*. To this end, let $g\colon \mathcal{D}' \to \mathcal{D}$ be an element in $\mathrm{Cat}_{\infty/\mathcal{D}}$. We have a diagram

$$
\begin{array}{ccccc}
\mathrm{Hom}_{\mathrm{Cat}_{\infty/\mathcal{C}}}\big(\mathcal{C}', F^*(\mathcal{D}')\big) & \longrightarrow & \mathrm{Hom}_{\mathrm{Cat}_\infty}\big(\mathcal{C}', F^*(\mathcal{D}')\big) & \longrightarrow & \mathrm{Hom}_{\mathrm{Cat}_\infty}(\mathcal{C}', \mathcal{D}') \\
\downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow \\
\{f\} & \longrightarrow & \mathrm{Hom}_{\mathrm{Cat}_\infty}(\mathcal{C}', \mathcal{C}) & \xrightarrow{\ F_*\ } & \mathrm{Hom}_{\mathrm{Cat}_\infty}(\mathcal{C}', \mathcal{D})
\end{array}
$$

in which the left square is a pullback by Corollary 5.15 and the right square is a pullback by Corollary 6.16. Hence the outer rectangle is a pullback too. Combining this with Corollary 5.15, we obtain

$$\mathrm{Hom}_{\mathrm{Cat}_{\infty/\mathcal{C}}}\big(\mathcal{C}', F^*(\mathcal{D}')\big) \simeq \mathrm{Hom}_{\mathrm{Cat}_{\infty/\mathcal{D}}}(\mathcal{C}', \mathcal{D}')\,.$$

Since every step in the argument can be made functorial in $g\colon \mathcal{D}' \to \mathcal{D}$, we have proved (a).

For (b), note that (a) combined with Theorem 6.18(b) immediately implies that $q\colon Y \to \mathcal{D}$ is a left adjoint object to $g\colon \mathcal{D}' \to \mathcal{D}$ under the inclusion $\mathrm{Right}(\mathcal{D}) \subseteq \mathrm{Cat}_{\infty/\mathcal{D}}$. Since left adjoints can be constructed pointwise by Lemma 6.2, we only need to check that such a factorisation always exists. But that's easy! For example, we could choose $\mathcal{D}' \to Y$ to be right anodyne by Lemma 3.12 and Example 6.20(a). If you'd like to avoid simplicial sets, we could also argue as follows: Choose a factorisation $\mathcal{D}' \to Y' \to \mathcal{D}$ into a right adjoint functor followed by a cartesian fibration $g'\colon Y' \to \mathcal{D}$ as in the proof of Theorem 6.18. Then put

$$(g\colon Y \to \mathcal{D}) := \mathrm{Un}^{\mathrm{right}}\left(\mathcal{D} \xrightarrow{\ \mathrm{St}^{\mathrm{cart}}(g)\ } \mathrm{Cat}_\infty \xrightarrow{\ |\cdot|\ } \mathrm{An}\right).$$

Finally, (c) follows from the combined powers of (a) and (b). $\qquad\square$

In the following, we let $\mathrm{PSh}(\mathcal{C}) := \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{An})$ denote the ∞-*category of presheaves on* $\mathcal{C}$. Note that this is *not* compatible with the previous definition of presheaves on ordinary categories, which would be $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Set})$.

6.24. Corollary. — *Let* $F\colon \mathcal{C} \to \mathcal{D}$ *be a functor of ∞-categories. Then the precomposition functor* $F^*\colon \mathrm{PSh}(\mathcal{D}) \to \mathrm{PSh}(\mathcal{C})$ *has a left adjoint* $F_!$ *such that the diagram*

$$
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{\ \ F\ \ } & \mathcal{D} \\
\downarrow{\scriptstyle \text{よ}_{\mathcal{C}}} & /\!/\!/ & \downarrow{\scriptstyle \text{よ}_{\mathcal{D}}} \\
\mathrm{PSh}(\mathcal{C}) & \xrightarrow{\ \ F_!\ \ } & \mathrm{PSh}(\mathcal{D})
\end{array}
$$

commutes in the ∞-category Cat_∞.

Proof. It's clear from Lemma 6.23 and the right straightening equivalence (the dual of Theorem 5.4(b)) that $F_!$ exists, so we only have to show that the diagram commutes. To this end, first note that the natural transformation $\mathrm{Hom}_{\mathcal{C}}(-,-) \Rightarrow \mathrm{Hom}_{\mathcal{D}}(F(-), F(-))$ gets transformed into $\text{よ}_{\mathcal{C}} \Rightarrow F^* \circ \text{よ}_{\mathcal{D}} \circ F$ under the equivalence in $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}} \times \mathcal{C}, \mathrm{An}) \simeq \mathrm{Fun}(\mathcal{C}, \mathrm{PSh}(\mathcal{C}))$. Using the adjunction $F_! \dashv F^*$ as well as Corollary 6.6, this transformation is adjoint to a natural transformation $F_! \circ \text{よ}_{\mathcal{C}} \Rightarrow \text{よ}_{\mathcal{D}} \circ F$.

So our diagram commutes up to natural transformation, and we have to show that said natural transformation is an equivalence. By Theorem 4.5, this can be done pointwise. So choose $x \in \mathcal{C}$. Under the straightening equivalence, the functor $\text{よ}_{\mathcal{C}}(x) \simeq \mathrm{Hom}_{\mathcal{C}}(-, x)$ corresponds to the right fibration $\mathcal{C}_{/x} \to \mathcal{C}$. Likewise, $\text{よ}_{\mathcal{D}}(F(x)) \simeq \mathrm{Hom}_{\mathcal{D}}(-, F(x))$ corresponds to $\mathcal{D}_{/F(x)} \to \mathcal{D}$. Using Lemma 6.23($c$), we only have to show that the top horizontal arrow in the diagram

$$
\begin{array}{ccc}
\mathcal{C}_{/x} & \longrightarrow & \mathcal{D}_{/F(x)} \\
\downarrow & /\!/\!/ & \downarrow \\
\mathcal{C} & \xrightarrow{\ \ F\ \ } & \mathcal{D}
\end{array}
$$

is coinitial. But that's easy! Both $\mathcal{C}_{/x}$ and $\mathcal{D}_{/F(x)}$ have terminal objects, hence there are adjunctions $\mathcal{C}_{/x} \rightleftarrows \{\mathrm{id}_x\}$ and $\mathcal{D}_{/F(x)} \rightleftarrows \{\mathrm{id}_{F(x)}\}$. Hence $* \to \mathcal{C}_{/x}$ and $* \to \mathcal{D}_{/F(x)}$ are both coinitial by Example 6.20(b). Since being coinitial is closed under 2-out-of-3 (for example, by the condition from Theorem 6.18(a)), $\mathcal{C}_{/x} \to \mathcal{D}_{/F(x)}$ must be coinitial too. $\qquad\square$

Corollary 6.24 allows us to compute Hom animae in functor ∞-categories!

6.25. Corollary. — *Given functors* $F, G\colon \mathcal{C} \to \mathcal{D}$ *of ∞-categories, the anima of natural transformations* $\mathrm{Hom}_{\mathrm{Fun}(\mathcal{C},\mathcal{D})}(F, G)$ *can be computed as the following limit:*

$$
\lim_{(x \to y) \in \mathrm{TwAr}(\mathcal{C})} \mathrm{Hom}_{\mathcal{D}}\big(F(x), G(y)\big) := \lim\left(\mathrm{TwAr}(\mathcal{C}) \xrightarrow{(s,t)} \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \xrightarrow{F^{\mathrm{op}} \times G} \mathcal{D}^{\mathrm{op}} \times \mathcal{D} \xrightarrow{\mathrm{Hom}_{\mathcal{D}}} \mathrm{An}\right).
$$

Proof. By Lemma 6.14, the right-hand side can be computed as $\mathrm{Hom}_{\mathrm{Cat}_{\infty / \mathrm{TwAr}(\mathcal{C})}}(\mathrm{TwAr}(\mathcal{C}), \mathcal{U})$, where $\mathcal{U}$ denotes the unstraightening of $\mathrm{Hom}_{\mathcal{D}} \circ (F^{\mathrm{op}} \times G) \circ (s,t)\colon \mathrm{TwAr}(\mathcal{C}) \to \mathrm{An}$. Since unstraightening transforms compositions into pullbacks and the unstraightening of $\mathrm{Hom}_{\mathcal{D}}$ is $\mathrm{TwAr}(\mathcal{D}) \to \mathcal{D}^{\mathrm{op}} \times \mathcal{D}$ by Construction 5.21 or 5.22, we have a pullback diagram

$$
\begin{array}{ccccc}
\mathcal{U} & \longrightarrow & \mathcal{U}' & \longrightarrow & \mathrm{TwAr}(\mathcal{D}) \\
\downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow \\
\mathrm{TwAr}(\mathcal{C}) & \xrightarrow{(s,t)} & \mathcal{C}^{\mathrm{op}} \times \mathcal{C} & \xrightarrow{F^{\mathrm{op}} \times G} & \mathcal{D}^{\mathrm{op}} \times \mathcal{D}
\end{array}
$$

Using Lemma 6.23(c), we see $\mathrm{Hom}_{\mathrm{Cat}_{\infty/\mathrm{TwAr}(\mathcal{C})}}(\mathrm{TwAr}(\mathcal{C}),\mathcal{U}) \simeq \mathrm{Hom}_{\mathrm{Cat}_{\infty/\mathcal{C}^{\mathrm{op}}\times\mathcal{C}}}(\mathrm{TwAr}(\mathcal{C}),\mathcal{U}')$. But these are both left fibrations over $\mathcal{C}^{\mathrm{op}} \times \mathcal{C}$, so the Hom anima on the right-hand side can be equivalently computed as $\mathrm{Hom}_{\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}\times\mathcal{C},\mathrm{An})}(\mathrm{Hom}_{\mathcal{C}}, \mathrm{Hom}_{\mathcal{D}} \circ (F^{\mathrm{op}} \times G))$. Now the "currying" equivalence $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}} \times \mathcal{C}, \mathrm{An}) \simeq \mathrm{Fun}(\mathcal{C}, \mathrm{PSh}(\mathcal{C}))$ sends $\mathrm{Hom}_{\mathcal{C}}$ to $\mathord{\natural}_{\mathcal{C}}$ and $\mathrm{Hom}_{\mathcal{D}} \circ (F^{\mathrm{op}} \times G)$ to $F^* \circ \mathord{\natural}_{\mathcal{D}} \circ G$, hence the Hom anima under consideration is given by

$$\mathrm{Hom}_{\mathrm{Fun}(\mathcal{C},\mathrm{PSh}(\mathcal{C}))}\big(\mathord{\natural}_{\mathcal{C}}, F^* \circ \mathord{\natural}_{\mathcal{D}} \circ G\big) \simeq \mathrm{Hom}_{\mathrm{Fun}(\mathcal{C},\mathrm{PSh}(\mathcal{D}))}\big(F_! \circ \mathord{\natural}_{\mathcal{C}}, \mathord{\natural}_{\mathcal{D}} \circ G\big)$$
$$\simeq \mathrm{Hom}_{\mathrm{Fun}(\mathcal{C},\mathrm{PSh}(\mathcal{D}))}\big(\mathord{\natural}_{\mathcal{D}} \circ F, \mathord{\natural}_{\mathcal{D}} \circ G\big)$$
$$\simeq \mathrm{Hom}_{\mathrm{Fun}(\mathcal{C},\mathcal{D})}(F, G)\,,$$

as claimed. For the first equivalence, we use that $F_! \circ -$ is an adjoint of $F^* \circ -$ by construction and Corollary 6.6, the second equivalence follows from Corollary 6.24, and the third one since $\mathord{\natural}_{\mathcal{D}} \colon \mathcal{D} \to \mathrm{PSh}(\mathcal{D})$ is fully faithful by Yoneda's lemma (Corollary 5.25). $\qquad\square$

We'll now define and construct Kan extensions in the ∞-categorical world.

6.26. Definition. — Let $f \colon \mathcal{C} \to \mathcal{C}'$ and $F \colon \mathcal{C} \to \mathcal{D}$ be functors of ∞-categories. A *left Kan extension of F along f*, denoted $\mathrm{Lan}_f F \colon \mathcal{C}' \to \mathcal{D}$, is a left adjoint object to F under the precomposition functor $f^* \colon \mathrm{Fun}(\mathcal{C}', \mathcal{D}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{D})$. Dually, a *right Kan extension of F along f*, denoted $\mathrm{Ran}_f F \colon \mathcal{C}' \to \mathcal{D}$, is a right adjoint object to F under f^*.

Kan extensions in the ∞-categorical world can be computed by the same formula as in the ordinary case (Lemma 1.15):

6.27. Lemma (Kan extension formula). — *In the situation of Definition 6.26, assume that for all $x' \in \mathcal{C}'$ the following colimits exist in $\mathcal{D}$:*

$$\operatorname*{colim}_{(x,f(x)\to x')\in\mathcal{C}_{/x'}} F(x) \coloneqq \mathrm{colim}\Big(\mathcal{C}_{/x'} \longrightarrow \mathcal{C} \xrightarrow{\ F\ } \mathcal{D}\Big)$$

Then $\mathrm{Lan}_f F$ exists and $\mathrm{Lan}_f F(x')$ is given by that colimit.

To prove this, we first show that taking colimits is functorial in both the indexing ∞-category and the functor. As it will turn out during the proof, this is equivalent to constructing a partial left adjoint to the Yoneda embedding $\mathord{\natural}_{\mathcal{D}} \colon \mathcal{D} \to \mathrm{PSh}(\mathcal{D})$.

6.28. Lemma ("Colimits are functorial"). — *Let $\mathcal{D}$ be an ∞-category. Let $\mathcal{T} \subseteq \mathrm{Cat}_{\infty/\mathcal{D}}$ be spanned by those $\alpha \colon \mathcal{I} \to \mathcal{D}$ that admit a colimit. Consider the functor $\mathcal{D}_{/-} \colon \mathcal{D} \to \mathrm{Cat}_{\infty/\mathcal{D}}$ that sends $y \in \mathcal{D}$ to $\mathcal{D}_{/y} \to \mathcal{D}$. Then $\mathcal{D}_{/-}$ lands in $\mathcal{T}$ and admits a left adjoint $\mathrm{colim} \colon \mathcal{T} \to \mathcal{D}$ that sends $\alpha \colon \mathcal{I} \to \mathcal{D}$ to $\mathrm{colim}_{i\in\mathcal{I}} \alpha(i) \in \mathcal{D}$.*

Proof. Formally, the functor $\mathcal{D}_{/-} \colon \mathcal{D} \to \mathrm{Cat}_{\infty/\mathcal{D}}$ is defined via

$$\mathcal{D} \xrightarrow{\ \mathord{\natural}_{\mathcal{D}}\ } \mathrm{PSh}(\mathcal{D}) \simeq \mathrm{Right}(\mathcal{D}) \longrightarrow \mathrm{Cat}_{\infty/\mathcal{D}}\,,$$

using the Yoneda embedding and the right straightening equivalence (the dual of Theorem 5.4(b)). It's clear that $\mathcal{D}_{/-}$ takes values in $\mathcal{T}$. Indeed, $\mathcal{D}_{/y}$ has a terminal object and so the colimit over $\mathcal{D}_{/y} \to \mathcal{D}$ is just y. To prove the second assertion, by Lemma 6.2, it's enough to prove that for every $\alpha \colon \mathcal{I} \to \mathcal{D}$, the colimit $\mathrm{colim}_{i\in\mathcal{I}} \alpha(i) \in \mathcal{D}$ is a left adjoint object to α under $\mathcal{D}_{/-} \colon \mathcal{D} \to \mathcal{T}$.

This can be seen as follows: If $c \simeq \operatorname{colim}_{i \in \mathcal{I}} \alpha(i)$, then the associated natural transformation $\alpha \Rightarrow \operatorname{const} c$ induces a functor $u_\alpha \colon \mathcal{I} \to \mathcal{D}_{/c}$ in $\operatorname{Cat}_{\infty/\mathcal{D}}$. We then get a natural transformation

$$\operatorname{Hom}_{\mathcal{D}}(c, -) \xRightarrow{\mathcal{D}_{/-}} \operatorname{Hom}_{\operatorname{Cat}_{\infty/\mathcal{D}}}\!\big(\mathcal{D}_{/c}, \mathcal{D}_{/-}\big) \xRightarrow{u_\alpha^*} \operatorname{Hom}_{\operatorname{Cat}_{\infty/\mathcal{D}}}\!\big(\mathcal{I}, \mathcal{D}_{/-}\big).$$

Equivalences can be checked pointwise by Theorem 4.5. So choose $y \in \mathcal{D}$. We compute

$$
\begin{aligned}
\operatorname{Hom}_{\operatorname{Cat}_{\infty/\mathcal{D}}}\!\big(\mathcal{I}, \mathcal{D}_{/y}\big) &\simeq \{\alpha\} \times_{\operatorname{Hom}_{\operatorname{Cat}_\infty}(\mathcal{I},\mathcal{D}),s} \operatorname{Hom}_{\operatorname{Cat}_\infty}\!\big(\mathcal{I}, \operatorname{Ar}(\mathcal{D}) \times_{t,\mathcal{D}} \{y\}\big) \\
&\simeq \{\alpha\} \times_{\operatorname{Hom}_{\operatorname{Cat}_\infty}(\mathcal{I},\mathcal{D}),s} \operatorname{Hom}_{\operatorname{Cat}_\infty}\!\big(\mathcal{I}, \operatorname{Ar}(\mathcal{D})\big) \times_{t,\operatorname{Hom}_{\operatorname{Cat}_\infty}(\mathcal{I},\mathcal{D})} \{\operatorname{const} y\} \\
&\simeq \operatorname{Hom}_{\operatorname{Fun}(\mathcal{I},\mathcal{D})}(\alpha, \operatorname{const} y),
\end{aligned}
$$

and this agrees with $\operatorname{Hom}_{\mathcal{D}}(c, y)$ by definition of c. In the first step we use Corollary 5.15 as well as $\mathcal{D}_{/y} \simeq \operatorname{Ar}(\mathcal{D}) \times_{t,\mathcal{D}} \{y\}$. In the second step we use Corollary 6.17. In the third step, we use "currying" in the form of $\operatorname{Hom}_{\operatorname{Cat}_\infty}(\mathcal{I}, \operatorname{Ar}(\mathcal{D})) \simeq \operatorname{Hom}_{\operatorname{Cat}_\infty}(\Delta^1, \operatorname{Fun}(\mathcal{I},\mathcal{D}))$ and then plug in the definition of $\operatorname{Hom}_{\operatorname{Fun}(\mathcal{I},\mathcal{D})}$ as in 2.11. □

Proof of Lemma 6.27. Consider the diagram of functors

$$
\begin{array}{ccc}
\operatorname{Fun}(\mathcal{C}, \mathcal{D}) & \xrightarrow{(\text{よ}_{\mathcal{D}})_*} \operatorname{Fun}\!\big(\mathcal{C}, \operatorname{PSh}(\mathcal{D})\big) \dashrightarrow & \operatorname{Fun}\!\big(\mathcal{C}', \operatorname{Right}(\mathcal{D})\big) \\
& \simeq \Big\downarrow & \Big\downarrow \simeq \\
& \operatorname{Right}(\mathcal{D} \times \mathcal{C}^{\operatorname{op}}) \xrightarrow{(\operatorname{id}_{\mathcal{D}} \times f^{\operatorname{op}})_!} & \operatorname{Right}\!\big(\mathcal{D} \times (\mathcal{C}')^{\operatorname{op}}\big)
\end{array}
$$

(the vertical equivalences follow from the right straightening equivalence, see the dual of Theorem 5.4(*b*)). Let $F' \colon \mathcal{C}' \to \operatorname{Right}(\mathcal{D})$ denote the image of F under the top row functors and let $\overline{\mathcal{T}} := \mathcal{T} \cap \operatorname{Right}(\mathcal{D})$, where $\mathcal{T}$ is defined as in Lemma 6.28. If we can show that F' is contained in the full sub-∞-category $\operatorname{Fun}(\mathcal{C}', \overline{\mathcal{T}}) \subseteq \operatorname{Fun}(\mathcal{C}', \operatorname{Right}(\mathcal{D}))$, then we can define $\operatorname{Lan}_f F := \operatorname{colim} \circ F' \in \operatorname{Fun}(\mathcal{C}', \mathcal{D})$. It's clear from the various equivalences and adjunctions involved (more precisely, from Corollary 5.25, Lemma 6.23(*c*), and Lemma 6.28 combined with Corollary 6.6) that $\operatorname{Lan}_f F$ is indeed a left adjoint object of F under the precomposition functor $f^* \colon \operatorname{Fun}(\mathcal{C}', \mathcal{D}) \to \operatorname{Fun}(\mathcal{C}, \mathcal{D})$.

So we have to check that F' is indeed contained in $\operatorname{Fun}(\mathcal{C}', \overline{\mathcal{T}})$. The image of F under $(\text{よ}_{\mathcal{D}})_*$ followed by the "currying" equivalence $\operatorname{Fun}(\mathcal{C}, \operatorname{Fun}(\mathcal{D}^{\operatorname{op}}, \operatorname{An})) \simeq \operatorname{Fun}(\mathcal{D}^{\operatorname{op}} \times \mathcal{C}, \operatorname{An})$ is $\operatorname{Hom}_{\mathcal{D}}(-, F(-)) \colon \mathcal{D}^{\operatorname{op}} \times \mathcal{C} \to \operatorname{An}$. Its right unstraightening is

$$\operatorname{TwAr}(\mathcal{D})^{\operatorname{op}} \times_{t^{\operatorname{op}}, \mathcal{D}^{\operatorname{op}}, F^{\operatorname{op}}} \mathcal{C}^{\operatorname{op}} \longrightarrow \mathcal{D} \times \mathcal{C}^{\operatorname{op}}.$$

Indeed, the right unstraightening of $\operatorname{Hom}_{\mathcal{D}} \colon \mathcal{D}^{\operatorname{op}} \times \mathcal{D} \to \operatorname{An}$ is $(s^{\operatorname{op}}, t^{\operatorname{op}}) \colon \operatorname{TwAr}(\mathcal{D})^{\operatorname{op}} \to \mathcal{D} \times \mathcal{D}^{\operatorname{op}}$ by definition (of either $\operatorname{Hom}_{\mathcal{D}}$ or $\operatorname{TwAr}(\mathcal{D})$, see Constructions 5.21 and 5.22), and precomposition with $F \colon \mathcal{C} \to \mathcal{D}$ corresponds to pullback along F^{op}.

By Lemma 6.23(*c*), the functor $(\operatorname{id}_{\mathcal{D}} \times f^{\operatorname{op}})_!$ sends $\operatorname{TwAr}(\mathcal{D})^{\operatorname{op}} \times_{t^{\operatorname{op}}, \mathcal{D}^{\operatorname{op}}, F^{\operatorname{op}}} \mathcal{C}^{\operatorname{op}} \to \mathcal{D} \times \mathcal{C}^{\operatorname{op}}$ to a coinitial replacement of $\operatorname{TwAr}(\mathcal{D})^{\operatorname{op}} \times_{t^{\operatorname{op}}, \mathcal{D}^{\operatorname{op}}, F^{\operatorname{op}}} \mathcal{C}^{\operatorname{op}} \to \mathcal{D} \times \mathcal{C}^{\operatorname{op}} \to \mathcal{D} \times (\mathcal{C}')^{\operatorname{op}}$ by a right fibration. To figure out how such a coinitial replacement looks like, we claim the following:

($\boxtimes_1$) *In the diagram below, both vertical arrows are coinitial:*

$$\operatorname{TwAr}(\mathcal{D})^{\operatorname{op}} \times_{t^{\operatorname{op}}, \mathcal{D}^{\operatorname{op}}, F \circ t^{\operatorname{op}}} \operatorname{TwAr}(\mathcal{C})^{\operatorname{op}} \times_{f \circ s^{\operatorname{op}}, \mathcal{C}', s^{\operatorname{op}}} \operatorname{TwAr}(\mathcal{C}')^{\operatorname{op}}$$

$$\mathcal{C} \times_{f, \mathcal{C}', s^{\operatorname{op}}} \operatorname{TwAr}(\mathcal{C}')^{\operatorname{op}} \qquad\qquad \operatorname{TwAr}(\mathcal{D})^{\operatorname{op}} \times_{t^{\operatorname{op}}, \mathcal{D}^{\operatorname{op}}, F^{\operatorname{op}}} \mathcal{C}^{\operatorname{op}}$$

To prove claim ($\boxtimes_1$), we first observe that for every ∞-category $\mathcal{I}$, both the source projection $s^{\mathrm{op}} \colon \mathrm{TwAr}(\mathcal{I})^{\mathrm{op}} \to \mathcal{I}$ and the target projection $t^{\mathrm{op}} \colon \mathrm{TwAr}(\mathcal{I})^{\mathrm{op}} \to \mathcal{I}^{\mathrm{op}}$ are coinitial cartesian fibrations. Indeed, cartesianness is clear since $\mathrm{TwAr}(\mathcal{I})^{\mathrm{op}} \to \mathcal{I} \times \mathcal{I}^{\mathrm{op}}$ is a right fibration and projection to either factor is cartesian. For coini, we use Lemma 6.21: The fibre of t^{op} over $i \in \mathcal{I}^{\mathrm{op}}$ is $(t^{\mathrm{op}})^{-1}\{i\} \simeq \mathcal{I}_{/i}$; this follows from Lemma 5.24, regardless of which construction of $\mathrm{TwAr}(\mathcal{I})$ you use. Now $\mathcal{I}_{/i}$ is weakly contractible since it has a terminal object. The same argument applies to s^{op}. To apply this observation, observe that in the diagram above, the left vertical arrow is a composition of a base change of $t^{\mathrm{op}} \colon \mathrm{TwAr}(\mathcal{C})^{\mathrm{op}} \to \mathcal{C}^{\mathrm{op}}$ and a base change of $s^{\mathrm{op}} \colon \mathrm{TwAr}(\mathcal{C}')^{\mathrm{op}} \to \mathcal{C}'$. Since the conditions from Lemma 6.21 are stable under base change, this proves that the left vertical arrow is indeed coinitial. Similarly, the right vertical arrow is a composition of a base change of $s^{\mathrm{op}} \colon \mathrm{TwAr}(\mathcal{C})^{\mathrm{op}} \to \mathcal{C}$ and a base change of $t^{\mathrm{op}} \colon \mathrm{TwAr}(\mathcal{D})^{\mathrm{op}} \to \mathcal{D}^{\mathrm{op}}$, whence the same argument applies.

So we may equivalently look for a coinitial replacement of $\mathcal{C} \times_{f,\mathcal{C}',s^{\mathrm{op}}} \mathrm{TwAr}(\mathcal{C}')^{\mathrm{op}} \to \mathcal{D} \times (\mathcal{C}')^{\mathrm{op}}$ by a right fibration. Once again, we won't do this directly; instead, we claim another claim:

($\boxtimes_2$) *In the diagram below, the vertical arrows are cartesian fibrations over $(\mathcal{C}')^{\mathrm{op}}$ and the horizontal arrows preserve cartesian lifts:*

$$
\begin{array}{ccccc}
\mathcal{D} \times (\mathcal{C}')^{\mathrm{op}} & \xleftarrow{\; F \times \mathrm{id}_{(\mathcal{C}')^{\mathrm{op}}} \;} & \mathcal{C} \times (\mathcal{C}')^{\mathrm{op}} & \xleftarrow{\; (s^{\mathrm{op}},\, t^{\mathrm{op}}) \;} & \mathcal{C} \times_{\mathcal{C}',s^{\mathrm{op}}} \mathrm{TwAr}(\mathcal{C}')^{\mathrm{op}} \\
& {}_{\mathrm{pr}_2}\searrow \quad {/\!/\!/} & \downarrow{}_{\mathrm{pr}_2} & {/\!/\!/} \quad \swarrow {}_{t^{\mathrm{op}}} & \\
& & (\mathcal{C}')^{\mathrm{op}} & &
\end{array}
$$

Indeed, by definition of $\mathrm{TwAr}(\mathcal{C}')$, the arrow labelled $(s^{\mathrm{op}}, t^{\mathrm{op}})$ is a right fibration, and it's clear that both arrows labelled pr_2 are cartesian fibrations (see Example 5.5(a)). Hence t^{op}, being a composition of cartesian fibrations, is cartesian too. Furthermore, by a simple unravelling, we see that t^{op}-cartesian lifts are precisely the $(s^{\mathrm{op}}, t^{\mathrm{op}})$-cartesian lifts of pr_2-cartesian lifts, which immediately proves that $(s^{\mathrm{op}}, t^{\mathrm{op}})$ preserves cartesian lifts. Finally, it's clear that $F \times \mathrm{id}_{(\mathcal{C}')^{\mathrm{op}}}$ preserves cartesian lifts, since these are given by those morphisms in $\mathcal{C} \times (\mathcal{C}')^{\mathrm{op}}$ and $\mathcal{D} \times (\mathcal{C}')^{\mathrm{op}}$ that are equivalences in the first component. This proves claim ($\boxtimes_2$).

The cartesian straightening $\mathrm{St}^{\mathrm{cart}}(t^{\mathrm{op}})$ is a functor $\mathcal{C}' \to \mathrm{Cat}_{\infty}$. By the diagram above, it comes with a natural transformation $\mathrm{St}^{\mathrm{cart}}(t^{\mathrm{op}}) \Rightarrow \mathrm{const}\,\mathcal{D}$, so that $\mathrm{St}^{\mathrm{cart}}(t^{\mathrm{op}})$ lifts to a functor $\mathcal{C}_{/-} \colon \mathcal{C}' \to \mathrm{Cat}_{\infty/\mathcal{D}}$. On objects, $\mathcal{C}_{/-}$ is given by sending $x' \in \mathcal{C}'$ to the slice category $\mathcal{C}_{/x'}$, which becomes an object in $\mathrm{Cat}_{\infty/\mathcal{D}}$ via

$$
\mathcal{C}_{/x'} \longrightarrow \mathcal{C} \xrightarrow{\; F \;} \mathcal{D} \,.
$$

Now that's something we've seen before! Our assumption that the functor above admits a colimit precisely tells us that $\mathcal{C}_{/-}$ restricts to a functor $\mathcal{C}_{/-} \colon \mathcal{C}' \to \mathcal{T}$. To finish the proof, let $c \colon \mathrm{Cat}_{\infty/\mathcal{D}} \to \mathrm{Right}(\mathcal{D})$ denote the left adjoint to $\mathrm{Right}(\mathcal{D}) \subseteq \mathrm{Cat}_{\infty/\mathcal{D}}$, which exists due to Lemma 6.23(b). It's clear from Theorem 6.18(a) that c sends $\mathcal{T}$ to $\overline{\mathcal{T}}$, hence we obtain a functor $c \circ \mathcal{C}_{/-} \colon \mathcal{C}' \to \overline{\mathcal{T}}$. We claim that this finally allows us to compute the desired coinitial replacement:

($\boxtimes_3$) *If $p \colon X \to \mathcal{D} \times (\mathcal{C}')^{\mathrm{op}}$ is a coinitial replacement of $\mathcal{C} \times_{f,\mathcal{C}',s^{\mathrm{op}}} \mathrm{TwAr}(\mathcal{C}')^{\mathrm{op}} \to \mathcal{D} \times (\mathcal{C}')^{\mathrm{op}}$ by a right fibration, then the image of p under $\mathrm{Right}(\mathcal{D} \times (\mathcal{C}')^{\mathrm{op}}) \simeq \mathrm{Fun}(\mathcal{C}', \mathrm{Right}(\mathcal{D}))$ will coincide with $c \circ \mathcal{C}_{/-}$.*

To prove claim ($\boxtimes_3$), consider the following diagram, in which the dashed arrows are left adjoints (whose existence we're going to prove below):

$$
\begin{array}{ccccc}
\big(\mathrm{Cat}_{\infty/(\mathcal{C}')^{\mathrm{op}}}\big)_{/\mathcal{D}\times(\mathcal{C}')^{\mathrm{op}}} & \longleftarrow & \mathrm{Cart}\big((\mathcal{C}')^{\mathrm{op}}\big)_{/\mathcal{D}\times(\mathcal{C}')^{\mathrm{op}}} & \xrightarrow{\;\simeq\;} & \mathrm{Fun}\big(\mathcal{C}',\mathrm{Cat}_{\infty/\mathcal{D}}\big) \\[4pt]
{\scriptstyle\simeq}\Big\downarrow & /\!/\!/ & c\Big\updownarrow & /\!/\!/ & c_{*}\Big\updownarrow \\[4pt]
\mathrm{Cat}_{\infty/\mathcal{D}\times(\mathcal{C}')^{\mathrm{op}}} & \xdashleftarrow{\quad c\quad} & \mathrm{Right}\big(\mathcal{D}\times(\mathcal{C}')^{\mathrm{op}}\big) & \xrightarrow{\;\simeq\;} & \mathrm{Fun}\big(\mathcal{C}',\mathrm{Right}(\mathcal{D})\big)
\end{array}
$$

The horizontal equivalences as well as commutativity of the square on the right follow from the cartesian straightening equivalence (the dual of Theorem 5.4). Furthermore, once we know that the left adjoints exist, they will also form a commutative square on the right, since taking left adjoints is always compatible with equivalences. The vertical equivalence on the left follows by inspection ("a slice of a slice is a slice"). The vertical left adjoint c_{*} exists by Corollary 6.6. The horizontal left adjoint $c\colon \mathrm{Cat}_{\infty/\mathcal{D}\times(\mathcal{C}')^{\mathrm{op}}} \to \mathrm{Right}(\mathcal{D}\times(\mathcal{C}')^{\mathrm{op}})$ exists by Lemma 6.23(b), and it we claim that it induces a left adjoint

$$
c\colon \mathrm{Cart}\big((\mathcal{C}')^{\mathrm{op}}\big)_{/\mathcal{D}\times(\mathcal{C}')^{\mathrm{op}}} \longrightarrow \mathrm{Right}\big(\mathcal{D}\times(\mathcal{C}')^{\mathrm{op}}\big)
$$

to the forgetful functor $\mathrm{Right}(\mathcal{D}\times(\mathcal{C}')^{\mathrm{op}}) \to \mathrm{Cart}((\mathcal{C}')^{\mathrm{op}})_{/\mathcal{D}\times(\mathcal{C}')^{\mathrm{op}}}$. Indeed, if $\mathcal{U}\to\mathcal{D}\times(\mathcal{C}')^{\mathrm{op}}$ and $\mathcal{U}'\to\mathcal{D}\times(\mathcal{C}')^{\mathrm{op}}$ are objects in $\mathrm{Cart}((\mathcal{C}')^{\mathrm{op}})_{/\mathcal{D}\times(\mathcal{C}')^{\mathrm{op}}}$, then

$$
\mathrm{Hom}_{\mathrm{Cart}((\mathcal{C}')^{\mathrm{op}})_{/\mathcal{D}\times(\mathcal{C}')^{\mathrm{op}}}}(\mathcal{U},\mathcal{U}') \longrightarrow \mathrm{Hom}_{\mathrm{Cat}_{\infty/\mathcal{D}\times(\mathcal{C}')^{\mathrm{op}}}}(\mathcal{U},\mathcal{U}')
$$

is usually *not* an equivalence, only an inclusion of path components, since on the left-hand side, cartesian lifts need to be preserved. However, if $\mathcal{U}'\to\mathcal{D}\times(\mathcal{C}')^{\mathrm{op}}$ happens to be a right fibration, then cartesian lifts are preserved automatically[(6.6)], so in this case we *do* get an equivalence, which proves that c is still a left adjoint when restricted along the non-fully faithful functor $\mathrm{Cart}((\mathcal{C}')^{\mathrm{op}})_{/\mathcal{D}\times(\mathcal{C}')^{\mathrm{op}}} \to \mathrm{Cat}_{\infty/\mathcal{D}\times(\mathcal{C}')^{\mathrm{op}}}$. So we've proved that the diagram above also commutes if we take the dashed left adjoints into account. This is precisely what we need to prove claim ($\boxtimes_3$).

Using claim ($\boxtimes_3$), we've now succeeded in proving that $F\colon \mathcal{C}'\to\mathrm{Right}(\mathcal{D})$ takes values in $\overline{\mathcal{T}}$, which proves that $\mathrm{Lan}_f F$ exists. Furthermore, for every $c'\in\mathcal{C}'$, the value $\mathrm{Lan}_f F(x')$ is given by a colimit over $c(\mathcal{C}_{/x'})$. Since the unit morphism $u_{\mathcal{C}_{/x'}}\colon \mathcal{C}_{/x'}\to c(\mathcal{C}_{/x'})$ is coinitial by Lemma 6.23(c), we may as well take the colimit over $\mathcal{C}_{/x'}$. This proves that $\mathrm{Lan}_f F(x')$ is given by the desired formula and we're finally done! $\qquad\square$

6.29. Corollary. — *In the situation from Definition 6.26, assume that $f\colon \mathcal{C}\to\mathcal{C}'$ is fully faithful and that the colimits from Lemma 6.27 exist in $\mathcal{D}$. Then the natural transformation $u_F\colon F \Rightarrow \mathrm{Lan}_f F \circ f$ is an equivalence.*

Proof. This follows from the same argument as in Corollary 1.16, plus the fact that equivalences can be checked pointwise by Theorem 4.5. $\qquad\square$

[(6.6)]It's easy to get confused here: $\mathcal{U}'\to(\mathcal{C}')^{\mathrm{op}}$ need not be a right fibration, so we can't appeal to (the dual of) Lemma 5.3 directly. But the argument is still straightforward: If $\mathcal{U}'\to\mathcal{D}\times(\mathcal{C}')^{\mathrm{op}}$ is a right fibration, then *any* lift of a cartesian morphism in $\mathcal{D}\times(\mathcal{C}')^{\mathrm{op}}$ will be cartesian again, thanks to (the dual of) Lemma 5.3. So a morphism $\mathcal{U}\to\mathcal{U}'$ in $\mathrm{Cat}_{\infty/\mathcal{D}\times(\mathcal{C}')^{\mathrm{op}}}$ preserves cocartesian lifts if and only if $\mathcal{U}\to\mathcal{D}\times(\mathcal{C}')^{\mathrm{op}}$ does. But the latter is true by definition, since $\mathcal{U}\to\mathcal{D}\times(\mathcal{C}')^{\mathrm{op}}$ is a morphism in $\mathrm{Cart}((\mathcal{C}')^{\mathrm{op}})$ if $\mathcal{U}$ is an object of the slice ∞-category $\mathrm{Cart}((\mathcal{C}')^{\mathrm{op}})_{/\mathcal{D}\times(\mathcal{C}')^{\mathrm{op}}}$.

We can now state the main result of this section: the ∞-categorical analogue of Theorem 1.17!

6.30. Theorem ("PSh($\mathcal{C}$) arises by freely adding colimits to $\mathcal{C}$."). — *Let $\mathcal{C}$ and $\mathcal{D}$ be ∞-categories, where $\mathcal{D}$ has all colimits. Then restriction along the Yoneda embedding $よ_\mathcal{C}$ induces an equivalence*

$$よ_\mathcal{C}^*\colon \operatorname{Fun}^{\operatorname{colim}}\big(\operatorname{PSh}(\mathcal{C}), \mathcal{D}\big) \xrightarrow{\simeq} \operatorname{Fun}(\mathcal{C}, \mathcal{D})\,.$$

Here $\operatorname{Fun}^{\operatorname{colim}}(\operatorname{PSh}(\mathcal{C}), \mathcal{D}) \subseteq \operatorname{Fun}(\operatorname{PSh}(\mathcal{C}), \mathcal{D})$ is the full sub-∞-category spanned by the colimit-preserving functors. Furthermore, every colimit-preserving functor $\operatorname{PSh}(\mathcal{C}) \to \mathcal{D}$ admits a right adjoint.

As it turns out, the proof will be exactly the same as for ordinary categories. Let's start with the two lemmas whose proofs where omitted in the ordinary case.

6.31. Lemma ("Every presheaf is a colimit of representables."). — *Let $\mathcal{C}$ be an ∞-category. For every $E \in \operatorname{PSh}(\mathcal{C})$, the natural morphism*

$$\operatorname*{colim}_{(y, \operatorname{Hom}_\mathcal{C}(-,y) \to E) \in \mathcal{C}_{/E}} \operatorname{Hom}_\mathcal{C}(-, y) \xrightarrow{\simeq} E$$

is an equivalence.

Proof. Since we get the natural transformation for free, we can check pointwise whether it is an equivalence (Theorem 4.5). So fix $x \in \mathcal{C}$. Since colimits in $\operatorname{PSh}(\mathcal{C})$ are computed pointwise (Lemma 6.12), what we need to show is

$$\operatorname{colim}\left(\mathcal{C}_{/E} \xrightarrow{s} \mathcal{C} \xrightarrow{\operatorname{Hom}_\mathcal{C}(x,-)} \operatorname{An}\right) \simeq E(x)\,.$$

By Lemma 6.14, the colimit on the left-hand side is given by $|\mathcal{U}|$, where $\mathcal{U}$ is the unstraightening of $\operatorname{Hom}_\mathcal{C}(x, -) \circ s$. Since precomposition transforms into pullbacks under unstraightening, we find that $\mathcal{U}$ sits inside a pullback

$$
\begin{array}{ccccc}
\mathcal{U} & \longrightarrow & \mathcal{C}_{/E} & \longrightarrow & \operatorname{PSh}(\mathcal{C})_{/E} \\
\downarrow & \lrcorner & \downarrow{\scriptstyle s} & \lrcorner & \downarrow{\scriptstyle s} \\
\mathcal{C}_{x/} & \longrightarrow & \mathcal{C} & \xrightarrow{よ_\mathcal{C}} & \operatorname{PSh}(\mathcal{C})
\end{array}
$$

Since $\operatorname{PSh}(\mathcal{C})_{/E} \to \operatorname{PSh}(\mathcal{C})$ is a right fibration, $\mathcal{U} \to \mathcal{C}_{x/}$ is one too. In particular, it is a cartesian fibration. Hence Lemma 6.22 shows $|\mathcal{U} \times_{\mathcal{C}_{x/}} \{\operatorname{id}_x\}| \simeq |\mathcal{U} \times_{\mathcal{C}_{x/}} (\mathcal{C}_{x/})_{(\operatorname{id}_x\colon x \to x)/}| \simeq |\mathcal{U}|$; here we use $(\mathcal{C}_{x/})_{(\operatorname{id}_x\colon x \to x)/} \simeq \mathcal{C}_{x/}$ ("a slice of a slice is a slice"). Now

$$\big|\mathcal{U} \times_{\mathcal{C}_{x/}} \{\operatorname{id}_x\}\big| \simeq \mathcal{U} \times_{\mathcal{C}_{x/}} \{\operatorname{id}_x\} \simeq \operatorname{PSh}(\mathcal{C})_{/E} \times_{\operatorname{PSh}(\mathcal{C})} \{よ_\mathcal{C}(x)\}$$
$$\simeq \operatorname{Hom}_{\operatorname{PSh}(\mathcal{C})}\big(よ_\mathcal{C}(x), E\big)$$
$$\simeq E(x)\,.$$

In the first step, we use that the fibre $\mathcal{U} \times_{\mathcal{C}_{x/}} \{\operatorname{id}_x\}$ is already an anima, since $\mathcal{U} \to \mathcal{C}_{x/}$ is a right fibration. The second equivalence follows from the pullback diagram above. In the third step, we use the definition of $\operatorname{Hom}_{\operatorname{PSh}(\mathcal{C})}$, and in the fourth step, we use Yoneda's lemma (Theorem 5.19). In total, we find $|\mathcal{U}| \simeq E(x)$, which is exactly what we wanted to prove. $\quad\square$

6.32. Lemma. — *For every $F\colon \mathcal{C} \to \mathcal{D}$, the left Kan extension $\mathrm{Lan}_{\sout_{\mathcal{C}}} F\colon \mathrm{PSh}(\mathcal{C}) \to \mathcal{D}$ (which exists due to Lemma 6.27) admits a right adjoint. The right adjoint sends $y \in \mathcal{D}$ to $\mathrm{Hom}_{\mathcal{D}}(F(-), y)\colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$.*

Proof. Fix $y \in \mathcal{D}$. Since adjoints can be constructed pointwise (Lemma 6.2), we only need to construct an equivalence

$$\mathrm{Hom}_{\mathcal{D}}\big(\mathrm{Lan}_{\sout_{\mathcal{C}}} F(-), y\big) \simeq \mathrm{Hom}_{\mathrm{PSh}(\mathcal{C})}\big(-, \mathrm{Hom}_{\mathcal{D}}(F(-), y)\big)$$

of functors $\mathrm{PSh}(\mathcal{C})^{\mathrm{op}} \to \mathrm{An}$. Restricting along $\sout_{\mathcal{C}}^{\mathrm{op}}\colon \mathcal{C}^{\mathrm{op}} \to \mathrm{PSh}(\mathcal{C})^{\mathrm{op}}$, both sides become $\mathrm{Hom}_{\mathcal{D}}(F(-), y)$: The left-hand side by Corollary 6.29, the right-hand side by Yoneda's lemma (Theorem 5.19; see also 5.31). By the universal property of right Kan extension, we thus obtain natural transformations

$$\mathrm{Hom}_{\mathcal{D}}\big(\mathrm{Lan}_{\sout_{\mathcal{C}}} F(-), y\big) \Longrightarrow \mathrm{Ran}_{\sout_{\mathcal{C}}^{\mathrm{op}}} \mathrm{Hom}_{\mathcal{D}}\big(F(-), y\big) \Longleftarrow \mathrm{Hom}_{\mathrm{PSh}(\mathcal{C})}\big(-, \mathrm{Hom}_{\mathcal{D}}\big(F(-), y\big)\big)\,.$$

We claim that they're both equivalences. In either case, this can be checked pointwise by Theorem 4.5. So plug in some $E \in \mathrm{PSh}(\mathcal{C})$. We obtain a diagram

$$\mathrm{Hom}_{\mathcal{D}}\big(\mathrm{Lan}_{\sout_{\mathcal{C}}} F(E), y\big) \longrightarrow \mathrm{Ran}_{\sout_{\mathcal{C}}^{\mathrm{op}}} \mathrm{Hom}_{\mathcal{D}}\big(F(E), y\big) \longleftarrow \mathrm{Hom}_{\mathrm{PSh}(\mathcal{C})}\big(E, \mathrm{Hom}_{\mathcal{D}}\big(F(-), y\big)\big)$$

with the lower convergence to

$$\varprojlim_{(x,\,\sout_{\mathcal{C}}(x) \to E) \in (\mathcal{C}_{/E})^{\mathrm{op}}} \mathrm{Hom}_{\mathcal{D}}\big(F(x), y\big)$$

The vertical arrow in the middle is an equivalence by the dual of Lemma 6.27. For the vertical arrow on the left, we plug in the left Kan extension formula from Lemma 6.27 and use Corollary 6.16 to see that $\mathrm{Hom}_{\mathcal{D}}(-, y)$ transforms the colimit into a limit. For the vertical arrow on the right, we plug in Lemma 6.31, use Corollary 6.16 again to see that $\mathrm{Hom}_{\mathrm{PSh}(\mathcal{C})}(-, \mathrm{Hom}_{\mathcal{D}}(F(-), y))$ transforms the colimit into a limit, and then use Yoneda's lemma. This proves that we obtain equivalences as desired. □

6.33. Lemma. — *Let $\mathcal{C}$ and $\mathcal{D}$ be categories and let $L\colon \mathcal{C} \rightleftarrows \mathcal{D} :R$ be an adjunction.*

(a) *The left adjoint L is fully faithful if and only if the unit transformation $u\colon \mathrm{id}_{\mathcal{C}} \Rightarrow RL$ is an equivalence.*

(b) *Suppose the condition from (a) is true. Furthermore, suppose that R is conservative (that is, if $\alpha\colon x \to y$ is a morphism in $\mathcal{D}$ such that $R(\alpha)$ is an equivalence, then α is an equivalence too). Then L and R are inverse equivalences of categories.*

Proof. The proof of Lemma 1.20 can be copied verbatim. □

Proof of Theorem 6.30. By Lemma 6.32 and Lemma 6.11, the adjunction $\mathrm{Lan}_{\sout_{\mathcal{C}}} \dashv \sout_{\mathcal{C}}^{*}$ restricts to an adjunction

$$\mathrm{Lan}_{\sout_{\mathcal{C}}}\colon \mathrm{Fun}(\mathcal{C}, \mathcal{D}) \rightleftarrows \mathrm{Fun}^{\mathrm{colim}}\big(\mathrm{PSh}(\mathcal{C}), \mathcal{D}\big) :\sout_{\mathcal{C}}^{*}\,.$$

By Lemma 6.33(b), to prove that $\mathrm{Lan}_{\sout_{\mathcal{C}}}$ and $\sout_{\mathcal{C}}^{*}$ are inverse equivalences, we need to show that the unit $u\colon \mathrm{id}_{\mathrm{Fun}(\mathcal{C}, \mathcal{D})} \Rightarrow \sout_{\mathcal{C}}^{*} \circ \mathrm{Lan}_{\sout_{\mathcal{C}}}$ is an equivalence and that $\sout_{\mathcal{C}}^{*}$ is conservative. That u is an equivalence can be checked object-wise by Theorem 4.5, where it follows from Corollary 6.29, since the Yoneda embedding $\sout_{\mathcal{C}}$ is fully faithful (Corollary 5.25). To see that

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$ょ_{\mathcal{C}}^*$ is conservative, we must show that a natural transformation $\eta\colon F \Rightarrow G$ between colimit-preserving functors $F, G\colon \mathrm{PSh}(\mathcal{C}) \to \mathcal{D}$ is an equivalence already if it is an equivalence when restricted to representable presheaves. But this is clear since every presheaf can be written as a colimit of representables (Lemma 6.31). $\qquad\square$

§6.5. Homology, cohomology, Eilenberg–MacLane animae

Theorem 6.30 is surprisingly powerful even in the special case $\mathcal{C} \simeq *$. In this case we have $\mathrm{PSh}(*) \simeq \mathrm{An}$ and so Theorem 6.30 says that a colimit-preserving functor $\mathrm{An} \to \mathcal{D}$ is uniquely determined by what it does on $* \in \mathrm{An}$.[(6.7)] Using this observation, our goal in this subsection is to give a purely abstract proof of the Eilenberg–MacLane theorem (Theorem 6.44).

The first step is to construct an interesting ∞-category $\mathcal{D}$ with all colimits: For a ring R (not necessarily commutative), we'll give a brief introduction to the *derived ∞-category $\mathcal{D}(R)$* and its variant $\mathcal{D}_{\geqslant 0}(R)$.

6.34. Crash course in derived ∞-categories I: Basic definitions. — Let $\mathrm{Ch}(R)$ be the category of chain complexes

$$M_* = \left(\cdots \xrightarrow{\partial} M_{n+1} \xrightarrow{\partial} M_n \xrightarrow{\partial} M_{n-1} \xrightarrow{\partial} \cdots \right)$$

of left R-modules and let $\mathrm{Ch}_{\geqslant 0}(R) \subseteq \mathrm{Ch}(R)$ be the full subcategory of those chain complexes that satisfy $M_n \cong 0$ for $n < 0$. We usually write $\mathrm{Z}_n(M_*) := \ker(\partial\colon M_n \to M_{n-1})$ and $\mathrm{B}_n(M_*) := \mathrm{im}(\partial\colon M_{n+1} \to M_n)$. The quotient $\mathrm{H}_n(M_*) := \mathrm{Z}_n(M_*)/\mathrm{B}_n(M_*)$ is called the nth *homology of M_**. A morphism $\alpha\colon M_* \to N_*$ in $\mathrm{Ch}(R)$ is called a *quasi-isomorphism* if $\mathrm{H}_n(\alpha)\colon \mathrm{H}_n(M_*) \xrightarrow{\cong} \mathrm{H}_n(N_*)$ is an isomorphism for all n. Then we put

$$\mathcal{D}(R) := \mathrm{Ch}(R)\big[\{\text{quasi-isomorphisms}\}^{-1}\big]$$
$$\mathcal{D}_{\geqslant 0}(R) := \mathrm{Ch}_{\geqslant 0}(R)\big[\{\text{quasi-isomorphisms}\}^{-1}\big],$$

where the localisations are taken in the ∞-categorical sense (see Construction 4.10). If you've seen the ordinary derived categories $D(R)$ and $D_{\geqslant 0}(R)$ before, then Corollary/Warning 4.12 will convince you that these are simply the homotopy categories $h\mathcal{D}(R)$ and $h\mathcal{D}_{\geqslant 0}(R)$.

This definition of $\mathcal{D}(R)$ is easy to state, but just from that it's nearly impossible to say anything about colimits in $\mathcal{D}(R)$, which is why, we will describe a more explicit construction of $\mathcal{D}(R)$ in crash course 6.36 below. However, already with the abstract definition one can get quite far. For example, let's show that $\mathcal{D}_{\geqslant 0}(R)$ is indeed a full sub-∞-category of $\mathcal{D}(R)$. To this end, observe that the inclusion $\mathrm{Ch}_{\geqslant 0}(R) \subseteq \mathrm{Ch}(R)$ has a right adjoint $\tau_{\geqslant 0}\colon \mathrm{Ch}(R) \to \mathrm{Ch}_{\geqslant 0}(R)$ given by *smart truncation*: For a chain complex M_* and an integer $i \in \mathbb{Z}$, we let $\tau_{\geqslant i}M_*$ be the chain complex given by

$$(\tau_{\geqslant i}M_*)_n := \begin{cases} M_n & \text{if } n > i \\ \mathrm{Z}_i(M_*) & \text{if } n = i \\ 0 & \text{if } n < i \end{cases},$$

so that $\mathrm{H}_n(\tau_{\geqslant i}M_*) \cong \mathrm{H}_n(M_*)$ if $n \geqslant i$ and $\mathrm{H}_n(\tau_{\geqslant i}M_*) \cong 0$ for $n < i$. It's clear that $\tau_{\geqslant i}$ preserves quasi-isomorphisms, hence it descends to a functor $\tau_{\geqslant i}\colon \mathcal{D}(R) \to \mathcal{D}_{\geqslant 0}(R)$ by Lemma 4.11. We

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[(6.7)]If you think about this fact for a bit, it becomes very natural: Theorem 3.26 says that animae are essentially CW complexes and every CW complex is glued together from topological disks D^n. But $D^n \simeq *$. So it makes sense that $*$ should generate all of An under colimits. Another way to see this is via Lemma 6.14: It's immediately clear that $X \simeq \mathrm{colim}(\mathrm{const}\, *\colon X \to \mathrm{An})$ for all $X \in \mathrm{An}$.

claim that $\tau_{\geqslant 0}$ is a right adjoint to $\mathcal{D}_{\geqslant 0}(R) \to \mathcal{D}(R)$. By Lemma 6.5, it's enough to provide a unit and a counit transformation and to verify the triangle identities. But Lemma 4.11 allows us to inherit all this data from the adjunction $i\colon \mathrm{Ch}_{\geqslant 0}(R) \rightleftarrows \mathrm{Ch}(R) :\tau_{\geqslant 0}$.[6.8] Now to show that $\mathcal{D}_{\geqslant 0}(R) \to \mathcal{D}(R)$ is fully faithful, it's enough to check that the unit is an equivalence (see Lemma 6.33(a)), which is obvious.

Apart from $\tau_{\geqslant i}\colon \mathcal{D}(R) \to \mathcal{D}(R)$, there are some more useful functors that can be constructed directly using our definition of $\mathcal{D}(R)$ and Lemma 4.11. For example, if M_* is a chain complex, its *shift by i* is the chain complex $M[i]_*$ given by $M[i]_n \cong M_{n-i}$; the differentials are those of M_*, but multiplied by $(-1)^i$ (for technical reasons). It's clear that $(-)[i]\colon \mathrm{Ch}(R) \to \mathrm{Ch}(R)$ preserves quasi-isomorphisms and so it defines a functor $(-)[i]\colon \mathcal{D}(R) \to \mathcal{D}(R)$. For an even more obvious example, consider $\mathrm{H}_n\colon \mathrm{Ch}(R) \to \mathrm{Mod}_R$ and $\mathrm{H}_n\colon \mathrm{Ch}_{\geqslant 0}(R) \to \mathrm{Mod}_R$. These functors send quasi-isomorphisms to isomorphisms (by definition), hence they define essentially unique functors

$$\mathrm{H}_n\colon \mathcal{D}(R) \longrightarrow \mathrm{Mod}_R \quad \text{and} \quad \mathrm{H}_n\colon \mathcal{D}_{\geqslant 0}(R) \longrightarrow \mathrm{Mod}_R\,.$$

by Lemma 4.11. It's probably clear to you, but let us mention that neither $\mathrm{Z}_n\colon \mathrm{Ch}(R) \to \mathrm{Mod}_R$ nor $\mathrm{B}_n\colon \mathrm{Ch}(R) \to \mathrm{Mod}_R$ preserves quasi-isomorphisms, so they don't extend to $\mathcal{D}(R)$, even though their quotient $\mathrm{H}_n \cong \mathrm{Z}_n/\mathrm{B}_n$ does.

For a chain complex M_*, we often write M for its image in $\mathcal{D}(R)$ to emphasise that this is no longer a "complex up to isomorphism", but a "complex up to quasi-isomorphism", so that for $M \in \mathcal{D}(R)$ there is no longer a well-defined notion of "M_n, the degree-n part of M".[6.9]

6.35. Crash course in derived ∞-categories II: Simplicialities. — The famous *Dold–Kan correspondence* (see [L-HA, Theorem 1.2.3.7] or [GJ99, §III.2] for example) states that there is an equivalence

$$\mathrm{Ch}_{\geqslant 0}(\mathbb{Z}) \xrightarrow{\simeq} \mathrm{sAb}$$

between the category of chain complexes in non-negative degrees and the category of simplicial abelian groups. We'll need following two facts about the Dold–Kan correspondence:

(a) *Every simplicial abelian groups is automatically a Kan complex and every degree-wise surjective morphism in* sAb *maps to a Kan fibration in* sSet.

(b) *If A is a simplicial abelian group and M_* is the associated chain complex, then there are isomorphisms $\pi_n(A, a) \cong \mathrm{H}_n(M_*)$ for all $a \in A$ and all $n \geqslant 0$.*

[6.8]We've seen a similar argument in Remark 4.15. The crucial observation to construct natural transformations via Lemma 4.11 is the following: For every ∞-category $\mathcal{C}$ and every collection of morphisms W in $\mathcal{C}$, the functor

$$\left(\mathcal{C} \times \Delta^1\right)\left[\left(W \times \{\mathrm{id}_0\} \cup W \times \{\mathrm{id}_1\}\right)^{-1}\right] \xrightarrow{\simeq} \mathcal{C}[W^{-1}] \times \Delta^1$$

(which is itself constructed via Lemma 4.11) is an equivalence of ∞-categories. Back in Remark 4.15, we appealed to the explicit simplicial construction, but there's also a model-independent way to see this fact. By Yoneda's lemma, Theorem 4.5, and Lemma 4.11, it's enough to check for every ∞-category $\mathcal{D}$ that the morphism of animae $\mathrm{Hom}_{\mathrm{Cat}_\infty}(\mathcal{C}[W^{-1}] \times \Delta^1, \mathcal{D}) \to \mathrm{Hom}_{\mathrm{Cat}_\infty}(\mathcal{C} \times \Delta^1, \mathcal{D})$ exhibits the left-hand side as the collection of path components of functors that send $W \times \{\mathrm{id}_0\} \cup W \times \{\mathrm{id}_1\}$ to equivalences. By Example 6.3(b), we can rewrite the morphism in question as $\mathrm{Hom}_{\mathrm{Cat}_\infty}(\mathcal{C}[W^{-1}], \mathrm{Ar}(\mathcal{D})) \to \mathrm{Hom}_{\mathrm{Cat}_\infty}(\mathcal{C}, \mathrm{Ar}(\mathcal{D}))$ and then Lemma 4.11 shows that, indeed, we get the correct inclusion of path components.

[6.9]On a related note, the inclusion $\mathrm{Ch}_{\geqslant 0}(R) \subseteq \mathrm{Ch}(R)$ also has a left adjoint, which simply replaces everything in negative degrees by 0. This is called the *stupid truncation*. It doesn't preserve quasi-isomorphisms (hence the name) and so we couldn't have used it to show that $\mathcal{D}_{\geqslant 0}(R) \to \mathcal{D}(R)$ is fully faithful.

Fact (a) not particularly difficult, but not completely obvious either; see [Stacks, Tags 08NZ and 08P0]. Let us sketch how to prove (b). First, we may assume $a = 0$, since $(-) + a\colon A \to A$ is an automorphism of A as a simplicial set and induces an isomorphism $\pi_n(A, 0) \cong \pi_n(A, a)$. Now $\pi_n(A, 0) \cong [(\Delta^n, \partial\Delta^n), (A, 0)]$ by Lemma 3.21, where $[-, -]$ denotes homotopy classes of maps of pairs. Since $(A, 0)$ is a group object, even in the homotopy category of pairs, $[(\Delta^n, \partial\Delta^n), (A, 0)]$ inherits a group structure. Using the Eckmann–Hilton trick (see the proof of Lemma 3.17(b)), we see that this group structure agrees with the one on $\pi_n(A, 0)$.

Using Corollary 1.5, the free-forgetful adjunction $\mathbb{Z}[-]\colon \mathrm{Set} \rightleftarrows \mathrm{Ab} \colon \mathrm{forget}$ induces a similar adjunction $\mathbb{Z}[-]\colon \mathrm{sSet} \rightleftarrows \mathrm{sAb} \colon \mathrm{forget}$. Then a map of pairs $(\Delta^n, \partial\Delta^n) \to (A, 0)$ is the same as a morphism $\mathbb{Z}[\Delta^n]/\mathbb{Z}[\partial\Delta^n] \to A$ in sAb. We are, however, not interested in maps, but *homotopy classes* of maps. Our analysis of the group structure on $\pi_n(A, 0)$ shows: Instead of quotienting out the equivalence relation generated by homotopies, we may as well quotient out the subgroup generated by the nullhomotopic maps. Using the results from §3, it's straightforward to show that, for any pointed Kan complex (X, x), a map of pairs $\sigma\colon (\Delta^n, \partial\Delta^n) \to (X, x)$ is nullhomotopic if and only if it can be extended to a map of pairs $\overline{\sigma}\colon (\Delta^{n+1}, \Lambda^{n+1}_{n+1}) \to (X, x)$ in such a way that $d^*_{n+1}(\overline{\sigma}) = \sigma$. By the same reasoning as above, such a map is the same as a morphism $\mathbb{Z}[\Delta^{n+1}]/\mathbb{Z}[\Lambda^{n+1}_{n+1}] \to A$. In total, this proves:

$$\pi_n(A, 0) \cong \mathrm{Hom}_{\mathrm{sAb}}\big(\mathbb{Z}[\Delta^n]/\mathbb{Z}[\partial\Delta^n], A\big)\big/ \mathrm{Hom}_{\mathrm{sAb}}\big(\mathbb{Z}[\Delta^{n+1}]/\mathbb{Z}[\Lambda^{n+1}_{n+1}], A\big)$$

A simple unravelling of the Dold–Kan correspondence shows that $\mathbb{Z}[\Delta^n]/\mathbb{Z}[\partial\Delta^n]$ is sent to $\mathbb{Z}[n]_*$, the chain complex consisting of a single $\mathbb{Z}$ in degree n and zeros everywhere else. Hence $\mathrm{Hom}_{\mathrm{sAb}}(\mathbb{Z}[\Delta^n]/\mathbb{Z}[\partial\Delta^n], A) \cong \mathrm{Hom}_{\mathrm{Ch}_{\geq 0}(\mathbb{Z})}(\mathbb{Z}[n]_*, M_*) \cong \mathrm{Z}_n(M_*)$. A similar analysis shows that $\mathrm{Hom}_{\mathrm{sAb}}(\mathbb{Z}[\Delta^{n+1}]/\mathbb{Z}[\Lambda^{n+1}_{n+1}], A) \cong \mathrm{B}_n(M_*)$. Hence $\pi_n(A, 0) \cong \mathrm{H}_n(M_*)$, as desired.

6.36. Crash course in derived ∞-categories III: Projective resolutions. — Recall the simplicial nerve from Construction 2.21. It's also possible to construct $\mathcal{D}(R)$ and $\mathcal{D}_{\geq 0}(R)$ in this way; this alternative construction will allow us to study colimits. We'll first explain how to equip $\mathcal{D}(R)$ and $\mathcal{D}_{\geq 0}(R)$ with a Kan enrichment: Let $\underline{\mathrm{Hom}}_R(M_*, N_*)$ be the chain complex of abelian groups given by

$$\underline{\mathrm{Hom}}_R(M_*, N_*)_n := \prod_{i \in \mathbb{Z}} \mathrm{Hom}_R(M_i, N_{i+n}).$$

The differentials send a family of morphisms $f = (f_i)_{i \in \mathbb{Z}} \in \prod_{i \in \mathbb{Z}} \mathrm{Hom}_R(M_i, N_{i+n})$ to the family $\partial f := (\partial_N \circ f_i - (-1)^n f_{i-1} \circ \partial_M)_{i \in \mathbb{Z}}$; here ∂_M and ∂_N denote the differentials of M_* and N_*, respectively. By unravelling the definitions, we see that the n-cycles and n-boundaries of $\underline{\mathrm{Hom}}_R(M_*, N_*)$ are given by

$$\mathrm{Z}_n\big(\underline{\mathrm{Hom}}_R(M_*, N_*)\big) \cong \mathrm{Hom}_{\mathrm{Ch}(R)}\big(M_*, N[-n]_*\big)$$
$$\mathrm{B}_n\big(\underline{\mathrm{Hom}}_R(M_*, N_*)\big) \cong \big\{f \in \mathrm{Hom}_{\mathrm{Ch}(R)}\big(M_*, N[-n]_*\big) \,\big|\, f \text{ nullhomotopic}\big\}.$$

Here $N[-n]_*$ denotes the shift from crash course 6.34. Since $\mathrm{H}_n \cong \mathrm{Z}_n/\mathrm{B}_n$, we deduce that $\mathrm{H}_n(\underline{\mathrm{Hom}}_R(M_*, N_*))$ is in bijection with the set of homotopy classes of maps $M_* \to N[-n]_*$.

The complexes $\underline{\mathrm{Hom}}_R(-, -)$ provide an enrichment of $\mathrm{Ch}(R)$ over $\mathrm{Ch}(\mathrm{Ab})$ (in fact, even an enrichment of $\mathrm{Ch}(R)$ over itself). To make this into a Kan enrichment, we let $\tau_{\geq 0} \underline{\mathrm{Hom}}_R(M_*, N_*)$ be the smart truncation from crash course 6.34 and let $\mathrm{F}_{\mathrm{Ch}(R)}(M_*, N_*)$ denote the simplicial abelian group corresponding to $\tau_{\geq 0} \underline{\mathrm{Hom}}_R(M_*, N_*)$ under the Dold–Kan correspondence. The

simplicial abelian groups $\mathrm{F}_{\mathrm{Ch}(R)}(-,-)$ provide an enrichment of $\mathrm{Ch}(R)$ in simplicial sets, which is automatically a Kan enrichment by crash course 6.35(a).

A complex P_* of R-modules is called *K-projective* if $\underline{\mathrm{Hom}}_R(P_*,-)\colon \mathrm{Ch}(R) \to \mathrm{Ch}(R)$ preserves quasi-isomorphisms. It was shown by Spaltenstein [Spa88] that every chain complex of R-modules M_* admits a quasi-isomorphism $P_* \to M_*$ from a K-projective complex. If P_* is K-projective, then it is *degree-wise projective* in the sense that every P_n is a projective R-module. Conversely, if P_* is degree-wise projective and bounded below in the sense that $P_n \cong 0$ for $n \ll 0$, then P_* is K-projective. These statements can be found in [Hov99, Lemma 2.3.6]; the second statement also appears (in dual form) in [Stacks, Tag 070J].

Let $K\text{-}\mathrm{Proj}(R) \subseteq \mathrm{Ch}(R)$ and $\mathrm{Proj}_{\geqslant 0}(R) \subseteq \mathrm{Ch}_{\geqslant 0}(R)$ be the full subcategories spanned by the K-projective complexes. Equip $K\text{-}\mathrm{Proj}(R)$ and $\mathrm{Proj}_{\geqslant 0}(R)$ with the Kan enrichment above. Then

$$\mathcal{D}(R) \simeq \mathrm{N}^{\Delta}\big(K\text{-}\mathrm{Proj}(R)\big) \quad \text{and} \quad \mathcal{D}_{\geqslant 0}(R) \simeq \mathrm{N}^{\Delta}\big(\mathrm{Proj}_{\geqslant 0}(R)\big).$$

The idea to prove this is, of course, similar to Theorem 4.13: One can construct a simplicial model structure on $\mathrm{Ch}(R)$ (and, by restriction, on $\mathrm{Ch}_{\geqslant 0}(R)$) in such a way that $K\text{-}\mathrm{Proj}(R) \simeq \mathrm{Ch}(R)^{\mathrm{cf}}$ are precisely the bifibrant objects; see [Hov99, §2.3]. Then the above equivalences follow from Remarks 4.14 and 4.15.[(6.10)]

This alternative construction is useful to compute $\mathrm{Hom}_{\mathcal{D}(R)}$. Let M_* and N_* be complexes and let $P_* \to M_*$ and $Q_* \to N_*$ be quasi-isomorphisms from K-projective complexes. Using Theorem 2.24, we get $\mathrm{Hom}_{\mathcal{D}(R)}(M,N) \simeq \mathrm{F}_{\mathrm{Ch}(R)}(P_*,Q_*)$. In particular, since the Dold–Kan correspondence transforms homotopy groups of simplicial abelian groups into homology groups of the associated chain complexes by crash course 6.35(b), we find

$$\pi_n \mathrm{Hom}_{\mathcal{D}(R)}(M,N) \cong \pi_n \mathrm{F}_{\mathrm{Ch}(R)}(P_*,Q_*) \cong \mathrm{H}_n\big(\underline{\mathrm{Hom}}_R(P_*,Q_*)\big)$$

for all $n \geqslant 0$ and all basepoints. Furthermore, $\mathrm{H}_n(\underline{\mathrm{Hom}}_R(P_*,Q_*)) \cong \mathrm{H}_n(\underline{\mathrm{Hom}}_R(P_*,N_*))$ by definition of P_* being K-projective, so we only need to resolve M_* by a K-projective complex. If you've seen derived categories before, you'll probably have noticed that $\mathrm{Hom}_{\mathcal{D}(R)}(M,N)$ looks suspiciously like the derived Hom functor $\mathrm{RHom}_R(M,N)$: We'll see in Corollary 7.50 how exactly these two are related. ∎

To be able to apply Theorem 6.30 to $\mathcal{D}(R)$ or $\mathcal{D}_{\geqslant 0}(R)$, we need to show that these ∞-categories have all colimits. In order to to this, we'll show some general results about colimits in ∞-categories; these results will also be very useful later on.

6.37. Lemma. — *An ∞-category $\mathcal{C}$ has all colimits if and only if $\mathcal{C}$ has pushouts and arbitrary coproducts. A functor $F\colon \mathcal{C} \to \mathcal{D}$ of ∞-categories preserves colimits if and only if it preserves pushouts and arbitrary coproducts. A dual assertion holds for limits.*

[(6.10)] It's worth pointing out that the process of choosing a cofibrant replacement in $\mathrm{Ch}_{\geqslant 0}(R)$ precisely recovers the method of *projective resolutions* that you may be familiar with from homological algebra. Indeed, the cofibrant objects in $\mathrm{Ch}_{\geqslant 0}(R)$ are precisely the degree-wise projective complexes in non-negative degrees. Now if M is a left R-module and we think of M as a complex $M[0]_*$ concentrated in degree 0 (see Construction 6.41, then a cofibrant replacement of $M[0]_*$, that is, a quasi-isomorphism $P_* \to M[0]_*$ from a degree-wise projective complex, is precisely a projective resolution of M. This begs the question how *injective resolutions* fit into the picture. There is another simplicial model structure on $\mathrm{Ch}(R)$ in which the bifibrant objects are the *K-injective* complexes, that is, those I_* for which $\underline{\mathrm{Hom}}_R(-,I_*)$ preserves quasi-isomorphisms. A K-injective complex is degree-wise an injective R-module and conversely any degree-wise injective and bounded above complex is K-injective. One then has similar equivalences $\mathcal{D}(R) \simeq \mathrm{N}^{\Delta}(K\text{-}\mathrm{Inj}(R))$ and $\mathcal{D}_{\leqslant 0}(R) \simeq \mathrm{N}^{\Delta}(\mathrm{Inj}_{\leqslant 0}(R))$.

The crucial point, and the reason why we get away with "ordinary" colimits like pushouts and coproducts, is that $\{n\} \to \Delta^n$ is coinitial (in fact, right anodyne, so Example 6.20(a) applies). Hence every functor $T\colon \Delta^n \to \mathcal{C}$ admits a colimit. For a general functor $T\colon \mathcal{I} \to \mathcal{C}$, we write $\mathcal{I}$ as a colimit of its skeleta to build $\mathrm{colim}_{i \in \mathcal{I}} T(i)$ "simplex-by-simplex": This needs pushouts (to attach n-simplices in the nth step) and coproducts (to attach arbitrarily many n-simplices at the same time).

To make this precise, we'll prove a lemma that will allow us to manipulate colimits: We can "slice a colimit into pieces" and "assemble colimits from subdiagrams":

6.38. Lemma. — *Let $\mathcal{I}$ and $\mathcal{C}$ be ∞-categories.*

(a) *Suppose $p\colon \mathcal{U} \to \mathcal{I}$ is a cocartesian fibration and $T\colon \mathcal{U} \to \mathcal{C}$ is a functor such that $T|_{p^{-1}\{i\}}\colon p^{-1}\{i\} \to \mathcal{C}$ admits a colimit for all $i \in \mathcal{I}$. Then these colimits assemble into a functor $\overline{T}\colon \mathcal{I} \to \mathcal{C}$ satisfying $\overline{T}(i) \simeq \mathrm{colim}_{u \in p^{-1}\{i\}} T(u)$. Furthermore,*

$$\mathrm{colim}_{u \in \mathcal{U}} T(u) \simeq \mathrm{colim}_{i \in \mathcal{I}} \overline{T}(i)\,,$$

provided that at least one of these colimits exists in $\mathcal{C}$ (in which case the other exists as well). Informally, we can rephrase this as $\mathrm{colim}_{u \in \mathcal{U}} T(u) \simeq \mathrm{colim}_{i \in \mathcal{I}} \mathrm{colim}_{u \in p^{-1}\{i\}} T(u)$.

(b) *Suppose $\mathcal{I} \simeq \mathrm{colim}_{j \in \mathcal{J}} \mathcal{I}_j$ in Cat_∞. Let $T\colon \mathcal{I} \to \mathcal{C}$ be a functor such that the restrictions $T|_{\mathcal{I}_j}\colon \mathcal{I}_j \to \mathcal{C}$ admit colimits for all $j \in \mathcal{J}$. Then these colimits assemble into a functor $\overline{T}\colon \mathcal{J} \to \mathcal{C}$ satisfying $\overline{T}(j) \simeq \mathrm{colim}_{i \in \mathcal{I}_j} T(i)$. Furthermore,*

$$\mathrm{colim}_{i \in \mathcal{I}} T(i) \simeq \mathrm{colim}_{j \in \mathcal{J}} \overline{T}(j)\,.$$

provided that at least one of these colimits exists in $\mathcal{C}$ (in which case the other exists as well). Informally, we can rephrase this as $\mathrm{colim}_{i \in \mathcal{I}} T(i) \simeq \mathrm{colim}_{j \in \mathcal{J}} \mathrm{colim}_{i \in \mathcal{I}_j} T(i)$.

In particular, "colimits commute with colimits": If $\mathcal{J}$ is an ∞-category and $T\colon \mathcal{I} \times \mathcal{J} \to \mathcal{C}$ is any functor, then

$$\mathrm{colim}_{i \in \mathcal{I}} \mathrm{colim}_{j \in \mathcal{J}} T(i,j) \simeq \mathrm{colim}_{(i,j) \in \mathcal{I} \times \mathcal{J}} T(i,j) \simeq \mathrm{colim}_{j \in \mathcal{J}} \mathrm{colim}_{i \in \mathcal{I}} T(i,j)\,.$$

Proof. To prove (a), first note that $\mathrm{Lan}_p T$ exists. Indeed, $p^{-1}\{i\} \to \mathcal{U}_{/i}$ is coinitial by the dual of Lemma 6.22 and Example 6.20(b), so the existence of the colimits over $p^{-1}\{i\}$ implies that the condition from Lemma 6.27 is satisfied. So we can put $\overline{T} \coloneqq \mathrm{Lan}_p T$. Now $\mathrm{colim}\, T$ corresponds to taking the left Kan extension of T along $\mathcal{U} \to *$ (see Example 1.12). But we may as well first left Kan extend along $\mathcal{U} \to \mathcal{I}$ and then left Kan extend along $\mathcal{I} \to *$. This proves $\mathrm{colim}\, T \simeq \mathrm{colim}\, \mathrm{Lan}_p T$ and we've finished the proof of (a). In the special case where p is the projection $\mathrm{pr}_1\colon \mathcal{I} \times \mathcal{J} \to \mathcal{J}$, we obtain the "in particular".

For (b), let $p\colon \mathcal{U} \to \mathcal{J}$ be the unstraightening of the functor $\mathcal{J} \to \mathrm{Cat}_\infty$ of which $\mathcal{I}$ is the colimit. Then $\mathcal{I}$ is a localisation of $\mathcal{U}$ by Lemma 6.14, so there's a natural functor $q\colon \mathcal{U} \to \mathcal{I}$. We have $p^{-1}\{j\} \simeq \mathcal{I}_j$, so we can apply ($a$) to the functor $q \circ T\colon \mathcal{U} \to \mathcal{C}$. This allows us to construct $\overline{T}$ and we obtain $\mathrm{colim}\, \overline{T} \simeq \mathrm{colim}\, q \circ T$. But q, being a localisation, is coinitial by Example 6.20(c), and so $\mathrm{colim}\, q \circ T \simeq \mathrm{colim}\, T$. This proves ($b$). $\qquad\square$

Proof sketch of Lemma 6.37. In simplicial sets, we can write $\mathcal{I} \cong \mathrm{colim}_{n \geqslant 0} \mathrm{sk}_n \mathcal{I}$, where $\mathrm{sk}_n \mathcal{I}$ is obtained from $\mathrm{sk}_{n-1} \mathcal{I}$ by attaching copies of Δ^n; that is, we take a pushout along some coproduct of the form $\coprod \partial \Delta^n \to \coprod \Delta^n$. Up to replacing everything by quasi-categories (using

Lemma 3.12), we can thus write $\mathcal{I} \simeq \operatorname{colim}_{n \geqslant 0} \mathcal{I}_n$ in Cat_∞ in such a way that $\mathcal{I}_n$ is obtained from $\mathcal{I}_{n-1}$ by a pushout along $\coprod \mathcal{B}^n \to \coprod \Delta^n$, where $\mathcal{B}^n$ is defined by choosing an inner anodyne map $\partial\Delta^n \to \mathcal{B}^n$ into a quasi-category. By an inductive argument (in which Lemma 6.38(b) powers the inductive step), we find that $\operatorname{colim}_{i \in \mathcal{I}_n} T(i)$ exists in $\mathcal{C}$ for all $n \geqslant 0$. Using Lemma 6.38(b) once again, it remains to show that $\operatorname{colim}_{n \geqslant 0} \operatorname{colim}_{i \in \mathcal{I}_n} T(i)$ exists in $\mathcal{C}$. But this colimit can be easily written as a suitable pushout of the disjoint union $\coprod_{n \geqslant 0} \operatorname{colim}_{i \in \mathcal{I}_n} T(i)$. $\qquad\square$

To study colimits in derived ∞-categories, we introduce the following convenient terminology.

6.39. Definition. — Let $\mathcal{C}$ be an ∞-category with a terminal object $*$ and let $\alpha\colon x \to y$ be a morphism in $\mathcal{C}$. The *cofibre of α* is defined as the pushout

$$
\begin{array}{ccc}
x & \xrightarrow{\ \alpha\ } & y \\
\downarrow & \ulcorner. & \downarrow \\
* & \longrightarrow & \operatorname{cofib}(\alpha)
\end{array}
$$

(provided this exists in $\mathcal{C}$). We say that $x \xrightarrow{\alpha} y \to z$ is a *cofibre sequence in $\mathcal{C}$* if the induced morphism $x \to z$ can be factored through $*$ in such a way that it exhibits z as the cofibre of α. There are dual notions of the *fibre of α* $\operatorname{fib}(\alpha)$ (given as the pullback against an initial object) and *fibre sequences*.

6.40. Lemma. — *Let R be any ring (not necessarily commutative). The ∞-category $\mathcal{D}(R)$ has all colimits. The full sub-∞-caegory $\mathcal{D}_{\geqslant 0}(R) \subseteq \mathcal{D}(R)$ is closed under colimits in $\mathcal{D}(R)$ and therefore has all colimits too. Coproducts and pushouts in $\mathcal{D}(R)$ can be described as follows:*

(a) *If I is any set and $M_{i,*}$ are chain complexes of left R-modules, then the chain complex $\bigoplus_{i \in I} M_{i,*}$ defines a coproduct of the objects $M_i \in \mathcal{D}(R)$.*

(b) *Let $\alpha\colon M_* \to N_*$ be a morphism of chain complexes of R-modules for some ring R. Then the cofibre of α in $\mathcal{D}(R)$ can be computed as the mapping cone*[(6.11)]

$$
\operatorname{cofib}(\alpha\colon M \to N) \simeq \operatorname{cone}(\alpha\colon M_* \to N_*)\,.
$$

More generally, if $\beta\colon M_* \to M'_*$ is another morphism of complexes, then the pushout of the span $N_* \leftarrow M_* \to M'_*$ in $\mathcal{D}(R)$ is given by $\operatorname{cone}((\alpha, -\beta)\colon M_* \to M'_* \oplus N_*)$.

Proof sketch. By Lemma 6.37, to show that $\mathcal{D}(R)$ has all colimits, it's enough to check that coproducts and pushouts in $\mathcal{D}(R)$ can be described as in (a) and (b). Furthermore, it's clear

---

[(6.11)] The mapping cone $\operatorname{cone}(\alpha)_*$ is the chain complex given by $\operatorname{cone}(\alpha)_n \cong N_n \oplus M_{n-1}$, with differentials given by the matrix

$$
\begin{pmatrix} \partial_N & \alpha \\ 0 & -\partial_M \end{pmatrix} \colon N_n \oplus M_{n-1} \to N_{n-1} \oplus M_{n-2}\,;
$$

here ∂_M and ∂_N denote the differentials of M_* and N_*, respectively. The mapping cone comes equipped with obvious maps $N_* \to \operatorname{cone}(\alpha)_*$ and $\operatorname{cone}(\alpha)_* \to M[1]_*$. The induced maps $\mathrm{H}_n(N_*) \to \mathrm{H}_n(\operatorname{cone}(\alpha)_*)$ and $\mathrm{H}_n(\operatorname{cone}(\alpha)_*) \to \mathrm{H}_{n-1}(M_*)$ on homology fit into a long exact sequence

$$
\cdots \longrightarrow \mathrm{H}_{n+1}\big(\operatorname{cone}(\alpha)_*\big) \xrightarrow{\ \partial\ } \mathrm{H}_n(M_*) \longrightarrow \mathrm{H}_n(N_*) \longrightarrow \mathrm{H}_n\big(\operatorname{cone}(\alpha)_*\big) \xrightarrow{\ \partial\ } \mathrm{H}_{n-1}(M_*) \longrightarrow \cdots
$$

called the *cone sequence*. In fact, this is nothing but the long exact homology sequence associated to the short exact sequence of complexes $0 \to N_* \to \operatorname{cone}(\alpha)_* \to M[1]_* \to 0$.

from these descriptions that $\mathcal{D}_{\geqslant 0}(R) \subseteq \mathcal{D}(R)$ is closed under formation of coproducts and pushouts, hence under all colimits.

To prove (a), choose quasi-isomorphisms $P_{i,*} \to M_{i,*}$ from K-projective complexes. Since quasi-isomorphisms are preserved under direct sums, it's enough to show that $\bigoplus_{i \in I} P_i$ is a coproduct of the P_i. Using Corollary 6.16, we must show that

$$\mathrm{Hom}_{\mathcal{D}(R)}\left(\bigoplus_{i \in I} P_i, T \right) \xrightarrow{\simeq} \prod_{i \in I} \mathrm{Hom}_{\mathcal{D}(R)}(P_i, T)$$

is an equivalence of animae for all $K \in \mathcal{D}(R)$. But if T_* is any chain complex representing T, then $\underline{\mathrm{Hom}}_R\left(\bigoplus_{i \in I} M_{P,*}, T_* \right) \cong \prod_{i \in I} \underline{\mathrm{Hom}}_R(P_{i,*}, T_*)$ holds in $\mathrm{Ch}(R)$. Both the functor $\tau_{\geqslant 0} \colon \mathrm{Ch}(R) \to \mathrm{Ch}_{\geqslant 0}(R)$ and the Dold–Kan equivalence $\mathrm{Ch}_{\geqslant 0}(R) \simeq \mathrm{sAb}$ preserve products. Hence we get an isomorphism of Kan complexes (in fact, of simplicial abelian groups) $\mathrm{F}_{\mathrm{Ch}(R)}\left(\bigoplus_{i \in I} P_{i,*}, T_* \right) \cong \prod_{i \in I} \mathrm{F}_{\mathrm{Ch}(R)}(P_{i,*}, T_*)$. By crash course 6.36, this is (stronger than) what we need.

The proof of (b) is similar, but needs a little more care. First, the assertion about pushouts is a formal consequence of the assertion about cofibres; we leave this to you (just verify the universal property). To show the assertion about cofibres, choose quasi-isomorphisms $P_* \to M_*$ and $Q_* \to N_*$ from K-projective complexes and replace α by a morphism $\alpha' \colon P_* \to Q_*$. Since the mapping cone construction preserves quasi-isomorphisms (by the long exact cone sequence and the five lemma), it suffices to show $\mathrm{cofib}(\alpha') \simeq \mathrm{cone}(\alpha')$. To this end, first note that the composition $P_* \to Q_* \to \mathrm{cone}(\alpha')_*$ is nullhomotopic as a map of complexes. Any choice of nullhomotopy defines a morphism $\mathrm{cofib}(\alpha') \to \mathrm{cone}(\alpha')$ in $\mathcal{D}(R)$. To see that this morphism is an equivalence, we can appeal again to Corollary 6.16 and the Yoneda lemma: It's enough to show that
$$\mathrm{Hom}_{\mathcal{D}(R)}\left(\mathrm{cone}(\alpha'), T \right) \longrightarrow \mathrm{Hom}_{\mathcal{D}(R)}(Q, T) \longrightarrow \mathrm{Hom}_{\mathcal{D}(R)}(P, T)$$
is a fibre sequence of animae for all $T \in \mathcal{D}(R)$. Now we claim:

($\boxtimes$) *Let $\varphi \colon K_* \to L_*$ be any morphism of chain complexes and consider the canonical sequence*

$$\mathrm{cone}(\varphi)[-1]_* \longrightarrow K_* \xrightarrow{\varphi} L_*$$

in $\mathrm{Ch}(R)$. Upon applying $\tau_{\geqslant 0} \colon \mathrm{Ch}(R) \to \mathrm{Ch}_{\geqslant 0}(R)$ and the Dold–Kan correspondence, this sequence is sent to a fibre sequence of animae.

Once we know ($\boxtimes$), we're done. Indeed, it's straightforward to check that the sequence $\underline{\mathrm{Hom}}_{\mathcal{D}(R)}(\mathrm{cone}(\alpha')_*, T_*) \to \underline{\mathrm{Hom}}_R(Q_*, T_*) \to \underline{\mathrm{Hom}}_R(P_*, T_*)$ is of the desired form. Then ($\boxtimes$) ensures that $\mathrm{F}_{\mathrm{Ch}(R)}(\mathrm{cone}(\alpha')_*, T_*) \to \mathrm{F}_{\mathrm{Ch}(R)}(Q_*, T_*) \to \mathrm{F}_{\mathrm{Ch}(R)}(P_*, T_*)$ is a fibre sequence; by crash course 6.36, this is what we need.

To prove ($\boxtimes$), we can restrict ourselves to the case where $\varphi \colon K_* \to L_*$ is degree-wise surjective. Indeed, we can always replace K_* by $K_* \oplus \mathrm{cone}(\mathrm{id}_{L_*} \colon L_* \to L_*)[-1]_*$; this doesn't change anything since $\mathrm{cone}(\mathrm{id}_{L_*})_* \to 0$ is a quasi-isomorphism, $\tau_{\geqslant 0}$ preserves quasi-isomorphisms, and Dold–Kan sends quasi-isomorphisms to homotopy equivalences of Kan complexes (because it sends H_n to π_n by crash course 6.35(b)). If $\varphi \colon K_* \to L_*$ is surjective, a well-known fact from homological algebra states that there is a quasi-isomorphism $\ker(\varphi)_* \simeq \mathrm{cone}(\varphi)[-1]_*$.[(6.12)] So it's enough to show that $\ker(\varphi)_* \to K_* \to L_*$ is sent to a fibre sequence.

[(6.12)]To see this, the first step is to construct a morphism $\ker(\varphi)_* \to \mathrm{cone}(\varphi)[-1]_*$: This is straightforward from the construction of $\mathrm{cone}(\varphi)[-1]_*$. Alternatively, we can invoke a universal property: It can be shown that maps $T_* \to \mathrm{cone}(\varphi)[-1]_*$ are in bijection with pairs (α, η), where $\alpha \colon T_* \to K_*$ is a morphism and η is

Since $\tau_{\geqslant 0}\colon \mathrm{Ch}(R) \to \mathrm{Ch}_{\geqslant 0}(R)$ is a right adjoint, we see that $\tau_{\geqslant 0}\ker(\varphi)_*$ is still the kernel of $\tau_{\geqslant 0}\varphi\colon \tau_{\geqslant 0}K_* \to \tau_{\geqslant 0}L_*$, but that map might not be surjective anymore in degree 0. So consider $\mathrm{im}(\tau_{\geqslant 0}\varphi)_* \to \tau_{\geqslant 0}L_*$. Under the Dold–Kan correspondence, this map is sent to the inclusion of a collection of path components. Indeed, $\mathrm{im}(\tau_{\geqslant 0}\varphi)_* \to \tau_{\geqslant 0}L_*$ is an isomorphism on H_n for all $n \geqslant 1$ and injective on H_0, so after Dold–Kan, we obtain an isomorphism on π_n for all $n \geqslant 1$ and an injection on π_0. Therefore, it's enough to show that $\tau_{\geqslant 0}\ker(\varphi)_* \to \tau_{\geqslant 0}K_* \to \mathrm{im}(\tau_{\geqslant 0}\varphi)_*$ is sent to a fibre sequence. But that's a short exact sequence in $\mathrm{Ch}_{\geqslant 0}(R)$, and so it's sent to a short exact sequence in sAb. As mentioned in crash course 6.35(a), surjections of simplicial abelian groups are Kan fibrations. By model category fact 6.13, this means that the kernel, that is, the fibre over 0 taken in simplicial sets, agrees with the homotopy fibre. So we're done. □

With all the preparatory stuff about $\mathcal{D}(R)$ out of the way, we can now finally get to the actual subject of §6.5: Homology, cohomology, and Eilenberg–MacLane animae.

6.41. Construction. — For all integers n and all abelian groups A define a chain complex

$$A[n]_* \coloneqq (\cdots \to 0 \to 0 \to A \to 0 \to 0 \to \cdots)$$

with A in degree n and 0 everywhere else. Consider the functor $\mathrm{Ab} \to \mathrm{Ch}_{\geqslant 0}(\mathbb{Z}) \to \mathcal{D}_{\geqslant 0}(\mathbb{Z})$ that sends A to $A[0]$. Since $\mathcal{D}_{\geqslant 0}(\mathbb{Z})$ has all colimits, so has $\mathrm{Fun}(\mathrm{Ab}, \mathcal{D}_{\geqslant 0}(\mathbb{Z}))$ by Lemma 6.12. Hence, by Theorem 6.30, there exists a unique colimit-preserving functor $\mathrm{An} \to \mathrm{Fun}(\mathrm{Ab}, \mathcal{D}_{\geqslant 0}(\mathbb{Z}))$ that sends $* \in \mathrm{An}$ to the functor $\mathrm{Ab} \to \mathrm{Ch}_{\geqslant 0}(\mathbb{Z}) \to \mathcal{D}_{\geqslant 0}(\mathbb{Z})$ discussed above. By "currying", we obtain a functor

$$\mathrm{C}(-,-)\colon \mathrm{An} \times \mathrm{Ab} \longrightarrow \mathcal{D}_{\geqslant 0}(\mathbb{Z})\,.$$

For every abelian group A, $\mathrm{C}(-,A)\colon \mathrm{An} \to \mathcal{D}_{\geqslant 0}(\mathbb{Z})$ is the unique colimit-preserving functor that sends $* \in \mathrm{An}$ to $A[0]$ as above. For an anima X, we call $\mathrm{C}(X,A)$ the *chains of X with coefficients in A* and we call $\widetilde{\mathrm{C}}(X,A) \coloneqq \mathrm{fib}(\mathrm{C}(X,A) \to \mathrm{C}(*,A))$ the *reduced chains of X with coefficients in A* (using the fibre construction from Definition 6.39). For all $n \geqslant 0$, we let

$$\mathrm{H}_n(X,A) \coloneqq \mathrm{H}_n\big(\mathrm{C}(X,A)\big)\,, \quad \widetilde{\mathrm{H}}_n(X,A) \coloneqq \mathrm{H}_n\big(\widetilde{\mathrm{C}}(X,A)\big)$$

denote the nth *homology of X with coefficients in A* and the nth *reduced homology of X with coefficients in A*; here $\mathrm{H}_n\colon \mathcal{D}_{\geqslant 0}(\mathbb{Z}) \to \mathrm{Ab}$ is the functor from crash course 6.34. Finally,

$$\mathrm{H}^n(X,A) \coloneqq \pi_0 \mathrm{Hom}_{\mathcal{D}_{\geqslant 0}(\mathbb{Z})}\big(\mathrm{C}(X,\mathbb{Z}), A[n]\big)\,, \quad \widetilde{\mathrm{H}}^n(X,A) \coloneqq \pi_0 \mathrm{Hom}_{\mathcal{D}_{\geqslant 0}(\mathbb{Z})}\big(\widetilde{\mathrm{C}}(X,\mathbb{Z}), A[n]\big)$$

denote the nth *cohomology of X with coefficients in A* and the nth *reduced cohomology of X with coefficients in A*. We'll verify in Lemma 6.42 below that this definition of homology and cohomology is compatible with the one you are familiar with.

But before we do that, let's give yet another unfamiliar formulation of a familiar definition! By Theorem 6.30, $\mathrm{C}(-,\mathbb{Z})\colon \mathrm{An} \to \mathcal{D}_{\geqslant 0}(\mathbb{Z})$ automatically acquires a right adjoint, which we denote $\mathrm{K}\colon \mathcal{D}_{\geqslant 0}(\mathbb{Z}) \to \mathrm{An}$. For $M \in \mathcal{D}_{\geqslant 0}(\mathbb{Z})$ we call $\mathrm{K}(M)$ the *generalised Eilenberg–MacLane anima of M*. In the special case $M \simeq A[n]$ we say that $\mathrm{K}(A,n) \coloneqq \mathrm{K}(A[n])$ is the *Eilenberg–MacLane anima of type (A,n)*. Again, we'll justify in Theorem 6.44 below that this recovers the definition you're familiar with.

---

a nullhomotopy of the composition $\varphi \circ \alpha\colon T_* \to L_*$. Since $\ker(\varphi)_* \to L_*$ is zero on the nose, we can just choose the trivial nullhomotopy. To show that the map $\ker(\varphi)_* \to \mathrm{cone}(\varphi)[-1]_*$ is a quasi-isomorphism, it's straightforward to check that this map induces a morphism between the long exact homology sequence associated to $0 \to \ker(\varphi)_* \to K_* \to L_* \to 0$ and the cone sequence for $\mathrm{cone}(\varphi)_*$. Then the five lemma does the rest.

6.42. Lemma. — *Let A be an arbitrary abelian group.*

(a) *The functors $H_*(-, A)$ and $\widetilde{H}_*(-, A)$ from Construction 6.41 satisfy the Eilenberg–Steenrod axioms. In particular, they are homotopy invariants; if $f\colon Y \to X$ is a morphism of animae with cofibre $X/Y \coloneqq \mathrm{cofib}(f)$, then there is a long exact sequence*

$$\cdots \longrightarrow H_n(Y, A) \longrightarrow H_n(X, A) \longrightarrow \widetilde{H}_n(X/Y, A) \xrightarrow{\ \partial\ } H_{n-1}(Y, A) \longrightarrow \cdots$$

(so $\widetilde{H}_*(X/Y, A)$ plays the role of the relative homology $H_*(X, Y, A)$); and $H_*(-, A)$ sends disjoint unions to direct sums. Similar assertions hold for $\widetilde{H}_*(-, A)$. Furthermore, the suspension isomorphism is satisfied and pushouts of animae yield long exact Mayer–Vietoris sequences.

(b) *Let X be a Kan complex with geometric realisation $|X| \in \mathrm{Top}$. Let $C_*^{\mathrm{sing}}(|X|, A)$ denote the singular chain complex of $|X|$ with coefficients in A. Then there is a natural quasi-isomorphism*

$$C(X, A) \xrightarrow{\ \simeq\ } C_*^{\mathrm{sing}}(|X|, \mathbb{Z}).$$

In particular, we get $H_*(X, A) \cong H_*^{\mathrm{sing}}(|X|, A)$, as well as similar isomorphisms for reduced homology and for cohomology (both unreduced and reduced).

Proof sketch. We begin with (a). Homotopy invariance follows from the definition of An. The other Eilenberg–Steenrod axioms all follow from the fact that $C(-, A)\colon \mathrm{An} \to \mathcal{D}_{\geqslant 0}(\mathbb{Z})$ preserves colimits. To demonstrate these kinds of arguments, we'll show the long exact sequence for $H_*(-, A)$; the disjoint union axiom as well as the Eilenberg–Steenrod axioms for $\widetilde{H}_*(-, A)$ will be left to you. We start with the following diagram:

$$
\begin{array}{ccccc}
C(Y, A) & \longrightarrow & C(*, A) & \longrightarrow & 0 \\
\downarrow & \ulcorner & \downarrow & \ulcorner & \downarrow \\
C(X, A) & \longrightarrow & C(X/Y, A) & \longrightarrow & \widetilde{C}(X/Y, A)
\end{array}
$$

The left square is a pushout since $C(-, A)$ preserves pushouts. To obtain the pushout square on the right, observe that X/Y is canonically a pointed anima via $* \simeq Y/Y \to X/Y$. In general, for any pointed anima $(Z, z) \in \mathrm{An}_{*/}$, the canonical morphism $Z \to *$ has a preferred section given by $\{z\} \to Z$. Thus $C(Z, \mathbb{Z}) \to C(*, \mathbb{Z})$ has a section and we obtain

$$C(Z, \mathbb{Z}) \simeq \widetilde{C}(Z, \mathbb{Z}) \oplus C(\{z\}, \mathbb{Z}).$$

Hence the reduced chains $\widetilde{C}(Z, \mathbb{Z})$ from Construction 6.41 can also be written as the cofibre $\widetilde{C}(Z, \mathbb{Z}) \simeq \mathrm{cofib}(C(\{z\}, \mathbb{Z}) \to C(Z, \mathbb{Z}))$, functorially in (Z, z). This explains the right square in the diagram above. Hence $C(Y, A) \to C(X, A) \to \widetilde{C}(X/Y, A)$ is a cofibre sequence in $\mathcal{D}_{\geqslant 0}(\mathbb{Z})$ and thus in $\mathcal{D}(\mathbb{Z})$. By Lemma 6.40(b), cofibre sequences in $\mathcal{D}(\mathbb{Z})$ can be represented by cone sequences, so the desired long exact sequence is simply the cone sequence.

The suspension isomorphism and the Mayer–Vietoris sequence are formal consequences of the Eilenberg–Steenrod axioms. Alternatively, they can be checked by hand. For example, for Mayer–Vietoris, use that $C(-, A)$ preserves pushouts and then apply the characterisation of pushouts in $\mathcal{D}(\mathbb{Z})$ from Lemma 6.40(b).

For (b), let's first describe how to get a functor $C^{\mathrm{sing}}(|-|, A)\colon \mathrm{An} \to \mathcal{D}_{\geqslant 0}(\mathbb{Z})$. By Theorem 4.13 and Lemma 4.11, it's enough to check that $C_*^{\mathrm{sing}}(|-|, A)\colon \mathrm{Kan} \to \mathrm{Ch}_{\geqslant 0}(\mathbb{Z})$ sends

homotopy equivalences to quasi-isomorphisms, which is obviously true. So we get our desired functor. We know from Lemma 6.31 and the Kan extension formula that $C(-, A)\colon \mathrm{An} \to \mathcal{D}_{\geqslant 0}(\mathbb{Z})$ is the left Kan extension of its restriction to $\{*\} \subseteq \mathrm{An}$. By the universal property of Kan extensions, the equivalence $C(*, A) \simeq A[0] \simeq C^{\mathrm{sing}}(|*|, A)$ extends to a natural transformation

$$C(-, A) \Longrightarrow C^{\mathrm{sing}}(|-|, A).$$

Whether this is an equivalence can be checked pointwise and on homology groups. Now we can use the well-known fact that any unreduced homology theory h_* with $h_0(*) \cong A$ and $h_n(*) \cong 0$ for $n \geqslant 1$ must necessarily coincide with singular homology with coefficients in A. See [Hat02, Theorem 4.59]. Alternatively, we can directly show that $C^{\mathrm{sing}}(|-|, A)$ preserves colimits. Using Lemma 6.40, this is straightforward (preservation of coproducts is trivial and preservation of pushouts is, essentially, the Mayer–Vietoris sequence).

The isomorphism $H_*(X, A) \cong H_*^{\mathrm{sing}}(|X|, A)$ is an immediate consequence, as is its variant for reduced homology. For cohomology, we can argue as follows: We already know that $C(X, \mathbb{Z}) \simeq C^{\mathrm{sing}}(|X|, \mathbb{Z})$. The chain complex $C_*^{\mathrm{sing}}(|X|, \mathbb{Z})$ is K-projective because it is degree-wise free over $\mathbb{Z}$ and bounded below. Using the computation from crash course 6.36, we deduce

$$\pi_0 \operatorname{Hom}_{\mathcal{D}_{\geqslant 0}(\mathbb{Z})}\big(C^{\mathrm{sing}}(|X|, \mathbb{Z}), A[n]\big) \cong H_0 \underline{\operatorname{Hom}}_{\mathbb{Z}}\big(C_*^{\mathrm{sing}}(|X|, \mathbb{Z}), A[n]\big)$$

Now $\underline{\operatorname{Hom}}_{\mathbb{Z}}(C_*^{\mathrm{sing}}(|X|, \mathbb{Z}), A) \cong C_{\mathrm{sing}}^{-*}(|X|, A)$ is the cochain complex that computes singular cohomology of $|X|$, placed in negative degrees (so that it becomes a chain complex). Taking the shifts into account, we get $H^n(X, A) \cong H_{\mathrm{sing}}^n(|X|, A)$, as claimed. The same argument also shows the assertion about reduced cohomology. $\qquad\square$

6.43. Remark. — Let's take a moment to appreciate the beauty of Lemma 6.42(a). With our definition of homology, the Eilenberg–Steenrod axioms and all the usual properties of homology are completely formal. The only input we need is of algebraic nature: namely, the description of colimits in $\mathcal{D}(\mathbb{Z})$ from Lemma 6.40. Now compare that to the classical construction of singular homology: To prove the Mayer–Vietoris sequence (or equivalently excision), one has to take sufficiently fine barycentric subdivisions, apply Lebesgue's covering theorem, and construct a bunch of chain homotopies by hand (see [Hat02, Proposition 2.21] for such an argument). Blargh! I find our approach much more enlightening and much less technical.[(6.13)] In fact, I'd argue that Construction 6.41 is the better definition of homology (and an even better one is Corollary 8.27)!

Although our definition is very abstract, for a given anima X, it's often easy to write down an explicit complex that computes $C(X, A)$. For example, assume we're given a "CW decomposition" of X; that is, a way to write X as a sequence of pushouts along $S^n \to *$ (the n-disk is contractible, so we may as well use $*$ instead). Then, $C(X, A)$ can be written as a similar sequence of pushouts in $\mathcal{D}_{\geqslant 0}(\mathbb{Z})$. Since we understand $C(S^n, A) \simeq A[0] \oplus A[n]$ as well as $C(*, A) \simeq A[0]$ and since we know how to compute pushouts in $\mathcal{D}_{\geqslant 0}(\mathbb{Z})$ by Lemma 6.40(b), we can compute $C(X, A)$. If you think about this, the complex we end up with is precisely the *cellular complex of X*, so we've just proved that homology agrees with cellular homology. Combining this with the classical fact that cellular and singular homology agree, we get an alternative proof of Lemma 6.42(b).

[(6.13)]The careful traditionalist will—rightfully—object that our theory doesn't really avoid barycentric subdivision, it just moves it to the proof of Theorem 3.26, conveniently hidden in a black box. Sure, but that doesn't undermine my point. Barycentric subdivision is a technical tool to compare animae to CW complexes—that's its natural place in the theory. Beyond that, it can be avoided.

We finish this subsection by proving the classical Eilenberg–MacLane theorem. As we'll see, once again, the proof is entirely formal.

6.44. Theorem ("Eilenberg–MacLane animae represent cohomology"). — *For every abelian group A and all $n \geqslant 0$, the Eilenberg–MacLane anima $\mathrm{K}(A, n)$ from Construction 6.41 satisfies $\pi_n \mathrm{K}(A, n) \cong A$ and $\pi_i \mathrm{K}(A, n) \cong 0$ for $i \neq n$. This condition determines $\mathrm{K}(A, n)$ uniquely up to homotopy equivalence. Furthermore, $\mathrm{K}(A, n)$ represents cohomology with coefficients in A (both unreduced and reduced) in the sense that the functors $\mathrm{H}^n(-, A)\colon \mathrm{An} \to \mathrm{Ab}$ and $\widetilde{\mathrm{H}}^n(-, A)\colon \mathrm{An}_{*/} \to \mathrm{Ab}$ are given by*

$$\left[-, \mathrm{K}(A, n)\right] := \pi_0 \operatorname{Hom}_{\mathrm{An}}\left(-, \mathrm{K}(A, n)\right) \quad and \quad \left[-, \mathrm{K}(A, n)\right]_* := \pi_0 \operatorname{Hom}_{\mathrm{An}_{*/}}\left(-, \mathrm{K}(A, n)\right),$$

respectively.

Proof. For every anima X and every $M_* \in \mathrm{Ch}_{\geqslant 0}(\mathbb{Z})$, the adjunction $\mathrm{C}(-, \mathbb{Z})\colon \mathrm{An} \rightleftarrows \mathcal{D}_{\geqslant 0}(\mathbb{Z}) : \mathrm{K}$ from Construction 6.41 shows

$$\operatorname{Hom}_{\mathrm{An}}\left(X, \mathrm{K}(M)\right) \cong \operatorname{Hom}_{\mathcal{D}_{\geqslant 0}(\mathbb{Z})}\left(\mathrm{C}(X, \mathbb{Z}), M\right).$$

In the case $M_* \simeq A[n]_*$, we immediately obtain $\pi_0 \operatorname{Hom}_{\mathrm{An}}(X, \mathrm{K}(A, n)) \simeq \mathrm{H}^n(X, A)$. This shows that $[-, \mathrm{K}(A, n)] \cong \mathrm{H}^n(-, A)$. The assertion about reduced cohomology follows analogously if we can show that the adjunction $\mathrm{C}(-, \mathbb{Z})\colon \mathrm{An} \rightleftarrows \mathcal{D}_{\geqslant 0}(\mathbb{Z}) : \mathrm{K}$ lifts to an adjunction

$$\widetilde{\mathrm{C}}(-, \mathbb{Z})\colon \mathrm{An}_{*/} \rightleftarrows \mathcal{D}_{\geqslant 0}(\mathbb{Z}) : \mathrm{K}.$$

In Lemma 6.45 below, we'll show a general fact about passing adjunctions to slice ∞-categories. Let's explain how this applies in our situation: Since K is a right adjoint, it preserves terminal objects, whence $\mathrm{K}(0) \simeq *$. But 0 is also an initial object in $\mathcal{D}_{\geqslant 0}(\mathbb{Z})$. Hence $\mathcal{D}_{\geqslant 0}(\mathbb{Z}) \simeq \mathcal{D}_{\geqslant 0}(\mathbb{Z})_{0/}$ and so the induced functor $\mathrm{K}\colon \mathcal{D}_{\geqslant 0}(\mathbb{Z}) \simeq \mathcal{D}_{\geqslant 0}(\mathbb{Z})_{0/} \to \mathrm{An}_{\mathrm{K}(0)/} \simeq \mathrm{An}_{*/}$ on slice ∞-categories is indeed of the form studied in Lemma 6.45. We've seen in the proof of Lemma 6.42(*a*) that for every pointed anima (X, x) one has $\widetilde{\mathrm{C}}(X, \mathbb{Z}) \simeq \operatorname{cofib}(\mathrm{C}(\{x\}, \mathbb{Z}) \to \mathrm{C}(X, \mathbb{Z}))$, so $\widetilde{\mathrm{C}}(-, \mathbb{Z})$ agrees with the left adjoint constructed in Lemma 6.45.

To compute the homotopy groups of $\mathrm{K}(A, n)$, let $S^i \in \mathrm{An}$ be the *i-sphere*.[6.14] Plugging in $(S^i, *)$ for any choice of basepoint yields

$$\pi_i \mathrm{K}(A, n) \cong \pi_0 \operatorname{Hom}_{\mathrm{An}_{*/}}\left((S^i, *), \mathrm{K}(A, n)\right) \cong \widetilde{\mathrm{H}}^n(S^i, A).$$

By Lemma 6.42, one has $\widetilde{\mathrm{H}}^n(S^n, A) \cong A$ and $\widetilde{\mathrm{H}}^n(S^i, A) \cong 0$ for $i \neq n$, and the desired description of $\pi_* \mathrm{K}(A, n)$ follows. By the usual argument from topology, $\mathrm{K}(A, n)$ is uniquely determined by this property up to homotopy equivalence. $\square$

The following lemma was used in the proof:

[6.14] There are many possible constructions for S^i. The most conceptual way would be to define $S^i := \Sigma^i(* \sqcup *)$ as the i-fold suspension of two points, using the upcoming definition Definition 7.1. But there are also many possible simplicial models. For example, if $\partial D^{i+1} \subseteq D^{i+1}$ is the boundary of the topological $(i+1)$-disk, we could take $\operatorname{Sing} \partial D^{i+1}$ as our model for S^i. Alternatively, we could choose anodyne maps from $\square^i/\partial\square^i$ or $\Delta^i/\partial\Delta^i$ or $\partial\Delta^{i+1}$ into Kan complexes. All constructions you could possibly come up with will be homotopy equivalent, so you can just choose your favourite option.

6.45. Lemma. — *Let $L\colon \mathcal{C} \rightleftarrows \mathcal{D} \colon R$ be an adjunction of ∞-categories and let $y \in \mathcal{D}$. If for every morphism $R(y) \to x$ in $\mathcal{C}$ the pushout*

$$
\begin{array}{ccc}
LR(y) & \longrightarrow & L(x) \\
{\scriptstyle c_y}\downarrow & \quad\ulcorner & \downarrow \\
y & \longrightarrow & y \sqcup_{LR(y)} L(x)
\end{array}
$$

exists in $\mathcal{D}$, then the functor $R\colon \mathcal{D}_{y/} \to \mathcal{C}_{R(y)/}$ on slice ∞-categories still has a left adjoint $L_y\colon \mathcal{C}_{R(y)/} \to \mathcal{D}_{y/}$. On objects, L_y is given by $L_y(R(y) \to x) \simeq (y \to y \sqcup_{LR(y)} L(x))$, constructed via the pushout square above. Moreover, the pushout square can be made functorial in an obvious way and this recovers L_y as a functor (not only pointwise).

Proof sketch. You can directly verify $\mathrm{Hom}_{\mathcal{D}_{y/}}(L_y(R(y) \to x), -) \simeq \mathrm{Hom}_{\mathcal{C}_{R(y)/}}(R(y) \to x, R(-))$. To do so, plug in Corollary 6.16, Corollary 5.15, and the given adjunction $L \dashv R$, then perform a formal manipulation of pullbacks. Since adjoints can be constructed pointwise (Lemma 6.2), this proves the existence of L_y. With a little more care, one can make the pullback manipulation functorial in $R(y) \to x$ as well, and then the claimed description of L_y follows. $\qquad\square$

Appendix to §6. Presentable ∞-categories

Suppose $\mathcal{C}$ is an ∞-category with all colimits and let $F\colon \mathcal{C} \to \mathcal{D}$ be a colimit-preserving functor of ∞-categories. Then the only thing preventing F from having a right adjoint is set theory. Indeed, the values of a hypothetical right adjoint $G\colon \mathcal{D} \to \mathcal{C}$ would be given by $G(y) \simeq \mathrm{colim}(\mathcal{C}_{/y} \to \mathcal{C})$ for all $y \in \mathcal{D}$ (as we'll see in the proof of Theorem 6.66(a)), except that this colimit usually doesn't exist, even though $\mathcal{C}$ has all colimits. The problem ist that $\mathcal{C}_{/y}$ is usually not an *essentially small* ∞-category in the sense of Definition 6.46(b) below. So far, we have ignored these smallness issues. Still, §§6.1–6.4 can be made set-theoretically sound. As a rule of thumb, whenever a limit or colimit is considered, the indexing ∞-category should be assumed small (or at least admit a coinitial/initial functor from an essentially small ∞-category) and whenever we consider $\mathrm{PSh}(\mathcal{C})$, we should assume that $\mathcal{C}$ is essentially small. The only time this gets hairy is in the proof of Lemma 6.27, where we should allow $\mathcal{D}$ to be large, but also consider $\mathrm{PSh}(\mathcal{D})$. Nevertheless, this can be fixed too.[(6.15)]

However, a more thorough analysis is needed to save our adjoint functor argument. In fact, ∞-categories $\mathcal{C}$ with all colimits are very seldomly essentially small, and so neither is $\mathcal{C}_{/y}$. However, often there exists an essentially small sub-∞-category $\mathcal{C}_0 \subseteq \mathcal{C}$ that generates $\mathcal{C}$ under colimits, and in this case one can replace $\mathcal{C}_{/y}$ by a coinitial essentially small sub-∞-category, so that the required colimits do exist. The theory of accessible and presentable ∞-categories makes these ideas precise and allows to prove an incredibly powerful *adjoint functor theorem*.

In §§6.6–6.9, we'll give the necessary definitions and prove Lurie's adjoint functor theorem (Theorem 6.66). After that, we'll discuss some supplements in §6.10. Naturally, this means that §§6.6–6.10 will be very technical. If you're mainly interested in spectra and willing to take the adjoint functor theorem on faith, you can safely skip ahead to §7 at this point. If instead you're looking for a much more detailed exposition, you should consult [L-HTT, §5].

§6.6. Essentially small and locally small ∞-categories

First we'll explain how to put cardinality bounds on ∞-categories.

6.46. Definition. — Let κ be a regular cardinal and let $\mathcal{C}$ be an ∞-category.

(a)　If $\kappa = \aleph_0$, then $\mathcal{C}$ is called *essentially $\aleph_0$-small* if it is contained in the full sub-∞-category of Cat_∞ generated under pushouts by $\emptyset$ and Δ^n for all $n \geqslant 0$. If κ is uncountable, then $\mathcal{C}$ is called *essentially κ-small* if $\pi_0 \operatorname{core}\mathcal{C}$ as well as $\pi_0 \operatorname{Hom}_{\mathcal{C}}(x,y)$ and $\pi_n(\operatorname{Hom}_{\mathcal{C}}(x,y), \alpha)$ are sets of cardinality $< \kappa$ for all $x, y \in \mathcal{C}$, all $\alpha\colon x \to y$, and all $n \geqslant 1$.

(b)　$\mathcal{C}$ is called *essentially small* if it is essentially κ-small for some regular cardinal κ, and *large* otherwise. $\mathcal{C}$ is called *locally small* if $\operatorname{Hom}_{\mathcal{C}}(x,y)$ is essentially small for all $x, y \in \mathcal{C}$.

(c)　A colimit or a limit over a functor $F\colon \mathcal{I} \to \mathcal{C}$ is called *κ-small* if $\mathcal{I}$ is essentially κ-small. Instead of $\aleph_0$-small, we often say that a limit or colimit is *finite*.

6.47. Remark. — If $\mathcal{C}$ is a small ∞-category and $\mathcal{D}$ is locally small, then $\operatorname{Fun}(\mathcal{C}, \mathcal{D})$ is again locally small, as can be seen by Corollary 6.25. In particular, $\mathrm{PSh}(\mathcal{C})$ and the ∞-categories $\mathrm{Ind}_\kappa(\mathcal{C})$ from Construction 6.59 below will be locally small.

[(6.15)] For example, by using universes, but Fabian proposed a trick to get away with ZFC: Instead of $\mathrm{PSh}(\mathcal{D})$, consider the ∞-category of right fibrations $\mathcal{U} \to \mathcal{D}$, for which $\mathcal{U}$ admits a coinitial functor from an essentially small ∞-category. This ∞-category contains $\mathcal{D}_{/y} \to \mathcal{D}$ for all $y \in \mathcal{D}$, so all Yoneda arguments go through.

For practical applications, it will, unfortunately, be necessary to translate our nice model independent Definition 6.46(a) into the language of simplicial sets.

6.48. Lemma. — *Let κ be an uncountable regular cardinal and let $\mathcal{C}$ be an ∞-category. Then the following are equivalent:*

(a) $\mathcal{C}$ *is essentially κ-small.*

(b) $\mathcal{C}$ *is equivalent to a quasi-category with $< \kappa$ simplices across all dimensions.*

(c) *There exists a simplicial set K with $< \kappa$ simplices across all dimensions and a Joyal equivalence $K \to \mathcal{C}$ (that is, a weak equivalence in the Joyal model structure from Example 3.29).*

Furthermore, if K is a finite simplicial set (that is, a simplicial set with only finitely many non-degenerate simplices) and $K \to \mathcal{C}$ is a Joyal equivalence, then $\mathcal{C}$ is $\aleph_0$-small.

Proof sketch. The implications (b) $\Rightarrow$ (a) and (b) $\Rightarrow$ (c) are trivial. For (c) $\Rightarrow$ (b) let $K \to \mathcal{C}'$ be the inner anodyne map into a quasi-category provided by the proof of Lemma 3.12. Then $\mathcal{C}'$ has again $< \kappa$ simplices across all dimensions, because we're attaching $< \kappa$ new simplices countably many times. For the additional assertion, use induction on the dimension and write K as a sequence of pushouts against $\coprod \partial \Delta^n \to \coprod \Delta^n$, where the disjoint union is finite. Replacing everything by quasi-categories and using model category fact 6.13, we conclude that $\mathcal{C}$ is contained in the full sub-∞-category of Cat_∞ generated under pushouts by $\emptyset$ and Δ^n for all $n \geqslant 0$, as desired.

It remains to show (a) $\Rightarrow$ (b). We build a sub-simplicial set $\mathcal{C}' \subseteq \mathcal{C}$ as follows: Start with $\mathcal{C}' = \emptyset$. Choose $< \kappa$ representatives for every equivalence class in $\pi_0 \operatorname{core}(\mathcal{C})$ and add them to $\mathcal{C}'$. For all $x, y \in \mathcal{C}'$ and every equivalence class in $\pi_0 \operatorname{Hom}_\mathcal{C}(x,y)$, we add a representative $\alpha \colon x \to y$. Furthermore, for every $n \geqslant 1$ and every class in $\pi_n(\operatorname{Hom}_\mathcal{C}(x,y), \alpha)$, we choose a representative $\Delta^n/\partial \Delta^n \to \operatorname{Hom}_\mathcal{C}(x,y)$ and add the simplices in the image of the corresponding map $\Delta^n/\partial \Delta^n \times \Delta^1 \to \mathcal{C}$ to $\mathcal{C}'$. Then $\mathcal{C}'$ still has $< \kappa$ simplices. Mimicking the proof of Lemma 3.12, we can add $< \kappa$ further simplices to $\mathcal{C}'$ to ensure that $\mathcal{C}'$ is a quasi-category. By construction, $\mathcal{C}' \to \mathcal{C}$ is essentially surjective and the map $\operatorname{Hom}_{\mathcal{C}'}(x,y) \to \operatorname{Hom}_\mathcal{C}(x,y)$ is a surjection on all π_n for all $x, y \in \mathcal{C}'$. To make it injective, for every class in the kernel, choose a homotopy $\Delta^n/\partial \Delta^n \times \Delta^1 \to \operatorname{Hom}_\mathcal{C}(x,y)$ to $\operatorname{const} \alpha$. This homotopy corresponds to a map $(\Delta^n/\partial \Delta^n \times \Delta^1) \times \Delta^1 \to \mathcal{C}$ and we add its image to $\mathcal{C}'$. Then we add $< \kappa$ simplices to make $\mathcal{C}'$ into a quasi-category again. Clearly, $\mathcal{C}' \to \mathcal{C}$ is still essentially surjective; furthermore, all elements in the previous kernel of $\pi_n(\operatorname{Hom}_{\mathcal{C}'}(x,y), \alpha) \to \pi_n(\operatorname{Hom}_\mathcal{C}(x,y), \alpha)$ have been killed now. But there could be new ones. So we simply repeat this process countably many times. Then $\mathcal{C}' \to \mathcal{C}$ is fully faithful too and thus an equivalence by Theorem 4.6. $\square$

6.49. Remark. — If κ is an uncountable regular cardinal, then pushouts or pullbacks of essentially κ-small ∞-categories are essentially κ-small again. Indeed, this follows from Lemma 6.48(b) together with 6.13 and a cardinality bound on Lemma 3.12: A functor between quasi-categories with $< \kappa$ simplices across all dimensions can be factored into a cofibration followed by a trivial fibration or into a Joyal equivalence followed by an isofibration in such a way that the new quasi-category in the middle has again $< \kappa$ simplices across all dimensions. Combining this observation with Lemma 6.50 below, we see that the full sub-∞-category $\mathrm{Cat}_\infty^{<\kappa}$ of essentially κ-small ∞-categories is closed under κ-small limits and colimits.

In the case $\kappa = \aleph_0$ it's obvious that $\aleph_0$-small ∞-categories are closed under pushouts and thus under finite colimits by Lemma 6.50 below. The same can be shown for finite products, but I don't know if it works for pullbacks too.

6.50. Lemma. — *Let κ be a regular cardinal. An ∞-category $\mathcal{C}$ has all κ-small colimits if and only if $\mathcal{C}$ has pushouts and κ-small coproducts. A functor $F\colon \mathcal{C} \to \mathcal{D}$ of ∞-categories preserves colimits if and only if it preserves pushouts and κ-small coproducts. A dual assertion holds for limits.*

Proof sketch. Repeat the proof of Lemma 6.37 and use Lemma 6.48 together with model category fact 6.13 to see that pushouts of κ-small ∞-categories are still κ-small. $\square$

§6.7. Filtered colimits

In this subsection, we'll study filtered colimits in ∞-categories and prove a version of the well-known fact that filtered colimits commute with finite limits (Theorem 6.54).

6.51. Construction. — Let $\mathcal{I}$ be an ∞-category. We define the *cone $\mathcal{I}^{\triangleleft}$ over $\mathcal{I}$* and the *cocone $\mathcal{I}^{\triangleright}$ under $\mathcal{I}$* as the following pushouts in Cat_∞:

$$
\begin{array}{ccc}
\mathcal{I} \times \{0\} \longrightarrow \mathcal{I} \times \Delta^1 \\
\downarrow \qquad \ulcorner. \qquad \downarrow \\
* \longrightarrow \mathcal{I}^{\triangleleft}
\end{array}
\qquad \text{and} \qquad
\begin{array}{ccc}
\mathcal{I} \times \{1\} \longrightarrow \mathcal{I} \times \Delta^1 \\
\downarrow \qquad \ulcorner. \qquad \downarrow \\
* \longrightarrow \mathcal{I}^{\triangleright}
\end{array}
$$

It's tempting to use the procedure from model category fact 6.13 to compute these pushouts explicitly, but this is a little tricky. Steps (*a*), (*b*), and (*c*) are easy though: $\mathcal{I} \times \{0\} \to \mathcal{I} \times \Delta^1$ and $\mathcal{I} \times \{1\} \to \mathcal{I} \times \Delta^1$ are already cofibrations, so we can simply take the pushout on the nose. The tricky step, however, is (*d*), in which one has to replace the pushout in sSet by a quasi-category. One can show that the *joins* $\{0\} \star \mathcal{I}$ and $\mathcal{I} \star \{1\}$, which we didn't introduce, are such replacements; see [L-HTT, Proposition 4.1.2.1] or [Lan21, Proposition 2.5.19]. We won't need this explicit description and work with the abstract construction exclusively.

Note that $* \to \mathcal{I}^{\triangleleft}$ is an initial object and $* \to \mathcal{I}^{\triangleright}$ is a terminal object. This is obvious in the simplicial models, but there's also a model-independent argument: We must show that $* \to \mathcal{I}^{\triangleleft}$ is left adjoint to the unique functor $\mathcal{I}^{\triangleleft} \to *$. This can be done via Lemma 6.5 by constructing the unit and counit by hand. The unit is clear, as there are not that many functors from $*$ to itself (in fact, there's only one). For the counit, we must construct a natural transformation $c\colon \mathcal{I}^{\triangleleft} \times \Delta^1 \to \mathcal{I}^{\triangleleft}$ from $\mathrm{const}\, *$ to $\mathrm{id}_{\mathcal{I}^{\triangleleft}}$. Using that $- \times \Delta^1\colon \mathrm{Cat}_\infty \to \mathrm{Cat}_\infty$ commutes with pushouts (since $\mathrm{Fun}(\Delta^1, -)$ is a right adjoint by Example 6.3(*b*)), this boils down to constructing a natural transformation $\Delta^1 \times \Delta^1 \to \Delta^1$ from $\mathrm{const}\, 0$ to id_{Δ^1}, which is easy. In the same way, verifying the triangle identities reduces to a question about Δ^1. In particular, we deduce $|\mathcal{I}^{\triangleleft}| \simeq *$ and $|\mathcal{I}^{\triangleright}| \simeq *$, as ∞-categories with an initial or terminal object are always weakly constractible.

As in ordinary category theory, cones and cocones are closely related to limits and colimits, respectively. Concretely, if $F\colon \mathcal{I} \to \mathcal{C}$ is a functor and $y \in \mathcal{C}$ is an object, then an easy calculation using Corollary 6.17 shows

$$\{F\} \times_{\mathrm{Hom}_{\mathrm{Cat}_\infty}(\mathcal{I},\mathcal{C})} \mathrm{Hom}_{\mathrm{Cat}_\infty}(\mathcal{I}^{\triangleright}, \mathcal{C}) \times_{\mathrm{Hom}_{\mathrm{Cat}_\infty}(*,\mathcal{C})} \{y\} \simeq \mathrm{Hom}_{\mathrm{Fun}(\mathcal{I},\mathcal{C})}(F, \mathrm{const}\, y)\,.$$

Informally, an extension of F to a functor $F^{\triangleright}\colon \mathcal{I}^{\triangleright} \to \mathcal{C}$ that sends the tip $* \in \mathcal{C}^{\triangleright}$ to y is the same as a natural transformation $F \Rightarrow \operatorname{const} y$. If F admits a colimit, such a natural transformation is the same as a morphsm $\operatorname{colim}_{i \in \mathcal{I}} F(i) \to y$, and the right-hand side above is equivalent to $\operatorname{Hom}_{\mathcal{C}}(\operatorname{colim}_{i \in \mathcal{I}} F(i), y)$.

6.52. Definition. — Let κ be a regular cardinal and let $\mathcal{J}, \mathcal{C}$ be ∞-categories.

(*a*) $\mathcal{J}$ is called κ-*filtered* if every functor $\mathcal{I} \to \mathcal{J}$ from an essentially κ-small ∞-category extends to a functor $\mathcal{I}^{\triangleright} \to \mathcal{J}$ from the cocone under $\mathcal{C}$, or in other words, if the restriction $\operatorname{Fun}(\mathcal{I}^{\triangleright}, \mathcal{J}) \to \operatorname{Fun}(\mathcal{I}, \mathcal{J})$ is essentially surjective. In the case $\kappa = \aleph_0$, we usually just say $\mathcal{J}$ is *filtered*.

(*b*)   A colimit over a functor $F\colon \mathcal{J} \to \mathcal{C}$ is called κ-*filtered* if $\mathcal{J}$ is κ-filtered, and *filtered* if $\mathcal{J}$ is filtered.

(*c*)   An object $x \in \mathcal{C}$ is called κ-*compact* or *compact* if $\operatorname{Hom}_{\mathcal{C}}(x, -)\colon \mathcal{C} \to \operatorname{An}$ commutes with κ-filtered or filtered colimits, respectively.

6.53. Remark. — We'll explain why Lurie's definition of κ-filteredness in [L-HTT, Definition 5.3.1.7] is equivalent to ours. Let $\mathcal{J}$ be a κ-filtered quasi-category as in Definition 6.52(*a*). Furthermore, let $\mathcal{I}$ be an essentially κ-small quasi-category and choose the simplicial model $\mathcal{I} \star \{1\}$ for $\mathcal{I}^{\triangleright}$ (as Lurie does). Then any functor $\mathcal{I} \to \mathcal{J}$ can not only be extended to $\mathcal{I}^{\triangleright} \to \mathcal{J}$ up to equivalence, but even on the nose. The reason is that $\mathcal{I} \to \mathcal{I}^{\triangleright}$ is a cofibration and thus $\operatorname{core} \operatorname{F}(\mathcal{I}^{\triangleright}, \mathcal{J}) \to \operatorname{core} \operatorname{F}(\mathcal{I}, \mathcal{J})$ has lifting against $\{0\} \to \Delta^1$ by claim ($\boxtimes$) in the proof of Theorem 4.6. Then Lemma 6.48 easily implies that Lurie's definition of κ-filteredness is equivalent to ours in the case where κ is uncountable.

If $\kappa = \aleph_0$, then Lemma 6.48 shows that any filtered ∞-category $\mathcal{J}$ in the sense of Definition 6.52(*a*) is also filtered in Lurie's sense. The converse is true as well, but not as obvious (at least to me), since I don't know if the converse of the additional assertion in Lemma 6.48 is true (I'd guess it's not). So here's a different argument: If $\mathcal{J}$ is filtered in Lurie's sense, then $\operatorname{colim}\colon \operatorname{Fun}(\mathcal{J}, \operatorname{An}) \to \operatorname{An}$ preserves finite limits (by [L-HTT, Proposition 5.3.3.3] or by observing that the proof of Theorem 6.54 still goes through). Hence Theorem 6.54, which we'll prove next, implies that $\mathcal{J}$ is filtered in our sense.

6.54. Theorem. — *Let κ be a regular cardinal. Then an ∞-category is κ-filtered if and only if the functor* $\operatorname{colim}\colon \operatorname{Fun}(\mathcal{J}, \operatorname{An}) \to \operatorname{An}$ *preserves κ-small limits.*

Before we can prove Theorem 6.54, we need to send four more lemmas in advance.

6.55. Lemma. — *Let κ be a regular cardinal and let $\mathcal{J}$ be a κ-filtered ∞-category. Then $|\mathcal{J}| \simeq *$. Furthermore, for every $j \in \mathcal{J}$ the slice $\mathcal{J}_{j/}$ is κ-filtered again and $\mathcal{J}_{j/} \to \mathcal{J}$ is coinitial.*

Proof sketch. To see $|\mathcal{J}| \simeq *$, unfortunately, we need to use simplicial methods. It's enough to show that every map $\sigma\colon \partial\Delta^n \to |\mathcal{J}|$ is nullhomotopic, because then the same argument as in the proof of Lemma 3.24 shows that $|\mathcal{J}| \to *$ is a trivial fibration. We'll show that for every σ there is a functor $\alpha\colon \mathcal{I} \to \mathcal{J}$ from $\aleph_0$-small ∞-category $\mathcal{I}$ such that σ factors through $|\alpha|\colon |\mathcal{I}| \to |\mathcal{J}|$. This will be enough since then σ also factors through $* \simeq |\mathcal{I}^{\triangleright}| \to |\mathcal{J}|$ by filteredness of $\mathcal{J}$. To construct α, recall that $\mathcal{J} \to |\mathcal{J}|$ can be constructed as an anodyne map into a Kan complex via Lemma 3.12. Accordingly, as simplicial sets, $|\mathcal{J}| \cong \operatorname{colim}_{i \geqslant 0} \mathcal{J}_i$, where $\mathcal{J}_0 = \mathcal{J}$ and $\mathcal{J}_{i+1}$ is obtained from $\mathcal{J}_i$ by attaching solutions to horn filling problems. All the finitely many simplices in the image of $\sigma\colon \partial\Delta^n \to |\mathcal{J}|$ must already be contained in $\mathcal{J}$ or occur in $\mathcal{J}_i$ as a solution

to some horn filling problem. If the latter is the case, all the finitely many simplices involved in that horn filling problem must already occur in $\mathcal{J}$ or in some $\mathcal{J}_k$ for $k < i$. Continuing in this way, we can trace back σ to a finite number of simplices in $\mathcal{J}$. Completing these finitely many simplices to a sub-quasi-category $\mathcal{I} \subseteq \mathcal{J}$ as in the proof of Lemma 6.48 yields the desired functor $\alpha\colon \mathcal{I} \to \mathcal{J}$.

For the other assertions, let $\mathcal{I} \to \mathcal{J}_{j/}$ be a map from an essentially κ-small ∞-category. By unravelling the respective universal properties, such a map is equivalent to a map $\mathcal{I}^{\triangleleft} \to \mathcal{J}$ sending the tip of the cone to j. Since $\mathcal{I}^{\triangleleft}$ is still essentially κ-small, we get an extension $(\mathcal{I}^{\triangleleft})^{\triangleright} \to \mathcal{J}$. Since $(\mathcal{I}^{\triangleleft})^{\triangleright} \simeq (\mathcal{I}^{\triangleright})^{\triangleleft}$, this defines a map $\mathcal{I}^{\triangleright} \to \mathcal{J}_{j/}$, proving that $\mathcal{J}_{j/}$ is κ-filtered. An analogous argument shows that $\mathcal{J}_{j/} \times_{\mathcal{J}} \mathcal{J}_{j'/}$ is κ-filtered for every $j' \in \mathcal{J}$. Hence $|\mathcal{J}_{j/} \times_{\mathcal{J}} \mathcal{J}_{j'/}| \simeq *$ by the first part. Thus $\mathcal{J}_{j/} \to \mathcal{J}$ is coinitial by Theorem 6.18(c). $\qquad\square$

6.56. Lemma. — *Let $\mathcal{D}$ be an ∞-category and $y \in \mathcal{D}$ an object.*

(a) *$\mathcal{D}_{y/} \to \mathcal{D}$ reflects and creates limits. That is, if $\alpha\colon \mathcal{I} \to \mathcal{D}_{y/}$ is a diagram such that the underlying diagram $\overline{\alpha}\colon \mathcal{I} \to \mathcal{D}_{y/} \to \mathcal{D}$ has a limit in $\mathcal{D}$, then α has a limit which is preserved under $\mathcal{D}_{y/} \to \mathcal{D}$.*

(b) *If $|\mathcal{I}| \simeq *$, then $\mathcal{D}_{y/} \to \mathcal{D}$ reflects and creates $\mathcal{I}$-shaped colimits. In particular, this applies to pushouts (since $|\Lambda_0^2| \simeq *$) and filtered colimits (by Lemma 6.55).*

(c) *In general, let $\alpha\colon \mathcal{I} \to \mathcal{D}_{y/}$ be a diagram in $\mathcal{D}_{y/}$ and let $\overline{\alpha}\colon \mathcal{I} \to \mathcal{D}_{y/} \to \mathcal{D}$ be the underlying diagram in $\mathcal{D}$. If the colimits $\operatorname{colim}_{i\in\mathcal{I}} \overline{\alpha}(i)$ and $\operatorname{colim}_{i\in\mathcal{I}} y$ as well as the pushout*

$$
\begin{array}{ccc}
\displaystyle\operatorname*{colim}_{i\in\mathcal{I}} y & \longrightarrow & \displaystyle\operatorname*{colim}_{i\in\mathcal{I}} \overline{\alpha}(i) \\
\downarrow & \ulcorner\!. & \downarrow \\
y & \longrightarrow & c
\end{array}
$$

exist in $\mathcal{D}$, then $(y \to c) \in \mathcal{D}_{y/}$ is the colimit of $\alpha\colon \mathcal{I} \to \mathcal{D}_{y/}$.

Proof sketch. For (a), first consider the case where $\mathcal{D}$ has all limits. The functor $\alpha\colon \mathcal{I} \to \mathcal{D}_{y/}$ defines a natural transformation $\operatorname{const} y \Rightarrow \overline{\alpha}$, hence a morphism $y \to \lim_{i\in\mathcal{I}} \overline{\alpha}(i)$. We claim that $(y \to \lim_{i\in\mathcal{I}} \overline{\alpha}(i))$ is the limit of α. Indeed, using Corollary 5.15 and the fact that limits commute with limits by the dual of Lemma 6.38, we immediately verify the condition from Corollary 6.16. This concludes the case where $\mathcal{D}$ has all limits. The general case can be reduced to this special case by considering a fully faithful limit-preserving functor $i\colon \mathcal{D} \to \mathcal{D}'$ into an ∞-category with all limits; for example, $\sharp_{\mathcal{D}}\colon \mathcal{D} \to \operatorname{Fun}(\mathcal{D}^{\mathrm{op}}, \mathrm{An})$ does it by Corollary 6.17.

Assertion (c) follows from Lemma 6.45, using $\operatorname{Fun}(\mathcal{I}, \mathcal{D}_{y/}) \simeq \operatorname{Fun}(\mathcal{I}, \mathcal{D})_{\operatorname{const} y/}$. To prove ($b$), first assume that $\mathcal{D}$ has all colimits. Then the assumptions from (c) are satisfied and $\operatorname{colim}_{i\in\mathcal{I}} \alpha(i)$ exists. If $|\mathcal{I}| \simeq *$, then Lemma 6.57 below implies that the canonical morphism $\operatorname{colim}_{i\in\mathcal{I}} y \to y$ is an equivalence. Hence the pushout from (c) becomes an equivalence $\operatorname{colim}_{i\in\mathcal{I}} \overline{\alpha}(i) \simeq c$. This proves ($b$) in the case where $\mathcal{D}$ has all colimits. For the general case, choose a fully faithful colimit-preserving functor $i\colon \mathcal{D} \to \mathcal{D}'$ into an ∞-category $\mathcal{D}$ with all colimits; for example, the mutilated Yoneda embedding $(\sharp_{\mathcal{D}^{\mathrm{op}}})^{\mathrm{op}}\colon (\mathcal{D}^{\mathrm{op}})^{\mathrm{op}} \to \operatorname{Fun}(\mathcal{D}, \mathrm{An})^{\mathrm{op}}$ does it by Corollary 6.17. $\quad\square$

6.57. Lemma. — *Let $\mathcal{D}$ be an ∞-category, $y \in \mathcal{D}$ an object, and $\mathcal{I}$ be an ∞-category satisfying $|\mathcal{I}| \simeq *$. Then $\operatorname{colim}_{i\in\mathcal{I}} y \simeq y$; in particular, this colimit always exists.*

Proof. Clearly, $\operatorname{const} y\colon \mathcal{I} \to \mathcal{D}$ factors through $\mathcal{I} \to |\mathcal{I}|$. This functor is coinitial by Example 6.20(c), and since $|\mathcal{I}| \simeq *$, it follows that the colimit is indeed given by y. $\qquad\square$

The following lemma is the crucial step in the proof of Theorem 6.54:

6.58. Lemma. — *The functor $\pi_0\colon \mathrm{An} \to \mathrm{Set}$ commutes with products and all colimits. The functors $\pi_1\colon \mathrm{An}_{*/} \to \mathrm{Grp}$, and $\pi_n\colon \mathrm{An}_{*/} \to \mathrm{Ab}$ for all $n \geqslant 2$ commute with products and filtered colimits.*

Proof sketch. Since $\pi_0\colon \mathrm{An} \to \mathrm{Set}$ is left adjoint to the inclusion $\mathrm{Set} \subseteq \mathrm{An}$, Lemma 6.11 shows that π_0 preserves colimits. By a simple inspection π_0 also preserves products. This immediately implies that π_n preserves products for all $n \geqslant 1$, since $\pi_n(X, x) \cong \pi_0 \mathrm{Hom}_{\mathrm{An}_{*/}}((S^n, *), (X, x))$ and $\mathrm{Hom}_{\mathrm{An}_{*/}}((S^n, *), -)\colon \mathrm{An}_{*/} \to \mathrm{An}$ preserves limits by Corollary 6.17.

The assertion about π_n needs simplicial methods (and two black boxes), unfortunately. Let $\mathcal{J}$ be a filtered ∞-category. For every ordinary category $\mathcal{C}$, we have $\mathrm{Fun}(\mathcal{J}, \mathcal{C}) \simeq \mathrm{Fun}(h(\mathcal{J}), \mathcal{C})$ by Lemma 2.14, and so $\mathcal{J}$-shaped colimits in $\mathcal{C}$ agree with $h(\mathcal{J})$-shaped colimits. It's straightforward to see that $h(\mathcal{J})$ is filtered in the usual sense. So for filtered colimits in an ordinary category we can replace the indexing diagram by an ordinary filtered category. But a stronger assertion is true, which we'll need later:

(■₁) *For every filtered ∞-category there exists a directed partially ordered set J and a coinitial functor $J \to \mathcal{J}$.*

For a proof of (■₁) see [L-HTT, Proposition 5.3.1.18] or [L-Ker, Tag 02QA] (the Kerodon proof is relatively short and only uses methods that we have already available).

Next, observe that $\pi_1\colon \mathrm{Kan}_{*/} \to \mathrm{Grp}$ and $\pi_n\colon \mathrm{Kan}_{*/} \to \mathrm{Ab}$ for $n \geqslant 2$ commute with filtered colimits in the ordinary category $\mathrm{Kan}_{*/}$. This follows essentially from the fact that $\square^n$ and $\partial\square^n$ are finite simplicial sets, using an argument as near the end of the proof of Lemma 3.12. It follows that for every filtered ∞-category $\mathcal{J}$, the functor $\mathrm{colim}\colon \mathrm{Fun}(\mathcal{J}, \mathrm{Kan}) \to \mathrm{Kan}$ sends pointwise homotopy equivalences to homotopy equivalences. Indeed, let $X_{(-)} \Rightarrow X'_{(-)}$ be a natural transformation in $\mathrm{Fun}(\mathcal{J}, \mathrm{Kan})$ such that $X_j \to X'_j$ is a homotopy equivalence for all $j \in \mathcal{J}$. We can check on homotopy groups whether $\mathrm{colim}_{j\in\mathcal{J}} X_j \to \mathrm{colim}_{j\in\mathcal{J}} X_j$ is a homotopy equivalence. By the argument above, we get a bijection on π_0. Now let $x \in \pi_0(\mathrm{colim}_{j\in\mathcal{J}} X_j)$ be a point. Since π_0 commutes with colimits, we must have $x \in \pi_0(X_{j_0})$ for some $j_0 \in \mathcal{J}$. By Lemma 6.55, we may replace $\mathcal{J}$ by $\mathcal{J}_{j_0/}$, so we may assume j_0 is initial in $\mathcal{J}$. Then $\{x\} \to X_{j_0} \to X_j$ for all $j \in \mathcal{J}$ turns $X_{(-)}$ into a functor $(X_{(-)}, x)\colon \mathcal{J} \to \mathrm{Kan}_{*/}$. The same works for $X'_{(-)}$. Since $X_{(-)} \Rightarrow X'_{(-)}$ is a pointwise homotopy equivalence and π_n commutes with filtered colimits in $\mathrm{Kan}_{*/}$, we conclude that $\pi_n(\mathrm{colim}_{j\in\mathcal{J}} X_j, x) \cong \pi_n(\mathrm{colim}_{j\in\mathcal{J}} X'_j, x)$. This finishes the proof that $\mathrm{colim}\colon \mathrm{Fun}(\mathcal{J}, \mathrm{Kan}) \to \mathrm{Kan}$ sends pointwise homotopy equivalences to homotopy equivalences. At this point, we need the second black box:

(■₂) *If J is a directed partially ordered set, then there is an equivalence of ∞-categories*

$$\mathrm{Fun}(J, \mathrm{Kan})\big[\{\text{pointwise homotopy equivalences}\}^{-1}\big] \xrightarrow{\simeq} \mathrm{Fun}(J, \mathrm{An})\,.$$

The proof of (■₂) is similar to that of Theorem 4.13: First, one defines a simplicial model structure on $\mathrm{Fun}(J, \mathrm{sSet})$ in such a way that $\mathrm{N}^\Delta((\mathrm{Fun}(J, \mathrm{sSet})^\Delta)^{\mathrm{cf}}) \simeq \mathrm{Fun}(J, \mathrm{An})$. In the proof of [L-HTT, Proposition 5.3.3.3], Lurie explains how to do this. Then one uses Remarks 4.14 and 4.15 to identify the simplicial nerve $\mathrm{N}^\Delta(\mathrm{Fun}(J, \mathrm{sSet})^{\mathrm{cf}}_\Delta)$ with the localisation above.

Now let $p\colon \mathrm{Kan} \to \mathrm{An}$ and $p_J\colon \mathrm{Fun}(J, \mathrm{Kan}) \to \mathrm{Fun}(J, \mathrm{An})$ denote the canonical functors. By Theorem 4.13 and (■₂), both p and p_J are localisations. As we've seen above, $\mathrm{colim}\colon \mathrm{Fun}(J, \mathrm{Kan}) \to \mathrm{Kan}$ sends pointwise homotopy equivalences to homotopy equivalences. Hence $p \circ \mathrm{colim}\colon \mathrm{Fun}(J, \mathrm{Kan}) \to \mathrm{An}$ factors uniquely through the localisation p_J by Lemma 4.11.

Let $c\colon \mathrm{Fun}(J, \mathrm{An}) \to \mathrm{An}$ be the induced functor; we claim that c is simply the colimit functor. To this end, consider $\mathrm{const}\colon \mathrm{Kan} \to \mathrm{Fun}(J, \mathrm{Kan})$; since it sends homotopy equivalences to pointwise homotopy equivalences, the same argument as above shows that $p_J \circ \mathrm{const}$ factors uniquely through the localisation p. That factorisation is necessarily $\mathrm{const}\colon \mathrm{An} \to \mathrm{Fun}(J, \mathrm{An})$. It's straightforward to verify that the adjunction $\mathrm{colim}\colon \mathrm{Fun}(J, \mathrm{Kan}) \rightleftarrows \mathrm{Kan} : \mathrm{const}$ descends to an adjunction $c\colon \mathrm{Fun}(J, \mathrm{An}) \rightleftarrows \mathrm{An} : \mathrm{const}$ on the localisations. Indeed, one can show using Lemma 4.11 that the unit and counit transformations as well as the triangle identities get inherited, so we may appeal to Lemma 6.5. This shows that c is left adjoint to const, hence it must be the colimit functor, as claimed.

Finally, we can finish the proof. Let $(X_{(-)}, x_{(-)})\colon \mathcal{J} \to \mathrm{An}_{*/}$ be a functor from a filtered ∞-category into pointed animae. By ($\blacksquare_1$), we may assume that $\mathcal{J} \simeq J$ is a directed partially ordered set. By Lemma 6.55, we may assume that J contains an initial object j_0. Then for every (X_j, x_j), the point $\{x_j\} \to X_j$ agrees with $\{x_{j_0}\} \to X_{j_0} \to X_j$. Since $\mathrm{An}_{*/} \to \mathrm{An}$ preserves filtered colimits by Lemma 6.56, it follows that the pointed anima $\mathrm{colim}_{j \in J}(X_j, x_j)$ is given by the unpointed colimit $\mathrm{colim}_{j \in J} X_j$ together with the point $x_{j_0} \to X_{j_0} \to \mathrm{colim}_{j \in J} X_j$. By ($\blacksquare_2$), $\mathrm{Fun}(J, \mathrm{Kan}) \to \mathrm{Fun}(J, \mathrm{An})$ is essentially surjective. So we may assume that $X_{(-)}$ comes from a functor $X_{(-)}\colon J \to \mathrm{Kan}$. As argued above, we may then as well take the colimit in Kan instead of An. So the fact that $\pi_n\colon \mathrm{An}_{*/} \to \mathrm{Set}$ preserves filtered colimits reduces to the same assertion about $\pi_n\colon \mathrm{Kan}_{*/} \to \mathrm{Set}$, which we already know. $\qquad\square$

Proof of Theorem 6.54. First assume $\mathcal{J}$ is κ-filtered. By Lemma 6.50, it's enough to show that $\mathrm{colim}\colon \mathrm{Fun}(\mathcal{J}, \mathrm{An}) \to \mathrm{An}$ preserves pullbacks and κ-small products. Using Lemma 3.19 and the five lemma (plus Remark 3.20), we can further reduce pullbacks to fibre sequences (in the sense of Definition 6.39).

Let's do κ-small products first. We have to show that for every set I of cardinality $< \kappa$, every κ-filtered ∞-category $\mathcal{J}$, and every functor $X_{(-,-)}\colon I \times \mathcal{J} \to \mathrm{An}$, the natural map

$$\mathrm{colim}_{j \in \mathcal{J}} \prod_{i \in I} X_{i,j} \longrightarrow \prod_{i \in I} \mathrm{colim}_{j \in \mathcal{J}} X_{i,j}$$

is an equivalence. This can be checked on homotopy groups. We get a bijection on π_0, since $\pi_0\colon \mathrm{An} \to \mathrm{Set}$ preserves products and colimits and in Set, κ-filtered colimits commute with κ-small products. For higher homotopy groups, fix some $x \in \pi_0\big(\mathrm{colim}_{j \in \mathcal{J}} \prod_{i \in I} X_{i,j}\big)$. Since π_0 commutes with colimits, we must have $x \in \pi_0\big(\prod_{i \in I} X_{i,j_0}\big)$ for some $j_0 \in \mathcal{J}$. By Lemma 6.55, we may replace $\mathcal{J}$ by $\mathcal{J}_{j_0/}$ to assume that j_0 is initial in $\mathcal{J}$. In this case, the composition $\{x\} \to \prod_{i \in I} X_{i,j_0} \to X_{i,j_0} \to X_{i,j}$ for all $(i,j) \in I \times \mathcal{J}$ turns $X_{(-,-)}$ into a functor $X_{(-,-)}\colon I \times \mathcal{J} \to \mathrm{An}_{*/}$. Then $\pi_n\big(\mathrm{colim}_{j \in \mathcal{J}} \prod_{i \in I} X_{i,j}, x\big) \cong \pi_n\big(\prod_{i \in I} \mathrm{colim}_{j \in \mathcal{J}} X_{i,j}, x\big)$ follows from Lemma 6.58 and the fact that κ-filtered colimits in Grp or Ab commute with κ-small products.

The case of fibre sequences is similar. Let $F_{(-)} \Rightarrow X_{(-)} \Rightarrow Y_{(-)}$ be a fibre sequence in $\mathrm{Fun}(\mathcal{J}, \mathrm{An})$; by Lemma 6.12, this is equivalent to $F_j \to X_j \to Y_j$ being a fibre sequences for every $j \in \mathcal{J}$. We must show that

$$\mathrm{colim}_{j \in \mathcal{J}} F_j \longrightarrow \mathrm{fib}\Big(\mathrm{colim}_{j \in \mathcal{J}} X_j \to \mathrm{colim}_{j \in \mathcal{J}} Y_j\Big)$$

is an equivalence. This follows from a comparison of long exact sequences, using Lemma 3.19 and the five lemma together with the fact that filtered colimits preserve exact sequences. This finishes the proof that $\mathrm{colim}\colon \mathrm{Fun}(\mathcal{J}, \mathrm{An}) \to \mathrm{An}$ preserves κ-small limits.

Conversely, assume this is the case; we must show that $\mathcal{J}$ is κ-filtered. Let $\alpha\colon \mathcal{I} \to \mathcal{J}$ be a functor from a essentially κ-small ∞-category. Consider the composition

$$\mathcal{I}^{\mathrm{op}} \xrightarrow{\alpha^{\mathrm{op}}} \mathcal{J}^{\mathrm{op}} \xrightarrow{\text{よ}_{\mathcal{J}^{\mathrm{op}}}} \mathrm{Fun}(\mathcal{J}, \mathrm{An})\,.$$

Since $\mathrm{Fun}(\mathcal{J}, \mathrm{An})$ has all limits by Lemma 6.12, we can put $E := \lim(\mathcal{I}^{\mathrm{op}} \to \mathrm{Fun}(\mathcal{J}, \mathrm{An}))$ and extend the functor above to a limit cone $(\alpha^{\mathrm{op}})^{\triangleleft}\colon (\mathcal{I}^{\mathrm{op}})^{\triangleleft} \to \mathrm{Fun}(\mathcal{J}, \mathrm{An})$. Suppose there is an object $j \in \mathcal{J}^{\mathrm{op}}$ together with a natural transformation $\eta\colon \mathrm{Hom}_{\mathcal{J}}(j, -) \Rightarrow E$ in $\mathrm{Fun}(\mathcal{J}, \mathrm{An})$. We may view $(\alpha^{\mathrm{op}})^{\triangleleft}$ as a natural transformation $\mathrm{const}\, E \Rightarrow \text{よ}_{\mathcal{J}^{\mathrm{op}}} \circ \alpha^{\mathrm{op}}$. Composing with $\mathrm{const}\, \eta\colon \mathrm{const}\, \mathrm{Hom}_{\mathcal{J}}(j, -) \Rightarrow \mathrm{const}\, E$ yields another natural transformation, which we may again view as a functor $(\beta^{\mathrm{op}})^{\triangleleft}\colon (\mathcal{I}^{\mathrm{op}})^{\triangleleft} \to \mathrm{Fun}(\mathcal{J}, \mathrm{An})$. Then $(\beta^{\mathrm{op}})^{\triangleleft}$ lands in the essential image of $\text{よ}_{\mathcal{J}^{\mathrm{op}}}$. Since the Yoneda embedding is fully faithful by Corollary 5.25, we obtain a functor $\beta^{\triangleright}\colon \mathcal{I}^{\triangleright} \to \mathcal{J}$, as desired.

So assume on the contrary that there exists no $\eta\colon \mathrm{Hom}_{\mathcal{J}}(j, -) \Rightarrow E$ as above. By Yoneda's lemma, Theorem 5.19, this implies $E(j) \simeq \emptyset$ for all $j \in \mathcal{J}$. Since initial objects are preserved under arbitrary colimits, $\mathrm{colim}_{j \in \mathcal{J}} E(j) \simeq \emptyset$. On the other hand, Lemma 6.14 implies $\mathrm{colim}_{j \in \mathcal{J}} \mathrm{Hom}_{\mathcal{J}}(j_0, j) \simeq |\mathcal{J}_{j_0/}| \simeq *$ for every $j_0 \in \mathcal{J}$. Since $\mathrm{colim}\colon \mathrm{Fun}(\mathcal{J}, \mathrm{An}) \to \mathrm{An}$ preserves κ-small limits by assumption, it follows that $\mathrm{colim}_{j \in \mathcal{J}} E(j) \simeq \lim_{i \in \mathcal{I}^{\mathrm{op}}} * \simeq *$, as terminal objects are preserved under arbitrary limits. Since $\emptyset \not\simeq *$, we get a contradiction. $\qquad\square$

§6.8. Accessible and presentable ∞-categories

We can now introduce a class of large ∞-categories that are generated by a small sub-∞-category.

6.59. Construction. — Let κ be a regular cardinal and let $\mathcal{C}$ be an essentially small ∞-category. We let $\mathrm{Ind}_{\kappa}(\mathcal{C}) \subseteq \mathrm{PSh}(\mathcal{C})$ be the full sub-∞-category spanned by those presheaves $E\colon \mathcal{C}^{\mathrm{op}} \to \mathrm{An}$ for which the unstraightening $\mathrm{Un}^{\mathrm{right}}(E)$ is κ-filtered. In the case $\kappa = \aleph_0$, we often write $\mathrm{Ind}(\mathcal{C}) := \mathrm{Ind}_{\aleph_0}(\mathcal{C})$.

Note that the Yoneda embedding $\text{よ}_{\mathcal{C}}\colon \mathcal{C} \to \mathrm{PSh}(\mathcal{C})$ factors through $\mathrm{Ind}_{\kappa}(\mathcal{C})$. Indeed, for every $x \in \mathcal{C}$, the unstraightening of $\mathrm{Hom}_{\mathcal{C}}(-, x)\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{C}$ is the right fibration $\mathcal{C}_{/x} \to \mathcal{C}$ by the dual of Example 5.5(b). Now $\mathcal{C}_{/x}$ has a terminal object $\mathrm{id}_x\colon x \to x$, hence it is κ-filtered for any κ. Indeed, composing any functor $\mathcal{I} \to \mathcal{C}_{/x}$ with $\mathrm{id}_{\mathcal{C}_{/x}} \Rightarrow \mathrm{const}\, \mathrm{id}_x$ yields an extension $\mathcal{I}^{\triangleright} \to \mathcal{C}_{/x}$, as desired. Alternatively, we could have used Theorem 6.54: Since $\mathcal{C}_{/x}$ has a terminal object, every colimit over $\mathcal{C}_{/x}$ is just given by evaluating at that object. Therefore, it follows from Lemma 6.12 that $\mathrm{colim}\colon \mathrm{Fun}(\mathcal{C}_{/x}, \mathrm{An}) \to \mathrm{An}$ preserves arbitrary limits. We'll denote the factorisation of $\text{よ}_{\mathcal{C}}$ by

$$\text{よ}_{\mathcal{C}}^{\kappa}\colon \mathcal{C} \longrightarrow \mathrm{Ind}_{\kappa}(\mathcal{C})\,.$$

If no confusion can occur, we'll usually drop the superscript and just write $\text{よ}_{\mathcal{C}}$.

More generally, we have $\mathrm{Un}^{\mathrm{right}}(E) \simeq \mathcal{C}_{/E}$ for all $E \in \mathrm{PSh}(\mathcal{C})$, so E is contained in $\mathrm{Ind}_{\kappa}(\mathcal{C})$ if and only if $\mathcal{C}_{/E}$ is filtered. Indeed, the right fibration $\mathrm{PSh}(\mathcal{C})_{/E} \to \mathrm{PSh}(\mathcal{C})$ is the unstraightening of $\mathrm{Hom}_{\mathrm{PSh}(\mathcal{C})}(-, E)\colon \mathrm{PSh}(\mathcal{C})^{\mathrm{op}} \to \mathrm{An}$. By Yoneda's lemma (combined with 5.31), we have an equivalence $E \simeq \mathrm{Hom}_{\mathrm{PSh}(\mathcal{C})}(\text{よ}_{\mathcal{C}}(-), E)$ of presheaves. Hence the unstraightening of E is the pullback of $\mathrm{PSh}(\mathcal{C})_{/E} \to \mathrm{PSh}(\mathcal{C})$ along $\text{よ}_{\mathcal{C}}\colon \mathcal{C} \to \mathrm{PSh}(\mathcal{C})$, which is $\mathcal{C}_{/E}$.

6.60. Definition. — Let κ be a regular cardinal. A (not necessarily essentially small) ∞-category $\mathcal{C}$ is κ-accessible if $\mathcal{C} \simeq \mathrm{Ind}_{\kappa}(\mathcal{C}_0)$ for some small ∞-category $\mathcal{C}_0$. We call $\mathcal{C}$ accessible if it is κ-accessible for some regular cardinal κ. We call $\mathcal{C}$ presentable if it is accessible and has all colimits.

6.61. Lemma. — *Let κ be a regular cardinal and let $\mathcal{C}$ be a small ∞-category.*

(a) *A presheaf $E \in \mathrm{PSh}(\mathcal{C})$ belongs to $\mathrm{Ind}_\kappa(\mathcal{C})$ if and only if E can be written as a κ-filtered colimit of representable presheaves. Furthermore, $\mathrm{Ind}_\kappa(\mathcal{C}) \subseteq \mathrm{PSh}(\mathcal{C})$ is closed under κ-filtered colimits.*

(b) *If $\mathcal{C}$ has κ-small colimits, then a presheaf $E \in \mathrm{PSh}(\mathcal{C})$ belongs to $\mathrm{Ind}_\kappa(\mathcal{C})$ if and only if $E \colon \mathcal{C}^{\mathrm{op}} \to \mathrm{An}$ preserves κ-small limits.*

(c) *If $\mathcal{D}$ is an ∞-category which has all κ-filtered colimits, then restriction along the Yoneda embedding induces an equivalence*

$$\sharp_{\mathcal{C}}^* \colon \mathrm{Fun}^{\kappa\text{-filt}}\big(\mathrm{Ind}_\kappa(\mathcal{C}), \mathcal{D}\big) \xrightarrow{\simeq} \mathrm{Fun}(\mathcal{C}, \mathcal{D}).$$

Here $\mathrm{Fun}^{\kappa\text{-filt}}(\mathrm{Ind}_\kappa(\mathcal{C}), \mathcal{D}) \subseteq \mathrm{Fun}(\mathrm{Ind}_\kappa(\mathcal{C}), \mathcal{D})$ is spanned by those functors that preserve κ-filtered colimits.

Proof. We begin with (*a*). By Lemma 6.31, every presheaf E can be written as a colimit of representables, with $\mathcal{C}_{/E}$ as indexing ∞-category. If $E \in \mathrm{Ind}_\kappa(\mathcal{C})$, then $\mathcal{C}_{/E}$ is κ-filtered by Construction 6.59, hence E is a κ-filtered colimit of representables. Conversely, assume E can be written as such a κ-filtered colimit, say, $E \simeq \mathrm{colim}_{j \in \mathcal{J}} \mathrm{Hom}_\mathcal{C}(-, x_j)$. Since the unstraightening $\mathrm{Un}^{\mathrm{right}} \colon \mathrm{PSh}(\mathcal{C}) \to \mathrm{Right}(\mathcal{C})$ is an equivalence of ∞-categories, it preserves colimits. Recall from Lemma 6.23(*b*) that the inclusion $\mathrm{Right}(\mathcal{C}) \subseteq \mathrm{Cat}_{\infty/\mathcal{C}}$ has a left adjoint $c \colon \mathrm{Cat}_{\infty/\mathcal{C}} \to \mathrm{Right}(\mathcal{C})$. Furthermore, by the dual of Lemma 6.56, $\mathrm{Cat}_{\infty/\mathcal{C}} \to \mathrm{Cat}_\infty$ preserves colimits. Hence a colimit in $\mathrm{Right}(\mathcal{C})$ is computed by taking the colimit in Cat_∞ and then applying c. Therefore

$$\mathrm{Un}^{\mathrm{right}}(E) \simeq c\Big(\mathrm{colim}_{j \in \mathcal{J}} \mathcal{C}_{/x_j}\Big).$$

Now a κ-filtered colimit of κ-filtered ∞-categories is κ-filtered again, which follows by combining Lemma 6.38(*b*) with the characterisation of κ-filteredness from Theorem 6.54. So $\mathrm{colim}_{j \in \mathcal{J}} \mathcal{C}_{/x_j}$ is κ-filtered. By Theorem 6.54 again it's clear that being κ-filtered is preserved under coinitial functors. Since $\mathrm{colim}_{j \in \mathcal{J}} \mathcal{C}_{/x_j} \to c(\mathrm{colim}_{j \in \mathcal{J}} \mathcal{C}_{/x_j})$ is coinitial by Lemma 6.23(*b*), we've shown that $\mathrm{Un}^{\mathrm{right}}(E)$ is κ-filtered, as desired. The same argument shows that $\mathrm{Ind}_\kappa(\mathcal{C}) \subseteq \mathrm{PSh}(\mathcal{C})$ is closed under κ-filtered colimits.

For (*b*), let's temporarily denote $\mathrm{PSh}^\kappa(\mathcal{C}) \subseteq \mathrm{PSh}(\mathcal{C})$ the full sub-∞-category of presheaves $E \colon \mathcal{C}^{\mathrm{op}} \to \mathrm{An}$ that preserve κ-small limits. Every representable presheaf preserves all limits by Corollary 6.17, in particular, κ-small ones. By Theorem 6.54, $\mathrm{PSh}^\kappa(\mathcal{C}) \subseteq \mathrm{PSh}(\mathcal{C})$ is stable under κ-filtered colimits. By (*a*), every $E \in \mathrm{Ind}_\kappa(\mathcal{C})$ is a κ-filtered colimit of representables, hence $E \in \mathrm{PSh}^\kappa(\mathcal{C})$. Conversely, assume $E \in \mathrm{PSh}^\kappa(\mathcal{C})$. To show that $\mathcal{C}_{/E}$ is κ-filtered, we claim:

(⊠) *The restricted Yoneda embedding $\sharp_\mathcal{C} \colon \mathcal{C} \to \mathrm{PSh}^\kappa(\mathcal{C})$ preserves κ-small colimits.*[(6.16)]

If $\alpha \colon \mathcal{I} \to \mathcal{C}_{/E}$ is any functor from an essentially κ-small ∞-category, and $\overline{\alpha} \colon \mathcal{I} \to \mathcal{C}_{/E} \to \mathcal{C}$ denotes the composition of α with the projection to $\mathcal{C}$, then $x := \mathrm{colim}(\overline{\alpha} \colon \mathcal{I} \to \mathcal{C})$ exists by assumption on $\mathcal{C}$. Now α corresponds to a natural transformation $\eta \colon \sharp_\mathcal{C} \circ \overline{\alpha} \Rightarrow \mathrm{const}\, E$ in $\mathrm{Fun}(\mathcal{I}, \mathrm{PSh}(\mathcal{C}))$. If (⊠) holds, then we can use the universal property of colimits to show that

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[(6.16)] Note that this is completely false for the unrestricted Yoneda embedding: $\sharp_\mathcal{C} \colon \mathcal{C} \to \mathrm{PSh}(\mathcal{C})$ preserves limits (by Corollary 6.17), but not colimits. That is why, whenever we want to choose a fully faithful colimit-preserving functor $i \colon \mathcal{C} \to \mathcal{C}'$ into an ∞-category with all colimits, we have to take the awkward construction $(\sharp_{\mathcal{C}^{\mathrm{op}}})^{\mathrm{op}} \colon (\mathcal{C}^{\mathrm{op}})^{\mathrm{op}} \to \mathrm{PSh}(\mathcal{C}^{\mathrm{op}})^{\mathrm{op}}$.

η factors uniquely through $よ_{\mathcal{C}} \circ \overline{\alpha} \Rightarrow \mathrm{const} \, よ_{\mathcal{C}}(x)$. Since $よ_{\mathcal{C}}$ is fully faithful, this yields an extension $\alpha^{\triangleright} \colon \mathcal{I} \to \mathcal{C}_{/E}$, as desired.

To prove ($\boxtimes$), let $x_{(-)} \colon \mathcal{I} \to \mathcal{C}$ be a functor from an essentially κ-small ∞-category $\mathcal{C}$ and let $E \in \mathrm{PSh}^{\kappa}(\mathcal{C})$. Then $\mathrm{Hom}_{\mathrm{PSh}(\mathcal{C})}(よ_{\mathcal{C}}(\mathrm{colim}_{i \in \mathcal{I}} x_i), E) \simeq E(\mathrm{colim}_{i \in \mathcal{I}} x_i)$ by Yoneda's lemma. Since E preserves κ-small limits (and limits in $\mathcal{C}^{\mathrm{op}}$ correspond to colimits in $\mathcal{C}$), we can use Yoneda's lemma again to see

$$E\left(\operatorname*{colim}_{i \in \mathcal{I}} x_i\right) \simeq \lim_{i \in \mathcal{I}} E(x_i) \simeq \lim_{i \in \mathcal{I}} \mathrm{Hom}_{\mathrm{PSh}(\mathcal{C})}\big(よ_{\mathcal{C}}(x_i), E\big).$$

Using Corollary 6.16, this proves ($\boxtimes$).

To prove (c), we can more or less copy the proof of Theorem 6.30: Let $F \colon \mathcal{C} \to \mathcal{D}$ be any functor. Since $\mathcal{D}$ has filtered colimits and $\mathcal{C}_{/E}$ is filtered for every $E \in \mathrm{Ind}_{\kappa}(\mathcal{C})$, the Kan extension $\mathrm{Lan}_{よ_{\mathcal{C}}^{\kappa}} F \colon \mathrm{Ind}_{\kappa}(\mathcal{C}) \to \mathcal{D}$ exists by Lemma 6.27. We must show that the Kan extension $\mathrm{Lan}_{よ_{\mathcal{C}}^{\kappa}} F$ preserves κ-filtered colimits. Let's first assume that $\mathcal{D}$ has all colimits. Consider the Kan extension $\mathrm{Lan}_{よ_{\mathcal{C}}} F \colon \mathrm{PSh}(\mathcal{C}) \to \mathcal{D}$. For formal reasons, $\mathrm{Lan}_{よ_{\mathcal{C}}} F$ is the left Kan extension of $\mathrm{Lan}_{よ_{\mathcal{C}}^{\kappa}} F$ along $\mathrm{Ind}_{\kappa}(\mathcal{C}) \subseteq \mathrm{PSh}(\mathcal{C})$. Since the latter is fully faithful, Corollary 6.29 shows $\mathrm{Lan}_{よ_{\mathcal{C}}^{\kappa}} F \simeq (\mathrm{Lan}_{よ_{\mathcal{C}}} F)|_{\mathrm{Ind}_{\kappa}(\mathcal{C})}$. Now $\mathrm{Lan}_{よ_{\mathcal{C}}} F \colon \mathrm{PSh}(\mathcal{C}) \to \mathcal{D}$ preserves colimits by Lemma 6.32 and $\mathrm{Ind}_{\kappa}(\mathcal{C}) \subseteq \mathrm{PSh}(\mathcal{C})$ preserves κ-filtered colimits by (a), so $\mathrm{Lan}_{よ_{\mathcal{C}}^{\kappa}} F$ preserves κ-filtered colimits, as desired. This concludes the case where $\mathcal{D}$ has all colimits. The general case can be reduced to this as follows: As in the proof of Lemma 6.56, we can choose a fully faithful colimit-preserving functor $i \colon \mathcal{D} \to \mathcal{D}'$ into an ∞-category with all colimits. The formula from Lemma 6.27 combined with Theorem 4.5 show that the canonical natural transformation $\mathrm{Lan}_{よ_{\mathcal{C}}^{\kappa}}(i \circ F) \Rightarrow i \circ \mathrm{Lan}_{よ_{\mathcal{C}}^{\kappa}} F$ is an equivalence, and so it suffices that $\mathrm{Lan}_{よ_{\mathcal{C}}^{\kappa}}(i \circ F)$ preserves κ-filtered colimits, which we did above.

So the Kan extension functor $\mathrm{Lan}_{よ_{\mathcal{C}}^{\kappa}} \colon \mathrm{Fun}(\mathcal{C}, \mathcal{D}) \to \mathrm{Fun}(\mathrm{Ind}_{\kappa}(\mathcal{C}), \mathcal{D})$ lands in the full sub-∞-category $\mathrm{Fun}^{\kappa}(\mathrm{Ind}_{\kappa}(\mathcal{C}), \mathcal{D})$. Therefore, we obtain an adjunction

$$\mathrm{Lan}_{よ_{\mathcal{C}}} \colon \mathrm{Fun}(\mathcal{C}, \mathcal{D}) \rightleftarrows \mathrm{Fun}^{\kappa\text{-filt}}\big(\mathrm{Ind}_{\kappa}(\mathcal{C}), \mathcal{D}\big) \colon よ_{\mathcal{C}}^{*}$$

By the same arguments as in the proof of Theorem 6.30, the unit $u \colon \mathrm{id}_{\mathrm{Fun}(\mathcal{C}, \mathcal{D})} \Rightarrow よ_{\mathcal{C}}^{*} \circ \mathrm{Lan}_{よ_{\mathcal{C}}}$ is an equivalence and $よ_{\mathcal{C}}^{*}$ is conservative, hence the adjunction above is a pair of inverse equivalences by Lemma 6.33. $\square$

It's surprisingly common for an ∞-category to be accessible.

6.62. Lemma. — *Let κ be a regular cardinal and let $\mathcal{D}$ be a locally small ∞-category. Then the following are equivalent:*

(a) *$\mathcal{D}$ is of the form $\mathcal{D} \simeq \mathrm{Ind}_{\kappa}(\mathcal{C})$ for some essentially small ∞-category $\mathcal{C}$.*

(b) *$\mathcal{D}$ admits κ-filtered colimits and there exists a set S of κ-compact objects such that every object from $\mathcal{D}$ can be written as a κ-filtered colimit of objects from S.*

In this case automatically $\mathcal{D} \simeq \mathrm{Ind}_{\kappa}(\mathcal{D}^{\kappa})$, where $\mathcal{D}^{\kappa} \subseteq \mathcal{D}$ is the full sub-∞-category spanned by the κ-compact objects. Furthermore, if $\mathcal{D}$ has κ-small colimits, then there is another equivalent condition:

(c) *$\mathcal{D}$ admits κ-filtered colimits and has a set of κ-compact generators; that is, a set $S \subseteq \mathcal{D}$ of κ-compact objects such that $\mathrm{Hom}_{\mathcal{D}}(s, -) \colon \mathcal{D} \to \mathrm{An}$, $s \in S$, are jointly conservative.*

In the case where $\mathcal{D}$ has κ-small colimits and (c) holds, $\mathcal{D}$ is automatically presentable.

Proof. Assume (a) is true. Then $\mathcal{D}$ has κ-filtered colimits by Lemma 6.61(a). We claim that $S \coloneqq \{ ょ_\mathcal{C}(x) \mid x \in \mathcal{C} \}$ generates $\mathcal{D}$ under κ-filtered colimits and forms a set of compact generators in the sense of (c). The first assertion is Lemma 6.61(a). For the second assertion, Yoneda's lemma says $\operatorname{Hom}_{\mathrm{PSh}(\mathcal{C})}(ょ_\mathcal{C}(x), E) \simeq E(x)$ for every $E \in \operatorname{Ind}_\kappa(\mathcal{C})$. Hence $\operatorname{Hom}_{\mathrm{PSh}(\mathcal{C})}(ょ_\mathcal{C}(x), -)$ for $x \in \mathcal{C}$ are jointly conservative by Theorem 4.5. Furthermore, $ょ_\mathcal{C}(x)$ is κ-compact since colimits in presheaf ∞-categories are computed pointwise by Lemma 6.12 and $\operatorname{Ind}_\kappa(\mathcal{C}) \subseteq \mathrm{PSh}(\mathcal{C})$ preserves κ-filtered colimits by Lemma 6.61(a). This proves (a) $\Rightarrow$ (b) and (a) $\Rightarrow$ (c) (even without the assumption that $\mathcal{D}$ has κ-small colimits).

Now assume (b). Let's first sketch why $\mathcal{D}^\kappa$ is essentially small. We'll show that every $x \in \mathcal{D}^\kappa$ is a retract of some $s \in S$ and then leave it to you to verify that S can't have "too many" retracts in the locally small ∞-category $\mathcal{D}$. Write $x \simeq \operatorname{colim}_{j \in \mathcal{J}} s_j$ for some $s_j \in S$ and some κ-filtered ∞-category $\mathcal{J}$. Since x is κ-compact and π_0 commutes with colimits by Lemma 6.58, we get $\operatorname{colim}_{j \in \mathcal{J}} \pi_0 \operatorname{Hom}_\mathcal{C}(x, s_j) \cong \pi_0 \operatorname{Hom}_\mathcal{D}(x, x)$. Choosing a preimage of id_x yields a morphism $x \to s_j$ for some $j \in \mathcal{J}$, which exhibits x as a retract of s_j, as desired.

By Lemma 6.61(c), the inclusion $\mathcal{D}^\kappa \subseteq \mathcal{D}$ extends uniquely to a functor $L^\kappa \colon \operatorname{Ind}_\kappa(\mathcal{D}^\kappa) \to \mathcal{D}$ that preserves κ-filtered colimits. Let's first construct a right adjoint R^κ. To this end, choose a fully faithful colimit-preserving functor $i \colon \mathcal{D} \to \mathcal{D}'$ into an ∞-category $\mathcal{D}'$ with all colimits; this can be done as in the proof of Lemma 6.61(c). Furthermore, $i \circ L^\kappa$ extends uniquely to a colimit-preserving functor $L \colon \mathrm{PSh}(\mathcal{D}^\kappa) \to \mathcal{D}'$, which has a right adjoint R by Theorem 6.30. We claim that $R \circ i \colon \mathcal{D} \to \mathrm{PSh}(\mathcal{D})$ lands in $\operatorname{Ind}_\kappa(\mathcal{D}^\kappa)$. Indeed, let $y \in \mathcal{D}$ and write $y \simeq \operatorname{colim}_{j \in \mathcal{J}} x_j$ where $\mathcal{J}$ is κ-filtered and $x_j \in \mathcal{D}^\kappa$; we could even choose $x_j \in S$. By the formula from Lemma 6.32, $R(i(y))$ is the presheaf $\operatorname{Hom}_\mathcal{D}(-, \operatorname{colim}_{j \in \mathcal{J}} x_j) \colon (\mathcal{D}^\kappa)^{\mathrm{op}} \to \mathrm{An}$. By definition of $\mathcal{D}^\kappa$, this presheaf agrees with $\operatorname{colim}_{j \in \mathcal{J}} \operatorname{Hom}_\mathcal{D}(-, x_j)$. Hence $R(i(y))$ is a κ-filtered colimit of representable presheaves and thus contained in $\operatorname{Ind}_\kappa(\mathcal{D}^\kappa)$ by Lemma 6.61(a). Thus, putting $R^\kappa \coloneqq R \circ i$, we obtain the desired adjunction $L^\kappa \colon \operatorname{Ind}_\kappa(\mathcal{D}^\kappa) \rightleftarrows \mathcal{D} \colon R^\kappa$. Moreover, our argument shows that R^κ commutes with κ-filtered colimits of objects from $\mathcal{D}^\kappa$. By inspection, the counit $c \colon L^\kappa \circ R^\kappa \Rightarrow \mathrm{id}_\mathcal{D}$ is an equivalence for objects from $\mathcal{D}^\kappa$. Since both sides commute with κ-filtered colimits of objects from $\mathcal{D}^\kappa$ and every object of $\mathcal{D}$ can be written as such a colimit, we see that c is an equivalence. An analogous argument shows that $u \colon \mathrm{id}_{\operatorname{Ind}_\kappa(\mathcal{D}^\kappa)} \Rightarrow R^\kappa \circ L^\kappa$ is an equivalence. This proves (b) $\Rightarrow$ (a).

It remains to show (c) $\Rightarrow$ (a). First observe that if $\mathcal{D}$ has κ-small and κ-filtered colimits, then $\mathcal{D}$ has all colimits. Indeed, according to Lemma 6.37, we only need to check that $\mathcal{D}$ has arbitrary coproducts. This follows from the following claim:

($\boxtimes$) *Let T be a discrete set and let $\mathcal{P}^\kappa(T) \subseteq \mathcal{P}(T)$ be the partially ordered set of all subsets $S \subseteq T$ of cardinality $|S| < \kappa$. Then $\mathcal{P}^\kappa(T)$ is κ-filtered and for every collection $(x_t)_{t \in T}$ of objects of $\mathcal{D}$ we have*

$$\operatorname*{colim}_{S \in \mathcal{P}^\kappa(T)} \coprod_{s \in S} x_s \xrightarrow{\simeq} \coprod_{t \in T} x_t \, .$$

Using Lemma 2.14, κ-filteredness of $\mathcal{P}^\kappa(T)$ reduces to a question about ordinary categories, which is easy. Now consider the tautological functor $U \colon \mathcal{P}^\kappa(T) \to \mathrm{Set}$ sending $S \mapsto S$ and let $\mathcal{U}$ be its unstraightening, which is an ordinary category and easy to describe.[(6.17)] Namely, $\mathcal{U}$ is the category of pairs (S, s), where $S \subseteq T$ is a subset and $s \in S$ is an element. Morphisms can be described as follows: Fix (S, s) and (S', s'). If $S \subseteq S'$ and $s = s'$, there exists a

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[(6.17)] Here we use that for functors into Set or $\mathrm{Grpd}^{(2)}$, Lurie's unstraightening recovers the *Grothendieck construction* from classical category theory. We've seen something similar in 5.7; in particular, compare the description of the unstraightening of $U \colon \mathcal{P}^\kappa(T) \to \mathrm{Set}$ to the unstraightening of $[S/G] \colon (\mathrm{Sch}_{/S})^{\mathrm{op}} \to \mathrm{Grpd}^{(2)}$.

unique morphism $(S, s) \to (S', s')$ in $\mathcal{U}$; otherwise $\mathrm{Hom}_{\mathcal{U}}((S, s), (S', s')) = \emptyset$. There is a tautological natural transformation $U \Rightarrow \mathrm{const}\, T$, which induces a functor $\mathcal{U} \to \mathcal{P}^\kappa(T) \times T$ on unstraightenings. Note that $\mathcal{U} \to \mathcal{P}^\kappa(T) \times T \to T$ is coinitial. Indeed, for every $t \in T$, the slice $\mathcal{U} \times_T T_{t/}$ can be identified with the sub-partially ordered set $\mathcal{P}^\kappa_t(T) \subseteq \mathcal{P}^\kappa(T)$ of those S such that $t \in S$. Then $\mathcal{P}^\kappa_t(T)$ has an initial object, namely $\{t\}$, and so $|\mathcal{P}^\kappa_t(T)| \simeq *$, whence Theorem 6.18(c) is satisfied. Thus, if $T \to \mathcal{D}$ corresponds to the collection $(x_t)_{t \in T}$, then $\mathrm{colim}(T \to \mathcal{D}) \simeq \mathrm{colim}(\mathcal{U} \to T \to \mathcal{D})$. Using Lemma 6.38(a), the right-hand side can be identified with $\mathrm{colim}_{S \in \mathcal{P}^\kappa(T)} \coprod_{s \in S} x_s$. This finishes the proof of (⊠)

Now let $S \subseteq \mathcal{D}$ be a set of κ-compact generators and let $\mathcal{C} \subseteq \mathcal{D}$ be the full sub-∞-category generated by S under κ-small colimits. Since $\mathcal{D}$ is locally small, one can verify that $\mathcal{C}$ is essentially small (there are "not too many" κ-small diagrams); we leave this to you. Since $\mathcal{D}$ has all colimits, we can apply Theorem 6.30 to see that $\mathcal{C} \subseteq \mathcal{D}$ extends uniquely to a colimit-preserving functor $L\colon \mathrm{PSh}(\mathcal{C}) \to \mathcal{D}$, which has a right adjoint R. Observe that R factors through $\mathrm{Ind}_\kappa(\mathcal{C})$. Indeed, according to Lemma 6.32, for every $y \in \mathcal{D}$, the presheaf $R(y)$ is given by $\mathrm{Hom}_{\mathcal{D}}(-, y)\colon \mathcal{C}^{\mathrm{op}} \to \mathrm{An}$. This functor preserves arbitrary limits by Corollary 6.17, in particular, κ-small ones, and so Lemma 6.61(b) implies $R(y) \in \mathrm{Ind}_\kappa(\mathcal{C})$. Restricting L, we thus obtain an adjunction $L\colon \mathrm{Ind}_\kappa(\mathcal{C}) \rightleftarrows \mathcal{D} \colon R$. Observe that R preserves κ-filtered colimits. Indeed, let $y_{(-)}\colon \mathcal{J} \to \mathcal{D}$ be a functor from a κ-filtered ∞-category. By Theorem 4.5 and Lemma 6.12, it suffices to show that $\mathrm{colim}_{j \in \mathcal{J}} \mathrm{Hom}_{\mathcal{D}}(x, y_j) \to \mathrm{Hom}_{\mathcal{D}}(x, \mathrm{colim}_{j \in \mathcal{J}} y_j)$ is an equivalence for all $x \in \mathcal{C}$. But Theorem 6.54 easily implies that κ-compact objects are closed under κ-small colimits and so x must be κ-compact, whence we get an equivalence as desired. Now we can apply the same argument as in the proof of (b) $\Rightarrow$ (a) to show that the unit $u\colon \mathrm{id}_{\mathrm{Ind}_\kappa(\mathcal{C})} \Rightarrow R \circ L$ is an equivalence. Furthermore, R is conservative. Indeed, if $\alpha\colon y \to z$ in $\mathcal{D}$ induces an equivalence $\alpha_*\colon \mathrm{Hom}_{\mathcal{D}}(-, y) \Rightarrow \mathrm{Hom}_{\mathcal{D}}(-, z)$ of presheaves, then, in particular, $\alpha_*\colon \mathrm{Hom}_{\mathcal{D}}(s, y) \to \mathrm{Hom}_{\mathcal{D}}(s, z)$ must be an equivalence for all $s \in S$. But $\mathrm{Hom}_{\mathcal{D}}(s, -)\colon \mathcal{D} \to \mathrm{An}$ for $s \in S$ are jointly conservative by assumption. Now Lemma 6.33(b) finishes the proof of the implication (c) $\Rightarrow$ (a). □

This finishes our discussion of accessibility. Next, we'll characterise presentable ∞-categories.

6.63. Lemma. — *For a locally small ∞-category $\mathcal{D}$, the following are equivalent:*

(a) *$\mathcal{D}$ is presentable.*

(b) *$\mathcal{D}$ is κ-accessible and has κ-small colimits for some regular cardinal κ.*

(c) *There exists an essentially small ∞-category $\mathcal{C}$ and an adjunction $L\colon \mathrm{PSh}(\mathcal{C}) \rightleftarrows \mathcal{D} \colon R$ such that R is fully faithful and preserves κ-filtered colimits for some regular cardinal κ.*

(d) *$\mathcal{D}$ is of the form $\mathcal{D} \simeq \mathrm{Ind}_\kappa(\mathcal{C})$ for some essentially small ∞-category $\mathcal{C}$ which has κ-small colimits.*

In this case, $\mathcal{D}^\kappa$ automatically has all κ-small colimits (so that we may choose $\mathcal{C} \simeq \mathcal{D}^\kappa$ in (d) by Lemma 6.62).

Proof. The implication (a) $\Rightarrow$ (b) is trivial and (a) $\Rightarrow$ (c) follows from the proof of Lemma 6.62. For (b) $\Rightarrow$ (d), we use Lemma 6.62 to see that $\mathcal{D} \simeq \mathrm{Ind}_\kappa(\mathcal{D}^\kappa)$. It follows easily from Theorem 6.54 and Corollary 6.17 that κ-compact objects are closed under κ-small colimits. Therefore, if $\mathcal{D}$ has all κ-small colimits, then so has $\mathcal{D}^\kappa$. This proves (b) $\Rightarrow$ (d).

For (c) $\Rightarrow$ (a), first note that $\mathcal{D}$ has all colimits. Indeed, given a diagram $\alpha\colon \mathcal{I} \to \mathcal{D}$, we can form the colimit $c \simeq \mathrm{colim}_{i \in \mathcal{I}} R(\alpha(i))$ in $\mathrm{PSh}(\mathcal{C})$. Since L preserves colimits by Lemma 6.11

and $L \circ R \simeq \mathrm{id}_{\mathcal{D}}$ by the dual of Lemma 6.33(a), we see that $L(c) \simeq \mathrm{colim}_{i \in \mathcal{I}} \alpha(i)$, as desired. Now consider the objects $L(\text{よ}_{\mathcal{C}}(x))$, where x runs through a set of representatives for every equivalence class in $\mathcal{C}$. Then $\mathrm{Hom}_{\mathcal{D}}(L(\text{よ}_{\mathcal{C}}(x)), -) \simeq \mathrm{Hom}_{\mathrm{PSh}(\mathcal{C})}(\text{よ}_{\mathcal{C}}(x), R(-))$. By Yoneda's lemma, $\mathrm{Hom}_{\mathrm{PSh}(\mathcal{C})}(\text{よ}_{\mathcal{C}}(x), -) \colon \mathrm{PSh}(\mathcal{C}) \to \mathrm{An}$ are jointly conservative, and they preserve all colimits by Lemma 6.12. Since R is fully faithful and preserves κ-filtered colimits, it follows that $\mathrm{Hom}_{\mathcal{D}}(L(\text{よ}_{\mathcal{C}}(x)), -) \colon \mathcal{D} \to \mathrm{An}$ are jointly conservative and preserve κ-filtered colimits. So the set $\{L(\text{よ}_{\mathcal{C}}(x))\}$ satisfies the conditions from Lemma 6.62(c) and it follows that $\mathcal{D}$ is presentable. This proves $(c) \Rightarrow (a)$

It remains to show $(d) \Rightarrow (a)$. We need to show that $\mathrm{Ind}_{\kappa}(\mathcal{C})$ has all colimits. We know from Lemma 6.61(a) that $\mathrm{Ind}_{\kappa}(\mathcal{C})$ has κ-filtered colimits, so by claim ($\boxtimes$) in the proof of Lemma 6.62, it's enough to show that $\mathrm{Ind}_{\kappa}(\mathcal{C})$ has κ-small colimits. By Lemma 6.50, it's enough to construct pushouts and κ-small coproducts. Also, we've seen in the proof of Lemma 6.61 that $\text{よ}_{\mathcal{C}} \colon \mathcal{C} \to \mathrm{Ind}_{\kappa}(\mathcal{C})$ preserves κ-small colimits. Since $\mathcal{C}$ itself has all κ-small colimits by assumption, we see that $\mathrm{Ind}_{\kappa}(\mathcal{C})$ has κ-small colimits of representable presheaves.

Let's first construct the coproduct $\coprod_{s \in S} y_s$ for a discrete set S of cardinality $|S| < \kappa$ and $y_s \in \mathrm{Ind}_{\kappa}(\mathcal{C})$. Write $y_s \simeq \mathrm{colim}_{j \in \mathcal{J}_s} x_{j,s}$ for some filtered ∞-category $\mathcal{J}_s$ and representable presheaves $x_{j,s}$. Observe that arbitrary products of κ-filtered ∞-categories is κ-filtered again. Furthermore, if $\mathcal{I}$ is any κ-filtered ∞-category, then $\mathrm{colim}_{i \in \mathcal{I}} y_s \simeq y_s$ by Lemma 6.57 and Lemma 6.55. Hence $\mathrm{colim}_{(i,j) \in \mathcal{I} \times \mathcal{J}_s} x_{j,s} \simeq y_s$ by Lemma 6.38. So we may replace $\mathcal{J}_s$ by $\mathcal{I} \times \mathcal{J}_s$ for any κ-filtered $\mathcal{I}$. In particular, we may replace $\mathcal{J}_s$ by $\mathcal{J} \coloneqq \prod_{s \in S} \mathcal{J}_s$ and thus we may assume that the diagrams $\mathcal{J}_s$ coincide for all $s \in S$. Then Lemma 6.38 shows

$$\mathrm{colim}_{j \in \mathcal{J}} \coprod_{s \in S} x_{j,s} \simeq \coprod_{s \in S} \mathrm{colim}_{j \in \mathcal{J}} x_{j,s} \simeq \coprod_{s \in S} y_s \,,$$

provided any of these colimits exists. But $\coprod_{s \in S} x_{j,s}$ exists for all $j \in \mathcal{J}$ because $\mathrm{Ind}_{\kappa}(\mathcal{C})$ has κ-small coproducts of representable presheaves, and then $\mathrm{colim}_{j \in \mathcal{J}} \coprod_{s \in S} x_{j,s}$ exists because $\mathrm{Ind}_{\kappa}(\mathcal{C})$ has all κ-filtered colimits. This shows that $\mathrm{Ind}_{\kappa}(\mathcal{C})$ has κ-small coproducts.

It remains to construct pushouts. Fix a span $y \leftarrow x \rightarrow z$. Let's first construct the pushout in the case where x is representable. Write $y \simeq \mathrm{colim}_{j \in \mathcal{J}} y_j$ and $z \simeq \mathrm{colim}_{k \in \mathcal{K}} z_k$, where $\mathcal{J}$ and $\mathcal{K}$ are κ-filtered and y_j, z_k are representable presheaves. Since x is representable and thus κ-compact, Lemma 6.58 implies $\pi_0 \mathrm{Hom}_{\mathrm{Ind}_{\kappa}(\mathcal{C})}(x, y) \simeq \mathrm{colim}_{j \in \mathcal{J}} \pi_0 \mathrm{Hom}_{\mathrm{Ind}_{\kappa}(\mathcal{C})}(x, y_j)$. Hence $x \to y$ factors through $x \to y_{j_0}$ for some $j_0 \in \mathcal{J}$. By Lemma 6.55, we can replace $\mathcal{J}$ by $\mathcal{J}_{j_0/}$ and thus assume that $\mathcal{J}$ contains an initial element j_0 such that $x \to y$ is induced by a map $x \to y_{j_0}$. The same argument applies to $x \to z$. Furthermore, as above, we can replace $\mathcal{J}$ and $\mathcal{K}$ by $\mathcal{J} \times \mathcal{K}$ and thus assume $\mathcal{J} = \mathcal{K}$. Finally, we have $\mathrm{colim}_{j \in \mathcal{J}} x \simeq x$ by Lemma 6.57 and Lemma 6.55. Hence, using Lemma 6.38, we can construct the desired pushout as

$$\mathrm{colim}_{j \in \mathcal{J}}(y_j \sqcup_x z_j) \simeq \mathrm{colim}_{j \in \mathcal{J}} y_j \sqcup_{\mathrm{colim}_{j \in \mathcal{J}} x} \mathrm{colim}_{j \in \mathcal{J}} z_j \simeq y \sqcup_x z \,.$$

Here $y_j \sqcup_x z_j$ exists since $\mathrm{Ind}_{\kappa}(\mathcal{C})$ has pushouts of representable presheaves, as we've noted above, and then $\mathrm{colim}_{j \in \mathcal{J}}(y_j \sqcup_x z_j)$ exists because $\mathrm{Ind}_{\kappa}(\mathcal{C})$ has κ-filtered colimits. This finishes the case where x is representable. In the general case, write $x \simeq \mathrm{colim}_{j \in \mathcal{J}} x_j$, where $\mathcal{J}$ is κ-filtered and x_j are representable presheaves. By an argument we've seen several times, $y \simeq \mathrm{colim}_{j \in \mathcal{J}} y$ and $z \simeq \mathrm{colim}_{j \in \mathcal{J}} z$. Then

$$\mathrm{colim}_{j \in \mathcal{J}}(y \sqcup_{x_j} z) \simeq \mathrm{colim}_{j \in \mathcal{J}} y \sqcup_{\mathrm{colim}_{j \in \mathcal{J}} x_j} \mathrm{colim}_{j \in \mathcal{J}} z \simeq y \sqcup_x z \,.$$

Here the pushouts $y \sqcup_{x_j} z$ exist by the representable case and then $\mathrm{colim}_{j \in \mathcal{J}}(y \sqcup_{x_j} z)$ exists because $\mathrm{Ind}_\kappa(\mathcal{C})$ has κ-filtered colimits. This finishes the proof that $\mathrm{Ind}_\kappa(\mathcal{C})$ has pushouts. □

6.64. Corollary. — *The ∞-categories* An *and* Cat_∞ *are presentable.*

Proof sketch. It's clear that An and Cat_∞ are locally small and they have all colimits by Lemma 6.14. So it suffices to check that both are accessible. In fact, we'll show that both are $\aleph_0$-accessible, by verifying the condition from Lemma 6.62(*c*). For An, it's clear that $*$ is a compact generator as $\mathrm{Hom}_{\mathrm{An}}(*, X) \simeq X$ for all $X \in \mathrm{An}$. For Cat_∞, we claim that the ∞-categories $*$ and Δ^1 are compact generators.

Let's first argue that $\mathrm{Hom}_{\mathrm{Cat}_\infty}(*, -)$ and $\mathrm{Hom}_{\mathrm{Cat}_\infty}(\Delta^1, -)$ are jointly conservative. To this end, recall from Theorem 2.24 that $\mathrm{Hom}_{\mathrm{Cat}_\infty}(*, \mathcal{C}) \simeq \mathrm{core}(\mathcal{C})$ and $\mathrm{Hom}_{\mathrm{Cat}_\infty}(\Delta^1, \mathcal{C}) \simeq \mathrm{core}\,\mathrm{Ar}(\mathcal{C})$ for every ∞-category $\mathcal{C}$. Now if $F \colon \mathcal{C} \to \mathcal{D}$ is a functor such that $\mathrm{core}(F) \colon \mathrm{core}(\mathcal{C}) \to \mathrm{core}(\mathcal{D})$ is an equivalence, then F is essentially surjective. If furthermore $\mathrm{core}\,\mathrm{Ar}(\mathcal{C}) \to \mathrm{core}\,\mathrm{Ar}(\mathcal{D})$ is an equivalence, then F is fully faithful. Indeed, for all $x, y \in \mathcal{C}$ we can write $\mathrm{Hom}_{\mathcal{C}}(x, y)$ as a pullback of $\mathrm{core}\,\mathrm{Ar}(\mathcal{C}) \to \mathrm{core}(\mathcal{C}) \times \mathrm{core}(\mathcal{C})$ by 2.11 plus the fact that $\mathrm{core} \colon \mathrm{Cat}_\infty \to \mathrm{An}$ preserves pullbacks, since it is a right adjoint by Example 6.3(*a*).[(6.18)] This proves that $\mathrm{Hom}_{\mathrm{Cat}_\infty}(*, -)$ and $\mathrm{Hom}_{\mathrm{Cat}_\infty}(\Delta^1, -)$ are jointly conservative.

We'll only sketch the argument why $*$ and Δ^1 are compact in Cat_∞. The crucial observation is that equivalences of quasi-categories are preserved under filtered colimits in the ordinary category QCat. Indeed, QCat $\subseteq$ sSet is closed under filtered colimits, because Λ_i^n and Δ^n are finite simplicial sets and so every horn filling problem in a filtered colimit can be solved at some finite stage. So filtered colimits in QCat can be computed in sSet instead. Then it's straightforward to check that a filtered colimit of fully faithful and essentially surjective maps of quasi-categories is again fully faithful and essentially surjective. Now we can use the same arguments as in the proof of Lemma 6.58 (including the black box ($\blacksquare_1$) and an analogue of ($\blacksquare_2$)) to see that filtered colimits in Cat_∞ can be computed as ordinary filtered colimits in QCat. So it remains to show that $\mathrm{colim}_{j \in J} \mathrm{core}\,\mathrm{F}(*, \mathcal{C}_j) \cong \mathrm{core}\,\mathrm{F}(*, \mathrm{colim}_{j \in \mathcal{J}} \mathcal{C}_j)$ and $\mathrm{colim}_{j \in J} \mathrm{F}(\Delta^1, \mathcal{C}_j) \cong \mathrm{core}\,\mathrm{F}(\Delta^1, \mathrm{colim}_{j \in J} \mathcal{C}_j)$ holds for every filtered category J and every diagram $\mathcal{C}_{(-)} \colon J \to \mathrm{QCat}$. This is straightforward. □

6.65. Corollary. — *If $\mathcal{D}$ is a presentable ∞-category, then for every $y \in \mathcal{D}$ the slice ∞-categories $\mathcal{D}_{y/}$ and $\mathcal{D}_{/y}$ are presentable again. Furthermore, if $\mathcal{C}$ is an essentially small ∞-category, then $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ is presentable. In particular, $\mathrm{PSh}(\mathcal{C})$ and $\mathrm{Fun}(\mathcal{C}, \mathrm{Cat}_\infty)$ are presentable.*

The same results are true for accessible ∞-categories, but this requires significantly more effort. In practice, the results about presentable ∞-categories are usually sufficient and so we refer to [L-HTT, §5.4] for the accessible case.

Proof of Corollary 6.65. It follows from Lemma 6.56 and its dual that if $\mathcal{D}$ has all colimits, then $\mathcal{D}_{y/}$ and $\mathcal{D}_{/y}$ have all colimits again. Lemma 6.12 shows the same for $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$. So it's enough to check accessibility in each case.

By Lemma 6.62(*c*), we can choose a set S of κ-compact generators for $\mathcal{D}$. Since $\mathcal{D}$ has coproducts, one easily verifies via Lemma 6.2 that $\mathcal{D}_{y/} \to \mathcal{D}$ has a left adjoint, sending $z \in \mathcal{D}$ to $(y \to y \sqcup z) \in \mathcal{D}_{y/}$. Then $\mathrm{Hom}_{\mathcal{D}_{y/}}(y \to y \sqcup s, y \to z) \simeq \mathrm{Hom}_{\mathcal{D}}(s, z)$ and so $\mathrm{Hom}_{\mathcal{D}_{y/}}(y \to y \sqcup s, -) \colon \mathcal{D}_{y/} \to \mathrm{An}$ for $s \in S$ are jointly conservative. Using the adjunction

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[(6.18)]We're also implicitly using that the pullback diagram from 2.11, which lived in simplicial sets, is also a pullback of ∞-categories. See model category fact 6.13.

property plus the fact that $\mathcal{D}_{y/} \to \mathcal{D}$ preserves κ-filtered colimits by Lemma 6.56(b), we see that every $(y \to y \sqcup s)$ is κ-compact again. So $\mathcal{D}_{y/}$ satisfies the condition from Lemma 6.62(c) and is therefore accessible.

For $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$, consider the functors $F_{x,s} \coloneqq \mathrm{Lan}_{\{x\} \to \mathcal{C}}(\mathrm{const}\, s) \colon \mathcal{C} \to \mathcal{D}$, where $s \in S$ and x runs through a set of representatives of the equivalence classes of objects in $\mathcal{C}$. These Kan extensions exist by Lemma 6.27 since $\mathcal{D}$ has all colimits. The universal property of Kan extensions shows $\mathrm{Hom}_{\mathrm{Fun}(\mathcal{C},\mathcal{D})}(F_{x,s}, G) \simeq \mathrm{Hom}_{\mathrm{Fun}(\{x\}, \mathcal{D})}(\mathrm{const}\, s, G|_{\{x\}}) \simeq \mathrm{Hom}_{\mathcal{D}}(s, G(x))$ for every functor $G \in \mathrm{Fun}(\mathcal{C}, \mathcal{D})$. Since colimits in functor categories are computed pointwise by Lemma 6.12 and s is κ-compact by assumption, it follows that $F_{x,s}$ is κ-compact. Since equivalences of functors can be detected pointwise by Theorem 4.5, it follows that $\mathrm{Hom}_{\mathrm{Fun}(\mathcal{C},\mathcal{D})}(F_{x,s}, -) \colon \mathrm{Fun}(\mathcal{C}, \mathcal{D}) \to \mathrm{An}$ for $s \in S$ and x running through all equivalence classes in $\mathcal{C}$ are jointly conservative. So $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ satisfies the condition from Lemma 6.62(c) and is therefore accessible.

For $\mathcal{D}_{/y}$, we will instead verify the condition from Lemma 6.62(b). First observe that $\mathcal{D}_{/y}^\kappa$ is essentially small. Indeed, $\mathcal{D}_{/y}^\kappa \simeq \mathcal{D}^\kappa \times_{\mathcal{D}} \mathcal{D}_{/y} \to \mathcal{D}_{/y}$ is fully faithful, hence $\mathcal{D}_{/y}^\kappa$ is locally small, because $\mathcal{D}_{/y}$ is locally small by the assumption on $\mathcal{D}$ and Corollary 5.15. So its enough to show that $\pi_0 \mathrm{core}(\mathcal{D}_{/y}^\kappa)$ is a set. This follows from $\mathcal{D}^\kappa$ being essentially small (as we've seen in the proof of Lemma 6.62) and $\mathcal{D}$ being locally small, so that there can't be "too many" equivalence classes of morphisms $z \to y$ where $z \in \mathcal{D}^\kappa$. Since $\mathcal{D}_{/y} \to \mathcal{D}$ preserves arbitrary colimits by the dual of Lemma 6.56(a) and κ-filtered colimits are preserved under pullbacks by Theorem 6.54, we can use Corollary 5.15 to show that the objects in $\mathcal{D}_{/y}^\kappa$ are κ-compact in $\mathcal{D}_{/y}$. It remains to show that they generate $\mathcal{D}_{/y}$ under κ-filtered colimits. Pick some $(z \to y) \in \mathcal{D}_{/y}$ and write $z \simeq \mathrm{colim}_{j \in \mathcal{J}} z_j$ for some κ-filtered ∞-category $\mathcal{J}$ and some diagram $z_{(-)} \colon \mathcal{J} \to \mathcal{D}^\kappa$. Composing the colimit transformation $u \colon z_{(-)} \Rightarrow \mathrm{const}\, z$ with $\mathrm{const}\, z \Rightarrow \mathrm{const}\, y$ yields a transformation $z_{(-)} \Rightarrow \mathrm{const}\, y$, which in turn defines a functor $(z_{(-)} \to y) \colon \mathcal{J} \to \mathcal{D}_{/y}$. Then $(z \to y) \simeq \mathrm{colim}_{j \in \mathcal{J}}(z_j \to y)$ in $\mathcal{D}_{/y}$, as desired. $\square$

§6.9. The adjoint functor theorem

Finally, we can state and prove the adjoint functor theorem. The original version is of course Lurie's [L-HTT, Corollary 5.5.2.9]. Our version is slightly more general and is taken from Markus Land's book [Lan21, Theorems 5.2.2 and 5.2.14], who in turn took them from [NRS20].

6.66. Theorem (Adjoint functor theorem). *Let $F \colon \mathcal{C} \to \mathcal{D}$ be a functor between locally small ∞-categories.*

(a) *Assume that $\mathcal{C}$ and $\mathcal{D}$ have all colimits and $\mathcal{C}$ is generated under colimits by an essentially small sub-∞-category $\mathcal{C}_0 \subseteq \mathcal{C}$. Then F admits a right adjoint $G \colon \mathcal{D} \to \mathcal{C}$ if and only if F preserves colimits.*

(b) *Assume that $\mathcal{C}$ and $\mathcal{D}$ have all limits, that $\mathcal{C}$ is accessible, and that for every object $y \in \mathcal{D}$ there exists a regular cardinal κ_y such that y is κ_y-compact. If there exists a regular cardinal κ such that F preserves limits as well as κ-filtered colimits, then F admits a left adjoint $G \colon \mathcal{D} \to \mathcal{C}$. The converse is true as well provided that $\mathcal{D}$ is accessible too.*

Furthermore, in both (a) *and* (b), *the conditions on $\mathcal{C}$ and $\mathcal{D}$ are automatically satisfied if $\mathcal{C}$ and $\mathcal{D}$ are presentable.*

Before we embark on the proof of Theorem 6.66, we'll draw a somewhat surprising corollary and discuss a useful supplement.

6.67. Corollary. — *Let $\mathcal{C}$ be a locally small ∞-category.*

(a) *If $\mathcal{C}$ has all colimits and is generated under colimits by an essentially small sub-∞-category $\mathcal{C}_0 \subseteq \mathcal{C}$, then $\mathcal{C}$ has all limits too. In particular, presentable ∞-categories have all limits.*

(b) *If $\mathcal{C}$ is accessible and has all limits, then $\mathcal{C}$ has all colimits too. In particular, $\mathcal{C}$ is presentable.*

Proof. Let $\mathcal{I}$ be an essentially small ∞-category. Then $\mathrm{Fun}(\mathcal{I}, \mathcal{C})$ is locally small by Remark 6.47. It follows from Lemma 6.12 that $\mathrm{const}\colon \mathcal{C} \to \mathrm{Fun}(\mathcal{I}, \mathcal{C})$ preserves all limits and colimits. In the situation of (*a*), we may apply Theorem 6.66(*a*) to see that const has a right adjoint $\lim_{\mathcal{I}}\colon \mathrm{Fun}(\mathcal{I}, \mathcal{C}) \to \mathcal{C}$, as desired.

In the situation of (*b*), we only need to check that every element of $\mathrm{Fun}(\mathcal{I}, \mathcal{C})$ is τ-compact for some sufficiently large regular cardinal τ, for then const will have a left adjoint $\mathrm{colim}_{\mathcal{I}}\colon \mathrm{Fun}(\mathcal{I}, \mathcal{C}) \to \mathcal{C}$ by Theorem 6.66(*b*). Say $\mathcal{C}$ is κ-compact. Then every $x \in \mathcal{C}$ can be written as a colimit of κ-compact objects by Lemma 6.62(*b*). If κ_x is larger than κ and the cardinality of the indexing diagram, then x will be κ_x-compact, because κ_x-compact objects are closed under κ_x-small colimits (we've seen this argument several times in the proofs of Lemmas 6.62 and 6.63). Now let $\alpha\colon \mathcal{I} \to \mathcal{C}$ be a functor. Let τ_α be a regular cardinal such that $\mathrm{TwAr}(\mathcal{C})$ is τ_α-small and $\tau_\alpha \geqslant \kappa_{\alpha(i)}$ for every $i \in \mathcal{I}$. Using that τ_α-small limits commute with τ_α-filtered colimits by Theorem 6.54 and that colimits in functor categories are computed pointwise by Lemma 6.12, the formula from Corollary 6.25 shows that α is τ_α-compact. $\square$

A useful supplement to the adjoint functor theorem is the *reflection theorem:*

6.68. Theorem (Reflection theorem). — *Let $\mathcal{D}$ be a presentable ∞-category and let $\mathcal{C} \subseteq \mathcal{D}$ be a full sub-∞-category such that $\mathcal{C}$ is closed under limits in $\mathcal{D}$ and there exists a regular cardinal κ such that $\mathcal{C}$ is closed under κ-filtered colimits in $\mathcal{D}$. Then the inclusion $\mathcal{C} \subseteq \mathcal{D}$ has a left adjoint and $\mathcal{C}$ is presentable too.* ∎

We won't prove the reflection theorem. A proof for ordinary categories can be found in Adamek and Rosicky's book [AR94, Reflection Theorem 2.48]; the ∞-categorical version was only recently proven in [RS22].[(6.19)] The only thing one has to show is that in the given situation $\mathcal{C}$ is automatically accessible. Indeed, if that's true, then $\mathcal{C}$ is presentable by Corollary 6.67(*b*).

[(6.19)]Interestingly, the proof for ordinary categories can not entirely be carried over. The step that fails is related to the following two important caveats:

(a) Let $\mathcal{C}$ be an ∞-category and suppose there are morphisms $\alpha\colon x \to y$ and $\beta\colon y \to x$ satisfying $\beta \circ \alpha \simeq \mathrm{id}_x$, so that x is a retract of y. If $\mathcal{C}$ is an ordinary category, then x can be expressed as the equaliser (and also as the coequaliser) of id_y and $\alpha \circ \beta$. However, this doesn't work in general ∞-categories—it already fails in An, and even more spectacularly in the ∞-category of spectra. We can still express x in terms of y, for example, as

$$x \simeq \mathrm{colim}\left(y \xrightarrow{\alpha \circ \beta} y \xrightarrow{\alpha \circ \beta} y \xrightarrow{\alpha \circ \beta} \cdots \right) \simeq \lim\left(\cdots \xrightarrow{\alpha \circ \beta} y \xrightarrow{\alpha \circ \beta} y \xrightarrow{\alpha \circ \beta} y \right)$$

(to see this, just observe that these diagrams can be (co)initially replaced by constant x-valued diagrams). But a finite limit or colimit will never suffice.

(b) There is a notion of monomorphism in ∞-categories (see [L-HTT, §5.5.6]) and these allow for manipulation of equalisers as in ordinary categories. But the inclusion of a retract is usually not a monomorphism! Again, this already fails in An—monomorphisms are inclusions of path components, but there are many more retracts—and even more spectacularly in Sp, where there are no monomorphisms at all.

If you'd like to read up on the proof of the ∞-categorical reflection theorem, I'd suggest you first read the proof for ordinary categories and identify where the above issues occur. Then check that all other arguments can be carried over. Finally, check out [RS22] to see how the issues can be circumvented.

Furthermore, Corollary 6.67(a) shows that Theorem 6.66(b) is applicable, producing the desired left adjoint. But proving that $\mathcal{C}$ is automatically accessible is surprisingly hard (and rather surprising altogether).

We start off the proof of Theorem 6.66 with two preparatory lemmas.

6.69. Lemma. — *Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor between ∞-categories and let $y \in \mathcal{D}$ be an object. Then y admits a right adjoint object $x \in \mathcal{C}$ under F if and only if the slice ∞-category $\mathcal{C}_{/y} \simeq \mathcal{C} \times_{\mathcal{D}} \mathcal{D}_{/y}$ has a terminal object.*

Proof sketch. Let $x \in \mathcal{C}$ be an object and $c\colon F(x) \to y$ a morphism in $\mathcal{C}$. Then x is a right adjoint object to y under F, with counit c, if and only if the composition

$$\operatorname{Hom}_{\mathcal{C}}(-, x) \overset{F}{\Longrightarrow} \operatorname{Hom}_{\mathcal{D}}\big(F(-), F(x)\big) \overset{c_*}{\Longrightarrow} \operatorname{Hom}_{\mathcal{D}}\big(F(-), y\big)$$

is an equivalence of functors. By Theorem 4.5, this can be checked on objects. So choose $x' \in \mathcal{C}$. To check that $\operatorname{Hom}_{\mathcal{C}}(x, x') \to \operatorname{Hom}_{\mathcal{D}}(F(x'), y)$, it's enough by Theorem 3.18 to check that the fibres over every $\alpha \in \operatorname{Hom}_{\mathcal{D}}(F(x'), y)$ are contractible. So fix some $\alpha\colon F(x') \to y$. Using the fact that Hom animae in pullbacks of ∞-categories are the pullbacks of the respective Hom animae (which we'll prove in more generality in Lemma 6.76(a)), one easily computes that the fibre $\operatorname{Hom}_{\mathcal{C}}(x', x) \times_{\operatorname{Hom}_{\mathcal{D}}(F(x'), y)} \{\alpha\}$ is equivalent to $\operatorname{Hom}_{\mathcal{C}_{/y}}((x', \alpha\colon F(x') \to y), (x, c\colon F(x) \to y))$. So the fibres are all contractible if and only if $(x, c\colon F(x) \to y)$ is a terminal object of $\mathcal{C}_{/y}$. $\qquad\square$

6.70. Lemma. — *Let $\mathcal{C}$ be any (possibly large) ∞-category. Then $\mathcal{C}$ has a terminal object if and only if $\operatorname{id}_{\mathcal{C}}\colon \mathcal{C} \to \mathcal{C}$ has a colimit, in which case the terminal object is that colimit.*

Proof sketch. If $y \in \mathcal{C}$ is terminal, then $\{y\} \to \mathcal{C}$ is a right adjoint, hence coinitial by Example 6.20(b). Hence $\operatorname{colim}(\operatorname{id}_{\mathcal{C}}\colon \mathcal{C} \to \mathcal{C}) \simeq y$; in particular, the colimit exists.

Conversely, assume the colimit exists, and let $u\colon \operatorname{id}_{\mathcal{C}} \Rightarrow \operatorname{const} y$ be the natural transformation exhibiting y has the colimit. We wish to prove that $\mathcal{C} \rightleftarrows \{y\}$ is an adjunction. To this end, by Lemma 6.5, it suffices to construct the unit and the counit as well as to verify the triangle identities. We take u to be our unit. The counit as well as the first triangle identity come for free since $\operatorname{Fun}(\mathcal{C}, \{y\}) \simeq *$ and $\operatorname{Fun}(\{y\}, \{y\}) \simeq *$. By a quick unravelling, the second triangle identity comes down to proving that $u_y\colon y \to y$ is the identity on y. To this end, consider u as a functor $u\colon \Delta^1 \times \mathcal{C} \to \mathcal{C}$ and consider the composition $\sigma := u \circ (\operatorname{id}_{\Delta^1} \times \mathcal{C})\colon \Delta^1 \times (\Delta^1 \times \mathcal{C}) \to \Delta^1 \times \mathcal{C} \to \mathcal{C}$. By "currying", σ corresponds to a functor $\Delta^1 \times \Delta^1 \to \operatorname{Fun}(\mathcal{C}, \mathcal{C})$, or in other words, to a commutative square in $\operatorname{Fun}(\mathcal{C}, \mathcal{C})$. By a somewhat confusing unravelling, that commutative square is

$$
\begin{array}{ccc}
\operatorname{id}_{\mathcal{C}} & \overset{u}{\Longrightarrow} & \operatorname{const} y \\
u \big\Downarrow & /\!/\!/ & \big\Downarrow \operatorname{const} u_y \\
\operatorname{const} y & \underset{\operatorname{id}_{\operatorname{const} y}}{\Longrightarrow} & \operatorname{const} y
\end{array}
$$

Thus, in the equivalence $\operatorname{Hom}_{\mathcal{C}}(y, y) \simeq \operatorname{Hom}_{\mathcal{C}}(\operatorname{colim}_{\mathcal{C}} \operatorname{id}_{\mathcal{C}}, y) \simeq \operatorname{Hom}_{\operatorname{Fun}(\mathcal{C}, \mathcal{C})}(\operatorname{id}_{\mathcal{C}}, y)$, both id_y and u_y are mapped to $u \in \operatorname{Hom}_{\operatorname{Fun}(\mathcal{C}, \mathcal{C})}(\operatorname{id}_{\mathcal{C}}, y)$. This proves $\operatorname{id}_y \simeq u_y$, as desired. $\qquad\square$

Proof sketch of Theorem 6.66(a). If F admits a right adjoint, then F preserves colimits by Lemma 6.11. Conversely, assume F preserves colimits. Adjoints can be constructed pointwise by Lemma 6.2, and thus by Lemma 6.69, it's enough to show that the slice ∞-category $\mathcal{C}_{/y} \simeq \mathcal{C} \times_{\mathcal{D}} \mathcal{D}_{/y}$ has a terminal object for every $y \in \mathcal{D}$. A straightforward generalisation of the

arguments in the proof of Corollary 6.65 shows that $\mathcal{C}_{/y}$ has again all (small) colimits and is generated under colimits by its full sub-∞-category $(\mathcal{C}_0)_{/y} \simeq \mathcal{C}_0 \times_{\mathcal{C}} \mathcal{C}_{/y}$; this is the only time we use that F preserves colimits. So we can replace $\mathcal{C}$ by $\mathcal{C}_{/y}$ and are thus reduced to showing that $\mathcal{C}$ has a terminal object.

By Lemma 6.70, we must show that $\mathrm{id}_{\mathcal{C}} \colon \mathcal{C} \to \mathcal{C}$ admits a colimit. Since $\mathcal{C}$ has all small colimits, it will be enough to show that $\mathcal{C}$ admits a coinitial functor from a small ∞-category. Note that this step requires some set-theoretic care, since it's not so clear why Theorem 6.18 would be applicable to colimits with potentially large indexing ∞-categories. This problem can be solved by considering universes, and with some more effort even in ZFC; we'll ignore it in the following.

Since $\mathcal{C}$ has all small colimits, the colimit $t \coloneqq \mathrm{colim}(\mathcal{C}_0 \to \mathcal{C})$ exists. For every $x \in \mathcal{C}$ there exists a morphism $x \to t$. Indeed, since we assume $\mathcal{C}$ to be generated under colimits by $\mathcal{C}_0$, we can write x as a colimit $x \simeq \mathrm{colim}(\mathcal{I} \to \mathcal{C}_0 \to \mathcal{C})$ and then we can consider the morphism $x \simeq \mathrm{colim}(\mathcal{I} \to \mathcal{C}_0 \to \mathcal{C}) \to \mathrm{colim}(\mathcal{C}_0 \to \mathcal{C}) \simeq t$ using functoriality of colimits, see Lemma 6.28. Now let $\mathcal{T} \subseteq \mathcal{C}$ be the full sub-∞-category spanned by t (note that $\mathcal{T}$ is not just $\{t\}$, since we include all non-identity endomorphisms of t as well). Since $\mathcal{C}$ is locally small, $\mathcal{T}$ must be essentially small. We claim that $\mathcal{T} \to \mathcal{C}$ is coinitial. To this end, we'll show that $\mathcal{T} \times_{\mathcal{C}} \mathcal{C}_{x/}$ is filtered; then Lemma 6.55 will show that the condition from Theorem 6.18(c) is satisfied. Let $\alpha \colon \mathcal{I} \to \mathcal{T} \times_{\mathcal{C}} \mathcal{C}_{x/}$ be a functor from any small ∞-category. If $\overline{\alpha} \colon \mathcal{I} \to \mathcal{T} \times_{\mathcal{C}} \mathcal{C}_{x/} \to \mathcal{C}$ denotes the underlying functor, then α is equivalently given by a natural transformation $\mathrm{const}\, x \Rightarrow \overline{\alpha}$ such that $\overline{\alpha}$ takes values in the full sub-∞-category $\mathcal{T} \subseteq \mathcal{C}$. Since $\mathcal{C}$ has small colimits, $x \simeq \mathrm{colim}_{i \in \mathcal{I}} \overline{\alpha}(i)$ exists in $\mathcal{C}$. As observed above, there exists a morphism $x \to t$. Composing the colimit transformation $\overline{\alpha} \Rightarrow \mathrm{const}\, x$ with $\mathrm{const}\, x \Rightarrow t$ yields a natural transformation $\overline{\alpha} \Rightarrow \mathrm{const}\, t$, or equivalently, a functor $\overline{\alpha}^{\triangleright} \colon \mathcal{I}^{\triangleright} \to \mathcal{C}$. By construction, $\overline{\alpha}^{\triangleright}$ takes values in the full sub-∞-category $\mathcal{T} \subseteq \mathcal{C}$. Composing with $\mathrm{const}\, x \Rightarrow \overline{\alpha}$ provides a functor $\alpha^{\triangleright} \colon \mathcal{I}^{\triangleright} \to \mathcal{T} \times_{\mathcal{C}} \mathcal{C}_{x/}$, as desired. This finishes the proof that $\mathcal{T} \times_{\mathcal{C}} \mathcal{C}_{x/}$ is filtered. □

Our proof of Theorem 6.66(b) will again be preceded by two preparatory lemmas.

6.71. Lemma ("Right adjoints preserve sufficiently filtered colimits"). — *Let $G \colon \mathcal{D} \to \mathcal{C}$ be a functor between accessible ∞-categories. If G admits a left adjoint F, then G preserves τ-filtered colimits for sufficiently large regular cardinals τ.*

Proof. Choose regular cardinals κ and λ such that $\mathcal{C}$ is κ-accessible and $\mathcal{D}$ is λ-accessible. By Lemma 6.62, we may identify $\mathcal{C}$ and $\mathcal{D}$ with $\mathrm{Ind}_{\kappa}(\mathcal{C}^{\kappa})$ and $\mathrm{Ind}_{\lambda}(\mathcal{D}^{\lambda})$, respectively. First note that for every $y \in \mathcal{D}$ there exists a regular cardinal λ_y such that y is λ_y-compact. Indeed, we may write y has a colimit of λ-compact objects, and then it suffices to choose $\lambda_y \geqslant \lambda$ sufficiently large so that the indexing diagram of the colimit is λ_y-small. Since $\mathcal{C}^{\kappa}$ is essentially small, as we've seen in the proof of Lemma 6.62, we may choose a regular cardinal $\tau \geqslant \kappa$ such that $F(x)$ is τ-compact for all $x \in \mathcal{C}^{\kappa}$. We claim that G preserves τ-filtered colimits. Since $\mathcal{C} \simeq \mathrm{Ind}_{\kappa}(\mathcal{C}^{\kappa}) \subseteq \mathrm{PSh}(\mathcal{C}^{\kappa})$, the functors $\mathrm{Hom}_{\mathcal{C}}(x, -) \colon \mathcal{C} \to \mathrm{An}$ for $x \in \mathcal{C}^{\kappa}$ are jointly conservative and preserve κ-filtered and thus also τ-filtered colimits. So it's enough to show that $\mathrm{Hom}_{\mathcal{C}}(x, G(-))$ preserves τ-filtered colimits. But $\mathrm{Hom}_{\mathcal{C}}(x, G(-)) \simeq \mathrm{Hom}_{\mathcal{D}}(F(x), -)$ and $F(x)$ is τ-compact by construction. □

6.72. Lemma. — *Let $\mathcal{C}$ be a κ-accessible ∞-category. Then $\mathcal{C}$ is also τ-accessible for every sufficiently large regular cardinal τ.*

6.73. Remark. — It's usually *not* true that a κ-accessible ∞-category $\mathcal{C}$ is τ-accessible for all $\tau > \kappa$. However, this works if $\mathcal{C}$ is presentable. Indeed, it's immediately clear from Lemma 6.62(*c*) that any set of κ-compact generators is also a set of τ-compact generators

Proof sketch of Lemma 6.72. By Lemma 6.62(*b*) will be enough to show that $\mathcal{C}$ is generated under τ-filtered colimits by $\mathcal{C}^\tau$, where τ is a sufficiently large regular cardinal that will be chosen at the end of the proof. Every $x \in \mathcal{C}$ can be written as $x \simeq \operatorname{colim}_{j \in \mathcal{J}} x_j$, where $\mathcal{J}$ is κ-filtered and $x_j \in \mathcal{C}^\kappa$. We'll rewrite this as a τ-filtered colimit of τ-compact objects. First, by ($\blacksquare_1$) in the proof of Lemma 6.58, we find a coinitial functor $J \to \mathcal{J}$ from a directed partially ordered set J. Note that J is automatically a κ-filtered ∞-category by the criterion from Theorem 6.54. We'll show that J can be written as a colimit $J \simeq \operatorname{colim}_{i \in I} J_i$ in Cat_∞, where I is a τ-filtered directed partially ordered set and $J_i \subseteq J$ are essentially τ-small κ-filtered partially ordered subsets. If we can do this, we're done. Indeed, by Lemma 6.38(*b*), we may then write $x \simeq \operatorname{colim}_{i \in I} \operatorname{colim}_{j \in J_i} x_j$. Each $\operatorname{colim}_{j \in J_i} x_j$ exists, as $\mathcal{C}$ admits κ-filtered colimits by Lemma 6.61(*a*). Furthermore, $\operatorname{colim}_{j \in J_i} x_j$ is τ-compact because each x_j is κ-compact, hence τ-compact, and τ-compact objects are stable under τ-small colimits by an easy application of Theorem 6.54.

To write J as such a colimit, let $\mathcal{P}^\tau(J)$ be the partially ordered set of subsets $S \subseteq J$ of cardinality $|S| < \tau$. Note that $\mathcal{P}^\tau(J)$ is τ-filtered as an ∞-category. Indeed, using Lemma 2.14, it's enough to show that $\mathcal{P}^\tau(J)$ is τ-filtered as an ordinary category, which is true since we can just take unions of $< \tau$ subsets of cardinality $< \tau$. Each $S \in \mathcal{P}^\tau(J)$ can be identified with the full subcategory $J[S] \subseteq J$ spanned by S and we have $J \simeq \operatorname{colim}_{S \in \mathcal{P}^\tau(J)} J[S]$ in Cat_∞. One way to prove this would be to use that filtered colimits in Cat_∞ can be computed on the level of simplicial sets (see the proof of Corollary 6.64); then the desired equivalence is straightforward. For an alternative, model-independent argument, let $\mathcal{U}$ be the unstraightening of the functor $\mathcal{P}^\tau(J) \to \mathrm{Cat}_\infty$ sending $S \mapsto J[S]$. Then $\mathcal{U}$ is an ordinary category and can be easily described explicitly. The same argument as in the proof of claim ($\boxtimes$) in the proof of Lemma 6.62 shows that $\mathcal{U} \to J$ is coinitial. By Lemma 6.14, $\operatorname{colim}_{S \in \mathcal{P}^\tau(J)} J[S]$ is a localisation of $\mathcal{U}$. Since localisations are coinitial by Example 6.20(*c*), we conclude that $\operatorname{colim}_{S \in \mathcal{P}^\tau(J)} J[S] \to J$ is coinitial too. This is not quite what we wanted, but it's enough for our purposes. Now we claim:

($\boxtimes$)   *There exists a partially ordered subset $I \subseteq \mathcal{P}^\tau(J)$ such that $J[S]$ is κ-filtered for every $S \in I$ and such that the inclusion $I \to \mathcal{P}^\tau(J)$ has a left adjoint $L \colon \mathcal{P}^\tau(J) \to I$.*

Since right adjoints are coinitial by Example 6.20(*b*), we also get $J \simeq \operatorname{colim}_{S \in I} J[S]$. Furthermore, this coinitiality implies that I is τ-filtered again, because it satisfies the criterion from Theorem 6.54. So once we know ($\boxtimes$), we're done.

For every equivalence class of functors $\alpha \colon \mathcal{I} \to J$ from an essentially κ-small ∞-category $\mathcal{I}$, choose an extension $\alpha^\triangleright \colon \mathcal{I}^\triangleright \to J$. Let $S_0 \in \mathcal{P}^\tau(J)$. Let $S_1 \subseteq J$ be obtained from S_0 by adjoining the "tip of the cone" for every $\alpha^\triangleright \colon \mathcal{I}^\triangleright \to J$ such that $\alpha \colon \mathcal{I} \to J$ factors through $J[S_0] \to J$. If τ is larger than the set of equivalence classes of essentially κ-small ∞-categories, then S_1 will have cardinality $|S_1| < \tau$ again. By transfinite induction, we can repeat this construction κ many times. The result is a subset $S_\kappa \subseteq J$ such that $|S_\kappa| < \tau$ and $J[S_\kappa]$ is κ-filtered. If we put $L(S_0) := S_\kappa$, then $L \colon \mathcal{P}^\tau(J) \to \mathcal{P}^\tau(J)$ is a functor satisfying $L \circ L = L$ (we really get an equality, not just an equivalence). Thus, if $I \subseteq \mathcal{P}^\tau(J)$ is the image of L, then an easy argument shows that $L \colon \mathcal{P}^\tau(J) \to I$ is indeed left adjoint to the inclusion (bear in mind that we're working with ordinary categories here, so constructing functors and adjunctions can be done by hand). Therefore, the conditions from ($\boxtimes$) are satisfied. $\qquad\square$

Proof sketch of Theorem 6.66(*b*). By the dual of Lemma 6.69, it's enough to show that the slice ∞-category $\mathcal{C}_{y/} \simeq \mathcal{C} \times_{\mathcal{D}} \mathcal{D}_{y/}$ has an initial object for all $y \in \mathcal{D}$. By the dual of Lemma 6.70, this is equivalent to showing that $\mathrm{id}_{\mathcal{C}_{y/}} \colon \mathcal{C}_{y/} \to \mathcal{C}_{y/}$ admits a limit. A straightforward generalisation of Lemma 6.56(*a*) shows that $\mathcal{C}_{y/}$ has all (small) limits; this argument crucially uses that F preserves limits. So it will be enough to construct an initial functor from an essentially small ∞-category into $\mathcal{C}_{y/}$.

By assumption and Lemma 6.72 we may choose a sufficiently large regular cardinal κ such that $\mathcal{C}$ is κ-accessible, F preserves κ-filtered colimits, and y is κ-compact. Let $\mathcal{T} \subseteq \mathcal{C}_{y/}$ be the full sub-∞-category spanned by those $(x, \alpha \colon y \to F(x))$ where x is κ-compact. By an easy argument, the likes of which we've seen several times by now, $\mathcal{T}$ is essentially small. We claim that for every $z \in \mathcal{C}_{y/}$ there is an element $t \in \mathcal{T}$ and a morphism $t \to z$ in $\mathcal{C}_{y/}$. If we can show this, then a similar argument as in the proof of (*a*) shows that $\mathcal{T} \to \mathcal{C}_{y/}$ is initial. Indeed, we'll show that $\mathcal{T} \times_{\mathcal{C}_{y/}} (\mathcal{C}_{y/})_{/w}$ is cofiltered for every $w \in \mathcal{C}_{y/}$, which will imply initiality by the dual of Theorem 6.18(*c*) and the dual of Lemma 6.55. So let $\alpha \colon \mathcal{I} \to \mathcal{T} \times_{\mathcal{C}_{y/}} (\mathcal{C}_{y/})_{/w}$ be a functor from a small ∞-category $\mathcal{I}$. Since $\mathcal{C}_{y/}$ admits small limits, the underlying functor $\overline{\alpha} \colon \mathcal{I} \to \mathcal{C}_{y/}$ admits a limit $z \simeq \lim_{i \in \mathcal{I}} \overline{\alpha}(i)$. Choosing a morphism $t \to z$ for some $t \in \mathcal{T}$, we get natural transformations $\mathrm{const}\, t \Rightarrow \mathrm{const}\, z \Rightarrow \overline{\alpha}$. The composition $\mathrm{const}\, t \Rightarrow \overline{\alpha}$ induces an extension $\alpha^{\triangleleft} \colon \mathcal{I}^{\triangleleft} \to \mathcal{T} \times_{\mathcal{C}_{y/}} (\mathcal{C}_{y/})_{/w}$ of α, as desired.

It remains to show our claim that for every $z \in \mathcal{C}_{y/}$ there exists a moprhism $t \to z$ for some $t \in \mathcal{T}$. Write z as a pair $(x, \beta \colon y \to F(x))$ for some $x \in \mathcal{C}$. Since $\mathcal{C}$ is κ-accessible, we can write x as a κ-filtered colimit $x \simeq \mathrm{colim}_{j \in \mathcal{J}} x_j$ for some $x_j \in \mathcal{C}^{\kappa}$. Since F preserves κ-filtered colimits, $F(x) \simeq \mathrm{colim}_{j \in \mathcal{J}} F(x_j)$. Since y is κ-compact by assumption and π_0 commutes with colimits by Lemma 6.58, the canonical map

$$\mathrm{colim}_{j \in \mathcal{J}} \pi_0 \mathrm{Hom}_{\mathcal{D}}\big(y, F(x_j)\big) \xrightarrow{\;\cong\;} \pi_0 \mathrm{Hom}_{\mathcal{D}}\big(y, F(x)\big)$$

is a bijection. Hence $\beta \colon y \to F(x)$ factors over a map $\beta_j \colon y \to F(x_j)$ for some $j \in \mathcal{J}$. Then $x_j \to \mathrm{colim}_{j \in \mathcal{J}} x_j \simeq x$ induces a morphism $(x_j, \beta_j \colon y \to F(x_j)) \to (x, \beta \colon y \to F(x))$ in $\mathcal{C}_{y/}$. Since x_j is κ-compact, we see that $(x_j, \beta_j \colon y \to F(x_j)) \in \mathcal{T}$. This proves that there exists a morphism $t \to z$ for some $t \in \mathcal{T}$ and thus we've proved that $F \colon \mathcal{C} \to \mathcal{D}$ has a left adjoint.

To prove the converse in the case where $\mathcal{C}$ and $\mathcal{D}$ are both accessible, just observe that if F admits a left adjoint, then F preserves all limits by Lemma 6.11 and also all sufficiently filtered colimits by Lemma 6.71. Finally, to show that the conditions in (*a*) and (*b*) are satisfied in the case where $\mathcal{C}$ and $\mathcal{D}$ are presentable, the only non-obvious assertion is that for every $y \in \mathcal{D}$ there exists a regular cardinal κ_y such that y is κ_y-compact. But we've seen this in the proof of Lemma 6.71 already. □

§6.10. Lurie's magic ∞-category Pr^{L}

To finish this appendix to §6, we'll introduce Lurie's ∞-category Pr^{L}. At first, it'll probably not be obvious to you why this construction is so useful, but hopefully you'll come to appreciate it more and more. Without further ado, here's the "definition".

6.74. "Definition". — The ∞-*category of presentable ∞-categories and left adjoint functors* Pr^{L} is the non-full sub-∞-category of Cat_{∞} spanned by the presentable ∞-categories and the left adjoint, or equivalently (by Theorem 6.66(*a*)), colimit-preserving functors.

As stated, "Definition" 6.74 doesn't make sense: Cat_{∞} only contains small ∞-categories, but presentable ∞-categories usually aren't small. So to make "Definition" 6.74 work, we would

need to assume two nested universes (of "small" and "large" sets) and define Pr^{L} as a non-full sub-∞-category of the ∞-category $\widehat{\mathrm{Cat}}_\infty$ of all large ∞-categories (which is neither small nor large). But there's an alternative construction of Pr^{L} that stays within ZFC; see 6.80 below. For the moment, let's work with universes and assume $\widehat{\mathrm{Cat}}_\infty$ exists. Also note that Pr^{L} is not even locally small: We have $\mathrm{Hom}_{\mathrm{Pr}^{\mathrm{L}}}(\mathcal{C}, \mathcal{D}) \simeq \mathrm{core}\,\mathrm{Fun}^{\mathrm{L}}(\mathcal{C}, \mathcal{D})$; this is usually not an essential small anima.[(6.20)]

Let's begin by studying limits and colimits in Pr^{L}. To this end, we also consider the ∞-category Pr^{R} of all presentable ∞-categories and right adjoint functors.

6.75. Lemma. — *The ∞-categories Pr^{L} and Pr^{R} have all small limits and colimits. The forgetful functors $\mathrm{Pr}^{\mathrm{L}} \to \widehat{\mathrm{Cat}}_\infty$ and $\mathrm{Pr}^{\mathrm{R}} \to \widehat{\mathrm{Cat}}_\infty$ preserve all small limits.*

To prove Lemma 6.75, we need two more technical lemmas. The first one has already been referenced several times before.

6.76. Lemma. — *Let $\mathcal{C}_{(-)} \colon \mathcal{I} \to \mathrm{Cat}_\infty$ (or $\mathcal{C}_{(-)} \colon \mathcal{I} \to \widehat{\mathrm{Cat}}_\infty$) be a diagram of ∞-categories.*

(a) *For every pair of objects $x, y \in \lim_{i \in \mathcal{I}} \mathcal{C}_i$ and their images $x_i, y_i \in \mathcal{C}_i$ under the projections $\mathrm{pr}_i \colon \lim_{i \in \mathcal{I}} \mathcal{C}_i \to \mathcal{C}_i$ there is a canonical equivalence*

$$\mathrm{Hom}_{\lim_{i \in \mathcal{I}} \mathcal{C}_i}(x, y) \xrightarrow{\simeq} \lim_{i \in \mathcal{I}} \mathrm{Hom}_{\mathcal{C}_i}(x_i, y_i).$$

(b) *Let $F \colon \mathcal{J} \to \lim_{i \in \mathcal{I}} \mathcal{C}_i$ be a functor. Assume that all compositions $\mathrm{pr}_i \circ F \colon \mathcal{J} \to \mathcal{C}_i$ admit a colimit and that these colimits are preserved under $\mathcal{C}_i \to \mathcal{C}_j$ for all morphisms $i \to j$ in $\mathcal{I}$. Then F admits a colimit and that colimit is preserved under $\mathrm{pr}_i \colon \lim_{i \in \mathcal{I}} \mathcal{C}_i \to \mathcal{C}_i$ for all $i \in \mathcal{I}$. A similar assertion is true for limits.*

Proof. Let $\mathcal{C}$ be an ∞-category and $x, y \in \mathcal{C}$. Using $\mathrm{Hom}_{\mathrm{Cat}_\infty}(-, \mathcal{C}) \simeq \mathrm{core}\,\mathrm{Fun}(-, \mathcal{C})$, we get a pullback diagram

$$
\begin{array}{ccc}
\mathrm{Hom}_{\mathcal{C}}(x, y) & \longrightarrow & \mathrm{Hom}_{\mathrm{Cat}_\infty}(\Delta^1, \mathcal{C}) \\
\downarrow & \ulcorner\!\lrcorner & \downarrow \\
\{x\} \times \{y\} & \longrightarrow & \mathrm{Hom}_{\mathrm{Cat}_\infty}(*\ *, \mathcal{C})
\end{array}
$$

in An. Now for every ∞-category $\mathcal{D}$, the functor $\mathrm{Hom}_{\mathrm{Cat}_\infty}(\mathcal{D}, -) \colon \mathrm{Cat}_\infty \to \mathrm{An}$ preserves arbitrary limits by Corollary 6.17. Applying this for $\mathcal{D} \simeq \Delta^1$ and $\mathcal{D} \simeq *\ *$ and using that pullbacks commute with limits by the dual of Lemma 6.38, we deduce (a).

For (b), we only have to show that the colimit cocones $\mathcal{J}^\triangleright \to \mathcal{C}_i$ for all $i \in \mathcal{I}$ assemble into a functor $F^\triangleright \colon \mathcal{J}^\triangleright \to \lim_{i \in \mathcal{I}} \mathcal{C}_i$. If we can do this, then (a) combined with Corollary 6.16 and the fact that limits commute with limits (by the dual of Lemma 6.38) will show that $F^\triangleright$ is a colimit cocone. To construct $F^\triangleright$, we'll show a slightly stronger assertion: Consider the slice-∞-category $(\mathrm{Cat}_\infty)_{\mathcal{J}/}$ and let $(\mathrm{Cat}_\infty)_{\mathcal{J}/}^{\mathrm{colim}}$ be the non-full sub-∞-category spanned by those objects $(\mathcal{J} \to \mathcal{C})$ that admit a colimit and those morphisms that preserve this colimit. We wish to show that $(\mathrm{Cat}_\infty)_{\mathcal{J}/}^{\mathrm{colim}}$ has all limits and that $(\mathrm{Cat}_\infty)_{\mathcal{J}/}^{\mathrm{colim}} \to (\mathrm{Cat}_\infty)_{\mathcal{J}/}$ preserves all limits. By Lemma 6.37, it's enough to check this for products and pullbacks. So we can

---

[(6.20)] However, $\mathrm{Fun}^{\mathrm{L}}(\mathcal{C}, \mathcal{D})$ is at least locally small. To see this, write $\mathcal{C} \simeq \mathrm{Ind}_\kappa(\mathcal{C}^\kappa)$ for some regular cardinal κ. Using a similar argument as in Lemma 6.81(a), $\mathrm{Fun}^{\mathrm{L}}(\mathcal{C}, \mathcal{D})$ can be identified with the full sub-∞-category of $\mathrm{Fun}(\mathcal{C}^\kappa, \mathcal{D})$ spanned by those functors that preserve κ-small colimits. This is clearly a locally small ∞-category.

reduce the construction of $F^{\triangleright}\colon \mathcal{J}^{\triangleright} \to \lim_{i\in\mathcal{I}}\mathcal{C}_i$ to the case where $\lim_{i\in\mathcal{I}}\mathcal{C}_i$ is a product or a pullback.[(6.21)]

In the product case it's clear how to construct $F^{\triangleright}\colon \mathcal{J}^{\triangleright} \to \prod_{i\in I}\mathcal{C}_i$. So let's consider a pullback $\mathcal{C}_0 \times_{\mathcal{C}_2} \mathcal{C}_1$ of $\alpha_0\colon \mathcal{C}_0 \to \mathcal{C}_2$ and $\alpha_1\colon \mathcal{C}_1 \to \mathcal{C}_2$. Choose colimit cocones $F_0^{\triangleright}\colon \mathcal{J}^{\triangleright} \to \mathcal{C}_0$ and $F_1^{\triangleright}\colon \mathcal{J}^{\triangleright} \to \mathcal{C}_1$. Choose a composition $F_2^{\triangleright} := \alpha_0 \circ F_0^{\triangleright}$ and a composition $\alpha_1 \circ F_1^{\triangleright}$. Then $F_2^{\triangleright}$ and $\alpha_1 \circ F_1^{\triangleright}$ are both colimit cocones of the given functor $\mathrm{pr}_2 \circ F\colon \mathcal{J} \to \mathcal{C}_2$. Since colimit cocones are unique up to equivalence, there must be a natural equivalence $\eta\colon F_2^{\triangleright} \overset{\simeq}{\Longrightarrow} \alpha_1 \circ F_1^{\triangleright}$. Thus, we obtain a commutative diagram

$$
\begin{array}{ccccc}
\mathcal{J}^{\triangleright} & =\!=\!= & \mathcal{J}^{\triangleright} & =\!=\!= & \mathcal{J}^{\triangleright} \\
{\scriptstyle F_0^{\triangleright}}\big\downarrow & \!/\!/\!/\! & {\scriptstyle F_2^{\triangleright}}\big\downarrow & \!/\!/\!/\! & \big\downarrow{\scriptstyle F_1^{\triangleright}} \\
\mathcal{C}_0 & \xrightarrow{\ \alpha_0\ } & \mathcal{C}_2 & \xleftarrow{\ \alpha_1\ } & \mathcal{C}_1
\end{array}
$$

in Cat_{∞} (η is precisely what makes the right square commute). This diagram constitutes a natural transformation $\mathrm{const}\,\mathcal{J}^{\triangleright} \Rightarrow \mathcal{C}_{(-)}$ in $\mathrm{Fun}(\Lambda_2^2, \mathcal{C})$, which induces the desired functor $\mathcal{J}^{\triangleright} \to \mathcal{C}_0 \times_{\mathcal{C}_2} \mathcal{C}_2$. As argued above, this is automatically a colimit cone. $\qquad\square$

6.77. Lemma. — *Let* $\mathrm{Acc} \subseteq \widehat{\mathrm{Cat}}_{\infty}$ *be the non-full sub-∞-category spanned by accessible ∞-categories and functors that preserve sufficiently filtered colimits. Then* Acc *has all limits and* $\mathrm{Acc} \to \widehat{\mathrm{Cat}}_{\infty}$ *preserves all limits.*

Proof sketch. By Lemma 6.37 we can reduce to products and pullbacks. We start with products. Let $(\mathcal{C}_i)_{i\in I}$ be an collection of accessible ∞-categories. We must to show that $\mathcal{C} := \prod_{i\in I}\mathcal{C}_i$ is accessible again and that the projections $\mathrm{pr}_i\colon \mathcal{C} \to \mathcal{C}_i$ preserve sufficiently filtered colimits. The latter is immediate from Lemma 6.76(*b*). To prove that $\mathcal{C}$ is accessible, we use Lemma 6.72 to choose a sufficiently large regular cardinal κ such that $\kappa > |I|$ and all $\mathcal{C}_i$ are κ-accessible. If $x = (x_i)_{i\in I}$ is an object of $\mathcal{C}$ such that each $x_i \in \mathcal{C}_i$ is κ-compact, then x is κ-compact too: Indeed, this follows from Lemma 6.76(*a*) and the fact that κ-filtered colimits commute with I-indexed products in An by Theorem 6.54. Now let $y = (y_i)_{i\in I}$ be another object of $\mathcal{C}$. For all $i \in I$, we can write $y_i \simeq \mathrm{colim}_{j\in\mathcal{J}_i} x_i(j)$ as a κ-filtered colimit of κ-compact objects. By the same trick as in the proof of Lemma 6.63, we can replace each $\mathcal{J}_i$ by $\mathcal{J} := \prod_{i\in I}\mathcal{J}_i$. Putting $x(j) := (x_i(j))_{i\in I} \in \mathcal{C}$ we deduce that $y \simeq \mathrm{colim}_{j\in\mathcal{J}} x(j)$ is expressible as a κ-filtered colimit of κ-compact objects by Lemma 6.76(*b*). This shows that $\mathcal{C}$ is accessible.

Now consider a pullback $\mathcal{C} := \mathcal{C}_0 \times_{\mathcal{C}_2} \mathcal{C}_1$ of accessible ∞-categories along functors $\alpha_0\colon \mathcal{C}_0 \to \mathcal{C}_2$ and $\alpha_1\colon \mathcal{C}_1 \to \mathcal{C}_2$ that preserve sufficiently filtered colimits. Again, Lemma 6.76(*b*) ensures that $\mathrm{pr}_i\colon \mathcal{C} \to \mathcal{C}_i$ preserve sufficiently filtered colimits, so it's enough to show that $\mathcal{C}$ is accessible. Choose a sufficiently large regular cardinal κ such that $\mathcal{C}_0$, $\mathcal{C}_1$, $\mathcal{C}_2$ are κ-accessible and α_0, α_1 preserve κ-filtered colimits. Then $\mathcal{C}_0^{\kappa}$ and $\mathcal{C}_1^{\kappa}$ are small ∞-categories. Choose τ large enough such that the images of $\alpha_0|_{\mathcal{C}_0^{\kappa}}\colon \mathcal{C}_0^{\kappa} \to \mathcal{C}_2$ and $\alpha_1|_{\mathcal{C}_1^{\kappa}}\colon \mathcal{C}_1^{\kappa} \to \mathcal{C}_2$ land inside τ-compact objects of $\mathcal{C}_2$. Let $\overline{\mathcal{C}}_0 \subseteq \mathcal{C}_0$ be the full sub-∞-category spanned by all retracts of objects that can be

---

[(6.21)]After constructing $F^{\triangleright}\colon \mathcal{J}^{\triangleright} \to \lim_{i\in\mathcal{I}}\mathcal{C}_i$ there's still something to show before we can conclude that $\lim_{i\in\mathcal{I}}\mathcal{C}_i$ (taken in Cat_{∞} or $(\mathrm{Cat}_{\infty})_{\mathcal{J}/}$, this doesn't matter by the dual of Lemma 6.56(*a*)) is also the limit in $(\mathrm{Cat}_{\infty})_{\mathcal{J}/}^{\mathrm{colim}}$. The problem is that $(\mathrm{Cat}_{\infty})_{\mathcal{J}/}^{\mathrm{colim}}$ is only a *non-full* sub-∞-category of $(\mathrm{Cat}_{\infty})_{\mathcal{J}/}$. But this is easily fixed. Using Lemma 4.9 and the universal property of $\lim_{i\in\mathcal{I}}\mathcal{C}_i$ in $(\mathrm{Cat}_{\infty})_{\mathcal{J}/}$, verifying the corresponding universal property in $(\mathrm{Cat}_{\infty})_{\mathcal{J}/}^{\mathrm{colim}}$ reduces to a matching of path components. For example, in the case of a product, we have to show that the projections $\mathrm{pr}_i\colon \prod_{i\in I}\mathcal{C}_i \to \mathcal{C}_i$ preserve the colimit over $\mathcal{J}$; and furthermore, that a functor $\mathcal{D} \to \prod_{i\in I}\mathcal{C}_i$ in $(\mathrm{Cat}_{\infty})_{\mathcal{J}/}$ preserves the colimit over $\mathcal{J}$ if and only if the same is true for each $\mathcal{D} \to \mathcal{C}_i$. A similar assertion would be to show for pullbacks. Both are straightforward.

written as a τ-small κ-filtered colimit of objects in $\mathcal{C}_0^\kappa$. Enlarging τ if necessary, we can ensure $\overline{\mathcal{C}}_0 \simeq \mathcal{C}_0^\tau$. Indeed, it's clear that all objects of $\overline{\mathcal{C}}_0$ are τ-compact in $\mathcal{C}_0$. Conversely, the proof of Lemma 6.72 shows that, for sufficiently large τ, every object of $\mathcal{C}_0$ can be written as a τ-filtered colimit of objects in $\overline{\mathcal{C}}_0$. By the argument in the proof of Lemma 6.62, all τ-compact objects of $\mathcal{C}_0$ must then be retracts of objects in $\overline{\mathcal{C}}_0$, as desired.

By assumption, $\alpha_0 \colon \mathcal{C}_0 \to \mathcal{C}_1$ preserves κ-filtered colimits and retracts. Furthermore, $\mathcal{C}_2^\tau$ is closed under τ-small colimits and under retracts. We deduce that for sufficiently large τ, the functor α_0 restricts to a functor $\alpha_0 \colon \mathcal{C}_0^\tau \to \mathcal{C}_2^\tau$ and likewise α_1 restricts to a functor $\alpha_1 \colon \mathcal{C}_1^\tau \to \mathcal{C}_2^\tau$. Now consider the pullback $\overline{\mathcal{C}} \coloneqq \mathcal{C}_0^\tau \times_{\mathcal{C}_2^\tau} \mathcal{C}_1^\tau$. We wish to show that $\mathcal{C} \simeq \mathrm{Ind}_\tau(\overline{\mathcal{C}})$. By construction, $\overline{\mathcal{C}}$ is a full sub-∞-category of $\mathcal{C}$, hence according to Lemma 6.62(b) it will be enough to show that every object in $\mathcal{C}$ can be written as a τ-filtered colimit of objects in $\overline{\mathcal{C}}$. So let $x \in \mathcal{C}$ and let $x_i \coloneqq \mathrm{pr}_i(x)$ be its projections to $\mathcal{C}_i$ for $i = 0, 1, 2$. We claim:

($\boxtimes_1$) $\overline{\mathcal{C}}_{/x}$ is τ-filtered.

($\boxtimes_2$) The projections $\mathrm{pr}_i \colon \overline{\mathcal{C}}_{/x} \to \mathcal{C}_{i/x_i}^\tau$ for $i = 0, 1$ are coinitial functors.

Together, these claims imply that the canonical morphism $\mathrm{colim}(s \colon \overline{\mathcal{C}}_{/x} \to \mathcal{C}) \to x$ is an equivalence, so that x can be written as a τ-filtered colimits of objects in $\overline{\mathcal{C}}$, as desired. Indeed, by definition of $\mathcal{C}$ as a pullback, the projections pr_i for $i = 0, 1$ are jointly conservative and they preserve κ-filtered colimits by Lemma 6.76(b). In particular, if ($\boxtimes_1$) holds, then pr_i preserves $\overline{\mathcal{C}}$-indexed colimits. Using ($\boxtimes_2$), we conclude

$$\mathrm{pr}_i\Big(\mathrm{colim}\big(\overline{\mathcal{C}}_{/x} \to \mathcal{C}\big)\Big) \simeq \mathrm{colim}\big(\mathcal{C}_{i/x_i}^\tau \to \mathcal{C}_i\big) \simeq x_i$$

for $i = 0, 1$, as desired.

It remains to prove the two claims. It's straightforward to check that taking slice-∞-categories commutes with pullbacks. Therefore we have a pullback diagram

$$
\begin{array}{ccc}
\overline{\mathcal{C}}_{/x} & \xrightarrow{\mathrm{pr}_1} & \mathcal{C}_{1/x_1}^\tau \\
{\scriptstyle \mathrm{pr}_0}\downarrow & \lrcorner & \downarrow{\scriptstyle \alpha_1} \\
\mathcal{C}_{0/x_0}^\tau & \xrightarrow{\alpha_0} & \mathcal{C}_{2/x_2}^\tau
\end{array}
$$

Since $\mathcal{C}_i \simeq \mathrm{Ind}_\tau(\mathcal{C}_i^\tau)$ for $i = 0, 1, 2$, the argument from Construction 6.59 shows that each $\mathcal{C}_{i/x_i}^\tau$ is τ-filtered. Furthermore, the legs of the pullback $\alpha_i \colon \mathcal{C}_{i/x_i}^\tau \to \mathcal{C}_{2/x_2}^\tau$ for $i = 0, 1$ are coinitial functors. Indeed, in general, if $F \colon \mathcal{U} \to \mathcal{V}$ is a functor of small ∞-categories, $u \in \mathrm{Ind}_\tau(\mathcal{U})$ and v is the image of u under $\mathrm{Ind}_\tau(\mathcal{U}) \to \mathrm{Ind}_\tau(\mathcal{V})$, then $\mathcal{U}_{/u} \to \mathcal{V}_{/v}$ is coinitial. To see this, observe that $\mathrm{Ind}_\tau(\mathcal{U}) \to \mathrm{Ind}_\tau(\mathcal{V})$ fits into a diagram

$$
\begin{array}{ccccc}
\mathcal{U} & \longrightarrow & \mathrm{Ind}_\tau(\mathcal{U}) & \longrightarrow & \mathrm{PSh}(\mathcal{U}) \\
{\scriptstyle F}\downarrow & & \downarrow & & \downarrow{\scriptstyle F_!} \\
\mathcal{V} & \longrightarrow & \mathrm{Ind}_\tau(\mathcal{V}) & \longrightarrow & \mathrm{PSh}(\mathcal{V})
\end{array}
$$

(this follows from Lemma 6.61(c), Theorem 6.30, and the fact that $\mathrm{Ind}_\tau(\mathcal{V}) \to \mathrm{PSh}(\mathcal{V})$ preserves τ-filtered colimits by Lemma 6.61(a)). By Construction 6.59, $\mathcal{U}_{/u} \to \mathcal{U}$ is the right fibration associated to the presheaf $u \colon \mathcal{U}^{\mathrm{op}} \to \mathrm{An}$. Hence coinitiality of $\mathcal{U}_{/u} \to \mathcal{V}_{/v}$ follows from Lemma 6.23(c).

To prove ($\boxtimes_1$), let $\varphi\colon \mathcal{I} \to \overline{\mathcal{C}}_{/x}$ be any functor from a τ-small ∞-category. Since $\mathcal{C}^\tau_{i/x_i}$ are τ-filtered, we may extend $\mathrm{pr}_i \circ \varphi$ to functors $\varphi_i\colon \mathcal{I}^\triangleright \to \mathcal{C}^\tau_{i/x_i}$ for $i = 0, 1$. The problem is that we don't necessarily have an equivalence $\alpha_1 \circ \varphi_0 \simeq \alpha_0^\tau \circ \varphi_1$, so φ_0 and φ_1 can't necessarily be combined into an extension $\varphi^\triangleright\colon \mathcal{I}^\triangleright \to \overline{\mathcal{C}}_{/x}$ of φ. Instead, $\alpha_1 \circ \varphi_0$ and $\alpha_0 \circ \varphi_1$ define a functor $\psi\colon \mathcal{I}^\triangleright \sqcup_{\mathcal{I}} \mathcal{I}^\triangleright \to \mathcal{C}^\tau_{2/x_2}$. Since the target is τ-filtered, we can extend ψ to a functor $\psi^\triangleright\colon (\mathcal{I}^\triangleright \sqcup_{\mathcal{I}} \mathcal{I}^\triangleright)^\triangleright \to \mathcal{C}^\tau_{2/x_2}$. Restricting along $\mathcal{I} \to \mathcal{I}^\triangleright \sqcup_{\mathcal{I}} \mathcal{I}^\triangleright$ yields a functor $\varphi_2\colon \mathcal{I}^\triangleright \to \mathcal{C}^\tau_{2/x_2}$ which extends $\mathrm{pr}_2 \circ \varphi$. The problem with this extension is that it doesn't necessarily factor through $\mathcal{C}^\tau_{0/x_0}$ and $\mathcal{C}^\tau_{1/x_1}$. So it seems that no matter what we do, we either end up with φ_1 and φ_2, which factor as desired but need not be compatible, or with ψ, which is compatible (with itself) but need not factor as desired. The crucial idea is now that we can get the best of both worlds by iterating!

To make this work, first observe that $\psi^\triangleright$ defines natural transformations $\eta_0\colon \alpha_1 \circ \varphi_0 \Rightarrow \eta_2$ and $\eta_1\colon \alpha_0 \circ \varphi_1 \Rightarrow \varphi_2$. For $i = 0, 1, 2$ let y_i be the image of the tip $* \in \mathcal{I}^\triangleright$ under φ_i. Then $\psi^\triangleright$ defines morphisms $\alpha_i(y_i) \to y_2$ for $i = 0, 1$ and the natural transformations η_i are necessarily given by postcomposition with these morphisms. Since $\alpha_i\colon \mathcal{C}^\tau_{i/x_i} \to \mathcal{C}^\tau_{2/x_2}$ is coinitial, we find $y_i' \in \mathcal{C}^\tau_{i/x_i}$ and a morphism $y_2 \to \alpha_i(y_i')$ in $\mathcal{C}^\tau_{2/x_2}$ for $i = 0, 1$. Applying the same argument to $\alpha_i\colon (\mathcal{C}^\tau_{i/x_i})_{y_i/} \to (\mathcal{C}^\tau_{2/x_2})_{\alpha_i(y_i)/}$, which is still coinitial by Lemma 6.55, we can even arrange that a morphism $y_i \to y_i'$ exists in such a way that its image under α_i is the composition $\alpha_i(y_i) \to y_2 \to \alpha_i(y_i')$. Postcomposition with $y_i \to y_i'$ yields another extension $\varphi_i'\colon \mathcal{I}^\triangleright \to \mathcal{C}^\tau_{i/x_i}$ of $\mathrm{pr}_i \circ \varphi$ together with a natural transformation $\vartheta_i\colon \varphi_i \Rightarrow \varphi_i'$. So now we are in the same situation as before, except that φ_1, φ_2 have been replaced by φ_1', φ_2'. Which means we can iterate!

Iterating the construction $\varphi_i \mapsto \varphi_i'$ transfinitely[(6.22)] many times, we obtain $\varphi_i^{(\gamma)}\colon \mathcal{I}^\triangleright \to \mathcal{C}^\tau_{i/x_i}$ for all ordinals $\gamma < \kappa$. Now define

$$\varphi_i^\triangleright := \operatorname*{colim}_{\gamma < \kappa} \varphi_i^{(\gamma)}\colon \mathcal{I}^\triangleright \to \mathcal{C}^\tau_{i/x_i}.$$

This colimit exists in $\mathrm{Fun}(\mathcal{I}^\triangleright, \mathcal{C}^\tau_{i/x_i})$ since $\{\gamma < \kappa\}$ is κ-filtered and τ-small. Furthermore, using that $\alpha_i\colon \mathcal{C}^\tau_{i/x_i} \to \mathcal{C}^\tau_{2/x_2}$ preserve κ-filtered τ-small colimits, it's straightforward to check that now indeed $\alpha_1 \circ \varphi_0^\triangleright \simeq \alpha_0 \circ \varphi_1^\triangleright$ holds, so we obtain our desired extension $\varphi^\triangleright\colon \mathcal{I}^\triangleright \to \overline{\mathcal{C}}_{/x}$. This finishes the proof of ($\boxtimes_1$).

To show that $\mathrm{pr}_0\colon \overline{\mathcal{C}}_{/x} \to \mathcal{C}^\tau_{0/x_0}$ is coinitial, we use Theorem 6.18(c): For all $z_0 \in \mathcal{C}^\tau_{0/x_0}$, we wish to show that $\overline{\mathcal{C}}_{/x} \times_{\mathcal{C}^\tau_{0/x_0}} (\mathcal{C}^\tau_{0/x_0})_{z_0/}$ is weakly contractible. We'll even show that it is τ-filtered, which is enough by Lemma 6.55. To this end, consider any functor

$$\varphi\colon \mathcal{I} \longrightarrow \overline{\mathcal{C}}_{/x} \times_{\mathcal{C}^\tau_{0/x_0}} \big(\mathcal{C}^\tau_{0/x_0}\big)_{z_0/}$$

from a τ-small ∞-category. Equivalently, φ is a functor $\overline{\varphi}\colon \mathcal{I} \to \overline{\mathcal{C}}_{/x}$ together with an extension of $\mathrm{pr}_0 \circ \overline{\varphi}$ to a functor $\varphi_0^\triangleleft\colon \mathcal{I}^\triangleleft \to \mathcal{C}^\tau_{0/x_0}$ that sends the tip $* \in \mathcal{I}^\triangleleft$ to z_0. By ($\boxtimes_1$), we find an extension $\overline{\varphi}^\triangleright\colon \mathcal{I}^\triangleright \to \overline{\mathcal{C}}_{/x}$. The pushout of $\mathrm{pr}_0 \circ \overline{\varphi}^\triangleright$ and $\overline{\varphi}_0^\triangleleft$ defines a functor $(\varphi_0^\triangleright)^\triangleleft\colon (\mathcal{I}^\triangleright)^\triangleleft \to \mathcal{C}^\tau_{0/x_0}$. Together, $\overline{\varphi}^\triangleright$ and $(\varphi_0^\triangleright)^\triangleleft$ define the desired extension $\varphi^\triangleright$ of φ. This shows that $\mathrm{pr}_0\colon \overline{\mathcal{C}}_{/x} \to \mathcal{C}^\tau_{0/x_0}$ is coinitial. The argument for $\mathrm{pr}_1\colon \overline{\mathcal{C}}_{/x} \to \mathcal{C}^\tau_{1/x_1}$ is analogous and thus we've shown ($\boxtimes_2$). $\square$

Proof sketch of Lemma 6.75. The equivalence from Corollary 6.8(b) (applied to $\widehat{\mathrm{Cat}}_\infty$ rather than Cat_∞) restricts to an equivalence $\mathrm{Pr}^{\mathrm{L}} \simeq (\mathrm{Pr}^{\mathrm{R}})^{\mathrm{op}}$. In particular, colimits in Pr^{L} are just

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[(6.22)]The above construction only takes care of the transfinite induction step for successor ordinals. But the case of limit ordinals is entirely analogous.

limits in Pr^{R} and vice versa. So it suffices to study limits in either case. By Lemma 6.37 we can reduce to products and pullbacks.

We start with products. Let $(\mathcal{C}_i)_{i \in I}$ be an collection of presentable ∞-categories and let $\mathcal{C} := \prod_{i \in I} \mathcal{C}_i$. From Lemma 6.77 we know that $\mathcal{C}$ is accessible and from Lemma 6.76(b) (and its dual) we know that $\mathcal{C}$ has all colimits and that the projections $\mathrm{pr}_i \colon \mathcal{C} \to \mathcal{C}_i$ preserve all limits and colimits. It's then straightforward to verify that $\mathcal{C}$ satisfies the universal property of the product in both Pr^{L} and Pr^{R} (compare this to footnote (6.21) on page page 124).

It remains to do pullbacks. If $\mathcal{C}_0 \to \mathcal{C}_2 \leftarrow \mathcal{C}_1$ are functors in Pr^{L} or Pr^{R}, then they preserve sufficiently filtered colimits (for Pr^{R} this needs Lemma 6.71) and so the pullback $\mathcal{D} := \mathcal{C}_0 \times_{\mathcal{C}_2} \mathcal{C}_1$ is accessible by Lemma 6.77. In the case of Pr^{L}, Lemma 6.76(b) shows that $\mathcal{C}$ has all colimits, so it is presentable, and that the projections $\mathrm{pr}_i \colon \mathcal{C} \to \mathcal{C}_i$ preserve all colimits, so they are functors in Pr^{L}. The required universal property in Pr^{L} is then straightforward to verify. In the case of Pr^{R}, Lemma 6.76(b) shows that $\mathcal{C}$ has all limits, hence it is presentable by Corollary 6.67(b), and that the projections $\mathrm{pr}_i \colon \mathcal{C} \to \mathcal{C}_i$ preserve all limits and sufficiently filtered colimits, so they are functors in Pr^{R} by Theorem 6.66(b). Again, the required universal property in Pr^{R} is then straightforward to verify. $\qquad\square$

Let us now explain how to construct Pr^{L} in ZFC. To this end, we need to introduce a variant of Pr^{L}. As we'll see in Theorem 6.83, this variant has a truly mindblowing property, which makes it quite interesting on its own.

6.78. Definition. — Let κ be a regular cardinal.

(a) A presentable ∞-category is called κ-*compactly generated* if it is κ-accessible, that is, of the form $\mathcal{C} \simeq \mathrm{Ind}_\kappa(\mathcal{C}^\kappa)$, where $\mathcal{C}^\kappa \subseteq \mathcal{C}$ is the full sub-∞-category spanned by the κ-compact objects.

(b) We let $\mathrm{Pr}^{\mathrm{L}}_\kappa$ be the non-full sub-∞-category of Pr^{L} spanned by the κ-compactly generated ∞-categories and those left adjoint functors that also preserve κ-compact objects.

6.79. Lemma. — *A left adjoint functor $F \colon \mathcal{C} \to \mathcal{D}$ between presentable ∞-categories preserves κ-compact objects if the right adjoint $G \colon \mathcal{D} \to \mathcal{C}$ preserves κ-filtered colimits. The converse is true as well, provided $\mathcal{C}$ is κ-compactly generated.*

Proof. Let $x \in \mathcal{C}$ be κ-compact and let $y_{(-)} \colon \mathcal{I} \to \mathcal{D}$ be a κ-filtered diagram. If G preserves κ-filtered colimits, then

$$\mathrm{Hom}_{\mathcal{D}}\Big(F(x), \operatorname*{colim}_{i \in \mathcal{I}} y_i\Big) \simeq \mathrm{Hom}_{\mathcal{C}}\Big(x, \operatorname*{colim}_{i \in \mathcal{I}} G(y_i)\Big)$$
$$\simeq \operatorname*{colim}_{i \in \mathcal{I}} \mathrm{Hom}_{\mathcal{C}}\big(x, G(y_i)\big)$$
$$\simeq \operatorname*{colim}_{i \in \mathcal{I}} \mathrm{Hom}_{\mathcal{D}}\big(F(x), y_i\big),$$

proving that $F(x)$ is κ-compact. Conversely, if F preserves κ-compact objects, then the same calculation run backwards shows that the canonical morphism $\operatorname{colim}_{i \in \mathcal{I}} G(y_i) \to G(\operatorname{colim}_{i \in \mathcal{I}} y_i)$ becomes an equivalence after applying the functors $\mathrm{Hom}_{\mathcal{C}}(x, -) \colon \mathcal{C} \to \mathrm{An}$ for every κ-compact object $x \in \mathcal{C}^\kappa$. These functors are jointly conservative if $\mathcal{C}$ is κ-compactly generated. $\qquad\square$

6.80. Constructing Pr^{L} in ZFC. — By Remark 6.73, we have an inclusion $\mathrm{Pr}^{\mathrm{L}}_\kappa \subseteq \mathrm{Pr}^{\mathrm{L}}_\lambda$ of non-full sub-∞-categories of Pr^{L} for all regular cardinals $\lambda > \kappa$. By Lemma 6.72, every presentable ∞-category is κ-compactly generated for all sufficiently large regular cardinals κ.

Furthermore, by Lemma 6.79 and Lemma 6.71, every left adjoint functor $F\colon \mathcal{C} \to \mathcal{D}$ between presentable ∞-categories preserves κ-compact objects for sufficiently large κ. Therefore we can write

$$\mathrm{Pr}^{\mathrm{L}} \simeq \bigcup_{\kappa} \mathrm{Pr}^{\mathrm{L}}_{\kappa},$$

where the union is taken over all cardinals which are small with respect to our two nested universes. The union can be made precise as a colimit in $\widehat{\mathrm{Cat}}_\infty$, the ∞-category of all large ∞-categories, but it can also be viewed as simply a union of simplices in every degree. In any case, we see that every functor $T\colon \mathcal{I} \to \mathrm{Pr}^{\mathrm{L}}$ from a small ∞-category $\mathcal{I}$ factors through $\mathrm{Pr}^{\mathrm{L}}_{\kappa}$ for all sufficiently large cardinals κ.

So it suffices to explain how $\mathrm{Pr}^{\mathrm{L}}_{\kappa}$ can be constructed in ZFC. In Lemma 6.81(c) below, we see that, for uncountable κ, $\mathrm{Pr}^{\mathrm{L}}_{\kappa}$ can be identified with a non-full sub-∞-category of Cat_∞ in a non-trivial way. This can be viewed as an alternative construction of $\mathrm{Pr}^{\mathrm{L}}_{\kappa}$, thus providing a construction within the confines of ZFC.

6.81. Lemma. — *Let κ be an uncountable regular cardinal and let $\mathrm{Cat}_\infty^{\kappa\text{-colim}} \subseteq \mathrm{Cat}_\infty$ be the non-full sub-∞-category spanned by those small ∞-categories that have all κ-small colimits and those functors that preserve all κ-small colimits. Let $(-)^{\kappa}$ be the functor that sends a κ-compactly generated presentable ∞-category $\mathcal{D}$ to its full sub-∞-category $\mathcal{D}^{\kappa}$ spanned by the κ-compact objects.*[(6.23)]

(a) *If $\mathcal{C}$ is a small ∞-category with all κ-small colimits and $\mathcal{D}$ is a presentable ∞-category, then restriction along $\mathcal{K}_\mathcal{C}\colon \mathcal{C} \to \mathrm{Ind}_\kappa(\mathcal{C})$ induces an equivalence of ∞-categories*

$$\mathcal{K}_\mathcal{C}^*\colon \ \mathrm{Fun}^{\mathrm{L}}_{\kappa}\big(\mathrm{Ind}_\kappa(\mathcal{C}), \mathcal{D}\big) \xrightarrow{\simeq} \mathrm{Fun}^{\kappa\text{-colim}}(\mathcal{C}, \mathcal{D}^{\kappa}).$$

  Here $\mathrm{Fun}^{\mathrm{L}}_{\kappa} \subseteq \mathrm{Fun}$ is the full sub-∞-category spanned by left adjoint functors that preserve κ-compact objects and $\mathrm{Fun}^{\kappa\text{-colim}} \subseteq \mathrm{Fun}$ is spanned by κ-small colimit-preserving functors.

(b) *If $\mathcal{C}$ is a small ∞-category with all κ-small colimits, then $\mathcal{C} \simeq \mathrm{Ind}_\kappa(\mathcal{C})^{\kappa}$.*

(c) *The functor $(-)^{\kappa}$ induces an equivalence of ∞-categories*

$$(-)^{\kappa}\colon \mathrm{Pr}^{\mathrm{L}}_{\kappa} \xrightarrow{\simeq} \mathrm{Cat}_\infty^{\kappa\text{-colim}}.$$

Proof. We begin with (b). It's clear that the objects $\{\mathcal{K}_\mathcal{C}(x) \mid x \in \mathcal{C}\}$ form a set of κ-compact generators of $\mathrm{Ind}_\kappa(\mathcal{C})$. As we've seen in the proof of Lemma 6.62, this means that every κ-compact object of $\mathrm{Ind}_\kappa(\mathcal{C})$ is a retract of an object in $\{\mathcal{K}_\mathcal{C}(x) \mid x \in \mathcal{C}\}$. As we've seen in footnote (6.19) on page 118, retracts can be written as countable colimits. Furthermore, we've

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[(6.23)]We should explain how to construct this functor. In general, given a functor $F\colon \mathcal{C} \to \mathrm{Cat}_\infty$, it's easy to construct *subfunctors* of F. Indeed, suppose for every $x \in \mathcal{C}$ we're given a sub-∞-category $F_0(x) \subseteq F(x)$ in the sense of 2.16. We don't assume $F_0(x)$ to be a full sub-∞-category, although in the case at hand it is. If, for every morphism $\alpha\colon x \to y$ in F, the functor $F(\alpha)\colon F(x) \to F(y)$ restricts to a functor $F_0(x) \to F_0(y)$, then we automatically get a functor $F_0\colon \mathcal{C} \to \mathrm{Cat}_\infty$ together with a natural transformation $F_0 \Rightarrow F$. Indeed, let $p\colon \mathcal{U} \to \mathcal{C}$ be the unstraightening of F and let S_1 be the set of morphisms that can be written as $\varphi_0 \circ \varphi$, where φ_0 is a morphism in some $F_0(x)$ and φ is a cocartesian morphism in $\mathcal{U}$. Then S_1 contains all identities and is closed under equivalences. Furthermore, our assumption on F_0 guarantees that S_1 is closed under compositions. Let $\mathcal{U}_0 := \mathcal{U}[S_1] \subseteq \mathcal{U}$ be the (not necessarily full) sub-∞-category spanned by S_1. Then $p_0\colon \mathcal{U}_0 \to \mathcal{C}$ given as the restriction of p is still a cocartesian fibration, since we've ensured that $\mathcal{U}_0$ is closed under p-cocartesian lifts in $\mathcal{U}$. Thus, we can define F_0 as the straightening of p_0.

In the case at hand, the identity on $\mathrm{Pr}^{\mathrm{L}}_{\kappa}$ can be viewed as a functor $\mathrm{Pr}^{\mathrm{L}}_{\kappa} \to \widehat{\mathrm{Cat}}_\infty$ and we can construct $(-)^{\kappa}$ as a subfunctor of it.

seen in the proof of Lemma 6.61 that $\yotimes_{\mathcal{C}} \colon \mathcal{C} \to \mathrm{Ind}_\kappa(\mathcal{C})$ preserves κ-small colimits; in particular, $\yotimes_{\mathcal{C}}$ preserves countable colimits, as κ is assumed uncountable. Since we assume that $\mathcal{C}$ has all κ-small colimits, it follows that $\{\yotimes_{\mathcal{C}}(x) \mid x \in \mathcal{C}\}$ is closed under retracts and thus comprises all κ-compact objects. The claim follows.

To prove (a), our starting point is the equivalence $\mathrm{Fun}^{\kappa\text{-filt}}(\mathrm{Ind}_\kappa(\mathcal{C}), \mathcal{D}) \simeq \mathrm{Fun}(\mathcal{C}, \mathcal{D})$ from Lemma 6.61(c). So we only have to match full sub-∞-categories on either side. Suppose a functor $F \colon \mathrm{Ind}_\kappa(\mathcal{C}) \to \mathcal{D}$ preserves κ-compact objects. Then the associated functor $\mathcal{C} \to \mathcal{D}$ factors through a functor $G \colon \mathcal{C} \to \mathcal{D}^\kappa$. Furthermore, $\mathcal{D}^\kappa \subseteq \mathcal{D}$ is closed under κ-small colimits and so is $\mathcal{C} \simeq \mathrm{Ind}_\kappa(\mathcal{C})^\kappa \subseteq \mathrm{Ind}_\kappa(\mathcal{C})$ by (b). Thus, if F preserves all colimits, then G preserves κ-small colimits. Conversely, suppose we're given a functor $G \colon \mathcal{C} \to \mathcal{D}^\kappa$ that preserves κ-small colimits. Let $F \colon \mathrm{Ind}_\kappa(\mathcal{C}) \to \mathcal{D}$ be the associated functor. By construction, F preserves κ-compact objects and κ-filtered colimits. Furthermore, we've seen in the proof of Lemma 6.63 that arbitrary colimits in $\mathrm{Ind}_\kappa(\mathcal{C})$ can be built from κ-filtered colimits as well as κ-small colimits of objects in the image of $\yotimes_{\mathcal{C}} \colon \mathcal{C} \to \mathrm{Ind}_\kappa(\mathcal{C})$. Thus F preserves all colimits. This finishes the proof of (a).

Finally, (c) is a formal consequence: (b) shows that $(-)^\kappa$ is essentially surjective, and (a), together with Lemma 6.63(d), shows that $(-)^\kappa$ is fully faithful. $\qquad\square$

6.82. Corollary. — *Let $\lambda > \kappa$ be regular cardinals.*

(a) *The inclusion $\mathrm{Pr}^{\mathrm{L}}_\kappa \subseteq \mathrm{Pr}^{\mathrm{L}}_\lambda$ admits a right adjoint. On objects, it sends $\mathcal{D} \in \mathrm{Pr}^{\mathrm{L}}_\lambda$ to $\mathrm{Ind}_\kappa(\mathcal{D}^\lambda)$.*

(b) *The ∞-category $\mathrm{Pr}^{\mathrm{L}}_\kappa$ has all small limits and colimits. The forgetful functor $\mathrm{Pr}^{\mathrm{L}}_\kappa \to \mathrm{Pr}^{\mathrm{L}}$ preserves all small colimits.*[(6.24)]

Proof. We start with (a). Let $\mathcal{D} \in \mathrm{Pr}^{\mathrm{L}}_\lambda$. By Lemma 6.61($c$), the identity functor $\mathrm{id}_{\mathcal{D}^\lambda} \colon \mathcal{D}^\lambda \to \mathcal{D}^\lambda$ induces a κ-filtered colimit-preserving functor $c_{\mathcal{D}} \colon \mathrm{Ind}_\kappa(\mathcal{D}^\lambda) \to \mathcal{D}$. Let's first argue why $c_{\mathcal{D}}$ is a functor in $\mathrm{Pr}^{\mathrm{L}}_\lambda$. Since $\mathcal{D}^\lambda$ has all κ-small colimits and $\mathrm{id}_{\mathcal{D}^\lambda}$ preserves them, the same argument as in the proof of Lemma 6.81(a) shows that $c_{\mathcal{D}}$ preserves all colimits. Furthermore, the λ-compact objects of $\mathrm{Ind}_\kappa(\mathcal{D}^\lambda)$ are precisely those generated under λ-small colimits by objects in the image of $\yotimes_{\mathcal{D}^\lambda} \colon \mathcal{D}^\lambda \to \mathrm{Ind}_\kappa(\mathcal{D}^\lambda)$. Indeed, if $\overline{\mathcal{D}}$ denotes the full sub-∞-category spanned by these objects, then the proof of Lemma 6.62(c) yields an equivalence $\mathrm{Ind}_\lambda(\overline{\mathcal{D}}) \simeq \mathrm{Ind}_\kappa(\mathcal{D}^\lambda)$; now apply Lemma 6.81($b$). Since $c_{\mathcal{D}}$ preserves all colimits and $\mathcal{D}^\lambda \subseteq \mathcal{D}$ is closed under λ-small colimits, it follows that all λ-compact objects of $\mathrm{Ind}_\kappa(\mathcal{D}^\lambda)$ land in $\mathcal{D}^\lambda$. This proves that $c_{\mathcal{D}}$ is indeed a functor in $\mathrm{Pr}^{\mathrm{L}}_\lambda$.

To construct the desired right adjoint, it's now enough by Lemma 6.2 to show that the functor $(c_{\mathcal{D}})_* \colon \mathrm{Fun}^{\mathrm{L}}_\kappa(\mathcal{C}, \mathrm{Ind}_\kappa(\mathcal{D}^\lambda)) \longrightarrow \mathrm{Fun}^{\mathrm{L}}_\lambda(\mathcal{C}, \mathcal{D})$ for all $\mathcal{C} \in \mathrm{Pr}^{\mathrm{L}}_\kappa$, given by postcomposition with $c_{\mathcal{D}}$, is an equivalence of ∞-categories. This functor fits into the following diagram:

$$
\begin{array}{ccc}
\mathrm{Fun}^{\mathrm{L}}_\kappa\big(\mathcal{C}, \mathrm{Ind}_\kappa(\mathcal{D}^\lambda)\big) & \xrightarrow{\;(c_{\mathcal{D}})_*\;} & \mathrm{Fun}^{\mathrm{L}}_\lambda(\mathcal{C}, \mathcal{D}) \\
\big\downarrow & /\!/\!/ & \big\downarrow \\
\mathrm{Fun}\big(\mathcal{C}^\kappa, \mathrm{Ind}_\kappa(\mathcal{D}^\lambda)^\kappa\big) & \xrightarrow{\;\simeq\;} & \mathrm{Fun}(\mathcal{C}^\kappa, \mathcal{D}^\lambda)
\end{array}
$$

The vertical arrows are given by restriction along $\mathcal{C}^\kappa \subseteq \mathcal{C}$. By Lemma 6.81, the left vertical arrow is fully faithful and the bottom arrow is an equivalence. The right vertical arrow is fully

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[(6.24)]It's true that any product of κ-compactly generated ∞-categories in Pr^{L} is κ-compactly generated again; we'll see this in the proof of Corollary 6.82(b). But, confusingly, the product in Pr^{L} is usually not the product in $\mathrm{Pr}^{\mathrm{L}}_\kappa$. The reason is that preservation of limits or colimits can be detected factor-wise, but not preservation of κ-compact objects.

faithful by Lemma 6.61(c), using $\mathcal{C} \simeq \mathrm{Ind}_\kappa(\mathcal{C}^\kappa)$. It follows that $(c_\mathcal{D})_*$ must be fully faithful too. Furthermore, by Lemma 6.81, the essential image of $\mathrm{Fun}^{\mathrm{L}}_\kappa(\mathcal{C}, \mathrm{Ind}_\kappa(\mathcal{D}^\lambda)) \to \mathrm{Fun}(\mathcal{C}^\kappa, \mathcal{D}^\lambda)$ is spanned by those functors that preserve κ-small colimits. Since $\mathcal{C}^\kappa \subseteq \mathcal{C}$ and $\mathcal{D}^\lambda \subseteq \mathcal{D}$ are closed under κ-small colimits, the essential image of $\mathrm{Fun}^{\mathrm{L}}_\lambda(\mathcal{C}, \mathcal{D}) \to \mathrm{Fun}(\mathcal{C}^\kappa, \mathcal{D}^\lambda)$ must also be contained in the κ-small colimit-preserving functors. This shows that $(c_\mathcal{D})_*$ is essentially surjective and we've finished the proof of (a).

The existence of limits in $\mathrm{Pr}^{\mathrm{L}}_\kappa$ follows from Theorem 6.83 below combined with Corollary 6.67(a). For colimits, let $\mathrm{Pr}^{\mathrm{R}}_\kappa$ be the ∞-category of all κ-compactly generated presentable ∞-categories and right adjoint functors that preserve κ-filtered colimits. Then Corollary 6.8(b) and Lemma 6.79 show $\mathrm{Pr}^{\mathrm{L}}_\kappa \simeq (\mathrm{Pr}^{\mathrm{R}}_\kappa)^{\mathrm{op}}$. Thus it's enough to check that $\mathrm{Pr}^{\mathrm{R}}_\kappa$ is closed under limits in Pr^{R}.[(6.25)] As usual, it's enough to do products and pullbacks. In either case, we know from Lemma 6.75 that the limit in $\widehat{\mathrm{Cat}}_\infty$ is also the limit in Pr^{R}, so by Lemma 6.62(c) we only need to construct a set of κ-compact generators. For a product $\prod_{i \in I} \mathcal{C}_i$, choose a set S_i of κ-compact generators of $\mathcal{C}_i$ for all $i \in I$. Furthermore, choose an initial object $\emptyset_i \in \mathcal{C}_i$. For all $s_i \in S_i$ let $e_i(s_i) \in \prod_{i \in I} \mathcal{C}_i$ be the object given by $e_i(s_i)_i = s_i$ and $e_i(s_i)_j = \emptyset_j$ for $j \neq i$. Then $\{e_i(s_i) \mid s_i \in S_i\}$ are κ-compact and jointly detect equivalences in the ith component. Hence the union $\bigcup_{i \in I}\{e_i(s_i) \mid s_i \in S_i\}$ is a set of κ-compact generators of $\prod_{i \in I} \mathcal{C}_i$.

The argument for pullbacks is similar. Let $\mathcal{C} := \mathcal{C}_0 \times_{\mathcal{C}_2} \mathcal{C}_1$ be a pullback in Pr^{R}, where the underlying diagram is already contained in $\mathrm{Pr}^{\mathrm{R}}_\kappa$. By definition of Pr^{R}, the pullback projections $\mathrm{pr}_0 \colon \mathcal{C} \to \mathcal{C}_0$ and $\mathrm{pr}_1 \colon \mathcal{C} \to \mathcal{C}_1$ admit left adjoints $L_0 \colon \mathcal{C}_0 \to \mathcal{C}$ and $L_1 \colon \mathcal{C}_1 \to \mathcal{C}$. By Lemma 6.76($b$), the projections pr_0 and pr_1 preserve κ-filtered colimits, hence Lemma 6.79 shows that L_0 and L_1 preserve κ-compact objects. Now choose sets S_0 and S_1 of κ-compact generators of $\mathcal{C}_0$ and $\mathcal{C}_1$. Then $\{L_0(s_0) \mid s_0 \in S_0\}$ are κ-compact objects of $\mathcal{C}$ and jointly detect equivalences in the first factor. Similarly, $\{L_1(s_1) \mid s_1 \in S_1\}$ jointly detect equivalences in the second factor. Hence the union $\{L_0(s_0) \mid s_0 \in S_0\} \cup \{L_1(s_1) \mid s_1 \in S_1\}$ is a set of κ-compact generators of $\mathcal{C}$. This finishes the proof of (b). □

To finish this rather lengthy subsection, we'll show the aforementioned mindblowing property of $\mathrm{Pr}^{\mathrm{L}}_\kappa$. If you're in a situation where you can fix an uncountable regular cardinal κ and only work with κ-compactly generated ∞-categories (for most practical applications, $\kappa = \aleph_1$ is enough), Theorem 6.83 allows you to bypass all set-theoretic problems.

6.83. Theorem ("Russel's paradox = skill issue"). — *Let κ be an uncountable regular cardinal. Then $\mathrm{Pr}^{\mathrm{L}}_\kappa$ is an object of $\mathrm{Pr}^{\mathrm{L}}_\kappa$.*

Proof sketch. We already know from Corollary 6.82(b) that $\mathrm{Pr}^{\mathrm{L}}_\kappa$ has all colimits. Thus, by Lemma 6.62(c) it's enough to find a set of κ-compact generators. We'll show that $\{\mathrm{An}, \mathrm{PSh}(\Delta^1)\}$ does it. Let's first show that the functors $\mathrm{Hom}_{\mathrm{Pr}^{\mathrm{L}}_\kappa}(\mathrm{An}, -)$ and $\mathrm{Hom}_{\mathrm{Pr}^{\mathrm{L}}_\kappa}(\mathrm{PSh}(\Delta^1), -)$ are jointly conservative. To this end, we claim more generally:

($\boxtimes_1$) *If $\mathcal{C}$ is a small ∞-category and $\mathcal{D}$ is a κ-compactly generated presentable ∞-category, then restriction along $\sethinspace \mathcal{C} \colon \mathcal{C} \to \mathrm{PSh}(\mathcal{C})$ induces an equivalence of ∞-categories*

$$\sethinspace_\mathcal{C}^* \colon \mathrm{Fun}^{\mathrm{L}}_\kappa\big(\mathrm{PSh}(\mathcal{C}), \mathcal{D}\big) \xrightarrow{\simeq} \mathrm{Fun}(\mathcal{C}, \mathcal{D}^\kappa) \,.$$

[(6.25)] As in the proof of Lemma 6.75, a straightforward extra-argument is needed since $\mathrm{Pr}^{\mathrm{L}}_\kappa$ is not a full sub-∞-category of Pr^{R}. As preservation of limits and κ-filtered colimits can be checked factor-wise, we don't run into the same issue as in footnote (6.24)

Believing ($\boxtimes_1$), we find $\mathrm{Hom}_{\mathrm{Pr}^{\mathrm{L}}_\kappa}(\mathrm{An}, \mathcal{D}) \simeq \mathrm{core}\, \mathcal{D}^\kappa$ and $\mathrm{Hom}_{\mathrm{Pr}^{\mathrm{L}}_\kappa}(\mathrm{PSh}(\Delta^1), \mathcal{D}) \simeq \mathrm{core}\, \mathrm{Ar}(\mathcal{D}^\kappa)$. Now $(-)^\kappa \colon \mathrm{Pr}^{\mathrm{L}}_\kappa \to \mathrm{Cat}_\infty$ is conservative by Lemma 6.81(c). Furthermore, $\mathrm{core}(-) \colon \mathrm{Cat}_\infty \to \mathrm{An}$ and $\mathrm{core}\, \mathrm{Ar}(-) \colon \mathrm{Cat}_\infty \to \mathrm{An}$ are jointly conservative, as we've seen in the proof of Corollary 6.64. It follows that An and $\mathrm{PSh}(\Delta^1)$ are generators of $\mathrm{Pr}^{\mathrm{L}}_\kappa$, as desired.

To prove ($\boxtimes_1$), recall $\mathrm{Fun}^{\mathrm{L}}(\mathrm{PSh}(\mathcal{C}), \mathcal{D}) \simeq \mathrm{Fun}(\mathcal{C}, \mathcal{D})$ from Theorem 6.30, so we only have to find out to which full sub-∞-category $\mathrm{Fun}^{\mathrm{L}}_\kappa \subseteq \mathrm{Fun}^{\mathrm{L}}$ corresponds in $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$. Since $\{ よ_{\mathcal{C}}(x) \mid x \in \mathcal{C} \}$ is a set of generators for $\mathrm{PSh}(\mathcal{C})$, the same argument as in the proof of Lemma 6.81(a) shows that the κ-compact objects in $\mathrm{PSh}(\mathcal{C})$ are precisely those generated under κ-small colimits from representable presheaves. Thus, a colimit-preserving functor $\mathrm{PSh}(\mathcal{C}) \to \mathcal{D}$ also preserves κ-compact objects if and only if it restricts to a functor $\mathcal{C} \to \mathcal{D}^\kappa$. This proves ($\boxtimes_1$).

It remains to show that An and $\mathrm{PSh}(\Delta^1)$ are κ-compact in $\mathrm{Pr}^{\mathrm{L}}_\kappa$. As explained above, ($\boxtimes_1$) shows $\mathrm{Hom}_{\mathrm{Pr}^{\mathrm{L}}_\kappa}(\mathrm{An}, -) \simeq \mathrm{core}((-)^\kappa)$ and $\mathrm{Hom}_{\mathrm{Pr}^{\mathrm{L}}_\kappa}(\mathrm{PSh}(\Delta^1), -) \simeq \mathrm{core}\, \mathrm{Ar}((-)^\kappa)$. We know from Lemma 6.81(c) that $(-)^\kappa \colon \mathrm{Pr}^{\mathrm{L}}_\kappa \to \mathrm{Cat}^{\kappa\text{-colim}}_\infty$ is an equivalence of ∞-categories, so it preserves all κ-filtered colimits, and we've sketched in the proof of Corollary 6.64 that $\mathrm{core}(-) \colon \mathrm{Cat}_\infty \to \mathrm{An}$ and $\mathrm{core}\, \mathrm{Ar}(-) \colon \mathrm{Cat}_\infty \to \mathrm{An}$ preserve filtered colimits. Therefore, to finish the proof it's enough to show the following claim about κ-filtered colimits in $\mathrm{Cat}^{\kappa\text{-colim}}_\infty$.

($\boxtimes_2$) *The forgetful functor* $\mathrm{Cat}^{\kappa\text{-colim}}_\infty \to \mathrm{Cat}_\infty$ *preserves κ-filtered colimits.*

To prove ($\boxtimes_2$), let $\mathcal{C}_{(-)} \colon \mathcal{J} \to \mathrm{Cat}^{\kappa\text{-colim}}_\infty$ be a κ-filtered diagram. We have to show that the colimit $\mathrm{colim}_{j \in \mathcal{J}} \mathcal{C}_j$ in Cat_∞ also has all κ-small colimits and constitutes a colimit in $\mathrm{Cat}^{\kappa\text{-colim}}_\infty$. So let $\mathcal{I}$ be a κ-small ∞-category and let $T \colon \mathcal{I} \to \mathrm{colim}_{j \in \mathcal{J}} \mathcal{C}_j$ be an $\mathcal{I}$-shaped diagram in $\mathrm{colim}_{j \in \mathcal{J}} \mathcal{C}_j$. It's not hard to check that for uncountable regular cardinals κ the κ-compact objects in Cat_∞ are precisely the κ-small ∞-categories.[(6.26)] Hence $\mathcal{I}$ is κ-compact and so T factors through a functor $T_0 \colon \mathcal{I} \to \mathcal{C}_{j_0}$ for some $j_0 \in \mathcal{J}$. By assumption, $\mathcal{C}_{j_0}$ has all κ-small colimits and so T_0 extends to a colimit cone $T_0^\triangleright \colon \mathcal{I}^\triangleright \to \mathcal{C}_{j_0}$. Furthermore, for every morphism $j_0 \to j$ in $\mathcal{J}$ the functor $\mathcal{C}_{j_0} \to \mathcal{C}_j$ preserves κ-small colimits, hence $\mathcal{I}^\triangleright \to \mathcal{C}_{j_0} \to \mathcal{C}_j$ is still a colimit cone. We claim that then also $\mathcal{I}^\triangleright \to \mathcal{C}_{j_0} \to \mathrm{colim}_{j \in \mathcal{J}} \mathcal{C}_j$ is a colimit cone. To show this, we need an analogue of Lemma 6.76(a) for filtered colimits; this can be obtained in the exact same way, using that $\mathrm{Hom}_{\mathrm{Cat}_\infty}(\Delta^1, -)$ and $\mathrm{Hom}_{\mathrm{Cat}_\infty}(*\, *, -)$ also preserve filtered colimits and that pullbacks in An commute with filtered colimits by Theorem 6.54. Then Corollary 6.16 shows that $\mathcal{I}^\triangleright \to \mathcal{C}_{j_0} \to \mathrm{colim}_{j \in \mathcal{J}} \mathcal{C}_j$ is indeed a colimit cone, as claimed. This proves that $\mathrm{colim}_{j \in \mathcal{J}} \mathcal{C}_j$ again has all κ-small colimits.

To finish the proof, it remains to argue why $\mathrm{colim}_{j \in \mathcal{J}} \mathcal{C}_j$ is also a colimit in the non-full sub-∞-category $\mathrm{Cat}^{\kappa\text{-colim}}_\infty \subseteq \mathrm{Cat}_\infty$. Unravelling the definitions (and using Lemma 4.9), we must check that for every natural transformation $\eta \colon \mathcal{C}_{(-)} \Rightarrow \mathrm{const}\, \mathcal{D}$ in $\mathrm{Cat}^{\kappa\text{-colim}}_\infty$ the induced functor $\mathrm{colim}_{j \in \mathcal{J}} \mathcal{C}_j \to \mathcal{D}$ in Cat_∞ preserves κ-small colimits. But we've seen above that every κ-small colimit in $\mathrm{colim}_{j \in \mathcal{J}} \mathcal{C}_j$ is inherited from $\mathcal{C}_{j_0}$ for some $j_0 \in \mathcal{J}$ and the functor $\eta_{j_0} \colon \mathcal{C}_{j_0} \to \mathcal{D}$ preserves κ-small colimits by assumption on η. $\qquad\square$

[(6.26)]Since Cat_∞ is generated by the compact objects $*$ and Δ^1, hence the κ-compact objects are precisely those ∞-categories generated under κ-small colimits by $*$ and Δ^1. Everyone of them is κ-small by Remark 6.49. Conversely, for all $n \geqslant 0$, the ∞-category Δ^n can be written as a finite colimit in $*$ and Δ^1 (namely, as the iterated pushout $\Delta^{\{0,1\}} \sqcup_{\{1\}} \cdots \sqcup_{\{n-1\}} \Delta^{\{n-1,n\}}$). Using Lemma 6.48($c$), every other κ-small ∞-category is contained in the full sub-∞-category of Cat_∞ generated under κ-small colimits by $\{ \Delta^n \mid n \geqslant 0 \}$.

§7. Towards spectra

The goal of this section is to introduce the stable ∞-category Sp of *spectra*. Along the way we'll be able to deduce many classical topological results.

§7.1. Suspensions and loop animae

7.1. Definition. — Let $X \in \mathrm{An}$ be an anima or $(X, x) \in \mathrm{An}_{*/}$ be a pointed anima. We define ΣX, the *suspension of X*, and $\Omega_x X$, the *loop anima of X with basepoint x*, via

$$
\begin{array}{ccc}
X \longrightarrow * & & \Omega_x X \longrightarrow \{x\} \\
\downarrow \quad \ulcorner \quad \downarrow & \text{and} & \downarrow \quad \lrcorner \quad \downarrow \\
* \longrightarrow \Sigma X & & \{x\} \longrightarrow X
\end{array}
$$

respectively, where the pushout and the pullback are taken in An. If the basepoint is clear from the context, we often simply write ΩX. Note that ΣX is canonically a pointed anima via $* \to \Sigma X$ and $\Omega_x X$ is canonically a pointed anima since the pullback can be taken in $\mathrm{An}_{*/}$ instead by Lemma 6.56(*a*).

7.2. Remark. — By model category fact 6.13, to compute ΣX, we have to replace one $X \to *$ by a cofibration, then take the usual pushout of simplicial sets, and finally replace the result by a Kan complex. Such a replacement by a cofibration could be $X \to X^{\triangleright} \to CX$, where $X^{\triangleright} \to CX$ is an anodyne map from the cone $X^{\triangleright}$ from Construction 6.51 into a Kan complex (which exists thanks to Lemma 3.12); then CX is contractible because $CX \simeq |X^{\triangleright}| \simeq *$. From this description, we see that Σ is compatible with the topological suspension functor $\Sigma^{\mathrm{Top}} \colon \mathrm{Top} \to \mathrm{Top}$ (reduced or unreduced doesn't matter) in the sense that

$$
\begin{array}{ccc}
h(\mathrm{An}) & \xrightarrow{\ \Sigma\ } & h(\mathrm{An}) \\
{\scriptstyle |\cdot|} \downarrow & {\scriptstyle ///} & \downarrow {\scriptstyle |\cdot|} \\
h(\mathrm{Top}) & \xrightarrow{\ \Sigma^{\mathrm{Top}}\ } & h(\mathrm{Top})
\end{array}
$$

commutes; here $|\cdot| \colon h(\mathrm{An}) \to h(\mathrm{Top})$ denotes the geometric realisation functor. So the suspension functor $\Sigma \colon \mathrm{An} \to \mathrm{An}$ deserves its name.

Next, we'll show that the loop functor $\Omega \colon \mathrm{An}_{*/} \to \mathrm{An}_{*/}$ deserves its name as well.

7.3. Lemma. — *Suspension and loop form an adjunction $\Sigma \colon \mathrm{An}_{*/} \rightleftarrows \mathrm{An}_{*/} \colon \Omega$. In particular, for every pointed anima (X, x), the following hold:*

(*a*) $\pi_n(\Omega_x X, x) \cong \pi_{n+1}(X, x)$ *for all $n \geqslant 0$.*

(*b*) $\Omega_x X \simeq \mathrm{Hom}_{\mathrm{An}_{*/}}((S^1, *), (X, x)) \simeq \mathrm{Hom}_X(x, x)$.

Proof sketch. The adjunction $\Sigma \dashv \Omega$ follows immediately from Corollary 6.16 and the fact that the pushout and pullback diagrams in Definition 7.1 can be taken in $\mathrm{An}_{*/}$ as well by Lemma 6.56.

Part (*a*) follows immediately from the suspension-loop adjunction and $S^{n+1} \simeq \Sigma S^n$. The latter is clear if we define $S^n := \Sigma(* \ *)$ as the n-fold suspension of two points; for any other construction, it is a straightforward check.

The first equivalence in (b) follows from $\Omega_x X \simeq \operatorname{Hom}_{\mathrm{An}}(*, \Omega_x X) \simeq \operatorname{Hom}_{\mathrm{An}_{*/}}(* \ *, (\Omega_x X, x))$ and $\Sigma(* \ *) \simeq S^1$. For the second equivalence, note $\operatorname{Ar}(X) \simeq X$. Indeed, $\operatorname{Ar}(X) \simeq \operatorname{Fun}(\Delta^1, X)$ is already an anima and so $\operatorname{Fun}(\Delta^1, X) \simeq \operatorname{core} \operatorname{Fun}(\Delta^1, X) \simeq \operatorname{Hom}_{\mathrm{Cat}_\infty}(\Delta^1, X)$. Note that $|\Delta^1| \simeq *$, since Δ^1 has an initial object, and so Example 6.3(a) implies the desired equivalence $\operatorname{Hom}_{\mathrm{Cat}_\infty}(\Delta^1, X) \simeq \operatorname{Hom}_{\mathrm{An}}(*, X) \simeq X$. Therefore, $(s,t)\colon \operatorname{Ar}(X) \to X \times X$ is homotopic to the diagonal $\Delta\colon X \to X \times X$ and so $\operatorname{Hom}_X(x,x) \simeq (\{x\} \times \{x\}) \times_{X \times X, \Delta} X$. Now consider the following diagram:

$$
\begin{array}{ccccc}
\{x\} & \longrightarrow & \{x\} \times X & \longrightarrow & \{x\} \\
\downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow \\
X & \xrightarrow{\ \Delta\ } & X \times X & \xrightarrow{\ \mathrm{pr}_1\ } & X
\end{array}
$$

The right square is a pullback by inspection and the outer rectangle is a pullback because the bottom row $\mathrm{pr}_1 \circ \Delta\colon X \to X$ is the identity on X. It follows formally that the left square must be a pullback as well. Finally, consider the following diagram:

$$
\begin{array}{ccccc}
\Omega_x X & \longrightarrow & \{x\} & \longrightarrow & X \\
\downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow{\scriptstyle\Delta} \\
\{x\} & \longrightarrow & \{x\} \times X & \longrightarrow & X \times X
\end{array}
$$

The right square is a pullback as argued above and the left square is a pullback by Definition 7.1. Now the outer square is a pullback again, which proves $\Omega_x X \simeq \operatorname{Hom}_X(x,x)$, as desired. $\qquad\square$

7.4. Example. — For every $n \geqslant 0$, the following is a pullback diagram in $\mathcal{D}_{\geqslant 0}(\mathbb{Z})$:

$$
\begin{array}{ccc}
A[n] & \longrightarrow & 0 \\
\downarrow & \lrcorner & \downarrow \\
0 & \longrightarrow & A[n+1]
\end{array}
$$

(this may seem weird at first, but will become more clear once we discuss stable ∞-categories in §7.4; the proof is similar to Lemma 6.40(b)). Since the Eilenberg–MacLane anima functor $\mathrm{K}\colon \mathcal{D}_{\geqslant 0}(\mathbb{Z}) \to \mathrm{An}$ from Construction 6.41 is a right adjoint, it preserves pullbacks by Lemma 6.11, which shows $\mathrm{K}(A,n) \simeq \Omega\mathrm{K}(A, n+1)$. This fits prefectly with the fact that the loop functor shifts homotopy groups down by Lemma 7.3(a).

§7.2. $\mathbb{E}_1$-monoids and $\mathbb{E}_1$-groups

7.5. Associahedra. — What's an associative monoid in the ∞-category An? Clearly, part of the data should be an anima M together with a multiplication $\mu\colon M \times M \to M$. We'll often write we put $a \cdot b := \mu(a,b)$ for convenience.

Intuitively, associativity means that for every $n \geqslant 3$ and all $a_1, \dots, a_n \in M$, every way of bracketing the product $a_1 \cdots a_n$ should be equivalent. What does this mean concretely? In the case $n = 3$, another part of the data should be a homotopy $\eta_3\colon \mu(-, \mu(-,-)) \Rightarrow \mu(\mu(-,-),-)$ in $\operatorname{Hom}_{\mathrm{An}}(M^3, M)$, witnessing $a \cdot (b \cdot c) \simeq (a \cdot b) \cdot c$ for all $a, b, c \in M$. If M were a monoid in Set (or in any ordinary category), then the case $n = 3$ would already guarantee associativity for arbitrary n. However, in an ∞-category, this no longer works. For example, in the case $n = 4$,

we need additional data—a homotopy η_4 in $\mathrm{Hom}_{\mathrm{An}}(M^4, M)$ that witnesses commutativity of the diagram

$$
\begin{array}{ccc}
& a \cdot (b \cdot (c \cdot d)) & \\
{\scriptstyle a \cdot \eta_{b,c,d}} \swarrow {\scriptstyle \simeq} & & {\scriptstyle \simeq} \searrow {\scriptstyle \eta_{a,b,(c \cdot d)}} \\
a \cdot ((b \cdot c) \cdot d) & & (a \cdot b) \cdot (c \cdot d) \\
{\scriptstyle \eta_{a,(b \cdot c),d}} \searrow {\scriptstyle \simeq} \quad \overset{\eta_4}{\Leftarrow} & & {\scriptstyle \simeq} \swarrow {\scriptstyle \eta_{(a \cdot b),c,d}} \\
(a \cdot (b \cdot c)) \cdot d & \xrightarrow[\eta_{a,b,c} \cdot d]{\simeq} & ((a \cdot b) \cdot c) \cdot d
\end{array}
$$

Then η_4 needs to satisfy another compatibility in $\mathrm{Hom}_{\mathrm{An}}(M^5, M)$ and so on. In general, Stasheff [Sta63] introduced $(d-2)$-dimensional polytopes K_d, called *associahedra*, such that associativity up to $n = d - 1$ induces a map $\partial K_d \to \mathrm{Hom}_{\mathrm{An}}(M^d, M)$ and associativity up to $n = d$ amounts to extending this to a map $K_d \to \mathrm{Hom}_{\mathrm{An}}(M^d, M)$.

A similar story exists for unitality. This leads to a notion of $\mathbb{A}_n$-*monoids*, and in the limit case, $\mathbb{A}_\infty$-*monoids*. Fortunately, ∞-category theory provides a way to package all this unwieldy data into a much cleaner definition.

7.6. Definition. — Let $\mathcal{C}$ be an ∞-category with finite products (so in particular, the empty product exists, so $\mathcal{C}$ has a terminal object $*$).

(a) An $\mathbb{A}_\infty$-*monoid* or $\mathbb{E}_1$-*monoid in $\mathcal{C}$* is a functor $M \colon \Delta^{\mathrm{op}} \to \mathcal{C}$ satisfying $M_0 \simeq *$ as well as the *Segal condition*: The *Segal maps* $e_i \colon [1] \to [n]$ that send $[1]$ bijectively to $\{i, i+1\}$ induce an equivalence

$$M_n \xrightarrow{\simeq} M_1^n.$$

We call M_1 the *underlying object of M*; we'll often don't distinguish between M and M_1. Let $\mathrm{Mon}(\mathcal{C}) \subseteq \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathcal{C})$ denote the full sub-∞-category spanned by the $\mathbb{E}_1$-monoids.

(b) For an $\mathbb{E}_1$-monoid M in $\mathcal{C}$, we get a multiplication map $\mu \colon M_1 \times M_1 \simeq M_2 \xrightarrow{d_1^*} M_1$ using the Segal condition. Then M is called an $\mathbb{E}_1$-*group in $\mathcal{C}$* if the *shearing map*

$$(\mathrm{pr}_1, \mu) \colon M_1 \times M_1 \xrightarrow{\simeq} M_1 \times M_1$$

is an equivalence. We let $\mathrm{Grp}(\mathcal{C}) \subseteq \mathrm{Mon}(\mathcal{C})$ denote the full sub-∞-category spanned by $\mathbb{E}_1$-groups.

7.7. Associahedra revisited. — Let's unravel what happens in Definition 7.6. Let $M \colon \Delta^{\mathrm{op}} \to \mathcal{C}$ be an $\mathbb{E}_1$-monoid in an ∞-category $\mathcal{C}$. We've already seen that $d_1^* \colon M_2 \to M_1$ encodes the multiplication on M. In general, if we identify $M_n \simeq M_1^n$ and $M_{n-1} \simeq M_1^{n-1}$ via Definition 7.6(a), then the face map $d_i^* \colon M_n \to M_{n-1}$ for $0 < i < n$ can be interpreted as the map that sends $(a_1, \dots, a_n)$ to $(a_1, \dots, a_{i-1}, a_i \cdot a_{i+1}, a_{i+2}, \dots, a_n)$. More precisely, if $e_{i,i+1}^{(2)} \colon [2] \to [n]$ is the map that sends $[2]$ bijectively to $\{i-1, i, i+1\}$, then the diagram

$$
\begin{array}{ccccc}
M_n & \xrightarrow[\simeq]{(e_1,\dots,e_{i-1}) \times e_{i,i+1}^{(2)} \times (e_{i+2},\dots,e_n)} & M_1^{i-1} \times M_2 \times M_1^{n-i-1} & \xrightarrow[\simeq]{\mathrm{id} \times (e_1, e_2) \times \mathrm{id}} & M_1^{i-1} \times M_1^2 \times M_1^{n-i-1} \\
{\scriptstyle d_i^*} \downarrow & /\!/\!/ & \downarrow {\scriptstyle \mathrm{id} \times d_1^* \times \mathrm{id}} & /\!/\!/ & \\
M_{n-1} & \xrightarrow[\simeq]{(e_1,\dots,e_{i-1}) \times e_i \times (e_{i+1},\dots,e_{n-1})} & M_1^{i-1} \times M_1 \times M_1^{n-i-1} & \xleftarrow{\quad \mathrm{id} \times \mu \times \mathrm{id} \quad} &
\end{array}
$$

commutes. Indeed, the square on the left can be reduced to certain commutative squares in the ordinary category Δ^{op}; we leave the details to you. The triangle on the right commutes by definition of μ. In a similar way, one can show that the "outer" face maps d_0^* and d_n^* simply forget a_1 and a_n, respectively.

So the face maps in Δ encode the multiplication, including its associativity, of the $\mathbb{E}_1$-monoid $M : \Delta^{\mathrm{op}} \to \mathcal{C}$. Likewise, the degeneracy maps encode unitality. The image of $* \simeq M_0$ under $s_0 : M_0 \to M_1$ is a point $1 \in M_1$ which plays the role of the identity element of M in the sense that the left and right multiplication maps

$$ M_1 \simeq \{1\} \times M_1 \xrightarrow{\mu} M_1 \quad \text{and} \quad M_1 \simeq M_1 \times \{1\} \xrightarrow{\mu} M_1 $$

are both homotopic to the identity $\mathrm{id}_{M_1} : M_1 \to M_1$. Indeed, this follows from the identities $s_0 \circ d_1 = \mathrm{id}_{[1]} = s_1 \circ d_1$ in Δ via the commutative diagrams

$$
\begin{array}{ccccc}
M_1 & \xrightarrow{s_0^*} & M_2 & \xrightarrow{d_1^*} & M_1 \\
{\scriptstyle\simeq}\downarrow & {\scriptstyle /\!/\!/ \ (e_1,e_2)}\downarrow & & {\scriptstyle /\!/\!/} & \nearrow \mu \\
\{1\} \times M_1 & \longrightarrow & M_1 \times M_1 & &
\end{array}
\quad \text{and} \quad
\begin{array}{ccccc}
M_1 & \xrightarrow{s_1^*} & M_2 & \xrightarrow{d_1^*} & M_1 \\
{\scriptstyle\simeq}\downarrow & {\scriptstyle /\!/\!/ \ (e_1,e_2)}\downarrow & & {\scriptstyle /\!/\!/} & \nearrow \mu \\
M_1 \times \{1\} & \longrightarrow & M_1 \times M_1 & &
\end{array}
$$

In general, $s_j^* : M_{n-1} \to M_n$ can be interpreted as the map that sends an $(n-1)$-tuple $(a_1, \ldots, a_n) \in M_1^{n-1} \simeq M_{n-1}$ to the n-tuple $(a_1, \ldots, a_{j-1}, 1, a_j, \ldots, a_n) \in M_1^n$.

These considerations lead to a nice conceptual description of Stasheff's associahedra K_d from 7.5. We've seen that the "inner" face maps $d_i : [n] \to [n-1]$ for $0 < i < n$ encode the multiplication on M. The (non-full) sub-category of Δ spanned by $d_i : [n] \to [n-1]$ for $0 < i < n$ and $1 < n \leqslant d$ is equivalent to $\square^{d-1} := (\Delta^1)^{d-1}$. Since $(\square^{d-1})^{\mathrm{op}} \simeq \square^{d-1}$, we get a (faithful but not fully faithful) functor $\square^{d-1} \to \Delta^{\mathrm{op}}$. The restriction $M|_{\square^{d-1}} : \square^{d-1} \to \mathcal{C}$ of M then encodes the multiplication μ on M plus the fact that μ is associative for up to d factors. But what does this have to do with Stasheff's associahedra? In the case $\mathcal{C} \simeq \mathrm{An} = \mathrm{N}_\Delta(\mathrm{Kan}_\Delta)$, a functor $\square^{d-1} \to \mathrm{N}^\Delta(\mathrm{Kan}^\Delta)$ is equivalently given by a simplicially enriched functor $\mathfrak{C}[\square^{d-1}] \to \mathrm{Kan}^\Delta$ by Construction 2.21. Thus, an anima M_1 together with a multiplication that's associative for up to d factors is encoded by a simplicially enriched functor $M^\Delta : \mathfrak{C}[\square^{d-1}] \to \mathrm{Kan}^\Delta$ such that M^Δ sends $(0, \ldots, 0)$ to M_1^d and $(1, \ldots, 1)$ to M_1. In particular, we get a morphism

$$ \mathrm{F}_{\mathfrak{C}[\square^{d-1}]}((0, \ldots, 0), (1, \ldots, 1)) \longrightarrow \mathrm{Hom}_{\mathrm{An}}\big(M_1^d, M_1\big) . $$

This is precisely the kind of structure we've seen in 7.5: a map from a polytope, modelled here as a simplicial set, into $\mathrm{Hom}_{\mathrm{An}}(M^d, M)$! And indeed, $\mathrm{F}_{\mathfrak{C}[\square^{d-1}]}((0, \ldots, 0), (1, \ldots, 1))$ turns out to be a model for Stasheff's associahedron K_d. In a similar way, ∂K_d arises as a Hom-simplicial set in $\mathfrak{C}[\partial \square^{d-1}]$. For a greatly expanded version of this explanation see [L-HA, §4.1.6].

7.8. Lemma ("Equivalences of $\mathbb{E}_1$-monoids can be checked on underlying objects"). — *Let $\mathcal{C}$ be an ∞-category with finite products. Then a morphism $f : M \to N$ in $\mathrm{Mon}(\mathcal{C})$ is an equivalence if and only if $f_1 : M_1 \to N_1$ is an equivalence.*

Proof. This follows immediately from Theorem 4.5 and the Segal condition. $\square$

7.9. Lemma. — *For an $\mathbb{E}_1$-monoid M in animae, the following conditions are equivalent:*

(a)   *M is an $\mathbb{E}_1$-group.*

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(b)   *For every $a \in M_1$, the "left multiplication map" $a \cdot (-)\colon M_1 \simeq \{a\} \times M_1 \xrightarrow{\mu} M_1$ is an equivalence.*

(c)   *For every $a \in M_1$, the "right multiplication map" $(-) \cdot a\colon M_1 \simeq M_1 \times \{a\} \xrightarrow{\mu} M_1$ is an equivalence.*

(d)   *The ordinary monoid $\pi_0(M) \in \mathrm{Mon}(\mathrm{Set})$ is a group.*

Proof. We prove $(a) \Leftrightarrow (b)$ first. Using Theorem 3.18, Lemma 3.19, and the five lemma (plus Remark 3.20), we see that the shearing map $(\mathrm{pr}_1, \mu)\colon M_1 \times M_1 \to M_1 \times M_1$ is an equivalence if and only if it induces equivalences on all fibres of $\mathrm{pr}_1\colon M_1 \times M_1 \to M_1$. The induced map on fibres over $a \in M_1$ is precisely $a \cdot (-)$. This already proves $(a) \Leftrightarrow (b)$.

The implication $(b) \Rightarrow (d)$ is clear, since the condition from (b) implies that for every equivalence class $[a] \in \pi_0(M)$, left multiplication with $[a]$ is a bijection. The same argument shows $(c) \Rightarrow (d)$. For $(d) \Rightarrow (b)$, note that associativity of the multiplication of M implies

$$\big(b \cdot (-)\big) \circ \big(c \cdot (-)\big) \simeq \big((b \cdot c) \cdot (-)\big)$$

for all $b, c \in M_1$. Since $\pi_0(M)$ is assumed to be a group, there exists an element $b \in M_1$ such that $a \cdot b \simeq 1 \simeq b \cdot a$, where $1 \in M_1$ is the identity element, that is, the image of $* \simeq M_0$ under $s_0\colon M_0 \to M_1$. Since $1 \cdot (-)\colon M_1 \to M_1$ is homotopic to id_{M_1}, as we've seen in 7.7, the equivalence above shows that $b \cdot (-)$ is both a left inverse and a right inverse to $a \cdot (-)$. So $a \cdot (-)\colon M_1 \to M_1$ is an equivalence. This finishes the proof of $(d) \Rightarrow (b)$. An analogous argument shows $(d) \Rightarrow (c)$. □

The main theorem of this subsection is Stasheff's *recognition principle* for loop spaces:

7.10. Theorem ("$\mathbb{E}_1$-groups are the same as loop animae"). — *Let $((\mathrm{Cat}_\infty)_{*/})_{\geqslant 1} \subseteq (\mathrm{Cat}_\infty)_{*/}$ be the full sub-∞-category of all (small) pointed ∞-categories $(\mathcal{C}, x)$ for which $\pi_0 \operatorname{core}(\mathcal{C}) \cong *$ and let $(\mathrm{An}_{*/})_{\geqslant 1} \subseteq ((\mathrm{Cat}_\infty)_{*/})_{\geqslant 1}$ be the full sub-∞-category spanned those pointed animae (X, x) where $\pi_0(X) \cong *$.*

(a)   *There is an equivalence of ∞-categories*

$$\mathrm{Mon}(\mathrm{An}) \xrightarrow{\simeq} \big((\mathrm{Cat}_\infty)_{*/}\big)_{\geqslant 1} \,.$$

(b)   *There is an adjunction $\mathrm{B}\colon \mathrm{Mon}(\mathrm{An}) \rightleftarrows \mathrm{An}_{*/} \colon \Omega$ which induces a pair of inverse equivalences*

$$\mathrm{B}\colon \mathrm{Grp}(\mathrm{An}) \underset{\simeq}{\overset{\simeq}{\rightleftarrows}} \big(\mathrm{An}_{*/}\big)_{\geqslant 1} \colon \Omega \,.$$

7.11. Remark. — The intuition behind Theorem 7.10 is easy to explain: If $(\mathcal{C}, x)$ is a pointed ∞-category, such that $\pi_0 \operatorname{core}(\mathcal{C}) \cong *$, then $\mathrm{Hom}_{\mathcal{C}}(x, x)$ is an $\mathbb{E}_1$-monoid via composition. Coversely, if M is an $\mathbb{E}_1$-monoid, then we can build an ∞-category $\mathrm{B}^+ M$ with only one object $*$ and $\mathrm{Hom}_{\mathrm{B}^+ M}(*, *) \simeq M$; the composition is dictated by the multiplication on M. Hence Theorem 7.10(a). Furthermore, $\mathcal{C}$ is an anima if and only if every morphism in $\mathrm{Hom}_{\mathcal{C}}(x, x)$ is invertible, which is equivalent to $\mathrm{Hom}_{\mathcal{C}}(x, x)$ being an $\mathbb{E}_1$-group by Lemma 7.9. Hence Theorem 7.10(b). Unfortunately, making this intuition formal requires a lot more work.

The proof of Theorem 7.10 will be rather lengthy. We'll first show Theorem 7.10(a), up to a pretty serious black box (Theorem 7.13). Theorem 7.10(b) could then be obtained as a simple consequence, but instead, we'll give a proof that avoids the aforementioned black box.

Our first goal on our way towards Theorem 7.10(a) is to construct an $\mathbb{E}_1$-monoid structure on the anima $\mathrm{End}_{\mathcal{C}}(x) \coloneqq \mathrm{Hom}_{\mathcal{C}}(x,x)$ of endomorphisms of x. This requires a construction which is quite interesting in its own right.

7.12. Construction. — Consider the functor $U\colon \Delta \to \mathrm{Cat}_\infty$ that sends $[n] \mapsto \Delta^n$ (or, if you want, $[n] \mapsto [n]$, since we suppress writing nerves). To construct U formally, observe that it already exists as a functor $\Delta \to \mathrm{QCat}$ of ordinary categories and use Theorem 4.13. Alternatively, one can write down the unstraightening explicitly; it will be an ordinary category over Δ. Using Theorem 6.30, U induces an adjunction

$$\mathrm{asscat}\colon \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{An}) \rightleftarrows \mathrm{Cat}_\infty : \mathrm{N}^{\mathrm{Rezk}}\ .$$

Here asscat stands for *associated category*[7.1], $\mathrm{N}^{\mathrm{Rezk}}$ is the *Rezk nerve.* According to Lemma 6.32, the Rezk nerve is given by $\mathrm{N}^{\mathrm{Rezk}}(\mathcal{C})_n \simeq \mathrm{Hom}_{\mathrm{Cat}_\infty}(\Delta^n, \mathcal{C})$ for every ∞-category $\mathcal{C}$ and all $n \geqslant 0$.

To prove Theorem 7.10(a), we'll need the following black box. Fortunately, a relatively short proof in model-independent language has recently been found by Fabian and Jan Steinebrunner [FS23]. The original proof due to Joyal and Tierney is in [JT07]; Lurie has given another proof in [Lur09].

7.13. Theorem. — *The Rezk nerve* $\mathrm{N}^{\mathrm{Rezk}}\colon \mathrm{Cat}_\infty \to \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{An})$ *is fully faithful and its image is given by the complete Segal animae. Here, a simplicial anima* $X\colon \Delta^{\mathrm{op}} \to \mathrm{An}$ *is called Segal if the Segal maps* $e_i\colon [1] \to [n]$ *induce equivalences*

$$X_n \xrightarrow{\simeq} \underbrace{X_1 \times_{X_0} \cdots \times_{X_0} X_1}_{n \text{ factors}}\ .$$

Furthermore, X is called complete, if $s_0^*\colon X_0 \to X_1$ *is an equivalence onto the collection of path components* $X_1^\sim \subseteq X_1$ *given by those* $\alpha \in X_1$ *for which there exist* $\sigma, \tau \in X_2$ *such that* $d_0^*(\sigma) \simeq \alpha \simeq d_2^*(\tau)$ *and both* $d_1^*(\sigma), d_1^*(\tau)$ *lie in the image of* $s_0^*\colon X_0 \to X_1$. ∎

Let us now construct the desired $\mathbb{E}_1$-monoid structure on $\mathrm{End}_{\mathcal{C}}(x)$.

7.14. Construction. — If $M \in \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{An})$ is an $\mathbb{E}_1$-monoid, then $M_0 \simeq *$. Via Yoneda's lemma, this induces a canonical morphism $よ_\Delta([0]) \simeq \mathrm{const}\, * \to M$ of $\mathbb{E}_1$-monoids. Accordingly, we get a canonical morphism $\mathrm{asscat}(\mathrm{const}\, *) \to M$. Since $\mathrm{asscat}(\mathrm{const}\, *) \simeq \mathrm{asscat}(よ_\Delta([0])) \simeq U([0]) \simeq *$, the morphism above canonically turns $\mathrm{asscat}(M)$ into a pointed ∞-category and so asscat upgrades to a functor $\mathrm{B}^+\colon \mathrm{Mon}(\mathrm{An}) \to (\mathrm{Cat}_\infty)_{*/}$.[7.2] For a pointed ∞-category $(\mathcal{C}, x)$, let, temporarily, $\mathrm{End}_{\mathcal{C}}(x) \in \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{An})$ be redefined as the pullback

$$\begin{array}{ccc}
\mathrm{End}_{\mathcal{C}}(x) & \Longrightarrow & \mathrm{N}^{\mathrm{Rezk}}(\mathcal{C}) \\
\Big\Vert & \llcorner & \Big\Vert u \\
\mathrm{const}\{x\} & \Longrightarrow & \mathrm{Ran}_{\{[0]\} \to \Delta^{\mathrm{op}}}\, \mathrm{N}^{\mathrm{Rezk}}(\mathcal{C})\big|_{\{[0]\}}
\end{array}$$

in $\mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{An})$. The right vertical arrow u is the unit transformation from a functor to the right Kan extension of its restriction. For the bottom horizontal arrow, note that since $(\mathcal{C}, x)$

[7.1]…and it has nothing to do with asinine felines (or worse). Why would you think that?!

[7.2]Note that B^+ is non-standard notation; there doesn't seem to be any standard notation.

is a pointed ∞-category, there is a canonical morphism $\{x\} \to \operatorname{core}(\mathcal{C}) \simeq \mathrm{N}^{\mathrm{Rezk}}(\mathcal{C})_0$; then the desired natural transformation $\operatorname{const}\{x\} \Rightarrow \operatorname{Ran}_{\{[0]\} \to \Delta^{\mathrm{op}}} \mathrm{N}^{\mathrm{Rezk}}(\mathcal{C})|_{\{[0]\}}$ is induced by the universal property of right Kan extension. It's straightforward to check that the right vertical and bottom horizontal arrows are functorial. Since taking pullbacks is functorial too, we get a functor $\operatorname{End} \colon (\mathrm{Cat}_\infty)_{*/} \to \operatorname{Fun}(\Delta^{\mathrm{op}}, \mathrm{An})$, as desired.

7.15. Lemma. — *The simplicial anima* $\operatorname{End}_{\mathcal{C}}(x)$ *from Construction* 7.14 *is an* $\mathbb{E}_1$-*monoid and its underlying anima* $\operatorname{End}_{\mathcal{C}}(x)_1$ *is the anima* $\operatorname{Hom}_{\mathcal{C}}(x,x)$ *of endomorphisms of* x. *Furthermore, the functors from Construction* 7.14 *fit into an adjunction*

$$\mathrm{B}^+ \colon \operatorname{Mon}(\mathrm{An}) \rightleftarrows (\mathrm{Cat}_\infty)_{*/} \colon \operatorname{End} .$$

Proof. Let's check first that End takes values in $\operatorname{Mon}(\mathrm{An}) \subseteq \operatorname{Fun}(\Delta^{\mathrm{op}}, \mathrm{An})$ and that the underlying anima of $\operatorname{End}_{\mathcal{C}}(x)$ is indeed $\operatorname{Hom}_{\mathcal{C}}(x,x)$. To this end, fix $n \geqslant 0$; we'll compute $\operatorname{End}_{\mathcal{C}}(x)_n$. Recall that $\mathrm{N}^{\mathrm{Rezk}}(\mathcal{C})_n \simeq \operatorname{Hom}_{\mathrm{Cat}_\infty}(\Delta^n, \mathcal{C})$. To compute the right-Kan extension, we use the formula from Lemma 6.27: The slice ∞-category $\{[0]\}_{/[n]}$ has $n+1$ objects, namely the morphisms $[0] \to \{j\} \to [n]$ for $0 \leqslant j \leqslant n$, and there are no non-identity morphisms in $\{[0]\}_{/[n]}$. So the Kan extension formula is just a limit over a discrete diagram with $n+1$ objects, which leads to $(\operatorname{Ran}_{\{[0]\} \to \Delta^{\mathrm{op}}} \mathrm{N}^{\mathrm{Rezk}}(\mathcal{C})|_{\{[0]\}})_n \simeq \operatorname{core}(\mathcal{C})^{n+1}$. Furthermore, a quick unravelling shows that the morphism u from Construction 7.14 can be identified with

$$\operatorname{Hom}_{\mathrm{Cat}_\infty}(\Delta^n, \mathcal{C}) \to \operatorname{Hom}_{\mathrm{Cat}_\infty}(\{0\} \sqcup \cdots \sqcup \{n\}, \mathcal{C}) \simeq \operatorname{core}(\mathcal{C})^{n+1} .$$

Now observe that Δ^n can be written as $\Delta^n \simeq \Delta^{\{0,1\}} \sqcup_{\{1\}} \Delta^{\{1,2\}} \sqcup_{\{2\}} \cdots \sqcup_{\{n-1\}} \Delta^{\{n-1,n\}}$ in Cat_∞.[(7.3)] Identifying $\operatorname{Hom}_{\mathrm{Cat}_\infty}(\Delta^{\{i-1,i\}}, \mathcal{C}) \simeq \operatorname{core} \operatorname{Ar}(\mathcal{C})$ via Theorem 2.24, we obtain

$$\operatorname{Hom}_{\mathrm{Cat}_\infty}(\Delta^n, \mathcal{C}) \simeq \underbrace{\operatorname{core} \operatorname{Ar}(\mathcal{C}) \times_{t,\operatorname{core}(\mathcal{C}),s} \cdots \times_{t,\operatorname{core}(\mathcal{C}),s} \operatorname{core} \operatorname{Ar}(\mathcal{C})}_{n \text{ factors}}$$

from Corollary 6.16. Recall from Lemma 6.12 that pullbacks in $\operatorname{Fun}(\Delta^{\mathrm{op}}, \mathrm{An})$ are computed degree-wise. So $\operatorname{End}_{\mathcal{C}}(x)_n$ is the pullback $\{x\} \times_{\operatorname{core}(\mathcal{C})^{n+1}} \operatorname{Hom}_{\mathrm{Cat}_\infty}(\Delta^n, \mathcal{C})$. Plugging in the formula above, we see

$$\operatorname{End}_{\mathcal{C}}(x)_n \simeq \operatorname{Hom}_{\mathcal{C}}(x,x)^n$$

by a simple manipulation of pullbacks. So we've achieved two things at once: We've shown that $\operatorname{End}_{\mathcal{C}}(x)$ satisfies the conditions from Definition 7.6, so that it is an $\mathbb{E}_1$-monoid, and that the underlying anima of that $\mathbb{E}_1$-monoid is indeed $\operatorname{Hom}_{\mathcal{C}}(x,x)$.

It remains to show that End is right adjoint to B^+. So let $M \in \operatorname{Mon}(\mathrm{An})$. The universal property of right Kan extensions combined with $M_0 \simeq *$ shows

$$\operatorname{Hom}_{\operatorname{Fun}(\Delta^{\mathrm{op}}, \mathrm{An})}\left(M, \operatorname{Ran}_{\{[0]\} \to \Delta^{\mathrm{op}}} \mathrm{N}^{\mathrm{Rezk}}(\mathcal{C})|_{\{[0]\}}\right) \simeq \operatorname{Hom}_{\mathrm{An}}(M_0, \operatorname{core}(\mathcal{C})) \simeq \operatorname{core}(\mathcal{C})$$

This allows us to compute

$$\begin{aligned}
\operatorname{Hom}_{\operatorname{Fun}(\Delta^{\mathrm{op}}, \mathrm{An})}\left(M, \operatorname{End}_{\mathcal{C}}(x)\right) &\simeq \operatorname{Hom}_{\operatorname{Fun}(\Delta^{\mathrm{op}}, \mathrm{An})}\left(M, \mathrm{N}^{\mathrm{Rezk}}(\mathcal{C})\right) \times_{\operatorname{core}(\mathcal{C})} \{x\} \\
&\simeq \operatorname{Hom}_{\mathrm{Cat}_\infty}\left(\operatorname{asscat}(M), \mathcal{C}\right) \times_{\operatorname{Hom}_{\mathrm{Cat}_\infty}(*, \mathcal{C})} \{x\} \\
&\simeq \operatorname{Hom}_{(\mathrm{Cat}_\infty)_{*/}}\left(\mathrm{B}^+ M, (\mathcal{C}, x)\right) .
\end{aligned}$$

[(7.3)]One way would to see this is to observe that the pushout in sSet would just be I^n from the proof of Theorem 4.6 and that $I^n \subseteq \Delta^n$ is inner anodyne, so that Δ^n is the pushout in Cat_∞ by model category fact 6.13. Another way would be to use Lemma 6.14 and think hard about the localisation.

In the first step we use that $\mathrm{Hom}_{\mathrm{Fun}(\Delta^{\mathrm{op}},\mathrm{An})}(M, -)$ commutes with pullbacks by Corollary 6.17 together with the above simplification. In the second step, we use the adjunction asscat $\dashv \mathrm{N}^{\mathrm{Rezk}}$ as well as $\mathrm{core}(\mathcal{C}) \simeq \mathrm{Hom}_{\mathrm{Cat}_\infty}(*, \mathcal{C})$. In the third step we use Corollary 5.15. It's easy to make all steps functorial in M and $(\mathcal{C}, x)$ and so the proof is finished. $\qquad\square$

Proof sketch of Theorem 7.10(*a*). Observe that a morphism $(\mathcal{C}, x) \to (\mathcal{D}, y)$ in $((\mathrm{Cat}_\infty)_{*/})_{\geqslant 1}$ is automatically essentially surjective. Hence any such morphism is an equivalence if and only if $\mathrm{Hom}_\mathcal{C}(x, x) \to \mathrm{Hom}_\mathcal{D}(y, y)$ is an equivalence. This immediately shows that the right adjoint $\mathrm{End}\colon ((\mathrm{Cat}_\infty)_{*/})_{\geqslant 1} \to \mathrm{Mon}(\mathrm{An})$ is conservative. It follows from Theorem 7.13, or more precisely, from [FS23, Corollary 3.15], that $\mathrm{Hom}_{\mathrm{B}^+M}(*, *) \simeq M$ holds for all $M \in \mathrm{Mon}(\mathrm{An})$. Using Lemma 7.8, it follows that the unit $u_M\colon M \to \mathrm{End}_{\mathrm{B}^+M}(*)$ is an equivalence. Hence $\mathrm{B}^+\colon \mathrm{Mon}(\mathrm{An}) \to ((\mathrm{Cat}_\infty)_{*/})_{\geqslant 1}$ is fully faithful by Lemma 6.33(*a*). Then Lemma 6.33(*b*) shows that B^+ and End are inverse equivalences. $\qquad\square$

Let us now turn to Theorem 7.10(*b*). The proof will consist of two parts: a formal part, in which we effortlessly deduce the adjunction $\mathrm{B}\colon \mathrm{Mon}(\mathrm{An}) \rightleftarrows \mathrm{An}_{*/} :\Omega$, and a hard part, in which we compute $\Omega \mathrm{B}G$ for every $\mathbb{E}_1$-group G to establish the equivalence $\mathrm{Grp}(\mathrm{An}) \simeq (\mathrm{An}_{*/})_{\geqslant 1}$.

Proof sketch of Theorem 7.10(*b*), *formal part.* Let $\Omega := \mathrm{End}|_{\mathrm{An}_{*/}}\colon \mathrm{An}_{*/} \to \mathrm{Mon}(\mathrm{An})$ denote the restriction of End from Construction 7.14 to $\mathrm{An}_{*/} \subseteq (\mathrm{Cat}_\infty)_{*/}$. It follows from Lemma 7.3(*b*) that the underlying anima of $\Omega X \in \mathrm{Mon}(\mathrm{An})$ is indeed the eponymous ΩX from Definition 7.1. Now we claim:

($\boxtimes_1$) *The functor* $\Omega\colon \mathrm{An}_{*/} \to \mathrm{Mon}(\mathrm{An})$ *factors through* $\mathrm{Grp}(\mathrm{An}) \subseteq \mathrm{Mon}(\mathrm{An})$. *Furthermore,* Ω *admits a left adjoint* $\mathrm{B}\colon \mathrm{Mon}(\mathrm{An}) \to \mathrm{An}_{*/}$, *which factors through* $(\mathrm{An}_{*/})_{\geqslant 1} \subseteq \mathrm{An}_{*/}$.

To see that ΩX is an $\mathbb{E}_\infty$-group, one can use Lemma 7.9(*d*) for example: $\pi_0(\Omega X) \cong \pi_1(X, x)$ is a group by Lemma 7.3(*a*).

Next, let's construct B. Using Corollary 5.15 and $|*| \simeq *$, it's straightforward to check that $|\cdot|\colon \mathrm{Cat}_\infty \to \mathrm{An}$ induces a functor $|\cdot|\colon (\mathrm{Cat}_\infty)_{*/} \to \mathrm{An}_{*/}$ which is left adjoint to the inclusion $\mathrm{An}_{*/} \subseteq (\mathrm{Cat}_\infty)_{*/}$. We then let $\mathrm{B} := |\mathrm{B}^+(-)|\colon \mathrm{Mon}(\mathrm{An}) \to \mathrm{An}_{*/}$ denote the *delooping* functor. From the diagram

$$\mathrm{Mon}(\mathrm{An}) \underset{\mathrm{End}}{\overset{\mathrm{B}^+}{\rightleftarrows}} (\mathrm{Cat}_\infty)_{*/}$$

it's immediate that B and Ω are adjoints. To show that B lands in $(\mathrm{An}_{*/})_{\geqslant 1}$, we need an alternative description of B. By construction, the composition of $\mathrm{B}\colon \mathrm{Mon}(\mathrm{An}) \to \mathrm{An}_{*/}$ with $\mathrm{An}_{*/} \to \mathrm{An}$ agrees with $|\mathrm{asscat}(-)|\colon \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{An}) \to \mathrm{An}$. Note that this functor preserves all colimits, because so do asscat$\colon \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{An}) \to \mathrm{Cat}_\infty$ and $|\cdot|\colon \mathrm{Cat}_\infty \to \mathrm{An}$. By Theorem 6.30, $|\mathrm{asscat}(-)|$ must be the unique colimit-preserving extension of the functor $\Delta \to \mathrm{An}$ sending $[n] \mapsto |\Delta^n| \simeq *$; that is, $|\mathrm{asscat}(-)|$ is the unique colimit-preserving extension of the constant functor $\mathrm{const}\,*\colon \Delta \to \mathrm{An}$. On the other hand, $\mathrm{colim}_{\Delta^{\mathrm{op}}}\colon \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{An}) \to \mathrm{An}$ also preserves colimits, since it is a left adjoint by definition. Moreover, $\mathrm{colim}_{\Delta^{\mathrm{op}}}\,✄_\Delta([n]) \simeq |\Delta_{/[n]}| \simeq *$ by Lemma 6.14 and the fact that $\Delta_{/[n]}$ has a terminal object. So $\mathrm{colim}_{\Delta^{\mathrm{op}}}\colon \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{An}) \to \mathrm{An}$ is also the unique colimit-preserving extension of $\mathrm{const}\,*$. It follows that if $M \in \mathrm{Mon}(\mathrm{An})$, then the underlying unpointed anima of $\mathrm{B}M$ is $\mathrm{colim}_{[n]\in\Delta^{\mathrm{op}}} M_n$, and the point $* \to \mathrm{B}M$ comes via $* \simeq M_0 \to \mathrm{colim}_{[n]\in\Delta^{\mathrm{op}}} M_n$. We've seen in Lemma 6.58 that $\pi_0\colon \mathrm{An} \to \mathrm{Set}$ commutes

with arbitrary colimits. So we get a bijection of sets $\pi_0(\mathrm{B}M) \cong \mathrm{colim}_{[n]\in\Delta^{\mathrm{op}}} \pi_0(M_n)$. Using $\pi_0(M_0) \cong *$, it's straightforward to check that the colimit must be $*$ as well.[(7.4)] This finishes the proof of ($\boxtimes_1$).

In particular, we obtain a restricted adjunction $\mathrm{B}\colon \mathrm{Grp}(\mathrm{An}) \rightleftarrows (\mathrm{An}_{*/})_{\geqslant 1} \colon \Omega$. To show that this is a pair of inverse equivalences, it's enough to show that B is fully faithful and that Ω is conservative; see Lemma 6.33(*b*). The latter is easy. If $\mathrm{ev}_{[1]}\colon \mathrm{Mon}(\mathrm{An}) \to \mathrm{An}$ is the functor that sends an $\mathbb{E}_1$-monoid to its underlying anima is conservative, then already

$$\mathrm{An}_{*/} \xrightarrow{\;\Omega\;} \mathrm{Mon}(\mathrm{An}) \xrightarrow{\;\mathrm{ev}_{[1]}\;} \mathrm{An}$$

is conservative. Indeed, this composition is the loop functor $\Omega\colon \mathrm{An}_{*/} \to \mathrm{An}$. Since any morphism $(X, x) \to (Y, y)$ in $(\mathrm{An}_{*/})_{\geqslant 1}$ is automatically a bijection on π_0, Theorem 3.18 and Lemma 7.3(*a*) show that such a morphism is an equivalence if and only if $\Omega_x X \to \Omega_y Y$ is an equivalence. This proves that Ω is indeed conservative. To prove that B is fully faithful, we will need another claim:

($\boxtimes_2$) *For every $\mathbb{E}_1$-group $G \in \mathrm{Grp}(\mathrm{An})$, the unit transformation $u_G\colon G \to \Omega\mathrm{B}G$ is an equivalence on underlying animae.*

If we can show ($\boxtimes_2$), then u_G will also be an equivalence of $\mathbb{E}_1$-groups by Lemma 7.8. So B is fully faithful by Lemma 6.33(*a*) and we would be done. The proof of ($\boxtimes_2$) requires some further tools, and we postpone it for now. $\qquad\square$

The main difficulty in the proof of ($\boxtimes_2$) is the fact that $\mathrm{B}G$ is defined as a colimit, whereas Ω is a pullback. So we need to commute pullbacks and (non-filtered) colimits. Fortunately, there's a relatively simple criterion due to Charles Rezk [Rez14, Proposition 2.4] that allows us to do this in certain situations.

7.16. Lemma. — *Let $\mathcal{J}$ be an ∞-category. A natural transformation $q\colon B \Rightarrow D$ in $\mathrm{Fun}(\mathcal{J}, \mathrm{An})$ is called equifibred if for every morphism $\alpha\colon i \to j$ in $\mathcal{J}$, the induced diagram*

$$
\begin{array}{ccc}
B_i & \xrightarrow{\;B(\alpha)\;} & B_j \\
{\scriptstyle q_i}\downarrow & \lrcorner & \downarrow{\scriptstyle q_j} \\
D_i & \xrightarrow{\;D(\alpha)\;} & D_j
\end{array}
$$

is a pullback square in An. Then the colimit functor $\mathrm{colim}_{\mathcal{J}}\colon \mathrm{Fun}(\mathcal{J}, \mathrm{An}) \to \mathrm{An}$ preserves pullback squares in which one leg is equifibred. That is, if we're given a pullback square

$$
\begin{array}{ccc}
A & \Longrightarrow & B \\
{\scriptstyle p}\Vert & \lrcorner & \Vert{\scriptstyle q} \\
C & \Longrightarrow & D
\end{array}
$$

in $\mathrm{Fun}(\mathcal{J}, \mathrm{An})$ such that $q\colon B \Rightarrow D$ is equifibred, then $\mathrm{colim}_{\mathcal{J}}\colon \mathrm{Fun}(\mathcal{J}, \mathrm{An}) \to \mathrm{An}$ sends this diagram to a pullback square in An.

[(7.4)]In fact, if $S\colon \Delta^{\mathrm{op}} \to \mathcal{C}$ is any functor into an ordinary category, then the colimit of S is given by the coequaliser

$$\mathrm{colim}_{[n]\in\Delta^{\mathrm{op}}} S_n \cong \mathrm{coeq}\left(S_1 \underset{d_1^*}{\overset{d_0^*}{\rightrightarrows}} S_0 \right).$$

(assuming either colimit exists). This formula is wildly false in general ∞-categories, as already evidenced by $\mathrm{B}M \simeq \mathrm{colim}_{[n]\in\Delta^{\mathrm{op}}} M_n$ in An.

The idea to prove Lemma 7.16 is to interpret $q_i\colon B_i \to D_i$ as the unstraightenings of certain functors $G_i\colon D_i \to \mathrm{An}$ and then to use Lemma 6.14 backwards. To this end, we need to study the straightening equivalence from Theorem 5.4 a little more.

7.17. The universal unstraightening. — Let κ be a regular cardinal and let $\mathrm{An}^{<\kappa} \subseteq \mathrm{An}$ be the full sub-∞-category of essentially κ-small animae as in Definition 6.46. Then $\mathrm{An}^{<\kappa}$ is essentially small itself (albeit not necessarily essentially κ-small) and so we can consider the unstraightening $p_{\mathrm{univ}}^{<\kappa}\colon \mathcal{U}_{\mathrm{univ}}^{<\kappa} \to \mathrm{An}^{<\kappa}$ of $\mathrm{An}^{<k} \to \mathrm{An}$ and we can regard $\mathrm{An}^{<\kappa}$ as an object in Cat_∞. If $p\colon \mathcal{U} \to \mathcal{C}$ is any left fibration with essentially κ-small fibres over an ∞-category $\mathcal{C}$ and $F \simeq \mathrm{St}^{(\mathrm{left})}(p)\colon \mathcal{C} \to \mathrm{An}$ is the associated functor, then F factors through $\mathrm{An}^{<\kappa}$. Since precompositions are sent to pullbacks by Theorem 5.4(a), it follows that there must be a pullback diagram

$$
\begin{array}{ccc}
\mathcal{U} & \longrightarrow & \mathcal{U}_{\mathrm{univ}}^{<\kappa} \\
{\scriptstyle p}\big\downarrow & \lrcorner & \big\downarrow{\scriptstyle p_{\mathrm{univ}}^{<\kappa}} \\
X & \xrightarrow{\ F\ } & \mathrm{An}^{<\kappa}
\end{array}
$$

in animae. So $p_{\mathrm{univ}}^{<\kappa}$ acts as a *universal unstraightening*, whence the notation. Of course, what we would really like to do here is to consider the unstraightening $p_{\mathrm{univ}}\colon \mathcal{U}_{\mathrm{univ}} \to \mathrm{An}$ of $\mathrm{id}_{\mathrm{An}}\colon \mathrm{An} \to \mathrm{An}$ and regard An as an object in Cat_∞. The only way to do this without any set theorist suffering a stroke would be to consider universes, which amounts to choosing a strongly inaccessible cardinal bound. It turns out that any cardinal bound κ does it, so we can get away without using universes.

7.18. Lemma. — *Consider the slice ∞-category $\mathrm{An}_{/\mathrm{An}^{<\kappa}} \simeq \mathrm{An} \times_{\mathrm{Cat}_\infty} (\mathrm{Cat}_\infty)_{/\mathrm{An}^{<\kappa}}$ and let $\mathrm{Ar}_{\lrcorner}^{<\kappa}(\mathrm{An}) \subseteq \mathrm{Ar}(\mathrm{An})$ be the (non-full) sub-∞-category, in the sense of 2.16, spanned by those objects $(\alpha\colon X \to Y) \in \mathrm{Ar}(\mathrm{An})$ for which the fibres of α are essentially κ-small and those morphisms $(\alpha\colon X \to Y) \to (\alpha'\colon X' \to Y')$ that represent pullback squares in An. Then there is an equivalence of ∞-categories*

$$
(p_{\mathrm{univ}}^{<\kappa})^*\colon \mathrm{An}_{/\mathrm{An}^{<\kappa}} \xrightarrow{\ \simeq\ } \mathrm{Ar}_{\lrcorner}^{<\kappa}(\mathrm{An})
$$

that sends an object $(F\colon X \to \mathrm{An}^{<\kappa}) \in \mathrm{An}_{/\mathrm{An}^{<\kappa}}$ to the pullback $(X \times_{\mathrm{An}^{<\kappa}} \mathcal{U}_{\mathrm{univ}}^{<\kappa} \to X)$, or equivalently (by 7.17) to the unstraightening of F.

Proof sketch. It's easy to construct $(p_{\mathrm{univ}}^{<\kappa})^*$ formally and we'll only sketch the necessary steps. First, one constructs a functor $\mathrm{An}_{/\mathrm{An}^{<\kappa}} \to \mathrm{Fun}(\Lambda_2^2, \mathrm{Cat}_\infty)$ that sends $(X \to \mathrm{An}^{<\kappa})$ to the span $(X \to \mathrm{An}^{<\kappa} \leftarrow \mathcal{U}_{\mathrm{univ}}^{<\kappa})$. To do so, let $\mathcal{X} \coloneqq \mathrm{Fun}(\Lambda_2^2, \mathrm{Cat}_\infty) \times_{\mathrm{Fun}(\Delta^{\{1,2\}}, \mathrm{Cat}_\infty)} \{p_{\mathrm{univ}}^{<\kappa}\}$ be the ∞-category of those spans whose second leg is $p_{\mathrm{univ}}^{<\kappa}$. There's a functor

$$
\mathcal{X} \to \mathrm{Fun}\big(\Delta^{\{0,2\}}, \mathrm{Cat}_\infty\big) \times_{\mathrm{Fun}(\{2\}, \mathrm{Cat}_\infty)} \{\mathrm{An}^{<\kappa}\} \simeq (\mathrm{Cat}_\infty)_{/\mathrm{An}^{<\kappa}}
$$

sending a span whose second leg is $p_{\mathrm{univ}}^{<\kappa}$ to its first leg. This functor is clearly essentially surjective, and one easily checks that it is fully faithful too, using the formulas from Corollary 5.15 and Lemma 6.76(a). Hence we get an equivalence by Theorem 4.6. Choosing an inverse of this equivalence yields the desired functor $\mathrm{An}_{/\mathrm{An}^{<\kappa}} \to \mathrm{Fun}(\Lambda_2^2, \mathrm{Cat}_\infty)$.

Next, one constructs a functor $\mathrm{Fun}(\Lambda_2^2, \mathrm{Cat}_\infty) \to \mathrm{Ar}(\mathrm{Cat}_\infty)$ that sends a span $(\mathcal{C} \to \mathcal{D} \leftarrow \mathcal{D}')$ to the pullback $\mathcal{C} \times_{\mathcal{D}} \mathcal{D}' \to \mathcal{C}$. To do so, let $\mathcal{Y} \subseteq \mathrm{Fun}(\square^2, \mathrm{Cat}_\infty)$, where $\square^2 \simeq \Delta^1 \times \Delta^1$, be the full sub-$\infty$-category spanned by the pullback squares. Then $\mathcal{Y} \to \mathrm{Fun}(\Lambda_2^2, \mathrm{Cat}_\infty)$ is essentially

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surjective, since Cat_∞ has pullbacks, and fully faithful by an easy application of Corollary 6.25 and Corollary 6.16. Hence it is an equivalence by Theorem 4.6. Choosing an inverse and composing it with the projection $\mathrm{Fun}(\square^2, \mathrm{Cat}_\infty) \to \mathrm{Fun}(\Delta^1 \times \{0\}, \mathrm{Cat}_\infty) \simeq \mathrm{Ar}(\mathrm{Cat}_\infty)$ yields the desired functor $\mathrm{Fun}(\Lambda_2^2, \mathrm{Cat}_\infty) \to \mathrm{Ar}(\mathrm{Cat}_\infty)$.

Putting everything together yields a functor $\mathrm{An}_{/\mathrm{An}^{<\kappa}} \to \mathrm{Ar}(\mathrm{Cat}_\infty)$, which, on objects, sends $(X \to \mathrm{An}^{<\kappa})$ to $(X \times_{\mathrm{An}^{<\kappa}} \mathcal{U}_{\mathrm{univ}}^{<\kappa} \to X)$. By inspection, our functor factors through the non-full sub-∞-category $\mathrm{Ar}_{\lrcorner}^{<\kappa}(\mathrm{An}) \to \mathrm{Ar}(\mathrm{Cat}_\infty)$ and we obtain a functor $(p_{\mathrm{univ}}^{<\kappa})^*$, as desired.

To show that $(p_{\mathrm{univ}}^{<\kappa})^*$ is an equivalence, we'll once again verify that it is essentially surjective and fully faithful. Essential surjectivity reduces to the assertion that every morphism $\alpha \colon X \to Y$ is equivalent to a left fibration in $\mathrm{Ar}(\mathrm{An})$. Using the dual of Lemma 6.23(b), this reduces to checking that every initial morphism $X \to X'$ of animae is an equivalence. For *coinitial* morphisms, this follows from $X \simeq |X| \simeq \mathrm{colim}_{x \in X} * \simeq \mathrm{colim}_{x' \in X'} * \simeq |X'| \simeq X'$ using Lemma 6.14. For initial morphisms, we can use the same argument to show that $X^{\mathrm{op}} \to (X')^{\mathrm{op}}$ is an equivalence and then $X \to X'$ must be an equivalence too.

To show that $p_{\mathrm{univ}}^{<\kappa}$ is fully faithful, let $(F \colon X \to \mathrm{An}^{<\kappa})$ and $(G \colon Y \to \mathrm{An}^{<\kappa})$ be elements in $\mathrm{An}_{/\mathrm{An}^{<\kappa}}$ and let $p \colon \mathcal{U} \to X$ and $q \colon \mathcal{V} \to Y$ be the unstraightenings of F and G, respectively. For brevity, let us put

$$\mathrm{Hom}(F, G) \coloneqq \mathrm{Hom}_{\mathrm{An}_{/\mathrm{An}^{<\kappa}}}\big((F \colon X \to \mathrm{An}^{<\kappa}), (G \colon Y \to \mathrm{An}^{<\kappa})\big)$$

$$\mathrm{Hom}(p, q) \coloneqq \mathrm{Hom}_{\mathrm{Ar}_{\lrcorner}^{<\kappa}(\mathrm{An})}\big((p \colon \mathcal{U} \to X), (q \colon \mathcal{V} \to Y)\big)$$

By Corollary 5.15, $\mathrm{Hom}(F, G)$ is the pullback $\mathrm{Hom}_{\mathrm{Cat}_\infty}(X, Y) \times_{\mathrm{Hom}_{\mathrm{Cat}_\infty}(X, \mathrm{An}^{<\kappa})} \{F\}$. By Lemma 5.13 and Lemma 4.9, $\mathrm{Hom}(p, q)$ is a collection of path components of the pullback $\mathrm{Hom}_{\mathrm{Cat}_\infty}(\mathcal{U}, \mathcal{V}) \times_{\mathrm{Hom}_{\mathrm{Cat}_\infty}(\mathcal{U}, Y)} \mathrm{Hom}_{\mathrm{Cat}_\infty}(X, Y)$. By Theorem 3.18, Lemma 3.19, and the five lemma (plus Remark 3.20), it's enough to check that $\mathrm{Hom}(F, G) \to \mathrm{Hom}(p, q)$ induces an equivalence on fibres over $\mathrm{Hom}_{\mathrm{Cat}_\infty}(X, Y)$. So fix $f \colon X \to Y$. If $F \not\simeq G \circ f$, then both fibres are empty by Theorem 5.4. If $F \simeq G \circ f$, then the fibre $\{f\} \times_{\mathrm{Hom}_{\mathrm{Cat}_\infty}(X, Y)} \mathrm{Hom}(F, G)$ is given by $\{f\} \times_{\mathrm{Hom}_{\mathrm{Cat}_\infty}(X, Y)} (\mathrm{Hom}_{\mathrm{Cat}_\infty}(X, Y) \times_{\mathrm{Hom}_{\mathrm{Cat}_\infty}(X, \mathrm{An}^{<\kappa})} \{F\})$, which can be simplified to

$$\{G \circ f\} \times_{\mathrm{Hom}_{\mathrm{Cat}_\infty}(X, \mathrm{An}^{<\kappa})} \{F\} \simeq \mathrm{Hom}_{\mathrm{core\,Fun}(X, \mathrm{An})}(G \circ f, F)$$

using Theorem 2.24 and Lemma 7.3(b). Likewise, $\mathrm{Hom}(p, q) \times_{\mathrm{Hom}_{\mathrm{Cat}_\infty}(X, Y)} \{f\}$ is a collection of path components in $(\mathrm{Hom}_{\mathrm{Cat}_\infty}(\mathcal{U}, \mathcal{V}) \times_{\mathrm{Hom}_{\mathrm{Cat}_\infty}(\mathcal{U}, Y)} \mathrm{Hom}_{\mathrm{Cat}_\infty}(X, Y)) \times_{\mathrm{Hom}_{\mathrm{Cat}_\infty}(X, Y)} \{f\}$. This pullback can be simplified to

$$\mathrm{Hom}_{\mathrm{Cat}_\infty}(\mathcal{U}, \mathcal{V}) \times_{\mathrm{Hom}_{\mathrm{Cat}_\infty}(\mathcal{U}, Y)} \{p \circ f\} \simeq \mathrm{Hom}_{\mathrm{Cat}_{\infty/Y}}(\mathcal{U}, \mathcal{V}) \simeq \mathrm{Hom}_{\mathrm{Left}(X)}(\mathcal{U}, X \times_Y \mathcal{V})\,.$$

In the first step we use Corollary 5.15 and in the second step we use the dual of Lemma 6.23(c) combined with the fact that $\mathrm{Left}(X) \subseteq \mathrm{Cat}_{\infty/X}$ is a full sub-∞-category. Thus, the fibre $\mathrm{Hom}(p, q) \times_{\mathrm{Hom}_{\mathrm{Cat}_\infty}(X, Y)} \{f\}$ is a collection of path components of $\mathrm{Hom}_{\mathrm{Left}(X)}(\mathcal{U}, X \times_Y \mathcal{V})$. A quick unravelling shows that the relevant path components are precisely those morphisms $\mathcal{U} \to X \times_Y \mathcal{V}$ that are equivalences. Since $\mathrm{Left}(X) \simeq \mathrm{Fun}(X, \mathrm{An})$ by Theorem 5.4(b), this agrees with the collection of path components of $\mathrm{Hom}_{\mathrm{Fun}(X, \mathrm{An})}(G \circ f, F)$ spanned by the equivalences. By Lemma 4.9, this is precisely $\mathrm{Hom}_{\mathrm{core\,Fun}(X, \mathrm{An})}(G \circ f, F)$. This finishes the proof that $(p_{\mathrm{univ}}^{<\kappa})^*$ is fully faithful. $\square$

Proof sketch of Lemma 7.16. Choose a cardinal κ in such a way that all A_j, B_j, C_j, D_j, and the colimits are essentially κ-small in the sense of Definition 6.46. The natural transformation

$q\colon B \Rightarrow D$ in $\mathrm{Fun}(\mathcal{J}, \mathrm{An})$ can be viewed as a functor $q\colon \mathcal{J} \to \mathrm{Ar}(\mathrm{An})$. The assumption that q is equifibred and our choice of κ guarantee that q factors through $\mathrm{Ar}_{\lrcorner}^{<\kappa}(\mathrm{An})$. Applying the equivalence of ∞-categories $\mathrm{Ar}_{\lrcorner}^{<\kappa}(\mathrm{An}) \simeq \mathrm{An}_{/\mathrm{An}^{<\kappa}}$ from Lemma 7.18, we see that q corresponds to a functor $F\colon \mathcal{J} \to \mathrm{An}_{/\mathrm{An}^{<\kappa}}$. On objects, F sends $j \in \mathcal{J}$ to $(F_j\colon D_j \to \mathrm{An}^{<\kappa})$ in $\mathrm{An}_{/\mathrm{An}^{<\kappa}}$ such that $(q_j\colon B_j \to D_j)$ is the unstraightening of F_j. By definition of the slice-∞-category $\mathrm{An}_{/\mathrm{An}^{<\kappa}}$ we can view F as a natural transformation $\eta\colon D \Rightarrow \mathrm{const}\,\mathrm{An}^{<\kappa}$ in $\mathrm{Fun}(\mathcal{J}, \mathrm{Cat}_\infty)$, hence it induces a functor $F_\infty\colon \mathrm{colim}_{i \in \mathcal{I}} D_i \to \mathrm{An}^{<\kappa}$. We claim:

($\boxtimes$) *The unstraightening of F_∞ is $\mathrm{colim}_{j \in \mathcal{J}} B_j \to \mathrm{colim}_{j \in \mathcal{J}} D_j$. In particular, we obtain the following pullback square:*

$$
\begin{array}{ccc}
\displaystyle\operatorname*{colim}_{j \in \mathcal{J}} B_j & \longrightarrow & \mathcal{U}_{\mathrm{univ}}^{<\kappa} \\
\downarrow & \lrcorner & \downarrow{\scriptstyle p_{\mathrm{univ}}^{<\kappa}} \\
\displaystyle\operatorname*{colim}_{j \in \mathcal{J}} D_j & \xrightarrow{\ F_\infty\ } & \mathrm{An}^{<\kappa}
\end{array}
$$

If we know ($\boxtimes$), then we're done. Indeed, our construction of F above exhibits $q\colon B \Rightarrow D$ as a pullback of $\mathrm{const}\, p_{\mathrm{univ}}^{<\kappa}\colon \mathrm{const}\,\mathcal{U}_{\mathrm{univ}}^{<\kappa} \Rightarrow \mathrm{const}\,\mathrm{An}^{<\kappa}$. Then $p\colon A \Rightarrow C$ must be a pullback of $\mathrm{const}\, p_{\mathrm{univ}}^{<\kappa}$ as well, hence p is equifibred again. The same reasoning as above then shows that $\mathrm{colim}_{j \in \mathcal{J}} A_j \to \mathrm{colim}_{j \in \mathcal{J}} C_j$ must too be a pullback of $p_{\mathrm{univ}}^{<\kappa}\colon \mathcal{U}_{\mathrm{univ}}^{<\kappa} \to \mathrm{An}^{<\kappa}$. Hence the square formed by the colimits must be a pullback as well.

To prove ($\boxtimes$), note that $B_j \simeq |B_j| \simeq \mathrm{colim}(F_j\colon D_j \to \mathrm{An}^{<\kappa})$ follows from Lemma 6.14. So Lemma 6.38(b) shows

$$
\operatorname*{colim}_{j \in \mathcal{J}} B_i \simeq \operatorname*{colim}_{j \in \mathcal{J}}\big(\mathrm{colim}(F_j\colon D_j \to \mathrm{An}^{<\kappa})\big) \simeq \mathrm{colim}\Big(F_\infty\colon \operatorname*{colim}_{j \in \mathcal{J}} D_j \to \mathrm{An}^{<\kappa}\Big)
$$

But the colimit on the right-hand side is the unstraightening of F_∞, again by Lemma 6.14. However, there's a subtlety: To make this argument work, we have to show that the *functor* $B\colon \mathcal{J} \to \mathrm{An}^{<\kappa}$ agrees with the *functor* $(j \mapsto \mathrm{colim}\, F_j)\colon \mathcal{J} \to \mathrm{An}^{<\kappa}$ constructed in the proof of Lemma 6.38(b); let's temporarily denote this functor by B'. So far, we've only verified that the values of B and B' coincide!

To fix this, let $\mathcal{B} \to \mathcal{J}$ and $\mathcal{D} \to \mathcal{J}$ be the unstraightenings of the functors $B\colon \mathcal{J} \to \mathrm{An}^{<\kappa}$ and $D\colon \mathcal{J} \to \mathrm{An}^{<\kappa}$. Let's first recall the construction of B': By the proof of Lemma 6.38(b), we have a diagram

$$
\begin{array}{ccc}
\mathcal{D} & \xrightarrow{\ d\ } |\mathcal{D}| \simeq \displaystyle\operatorname*{colim}_{j \in \mathcal{J}} D_j \xrightarrow{\ F_\infty\ } \mathrm{An}^{<\kappa} \\
\downarrow & \qquad\qquad\nearrow \\
\mathcal{J} & \cdots\cdots\cdots\cdots\!\!\!\!\!\!\!\!\!_{B'}
\end{array}
$$

that exhibits B' as the left Kan extension of $F_\infty \circ d\colon \mathcal{D} \to \mathrm{An}^{<\kappa}$ along $\mathcal{D} \to \mathcal{J}$. We know from the dual of Lemma 6.23(c) how left Kan extensions for functors into An interact with unstraightening. Namely, the unstraightening $\mathcal{B}' \to \mathcal{J}$ of $B'\colon \mathcal{J} \to \mathrm{An}^{<\kappa}$ is given by factoring the unstraightening of $F_\infty \circ d\colon \mathcal{D} \to \mathrm{An}^{<\kappa}$ into an initial functor followed by a left fibration. In particular, if we can show that $\mathcal{B} \to \mathcal{D}$ is the unstraightening of $F_\infty \circ d\colon \mathcal{D} \to \mathrm{An}^{<\kappa}$, then we're done, because $\mathcal{B} \to \mathcal{J}$ is already a left fibration and so we would be able to deduce $\mathcal{B} \simeq \mathcal{B}'$.

To show that $\mathcal{B} \to \mathcal{D}$ is the desired unstraightening, recall that $F\colon \mathcal{J} \to \mathrm{An}_{/\mathrm{An}^{<\kappa}}$ is equivalently given by a natural transformation $\eta\colon D \Rightarrow \mathrm{const}\,\mathrm{An}^{<\kappa}$ in $\mathrm{Fun}(\mathcal{J}, \mathrm{An})$. So we

obtain a morphism $\mathcal{D} \to \mathcal{J} \times \mathrm{An}^{<\kappa}$ of cocartesian fibrations over $\mathcal{J}$. Now consider the diagram

$$
\begin{array}{ccccc}
\mathcal{B} & \longrightarrow & \mathcal{J} \times \mathcal{U}_{\mathrm{univ}}^{<\kappa} & \longrightarrow & \mathcal{U}_{\mathrm{univ}}^{<\kappa} \\
\downarrow & {\scriptstyle \cdot}\lrcorner & \downarrow & {\scriptstyle \cdot}\lrcorner & \downarrow{\scriptstyle p_{\mathrm{univ}}^{<\kappa}} \\
\mathcal{D} & \longrightarrow & \mathcal{J} \times \mathrm{An}^{<\kappa} & \longrightarrow & \mathrm{An}^{<\kappa}
\end{array}
$$

The right square is a pullback for obvious reasons. To see that the left square is a pullback too, observe that the natural transformation $q\colon B \Rightarrow D$ is the pullback of $\eta\colon D \Rightarrow \mathrm{const}\,\mathrm{An}^{<\kappa}$ along $\mathrm{const}\,p_{\mathrm{univ}}^{<\kappa}\colon \mathrm{const}\,\mathcal{U}_{\mathrm{univ}}^{<\kappa} \Rightarrow \mathrm{const}\,\mathrm{An}^{<\kappa}$; this follows by construction of η, unravelling the proof of Lemma 7.18 and using that pullbacks in functor categories can be computed pointwise by Lemma 6.12. Since unstraightening is an equivalence of ∞-categories, it preserves pullbacks, and so the left square must be a pullback too.[(7.5)] It follows that the outer rectangle in the diagram above must be a pullback too. But then $\mathcal{B} \to \mathcal{D}$ is a pullback of the universal unstraightening and thus $\mathcal{B}$ is the unstraightening of the bottom composition $\mathcal{D} \to \mathcal{J} \times \mathrm{An}^{<\kappa} \to \mathrm{An}^{<\kappa}$ by 7.17. So it remains to identify that composition with $F_\infty \circ d\colon \mathcal{D} \to \mathrm{An}^{<\kappa}$. This follows from a closer investigation of the proof of Lemma 6.14. $\hfill\square$

With Rezk's equifibrancy condition from Lemma 7.16, we have obtained one of the two ingredients in the proof of ($\boxtimes_2$). The other one is a general construction for simplicial objects.

7.19. Construction. — Let $X\colon \Delta^{\mathrm{op}} \to \mathcal{C}$ be a simplicial object in an arbitrary ∞-category $\mathcal{C}$. We picture X as

$$
X \simeq \left(X_0 \underset{d_1^*}{\overset{d_0^*}{\rightleftarrows}} X_1 \underset{d_2^*}{\overset{d_0^*}{\rightleftarrows}} X_2 \ \cdots \right)
$$

(for typographical reasons, we couldn't label the degeneracy maps nor the inner face maps). The *décalage of X* is another simplicial object $\mathrm{déc}(X)\colon \Delta^{\mathrm{op}} \to \mathcal{C}$ given by "shifting" X, thus "forgetting" X_0 as well as all the face maps d_0^* and all the degeneracy maps s_0^*. In pictures:

$$
\mathrm{déc}(X) \simeq \left(X_1 \underset{d_2^*}{\overset{d_1^*}{\rightleftarrows}} X_2 \underset{d_3^*}{\overset{d_1^*}{\rightleftarrows}} X_3 \ \cdots \right).
$$

More precisely, there's a functor $\sigma\colon \Delta \to \Delta$ given by $\sigma([n]) := [n+1]$ on objects. A morphism $\alpha\colon [m] \to [n]$ is sent to $\sigma(\alpha)\colon [m+1] \to [n+1]$ given by $\sigma(\alpha)(0) := 0$ and $\sigma(\alpha)(i) := \alpha(i-1)+1$ for all $1 \leqslant i \leqslant n+1$. Then $\mathrm{déc}(X)$ is simply the composition $X \circ \sigma^{\mathrm{op}}\colon \Delta^{\mathrm{op}} \to \mathcal{C}$. The décalage sits inside a diagram

$$
\begin{array}{ccc}
\mathrm{const}\,X_0 & \Longrightarrow & \mathrm{déc}(X) \overset{d_0^*}{\Longrightarrow} X \\
& \underset{\mathrm{id}_{\mathrm{const}\,X_0}}{\searrow} \quad {\scriptstyle /\!/\!/} \quad \Big\Vert{\scriptstyle d_{\mathrm{last}}^*} & \\
& \mathrm{const}\,X_0 &
\end{array}
$$

The transformation $\mathrm{const}\,X_0 \Rightarrow \mathrm{déc}(X)$ is induced by the unique transformation $\sigma \Rightarrow \mathrm{const}\,[0]$ in $\mathrm{Fun}(\Delta, \Delta)$. This transformation has a left inverse $d_{\mathrm{last}}\colon \mathrm{const}\,[0] \Rightarrow \sigma$ given object-wise by

---

[(7.5)]We've used similar arguments in the proofs of Lemmas 5.24 and 5.26, except back then we couldn't talk about pullbacks in ∞-categories yet

the maps $[0] \to \sigma([n]) = [n+1]$ that send $0 \mapsto 0$. These maps can be written as compositions $d_{n+1} \circ d_n \circ \cdots \circ d_1 \colon [0] \to [1] \to \cdots \to [n] \to [n+1]$, whence the notation d_{last}. The natural transformation d_{last} induces a transformation $d_{\mathrm{last}}^* \colon \mathrm{déc}(X) \Rightarrow \mathrm{const}\, X_0$. Finally, the maps $d_0 \colon [n] \to [n+1] = \sigma([n])$ induce a natural transformation $d_0 \colon \mathrm{id}_\Delta \Rightarrow \sigma$, which in turn induces a transformation $d_0^* \colon \mathrm{déc}(X) \Rightarrow X$.

7.20. Lemma. — *If $\mathcal{C}$ is any ∞-category and $X \colon \Delta^{\mathrm{op}} \to \mathcal{C}$ is a simplicial object in $\mathcal{C}$, then the diagram from Construction 7.19 induces equivalences*

$$\operatorname*{colim}_{[n] \in \Delta^{\mathrm{op}}} \mathrm{déc}(X)_n \xrightarrow{\simeq} \operatorname*{colim}_{[n] \in \Delta^{\mathrm{op}}} X_0 \simeq X_0 \,.$$

In particular, these colimits always exist in $\mathcal{C}$.

Proof sketch. The equivalence $\operatorname{colim}_{[n] \in \Delta^{\mathrm{op}}} X_0 \simeq X_0$ follows from Lemma 6.57 and the fact that $|\Delta^{\mathrm{op}}| \simeq *$, since Δ has a terminal object, namely $[0]$. To show $\operatorname{colim}_{[n] \in \Delta^{\mathrm{op}}} \mathrm{déc}(X)_n \simeq X_0$, note that $\sigma \colon \Delta \to \Delta$ can be identified with the inclusion $\Delta_{\geqslant 1} \to \Delta$ of the (non-full) subcategory spanned by $[n+1]$ for all $n \geqslant 0$ and all morphisms $\alpha \colon [m+1] \to [n+1]$ satisfying $\alpha^{-1}\{0\} = \{0\}$. Furthermore, let $\Delta_0 \to \Delta$ be the (non-full) subcategory spanned by all objects but only those morphisms that send $0 \mapsto 0$.

Via this reinterpretation, $\operatorname{colim}_{[n] \in \Delta^{\mathrm{op}}} \mathrm{déc}(X)_n \simeq \operatorname{colim}_{[n] \in \Delta_{\geqslant 1}^{\mathrm{op}}} X_n$. On the other hand, it's straightforward to check that $[0] \in \Delta_0$ is an initial object; therefore, $\operatorname{colim}_{[n] \in \Delta_0^{\mathrm{op}}} X_n \simeq X_0$. So it would be enough to show that $\Delta_{\geqslant 1}^{\mathrm{op}} \to \Delta_0^{\mathrm{op}}$ is coinitial, or equivalently, that $\Delta_{\geqslant 1} \to \Delta_0$ is initial. By the dual of Theorem 6.18(*c*), we must show that $|\Delta_{\geqslant 1} \times_{\Delta_0} \Delta_{0/[n]}| \simeq 0$ for all $n \geqslant 0$. The case $n = 0$ is clear: It's straightforward to see that $[0] \in \Delta_0$ is also a terminal object, so that $\Delta_{0/[0]} \simeq \Delta_0$ and thus $|\Delta_{\geqslant 1} \times_{\Delta_0} \Delta_{0/[0]}| \simeq |\Delta_{\geqslant 1}| \simeq *$, since $[1] \in \Delta_{\geqslant 1}$ is terminal. Now let $n \geqslant 1$ and consider the full subcategory $\mathcal{X} \subseteq \Delta_{\geqslant 1} \times_{\Delta_0} \Delta_{0/[n]}$ spanned by those $\alpha \colon [m+1] \to [n]$ such that α maps $\alpha^{-1}\{1, \ldots, n\}$ bijectively to $\{1, \ldots, n\}$. It's straightforward to check that this inclusion has a left adjoint $\Delta_{\geqslant 1} \times_{\Delta_0} \Delta_{0/[n]} \to \mathcal{X}$.[(7.6)] Since adjunctions induce equivalences after $|\cdot|$, it's enough to show $|\mathcal{X}| \simeq *$. But now it's straightforward to check that $(\mathrm{id}_{[n]} \colon [n] \to [n])$ is an inital object of $\mathcal{X}$. $\qquad\square$

Now we can finally finish the proof of Theorem 7.10(*b*).

Proof sketch of Theorem 7.10(b), claim ($\boxtimes_2$). Let $G \in \mathrm{Grp}(\mathrm{An})$ be an $\mathbb{E}_1$-group. Using the Segal condition from Definition 7.6(*a*), one verifies that the following is a the pullback square in $\mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{An})$:

$$\begin{array}{ccc} \mathrm{const}\, G_1 & \Longrightarrow & \mathrm{déc}(G) \\ \big\Vert & \lrcorner & \big\Vert{\scriptstyle d_0^*} \\ \mathrm{const}\, * & \Longrightarrow & G \end{array}$$

Note that G being an $\mathbb{E}_1$-group as opposed to merely an $\mathbb{E}_1$-monoid implies that $d_0^* \colon \mathrm{déc}(G) \Rightarrow G$ from Construction 7.19 is equifibred in the sense of Lemma 7.16. Indeed, being an $\mathbb{E}_1$-group means that $(\mathrm{pr}_1, \mu) \colon G_1 \times G_1 \to G_1 \times G_1$ is an equivalence, so that all occurences of the

---

[(7.6)] The left adjoint can be constructed as follows: Let $(\alpha \colon [m+1] \to [n]) \in \Delta_{\geqslant 1} \times_{\Delta_0} \Delta_{0/[n]}$. Then there exists some $0 \leqslant k \leqslant m$ such that $\alpha^{-1}\{0\} = \{0, 1, \ldots, k+1\}$. Let $\bar{\alpha} \colon [k+n+1] \to [n]$ be defined by $\bar{\alpha}(i) = 0$ for $i = 0, 1, \ldots, k+1$ and $\bar{\alpha}(i) = i - (k+1)$ for $i \geqslant k+2$. Then $\bar{\alpha} \in \mathcal{X}$. Furthermore, there's a canonical morphism $u_\alpha \colon \alpha \to \bar{\alpha}$ in $\Delta_{\geqslant 1} \times_{\Delta_0} \Delta_{0/[n]}$, given by the identity on $\{0, 1, \ldots, k+1\}$ and $u_\alpha(i) = \alpha(i) + k + 1$ for $i \geqslant k+2$. Then $\alpha \mapsto \bar{\alpha}$ is the desired left adjoint and u_α is the unit of the adjunction.

multiplication map μ in 7.7 can be replaced by simple projections, and then equifibrancy is straightforward to check. Lemma 7.16 now implies that the central square of the diagram

$$
\begin{array}{ccccccc}
G_1 & \xrightarrow{\simeq} & \underset{[n]\in\Delta^{\mathrm{op}}}{\operatorname{colim}} G_1 & \longrightarrow & \underset{[n]\in\Delta^{\mathrm{op}}}{\operatorname{colim}} \operatorname{déc}(G)_n & \xleftarrow{\simeq} & G_0 \\
\downarrow & {\scriptstyle ///} & \downarrow & {\scriptstyle \lrcorner} & \downarrow & {\scriptstyle ///} & \downarrow \\
* & \xrightarrow{\simeq} & \underset{[n]\in\Delta^{\mathrm{op}}}{\operatorname{colim}} * & \longrightarrow & \underset{[n]\in\Delta^{\mathrm{op}}}{\operatorname{colim}} G_n & \xleftarrow{\simeq} & \mathrm{B}G
\end{array}
$$

is a pullback. The equivalences on the left follows from Lemma 6.57. The top right equivalence is due to Lemma 7.20. The bottom right equivalence is the definition of $\mathrm{B}G$. Since $G_0 \simeq *$, this diagram shows $G_1 \simeq \Omega\mathrm{B}G$, which is precisely what we claimed in ($\boxtimes_2$). $\qquad\square$

Here are some immediate consequences of Theorem 7.10(b):

7.21. Corollary ("$\Omega\mathrm{B}$ is group completion"). — *The inclusion* $\mathrm{Grp}(\mathrm{An}) \subseteq \mathrm{Mon}(\mathrm{An})$ *of* $\mathbb{E}_1$*-groups into* $\mathbb{E}_1$*-monoids has a left adjoint, given by* $\Omega\mathrm{B}\colon \mathrm{Mon}(\mathrm{An}) \to \mathrm{Grp}(\mathrm{An})$. $\qquad\square$

7.22. Corollary ("$\Omega\Sigma X_+$ is the free $\mathbb{E}_1$-group on X"). — *The functor* $\mathrm{ev}_{[1]}\colon \mathrm{Grp}(\mathrm{An}) \to \mathrm{An}$ *sending an* $\mathbb{E}_1$*-group to its underlying anima has a left adjoint, sending an anima* X *to* $\Omega\Sigma X_+$, *where* $X_+ \coloneqq X \sqcup *$, *regarded as a pointed anima.*

Proof. It's straightforward to check (for example, using Corollary 6.17 and Corollary 5.15) that $(-)_+ \coloneqq (-) \sqcup *\colon \mathrm{An} \to \mathrm{An}_{*/}$ is a left adjoint to the forgetful functor $\mathrm{An}_{*/} \to \mathrm{An}$. Combining this observation with Lemma 7.3 and Theorem 7.10(b) yields a diagram of adjunctions

$$
\mathrm{An} \underset{\longleftarrow}{\xrightarrow{(-)_+}} \mathrm{An}_{*/} \underset{\xleftarrow{\Omega}}{\xrightarrow{\Sigma}} (\mathrm{An}_{*/})_{\geqslant 1}
$$

which shows that $\Omega\Sigma(-)_+\colon \mathrm{An} \rightleftarrows \mathrm{Grp}(\mathrm{An}) : \mathrm{ev}_{[1]}$ must be an adjunction too. $\qquad\square$

Another immediate consequence of Theorem 7.10(b) is the Seifert–van Kampen theorem.

7.23. Theorem (Seifert–van Kampen). — *The functor* $\pi_1\colon (\mathrm{An}_{*/})_{\geqslant 1} \to \mathrm{Grp}$ *preserves pushouts. That is, the fundamental group of a pushout of pointed connected animae is given by the pushout of fundamental groups, taken in the category* Grp *of groups.*

Proof. Let (X, x) be a pointed anima. By Lemma 7.3, we have $\pi_1(X, x) \cong \pi_0(\Omega X)$. By Theorem 7.10($b$), the functor $\Omega\colon (\mathrm{An}_{*/})_{\geqslant 1} \to \mathrm{Grp}(\mathrm{An})$ is an equivalence of ∞-categories, so it preserves pushouts. The functor $\pi_0\colon \mathrm{An} \to \mathrm{Set}$ is left adjoint to the inclusion $i\colon \mathrm{Set} \to \mathrm{An}$ given by regarding sets as discrete animae. By Corollary 6.6, this implies that there is an adjunction $(\pi_0)_*\colon \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{An}) \rightleftarrows \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{Set}) : i_*$. Note that π_0 and i both preserve products. Hence $(\pi_0)_*$ and i_* preserve the conditions from Definition 7.6 and so the adjunction above restricts to an adjunction $\pi_0\colon \mathrm{Grp}(\mathrm{An}) \rightleftarrows \mathrm{Grp}(\mathrm{Set}) : i$. In particular, $\pi_0\colon \mathrm{Grp}(\mathrm{An}) \to \mathrm{Grp}(\mathrm{Set}) \simeq \mathrm{Grp}$ is a left adjoint and so it preserves pushouts. It follows that $\pi_1 \cong \pi_0 \circ \Omega$ preserves pushouts too, as claimed. $\qquad\square$

§7.3. $\mathbb{E}_\infty$-**monoids and** $\mathbb{E}_\infty$-**groups**

Our next goal is to study the analogue of commutative monoids and commutative groups in animae. The definition is quite similar to Definition 7.6, except that we have to replace Δ^{op} by a category that encodes commutativity as well.

7.24. Definition. — Let Fin be the ordinary category of finite sets $\langle n \rangle = \{1, \ldots, n\}$ for $n \geqslant 0$ and partially defined (!) maps. Let $\mathcal{C}$ be an ∞-category with finite products.

(a) An $\mathbb{E}_\infty$-*monoid in* $\mathcal{C}$ is a functor $M \colon \mathrm{Fin} \to \mathcal{C}$ satisfying $M_0 \simeq *$ as well as the *Segal condition*: The *Segal maps* $e_i \colon \langle n \rangle \to \langle 1 \rangle$, where e_i is everywhere undefined except at i, induce an equivalence
$$M_n \xrightarrow{\simeq} M_1^n \,.$$
We call M_1 the *underlying object of M*; we'll often don't distinguish between M and M_1. Let $\mathrm{CMon}(\mathcal{C}) \subseteq \mathrm{Fun}(\mathrm{Fin}, \mathcal{C})$ denote the full sub-∞-category spanned by the $\mathbb{E}_\infty$-monoids.

(b) An $\mathbb{E}_\infty$-monoid M in $\mathcal{C}$ is called an $\mathbb{E}_\infty$-*group* if its underlying $\mathbb{E}_1$-monoid in the sense of Construction 7.26 below is an $\mathbb{E}_1$-group. We let $\mathrm{CGrp}(\mathcal{C}) \subseteq \mathrm{CMon}(\mathcal{C})$ denote the full sub-∞-category spanned by $\mathbb{E}_\infty$-groups.

7.25. Construction. — Let's unravel how Definition 7.24(a) encodes a commutative multiplication on M_1. The unique everywhere defined map $f_2 \colon \langle 2 \rangle \to \langle 1 \rangle$ induces a morphism
$$\mu \colon M_1 \times M_1 \simeq M_2 \longrightarrow M_1 \,.$$

This is our multiplication. Now let's see why it is commutative: Let $\mathrm{flip} \colon \langle 2 \rangle \to \langle 2 \rangle$ be the everywhere defined map that sends $1 \mapsto 2$ and $2 \mapsto 1$. Then $f_2 \circ \mathrm{flip} = f_2$ and so the following diagram commutes in $\mathcal{C}$:

Here $\mathrm{flip} \colon M_1 \times M_1 \to M_1 \to M_1$ is the morphism that flips the two factors; under the Segal isomorphism $M_1 \times M_1 \simeq M_2$, this really corresponds to $\mathrm{flip} \colon M_2 \to M_2$, so the notational overload checks out.

7.26. Construction. — Let us construct an underlying $\mathbb{E}_1$-monoid to every $\mathbb{E}_\infty$-monoid M. To this end, we'll construct a functor $\mathrm{Cut} \colon \Delta^{\mathrm{op}} \to \mathrm{Fin}$. On objects, Cut is given by $\mathrm{Cut}([n]) \coloneqq \langle n \rangle$. A map $\alpha \colon [m] \to [n]$ in Δ, which corresponds to a morphism $[n] \to [m]$ in Δ^{op}, is sent to $\mathrm{Cut}(\alpha) \colon \langle n \rangle \to \langle m \rangle$ given by the formula
$$\mathrm{Cut}(\alpha)(i) \coloneqq \begin{cases} j & \text{if } \alpha(j-1) < i \leqslant \alpha(j) \\ \text{undefined} & \text{else} \end{cases} \,.$$

A more conceptual way of saying this is that Cut sends $[n]$ to its set of *Dedekind cuts*, that is, to the set of all partitions of $[n]$ into two non-empty intervals (of which there are exactly n, so $\mathrm{Cut}([n]) = \langle n \rangle$). The map $\mathrm{Cut}(\alpha) \colon \mathrm{Cut}([n]) \to \mathrm{Cut}([m])$ sends such a partition of $[n]$ to its preimage under α, which is again a partition of $[m]$ into intervals. However, it may happen that one of the intervals is empty; if this is the case, we define the value of $\mathrm{Cut}(\alpha)$ as undefined.

Now Cut induces a precomposition functor $\mathrm{Cut}^* \colon \mathrm{Fun}(\mathrm{Fin}, \mathcal{C}) \to \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathcal{C})$. It's straightforward to check that $\mathrm{Cut}(e_i) = e_i$, that is, Cut sends the Segal maps in Δ^{op} to the Segal maps in Fin. Hence Cut^* preserves the Segal condition from and therefore restricts to a functor

$$\mathrm{Cut}^* \colon \mathrm{CMon}(\mathcal{C}) \longrightarrow \mathrm{Mon}(\mathcal{C}) \,.$$

For an $\mathbb{E}_\infty$-monoid M, we call $\mathrm{Cut}^*(M)$ the *underlying $\mathbb{E}_1$-monoid of M*. As with the underlying object, we often abuse notation and identify M with its underlying $\mathbb{E}_1$-monoid.

Our eventual goal in this subsection is to prove an analogue of Theorem 7.10(b) for $\mathbb{E}_\infty$-monoids/groups. This needs some preparations.

7.27. Definition. — Let $\mathcal{C}$ be an ∞-category with finite coproducts and finite products (in particular, it has both an initial and a terminal object).

(a) $\mathcal{C}$ is called *semi-additive* if the initial object, which we denote $0 \in \mathcal{C}$, is also terminal, and for all $x, y \in \mathcal{C}$ the natural map

$$\begin{pmatrix} \mathrm{id}_x & 0 \\ 0 & \mathrm{id}_y \end{pmatrix} \colon x \sqcup y \xrightarrow{\simeq} x \times y$$

is an equivalence. Here $0 \colon x \to 0 \to y$ denotes the unique (up to contractible choice) morphism in $\mathrm{Hom}_{\mathcal{C}}(x, y)$ factoring through 0. If $\mathcal{C}$ is semi-additive, we usually write $x \sqcup y \simeq x \oplus y \simeq x \times y$.

(b) $\mathcal{C}$ is called *additive* if it is semi-additive and additionally for all $x \in \mathcal{C}$ the *shearing morphism* is an equivalence:

$$\begin{pmatrix} \mathrm{id}_x & \mathrm{id}_x \\ 0 & \mathrm{id}_x \end{pmatrix} \colon x \oplus x \xrightarrow{\simeq} x \oplus x \,.$$

7.28. Lemma ("Every object in an additive ∞-category is canonically an $\mathbb{E}_\infty$-group"). — *If $\mathcal{C}$ is a semi-additive ∞-category, then $\mathrm{CMon}(\mathcal{C}) \simeq \mathrm{Mon}(\mathcal{C}) \simeq \mathcal{C}$. If $\mathcal{C}$ is an additive ∞-category, then also $\mathrm{CGrp}(\mathcal{C}) \simeq \mathrm{Grp}(\mathcal{C}) \simeq \mathcal{C}$.*

Proof sketch. Let $\mathrm{Fin}_{\leqslant 1} \simeq \{\langle 0 \rangle \rightrightarrows \langle 1 \rangle\}$ denotes the full subcategory of Fin spanned by $\langle 0 \rangle$ and $\langle 1 \rangle$ and let $\mathrm{Fin}_{\leqslant 1}^\circ \subseteq \mathrm{Fin}_{\leqslant 1}$ denote the non-full subcategory given by $\{\langle 0 \rangle \leftarrow \langle 1 \rangle\}$. The proof rests upon the following two crucial observations:

($\boxtimes_1$) *A functor $F \colon \mathrm{Fin} \to \mathcal{C}$ with $F(\langle 0 \rangle) \simeq 0$ satisfies the Segal condition from Definition 7.24(a) if and only if F is the left Kan extension of its own restriction along $i \colon \mathrm{Fin}_{\leqslant 1} \to \mathrm{Fin}$.*

($\boxtimes_2$) *If $\mathrm{Fun}_* \subseteq \mathrm{Fun}$ denotes the full sub-∞-category spanned by those functors that send $\langle 0 \rangle \mapsto 0$, then restriction along $j \colon \mathrm{Fin}_{\leqslant 1}^\circ \to \mathrm{Fin}_{\leqslant 1}$ and evaluation at $\langle 1 \rangle$ induces equivalences*

$$\mathrm{Fun}_*(\mathrm{Fin}_{\leqslant 1}, \mathcal{C}) \xrightarrow{\simeq} \mathrm{Fun}_*\big(\mathrm{Fin}_{\leqslant 1}^\circ, \mathcal{C}\big) \xrightarrow{\simeq} \mathcal{C} \,.$$

We begin with ($\boxtimes_2$). Since $0 \in \mathcal{C}$ is terminal, it's clear that $\mathrm{ev}_{\langle 1 \rangle} \colon \mathrm{Fun}_*\big(\mathrm{Fin}_{\leqslant 1}^\circ, \mathcal{C}\big) \to \mathcal{C}$ is essentially surjective. By an easy application of Corollary 6.25, using that $\mathrm{Fin}_{\leqslant 1}^\circ$ is an ordinary category and so we understand its twisted arrow category $\mathrm{TwAr}(\mathrm{Fin}_{\leqslant 1})$, $\mathrm{ev}_{\langle 1 \rangle}$ is also fully faithful. Alternatively one could also use Lemma 5.13 (which amounts to the same). Hence $\mathrm{ev}_{\langle 1 \rangle}$ is an equivalence by Theorem 4.6.

To show that $j^* \colon \mathrm{Fun}_*(\mathrm{Fin}_{\leqslant 1}, \mathcal{C}) \to \mathrm{Fun}_*(\mathrm{Fin}_{\leqslant 1}^\circ, \mathcal{C})$ is an equivalence, we consider left Kan extension along j. To this end, let $F \colon \mathrm{Fin}_{\leqslant 1}^\circ \to \mathcal{C}$ be a functor satisfying $F(\langle 0 \rangle) \simeq 0$. We unravel the Kan extension formula from Lemma 6.27: We have $\mathrm{Fin}_{\leqslant 1}^\circ \times_{\mathrm{Fin}_{\leqslant 1}} (\mathrm{Fin}_{\leqslant 1})_{/\langle 0 \rangle} \simeq (\mathrm{Fin}_{\leqslant 1}^\circ)_{/\langle 0 \rangle}$, and so the colimit describing $\mathrm{Lan}_j F(\langle 0 \rangle)$ exists and is given by evaluating at the terminal object $(\mathrm{id}_{\langle 0 \rangle} \colon \langle 0 \rangle \to \langle 0 \rangle) \in (\mathrm{Fin}_{\leqslant 1}^\circ)_{/\langle 0 \rangle}$. Hence $\mathrm{Lan}_j F(\langle 0 \rangle) \simeq F(\langle 0 \rangle) \simeq 0$. In a similar way, we can analyse $\mathrm{Fin}_{\leqslant 1}^\circ \times_{\mathrm{Fin}_{\leqslant 1}} (\mathrm{Fin}_{\leqslant 1})_{/\langle 1 \rangle}$. This category is a disjoint union $\mathcal{T}_0 \sqcup \mathcal{T}_1$ of two components: $\mathcal{T}_1$ is simply $\{\mathrm{id}_{\langle 1 \rangle} \colon \langle 1 \rangle \to \langle 1 \rangle\}$. On the other hand, $\mathcal{T}_0$ is a category with two objects, namely the nowhere defined maps $(\langle 1 \rangle \to \langle 1 \rangle)$ and $(\langle 0 \rangle \to \langle 1 \rangle)$, as well as precisely one non-identity morphism $(\langle 1 \rangle \to \langle 1 \rangle) \to (\langle 0 \rangle \to \langle 1 \rangle)$. In particular, $(\langle 0 \rangle \to \langle 1 \rangle)$ is terminal in $\mathcal{T}_0$. Hence the colimit describing $\mathrm{Lan}_j F(\langle 1 \rangle)$ exists and is given by $F(\langle 1 \rangle) \oplus F(\langle 0 \rangle) \simeq F(\langle 1 \rangle) \oplus 0 \simeq F(\langle 1 \rangle)$.

In summary, Lemma 6.27 shows that $\mathrm{Lan}_j F$ exists and satisfies $\mathrm{Lan}_j F(\langle 0 \rangle) \simeq 0$ and so we get an adjunction

$$\mathrm{Lan}_j \colon \mathrm{Fun}_*(\mathrm{Fin}_{\leqslant 1}^\circ, \mathcal{C}) \rightleftarrows \mathrm{Fun}_*(\mathrm{Fin}_{\leqslant 1}, \mathcal{C}) \colon j^*.$$

It follows from our calculations above that for all functors $G \in \mathrm{Fun}_*(\mathrm{Fin}_{\leqslant 1}, \mathcal{C})$ the counit $c_G \colon \mathrm{Lan}_j(G \circ j) \Rightarrow G$ is a pointwise equivalence and thus an equivalence by Theorem 4.5. By Lemma 6.33(a), this implies that Lan_j is fully faithful, even though j itself is not. Furthermore, since j is essentially surjective, it's clear that j^* must be conservative. Hence Lan_j and j^* are inverse equivalences by Lemma 6.33(b) and we've finished the proof of ($\boxtimes_2$).

To prove ($\boxtimes_1$), first observe that by ($\boxtimes_2$) we can replace i by $i \circ j$. Then we use Lemma 6.27 once again to compute the values of $\mathrm{Lan}_{i \circ j}(F \circ i \circ j)$. To this end, one analyses the category $\mathrm{Fin}_{\leqslant 1}^\circ \times_{\mathrm{Fin}} \mathrm{Fin}_{/\langle n \rangle}$: This category is a disjoint union $\mathcal{T}_0 \sqcup \mathcal{T}_1 \sqcup \cdots \sqcup \mathcal{T}_n$, where $\mathcal{T}_0$ is as above and $\mathcal{T}_i$ is given by $\{s_i \colon \langle 1 \rangle \to \langle n \rangle\}$, where $s_i(1) := i$. Hence the colimit describing $\mathrm{Lan}_{i \circ j}(F \circ i \circ j)(\langle n \rangle)$ evaluates to $F(\langle 0 \rangle) \oplus F(\langle 1 \rangle) \oplus \cdots \oplus F(\langle 1 \rangle) \simeq 0 \oplus F(\langle 1 \rangle)^{\oplus n} \simeq F(\langle 1 \rangle)^{\oplus n}$. This shows that F satisfies the Segal condition if and only if $c_F \colon \mathrm{Lan}_{i \circ j}(F \circ i \circ j) \Rightarrow F$ is an equivalence of functors and thus proves ($\boxtimes_1$).

To finish the proof, observe that since $i \colon \mathrm{Fin}_{\leqslant 1} \to \mathrm{Fin}$ is fully faithful, the left Kan extension functor $\mathrm{Lan}_i \colon \mathrm{Fun}(\mathrm{Fin}_{\leqslant 1}, \mathcal{C}) \to \mathrm{Fun}(\mathrm{Fin}, \mathcal{C})$ must be fully faithful too by Corollary 6.29. So

$$\mathcal{C} \simeq \mathrm{Fun}_*(\mathrm{Fin}_{\leqslant 1}, \mathcal{C}) \to \mathrm{Fun}(\mathrm{Fin}_{\leqslant 1}, \mathcal{C}) \xrightarrow{\mathrm{Lan}_i} \mathrm{Fun}(\mathrm{Fin}, \mathcal{C})$$

is fully faithful, and its essential image is $\mathrm{CMon}(\mathcal{C})$ by ($\boxtimes_1$). It follows that $\mathrm{CMon}(\mathcal{C}) \simeq \mathcal{C}$. Replacing Fin by Δ^{op} everywhere, the same argument shows $\mathrm{Mon}(\mathcal{C}) \simeq \mathcal{C}$. Finally, if $\mathcal{C}$ is additive, then Definition 7.27(b) shows that the $\mathbb{E}_\infty$-monoid in $\mathcal{C}$ associated to $x \in \mathcal{C}$ is automatically an $\mathbb{E}_\infty$-group, so that $\mathrm{CGrp}(\mathcal{C}) \simeq \mathrm{CMon}(\mathcal{C})$ and $\mathrm{Grp}(\mathcal{C}) \simeq \mathrm{Mon}(\mathcal{C})$. $\qquad \square$

7.29. Lemma. — *If $\mathcal{C}$ is any ∞-category with finite products, then $\mathrm{CMon}(\mathcal{C})$ is semi-additive and $\mathrm{CGrp}(\mathcal{C})$ is an additive ∞-category.*

For the proof we need a criterion to decide when an ∞-category is semi-additive.

7.30. Lemma. — *Let $\mathcal{C}$ be an ∞-category with finite products. Then $\mathcal{C}$ is semi-additive if the following two conditions are satisfied:*

(a) *The terminal object $* \in \mathcal{C}$ is also initial.*

(b) *Let $\Delta \colon \mathcal{C} \to \mathcal{C}$ be the functor that sends $x \mapsto x \times x$. Then there exists a natural transformation $\mu \colon \Delta \Rightarrow \mathrm{id}_{\mathcal{C}}$ such that both compositions*

$$x \simeq x \times * \xrightarrow{\mathrm{id}_x \times 0} x \times x \xrightarrow{\mu_x} x$$

$$x \simeq * \times x \xrightarrow{0 \times \mathrm{id}_x} x \times x \xrightarrow{\mu_x} x$$

are homotopic to id_x *for all* $x \in \mathcal{C}$, *and the following diagram commutes for all* $x, y \in \mathcal{C}$:

$$
\begin{array}{ccc}
(x \times x) \times (y \times y) & \xrightarrow{\mathrm{id}_x \times \mathrm{flip} \times \mathrm{id}_y} & (x \times y) \times (x \times y) \\
& /\!/\!/ & \\
\quad \searrow \mu_x \times \mu_y & & \mu_{x \times y} \swarrow \quad \\
& x \times y &
\end{array}
$$

All conditions on μ *in* (b) *are pointwise; so for example, we don't need to assume that the diagram above commutes functorially in* x *and* y. ∎

The proof of Lemma 7.30 is rather straightforward: One proves that the morphisms $x \simeq x \times * \to x \times y$ and $y \simeq * \times y \to x \times y$ exhibit $x \times y$ as a coproduct of x and y. However, the details become rather tedious, and so we skip the proof. You can find a full argument in [F-KTh, Lemma II.20] and another variant in [L-HA, Proposition 2.4.3.19].

Proof sketch of Lemma 7.29. We use the criterion from Lemma 7.30 to check that $\mathrm{CMon}(\mathcal{C})$ is semi-additive. First observe that if $* \in \mathcal{C}$ is a terminal object, then $\mathrm{const} *$ is terminal in $\mathrm{CMon}(\mathcal{C})$ (even in $\mathrm{Fun}(\mathrm{Fin}, \mathcal{C})$ by Lemma 6.12). But it is also initial. Indeed, if $M \colon \mathrm{Fin} \to \mathcal{C}$ is any functor, then $\mathrm{Hom}_{\mathrm{Fun}(\mathrm{Fin},\mathcal{C})}(\mathrm{const} *, M) \simeq \mathrm{Hom}_{\mathcal{C}}(*, \lim_{\langle n \rangle \in \mathrm{Fin}} M_n)$. However, $\langle 0 \rangle \in \mathrm{Fin}$ is an initial object and so $\lim_{\langle n \rangle \in \mathrm{Fin}} M_n \simeq M_0$; in particular, the limit always exists. Now if $M \in \mathrm{CMon}(\mathcal{C})$, then $M_0 \simeq *$. Thus

$$
\mathrm{Hom}_{\mathrm{CMon}(\mathcal{C})}(\mathrm{const} *, M) \simeq \mathrm{Hom}_{\mathcal{C}}(*, *) \simeq *,
$$

as desired. So Lemma 7.30(a) is satisfied.[(7.7)]

To construct μ, consider the functor $\times \colon \mathrm{Fin} \times \mathrm{Fin} \to \mathrm{Fin}$ sending a pair $(\langle m \rangle, \langle n \rangle)$ to the product $\langle m \rangle \times \langle n \rangle := \langle mn \rangle$.[(7.8)] Precomposition with $\times$ then induces a functor $\mathrm{Fun}(\mathrm{Fin}, \mathcal{C}) \to \mathrm{Fun}(\mathrm{Fin} \times \mathrm{Fin}, \mathcal{C}) \simeq \mathrm{Fun}(\mathrm{Fin}, \mathrm{Fun}(\mathrm{Fin}, \mathcal{C}))$. It's straightforward to check that the Segal condition is preserved, and so we obtain a functor

$$
\mathrm{Double} \colon \mathrm{CMon}(\mathcal{C}) \longrightarrow \mathrm{CMon}(\mathrm{CMon}(\mathcal{C})).
$$

Unravelling the definitions, we find that

$$
\mathrm{Double}(-)_1 \colon \mathrm{CMon}(\mathcal{C}) \longrightarrow \mathrm{CMon}(\mathrm{CMon}(\mathcal{C})) \xrightarrow{\mathrm{ev}_{\langle 1 \rangle}} \mathrm{CMon}(\mathcal{C})
$$

$$
\mathrm{Double}(-)_2 \colon \mathrm{CMon}(\mathcal{C}) \longrightarrow \mathrm{CMon}(\mathrm{CMon}(\mathcal{C})) \xrightarrow{\mathrm{ev}_{\langle 2 \rangle}} \mathrm{CMon}(\mathcal{C})
$$

are equivalent to $\mathrm{id}_{\mathrm{CMon}(\mathcal{C})}$ and Δ, respectively. The everywhere defined map $f_2 \colon \langle 2 \rangle \to \langle 1 \rangle$ from Construction 7.25 induces a natural transformation $\mathrm{ev}_{\langle 2 \rangle} \Rightarrow \mathrm{ev}_{\langle 1 \rangle}$, which yields a natural transformation $\mu \colon \Delta \Rightarrow \mathrm{id}_{\mathrm{CMon}(\mathcal{C})}$ in $\mathrm{Fun}(\mathrm{CMon}(\mathcal{C}), \mathrm{CMon}(\mathcal{C}))$, as desired. It's straightforward to check that μ satisfies the conditions from Lemma 7.30(b). This finishes the proof that $\mathrm{CMon}(\mathcal{C})$ is semi-additive.

Since $\mathrm{CGrp}(\mathcal{C}) \subseteq \mathrm{CMon}(\mathcal{C})$ is closed under products, it follows that $\mathrm{CGrp}(\mathcal{C})$ must be semi-additive too. But then every $G \in \mathrm{CGrp}(\mathcal{C})$ also satisfies the condition from Definition 7.27(b), by definition of G being an $\mathbb{E}_\infty$-group. Hence $\mathrm{CGrp}(\mathcal{C})$ is additive. □

[(7.7)]The same argument works for $\mathrm{Mon}(\mathcal{C})$, since $[0] \in \Delta^{\mathrm{op}}$ is initial too. So $\mathrm{Mon}(\mathcal{C})$ also satisfies Lemma 7.30(a).
[(7.8)]Here we crucially use that we're working with Fin; for Δ^{op}, such a functor wouldn't exist. Thus, $\mathrm{Mon}(\mathcal{C})$ doesn't satisfy Lemma 7.30(b).

Now we're ready to approach the desired analogue of Theorem 7.10(*b*).

7.31. Construction. — By Corollary 6.6, the adjunction $B\colon \mathrm{Mon}(\mathrm{An}) \rightleftarrows \mathrm{An}_{*/} \colon \Omega$ from Theorem 7.10(*b*) induces an adjunction $B_*\colon \mathrm{Fun}(\mathrm{Fin}, \mathrm{Mon}(\mathrm{An})) \rightleftarrows \mathrm{Fun}(\mathrm{Fin}, \mathrm{An}_{*/}) \colon \Omega_*$. We claim that this restricts to another adjunction

$$B\colon \mathrm{CMon}(\mathrm{Mon}(\mathrm{An})) \rightleftarrows \mathrm{CMon}\big(\mathrm{An}_{*/}\big) \colon \Omega.$$

To see this, we must show that the Segal condition is preserved under B_* and Ω_*. This in turn reduces to checking that $B\colon \mathrm{Mon}(\mathrm{An}) \to \mathrm{An}_{*/}$ and $\Omega\colon \mathrm{An}_{*/} \to \mathrm{Mon}(\mathrm{An})$ preserve finite products. For Ω, this is obvious, since right adjoints preserve all limits. For B, this follows from Lemma 7.32 below (plus Lemma 6.56(*a*)).

Now the currying equivalence $\mathrm{Fun}(\mathrm{Fin}, \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{An})) \simeq \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{Fun}(\mathrm{Fin}, \mathrm{An}))$ restricts to an equivalence $\mathrm{CMon}(\mathrm{Mon}(\mathrm{An})) \simeq \mathrm{Mon}(\mathrm{CMon}(\mathrm{An}))$ by a straightforward check. Furthermore, Lemmas 7.28 and 7.29 show $\mathrm{Mon}(\mathrm{CMon}(\mathrm{An})) \simeq \mathrm{CMon}(\mathrm{An})$. In a similar way, the equivalence $\mathrm{Fun}(\mathrm{Fin}, \mathrm{An}_{*/}) \simeq \mathrm{Fun}(\mathrm{Fin}, \mathrm{An})_{\mathrm{const}\,*/}$ restricts to $\mathrm{CMon}(\mathrm{An}_{*/}) \simeq \mathrm{CMon}(\mathrm{An})_{\mathrm{const}\,*/}$. But $\mathrm{const}\,* \in \mathrm{CMon}(\mathrm{An})$ is an initial object, as we've seen in the proof of Lemma 7.29. Thus $\mathrm{CMon}(\mathrm{An})_{\mathrm{const}\,*/} \simeq \mathrm{CMon}(\mathrm{An})$. Putting everything together, we can rewrite the adjunction above as

$$B\colon \mathrm{CMon}(\mathrm{An}) \rightleftarrows \mathrm{CMon}(\mathrm{An}) \colon \Omega.$$

7.32. Lemma. — *The functor* $\mathrm{colim}_{\Delta^{\mathrm{op}}}\colon \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{An}) \to \mathrm{An}$ *preserves finite products.*

Proof sketch. The crucial step is to show that the diagonal $\Delta^{\mathrm{op}} \to \Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}}$ is coinitial. This is another application of Theorem 6.18(*c*), of course, but it's not completely obvious and we leave it as a not quite easy exercise. For a full proof, consult [L-Ker, Tag 02QP] or [F-KTh, Exercise II.18*a*].

It will be enough to show that $\mathrm{colim}_{\Delta^{\mathrm{op}}}$ preserves empty products and binary products. First note that $\mathrm{colim}_{[n]\in\Delta^{\mathrm{op}}} * \simeq *$ follows from Lemma 6.57, since $|\Delta^{\mathrm{op}}| \simeq *$ (which follows, for example, from the fact that $[0] \in \Delta^{\mathrm{op}}$ is an initial object). This shows that $\mathrm{colim}_{\Delta^{\mathrm{op}}}$ preserves empty products. For binary products, let $X, Y\colon \Delta^{\mathrm{op}} \to \mathrm{An}$ be functors. Since $\Delta^{\mathrm{op}} \to \Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}}$ is coinitial, we can rewrite $\mathrm{colim}_{[n]\in\Delta^{\mathrm{op}}}(X_n \times Y_n)$ as

$$\operatorname*{colim}_{([m],[n])\in\Delta^{\mathrm{op}}\times\Delta^{\mathrm{op}}}(X_m \times Y_n) \simeq \operatorname*{colim}_{[m]\in\Delta^{\mathrm{op}}}\left(X_m \times \operatorname*{colim}_{[n]\in\Delta^{\mathrm{op}}} Y_n\right) \simeq \left(\operatorname*{colim}_{[m]\in\Delta^{\mathrm{op}}} X_m\right) \times \left(\operatorname*{colim}_{[n]\in\Delta^{\mathrm{op}}} Y_n\right).$$

The first equivalence follows from Lemma 6.38 together with the fact that $X_m \times -\colon \mathrm{An} \to \mathrm{An}$ commutes with arbitrary colimits, because it is a left adjoint by Example 6.3(*b*). Applying the same argument to $- \times \mathrm{colim}_{\Delta^{\mathrm{op}}} Y_m$ gives the third equivalence. □

7.33. Theorem. — *The adjunctions from Construction 7.31 and Theorem 7.10(*b*) fit into a commutative diagram*

(∗)
$$
\begin{array}{ccc}
\mathrm{CMon}(\mathrm{An}) & \underset{\Omega}{\overset{B}{\rightleftarrows}} & \mathrm{CMon}(\mathrm{An}) \\
{\scriptstyle\mathrm{Cut}*}\big\downarrow & /\!/\!/ & \big\downarrow{\scriptstyle\mathrm{ev}_{\langle 1\rangle}} \\
\mathrm{Mon}(\mathrm{An}) & \underset{\Omega}{\overset{B}{\rightleftarrows}} & \mathrm{An}_{*/}
\end{array}
$$

(note that $\mathrm{ev}_{\langle 1\rangle}\colon \mathrm{CMon}(\mathrm{An}) \to \mathrm{An}$ *factors canonically over* $\mathrm{An}_{*/} \to \mathrm{An}$ *since we have* $\mathrm{CMon}(\mathrm{An}) \simeq \mathrm{CMon}(\mathrm{An}_{*/})$ *by Construction 7.31). Furthermore:*

(a) *Both* $\mathrm{B}\colon \mathrm{CMon}(\mathrm{An}) \to \mathrm{CMon}(\mathrm{An})$ *and* $\Omega\colon \mathrm{CMon}(\mathrm{An}) \to \mathrm{CMon}(\mathrm{An})$ *factor through the full sub-∞-category* $\mathrm{CGrp}(\mathrm{An}) \subseteq \mathrm{CMon}(\mathrm{An})$ *and they induce inverse equivalences*

$$\mathrm{B}\colon \mathrm{CGrp}(\mathrm{An}) \underset{\simeq}{\overset{\simeq}{\rightleftarrows}} \mathrm{CGrp}(\mathrm{An})_{\geqslant 1} :\Omega\,.$$

Here $\mathrm{CGrp}(\mathrm{An})_{\geqslant 1} \subseteq \mathrm{CGrp}(\mathrm{An})$ *is the full sub-∞-category spanned by those $\mathbb{E}_\infty$-groups G for which* $\pi_0(G) \cong *$.

(b) *The inclusion* $\mathrm{CGrp}(\mathrm{An}) \subseteq \mathrm{CMon}(\mathrm{An})$ *has a left adjoint, namely* $\Omega\mathrm{B}$. *So $\Omega\mathrm{B}$ is not only the "group completion" for $\mathbb{E}_1$-monoids (see Corollary 7.21), but for $\mathbb{E}_\infty$-monoids too.*

Proof sketch. Commutativity of ($*$) is a straightforward unravelling of definitions. Let's proceed with (a). Let $M \in \mathrm{CMon}(\mathrm{An})$. To show that B factors through $\mathrm{CGrp}(\mathrm{An}) \subseteq \mathrm{CMon}(\mathrm{An})$, simply observe $\pi_0(\mathrm{B}M) \cong *$. This ordinary monoid is a group and so the underlying $\mathbb{E}_1$-monoid of $\mathrm{B}M$ must be an $\mathbb{E}_1$-group by Lemma 7.9(d). To show that Ω factors through $\mathrm{CGrp}(\mathrm{An}) \subseteq \mathrm{CMon}(\mathrm{An})$, we must show that the underlying $\mathbb{E}_1$-monoid $\mathrm{Cut}^*(\Omega M)$ of ΩM is an $\mathbb{E}_1$-group. But commutativity of ($*$) shows $\mathrm{Cut}^*(\mathrm{B}M) \simeq \Omega M_1 \ \mathrm{Cut}^*(\Omega M) \simeq \Omega M_1$ and $\Omega\colon \mathrm{An}_{*/} \to \mathrm{Mon}(\mathrm{An})$ factors through $\mathrm{Grp}(\mathrm{An}) \subseteq \mathrm{Mon}(\mathrm{An})$, as we've seen in the proof of Theorem 7.10(b).

It remains to show that B and Ω induce inverse equivalences $\mathrm{CGrp}(\mathrm{An}) \simeq \mathrm{CGrp}(\mathrm{An})_{\geqslant 1}$. We've already seen that the functor $\mathrm{B}\colon \mathrm{CGrp}(\mathrm{An}) \to \mathrm{CMon}(\mathrm{An})$ factors through the inclusion $\mathrm{CGrp}(\mathrm{An})_{\geqslant 1} \subseteq \mathrm{CMon}(\mathrm{An})$, so at least we get an adjunction

$$\mathrm{B}\colon \mathrm{CGrp}(\mathrm{An}) \rightleftarrows \mathrm{CGrp}(\mathrm{An})_{\geqslant 1} :\Omega\,.$$

We use the criterion from Lemma 6.33(b). Observe that equivalences of $\mathbb{E}_\infty$-monoids can be checked on underlying animae by the same argument as in Lemma 7.8. Thus, the questions whether the unit $u\colon \mathrm{id}_{\mathrm{CGrp}(\mathrm{An})} \Rightarrow \Omega\mathrm{B}$ is an equivalence and whether Ω is conservative can be reduced to the analogous questions for the adjunction $\mathrm{B}\colon \mathrm{Grp}(\mathrm{An}) \rightleftarrows (\mathrm{An}_{*/})_{\geqslant 1} :\Omega$. But Theorem 7.10(b) says that this adjunction is a pair of inverse equivalences. This finishes the proof of (a). Part (b) is a formal consequence of (a). $\qquad\square$

§7.4. Spectra and stable ∞-categories

We've seen in Theorem 7.10(b) that $\mathbb{E}_1$-groups in An are essentially the same as loop animae. Furthermore, we've seen in Corollary 7.22 that $\Omega\Sigma X_+$ is the free $\mathbb{E}_1$-group on an anima X. Of course, these observations should have analogues for $\mathbb{E}_\infty$-groups, but it's not immediately clear how such analogues would look like, nor how they would follow from Theorem 7.33. In this subsection, we'll introduce the ∞-category of *spectra*, which will eventually lead us to answers for both questions (Remark 7.41 and Corollary 7.58), but also to many more applications.

7.34. Construction. — We've seen in Theorem 7.33(a) that $\Omega\mathrm{B}\colon \mathrm{CGrp}(\mathrm{An}) \to \mathrm{CGrp}(\mathrm{An})$ is homotopic to the identity. Therefore, the following diagram commutes in Cat_∞ (or really, in $\widehat{\mathrm{Cat}}_\infty$, since we're dealing with large ∞-categories, but we'll ignore this issue here):

$$
\begin{array}{ccccc}
\cdots =\!=\!= \mathrm{CGrp}(\mathrm{An}) =\!=\!= \mathrm{CGrp}(\mathrm{An}) =\!=\!= \mathrm{CGrp}(\mathrm{An}) \\
\Big\downarrow{\scriptstyle \mathrm{B}\circ\mathrm{B}} \quad /\!/\!/ \quad \Big\downarrow{\scriptstyle \mathrm{B}} \quad /\!/\!/ \quad \Big\| \\
\cdots \xrightarrow{\ \Omega\ } \mathrm{CGrp}(\mathrm{An}) \xrightarrow{\ \Omega\ } \mathrm{CGrp}(\mathrm{An}) \xrightarrow{\ \Omega\ } \mathrm{CGrp}(\mathrm{An})
\end{array}
$$

This diagram yields a functor[7.9]

$$\mathrm{B}^\infty \colon \mathrm{CGrp}(\mathrm{An}) \longrightarrow \lim\left(\cdots \xrightarrow{\Omega} \mathrm{CGrp}(\mathrm{An}) \xrightarrow{\Omega} \mathrm{CGrp}(\mathrm{An}) \xrightarrow{\Omega} \mathrm{CGrp}(\mathrm{An}) \right).$$

Observe that for all $n \geqslant 0$, B and Ω induce inverse equivalences $\mathrm{CGrp}(\mathrm{An})_{\geqslant n+1} \simeq \mathrm{CGrp}(\mathrm{An})_{\geqslant n}$, where $\mathrm{CGrp}(\mathrm{An})_{\geqslant n} \subseteq \mathrm{CGrp}(\mathrm{An})$ is the full sub-∞-category spanned by those $\mathbb{E}_\infty$-groups G that satisfy $\pi_i(G) \cong 0$ for all $0 \leqslant i < n$. Indeed, the case $n = 0$ follows from Theorem 7.33(a). Since Ω "shifts homotopy groups down by one" (see Lemma 7.3(a)), its inverse B must "shift homotopy groups up by one". This implies that $\mathrm{B} \colon \mathrm{CGrp}(\mathrm{An}) \to \mathrm{CGrp}(\mathrm{An})_{\geqslant 1}$ must map $\mathrm{CGrp}(\mathrm{An})_{\geqslant n}$ into $\mathrm{CGrp}(\mathrm{An})_{\geqslant n+1}$; similarly, Ω must map $\mathrm{CGrp}(\mathrm{An})_{\geqslant n+1}$ into $\mathrm{CGrp}(\mathrm{An})_{\geqslant n}$. Hence the equivalence from Theorem 7.33(a) must restrict to an equivalence $\mathrm{CGrp}(\mathrm{An})_{\geqslant n+1} \simeq \mathrm{CGrp}(\mathrm{An})_{\geqslant n}$ for all $n \geqslant 0$, as claimed.

Combining this observation with Lemma 6.76(a) shows that B^∞ is fully faithful, with essential image given by

$$\mathrm{B}^\infty \colon \mathrm{CGrp}(\mathrm{An}) \xrightarrow{\simeq} \lim\left(\cdots \xrightarrow{\Omega} \mathrm{CGrp}(\mathrm{An})_{\geqslant 2} \xrightarrow{\Omega} \mathrm{CGrp}(\mathrm{An})_{\geqslant 1} \xrightarrow{\Omega} \mathrm{CGrp}(\mathrm{An}) \right).$$

Let us now turn this construction into a definition.

7.35. Definition. — Let $\mathcal{C}$ be an ∞-category with finite limits (in the sense of Definition 6.46(c)); in particular, $\mathcal{C}$ has a terminal object $* \in \mathcal{C}$. The ∞-*category of spectra in* $\mathcal{C}$ is defined as the following limit in Cat_∞:

$$\mathrm{Sp}(\mathcal{C}) \coloneqq \lim\left(\cdots \xrightarrow{\Omega_{\mathcal{C}}} \mathcal{C}_{*/} \xrightarrow{\Omega_{\mathcal{C}}} \mathcal{C}_{*/} \xrightarrow{\Omega_{\mathcal{C}}} \mathcal{C}_{*/} \right).$$

Here $\Omega_{\mathcal{C}} \colon \mathcal{C}_{*/} \to \mathcal{C}_{*/}$ is defined by the same pullback diagram as in Definition 7.1. In the case $\mathcal{C} \simeq \mathrm{An}$, we write $\mathrm{Sp} \coloneqq \mathrm{Sp}(\mathrm{An})$ for brevity, and we call Sp simply the ∞-*category of spectra*.

7.36. Lemma. — *Let $\mathcal{C}$ be an ∞-category with finite limits. Then $\mathrm{ev}_{\langle 1 \rangle} \colon \mathrm{CGrp}(\mathcal{C}) \to \mathcal{C}$ induces an equivalence $\mathrm{Sp}(\mathrm{CGrp}(\mathcal{C})) \simeq \mathrm{Sp}(\mathcal{C})$. In particular, the first limit from Construction 7.34 agrees with Sp.*

Proof sketch. Let's address the "in particular" first. Since $0 \coloneqq \mathrm{const}\, * \in \mathrm{CGrp}(\mathrm{An})$ is both initial and terminal by Lemma 7.29, we have $\mathrm{CGrp}(\mathrm{An}) \simeq \mathrm{CGrp}(\mathrm{An})_{0/}$. Hence the first limit from Construction 7.34 agrees with $\mathrm{Sp}(\mathrm{CGrp})(\mathrm{An})$ and thus with Sp, as claimed.

To show $\mathrm{Sp}(\mathrm{CGrp}(\mathcal{C})) \simeq \mathrm{Sp}(\mathcal{C})$ in general, first observe that $\mathrm{Fun}(\mathrm{Fin}, -) \colon \mathrm{Cat}_\infty \to \mathrm{Cat}_\infty$ commutes with limits, since it is a right adjoint by Example 6.3(b). Hence we obtain

---

[7.9]Here we would like to point out a subtlety that only the extraordinarily careful reader will have noticed: Let $\mathbb{N}$ denote the partially ordered set $(\cdots \to 2 \to 1 \to 0)$. Then to construct a functor $\mathbb{N} \to \mathcal{D}$ into an arbitrary ∞-category $\mathcal{D}$, it's enough to specify objects $y_n \in \mathcal{D}$ together with morphisms $y_{n+1} \to y_n$ for all $n \in \mathbb{N}$. This is because the inclusion $\mathrm{sk}_1\,\mathrm{N}(\mathbb{N}) \to \mathrm{N}(\mathbb{N})$ of the 1-skeleton of the nerve of $\mathbb{N}$ is inner anodyne, so that $\mathrm{F}(\mathrm{N}(\mathbb{N}), \mathcal{D}) \to \mathrm{F}(\mathrm{sk}_1\,\mathrm{N}(\mathbb{N}), \mathcal{D})$ is a trivial fibration (and thus an equivalence of quasi-categories) by Corollary 3.10. In the situation above, we implicitly used this observation in the case $\mathcal{D} \simeq \mathrm{Ar}(\mathrm{Cat}_\infty)$ to turn the commutative diagram, which a priori only encodes a sequence of morphisms

$$\left(\mathrm{B}^{\circ(n+1)} \colon \mathrm{CGrp}(\mathrm{An}) \to \mathcal{C}_{*/} \right) \longrightarrow \left(\mathrm{B}^{\circ n} \colon \mathrm{CGrp}(\mathrm{An}) \to \mathcal{C}_{*/} \right)$$

in $\mathrm{Ar}(\mathrm{Cat}_\infty)$, into a functor $\mathbb{N} \to \mathrm{Ar}(\mathrm{Cat}_\infty)$, which by currying encodes a natural transformation in $\mathrm{Fun}(\mathbb{N}, \mathrm{Cat}_\infty)$ and thus a functor B^∞ by the universal property of limits. Also note that this subtle observation was implicitly used to even write down the limit above: We can't just take the limit of a sequence of morphisms, we must turn that sequence into a functor $\mathbb{N} \to \mathrm{Cat}_\infty$!

$\mathrm{Fun}(\mathrm{Fin}, \mathcal{C}_{*/}) \simeq \mathrm{Fun}(\mathrm{Fin}, \mathcal{C})_{\mathrm{const}\,*/}$ and $\mathrm{Fun}(\mathrm{Fin}, \mathrm{Sp}(\mathcal{C})) \simeq \mathrm{Sp}(\mathrm{Fun}(\mathrm{Fin}, \mathcal{C}))$. It's straightforward to check that the latter equivalence restricts to $\mathrm{Sp}(\mathrm{CGrp}(\mathcal{C})) \simeq \mathrm{CGrp}(\mathrm{Sp}(\mathcal{C}))$. Thus, by Lemma 7.28, it's enough to show that $\mathrm{Sp}(\mathcal{C})$ is additive. We'll use Lemma 7.30. Note that $* \in \mathcal{C}_{*/}$ is both initial and terminal. By Lemma 6.76(a), it follows that $0 := (\dots, *, *, *) \in \mathrm{Sp}(\mathcal{C})$ is initial and terminal too. So the condition from Lemma 7.30(a) is satisfied. To construct a natural transformation $\mu \colon \Delta \Rightarrow \mathrm{id}_{\mathrm{Sp}(\mathcal{C})}$ as in Lemma 7.30(b), we observe:

($\boxtimes_1$) $\Omega_{\mathcal{C}}$ *induces an equivalence* $\Omega_{\mathcal{C}} \colon \mathrm{Sp}(\mathcal{C}) \xrightarrow{\simeq} \mathrm{Sp}(\mathcal{C})$.

($\boxtimes_2$) $\Omega_{\mathcal{C}}$ *can be factored into* $\Omega_{\mathcal{C}} \colon \mathrm{Sp}(\mathcal{C}) \to \mathrm{Grp}(\mathrm{Sp}(\mathcal{C})) \xrightarrow{\mathrm{ev}_{[1]}} \mathrm{Sp}(\mathcal{C})$. *More generally, the same is true if* $\mathrm{Sp}(\mathcal{C})$ *is replaced by any* ∞-*category* $\mathcal{D}$ *with finite limits, whose terminal object* $* \in \mathcal{D}$ *is also initial.*

Observation ($\boxtimes_1$) is clear from Definition 7.35. Observation ($\boxtimes_2$) can be shown by hand. Alternatively, first observe ($\boxtimes_2$) is true in the case $\mathcal{D} \simeq \mathrm{An}_{*/}$ by Theorem 7.10(b), using that $\mathrm{Grp}(\mathrm{An}_{*/}) \simeq \mathrm{Grp}(\mathrm{An})_{*/} \simeq \mathrm{Grp}(\mathrm{An})$ since $* \in \mathrm{Grp}(\mathrm{An})$ is both initial and terminal by the arguments from the proof of Lemma 7.29. The general case can be reduced to this special case using the Yoneda embedding $\text{よ}_{\mathcal{D}} \colon \mathcal{D} \to \mathrm{Fun}(\mathcal{D}^{\mathrm{op}}, \mathrm{An})$, which preserves all limits by Corollary 6.17 and thus factors through $\mathrm{Fun}(\mathcal{D}^{\mathrm{op}}, \mathrm{An})_{\mathrm{const}\,*/} \simeq \mathrm{Fun}(\mathcal{D}^{\mathrm{op}}, \mathrm{An}_{*/})$. A full argument can be found in [F-KTh, Remark* II.23a].

Now ($\boxtimes_1$) and ($\boxtimes_2$) imply that every $X \in \mathrm{Sp}(\mathcal{C})$ can be written as $X \simeq \Omega_{\mathcal{C}}(\Omega_{\mathcal{C}}^{-1}(X))$, and so X can be functorially upgraded to an $\mathbb{E}_1$-group $X \in \mathrm{Grp}(\mathrm{Sp}(\mathcal{C}))$. We then define $\mu \colon X \times X \to X$ to be the multiplication on X. All conditions from Lemma 7.30(a) are easily verified. $\qquad\square$

7.37. Construction. — We regard $\mathbb{N}$ and $\mathbb{Z}$ as partially ordered sets and $\mathbb{N} \subseteq \mathbb{Z}$ as the inclusion $(\dots \to 2 \to 1 \to 0) \subseteq (\dots \to 2 \to 1 \to 0 \to (-1) \to (-2) \to \dots)$. Then $\mathbb{N} \to \mathbb{Z}$ is an initial functor of ∞-categories. Indeed, this is immediate from the dual of Theorem 6.18(c), or from the dual of Example 6.20(a). Hence we can rewrite $\mathrm{Sp}(\mathcal{C})$ as

$$\mathrm{Sp}(\mathcal{C}) \simeq \lim\left(\dots \xrightarrow{\Omega_{\mathcal{C}}} \mathcal{C}_{*/} \xrightarrow{\Omega_{\mathcal{C}}} \mathcal{C}_{*/} \xrightarrow{\Omega_{\mathcal{C}}} \mathcal{C}_{*/} \xrightarrow{\Omega_{\mathcal{C}}} \dots \right).$$

For all $n \in \mathbb{Z}$, we let $\Omega_{\mathcal{C}}^{\infty-n} \colon \mathrm{Sp}(\mathcal{C}) \to \mathcal{C}_{*/}$ denote the projection to the nth component of the limit. This notation is chosen in such a way that $\Omega_{\mathcal{C}}(\Omega_{\mathcal{C}}^{\infty-n} X) \simeq \Omega_{\mathcal{C}}^{\infty-(n-1)} X$, as one would expect. In the case $\mathcal{C} \simeq \mathrm{An}$, we drop the index and just write $\Omega^{\infty-n}$.

In the case $\mathcal{C} \simeq \mathrm{An}$, we define $\pi_n(X) := \pi_0(\Omega^{\infty+n} X)$, the nth *homotopy group of the spectrum* X. Since $\Omega^{\infty+n} X \simeq \Omega^i(\Omega^{\infty+n-i} X)$ and Ω shifts homotopy groups down by Lemma 7.3(a), we see $\pi_n(X) \cong \pi_i(\Omega^{\infty+n-i} X)$ for all $i \geqslant 0$ (we don't have to specify a base point since $\Omega^{\infty+n-i} X \in \mathrm{An}_{*/}$ by construction). In particular, choosing $i \geqslant 2$ and using Lemma 3.17(b), we see that $\pi_n(X)$ is an abelian group for all $n \in \mathbb{Z}$.

A spectrum X is called *connective* if $\pi_n(X) \cong 0$ for all $n < 0$, and *coconnective* if $\pi_n(X) \cong 0$ for all $n > 0$. It's customary to denote by $\mathrm{Sp}_{\geqslant 0} \subseteq \mathrm{Sp}$ and $\mathrm{Sp}_{\leqslant 0} \subseteq \mathrm{Sp}$ the full sub-∞-categories spanned by the connective and the coconnective spectra, respectively.

7.38. Lemma. — *Let* $\mathcal{C}$ *be an* ∞-*category with finite limits. Then* $\mathrm{Sp}(\mathcal{C})$ *has all finite limits and* $\Omega_{\mathcal{C}}^{\infty-n} \colon \mathrm{Sp}(\mathcal{C}) \to \mathcal{C}_{*/}$ *preserves all finite limits for all* $n \in \mathbb{Z}$. *Furthermore, in the special case* $\mathcal{C} \simeq \mathrm{An}$, *the following is true:*

(a) *The* ∞-*category* Sp *has all small limits and colimits. For all* $n \in \mathbb{Z}$, *the functors* $\Omega^{\infty-n} \colon \mathrm{Sp} \to \mathrm{An}_{*/}$ *commute with all limits and with filtered colimits.*

(b)   *For all $n \in \mathbb{Z}$, the functors $\pi_n \colon \mathrm{Sp} \to \mathrm{Ab}$ commute with all products (in particular, with finite products and thus with finite coproducts too) and with filtered colimits.*

(c)   *A morphism $f \colon X \to Y$ of spectra is an equivalence if and only if it induces isomorphisms $\pi_n(X) \cong \pi_n(Y)$ for all $n \in \mathbb{Z}$. Furthermore, if $X \to Y \to Z$ is a fibre sequence in Sp (in the sense of Definition 6.39), then there is a long exact sequence of abelian groups*

$$\cdots \longrightarrow \pi_{n+1}(Z) \xrightarrow{\ \partial\ } \pi_n(X) \longrightarrow \pi_n(Y) \longrightarrow \pi_n(Z) \xrightarrow{\ \partial\ } \pi_{n-1}(X) \longrightarrow \cdots .$$

Proof. The first assertion is an immediate consequence of Lemma 6.76(*b*). The same argument also proves that Sp has all limits and that $\Omega^{\infty-n} \colon \mathrm{Sp} \to \mathrm{An}_{*/}$ commutes with limits. To prove the existence of colimits in Sp, it's enough to show that pushouts, finite coproducts, and filtered colimits exist, because infinite coproducts can be written as filtered colimits of finite coproducts (see claim (⊠) in the proof of Lemma 6.62). Since $\Omega \colon \mathrm{An}_{*/} \to \mathrm{An}_{*/}$ preserves filtered colimits by Theorem 6.54, we can apply Lemma 6.76(*b*) again to deduce that Sp has all filtered colimits and that $\Omega^{\infty-n} \colon \mathrm{Sp} \to \mathrm{An}_{*/}$ commutes with filtered colimits. The existence of finite coproducts follows from the fact that Sp is additive, as observed in Lemma 7.36. Finally, pushouts will be constructed in Lemma 7.43 below. This finishes the proof of (*a*).

Part (*b*) follows immediately from (*a*) and the fact that $\pi_0 \colon \mathrm{An} \to \mathrm{Set}$ preserves all products and filtered colimits by Lemma 6.58 (plus the fact that $\mathrm{Ab} \to \mathrm{Set}$ is conservative and preserves all products and filtered colimits).

The long exact sequence from (*c*) follows immediately from Lemma 3.19 and the fact that $\Omega^{\infty-n}X \to \Omega^{\infty-n}Y \to \Omega^{\infty-n}Z$ is a fibre sequence in $\mathrm{An}_{*/}$ for all $n \in \mathbb{Z}$ by (*a*). It's clear that any equivalence $f \colon X \to Y$ induces isomorphisms $\pi_n(X) \cong \pi_n(Y)$ for all $n \in \mathbb{Z}$. The converse follows essentially from Theorem 3.18; the only non-obvious point is that Theorem 3.18 requires isomorphisms on homotopy groups *for all basepoints*, whereas $\pi_n(X) \cong \pi_i(\Omega^{\infty+n-i}X)$ only uses the preferred base point of $\Omega^{\infty+n-i}X \in \mathrm{An}_{*/}$. However, $\Omega^{\infty+n-i}X$ upgrades canonically to an $\mathbb{E}_\infty$-group in An by Lemma 7.36. In an $\mathbb{E}_\infty$-group, all path components are homotopy equivalent, and so it doesn't matter which basepoint we use.   □

7.39. Remark. — Using the formalism from §6.10, we can give a slick proof of Lemma 7.38(*a*): Suppose $\mathcal{C}$ is a presentable ∞-category. The loop functor $\Omega_{\mathcal{C}} \colon \mathcal{C}_{*/} \to \mathcal{C}_{*/}$ admits a left adjoint $\Sigma_{\mathcal{C}} \colon \mathcal{C}_{*/} \to \mathcal{C}_{*/}$ given by the same pushout diagram as in Definition 7.1. Thus $\Omega_{\mathcal{C}}$ is a functor in Pr^{R}. We know from Lemma 6.75 that $\mathrm{Pr}^{\mathrm{R}} \to \widehat{\mathrm{Cat}}_\infty$ preserves limits, and so the limit defining $\mathrm{Sp}(\mathcal{C})$ can also be viewed as a limit in Pr^{R}.[(7.10)] This immediately shows that $\mathrm{Sp}(\mathcal{C})$ is presentable, so in particular, it has all colimits. If $\mathcal{C}_{*/}$ is $\aleph_0$-compactly generated (which is true for $\mathcal{C} \simeq \mathrm{An}$, as the pointed 0-dimensional sphere $(S^0, *)$ is a compact generator of $\mathrm{An}_{*/}$; this is clear from Lemma 6.62(*c*)), the limit defining $\mathrm{Sp}(\mathcal{C})$ can also be interpreted as a limit in $\mathrm{Pr}^{\mathrm{R}}_{\aleph_0}$, because $\mathrm{Pr}^{\mathrm{R}}_{\aleph_0} \to \mathrm{Pr}^{\mathrm{R}}$ also preserves all limits by the dual of Corollary 6.82(*b*). This shows that the projections $\Omega^{\infty-n}_{\mathcal{C}} \colon \mathrm{Sp}(\mathcal{C}) \to \mathcal{C}_{*/}$ preserve filtered colimits.

We can take these considerations one step further: Recall that there's an equivalence of ∞-categories $\mathrm{Pr}^{\mathrm{L}} \simeq (\mathrm{Pr}^{\mathrm{R}})^{\mathrm{op}}$ given by extracting adjoints. Thus, $\mathrm{Sp}(\mathcal{C})$ can also be described as a *colimit* in Pr^{L}, namely

$$\mathrm{Sp} \simeq \mathrm{colim}\left(\mathcal{C}_{*/} \xrightarrow{\Sigma_{\mathcal{C}}} \mathcal{C}_{*/} \xrightarrow{\Sigma_{\mathcal{C}}} \mathcal{C}_{*/} \xrightarrow{\Sigma_{\mathcal{C}}} \cdots \right) .$$

[(7.10)]Since we can construct Pr^{R} in ZFC, see 6.80, this also provides a way to construct $\mathrm{Sp}(\mathcal{C})$ without enlarging our universe and talking about $\widehat{\mathrm{Cat}}_\infty$.

Thus, $\mathrm{Sp}(\mathcal{C})$ is the terminal ∞-category over $\mathcal{C}_{*/}$ such that $\Omega_{\mathcal{C}}$ becomes invertible, but it's also the initial *presentable* ∞-category under $\mathcal{C}_{*/}$ such that $\Sigma_{\mathcal{C}}$ becomes invertible. And we get for free that $\Omega_{\mathcal{C}}^{\infty} \colon \mathrm{Sp}(\mathcal{C}) \to \mathcal{C}_{*/}$ admits a left adjoint $\Sigma_{\mathcal{C}}^{\infty} \colon \mathcal{C}_{*/} \to \mathrm{Sp}(\mathcal{C})$. In Lemma 7.55 and Corollary 7.57, we'll give an explicit construction of $\Sigma_{\mathcal{C}}^{\infty}$ that works in greater generality (in particular, not only for presentable $\mathcal{C}$), but it's nice to see a first instance where Lurie's magical ∞-category Pr^{L} becomes really useful.

7.40. Corollary ("$\mathbb{E}_{\infty}$-groups are connective spectra"). — *The functor* B^{∞} *from Construction 7.34 fits into an adjunction*

$$\mathrm{B}^{\infty} \colon \mathrm{CGrp}(\mathrm{An}) \rightleftarrows \mathrm{Sp} \colon \Omega^{\infty}.$$

Furthermore, B^{∞} *is fully faithful and its essential image is the full sub-∞-category* $\mathrm{Sp}_{\geqslant 0} \subseteq \mathrm{Sp}$ *of connective spectra.*

Proof. This follows immediately from Construction 7.34 together with Lemma 7.36 and Construction 7.37. $\qquad\square$

7.41. Remark. — Corollary 7.40 implies that an anima Y can be equipped with an $\mathbb{E}_{\infty}$-group structure if and only if Y can be written as $\Omega^{\infty} X$ for some spectrum X. Equivalently, Y must admit a compatible sequence $(\dots, Y_2, Y_1, Y_0)$ of *deloopings*, satisfying $Y_0 \simeq Y$ and $\Omega Y_{n+1} \simeq Y_n$ for all $n \geqslant 0$. This can be regarded as an analogue of Theorem 7.10(b): Just as $\mathbb{E}_1$-groups are precisely the loop animae, that is, those animae that can be delooped once, $\mathbb{E}_{\infty}$-groups are precisely the *infinite loop animae*, that is, those animae that can be delooped arbitrarily often, in a compatible way. This is the *recognition principle* for infinite loop spaces due to Boardman–Vogt, May, and Segal.

Furthermore, Corollary 7.40 implies that if $Y \simeq \Omega^{\infty} X$, then the spectrum X can always be chosen to be connective. In other words, if Y admits a compatible sequence $(\dots, Y_2, Y_1, Y_0)$ of deloopings, then we may always assume that Y_n is n-connected for all $n \geqslant 0$, that is, $\pi_i(Y_n) = 0$ for all $i < n$ and all basepoints. The intuitive reason for this is that upon writing $Y \simeq \Omega^n Y_n$, all information about $\pi_*(Y_n)$ below degree n will be lost, so we may as well assume these homotopy groups vanish. If we work with spectra (not necessarily connective), this information is instead remembered in the form of negative homotopy groups. In general, working with Sp rather than $\mathrm{CGrp}(\mathrm{An})$ has a number of advantages, due to the excellent categorical properties of Sp. These properties are axiomatised in the notion of a *stable ∞-category*.

7.42. Definition. — An ∞-category $\mathcal{C}$ is called *stable* if it satisfies the equivalent conditions from Lemma 7.43 below.

7.43. Lemma. — *Suppose $\mathcal{C}$ is an ∞-category with an object $0 \in \mathcal{C}$ that's both initial and terminal. Then the following conditions are equivalent:*

(a) *$\mathcal{C}$ has finite limits and $\Omega_{\mathcal{C}} \colon \mathcal{C} \to \mathcal{C}$ is an equivalence of ∞-categories. Here $\Omega_{\mathcal{C}}$ is defined by an analogous pullback diagram as in Definition 7.1.*

(b) *$\mathcal{C}$ has finite colimits and $\Sigma_{\mathcal{C}} \colon \mathcal{C} \to \mathcal{C}$ is an equivalence of ∞-categories. Here $\Sigma_{\mathcal{C}}$ is defined by an analogous pushout diagram as in Definition 7.1.*

(c) *$\mathcal{C}$ has finite limits and finite colimits and a commutative square in $\mathcal{C}$ is a pushout square if and only if it is a pullback square.*

(d) *The functor $\Omega_{\mathcal{C}}^{\infty} \colon \mathrm{Sp}(\mathcal{C}) \to \mathcal{C}$ is an equivalence of ∞-categories.*

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(e)   *There exists an ∞-category $\mathcal{D}$ and an equivalence of ∞-categories $\mathrm{Sp}(\mathcal{D}) \simeq \mathcal{C}$.*

Proof. The implication $(a) \Rightarrow (d)$ is clear: Since $0 \in \mathcal{C}$ is both initial and terminal, we have $\mathcal{C}_{0/} \simeq \mathcal{C}$, and so $\Omega_{\mathcal{C}} \colon \mathcal{C}_{0/} \to \mathcal{C}_{0/}$ is an equivalence too. It follows that the limit defining $\mathrm{Sp}(\mathcal{C})$ is taken along equivalences and thus equivalent to $\mathcal{C}$ by (a dual variant of) Lemma 6.57. The implication $(d) \Rightarrow (e)$ is trivial, as is the implication $(e) \Rightarrow (a)$: $\Omega_{\mathrm{Sp}(\mathcal{D})} \colon \mathrm{Sp}(\mathcal{D}) \to \mathrm{Sp}(\mathcal{D})$ is an equivalence for obvious reasons (for example, using Construction 7.37, $\Omega_{\mathrm{Sp}(\mathcal{D})}$ just corresponds to a shift in the index category $\mathbb{Z}$, which is clearly an equivalence). Furthermore, the implications $(c) \Rightarrow (a)$, (b) are also clear: Applying the pushout-pullback condition to the pushout square defining $\Sigma_{\mathcal{C}}$ and the pullback square defining $\Omega_{\mathcal{C}}$ shows that the unit $u \colon \mathrm{id}_{\mathcal{C}} \Rightarrow \Omega_{\mathcal{C}} \Sigma_{\mathcal{C}}$ and counit $c \colon \Sigma_{\mathcal{C}} \Omega_{\mathcal{C}} \Rightarrow \mathrm{id}_{\mathcal{C}}$ are natural equivalences, so $\Sigma_{\mathcal{C}}$ and $\Omega_{\mathcal{C}}$ must be equivalences of ∞-categories.

It remains to show $(a) \Rightarrow (c)$; the implication $(b) \Rightarrow (c)$ will follow from a dual argument. The same argument as in the proof of Lemma 7.36 shows that $\mathcal{C}$ is additive (write $X \simeq \Omega_{\mathcal{C}}(\Omega_{\mathcal{C}}^{-1} X)$ to lift X to an $\mathbb{E}_1$-group in $\mathcal{C}$ and then apply Lemma 7.30). So we only need to check that pushouts exist and coincide with pullbacks. Let $\mathcal{X} \subseteq \mathrm{Fun}(\square^2, \mathcal{C})$, where $\square^2 \simeq \Delta^1 \times \Delta^1$, be the full subcategory spanned by pullback squares. We claim:

($\boxtimes$)   *The restriction $r \colon \mathcal{X} \longrightarrow \mathrm{Fun}(\Lambda_0^2, \mathcal{C})$ is an equivalence of ∞-categories.*

To prove ($\boxtimes$), we construct a functor $s \colon \mathrm{Fun}(\Lambda_0^2, \mathcal{C}) \to \mathcal{X}$ satisfying $r \circ s \simeq (\Omega_{\mathcal{C}})_*$ and $s \circ r \simeq (\Omega_{\mathcal{C}})_*$, where $(\Omega_{\mathcal{C}})_* \colon \mathrm{Fun}(\square^2, \mathcal{C}) \to \mathrm{Fun}(\square^2, \mathcal{C})$ is postcomposition with $\Omega_{\mathcal{C}}$. Since $\Omega_{\mathcal{C}}$ is an equivalence, so is $(\Omega_{\mathcal{C}})_*$, and ($\boxtimes$) will be proved. Given a functor $F \colon \Lambda_0 \to \mathcal{C}$, which we can view as a span $c \leftarrow a \to b$ in $\mathcal{C}$, we construct the following moderately large diagram:

$$
\begin{array}{ccccc}
\Omega_{\mathcal{C}}(a) & \longrightarrow & \Omega_{\mathcal{C}}(c) & \longrightarrow & 0 \\
\downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow \\
\Omega_{\mathcal{C}}(b) & \longrightarrow & x & \longrightarrow & f & \longrightarrow & 0 \\
\downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow \\
0 & \longrightarrow & g & \longrightarrow & a & \longrightarrow & b \\
& & \downarrow & \lrcorner & \downarrow \\
& & 0 & \longrightarrow & c
\end{array}
$$

All squares are pullbacks as indicated. The fact that $\Omega_{\mathcal{C}}(a)$, $\Omega_{\mathcal{C}}(b)$, and $\Omega_{\mathcal{C}}(c)$ appear in the top left corner follows by combining suitable pullback squares into larger pullback rectangles. The functor $s \colon \mathrm{Fun}(\Lambda_0^2, \mathcal{C}) \to \mathcal{X}$ now sends

$$
F = \left(\begin{array}{ccc} a & \longrightarrow & b \\ \downarrow & & \\ c & & \end{array} \right) \longmapsto \left(\begin{array}{ccc} \Omega_{\mathcal{C}}(a) & \longrightarrow & \Omega_{\mathcal{C}}(b) \\ \downarrow & \lrcorner & \downarrow \\ \Omega_{\mathcal{C}}(c) & \longrightarrow & x \end{array} \right)
$$

(technically we have only defined s on objects, but its clear how to make it functorial since limits are functorial). This proves ($\boxtimes$).

To construct pushouts, let $F \colon \Lambda_0^2 \to \mathcal{C}$ be a span $c \leftarrow a \to b$ as above. We know from ($\boxtimes$) that F can be uniquely (up to contractible choice) extended to a pullback square, where the bottom right corner is some object $d \in \mathcal{C}$. The same goes for the trivial span consisting of identities $y = y = y$ for some $y \in \mathcal{C}$, and in this case the object we have to add in the bottom

right corner has to be y by uniqueness. Hence

$$\mathrm{Hom}_{\mathrm{Fun}(\Lambda_0^2, \mathcal{C})}\begin{pmatrix} \begin{matrix} a \longrightarrow b \\ \downarrow \qquad \\ c \qquad \end{matrix} \;,\; \begin{matrix} y = y \\ \| \qquad \\ y \qquad \end{matrix} \end{pmatrix} \simeq \mathrm{Hom}_{\mathrm{Fun}(\Delta^1 \times \Delta^1, \mathcal{C})}\begin{pmatrix} \begin{matrix} a \longrightarrow b \\ \downarrow \;\lrcorner\; \downarrow \\ c \longrightarrow d \end{matrix} \;,\; \begin{matrix} y = y \\ \| \;\lrcorner\; \| \\ y = y \end{matrix} \end{pmatrix}.$$

By the universal property of colimits, this means that the colimit over the span $c \leftarrow a \rightarrow b$ agrees with the colimit over the commutative square formed by a, b, c, and d, provided that at least one of these colimits exists. But $\Delta^1 \times \Delta^1$ has a terminal object, namely the vertex $\{1\} \times \{1\}$, and so the colimit over any commutative square exists and is given by the bottom right corner. This shows that d is a pushout of the span $c \leftarrow a \rightarrow b$. Simultaneously, we've also shown that pushout squares agree with pullback squares. This finishes the proof of $(a) \Rightarrow (c)$ and so we're done. $\qquad \square$

7.44. Corollary. — *Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor between stable ∞-categories. Then F preserves finite colimits if and only if it preserves finite limits.*

Proof. This is an immediate consequence of Lemma 6.50: Since $\mathcal{C}$ and $\mathcal{D}$ are additive (as we've seen in the proof of Lemma 7.36), F preserves finite coproducts if and only it preserves finite products. By Lemma 7.43(c), F preserves pushouts if and only if it preserves pullbacks. $\qquad \square$

7.45. Definition. — A functor $F\colon \mathcal{C} \to \mathcal{D}$ between stable ∞-categories is called *exact* if it preserves finite colimits, or equivalently, finite limits. We let $\mathrm{Cat}_\infty^{\mathrm{st}} \subseteq \mathrm{Cat}_\infty$ denote the (non-full) sub-∞-category spanned by stable ∞-categories and exact functors between them.

In the remainder of this subsection, we'll explain how the *derived ∞-category* $\mathcal{D}(R)$ and its variant $\mathcal{D}_{\geqslant 0}(R)$ from crash course 6.34 fit into the framework of stable ∞-categories.

7.46. Lemma. — *Let R be a (not necessarily commutative) ring. Then there exists an equivalence of ∞-categories $\mathcal{D}(R) \simeq \mathrm{Sp}(\mathcal{D}_{\geqslant 0}(R))$, given on objects by*

$$M \longmapsto \left(\dots, (\tau_{\geqslant -2}M)[2], (\tau_{\geqslant -1}M)[1], \tau_{\geqslant 0}M \right).$$

(here $\tau_{\geqslant -n}(-)$ are the smart truncations and $(-)[n]$ are the shift functors from crash course 6.34). In particular, $\mathcal{D}(R)$ is a stable ∞-category.

Proof sketch. Let's first explain how to construct the desired functor $\mathcal{D}(R) \to \mathrm{Sp}(\mathcal{D}_{\geqslant 0}(R))$ formally. The crucial observation is that $\Omega_{\mathcal{D}(R)}\colon \mathcal{D}(R) \to \mathcal{D}(R)$ can be identified with the shift functor $(-)[-1]$; we've seen an instance of this Example 7.4, the general case follows from similar arguments as in Lemma 6.40(b). Since $\tau_{\geqslant 0}\colon \mathcal{D}(R) \to \mathcal{D}_{\geqslant 0}(R)$ is right adjoint to the inclusion $\mathcal{D}_{\geqslant 0}(R) \subseteq \mathcal{D}(R)$, it follows formally that $\Omega_{\mathcal{D}_{\geqslant 0}(R)}\colon \mathcal{D}_{\geqslant 0}(R) \to \mathcal{D}_{\geqslant 0}(R)$ is given by $\tau_{\geqslant 0}((-)[-1])$. Then we get an equivalence of functors

$$\tau_{\geqslant -n}(-)[n] \simeq \Omega_{\mathcal{D}_{\geqslant 0}(R)} \circ \tau_{\geqslant -(n+1)}(-)[n+1]$$

in $\mathrm{Fun}(\mathcal{D}(R), \mathcal{D}_{\geqslant 0}(R))$ for all $n \geqslant 0$. Indeed, substituting $\tau_{\geqslant 0}((-)[-1])$ for $\Omega_{\mathcal{D}_{\geqslant 0}(R)}$, this equivalence is straightforward to verify in $\mathrm{Fun}(\mathrm{Ch}(R), \mathrm{Ch}_{\geqslant 0}(R))$; after that, Lemma 4.11 does the rest. Thus, the functors $\tau_{\geqslant -n}(-)[n]\colon \mathcal{D}(R) \to \mathcal{D}_{\geqslant 0}(R)$ for all $n \geqslant 0$ assemble into a functor $\mathcal{D}(R) \to \mathrm{Sp}(\mathcal{D}_{\geqslant 0}(R))$, as desired.

Now we'll verify that this functor is fully faithful and essentially surjective. For fully faithfulness, we employ Lemma 6.76(*a*) to compute $\mathrm{Hom}_{\mathrm{Sp}(\mathcal{D}_{\geqslant 0}(R))}$; we must then show that $\mathrm{Hom}_{\mathcal{D}(R)}(M, N) \simeq \lim_{n \in \mathbb{N}} \mathrm{Hom}_{\mathcal{D}_{\geqslant 0}(R)}((\tau_{\geqslant -n}M)[n], (\tau_{\geqslant -n}N)[n])$ for all $M, N \in \mathcal{D}(R)$. Clearly, we can get rid of the shifts and instead write $\lim_{n \in \mathbb{N}} \mathrm{Hom}_{\mathcal{D}_{\geqslant -n}(R)}(\tau_{\geqslant -n}M, \tau_{\geqslant -n}N)$, where the transition morphisms are induced by applying the functor $\tau_{\geqslant -n}$. This functor can also be viewed as a right adjoint $\tau_{\geqslant -n} \colon \mathcal{D}(R) \to \mathcal{D}_{\geqslant -n}(R)$ of the inclusion $\mathcal{D}_{\geqslant -n}(R) \subseteq \mathcal{D}(R)$. Therefore, we get an adjunction equivalence $\mathrm{Hom}_{\mathcal{D}_{\geqslant -n}(R)}(\tau_{\geqslant -n}M, \tau_{\geqslant -n}N) \simeq \mathrm{Hom}_{\mathcal{D}(R)}(\tau_{\geqslant -n}M, N)$. We claim:

(⊠) *Under these adjunction equivalences, the transition morphisms, which were originally induced by $\tau_{\geqslant -n}$, get identified with the precomposition morphisms*

$$c_n^* \colon \mathrm{Hom}_{\mathcal{D}(R)}\big(\tau_{\geqslant -(n+1)}M, N\big) \longrightarrow \mathrm{Hom}_{\mathcal{D}(R)}(\tau_{\geqslant -n}M, N)$$

induced by the canonical morphisms $c_n \colon \tau_{\geqslant -n}M \to \tau_{\geqslant -(n+1)}M$.

To prove (⊠), recall from the proof of Lemma 6.5 that any adjunction equivalence can, at least pointwise, be obtained by applying the right adjoint and then precomposing with the unit transformation. In our case, we see that $\mathrm{Hom}_{\mathcal{D}_{\geqslant -n}(R)}(\tau_{\geqslant -n}M, \tau_{\geqslant -n}N) \simeq \mathrm{Hom}_{\mathcal{D}(R)}(\tau_{\geqslant -n}M, N)$ is simply given by applying $\tau_{\geqslant -n}$, since the unit $u_{\tau_{\geqslant -n}M} \colon \tau_{\geqslant -n}M \to \tau_{\geqslant -n}(\tau_{\geqslant -n}M)$ is just the identity. To show (⊠), we now simply observe that $\tau_{\geqslant -n} \circ \tau_{\geqslant -(n+1)} \simeq \tau_{\geqslant -n}$ and that $\tau_{\geqslant -n}(c_n)$ is the identity on $\tau_{\geqslant -n}M$.

Using (⊠) and Corollary 6.16, we see that to show the desired equivalence

$$\mathrm{Hom}_{\mathcal{D}(R)}(M, N) \xrightarrow{\simeq} \lim_{n \in \mathbb{N}} \mathrm{Hom}_{\mathcal{D}(R)}(\tau_{\geqslant -n}M, N),$$

it'll be enough to show $\mathrm{colim}_{n \in \mathbb{N}} \tau_{\geqslant -n}M \simeq M$. To prove this, observe that filtered colimits in $\mathrm{Ch}(R)$ preserve quasi-isomorphisms. Through Lemma 4.11, this formally implies that $\mathrm{Ch}(R) \to \mathcal{D}(R)$ preserves filtered colimits (we've seen analogous arguments in the proofs of Lemma 6.58 and Corollary 6.64). So $\mathrm{colim}_{n \in \mathbb{N}} \tau_{\geqslant -n}M \simeq M$ can be checked on the level of chain complexes, where it becomes obvious. This finishes the proof that $\mathcal{D}(R) \to \mathrm{Sp}(\mathcal{D}_{\geqslant 0}(R))$ is fully faithful.

To show essential surjectivity, observe that objects in $\mathrm{Sp}(\mathcal{D}_{\geqslant 0}(R))$ are given by sequences $(\ldots, M_2, M_1, M_0)$ in $\mathcal{D}_{\geqslant 0}(R)$ together with equivalences $M_n \simeq \Omega_{\mathcal{D}_{\geqslant 0}(R)} M_{n+1} \simeq \tau_{\geqslant 0}(M_{n+1}[-1])$. These equivalences induce morphisms $M_n \to M_{n+1}[-1]$ in $\mathcal{D}(R)$ and we can form the colimit $M \coloneqq \mathrm{colim}_{n \in \mathbb{N}} M_n[-n]$. Using once again that filtered colimits in $\mathcal{D}(R)$ are well-understood, one checks that M is a preimage of $(\ldots, M_2, M_1, M_0)$ up to equivalence. □

7.47. Eilenberg–MacLane spectra — In the special case $R = \mathbb{Z}$, we have the Eilenberg–MacLane functor $\mathrm{K} \colon \mathcal{D}_{\geqslant 0}(\mathbb{Z}) \to \mathrm{An}$ from Construction 6.41, which preserves all limits (being a right adjoint) and thus commutes with Ω. Therefore, K induces a functor $\mathrm{H} \colon \mathcal{D}(\mathbb{Z}) \to \mathrm{Sp}$ via the commutative diagram

$$
\begin{array}{ccc}
\mathrm{Sp}\big(\mathcal{D}_{\geqslant 0}(\mathbb{Z})\big) & \xrightarrow{\mathrm{Sp}(\mathrm{K})} & \mathrm{Sp}(\mathrm{An}) \\
{\scriptstyle \simeq}\Big\downarrow & {\scriptstyle /\!/\!/} & \Big\| \\
\mathcal{D}(\mathbb{Z}) & \xrightarrow{\quad \mathrm{H} \quad} & \mathrm{Sp}
\end{array}
$$

We call H the *Eilenberg–MacLane spectrum functor*. Unravelling the equivalence $\mathcal{D}(\mathbb{Z}) \simeq \mathrm{Sp}(\mathcal{D}_{\geqslant 0}(\mathbb{Z}))$ from Lemma 7.46 and the construction of $\mathrm{Sp}(\mathrm{K}) \colon \mathrm{Sp}(\mathcal{D}_{\geqslant 0}(\mathbb{Z})) \to \mathrm{Sp}$, we see that

for all abelian groups A and all $n \geqslant 0$, the spectrum $\mathrm{H}A := \mathrm{H}(A[0])$ is explicitly given by the sequence of animae

$$\mathrm{H}A \simeq \left(\ldots, \mathrm{K}(A, 2), \mathrm{K}(A, 1), \mathrm{K}(A, 0)\right).$$

This fits perfectly with the homotopy equivalences $\mathrm{K}(A, n) \simeq \Omega\mathrm{K}(A, n+1)$ from Example 7.4. We call $\mathrm{H}A$ the *Eilenberg–MacLane spectrum of A*.

The Eilenberg–MacLane functor induces an equivalence of ∞-categories $\mathrm{H}\colon \mathrm{Ab} \xrightarrow{\simeq} \mathrm{Sp}^{\heartsuit}$ from the (ordinary) category of abelian groups onto the ∞-category $\mathrm{Sp}^{\heartsuit} := \mathrm{Sp}_{\geqslant 0} \cap \mathrm{Sp}_{\leqslant 0}$ of spectra concentrated in degree 0. Indeed, an inverse functor is provided via $\mathrm{Sp}^{\heartsuit} \to \mathrm{Sp}_{\geqslant 0} \simeq \mathrm{CGrp}(\mathrm{An})$ (Corollary 7.40) and $\pi_0\colon \mathrm{CGrp}(\mathrm{An}) \to \mathrm{CGrp}(\mathrm{Set}) \simeq \mathrm{Ab}$. In the modern point of view, abelian groups *are* just spectra concentrated in degree 0. Following this, we'll often suppress H in the notation and write the Eilenberg–MacLane spectrum just as A.[(7.11)]

In the classical theory of derived categories, much emphasis is placed on the fact that $D(R)$ can be equipped with a *triangulated structure*. Let us now explain how this structure is captured and radically simplified by the fact that the derived ∞-category $\mathcal{D}(R)$ is stable.

7.48. Stable ∞-categories and triangulated categories. — One striking feature of stable ∞-categories is that their homotopy category admits a canonical triangulated structure. If you haven't see triangulated categories before, [L-HA, Definition 1.1.2.5] has a nice review (but you can also safely skip this remark). Moreover, [L-HA, Theorem 1.1.2.14] explains the triangulated structure in much more detail than we'll do below.

Let $\mathcal{C}$ be a stable ∞-category. We choose $(-)[1] := h(\Sigma_{\mathcal{C}})\colon h(\mathcal{C}) \to h(\mathcal{C})$ to be the shift functor in our emerging triangulated structure. By Lemma 7.43(*b*), $(-)[1]$ is an equivalence of categories. We say that $x \to y \to z \to x[1]$ is a *distinguished triangle* if $x \to y \to z$ is a cofibre sequence in $\mathcal{C}$ in the sense of Definition 6.39. Then we can form the following pushout diagram

$$
\begin{array}{ccccc}
x & \longrightarrow & y & \longrightarrow & 0 \\
\downarrow & \ulcorner & \downarrow & \ulcorner & \downarrow \\
0 & \longrightarrow & z & \longrightarrow & \Sigma_{\mathcal{C}}(x)
\end{array}
$$

(to see why $\Sigma_{\mathcal{C}}(x)$ appears in the bottom left corner, just observe that the outer rectangle must be a pushout too). This shows several things at once: First it explains where the morphism $z \to x[1]$ in a distinguished triangle comes from. Second, a closer investigation of the diagram shows that $x \to y \to z$ is a cofibre sequence if and only if $y \to z \to \Sigma_{\mathcal{C}}(x)$ is a cofibre sequence.[(7.12)] Hence $x \to y \to z \to x[1]$ is a distinguished triangle if and only if $y \to z \to x[1] \to y[1]$ is a distinguished triangle. In other words, Verdier's axiom (TR$_2$) is satisfied.

Furthermore, it's immediately clear that every morphism $x \to y$ can be extended to a distinguished triangle (just form the cofibre), that distinguished triangles are closed under isomorphisms in $\mathrm{Ho}(\mathcal{C})$, and that for every $x \in \mathcal{C}$, the identity $\mathrm{id}_x\colon x \to x$ fits into a distinguished triangle $x \to x \to 0 \to x[1]$. So (TR$_1$) is satisfied.

[(7.11)] In the words of Robert Burklund: "Why would you give a name to the functor that sends an abelian group to itself?"

[(7.12)] Here's the argument: We've already seen that $x \to y \to z$ being a cofibre sequence implies the same for $y \to z \to \Sigma_{\mathcal{C}}(x)$. Conversely, if $y \to z \to \Sigma_{\mathcal{C}}(x)$ is a cofibre sequence, then the right square in the diagram is a pushout, hence a pullback by Lemma 7.43(*c*). Similarly, the outer square must be a pullback. It follows formally that the left square must be a pullback too, hence a pushout, and so $x \to y \to z$ is a cofibre sequence too.

Next, we'll tackle axiom (TR$_3$). Since taking cofibres is functorial, for every commutative diagram in $\mathcal{C}$ as below there is a unique dashed arrow (up to contractible choice):

$$
\begin{array}{ccccc}
x & \xrightarrow{\ \alpha\ } & y & \longrightarrow & \mathrm{cofib}(\alpha) \\
\beta\downarrow & /\!/\!/ & \downarrow\gamma & & \vdots \\
x' & \xrightarrow{\ \alpha'\ } & y' & \longrightarrow & \mathrm{cofib}(\alpha')
\end{array}
$$

The crucial detail here is "$/\!/\!/$": A commutative diagram in $\mathcal{C}$ is a functor $\bar{\sigma}\colon \square^2 \to \mathcal{C}$, whereas a commutative diagram in $h(\mathcal{C})$ is a functor $\sigma\colon \partial\square^2 \to \mathcal{C}$, which can be extended to a functor $\bar{\sigma}\colon \square \to \mathcal{C}$; however, *the choice of $\bar{\sigma}$ is not part of the data!* In particular, there could be several non-homotopic choices, corresponding to the fact that $\pi_1(\mathrm{Hom}_{\mathcal{C}}(x, y'), \gamma \circ \alpha)$ may not be trivial. So taking cofibres is *not* functorial in commutative diagrams in $h(\mathcal{C})$. If we start with a commutative diagram in $h(\mathcal{C})$, then a dashed arrow will exist, but it will not necessarily be unique; the uniqueness only comes about once a filler $\bar{\sigma}\colon \square^2 \to \mathcal{C}$ has been chosen, which we indicate by writing "$/\!/\!/$" in a diagram as above. This shows axiom (TR$_3$) and it offers a nice conceptual explanation of the non-uniqueness statement in that axiom.

Finally, let's talk about the dreaded *octahedron axiom* (TR$_4$): Given morphisms $\alpha\colon x \to y$ and $\beta\colon y \to z$ in $\mathcal{C}$, we can form a pushout diagram

$$
\begin{array}{ccccc}
x & \xrightarrow{\ \alpha\ } & y & \xrightarrow{\ \beta\ } & z \\
\downarrow & \ulcorner. & \downarrow & \ulcorner. & \downarrow \\
0 & \longrightarrow & \mathrm{cofib}(\alpha) & \longrightarrow & \mathrm{cofib}(\beta \circ \alpha) \\
& & \downarrow & \ulcorner. & \downarrow \\
& & 0 & \longrightarrow & \mathrm{cofib}(\beta)
\end{array}
$$

which shows that $\mathrm{cofib}(\alpha) \to \mathrm{cofib}(\beta \circ \alpha) \to \mathrm{cofib}(\beta)$ is a cofibre sequence in $\mathcal{C}$. And that's already the octahedron axiom!

Not every triangulated category arises as the homotopy category of a stable ∞-category. However, every triangulated category encountered in nature does, the primordial example being the derived category $D(R)$ of a ring R, which arises as the homotopy category of $\mathcal{D}(R)$, which is stable by 7.47 and Lemma 7.43(e). The point we're trying to make here is that whenever you would work with triangulated categories, you should use stable ∞-categories instead: It is both conceptually simpler and more powerful! For a concrete example, you might have seen the *filtered derived category* of a ring R before. In the classical approach, you run into annoying technical subtleties when you try to define it in full generality; this is the reason why the Stacks Project only considers degree-wise finite filtrations in [Stacks, Tag 05RX]. But on the level of ∞-categories, everything works as expected: We simply define $\mathrm{Fil}(\mathcal{D}(R)) \coloneqq \mathrm{Fun}(\mathbb{Z}, \mathcal{D}(R))$. This is a stable ∞-category again[(7.13)] and so its homotopy category $h\,\mathrm{Fil}(\mathcal{D}(R))$ is canonically a triangulated category. This is the "right" definition of the filtered derived category. It also explains where the technical subtleties come from: The homotopy category $h\,\mathrm{Fun}(\mathbb{Z}, \mathcal{D}(R))$ is in general not the same as $\mathrm{Fun}(\mathbb{Z}, h\mathcal{D}(R))$

[(7.13)] In general, if $\mathcal{C}$ is stable, then so is $\mathrm{Fun}(\mathcal{I}, \mathcal{C})$ for any ∞-category $\mathcal{I}$; to see this, use that limits and colimits in functor ∞-categories can be computed pointwise (Lemma 6.12) to verify your favourite condition from Lemma 7.43.

As our final application to the theory of derived ∞-categories, we would like to explain the relationship between $\mathrm{Hom}_{\mathcal{D}(R)}(M,N)$ and $\mathrm{RHom}_R(M,N)$. This needs a general construction, which is pretty important on its own.

7.49. Lemma. — *If $\mathcal{C}$ is a stable ∞-category, then the* Hom *animae in $\mathcal{C}$ can be refined to spectra. More precisely, there is a unique (up to equivalence) functor* $\mathrm{hom}_{\mathcal{C}}\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathrm{Sp}$ *fitting into the following diagram:*

$$
\begin{array}{ccc}
 & & \mathrm{Sp} \\
 & \overset{\mathrm{hom}_{\mathcal{C}}}{\nearrow} & \downarrow{\scriptstyle\Omega^{\infty}} \\
\mathcal{C}^{\mathrm{op}} \times \mathcal{C} & \xrightarrow{\mathrm{Hom}_{\mathcal{C}}} & \mathrm{An}
\end{array}
$$

Proof. The Yoneda embedding $\maltese_{\mathcal{C}}\colon \mathcal{C} \to \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{An})$ preserves limits by Corollary 6.17. In particular, it commutes with Ω and thus induces a functor $\mathrm{Sp}(\maltese_{\mathcal{C}})\colon \mathrm{Sp}(\mathcal{C}) \to \mathrm{Sp}(\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{An}))$, which is uniquely (up to equivalence) characterised by the fact that $\Omega^{\infty}\maltese_{\mathcal{C}}^{\mathrm{st}} \simeq \maltese_{\mathcal{C}}$. Now $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, -)\colon \mathrm{Cat}_{\infty} \to \mathrm{Cat}_{\infty}$ commutes with limits, since it has a left adjoint given by $- \times \mathcal{C}^{\mathrm{op}}$. Furthermore, $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{An}_{*/}) \simeq \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{An})_{\mathrm{const}\,*/}$. Hence $\mathrm{Sp}(\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{An})) \simeq \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Sp})$ and so we've upgraded the Yoneda embedding to a functor

$$
\maltese_{\mathcal{C}}^{\mathrm{st}}\colon \mathcal{C} \longrightarrow \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Sp})\,.
$$

After currying, this induces the desired functor $\mathrm{hom}_{\mathcal{C}}\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathrm{Sp}$. Uniqueness follows from uniqueness of $\maltese_{\mathcal{C}}^{\mathrm{st}}$. $\qquad\square$

7.50. Corollary. — *For any ring R, the spectra-enriched hom in the derived ∞-category $\mathcal{D}(R)$ is given by*

$$
\mathrm{hom}_{\mathcal{D}(R)}(M,N) \simeq \mathrm{RHom}_R(M,N)\,,
$$

that is, the Eilenberg–MacLane spectrum associated to $\mathrm{RHom}_R(M,N) \in \mathcal{D}(R)$ as in 7.47, but we suppress writing H.

Proof sketch. By the uniqueneness statement from Lemma 7.49, it's enough to construct a functorial equivalence $\Omega^{\infty}\mathrm{RHom}_R(M,N) \simeq \mathrm{Hom}_{\mathcal{D}(R)}(M,N)$. Unravelling the construction in 7.47, we see that $\Omega^{\infty}\mathrm{RHom}_R(M,N) \simeq \mathrm{K}(\tau_{\geqslant 0}\mathrm{RHom}_R(M,N))$ is the Eilenberg–MacLane anima associated to the truncation $\tau_{\geqslant}\mathrm{RHom}_R(M,N)$. Now any anima X satisfies $X \simeq \mathrm{Hom}_{\mathrm{An}}(*,X)$ and then we can compute

$$
\begin{aligned}
\mathrm{Hom}_{\mathrm{An}}(*, \mathrm{K}(\tau_{\geqslant 0}\mathrm{RHom}_R(M,N))) &\simeq \mathrm{Hom}_{\mathcal{D}_{\geqslant 0}(\mathbb{Z})}(\mathbb{Z}[0], \tau_{\geqslant 0}\mathrm{RHom}_R(M,N)) \\
&\simeq \mathrm{Hom}_{\mathcal{D}(\mathbb{Z})}(\mathbb{Z}[0], \mathrm{RHom}_R(M,N)) \\
&\simeq \mathrm{Hom}_{\mathcal{D}(R)}\big(\mathbb{Z}[0] \otimes_{\mathbb{Z}}^{\mathrm{L}} M, N\big) \\
&\simeq \mathrm{Hom}_{\mathcal{D}(R)}(M,N)\,.
\end{aligned}
$$

In the first step, we use the adjunction $\mathrm{C}\colon \mathrm{An} \rightleftarrows \mathcal{D}_{\geqslant 0}(\mathbb{Z}) : \mathrm{K}$ from Construction 6.41. In the second step, we use that $\tau_{\geqslant 0}\colon \mathcal{D}(\mathbb{Z}) \to \mathcal{D}_{\geqslant 0}(\mathbb{Z})$ is right adjoint to the inclusion $\mathcal{D}_{\geqslant 0}(\mathbb{Z}) \subseteq \mathcal{D}(\mathbb{Z})$, as we've seen in crash course 6.34. In the third step, we use the "derived tensor-Hom adjunction". To construct this, using the description from crash course 6.36, we only need to verify that the ordinary tensor-Hom adjunction refines to an adjunction of Kan-enriched categories, which is straightforward. The fourth step is obvious.

Putting everything together, we get $\Omega^{\infty}\mathrm{RHom}_R(M,N) \simeq \mathrm{Hom}_{\mathcal{D}(R)}(M,N)$, as desired. Since all steps can easily be made functorial, we're done. $\qquad\square$

§7.5. Spectra and excisive functors

In this subsection, we'll explain an alternative construction of $\mathrm{Sp}(\mathcal{C})$ that more closely resembles the "Segal models" for $\mathbb{E}_1$-groups and $\mathbb{E}_\infty$-groups from Definitions 7.6 and 7.24. This alternative model will be needed in §8 to construct the tensor product of spectra, but we'll also use it to show that $\Omega^\infty\colon \mathrm{Sp} \to \mathrm{An}$ has a left adjoint and to construct the famous *sphere spectrum* $\mathbb{S}$.

7.51. Definition. — Let $\mathrm{An}_{*/}^{\mathrm{fin}} \subseteq \mathrm{An}_{*/}$ be the ∞-category of *finite pointed animae*, defined as smallest full sub-∞-category that contains $S^0 \simeq * \amalg *$ and is closed under finite colimits.[(7.14)] Furthermore, let $\mathcal{C}$ be an ∞-category with all finite limits, so that, in particular, $\mathcal{C}$ contains a terminal object $* \in \mathcal{C}$.

(*a*) A functor $F\colon \mathrm{An}_{*/}^{\mathrm{fin}} \to \mathcal{C}_{*/}$ is called *reduced* if $F(*) \simeq *$.

(*b*) A functor $F\colon \mathrm{An}_{*/}^{\mathrm{fin}} \to \mathcal{C}_{*/}$ is called *excisive* if F sends pushout squares to pullback squares.

Furthermore, we let $\mathrm{Fun}_*(\mathrm{An}_{*/}^{\mathrm{fin}}, \mathcal{C}_{*/}) \subseteq \mathrm{Fun}_*^{\mathrm{exc}}(\mathrm{An}_{*/}^{\mathrm{fin}}, \mathcal{C}_{*/}) \subseteq \mathrm{Fun}(\mathrm{An}_{*/}^{\mathrm{fin}}, \mathcal{C}_{*/})$ denote full sub-∞-categories spanned by the reduced functors or the reduced excisive functors, respectively.

7.52. Lemma. — *If $\mathcal{C}$ has finite limits, then $\mathrm{Fun}_*^{\mathrm{exc}}(\mathrm{An}_{*/}^{\mathrm{fin}}, \mathcal{C}_{*/})$ is a stable ∞-category.*

Proof. We'll verify the conditions from Lemma 7.43(*a*). Since limits in colimits in functor categories are computed pointwise by Lemma 6.12, it follows that $\mathrm{Fun}(\mathrm{An}_{*/}^{\mathrm{fin}}, \mathcal{C}_{*/})$ has all finite limits and that the terminal object $\mathrm{const}\, *$ is also initial. Furthermore, reduced excisive functors are closed under all limits, and so $\mathrm{Fun}_*^{\mathrm{exc}}(\mathrm{An}_{*/}^{\mathrm{fin}}, \mathcal{C}_{*/})$ still has all finite limits and its terminal object is initial too. It remains to show that the loop functor Ω on $\mathrm{Fun}_*^{\mathrm{exc}}(\mathrm{An}_{*/}^{\mathrm{fin}}, \mathcal{C}_{*/})$ is an equivalence. We'll show that precomposition with $\Sigma\colon \mathrm{An}_{*/}^{\mathrm{fin}} \to \mathrm{An}_{*/}^{\mathrm{fin}}$ provides an inverse. To this end, let $F\colon \mathrm{An}_{*/}^{\mathrm{fin}} \to \mathcal{C}_{*/}$ be a reduced excisive functor. By definition of Σ, we have a diagram of natural transformations

$$
\begin{array}{ccc}
F(-) & \Longrightarrow & F(\mathrm{const}\, *) \\
\big\Downarrow & /\!/\!/ & \big\Downarrow \\
F(\mathrm{const}\, *) & \Longrightarrow & F(\Sigma(-))
\end{array}
$$

Since $F(*) \simeq *$, we have $F(\mathrm{const}\, *) \simeq \mathrm{const}\, *$. Therefore, this diagram induces a natural transformation $\eta_F\colon F(-) \Rightarrow \Omega F(\Sigma(-))$. Since F sends pushout squares to pullbacks, η_F is a pointwise equivalence, hence an equivalence of functors by Theorem 4.5. Furthermore, it's clear from the construction that η_F is also functorial in F.

Now observe that the loop functor Ω on $\mathrm{Fun}_*^{\mathrm{exc}}(\mathrm{An}_{*/}^{\mathrm{fin}}, \mathcal{C}_{*/})$ is given by postcomposition with the loop functor $\Omega_\mathcal{C}\colon \mathcal{C}_{*/} \to \mathcal{C}_{*/}$, as follows from Lemma 6.12. Since, in general, postcomposition commutes with precomposition, our functorial equivalence $F(-) \simeq \Omega F(\Sigma(-))$ thus shows that precomposition with Σ is both a left and a right inverse of Ω. $\square$

7.53. Lemma. — *For every $n \geqslant 0$, let $\mathrm{ev}_{S^n}\colon \mathrm{Fun}_*^{\mathrm{exc}}(\mathrm{An}_{*/}^{\mathrm{fin}}, \mathcal{C}_{*/}) \to \mathcal{C}_{*/}$ be given by evaluation at the n-sphere S^n. Then $\mathrm{ev}_{S^{n+1}} \simeq \Omega_\mathcal{C} \circ \mathrm{ev}_{S^n}$ holds for all $n \geqslant 0$ and the induced functor*

$$
\mathrm{Fun}_*^{\mathrm{exc}}\big(\mathrm{An}_{*/}^{\mathrm{fin}}, \mathcal{C}_{*/}\big) \xrightarrow{\ \simeq\ } \mathrm{Sp}(\mathcal{C})
$$

is an equivalence of ∞-categories.

[(7.14)] $\mathrm{An}_{*/}^{\mathrm{fin}}$ looks like it could be the full sub-∞-category $(\mathrm{An}_{*/})^{\aleph_0} \subseteq \mathrm{An}_{*/}$ spanned by the compact objects (in the sense of Definition 6.52(*c*)), but it's not: $(\mathrm{An}_{*/})^{\aleph_0}$ also contains all retracts of objects in $\mathrm{An}_{*/}^{\mathrm{fin}}$.

Proof sketch. The condition $\mathrm{ev}_{S^{n+1}} \simeq \Omega_{\mathcal{C}} \circ \mathrm{ev}_{S^n}$ follows immediately from $S^{n+1} \simeq \Sigma S^n$ and the fact that excisive functors send pushouts to pullbacks, so the hard part will be to show that we get an equivalence. Let's first consider the case where $\mathcal{C}$ is a stable ∞-category. We'll show that

$$\mathrm{ev}_{S^0} \colon \mathrm{Fun}_*^{\mathrm{exc}}\big(\mathrm{An}_{*/}^{\mathrm{fin}}, \mathcal{C}_{*/}\big) \xrightarrow{\simeq} \mathcal{C}$$

is an equivalence of ∞-categories. This special case will occupy the majority of the proof; the general case is an easy consequence, as we'll see below. If $\mathcal{C}$ is stable, then $\mathcal{C}_{*/} \simeq \mathcal{C}$, since the terminal object is also initial. This also shows that a functor is reduced if and only if it preserves initial objects. Furthermore, since pushout squares and pullback squares agree in any stable ∞-category, a functor $F \colon \mathrm{An}_{*/}^{\mathrm{fin}} \to \mathcal{C}$ is excisive if and only if it preserves pushouts. Any finite colimit can be built from initial objects and pushouts. Indeed, by Lemma 6.50, we only have to show that finite coproducts can be built that way, but coproducts are pushouts over the initial object. In summary, we obtain

$$\mathrm{Fun}_*^{\mathrm{exc}}\big(\mathrm{An}_{*/}^{\mathrm{fin}}, \mathcal{C}_{*/}\big) \simeq \mathrm{Fun}^{\mathrm{fin\text{-}colim}}\big(\mathrm{An}_{*/}^{\mathrm{fin}}, \mathcal{C}\big) \,,$$

where $\mathrm{Fun}^{\mathrm{fin\text{-}colim}} \subseteq \mathrm{Fun}$ denotes the full sub-∞-category of functors that preserve finite colimits.

Now let $\mathcal{I} := \{* \rightleftarrows S^0\}$ be the full sub-∞-category of $\mathrm{An}_{*/}^{\mathrm{fin}}$ spanned by $*$ and S^0. Observe that $\mathcal{I} \simeq \mathrm{Fin}_{\leqslant 1}$, where $\mathrm{Fin}_{\leqslant 1}$ is the (ordinary) category from the proof of Lemma 7.28.[(7.15)] Applying claim ($\boxtimes_2$) from the proof of Lemma 7.28, we obtain that evaluation at S^0 induces an equivalence of ∞-categories

$$\mathrm{ev}_{S^0} \colon \mathrm{Fun}_*(\mathcal{I}, \mathcal{C}) \xrightarrow{\simeq} \mathcal{C} \,.$$

So it remains to show that restriction along the inclusion $i \colon \mathcal{I} \to \mathrm{An}_{*/}^{\mathrm{fin}}$ induces an equivalence of ∞-categories

$$i^* \colon \mathrm{Fun}^{\mathrm{fin\text{-}colim}}\big(\mathrm{An}_{*/}^{\mathrm{fin}}, \mathcal{C}\big) \xrightarrow{\simeq} \mathrm{Fun}_*(\mathcal{I}, \mathcal{C}) \,.$$

To prove this, we'll study left Kan extension along the inclusion i. By Lemma 7.54 below, every reduced functor $F \colon \mathcal{I} \to \mathcal{C}$ admits a left Kan extension; furthermore, that lemma provides an explicit formula (in form of a pushout diagram) for $\mathrm{Lan}_i F(X, x)$. Combining this formula with Lemma 6.38(*b*) shows that $\mathrm{Lan}_i F \colon \mathrm{An}_{*/}^{\mathrm{fin}} \to \mathcal{C}$ preserves pushouts. $\mathrm{Lan}_i F$ also preserves initial objects since F was assumed to be reduced. Hence $\mathrm{Lan}_i F$ preserves all finite colimits. Therefore, usual left Kan extension adjunction $\mathrm{Lan}_i \dashv i^*$ restricts to an adjunction

$$\mathrm{Lan}_i \colon \mathrm{Fun}_*(\mathcal{I}, \mathcal{C}) \rightleftarrows \mathrm{Fun}^{\mathrm{fin\text{-}colim}}\big(\mathrm{An}_{*/}^{\mathrm{fin}}, \mathcal{C}\big) \colon i^* \,.$$

Since i is fully faithful, so is Lan_i by Corollary 6.29. Furthermore, since $\mathrm{An}_{*/}^{\mathrm{fin}}$ is generated under finite colimits by $*$ and S^0, it's clear that i^* is conservative. So Lan_i and i^* are inverse equivalences by Lemma 6.33(*b*), which is what we wanted to show.

It remains to deduce the general case. So let $\mathcal{C}$ again be an arbitrary ∞-category with finite limits. Then an equivalence $\mathrm{Sp}(\mathcal{C}) \simeq \mathrm{Fun}_*^{\mathrm{exc}}(\mathrm{An}_{*/}^{\mathrm{fin}}, \mathcal{C}_{*/})$ can be obtained as follows:

$$\mathrm{Sp}(\mathcal{C}) \simeq \mathrm{Fun}_*^{\mathrm{exc}}\big(\mathrm{An}_{*/}^{\mathrm{fin}}, \mathrm{Sp}(\mathcal{C})\big) \simeq \mathrm{Sp}\Big(\mathrm{Fun}_*^{\mathrm{exc}}\big(\mathrm{An}_{*/}^{\mathrm{fin}}, \mathcal{C}_{*/}\big)\Big) \simeq \mathrm{Fun}_*^{\mathrm{exc}}\big(\mathrm{An}_{*/}^{\mathrm{fin}}, \mathcal{C}_{*/}\big)$$

[(7.15)]Indeed, it's straightforward to verify that $h(\mathcal{I}) \simeq \mathrm{Fin}_{\leqslant 1}$. But one also easily verifies that the Hom animae in $\mathcal{I}$ are discrete, and so the canonical functor $\mathcal{I} \to h(\mathcal{I})$ is an equivalence by Theorem 4.6.

The first equivalence follows from what we've just shown. The third equivalence follows from $\mathrm{Fun}_*^{\mathrm{exc}}(\mathrm{An}_{*/}^{\mathrm{fin}}, \mathcal{C}_{*/})$ being stable by Lemma 7.52. So let's explain where the second equivalence comes from: The functor $\mathrm{Fun}(\mathrm{An}_{*/}^{\mathrm{fin}}, -)\colon \mathrm{Cat}_\infty \to \mathrm{Cat}_\infty$ commutes with limits since it is a right adjoint by Example 6.3(b). Hence $\mathrm{Fun}(\mathrm{An}_{*/}^{\mathrm{fin}}, \mathrm{Sp}(\mathcal{C})) \simeq \mathrm{Sp}(\mathrm{Fun}(\mathrm{An}_{*/}^{\mathrm{fin}}, \mathcal{C}_{*/}))$. Now $\mathrm{Fun}_*^{\mathrm{exc}}(\mathrm{An}_{*/}^{\mathrm{fin}}, \mathrm{Sp}(\mathcal{C}))$ and $\mathrm{Sp}(\mathrm{Fun}_*^{\mathrm{exc}}(\mathrm{An}_{*/}^{\mathrm{fin}}, \mathcal{C}_{*/}))$ can be regarded as full sub-∞-categories of the left- and the right-hand side, respectively, and we only have to check that they match. To see this, recall that limits in $\mathrm{Sp}(\mathcal{C})$ are formed degree-wise by Lemma 6.76(b), and so a functor $F\colon \mathrm{An}_{*/}^{\mathrm{fin}} \to \mathrm{Sp}(\mathcal{C})$ is reduced and excisive if and only if $\Omega_{\mathcal{C}}^{\infty-n} \circ F\colon \mathrm{An}_{*/}^{\mathrm{fin}} \to \mathcal{C}_{*/}$ is reduced and excisive for all $n \in \mathbb{Z}$. This is precisely what we need.

So we've constructed an equivalence $\mathrm{Sp}(\mathcal{C}) \simeq \mathrm{Fun}_*^{\mathrm{exc}}(\mathrm{An}_{*/}^{\mathrm{fin}}, \mathcal{C}_{*/})$. By a straightforward unravelling, this equivalence is really induced by ev_{S^n} for all $n \geqslant 0$. $\qquad\square$

7.54. Lemma. — *Let $\mathcal{C}$ be an ∞-category with finite colimits; in particular, $\mathcal{C}$ contains an initial object $0 \in \mathcal{C}$. Let $i\colon \mathcal{I} \to \mathrm{An}_{*/}^{\mathrm{fin}}$ be as in the proof of Lemma 7.53 and let $F\colon \mathcal{I} \to \mathcal{C}$ be a functor such that $F(*) \simeq 0$. Then $\mathrm{Lan}_i F\colon \mathrm{An}_{*/}^{\mathrm{fin}} \to \mathcal{C}$ exists and its value on a pointed anima (X, x) is given as the pushout*

$$
\begin{array}{ccc}
F(S^0) & \longrightarrow & \mathrm{colim}\big(\mathrm{const}\,F(S^0)\colon X \to \mathcal{C}\big) \\
\downarrow & \quad \ulcorner. & \downarrow \\
0 & \longrightarrow & \mathrm{Lan}_i F(X, x)
\end{array}
$$

in $\mathcal{C}$ (where the top horizontal arrow is induced by $\{x\} \to X$).

Proof. First note that the pushout above exists in $\mathcal{C}$. Indeed, since $\mathcal{C}$ is stable, it has all finite colimits by Lemma 7.43(c). In particular, since X is a finite anima, $\mathrm{colim}(\mathrm{const}\,F(S^0)\colon X \to \mathcal{C})$ exists, and then so does the pushout.

Showing that $\mathrm{Lan}_i F(X, x)$ is indeed given by the pushout in question is essentially a lengthy unravelling of the Kan extension formula from Lemma 6.27. We've seen in the proof of Lemma 7.53 that $\mathcal{I} \simeq \mathrm{Fin}_{\leqslant 1}$. Under this equivalence, $\mathrm{Fin}_{\leqslant 1}^\circ$ corresponds to the non-full sub-∞-category $\mathcal{J} := \{* \longleftarrow S^0\}$ of $\mathrm{An}_{*/}^{\mathrm{fin}}$. Let $j\colon \mathcal{J} \to \mathrm{An}_{*/}^{\mathrm{fin}}$ be the inclusion of $\mathcal{J}$. By claim ($\boxtimes_2$) in the proof of Lemma 7.28 we may replace $\mathcal{I}$ by $\mathcal{J}$ and analyse the left Kan extension $\mathrm{Lan}_j F$ of a reduced functor $F\colon \mathcal{J} \to \mathrm{An}_{*/}^{\mathrm{fin}}$ instead. This will make our life much easier.

Fix some pointed anima (X, x) and consider the slice ∞-category

$$
\mathcal{Y} := \mathcal{J} \times_{\mathrm{An}_{*/}^{\mathrm{fin}}} \big(\mathrm{An}_{*/}^{\mathrm{fin}}\big)_{/(X,x)}
$$

together with its usual slice projection $s\colon \mathcal{Y} \to \mathcal{J}$. The Kan extension formula from Lemma 6.27 asserts that $\mathrm{Lan}_j F(X, x) \simeq \mathrm{colim}(F \circ s\colon \mathcal{Y} \to \mathcal{C})$, provided this colimits exists. So let's analyse the ∞-category $\mathcal{Y}$. The objects of $\mathcal{Y}$ come in two flavours: First there are pointed morphisms $* \to (X, x)$, of which there's only one, which by abuse of notation we'll also denote $*$. Second, there are pointed morphisms $S^0 \to (X, x)$. Every such morphism is uniquely given by where it sends the non-basepoint, and we let $y\colon S^0 \to (X, x)$ denote the morphism that sends the non-basepoint to $y \in X$. Next, let's compute morphism animae. For $y, z \in X$, we can use Corollary 5.15 and Lemma 6.76(a) to see that $\mathrm{Hom}_{\mathcal{Y}}(y, z)$ sits in a pullback square

$$
\begin{array}{ccc}
\mathrm{Hom}_{\mathcal{Y}}(y, z) & \longrightarrow & \{y\} \\
\downarrow & \quad \lrcorner & \downarrow \\
\mathrm{Hom}_{\mathcal{J}}\big(S^0, S^0\big) & \xrightarrow{\ z_*\ } & \mathrm{Hom}_{\mathrm{An}_{*/}}\big(S^0, (X, x)\big)
\end{array}
$$

Since $\mathrm{Hom}_{\mathcal{J}}(S^0, S^0) \simeq \mathrm{id}_{S^0}$ and $\mathrm{Hom}_{\mathrm{An}_{*/}}(S^0, (X, x)) \simeq \mathrm{Hom}_{\mathrm{An}}(*, X) \simeq X$, this pullback can be identified with $\{y\} \times_X \{z\}$, and then an argument as in Lemma 7.3(*b*) shows

$$\mathrm{Hom}_{\mathcal{Y}}(y, z) \simeq \mathrm{Hom}_X(y, z).$$

In a similar way, we obtain $\mathrm{Hom}_{\mathcal{Y}}(y, *) \simeq \mathrm{Hom}_X(y, x)$ as well as $\mathrm{Hom}_{\mathcal{Y}}(*, *) \simeq *$ and $\mathrm{Hom}_{\mathcal{Y}}(*, z) \simeq \emptyset$. In total, we may picture $\mathcal{Y}$ as follows:
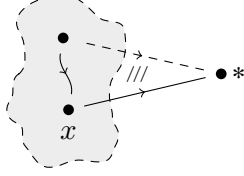
Now let $\mathcal{X}$ be the pushout in Cat_∞ of $\{x\} \to X$ along $\{x\} \simeq \{0\} \to \Delta^1$. We wish to construct a functor $\vartheta \colon \mathcal{X} \to \mathcal{Y}$ and then to show that ϑ is an equivalence of ∞-categories. To this end, first consider the functor

$$\varphi \colon X \simeq \{S^0\} \times_{\mathrm{An}^{\mathrm{fin}}_{*/}} \left(\mathrm{An}^{\mathrm{fin}}_{*/}\right)_{/(X, x)} \longrightarrow \mathcal{J} \times_{\mathrm{An}^{\mathrm{fin}}_{*/}} \left(\mathrm{An}^{\mathrm{fin}}_{*/}\right)_{/(X, x)} \simeq \mathcal{Y};$$

the equivalence on the left follows from the fact that the right fibration $\left(\mathrm{An}^{\mathrm{fin}}_{*/}\right)_{/(X, x)} \to \mathrm{An}^{\mathrm{fin}}_{*/}$ parametrises the functor $\mathrm{Hom}_{\mathrm{An}^{\mathrm{fin}}_{*/}}(-, (X, x)) \colon \left(\mathrm{An}^{\mathrm{fin}}_{*/}\right)^{\mathrm{op}} \to \mathrm{An}$ and so its fibre over S^0 is given by $\mathrm{Hom}_{\mathrm{An}_{*/}}(S^0, X) \simeq X$. Secondly, consider the functor $\psi \colon \Delta^1 \to \mathcal{Y}$ corresponding to the morphism $\psi \colon x \to *$ in $\mathcal{Y}$ which in turn corresponds to $\mathrm{id}_x \in \mathrm{Hom}_X(x, x) \simeq \mathrm{Hom}_{\mathcal{Y}}(x, *)$. By construction, $\varphi|_{\{x\}} \simeq \psi|_{\{0\}}$ and so by the universal property of pushouts, φ and ψ together determine a functor $\vartheta \colon \mathcal{X} \to \mathcal{Y}$.[(7.16)]

If we can show that ϑ is an equivalence, we're done. Indeed, using Lemma 6.38(*b*) and our assumption $F(*) \simeq 0$, we see that $\mathrm{colim}(F \circ s \circ \vartheta \colon \mathcal{X} \to \mathcal{C})$ is precisely the pushout we're looking for! It's obvious that ϑ is essentially surjective, so we only need to prove that ϑ is fully faithful, and for that, we must understand Hom animae in $\mathcal{X}$. In general, there's no nice way to describe Hom in a pushout[(7.17)], but here we can use a trick: The inclusion $\iota \colon X \to \mathcal{X}$ is left adjoint to the functor $r \colon \mathcal{X} \simeq X \sqcup_{\{x\}} \Delta^1 \to X \sqcup_{\{x\}} \{x\} \simeq X$ defined by $\Delta^1 \to \{x\}$! To see this, first note that $\{0\} \rightleftarrows \Delta^1$ is an adjunction (which is obvious, as these are ordinary categories), and recall from Lemma 6.5 that to construct an adjunction, it's enough to construct unit and counit as well as the triangle identities. Since $- \times \Delta^1 \colon \mathrm{Cat}_\infty \to \mathrm{Cat}_\infty$ commutes with pushouts, as it is a left adjoint by Example 6.3(*b*), we can construct the counit $c \colon \mathrm{id}_{\mathcal{X}} \to r \circ \iota$ by taking the pushout of the identity transformation on X with the counit of the adjunction $\{0\} \rightleftarrows \Delta^1$ (this crucially uses that the counit is already the identity when restricted to $\{0\}$; in general, extending adjunctions along pushouts doesn't work). In the same way, we can construct the unit, and the triangle identities will still be satisfied.

Using this adjunction, we see that ϑ induces equivalences $\mathrm{Hom}_{\mathcal{X}}(y, z) \simeq \mathrm{Hom}_{\mathcal{Y}}(y, z)$ for all $y, z \in X$.[(7.18)] Furthermore, if $* \in \mathcal{X}$ denotes the image of $1 \in \Delta^1$, then $\vartheta(*) \simeq *$ and

---

[(7.16)]More precisely, every *choice* of an equivalence $\varphi|_{\{x\}} \simeq \psi|_{\{0\}}$ determines a natural transformation between the span $X \leftarrow \{x\} \simeq \{0\} \to \Delta^1$ and the span $\mathrm{const}\, \mathcal{Y}$ in $\mathrm{Fun}(\Lambda_0^2, \mathrm{Cat}_\infty)$. And every such transformation determines a viable ϑ by the universal property of colimits.

[(7.17)]You could try to use Lemma 6.14 as well as a general formula for Hom in localisations, but usually this won't get you very far.

[(7.18)]In particular, it follows that $X \to \mathcal{X}$ is fully faithful. By a rather surprising result of Maxime Ramzi [Ram], it's true in general that fully faithful functors are preserved under pushouts in Cat_∞.

we have $\mathrm{Hom}_{\mathcal{X}}(y,*) \simeq \mathrm{Hom}_X(y,r(*)) \simeq \mathrm{Hom}_X(y,x)$, so $\mathrm{Hom}_{\mathcal{X}}(y,*) \simeq \mathrm{Hom}_{\mathcal{Y}}(y,*)$ for all $y \in X$. Finally, we have $\mathrm{Hom}_{\mathcal{X}}(*,*) \simeq *$ and $\mathrm{Hom}_{\mathcal{X}}(*,z) \simeq \emptyset$ for all $z \in X$. For the latter, simply note that $X \to \{x\}$ defines a functor $\mathcal{X} \simeq X \sqcup_{\{x\}} \Delta^1 \to \{x\} \sqcup_{\{x\}} \Delta^1 \simeq \Delta^1$ and then there's a morphism $\mathrm{Hom}_{\mathcal{X}}(*,z) \to \mathrm{Hom}_{\Delta^1}(1,0) \simeq \emptyset$. For the former, the easiest argument uses the simplicial model: By model category fact 6.13, $\mathcal{X}$ is given by choosing an inner anodyne map of the pushout $X \sqcup_{\{x\}} \Delta^1$ in sSet into a quasi-category. If we use the recipe from the proof of Lemma 3.12, we won't ever add any simplex whose vertices are all $*$, hence $\mathrm{Hom}_{\mathcal{X}}(*,*) \simeq \mathrm{Hom}_{\Delta^1}(1,1)$. Alternatively, for a model-independent argument, consider the Yoneda embedding $よ_{\mathcal{X}^{\mathrm{op}}} \colon \mathcal{X}^{\mathrm{op}} \to \mathrm{Fun}(\mathcal{X},\mathrm{An})$. If $E \colon \mathcal{X} \to \mathrm{An}$ is any functor, then $\mathrm{Hom}_{\mathrm{Fun}(\mathcal{X},\mathrm{An})}(\mathrm{Hom}_{\mathcal{X}}(*,-),E) \simeq E(*)$. On the other hand, since $\mathrm{Fun}(-,\mathrm{An})$ sends pushouts to pullbacks, we get $\mathrm{Fun}(\mathcal{X},\mathrm{An}) \simeq \mathrm{Fun}(X,\mathrm{An}) \times_{\mathrm{Fun}(\{x\},\mathrm{An})} \mathrm{Fun}(\Delta^1,\mathrm{An})$. So we can compute $\mathrm{Hom}_{\mathrm{Fun}(\mathcal{X},\mathrm{An})}(\mathrm{Hom}_{\mathcal{X}}(*,-),E)$ using Lemma 6.76(a). As we've already seen, when $\mathrm{Hom}_{\mathcal{X}}(*,-)$ is restricted to X or $\{x\}$, we get the constant $\emptyset$-valued functor. Furthermore, the restriction to Δ^1 sends $0 \mapsto \emptyset$ as well. It follows formally that $\mathrm{Hom}_{\mathcal{X}}(*,-)|_{\Delta^1}$ must be the left Kan extension along $\{1\} \to \Delta^1$ of the functor $\{1\} \to \mathrm{An}$ that sends $1 \mapsto \mathrm{Hom}_{\mathcal{X}}(*,*)$. Hence

$$\mathrm{Hom}_{\mathrm{Fun}(\mathcal{X},\mathrm{An})}\big(\mathrm{Hom}_{\mathcal{X}}(*,-),E\big) \simeq \mathrm{Hom}_{\mathrm{An}}\big(\mathrm{Hom}_{\mathcal{X}}(*,*),E(*)\big).$$

Since $E(*)$ can be any anima (just take E to be a constant functor), it follows that the inclusion $\{\mathrm{id}_*\} \to \mathrm{Hom}_{\mathcal{X}}(*,*)$ induces an equivalence $\mathrm{Hom}_{\mathrm{An}}(\mathrm{Hom}_{\mathcal{X}}(*,*),T) \simeq T$ for all animae T, and so $\{\mathrm{id}_*\} \simeq \mathrm{Hom}_{\mathcal{X}}(*,*)$, as desired. $\qquad\square$

This finishes the proof that $\mathrm{Sp}(\mathcal{C}) \simeq \mathrm{Fun}^{\mathrm{exc}}_*(\mathrm{An}^{\mathrm{fin}}_{*/},\mathcal{C}_{*/})$. Now we'll use this alternative description to define a left adjoint of $\Omega^\infty \colon \mathrm{Sp} \to \mathrm{An}$ and to construct the sphere spectrum $\mathbb{S}$.

7.55. Lemma. — *Let $\mathcal{C}$ be an ∞-category with finite limits; in particular, $\mathcal{C}$ has a terminal object $* \in \mathcal{C}$. Assume furthermore that $\mathcal{C}_{*/}$ admits sequential colimits and that $\Omega_{\mathcal{C}} \colon \mathcal{C}_{*/} \to \mathcal{C}_{*/}$ commutes with them. Then $\mathrm{Fun}^{\mathrm{exc}}_*(\mathrm{An}^{\mathrm{fin}}_{*/},\mathcal{C}_{*/}) \subseteq \mathrm{Fun}_*(\mathrm{An}^{\mathrm{fin}}_{*/},\mathcal{C}_{*/})$ has a left adjoint, which sends a reduced functor $F \colon \mathrm{An}^{\mathrm{fin}}_{*/} \to \mathcal{C}_{*/}$ to*

$$F^{\mathrm{sp}} := \operatorname*{colim}_{n \geqslant 0} \Omega^n_{\mathcal{C}} F\big(\Sigma^n(-)\big)$$

(*in the proof of Lemma 7.52 we've constructed a transformation $F \Rightarrow \Omega_{\mathcal{C}} F(\Sigma(-))$; the colimit on the right-hand side is given by iterating this construction*).

To prove Lemma 7.55, we need a general lemma about adjunctions:

7.56. Lemma. — *Let $L \colon \mathcal{C} \to \mathcal{C}$ be an endofunctor of an ∞-category and $u \colon \mathrm{id}_{\mathcal{C}} \Rightarrow L$ be a natural transformation. Suppose that both $Lu \colon L \Rightarrow L \circ L$ and $uL \colon L \Rightarrow L \circ L$ are equivalences. Then, if $i \colon \mathcal{C}_L \to \mathcal{C}$ denotes the inclusion of the full sub-∞-category spanned by the essential image of L, we have an adjunction*

$$L \colon \mathcal{C} \rightleftharpoons \mathcal{C}_L : i.$$

Proof. By Lemma 6.5, it's enough to construct the unit as well as the counit and to verify the triangle identities. This will be so tautological that it becomes confusing again. As the notation suggests, we take u to be our unit. Restricting u along $i \colon \mathcal{C}_L \to \mathcal{C}$ defines a natural transformation $ui \colon i \Rightarrow L \circ i$ in $\mathrm{Fun}(\mathcal{C}_L,\mathcal{C})$. By assumption, $u_{L(x)} \colon L(x) \to L(L(x))$ is an equivalence for all $x \in \mathcal{C}$. This shows that ui is a pointwise equivalence, hence it admits an

inverse by Theorem 4.5. Furthermore ui takes values in $\mathcal{C}_L$, so we can regard it as a natural transformation $\mathrm{id}_{\mathcal{C}_L} \Rightarrow L \circ i$ in $\mathrm{Fun}(\mathcal{C}_L, \mathcal{C}_L)$. Its inverse can then also be regarded as a natural transformation $c \colon L \circ i \Rightarrow \mathrm{id}_{\mathcal{C}_L}$ in $\mathrm{Fun}(\mathcal{C}_L, \mathcal{C}_L)$. This will be our counit.

Let's now verify the triangle identities. The second one from Lemma 6.5 is trivially satisfied, since, by construction, ic is an inverse of ui and so $ic \circ ui \simeq \mathrm{id}_i$. For the first triangle identity (in its weak form, where we only require $cL \circ uL$ to be an equivalence), we use that $Lu \colon L \Rightarrow L \circ L = L \circ i \circ L$ is an equivalence by assumption, so we only need to check that cL is an equivalence. But c itself is, by construction, already an equivalence. $\square$

Proof sketch of Lemma 7.55. It's clear that the construction of F^{sp} can be made into an endofunctor $(-)^{\mathrm{sp}} \colon \mathrm{Fun}_*(\mathrm{An}^{\mathrm{fin}}_{*/}, \mathcal{C}_{*/}) \to \mathrm{Fun}_*(\mathrm{An}^{\mathrm{fin}}_{*/}, \mathcal{C}_{*/})$. By construction, for every F there is a natural transformation $u_F \colon F \Rightarrow F^{\mathrm{sp}}$ in $\mathrm{Fun}_*(\mathrm{An}^{\mathrm{fin}}_{*/}, \mathcal{C}_{*/})$. This is clearly natural in F as well, hence defines a natural transformation $\mathrm{id}_{\mathrm{Fun}_*(\mathrm{An}^{\mathrm{fin}}_{*/}, \mathcal{C}_{*/})} \Rightarrow (-)^{\mathrm{sp}}$. We'll verify the conditions from Lemma 7.56 and show that the image of $(-)^{\mathrm{sp}}$ are precisely the reduced and excisive functors.

Let's start with the first condition: To show that $u^{\mathrm{sp}} \colon (-)^{\mathrm{sp}} \Rightarrow ((-)^{\mathrm{sp}})^{\mathrm{sp}}$ is an equivalence, we must show that

$$u_F^{\mathrm{sp}} \colon \operatorname*{colim}_{m \geqslant 0} \Omega_{\mathcal{C}}^m F\big(\Sigma^m(-)\big) \overset{\simeq}{\Longrightarrow} \operatorname*{colim}_{m \geqslant 0} \operatorname*{colim}_{n \geqslant 0} \Omega_{\mathcal{C}}^{m+n} F\big(\Sigma^{m+n}(-)\big)$$

is an equivalence for all F. This follows from a formal manipulation of colimits.

To show the second condition, observe that if F is already reduced and excisive, then $F \Rightarrow \Omega_{\mathcal{C}} F(\Sigma(-))$ is an equivalence, and so $u_F \colon F \Rightarrow F^{\mathrm{sp}}$ must be an equivalence too. Thus, to show that $u(-)^{\mathrm{sp}} \colon (-)^{\mathrm{sp}} \Rightarrow ((-)^{\mathrm{sp}})^{\mathrm{sp}}$ is a pointwise equivalence, it's enough to check that $(-)^{\mathrm{sp}}$ takes values in reduced and excisive functors. This has to be done anyway, since we have to identify the image of $(-)^{\mathrm{sp}}$. Also, our observation that u_F is an equivalence whenever F is reduced and excisive already shows that the essential image of $(-)^{\mathrm{sp}}$ contains all reduced and excisive functors. Thus, once we show that the F^{sp} is reduced and excisive, we'll be done.

To show this, it's clear that F^{sp} is reduced again. For excisivity, observe $F^{\mathrm{sp}} \simeq \Omega_{\mathcal{C}} F^{\mathrm{sp}}(\Sigma(-))$. Indeed, precomposition with Σ commutes with all colimits, and postcomposition with $\Omega_{\mathcal{C}}$ commutes with sequential colimits by our assumption, so the colimit defining F^{sp} just gets transformed into itself. Now consider an arbitrary pushout diagram in $\mathrm{An}^{\mathrm{fin}}_{*/}$ and extend it as follows:

$$
\begin{array}{ccccc}
A & \longrightarrow & C & \longrightarrow & * \\
\downarrow & \ulcorner \, . & \downarrow & \ulcorner \, . & \downarrow \\
B & \longrightarrow & D & \longrightarrow & Q & \longrightarrow & * \\
\downarrow & \ulcorner \, . & \downarrow & \ulcorner \, . & \downarrow & \ulcorner \, . & \downarrow \\
* & \longrightarrow & P & \longrightarrow & \Sigma A & \longrightarrow & \Sigma B \\
& & \downarrow & \ulcorner \, . & \downarrow & \ulcorner \, . & \downarrow \\
& & * & \longrightarrow & \Sigma C & \longrightarrow & \Sigma D
\end{array}
$$

The top left 2×2-square induces a morphism $F^{\mathrm{sp}}(B) \times_{F^{\mathrm{sp}}(D)} F^{\mathrm{sp}}(C) \to \Omega_{\mathcal{C}} F^{\mathrm{sp}}(\Sigma A) \simeq F^{\mathrm{sp}}(A)$. It's straightforward to check that this morphism is an inverse to the canonical morphism in the other direction. This proves that F^{sp} turns pushouts into pullbacks, as required. $\square$

7.57. Corollary. — *The functor $\Omega^\infty\colon \mathrm{Sp} \to \mathrm{An}_{*/}$ admits a left adjoint $\Sigma^\infty\colon \mathrm{An}_{*/} \to \mathrm{Sp}$. If (X, x) is a pointed anima, then $\Omega^\infty\Sigma^\infty(X, x) \simeq \mathrm{colim}_{n \geqslant 0} \Omega^n\Sigma^n X$ together with its basepoint x. In particular,*

$$\pi_*\Sigma^\infty(X, x) \cong \operatorname*{colim}_{n \geqslant 0} \pi_*(\Omega^n\Sigma^n X) \cong \operatorname*{colim}_{n \geqslant 0} \pi_{*+n}(\Sigma^n X)$$

are the stable homotopy groups of X.

Proof. To prove that Σ^∞ exists and is given as above, let $\mathcal{I} := \{* \underset{\smile}{\overset{\frown}{\,}} S^0\}$ be the full sub-∞-category of $\mathrm{An}^{\mathrm{fin}}_{*/}$ spanned by $*$ and S^0 and recall the chain of equivalences and adjunctions

$$\mathrm{An}_{*/} \xleftarrow[\mathrm{ev}_{S_0}]{\simeq} \mathrm{Fun}_*\big(\mathcal{I}, \mathrm{An}_{*/}\big) \underset{i^*}{\overset{\mathrm{Lan}_i}{\rightleftarrows}} \mathrm{Fun}_*\big(\mathrm{An}^{\mathrm{fin}}_{*/}, \mathrm{An}_{*/}\big)$$

from the proof of Lemma 7.53. Thus, $\mathrm{ev}_{S_0}\colon \mathrm{Fun}_*(\mathrm{An}^{\mathrm{fin}}_{*/}, \mathrm{An}_{*/}) \to \mathrm{An}_{*/}$ has a left adjoint. Furthermore, according to Lemmas 7.53 and 7.55, $\mathrm{Sp} \simeq \mathrm{Fun}^{\mathrm{exc}}_*(\mathrm{An}^{\mathrm{fin}}_{*/}, \mathrm{An}_{*/}) \subseteq \mathrm{Fun}_*(\mathrm{An}^{\mathrm{fin}}_{*/}, \mathrm{An}_{*/})$ has a left adjoint too. This shows that Σ^∞ exists.

To show the desired formula for Σ^∞, fix a pointed anima (Y, y) and let $- \wedge Y\colon \mathrm{An}^{\mathrm{fin}}_{*/} \to \mathrm{An}_{*/}$ denote the associated functor. Let (X, x) be a finite pointed anima; we wish to compute the value $X \wedge Y$ of $- \wedge Y$ on (X, x). It will turn out that $X \wedge Y$ agrees with the smash product you know from topology, so the suggestive notation is justified. But for the moment, let's forget what we know about smash products and regard $X \wedge Y$ as the value of our functor. We use Lemma 7.54 to compute it. By definition, $S^0 \wedge Y \simeq Y$. The colimit of the constant functor $\mathrm{const}\, S^0 \wedge Y\colon X \to \mathrm{An}$ is therefore $X \times Y$ by Lemma 6.14. If we take the colimit of $\mathrm{const}\, S^0 \wedge Y\colon X \to \mathrm{An}_{*/}$ in pointed animae, we get $(X \times Y)/(X \times \{y\})$ instead, see Lemma 6.56(c). Plugging this into Lemma 7.54, we get a pushout diagram

$$
\begin{array}{ccc}
X \times \{y\} & \longrightarrow & (X \times Y)/\big(X \times \{y\}\big) \\
\downarrow & \ulcorner\,. & \downarrow \\
* & \longrightarrow & X \wedge Y
\end{array}
$$

(in An or $\mathrm{An}_{*/}$, this doesn't matter by Lemma 6.56(b)). So $X \wedge Y$ is indeed the usual smash product from topology.[(7.19)]

Now $\Omega^\infty\Sigma^\infty(Y, y)$ can be described as the value of $(- \wedge Y)^{\mathrm{sp}}$ on S^0. According to the formula from Lemma 7.55, this value is given by

$$\operatorname*{colim}_{n \geqslant 0} \Omega^n(\Sigma^n S^0 \wedge Y) \simeq \operatorname*{colim}_{n \geqslant 0} \Omega^n(S^n \wedge Y) \simeq \operatorname*{colim}_{n \geqslant 0} \Omega^n\Sigma^n Y\,.$$

It remains to show the "in particular" about the homotopy groups of the spectrum $\Sigma^\infty(Y, y)$. The same argument as above shows that $\Omega^{\infty - i}\Sigma^\infty(Y, y)$ is given by the value of $(- \wedge Y)^{\mathrm{sp}}$ on S^i, which is $\mathrm{colim}_{n \geqslant 0} \Omega^n(\Sigma^n S^i \wedge Y) \simeq \mathrm{colim}_{n \geqslant 0} \Omega^n\Sigma^{n+i}Y$. Hence Lemmas 6.58 and 7.3(a) show $\pi_*\Sigma^\infty(Y, y) \cong \pi_0(\Omega^{\infty - *}\Sigma^\infty(Y, y)) \cong \mathrm{colim}_{n \geqslant 0} \pi_{*+n}(\Sigma^n X)$, as desired. $\qquad\square$

As an immediate consequence, we get an analogue of Corollary 7.22 for $\mathbb{E}_\infty$-groups.

[(7.19)]It's easy to turn the usual definition from topology into a functor $- \wedge Y\colon \mathrm{An}_{*/} \to \mathrm{An}_{*/}$ (more on that in 8.23). This functor agrees with the functor we've constructed above. So far, we only know this on objects, but the equivalence as functors is not hard to check. Since both definitions of $S^0 \wedge X$ agree, both functors must agree in $\mathrm{Fun}_*(\mathcal{I}, \mathrm{An}_{*/})$. From the universal property of left Kan extension, we then get a natural transformation between them for free. So knowing that they agree object-wise is enough by Theorem 4.5.

7.58. Corollary ("$\Omega^\infty \Sigma^\infty X_+$ is the free $\mathbb{E}_\infty$-group on X"). — *The forgetful functor* $\mathrm{ev}_{\langle 1 \rangle} \colon \mathrm{CGrp}(\mathrm{An}) \to \mathrm{An}$ *sending an $\mathbb{E}_\infty$-group to its underlying anima has a left adjoint, sending an anima X to $\Omega^\infty \Sigma^\infty X_+$, where $X_+ := X \sqcup *$, regarded as a pointed anima.*

Proof. Since $\pi_* \Sigma^\infty (X, x)$ is given by the stable homotopy groups of X, Σ^∞ takes values in the full sub-∞-category $\mathrm{Sp}_{\geqslant 0}$ of connective spectra. Therefore, we get a diagram of adjunctions

$$
\begin{array}{ccccc}
\mathrm{An} & \underset{(-)_+}{\rightleftarrows} & \mathrm{An}_{*/} & \underset{\Omega^\infty}{\overset{\Sigma^\infty}{\rightleftarrows}} & \mathrm{Sp}_{\geqslant 0} \\
 & & & & \\
 & \Omega^\infty \Sigma^\infty (-)_+ \quad /\!/\!/ & & \mathrm{B}^\infty \;\big\uparrow\!\big\downarrow\; \Omega^\infty & \\
 & \mathrm{ev}_{[1]} & & \mathrm{CGrp}(\mathrm{An}) &
\end{array}
$$

which shows that $\Omega^\infty \Sigma^\infty (-)_+ \colon \mathrm{An} \rightleftarrows \mathrm{CGrp}(\mathrm{An}) \colon \mathrm{ev}_{\langle 1 \rangle}$ must be an adjunction too. $\qquad\square$

And finally, we can define the legendary *sphere spectrum*.

7.59. Definition. — The *reduced suspension spectrum functor* functor $\Sigma^\infty \colon \mathrm{An}_{*/} \to \mathrm{Sp}$. The (*unreduced*) *suspension spectrum functor*[(7.20)] is the composition

$$
\mathbb{S}[-] \colon \mathrm{An} \xrightarrow{(-)_+} \mathrm{An}_{*/} \xrightarrow{\Sigma^\infty} \mathrm{Sp} \, ;
$$

it is a left adjoint of $\Omega^\infty \colon \mathrm{Sp} \to \mathrm{An}$. The spectrum $\mathbb{S} := \mathbb{S}[*]$ is called the *sphere spectrum*.

[(7.20)] In the old literature, and still in much of the modern one, the (unreduced) suspension spectrum of X is denoted $\Sigma^\infty_+ X$ rather than $\mathbb{S}[X]$. However, in the modern mathematics, we think of spectra as "modules over the sphere spectrum" (a point of view that will be much elaborated on in §8), and so it seems only natural that the "free $\mathbb{S}$-module on X" should be denoted $\mathbb{S}[X]$, just as $\mathbb{Z}[S]$ usually denotes the free abelian group on a set S.

§8. The tensor product of spectra

The main theme of these notes is to do topology without doing topology. So far, we've seen that many classical results are formal consequences of abstract ∞-category theory. We'll now up our game and allow *algebra* into the mix. To set the stage, we'll explain in this section how spectra are the correct homotopical generalisation of abelian groups. We already know that the stable ∞-category Sp is additive (see the proof of Lemma 7.36). What we'll now explain is that Sp can be equipped with a *tensor product*, which will allow us to talk about *algebras* and *modules* in Sp.

In §8.1, we'll study symmetric monoidal structures on arbitrary ∞-categories. In §8.2, we'll construct many interesting examples, including the tensor product of spectra. In §8.3, we'll take the theory of algebras and modules in Sp for granted and use it to give the "correct" construction of homology and cohomology. Finally, there'll be a lengthy appendix. In §8.4, we'll sketch the missing theory of algebras and modules. In §8.5, we'll introduce the notion of $\mathbb{E}_n$-*algebras* for all $0 \leqslant n \leqslant \infty$, which generalises the notions of $\mathbb{E}_1$- and $\mathbb{E}_\infty$-monoids that we already know. Finally, in §8.6, we'll prove more cool stuff about Lurie's magical ∞-category Pr^{L} and sketch another construction of the tensor product on Sp.

§8.1. Symmetric monoidal ∞-categories

8.1. Definition. — A *symmetric monoidal ∞-category* is an object of $\mathrm{CMon}(\mathrm{Cat}_\infty)$, that is, a functor $\mathcal{C}\colon \mathrm{Fin} \to \mathrm{Cat}_\infty$ satisfying $\mathcal{C}_0 \simeq *$ as well as the Segal condition from Definition 7.24(a). We often abusively identify $\mathcal{C}$ with its *underlying ∞-category $\mathcal{C}_1$*.

Unfortunately, the ∞-category $\mathrm{CMon}(\mathrm{Cat}_\infty)$ is *not* a good framework to study symmetric monoidal ∞-categories! This is because $\mathrm{CMon}(\mathrm{Cat}_\infty)$ only keeps track of functors $F\colon \mathcal{C} \to \mathcal{D}$ that satisfy $F(x \otimes_\mathcal{C} y) \simeq F(x) \otimes_\mathcal{D} F(y)$. In the classical theory, these are called *strictly symmetric monoidal*. However, many functors occuring in nature are only *lax symmetric monoidal*: They only admit functorial morphisms $F(x) \otimes_\mathcal{D} F(y) \to F(x \otimes_\mathcal{C} y)$, which need not be equivalences. For example, if $R \to S$ is a morphism of ordinary rings, then the base change $- \otimes_R S\colon \mathrm{Mod}_R \to \mathrm{Mod}_S$ is strictly symmetric monoidal, but its right adjoint, the forgetful functor $\mathrm{Mod}_S \to \mathrm{Mod}_R$ is usually only lax monoidal.

The solution to this problem is to think about symmetric monoidal ∞-categories not as functors[8.1] $\mathcal{C}\colon \mathrm{Fin} \to \mathrm{Cat}_\infty$, but instead as cocartesian fibrations $\mathcal{C}^\otimes \to \mathrm{Fin}$, which is made possible via Lurie's straightening/unstraightening equivalence (Theorem 5.4).

8.2. Definition. — Let $\mathcal{C}$ and $\mathcal{D}$ be symmetric monoidal ∞-categories, with associated cocartesian fibrations $\mathcal{C}^\otimes \to \mathrm{Fin}$ and $\mathcal{D}^\otimes \to \mathrm{Fin}$.

(a) A morphism $\alpha\colon \langle n \rangle \to \langle m \rangle$ in Fin is called *active* if α is everywhere defined, and *inert* if α is bijective when restricted to the subset of $\langle m \rangle$ where it is defined. We denote by $\mathrm{Fin}^{\mathrm{act}}, \mathrm{Fin}^{\mathrm{int}} \subseteq \mathrm{Fin}$ the non-full subcategories spanned by the active or inert morphisms, respectively. We also call a morphism $\varphi\colon x \to y$ in $\mathcal{C}^\otimes$ an *active* or *inert lift* if φ is a cocartesian lift of an active or inert morphism, respectively.

In particular, the Segal maps $e_i\colon \langle n \rangle \to \langle 1 \rangle$ are inert and for every n there is precisely one active morphism $f_n\colon \langle n \rangle \to \langle 1 \rangle$.

[8.1]It's possible to make the functor picture work, but this would need $(\infty, 2)$-categories.

(b) A *lax symmetric monoidal functor* $F^\otimes \colon \mathcal{C}^\otimes \to \mathcal{D}^\otimes$ is a functor in $\mathrm{Cat}_{\infty/\mathrm{Fin}}$ that preserves inert lifts. We call $F^\otimes$ *symmetric monoidal* (or sometimes *strictly symmetric monoidal* for emphasis) if $F^\otimes$ also preserves active lifts. We'll denote by $\mathrm{Fun}^{\mathrm{lax}\text{-}\otimes}(\mathcal{C}, \mathcal{D})$ and $\mathrm{Fun}^\otimes(\mathcal{C}, \mathcal{D})$ the full sub-∞-categories of $\mathrm{Fun}_{\mathrm{Fin}}(\mathcal{C}^\otimes, \mathcal{D}^\otimes) \coloneqq \mathrm{Fun}(\mathcal{C}^\otimes, \mathcal{D}^\otimes) \times_{\mathrm{Fun}(\mathcal{C}^\otimes, \mathrm{Fin})} \{\mathcal{C}^\otimes \to \mathrm{Fin}\}$ spanned by the (lax) symmetric monoidal functors.

We'll often abusively identify a (lax) symmetric monoidal functor $F^\otimes$ with its *underlying functor* $F \colon \mathcal{C} \to \mathcal{D}$ (which formally would be $F_1^\otimes \colon \mathcal{C}_1^\otimes \to \mathcal{D}_1^\otimes$).

8.3. Active and inert morphisms. — Let $\mathcal{C}$ be a symmetric monoidal ∞-category with associated cocartesian fibration $\mathcal{C}^\otimes \to \mathrm{Fin}$. Here and in the following, we let $\mathcal{C}_n^\otimes \coloneqq \{\langle n \rangle\} \times_{\mathrm{Fin}} \mathcal{C}^\otimes$ denote the fibre over some $\langle n \rangle \in \mathrm{Fin}$, and for a morphism $\alpha \colon \langle n \rangle \to \langle m \rangle$ in Fin, we let $\mathcal{C}_\alpha^\otimes$ denote the base change of $\mathcal{C}^\otimes$ along $\alpha \colon \Delta^1 \to \mathrm{Fin}$. The straightening of $\mathcal{C}_\alpha^\otimes \to \Delta^1$ defines a functor $\Delta^1 \to \mathrm{Cat}_\infty$, that is, a morphism in Cat_∞, which we'll denote by $\alpha_{\mathcal{C}\otimes} \colon \mathcal{C}_n^\otimes \to \mathcal{C}_m^\otimes$. According to the Segal condition, we must have $\mathcal{C}_n^\otimes \simeq \mathcal{C}^n$ for all n, hence $\alpha_{\mathcal{C}\otimes}$ can also be viewed as a functor $\alpha_{\mathcal{C}\otimes} \colon \mathcal{C}^n \to \mathcal{C}^m$.

Now let's see how active and inert morphisms encode the symmetric monoidal structure. Under the identifications above, the unique active morphism $f_2 \colon \langle 2 \rangle \to \langle 1 \rangle$ induces a functor $f_{2,\mathcal{C}\otimes} \colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$. We usually denote $f_{2,\mathcal{C}\otimes}$ by $- \otimes_{\mathcal{C}} -$ and call it the *tensor product on $\mathcal{C}$*. Similarly, $\langle 0 \rangle \to \langle 1 \rangle$ defines a morphism $* \to \mathcal{C}$, which classifies the *tensor unit* $\mathbb{1}_{\mathcal{C}}$. If no confusion can occur, we'll often drop the indices and just write $- \otimes -$ and $\mathbb{1}$. In general, $\alpha_{\mathcal{C}\otimes} \colon \mathcal{C}^n \to \mathcal{C}^m$ is given by

$$\alpha_{\mathcal{C}\otimes}(x_1, \ldots, x_n) \simeq \left(\bigotimes_{j \in \alpha^{-1}(i)} x_j \right)_{i=1,\ldots,m}$$

(where empty tensor products are defined to be $\mathbb{1}$). Thus, if α is active, then the tuple $(x_1, \ldots, x_n)$ gets partitioned into m collections, according to $\langle n \rangle = \coprod_{i \in \langle m \rangle} \alpha^{-1}(i)$, and then the entries in each collection get tensored together. The other extreme is when α is inert: In this case we simply forget those entries x_j where $\alpha(j)$ is undefined. In particular, if $e_i \colon \langle n \rangle \to \langle 1 \rangle$ is the ith Segal map, then $e_{i,\mathcal{C}\otimes} \colon \mathcal{C}^n \to \mathcal{C}$ is the projection to the ith factor, as it should be. In total, the situation can be summarised in the following picture:
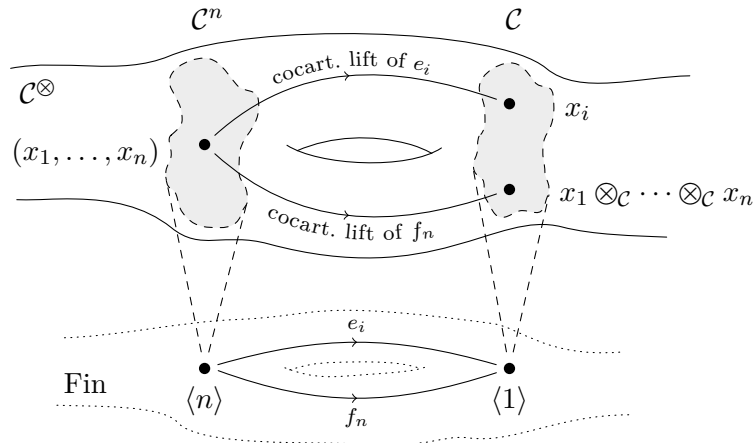

8.4. Lax symmetric monoidal functors. — First, observe that any morphism in Fin can be factored into an inert morphism followed by an active one. Using that cocartesian lifts

compose (Corollary 5.18), we see that if a functor $F^\otimes\colon \mathcal{C}^\otimes \to \mathcal{D}^\otimes$ preserves both inert and active lifts, then $F^\otimes$ must preserve arbitrary cocartesian lifts. Thus $F^\otimes$ is a functor in $\mathrm{Cocart}(\mathrm{Fin})$ and so strictly symmetric monoidal functors correspond to morphisms in $\mathrm{CAlg}(\mathrm{Cat}_\infty)$, as we would expect.

Now assume $F^\otimes\colon \mathcal{C}^\otimes \to \mathcal{D}^\otimes$ is only a lax symmetric monoidal functor and let $F\colon \mathcal{C} \to \mathcal{D}$ be the underlying functor of ∞-categories. By our considerations in 8.3, if φ is a cocartesian lift in $\mathcal{C}^\otimes$ of an inert morphism $\alpha\colon \langle n\rangle \to \langle m\rangle$, then φ connects a tuple $(x_1, \dots, x_n)$ to $(x_{\alpha(j)})_{\alpha(j)\ \mathrm{defined}}$. Since this is just forgetting some entries, $F^\otimes(\varphi)$ should still connect $(F(x_1), \dots, F(x_n))$ to $(F(x_{\alpha(j)}))_{\alpha(j)\ \mathrm{defined}}$. This is precisely the condition that $F^\otimes$ preserves inert lifts.

More interesting is what happens for active morphisms. For simplicity, let's only consider the unique active morphims $f_n\colon \langle n\rangle \to \langle 1\rangle$. If φ is a cocartesian lift of f_n in $\mathcal{C}^\otimes$, then φ connects a tuple $(x_1, \dots, x_n) \in \mathcal{C}^n$ to $x_1 \otimes_{\mathcal{C}} \cdots \otimes_{\mathcal{C}} x_n \in \mathcal{C}$. The image $F^\otimes(\varphi)$ is still a lift of f_n, but it doesn't need to be cocartesian in $\mathcal{D}^\otimes$. Instead, we get the following picture:
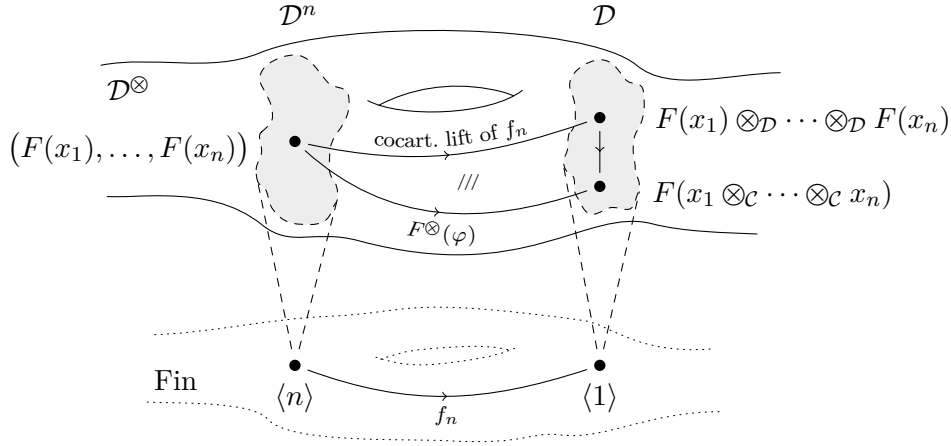
Together, $F^\otimes(\varphi)$ and the cocartesian lift of f_n define a horn $\sigma\colon \Lambda_0^2 \to \mathcal{D}^\otimes$, which admits a filler $\overline{\sigma}\colon \Delta^2 \to \mathcal{D}^\otimes$ by definition of being a cocartesian lift (as usual, we've depicted the filler as "///"). Restricting to $\overline{\sigma}|_{\Delta^{\{1,2\}}}$ yields a morphism

$$F(x_1) \otimes_{\mathcal{D}} \cdots \otimes_{\mathcal{D}} F(x_n) \longrightarrow F(x_1 \otimes_{\mathcal{C}} \cdots \otimes_{\mathcal{C}} x_n),$$

as we would expect from a lax symmetric monoidal functor.

With a little more effort, we can even make this morphism functorial.

8.5. Lemma. — *Let $p\colon \mathcal{U} \to \Delta^1$ and $q\colon \mathcal{V} \to \Delta^1$ be cocartesian fibrations, whose straightenings are functors $\alpha_\mathcal{U}\colon \mathcal{U}_0 \to \mathcal{U}_1$ and $\alpha_\mathcal{V}\colon \mathcal{V}_0 \to \mathcal{V}_1$ (regarded as objects in $\mathrm{Fun}(\Delta^1, \mathrm{Cat}_\infty)$). Suppose we're given functors $F_0\colon \mathcal{U}_0 \to \mathcal{V}_0$ and $F_1\colon \mathcal{U}_1 \to \mathcal{V}_1$. Then there exists a pullback diagram*

$$
\begin{array}{ccc}
\mathrm{Hom}_{\mathrm{Fun}(\mathcal{U}_0, \mathcal{V}_1)}(\alpha_\mathcal{V} \circ F_0, F_1 \circ \alpha_\mathcal{U}) & \longrightarrow & \mathrm{Hom}_{\mathrm{Cat}_{\infty/\Delta^1}}\big((p\colon \mathcal{U} \to \Delta^1), (q\colon \mathcal{V} \to \Delta^1)\big) \\
\downarrow & & \downarrow \\
\{F_0\} \times \{F_1\} & \longrightarrow & \mathrm{Hom}_{\mathrm{Cat}_\infty}(\mathcal{U}_0, \mathcal{V}_0) \times \mathrm{Hom}_{\mathrm{Cat}_\infty}(\mathcal{U}_1, \mathcal{V}_1)
\end{array}
$$

In particular, every lax symmetric monoidal functor $F^\otimes\colon \mathcal{C}^\otimes \to \mathcal{D}^\otimes$ defines a natural transformation $F(-) \otimes_{\mathcal{D}} F(-) \Rightarrow F(- \otimes_{\mathcal{C}} -)$ and a morphism $\mathbb{1}_\mathcal{D} \to F(\mathbb{1}_\mathcal{C})$.

8.6. Remark. — Roughly, Lemma 8.5 is saying that a functor $\mathcal{U} \to \mathcal{V}$ in $\mathrm{Cat}_{\infty/\Delta^1}$, which not necessarily preserves cocartesian lifts, is given by the data of $(\alpha_0, \alpha_1, \eta)$ fitting into a square

$$
\begin{array}{ccc}
\mathcal{U}_0 & \xrightarrow{F_0} & \mathcal{V}_0 \\
\alpha_{\mathcal{U}} \downarrow & \overset{\eta}{\mathbin{\not\Rightarrow}} & \downarrow \alpha_{\mathcal{V}} \\
\mathcal{U}_1 & \xrightarrow{F_1} & \mathcal{V}_1
\end{array}
$$

The functor $\mathcal{U} \to \mathcal{V}$ preserves cocartesian lifts (in other words, $\mathcal{U} \to \mathcal{V}$ is a morphism in $\mathrm{Cocart}(\Delta^1) \simeq \mathrm{Fun}(\Delta^1, \mathrm{Cat}_\infty)$) if and only if η is an equivalence, so that the square commutes on the nose in Cat_∞.

The proof of Lemma 8.5 is a little lengthy (and the reader may skip ahead to Lemma 8.9)—after all, we're trying to improve upon straightening/unstraightening, by considering functors $\mathcal{U} \to \mathcal{V}$ in $\mathrm{Cat}_{\infty/\Delta^1}$ that don't necessarily preserve cocartesian lifts. Still, we can get away with just the usual straightening/unstraightening equivalence. The first step is a general lemma, which will be useful later on as well. This is originally due to Ayala–Francis [AF20, Theorem 2.30].

8.7. Lemma. — *For any ∞-category $\mathcal{D}$, the (non-fully faithful) functor $\mathrm{Cocart}(\mathcal{D}) \to \mathrm{Cat}_{\infty/\mathcal{D}}$ admits a left adjoint, which sends an object $(\mathcal{C} \to \mathcal{D})$ to the cocartesian fibration*

$$
\mathcal{C} \times_{\mathcal{D},s} \mathrm{Ar}(\mathcal{D}) \xrightarrow{t} \mathcal{D}.
$$

An analogous assertion holds for $\mathrm{Cart}(\mathcal{D}) \to \mathrm{Cat}_{\infty/\mathcal{D}}$.

Proof sketch. By Lemma 6.5, it will be enough to construct a unit transformation, pointwise counits, and to verify the pointwise triangle identities. For the unit u, we use $\mathrm{const} \colon \mathcal{D} \to \mathrm{Ar}(\mathcal{D})$ to obtain a functor

$$
u_{(\mathcal{C} \to \mathcal{D})} \colon \mathcal{C} \simeq \mathcal{C} \times_{\mathcal{D}} \mathcal{D} \longrightarrow \mathcal{C} \times_{\mathcal{D},s} \mathrm{Ar}(\mathcal{D})
$$

in $\mathrm{Cat}_{\infty/\mathcal{D}}$. It's clear that this can be made functorial in $(\mathcal{C} \to \mathcal{D}) \in \mathrm{Cat}_{\infty/\mathcal{D}}$, so we've constructed the unit transformation u.

To construct pointwise counits, let $\mathcal{U} \to \mathcal{D}$ be a cocartesian fibration. Observe that $u_{(\mathcal{U} \to \mathcal{D})} \colon \mathcal{U} \to \mathcal{U} \times_{\mathcal{D},s} \mathrm{Ar}(\mathcal{D})$ admits a left adjoint. Indeed, by Lemma 6.2 this can be checked pointwise. Now given any object $(u, (\alpha \colon y \to y')) \in \mathcal{U} \times_{\mathcal{D},s} \mathrm{Ar}(\mathcal{D})$, choose a cocartesian lift $\varphi \colon u \to u'$ of α. Using Lemmas 5.13 and 5.16, we see that $u' \in \mathcal{U}$ is a left adjoint object of $(u, (\alpha \colon y \to y'))$, as desired.

Let us denote the left adjoint of $u_{(\mathcal{U} \to \mathcal{D})}$ by $c_{(\mathcal{U} \to \mathcal{D})} \colon \mathcal{U} \times_{\mathcal{D},s} \mathrm{Ar}(\mathcal{D}) \to \mathcal{U}$. It's straightforward to check that $c_{(\mathcal{U} \to \mathcal{D})}$ preserves cocartesian lifts, hence it is a functor in $\mathrm{Cocart}(\mathcal{D})$. We can thus use $c_{(\mathcal{U} \to \mathcal{D})}$ as our pointwise counit. To check the pointwise triangle identities, we must verify that the compositions

$$
\mathcal{C} \times_{\mathcal{D},s} \mathrm{Ar}(\mathcal{D}) \xrightarrow{u_{(\mathcal{C} \to \mathcal{D})} \times_{\mathcal{D},s} \mathrm{Ar}(\mathcal{D})} \mathcal{C} \times_{\mathcal{D},s} \mathrm{Ar}(\mathcal{D}) \times_{t,\mathcal{D},s} \mathrm{Ar}(\mathcal{D})
$$

with the diagonal maps $\simeq$ down to $\mathcal{C} \times_{\mathcal{D},s} \mathrm{Ar}(\mathcal{D})$ and the vertical $c_{(\mathcal{C} \times_{\mathcal{D},s} \mathrm{Ar}(\mathcal{D}) \to \mathcal{D})}$

and

$$
\mathcal{U} \xrightarrow{u_{(\mathcal{U} \to \mathcal{D})}} \mathcal{U} \times_{\mathcal{D},s} \mathrm{Ar}(\mathcal{D})
$$

with the diagonal $\simeq$ down to $\mathcal{U}$ and the vertical $c_{(\mathcal{U} \to \mathcal{D})}$

are equivalences. In both cases, we can directly verify that the composition is object-wise the identity on $\mathcal{C} \times_{\mathcal{D},s} \mathrm{Ar}(\mathcal{D})$ and $\mathcal{U}$, respectively, which is enough by Theorem 4.5. $\square$

Proof sketch of Lemma 8.5. To obtain a natural transformation $F(-) \otimes_\mathcal{D} F(-) \Rightarrow F(- \otimes_\mathcal{C} -)$, apply the general assertion to the restrictions $\mathcal{C}^\otimes_{f_2}$, $\mathcal{D}^\otimes_{f_2}$, which are cocartesian fibrations over Δ^1. Similarly, the restrictions $\mathcal{C}^\otimes_{\{\langle 0 \rangle \to \langle 1 \rangle\}}$, $\mathcal{D}^\otimes_{\{\langle 0 \rangle \to \langle 1 \rangle\}}$ yield the desired morphism $\mathbb{1}_\mathcal{D} \to F(\mathbb{1}_\mathcal{C})$.

To prove the general statement, we first claim that unstraightening over Δ^1 can be described as a "mapping cylinder":

($\boxtimes_1$) *We have $\mathcal{U} \simeq \mathcal{U}_0 \times \Delta^1 \sqcup_{\mathcal{U}_0 \times \{1\}, \alpha_\mathcal{U}} \mathcal{U}_1 \times \{1\}$, where the pushout is taken in Cat_∞ (or in $\mathrm{Cat}_{\infty/\Delta^1}$, this doesn't matter by the dual of Lemma 6.56(a)).*

To prove ($\boxtimes_1$), first observe that $F \colon \mathcal{U}_0 \to \mathcal{U}_1$ can be written as the pushout of the span $(\mathrm{id}_{\mathcal{U}_0} \colon \mathcal{U}_0 \to \mathcal{U}_0) \Leftarrow (\emptyset \to \mathcal{U}_0) \Rightarrow (\emptyset \to \mathcal{U}_1)$ in $\mathrm{Fun}(\Delta^1, \mathrm{Cat}_\infty)$. Since $\mathrm{Un}^{(\mathrm{cocart})}$ is an equivalence of ∞-categories, it must preserve pushouts. The unstraightening of $\mathrm{id}_{\mathcal{U}_0} \colon \mathcal{U}_0 \to \mathcal{U}_0$ is the projection $\mathrm{pr}_2 \colon \mathcal{U}_0 \times \Delta^1 \to \Delta^1$ by Example 5.5(a). The unstraightening of an arrow of the form $\emptyset \to \mathcal{X}$ is $\mathcal{X} \times \{1\} \to \Delta^1$. To see this, observe that $\emptyset \to \mathcal{X}$ is the left Kan extension along $\{1\} \to \Delta^1$ of the functor $\{1\} \to \mathrm{Cat}_\infty$ sending $1 \mapsto \mathcal{X}$. By definition, the left Kan extension functor $\mathrm{Lan}_{\{1\} \to \Delta^1} \colon \mathrm{Fun}(\{1\}, \mathrm{Cat}_\infty) \to \mathrm{Fun}(\Delta^1, \mathrm{Cat}_\infty)$ is left adjoint to restriction along $\{1\} \to \mathrm{Cat}_\infty$. Since restrictions correspond to fibre products under straightening/unstraightening, we must check that the left adjoint of the fibre functor $- \times_{\Delta^1} \{1\} \colon \mathrm{Cocart}(\Delta^1) \to \mathrm{Cocart}(\{1\})$ sends $\mathcal{X} \to \{1\}$ to $\mathcal{X} \times \{1\} \to \Delta^1$. This is straightforward (and we've seen a similar assertion in Lemma 6.23(a)).

To finish the proof of ($\boxtimes_1$), it remains to check that the pushout $\mathcal{U}_0 \times \Delta^1 \sqcup_{\mathcal{U}_0 \times \{1\}} \mathcal{U}_1 \times \{1\} \to \Delta^1$ is a cocartesian fibration, because then it's straightforward to verify that it must also be the pushout in $\mathrm{Cocart}(\Delta^1)$.[8.2] To this end, we can extend the adjunction $\Delta^1 \rightleftarrows \{1\}$ first to $\mathcal{U}_0 \times \Delta^1 \rightleftarrows \mathcal{U}_0 \times \{1\}$ and then to an adjunction $\mathcal{U}_0 \times \Delta^1 \sqcup_{\mathcal{U}_0 \times \{1\}} \mathcal{U}_1 \times \{1\} \rightleftarrows \mathcal{U}_1 \times \{1\}$ by the same pushout argument as in the proof of Lemma 7.54. Using this adjunction and the criterion from Lemma 5.16, we see that for all $u_0 \in \mathcal{U}_0$, the morphism represented by $\{u_0\} \times \Delta^1 \to \mathcal{U}_0 \times \Delta^1 \sqcup_{\mathcal{U}_0 \times \{1\}} \mathcal{U}_1 \times \{1\}$ is a cocartesian lift of $0 \to 1$ in Δ^1. Since this is the only non-identity morphism in Δ^1, we've verified that $\mathcal{U}_0 \times \Delta^1 \sqcup_{\mathcal{U}_0 \times \{1\}} \mathcal{U}_1 \times \{1\} \to \Delta^1$ is indeed a cocartesian fibration, and ($\boxtimes_1$) follows.

($\boxtimes_2$) *$\mathcal{U} \times_{\Delta^1, s} \mathrm{Ar}(\Delta^1)$ is the unstraightening of the functor $\Delta^1 \to \mathrm{Cat}_\infty$ given by $\mathcal{U}_0 \to \mathcal{U}$.*

If we believe ($\boxtimes_2$) for the moment, we can finish the proof as follows: First, by Lemma 8.7 and the straightening equivalence $\mathrm{Cocart}(\Delta^1) \simeq \mathrm{Ar}(\mathrm{Cat}_\infty)$, we can rewrite

$$\mathrm{Hom}_{\mathrm{Cat}_{\infty/\Delta^1}}(\mathcal{U}, \mathcal{V}) \simeq \mathrm{Hom}_{\mathrm{Ar}(\mathrm{Cat}_\infty)}\big((\mathcal{U}_0 \to \mathcal{U}), (\alpha_\mathcal{V} \colon \mathcal{V}_0 \to \mathcal{V})\big)$$

Now one can plug in the pushout formula for $\mathcal{U}$ from ($\boxtimes_1$) and use Lemma 5.13 to write $\mathrm{Hom}_{\mathrm{Ar}(\mathrm{Cat}_\infty)}((\mathcal{U}_0 \to \mathcal{U}), (\alpha_\mathcal{V} \colon \mathcal{V}_0 \to \mathcal{V}))$ as an iterated pullback. Via elementary pullback manipulations, we can then deduce the desired pullback diagram from the pullback

$$
\begin{array}{ccc}
\mathrm{Hom}_{\mathrm{Fun}(\mathcal{U}_0, \mathcal{V}_1)}(\alpha_\mathcal{V} \circ F_0, F_1 \circ \alpha_\mathcal{U}) & \longrightarrow & \mathrm{Hom}_{\mathrm{Cat}_\infty}(\mathcal{U}_0 \times \Delta^1, \mathcal{U}_1) \\
\downarrow & \lrcorner & \downarrow \\
\{(\alpha_\mathcal{V} \circ F_0, F_1 \circ \alpha_\mathcal{U})\} & \longrightarrow & \mathrm{Hom}_{\mathrm{Cat}_\infty}(\mathcal{U}_0 \times \{0, 1\}, \mathcal{V}_1)
\end{array}
$$

[8.2] By Lemma 4.9, $\mathrm{Hom}_{\mathrm{Cocart}(\Delta^1)}(-, -)$ is always a collection of path components of $\mathrm{Hom}_{\mathrm{Cat}_{\infty/\Delta^1}}(-, -)$. Thus, once the universal property holds in $\mathrm{Cat}_{\infty/\Delta^1}$, verifying the universal property in $\mathrm{Cocart}(\Delta^1)$ boils down to a matching of path components, which is straightforward.

To show this pullback (which also seems intuitively obvious), use $\mathrm{Hom}_{\mathrm{Cat}_\infty} \simeq \mathrm{core}\,\mathrm{Fun}$, write out the pullback defining $\mathrm{Hom}_{\mathrm{Fun}(\mathcal{U}_0, \mathcal{V}_1)}$ from 2.11, and use $\mathrm{Ar}(\mathrm{Fun}(\mathcal{U}_0, \mathcal{V}_1)) \simeq \mathrm{Fun}(\mathcal{U}_0 \times \Delta^1, \mathcal{V}_1)$ as well as $\mathrm{Fun}(\mathcal{U}_0, \mathcal{V}_1) \times \mathrm{Fun}(\mathcal{U}_0, \mathcal{V}_1) \simeq \mathrm{Fun}(\mathcal{U}_0 \times \{0, 1\}, \mathcal{V}_1)$.

It remains to show claim ($\boxtimes_2$). To this end write $\mathcal{U} \simeq \mathcal{U}_0 \times \Delta^1 \sqcup_{\mathcal{U}_0 \times \{1\}, \alpha_\mathcal{U}} \mathcal{U}_1 \times \{1\}$ by claim ($\boxtimes_1$) and observe that $- \times_{\Delta^1, s} \mathrm{Ar}(\Delta^1)$ preserves this pushout. Indeed, this follows from the dual of Lemma 8.8, since $s \colon \mathrm{Ar}(\Delta^1) \to \Delta^1$ is a cartesian fibration. Consequently,

$$\mathcal{U} \times_{\Delta^1, s} \mathrm{Ar}(\Delta^1) \simeq \mathcal{U}_0 \times \mathrm{Ar}(\Delta^1) \sqcup_{\mathcal{U}_0 \times \{\mathrm{id}_1\}} \mathcal{U}_1 \times \{\mathrm{id}_1\},$$

where we use $\{1\} \times_{\Delta^1, s} \mathrm{Ar}(\Delta^1) \simeq (\Delta^1)_{1/} \simeq \{\mathrm{id}_1\}$. Using $\mathrm{Ar}(\Delta^1) \simeq \Delta^2 \simeq \Delta^{\{0,1\}} \sqcup_{\{1\}} \Delta^{\{1,2\}}$, we can finally rewrite the pushout above as

$$\mathcal{U}_0 \times \Delta^{\{0,1\}} \sqcup_{\mathcal{U}_0 \times \{1\}} \left(\mathcal{U}_0 \times \Delta^{\{1,2\}} \sqcup_{\mathcal{U}_0 \times \{2\}} \mathcal{U}_1 \times \{2\} \right) \simeq \mathcal{U}_0 \times \Delta^1 \sqcup_{\mathcal{U}_0 \times \{1\}} \mathcal{U} \times \{1\},$$

which by ($\boxtimes_1$) is indeed the unstraightening of the functor $\Delta^1 \to \mathrm{Cat}_\infty$ given by $\mathcal{U}_0 \to \mathcal{U}$. $\square$

In the proof above, we've used a general lemma about commutation of pushouts and pullbacks, which is pretty useful on its own. Functors $p \colon \mathcal{U} \to \mathcal{C}$ with the property that pullback along them preserves colimits are called *exponentiable fibrations* (by Ayala–Francis [AF20]) or *flat fibrations* (by Lurie).

8.8. Lemma. — *For a functor of ∞-categories $p \colon \mathcal{U} \to \mathcal{C}$, the pullback functor*

$$- \times_\mathcal{C} \mathcal{U} \colon \mathrm{Cat}_{\infty/\mathcal{C}} \longrightarrow \mathrm{Cat}_{\infty/\mathcal{U}}$$

preserves arbitrary colimits if and only if it preserves pushouts of the form $\Delta^{\{0,1\}} \sqcup_{\{1\}} \Delta^{\{1,2\}} \simeq \Delta^2$ for all compatible pairs of functors $\Delta^{\{0,1\}}, \Delta^{\{1,2\}} \to \mathcal{C}$. This condition is satisfied, for example, if p is a cocartesian fibration.

Proof of Lemma 8.8. Unfortunately, our proof needs simplicial methods. The idea is that pullbacks in simplicial sets already commute with colimits, and the only difference between colimits in sSet and Cat_∞ is that "compositions need to be added". This addition of compositions is completely captured in the fact that the pushout $\Delta^{\{0,1\}} \sqcup_{\{1\}} \Delta^{\{1,2\}}$ is Λ_1^2 in sSet, but Δ^2 in Cat_∞, which leads to the criterion above.

To make this idea into a precise argument, observe that $- \times_\mathcal{C} \mathcal{U} \colon \mathrm{sSet}_{/\mathcal{C}} \to \mathrm{sSet}$ preserves colimits. Hence we can use the 1-categorical version of the adjoint functor theorem (Theorem 6.66(a)) to construct a right adjoint $R \colon \mathrm{sSet} \to \mathrm{sSet}_{/\mathcal{C}}$.[8.3] For all $(X \to \mathcal{C}) \in \mathrm{sSet}_{/\mathcal{C}}$ and all $Y \in \mathrm{sSet}_{/\mathcal{U}}$, the adjunction bijection $\mathrm{Hom}_{\mathrm{sSet}}(X \times_\mathcal{C} \mathcal{U}, Y) \cong \mathrm{Hom}_{\mathrm{sSet}_{/\mathcal{C}}}(X, R(Y))$ upgrades to an isomorphism of simplicial sets

$$\mathrm{F}(X \times_\mathcal{C} \mathcal{U}, Y) \cong \mathrm{F}_\mathcal{C}\big(X, R(Y)\big),$$

where $\mathrm{F}_\mathcal{C}(X, -) := \mathrm{F}(X, -) \times_{\mathrm{F}(X, \mathcal{C})} \{X \to \mathcal{C}\}$. Indeed, this follows by plugging $\Delta^n \times X$ for all $n \geqslant 0$ into the original adjunction bijection. Thus, if we can show that R sends quasi-categories to isofibrations over $\mathcal{C}$, we'll be done. Indeed, for every quasi-category $\mathcal{D}$, the counit $c_\mathcal{D} \colon R(\mathcal{D}) \times_\mathcal{C} \mathcal{U} \to \mathcal{D}$ induces a natural transformation

$$\mathrm{Fun}_\mathcal{C}\big(-, R(\mathcal{D})\big) \xRightarrow{-\times_\mathcal{C} \mathcal{U}} \mathrm{Fun}\big(- \times_\mathcal{C} \mathcal{U}, R(\mathcal{D}) \times_\mathcal{C} \mathcal{U}\big) \xRightarrow{(c_\mathcal{D})_*} \mathrm{Fun}(- \times_\mathcal{C} \mathcal{U}, \mathcal{D}).$$

[8.3] In fact, one can show $\mathrm{sSet}_{/\mathcal{C}} \simeq \mathrm{PSh}(\Delta_{/\mathcal{C}})$, so Theorem 1.17 is already enough and we don't need the full power of the 1-categorical adjoint functor theorem.

If $R(\mathcal{D}) \to \mathcal{C}$ are isofibrations, then the pullback defining $\mathrm{F}_{\mathcal{C}}(-, R(\mathcal{D}))$ is also a homotopy pullback by "Definition" 5.11(b) and we can conclude that the composition above is a pointwise equivalence, hence an equivalence by Theorem 4.5. Using Lemma 6.2, it follows that $- \times_{\mathcal{C}} \mathcal{U} \colon \mathrm{Cat}_{\infty/\mathcal{C}} \to \mathrm{Cat}$ admits a right adjoint, hence it must preserve all colimits. By the dual of Lemma 6.56(a), the same must then be true for $- \times_{\mathcal{C}} \mathcal{U} \to \mathrm{Cat}_{\infty/\mathcal{C}} \to \mathrm{Cat}_{\infty/\mathcal{U}}$ and so we would be done.

If $- \times_{\mathcal{C}} \mathcal{U}$ preserves pushouts of the form $\Delta^{\{0,1\}} \sqcup_{\{1\}} \Delta^{\{1,2\}} \simeq \Delta^2$, then it also preserves pushouts of the form $\Delta^{\{0,1\}} \sqcup_{\{1\}} \cdots \sqcup_{\{n-1\}} \Delta^{\{n-1,n\}} \simeq \Delta^n$ for all $n \geqslant 2$. Indeed, this can be shown via induction on n. The case $n = 2$ is clear. For the inductive step, let $n \geqslant 3$; we need to show that $- \times_{\mathcal{C}} \mathcal{U}$ preserves the pushout $\Delta^{n-1} \sqcup_{\{n-1\}} \Delta^{\{n-1,n\}} \simeq \Delta^n$. We can write Δ^{n-1} as a retract of the "cylinder" $\Delta^{n-2} \times \Delta^{\{0,n-1\}}$. Since $\Delta^{n-2} \times -$ commutes with all colimits, as it admits a left adjoint given by $\mathrm{Fun}(\Delta^{n-2}, -)$, it will be enough to check that $- \times_{\mathcal{C}} \mathcal{U}$ preserves the pushout $\Delta^{\{0,n-1\}} \sqcup_{\{n-1\}} \Delta^{\{n-1,n\}} \simeq \Delta^{\{0,n-1,n\}}$, which is precisely our assumption. This finishes the induction.

It follows that for all $\Delta^n \to \mathcal{C}$ the map $I^n \times_{\mathcal{C}} \mathcal{U} \to \Delta^n \times_{\mathcal{C}} \mathcal{U}$ is a Joyal equivalence. Indeed, $I^n \times_{\mathcal{C}} \mathcal{U}$ would be the pushout $(\Delta^{\{0,1\}} \times_{\mathcal{C}} \mathcal{U}) \times_{\{1\} \times_{\mathcal{C}} \mathcal{U}} \cdots \sqcup_{\{n-1\} \times \mathcal{U}} (\Delta^{\{n-1,n\}} \times_{\mathcal{C}} \mathcal{U})$ taken in sSet and then the corresponding pushout in Cat_{∞} is given by choosing a Joyal equivalence into a quasi-category by model category fact 6.13. Since $I^n \times_{\mathcal{C}} \mathcal{U} \to \Delta^n \times_{\mathcal{C}} \mathcal{U}$ is also a cofibration, we deduce that $\mathrm{F}(\Delta^n \times_{\mathcal{C}} \mathcal{U}, \mathcal{D}) \to \mathrm{F}(I^n \times_{\mathcal{C}} \mathcal{U}, \mathcal{D})$ is an equivalence of quasi-categories and an inner fibration, hence a trivial fibration. Thus $\mathrm{F}_{\mathcal{C}}(\Delta^n, R(\mathcal{D})) \to \mathrm{F}_{\mathcal{C}}(I^n, R(\mathcal{D}))$ is a trivial fibration. Via induction on n, this implies that

$$\mathrm{F}_{\mathcal{C}}\big(\Delta^n, R(\mathcal{D})\big) \longrightarrow \mathrm{F}_{\mathcal{C}}\big(\Lambda_i^n, R(\mathcal{D})\big)$$

must be a trivial fibration for all $0 < i < n$. Indeed, the case $n = 2$ is clear, as $\Lambda_1^2 = I^2$. For $n \geqslant 3$, observe that $I^n \subseteq \Lambda_i^n$ can be written as a sequence of pushouts against inner horn inclusions of lower dimension. By the inductive hypothesis, $\mathrm{F}_{\mathcal{C}}(\Lambda_i^n, R(\mathcal{D})) \to \mathrm{F}_{\mathcal{C}}(I^n, R(\mathcal{D}))$ is then a sequence of pullbacks of trivial fibrations, hence a trivial fibration itself. Then $\mathrm{F}_{\mathcal{C}}(\Delta^n, R(\mathcal{D})) \to \mathrm{F}_{\mathcal{C}}(\Lambda_i^n, R(\mathcal{D}))$ must be a trivial fibration too, finishing the inductive step.

In particular, the map $\mathrm{F}_{\mathcal{C}}(\Delta^n, R(\mathcal{D})) \to \mathrm{F}_{\mathcal{C}}(\Lambda_i^n, R(\mathcal{D}))$ is surjective on 0-simplices, which implies that $R(\mathcal{D}) \to \mathcal{C}$ has lifting against $\Lambda_i^n \to \Delta^n$. Hence it is an inner fibration. To show that it is an isofibration, we only need that for all maps $\mathrm{N}(J) \to \mathcal{C}$, the pullback $\{0\} \times_{\mathcal{C}} \mathcal{U} \to \mathrm{N}(J) \times_{\mathcal{C}} \mathcal{U}$ is still a Joyal equivalence, as then the same argument as above can be run. But we're free to choose $\mathcal{U} \to \mathcal{C}$ to be an isofibration itself, so that the pullbacks are homotopy pullbacks as well, and then the equivalence of quasi-categories $\{0\} \simeq \mathrm{N}(J)$ will be preserved.

To check the condition for cocartesian fibrations, a straightforward generalisation of claim ($\boxtimes_1$) from the proof of Lemma 8.5 allows us to write $\Delta^n \times_{\mathcal{C}} \mathcal{U}$ as a "telescope"

$$\big(\Delta^{\{0,1\}} \times \mathcal{U}_0\big) \sqcup_{\{1\} \times \mathcal{U}_0, F_0} \big(\Delta^{\{1,2\}} \times \mathcal{U}_1\big) \sqcup_{\{2\} \times \mathcal{U}_1, F_1} \cdots \sqcup_{\{n\} \times \mathcal{U}_{n-1}, F_{n-1}} \big(\{n\} \times \mathcal{U}_n\big).$$

The desired property is then immediate. $\qquad\square$

This finally finishes our digreesion and we can get back to symmetric monoidal ∞-categories. As is apparent from Definition 8.2(b), a functor $F \colon \mathcal{C} \to \mathcal{D}$ being (lax) symmetric monoidal is not a property, but extra structure. Such structure is usually impossible to construct by hand. The following lemmas allow us to construct (lax) symmetric monoidal structures by checking only pointwise properties.

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8.9. Lemma. — *Let $\mathcal{C}$ be a symmetric monoidal ∞-category, let $\mathcal{D} \subseteq \mathcal{C}$ be a full sub-∞-category of (the underlying ∞-category of) $\mathcal{C}$ and let $\mathcal{D}^{\otimes} \subseteq \mathcal{C}^{\otimes}$ be the full sub-∞-category spanned by those $(y_1, \ldots, y_n) \in \mathcal{C}^{\otimes}$ that satisfy $y_1, \ldots, y_n \in \mathcal{D}$.*

(a) *Suppose $\mathbb{1}_{\mathcal{C}} \in \mathcal{D}$ and for all $y_1, y_2 \in \mathcal{D}$ one has $y_1 \otimes_{\mathcal{C}} y_2 \in \mathcal{D}$. Then $\mathcal{D}^{\otimes} \to \mathrm{Fin}$ is a cocartesian fibration, defining a symmetric monoidal structure on $\mathcal{D}$ in such a way that the inclusion $\mathcal{D}^{\otimes} \subseteq \mathcal{C}^{\otimes}$ is symmetric monoidal.*

(b) *Suppose the inclusion $\mathcal{D} \subseteq \mathcal{C}^{\otimes}$ admits a left adjoint $L \colon \mathcal{C} \to \mathcal{D}$ in such a way that for all $x_1, x_2 \in \mathcal{C}$ the unit $u_{x_1} \colon x_1 \to L(x_1)$ induces an equivalence*

$$L(x_1 \otimes_{\mathcal{C}} x_2) \xrightarrow{\simeq} L\big(L(x_1) \otimes_{\mathcal{C}} x_2\big)$$

Then $\mathcal{D}^{\otimes} \to \mathrm{Fin}$ is a cocartesian fibration, defining a symmetric monoidal structure on $\mathcal{D}$, the functor L can be equipped with a symmetric monoidal structure $L^{\otimes} \colon \mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$, and the inclusion $\mathcal{D}^{\otimes} \subseteq \mathcal{C}^{\otimes}$ is lax monoidal.

8.10. Lemma. — *Let $\mathcal{C}, \mathcal{D}$ be symmetric monoidal ∞-categories and let $L \colon \mathcal{C} \rightleftarrows \mathcal{D} \colon R$ be an adjunction on their underlying ∞-categories.*

(a) *If L can be equipped with a symmetric monoidal structure $L^{\otimes} \colon \mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$, then R can be canonically equipped with a lax symmetric monoidal structure $R^{\otimes} \colon \mathcal{D}^{\otimes} \to \mathcal{C}^{\otimes}$.*

(b) *Suppose R can be equipped with a lax symmetric monoidal structure $R^{\otimes} \colon \mathcal{D}^{\otimes} \to \mathcal{C}^{\otimes}$. Suppose furthermore that L satisfies pointwise equivalences*

$$L(\mathbb{1}_{\mathcal{C}}) \simeq \mathbb{1}_{\mathcal{D}} \quad and \quad L(x_1 \otimes_{\mathcal{C}} x_2) \simeq L(x_1) \otimes_{\mathcal{D}} L(x_2)$$

for all $x_1, x_2 \in \mathcal{C}$. Then L can be canonically equipped with a symmetric monoidal structure $L^{\otimes} \colon \mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$.

In both cases, $L^{\otimes} \colon \mathcal{C}^{\otimes} \rightleftarrows \mathcal{D}^{\otimes} \colon R^{\otimes}$ is again an adjunction.

Proof sketch of Lemmas 8.9 and 8.10. All assertions are rather straightforward consequences of Lemma 6.2, which allows us to construct adjunctions pointwise. We'll only explain the proof of Lemma 8.9(*b*) in detail and leave it to the reader to figure out the other parts.

Let $p \colon \mathcal{C}^{\otimes} \to \mathrm{Fin}$ and $q \colon \mathcal{D}^{\otimes} \to \mathrm{Fin}$ denote the projections and $i \colon \mathcal{D} \to \mathcal{C}$ and $i^{\otimes} \colon \mathcal{D}^{\otimes} \to \mathcal{C}^{\otimes}$ the inclusions. To show that q is a cocartesian fibration, let $\alpha \colon \langle n \rangle \to \langle m \rangle$ be a morphism in Fin and let $(y_1, \ldots, y_n) \in \mathcal{D}_n^{\otimes}$. Choose a p-cocartesian lift $\varphi \colon (y_1, \ldots, y_n) \to (x_1, \ldots, x_m)$ in $\mathcal{C}^{\otimes}$. Composing with the units $u_{x_i} \colon x_i \to L(x_i)$ for $i = 1, \ldots, m$, we obtain a morphism $\overline{\varphi} \colon (y_1, \ldots, y_n) \to (L(x_1), \ldots, L(x_m))$ in $\mathcal{D}^{\otimes}$. We first check that $\overline{\varphi}$ is a *locally q-cocartesian* lift of α in the sense of Definition 5.17(*a*). That is, we wish to check that $\overline{\varphi}$ is q_α-cocartesian, where $q_\alpha \colon \mathcal{D}_\alpha^{\otimes} \to \Delta^1$ is the base change of q along $\alpha \colon \Delta^1 \to \mathrm{Fin}$. To this end, we verify the criterion from Lemma 5.16. Using the adjunction $L^m \colon \mathcal{C}^m \rightleftarrows \mathcal{D}^m \colon i^m$, the criterion for $\overline{\varphi}$ can be reduced to the criterion for φ to be p_α-cocartesian, which is clearly satisfied. This shows that q is a locally cocartesian fibration. Our assumption $L(x_1 \otimes_{\mathcal{C}} x_2) \simeq L(L(x_1) \otimes_{\mathcal{C}} x_2)$ guarantees that locally q-cocartesian lifts are closed under composition, whence q is cocartesian by Corollary 5.18.

Since we wish to construct $L^{\otimes} \colon \mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$ as an adjoint of $i^{\otimes}$, it's enough to do so pointwise by Lemma 6.2. For $x = (x_1, \ldots, x_n) \in \mathcal{C}^{\otimes}$, we simply define $L^{\otimes}(x) \coloneqq (L(x_1), \ldots, L(x_n))$, and we let $u_x \colon x \to L^{\otimes}(x)$ be induced by $u_{x_j} \colon x_j \to L(x_j)$ for all $j = 1, \ldots, n$. It's enough to check that the natural transformation $u_x^* \colon \mathrm{Hom}_{\mathcal{C}^{\otimes}}(x, -) \Rightarrow \mathrm{Hom}_{\mathcal{D}^{\otimes}}(L^{\otimes}(x), -)$, given by

precomposition with u_x, is an equivalence in $\mathrm{Fun}(\mathcal{D}^\otimes, \mathrm{An})$. This can again be checked pointwise. So fix some $y = (y_1, \ldots, y_m) \in \mathcal{D}^\otimes$. Whether $u_x^* \colon \mathrm{Hom}_{\mathcal{C}^\otimes}(x, y) \to \mathrm{Hom}_{\mathcal{D}^\otimes}(L^\otimes(x), y)$ is an equivalence can be checked on fibres over $\mathrm{Hom}_{\mathrm{Fin}}(\langle n \rangle, \langle m \rangle)$ by Theorem 3.18 and Lemma 3.19 (plus Remark 3.20). So fix some $\alpha \colon \langle n \rangle \to \langle m \rangle$. Restricting to fibres over α amounts to replacing $L^\otimes$ by $L_\alpha^\otimes \colon \mathcal{C}_\alpha^\otimes \to \mathcal{D}_\alpha^\otimes$; we must check that $L_\alpha^\otimes \colon \mathcal{C}_\alpha^\otimes \rightleftarrows \mathcal{D}_\alpha^\otimes : i_\alpha^\otimes$ is an adjunction. Let $\varphi \colon x \to x'$ be a p_α-cocartesian lift of α. Then $\mathrm{Hom}_{\mathcal{C}_\alpha^\otimes}(x, y) \simeq \mathrm{Hom}_{\mathcal{C}_\alpha^\otimes}(x', y) \simeq \mathrm{Hom}_{\mathcal{C}_n^\otimes}(x', y)$ follows from Lemma 5.16 and the fact that $\mathcal{C}_n^\otimes \to \mathcal{C}_\alpha^\otimes$ is fully faithful (as the same is true for $\{1\} \to \Delta^1$). By construction, $L^\otimes(\varphi) \colon L^\otimes(x) \to L^\otimes(x')$ is q_α-cocartesian. Hence the same argument shows $\mathrm{Hom}_{\mathcal{D}_\alpha^\otimes}(L^\otimes(x), y) \simeq \mathrm{Hom}_{\mathcal{D}_\alpha^\otimes}(L^\otimes(x'), y) \simeq \mathrm{Hom}_{\mathcal{D}_n^\otimes}(L^\otimes(x'), y)$. So it will be enough to show that $L_n^\otimes \colon \mathcal{C}_n^\otimes \rightleftarrows \mathcal{D}_n^\otimes : i_n^\otimes$ is an adjunction, which is clear since $\mathcal{C}_n^\otimes \simeq \mathcal{C}^n$, $\mathcal{D}_n^\otimes \simeq \mathcal{D}^n$.

So we've successfully constructed $L^\otimes$ as a left adjoint of $i^\otimes$. But we still need to check that $L^\otimes$ is a functor in $\mathrm{Cat}_{\infty/\mathrm{Fin}}$! That is, we need an equivalence $q \circ L^\otimes \simeq p$. The unit $u \colon \mathrm{id}_{\mathcal{C}^\otimes} \Rightarrow i^\otimes \circ L^\otimes$ induces a natural transformation $p \Rightarrow p \circ i^\otimes \circ L^\otimes \simeq q \circ L^\otimes$. Whether this is an equivalence can indeed be checked pointwise, where it holds by construction. $\qquad\square$

8.11. Example. — Finally, we can give some well-earned examples.

(a) If $\mathcal{C}$ is an ordinary symmetric monoidal category, then it can also be equipped with a symmetric monoidal structure in the ∞-categorical sense. To see this, let us construct an ordinary category $\mathcal{C}^\otimes$ together with a cocartesian fibration $\mathcal{C}^\otimes \to \mathrm{Fin}$. Thanks to 8.3, it's clear what to do: Objects of $\mathcal{C}$ are pairs $(\langle n \rangle, x)$, where $x = (x_1, \ldots, x_n) \in \mathcal{C}^m$, morphisms are defined by

$$\mathrm{Hom}_{\mathcal{C}^\otimes}\big((\langle n \rangle, x), (\langle m \rangle, y)\big) \coloneqq \coprod_{\alpha \in \mathrm{Hom}_{\mathrm{Fin}}(\langle n \rangle, \langle m \rangle)} \prod_{i \in \langle m \rangle} \mathrm{Hom}_{\mathcal{C}}\left(\bigotimes_{j \in \alpha^{-1}(i)} x_j, y_i\right),$$

and the functor $p \colon \mathcal{C}^\otimes \to \mathrm{Fin}$ sends $(\langle n \rangle, x) \mapsto \langle n \rangle$. Using Lemma 5.16 (plus the fact that homotopy pullbacks of discrete sets are just ordinary pullbacks) it's straightforward to check that p is indeed a cocartesian fibration.

(b) More generally, if $\mathcal{C}$ is a symmetric monoidal Kan-enriched category, then its simplicial nerve $\mathrm{N}^\Delta(\mathcal{C})$ is canonically a symmetric monoidal ∞-category. To see this, one constructs a functor $\mathcal{C}^\otimes \to \mathrm{Fin}$ of Kan-enriched categories (where Fin is given the trivial enrichment in which all $\mathrm{F}_{\mathrm{Fin}}(-, -)$ are discrete sets) in the exact same way as in (a). Again, using Lemma 5.16 and Theorem 2.24, it's straightforward to check that $\mathrm{N}^\Delta(\mathcal{C}^\otimes) \to \mathrm{N}^\Delta(\mathrm{Fin})$ is a cocartesian fibration, which equips $\mathrm{N}^\Delta(\mathcal{C})$ with the desired structure of a symmetric monoidal ∞-category.

(c) In particular, since Kan and QCat can be turned into symmetric monoidal Kan-enriched categories via the usual product $- \times -$, we can use (b) to obtain symmetric monoidal structures $\mathrm{An}^\times$ and $\mathrm{Cat}_\infty^\times$ on An and Cat_∞. In a similar way, one can equip the derived ∞-category $\mathcal{D}(R)$ of a commutative ring R with a symmetric monoidal structure given by the derived tensor product $- \otimes_R^{\mathrm{L}} -$. This construction will be explained in detail in 8.25.

(d) In general, if $\mathcal{C}$ is any ∞-category with finite products (hence, in particular, a terminal object $* \in \mathcal{C}$), then $\mathcal{C}$ can be given a canonical symmetric monoidal structure in which the tensor product is given by $- \times -$ and the tensor unit by $*$. This is called the *cartesian symmetric monoidal structure*. We'll explain the construction in [TODO]. The same works, of course, for coproducts. In this case one obtains the *cocartesian symmetric monoidal structure*.

The main reason for considering symmetric monoidal ∞-categories is that one can do algebra in them. In particular, we have the following notions of commutative and associative algebras.

8.12. Definition. — Let $\mathcal{C}$ be a symmetric monoidal ∞-category, with associated cocartesian fibration $\mathcal{C}^{\otimes} \to \mathrm{Fin}$.

(a) A morphism $[n] \to [m]$ in Δ^{op} is called *inert* if the corresponding morphism $\alpha \colon [m] \to [n]$ in Δ is the inclusion of an interval.[(8.4)]

(b) An $\mathbb{E}_{\infty}$-*algebra in* $\mathcal{C}$ or an $\mathbb{E}_1$-*algebra in* $\mathcal{C}$ is a functor

$$
\begin{array}{ccc}
 & \mathcal{C}^{\otimes} & \\
\stackrel{A}{\nearrow} & \downarrow & \\
\mathrm{Fin} =\!=\!= \mathrm{Fin} &
\end{array}
\quad \text{or} \quad
\begin{array}{ccc}
 & \mathcal{C}^{\otimes} & \\
\stackrel{A}{\nearrow} & \downarrow & \\
\Delta^{\mathrm{op}} \xrightarrow[\mathrm{Cut}]{} \mathrm{Fin} &
\end{array}
$$

respectively, such that A sends inert morphisms (in Fin or Δ^{op}, respectively) to inert lifts. We let $\mathrm{CAlg}(\mathcal{C}) \subseteq \mathrm{Fun}_{\mathrm{Fin}}(\mathrm{Fin}, \mathcal{C}^{\otimes})$ and $\mathrm{Alg}(\mathcal{C}) \subseteq \mathrm{Fun}_{\mathrm{Fin}}(\Delta^{\mathrm{op}}, \mathcal{C}^{\otimes})$ denote the full sub-∞-categories spanned by the $\mathbb{E}_{\infty}$- or $\mathbb{E}_1$-algebras, respectively.

For convenience, let us also introduce the following much weaker notions:

(c) A *homotopy-commutative* or *homotopy-associative algebra in* $\mathcal{C}$ is an object of $\mathrm{CAlg}(h(\mathcal{C}))$ or $\mathrm{Alg}(h(\mathcal{C}))$, respectively. Here $h(\mathcal{C})$ is equipped with its induced symmetric monoidal structure.

8.13. Monoidal ∞-categories. — There's also a notion of *monoidal ∞-categories*: These are simply objects in $\mathrm{Mon}(\mathrm{Cat}_{\infty})$. Using the notion of inertness from Definition 8.12(*a*), we can then define *lax monoidal functors* analogously to Definition 8.2(*b*). Also, the notion of an $\mathbb{E}_1$-*algebra* from Definition 8.12(*b*) already makes sense in a monoidal ∞-category. In fact, all of this works in much greater generality, as we'll see in §8.4

In general, any symmetric monoidal ∞-category defines an *underlying monoidal ∞-category* via the forgetful functor $\mathrm{CMon}(\mathrm{Cat}_{\infty}) \to \mathrm{Mon}(\mathrm{Cat}_{\infty})$.[(8.5)] Similarly, if $\mathcal{C}$ is symmetric monoidal, then precomposition with $\mathrm{Cut} \colon \Delta^{\mathrm{op}} \to \mathrm{Fin}$ induces a forgetful functor $\mathrm{CAlg}(\mathcal{C}) \to \mathrm{Alg}(\mathcal{C})$ that assigns to any $\mathbb{E}_{\infty}$-algebra an *underlying $\mathbb{E}_1$-algebra*. The ∞-category $\mathrm{Alg}(\mathcal{C})$ then only depends on the underlying monoidal structure of $\mathcal{C}$.

8.14. $\mathbb{E}_{\infty}$- and $\mathbb{E}_1$-algebras. — Note that the identity $\mathrm{id}_{\mathrm{Fin}} \colon \mathrm{Fin} \to \mathrm{Fin}$ can be viewed as the cocartesian fibration that classifies the symmetric monoidal ∞-category $\{\mathbb{1}\}$ consisting of a single object (which is necessarily the tensor unit). Then $\mathrm{CAlg}(\mathcal{C}) \simeq \mathrm{Fun}^{\mathrm{lax}\text{-}\otimes}(\{\mathbb{1}\}, \mathcal{C})$ is the ∞-category of lax symmetric monoidal functors $\{\mathbb{1}\} \to \mathcal{C}$, which makes some sense intuitively. Similarly, $\mathrm{Alg}(\mathcal{C})$ is the ∞-category of lax monoidal functors $\{\mathbb{1}\} \to \mathcal{C}$.

If $\mathcal{C}$ is an ordinary symmetric monoidal category, then by a simple unravelling of Example 8.11(*a*), we see that an $\mathbb{E}_{\infty}$-*algebra in* $\mathcal{C}$ is precisely a *commutative algebra object*: That is,

---

[(8.4)]By unravelling Construction 7.26, this is equivalent to $\mathrm{Cut}(\alpha) \colon \langle n \rangle \to \langle m \rangle$ being inert in Fin in the sense of Definition 8.1(*a*). So the terminology makes sense.

[(8.5)]Even better, if $\mathrm{SymMonCat}^{\mathrm{lax}}_{\infty} \subseteq \mathrm{Cat}_{\infty/\mathrm{Fin}}$ and $\mathrm{MonCat}^{\mathrm{lax}}_{\infty} \subseteq \mathrm{Cat}_{\infty/\Delta^{\mathrm{op}}}$ denote the (non-full) sub-∞-categories spanned by the (symmetric) monoidal ∞-categories and the lax (symmetric) monoidal functors, then the functor $\mathrm{Cat}_{\infty/\mathrm{Fin}} \to \mathrm{Cat}_{\infty/\Delta^{\mathrm{op}}}$ given by pullback along $\mathrm{Cut} \colon \Delta^{\mathrm{op}} \to \mathrm{Fin}$ induces a forgetful functor $\mathrm{SymMonCat}^{\mathrm{lax}}_{\infty} \to \mathrm{MonCat}^{\mathrm{lax}}_{\infty}$. Hence we may functorially assign an *underlying lax monoidal functor* to any symmetric monoidal functor.

an object $A \in \mathcal{C}$ together with a unit morphism $\mathbb{1} \to A$ and a multiplication $\mu \colon A \otimes A \to A$ such that μ is unital, associative, and commutative. Similarly, $\mathbb{E}_1$-algebras are *associative algebra objects*, given by the same data except that μ doesn't need to be commutative. For general symmetric monoidal ∞-categories $\mathcal{C}$, any $\mathbb{E}_\infty$-algebra or $\mathbb{E}_1$-algebra still has an *underlying object $A \in \mathcal{C}$* together with a unit $\mathbb{1} \to A$ and a multiplication $\mu \colon A \otimes A \to A$, which is unital, associative, and commutative (for $\mathbb{E}_\infty$-algebras) up to homotopy. In particular, A becomes a homotopy-commutative or homotopy-associative algebra in $\mathcal{C}$ as defined in Definition 8.12(*c*). However, A being an $\mathbb{E}_\infty$-algebra or $\mathbb{E}_1$-algebra is *much more structure*, since it also involves a coherent choice of homotopies (and higher homotopies between them) that make μ unital, associative, or commutative (compare this to the discussion in 7.5).

It can be shown that our notions of $\mathbb{E}_1$-*monoids* and $\mathbb{E}_\infty$-*monoids* from Definitions 7.6 and 7.24 are recovered by Definition 8.12(*b*). This is the content of the following theorem, whose proof we'll sketch in §8.4.

8.15. Theorem. — *If $\mathcal{C}$ is equipped with the cartesian monoidal structure from Example 8.11(d), then there are canonical equivalences* $\mathrm{CAlg}(\mathcal{C}) \simeq \mathrm{CMon}(\mathcal{C})$ *and* $\mathrm{Alg}(\mathcal{C}) \simeq \mathrm{Mon}(\mathcal{C})$.

§8.2. Day convolution

In this subsection, we'll show that for symmetric monoidal ∞-categories $\mathcal{C}$ and $\mathcal{D}$, one can often construct a symmetric monoidal structure on $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$, called *Day convolution*. We'll first prove existence, which is going to take some time, and then use Day convolution to construct the tensor product of spectra.

8.16. Theorem (Day convolution). — *Let $\mathcal{C}$ and $\mathcal{D}$ be symmetric monoidal ∞-categories such that $\mathcal{D}$ has all[(8.6)] colimits and $- \otimes_\mathcal{D} -$ preserves colimits in either variable. Then there exists a symmetric monoidal structure* $\mathrm{Fun}(\mathcal{C}, \mathcal{D})^{\otimes_{\mathrm{Day}}}$ *on* $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ *with the following properties:*

(a) *One has* $(F_1 \otimes_{\mathrm{Day}} F_2)(x) \simeq \mathrm{colim}_{x_1 \otimes_\mathcal{C} x_2 \to x} F_1(x_1) \otimes_\mathcal{D} F_2(x_2)$ *and* $\mathbb{1}_{\mathrm{Day}}(x) \simeq \mathrm{colim}_{\mathbb{1}_\mathcal{C} \to x} \mathbb{1}_\mathcal{D}$ *for all $x \in \mathcal{C}$ and all functors $F_1, F_2 \colon \mathcal{C} \to \mathcal{D}$. In other words,*

$$
\begin{array}{ccc}
\mathcal{C} \times \mathcal{C} \xrightarrow{F_1 \times F_2} \mathcal{D} \times \mathcal{D} \xrightarrow{- \otimes_\mathcal{D} -} \mathcal{D} & & * \xrightarrow{\mathbb{1}_\mathcal{D}} \mathcal{D} \\
{\scriptstyle -\otimes_\mathcal{C}-}\Big\downarrow \quad {\scriptstyle \diagup\!\!\diagup} & \text{and} & {\scriptstyle \mathbb{1}_\mathcal{C}}\Big\downarrow \quad {\scriptstyle \diagup\!\!\diagup} \\
\mathcal{C} \dashrightarrow[F_1 \otimes_{\mathrm{Day}} F_2] & & \mathcal{C} \dashrightarrow[\mathbb{1}_{\mathrm{Day}}]
\end{array}
$$

are left Kan extensions. Furthermore, $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ has again all colimits and $- \otimes_{\mathrm{Day}} -$ preserves colimits in either variable.

(b) *Lax symmetric monoidal functors $q \colon \mathcal{C} \to \mathcal{C}'$ and $p \colon \mathcal{D} \to \mathcal{D}'$ induce lax symmetric monoidal functors $q^* \colon \mathrm{Fun}(\mathcal{C}', \mathcal{D})^{\otimes_{\mathrm{Day}}} \to \mathrm{Fun}(\mathcal{C}, \mathcal{D})^{\otimes_{\mathrm{Day}}}$ and $p_* \colon \mathrm{Fun}(\mathcal{C}, \mathcal{D})^{\otimes_{\mathrm{Day}}} \to \mathrm{Fun}(\mathcal{C}, \mathcal{D}')^{\otimes_{\mathrm{Day}}}$. If q is symmetric monoidal, then q^* has a symmetric monoidal left adjoint*

$$
\mathrm{Lan}_q \colon \mathrm{Fun}(\mathcal{C}, \mathcal{D})^{\otimes_{\mathrm{Day}}} \longrightarrow \mathrm{Fun}(\mathcal{C}', \mathcal{D})^{\otimes_{\mathrm{Day}}}.
$$

[(8.6)]If $\mathcal{C}$ is essentially κ-small in the sense of Definition 6.46, then it's enough for $\mathcal{D}$ to have κ-small colimits and for $- \otimes_\mathcal{D} -$ to preserve κ-small colimits in either variable. Indeed, as we'll see during the proof, the colimit assumption will only be needed to apply the Kan extension formula from Lemma 6.27. This already works if the indexing ∞-category is κ-small and $\mathcal{D}$ has κ-small colimits.

(c) $\mathrm{CAlg}(\mathrm{Fun}(\mathcal{C}, \mathcal{D})) \simeq \mathrm{Fun}^{\mathrm{lax}\text{-}\otimes}(\mathcal{C}, \mathcal{D})$. *More generally, for any symmetric monoidal ∞-category $\mathcal{X}$,*

$$\mathrm{Fun}^{\mathrm{lax}\text{-}\otimes}(\mathcal{X}, \mathrm{Fun}(\mathcal{C}, \mathcal{D})) \simeq \mathrm{Fun}^{\mathrm{lax}\text{-}\otimes}(\mathcal{X} \times \mathcal{C}, \mathcal{D}).$$

The proof of Theorem 8.16 will be a little lengthy (if not to say *convoluted*) and you can safely skip ahead to 8.22. Our construction of $\mathrm{Fun}(\mathcal{C}, \mathcal{D})^{\otimes_{\mathrm{Day}}}$ follows [L-HA, Construction 2.2.6.18], except that we're not aiming for Lurie's maximum level of generality and also all reliance on simplicial methods has been put into the proof of Lemma 8.8.

8.17. Lemma. — *Let $\mathrm{Ar}^{\mathrm{int}}(\mathrm{Fin}) \subseteq \mathrm{Ar}(\mathrm{Fin})$ be the full sub-∞-category spanned by the inert arrows.[(8.7)] Then the source projection $s\colon \mathrm{Ar}^{\mathrm{int}}(\mathrm{Fin}) \to \mathrm{Fin}$ is a flat fibration, that is, it satisfies the condition of Lemma 8.8.*

Proof sketch. We make the following two crucial observations:

($\boxtimes_1$) *Let $\alpha\colon \langle n \rangle \to \langle m \rangle$ be a morphism in Fin and let $i\colon \langle n \rangle \to \langle n' \rangle$ and $j\colon \langle m \rangle \to \langle m' \rangle$ be inert arrows. Then there can be at most one lift $i \to j$ of α in $\mathrm{Ar}^{\mathrm{int}}(\mathrm{Fin})$.*

($\boxtimes_2$) *Let $\beta\colon \langle m \rangle \to \langle \ell \rangle$ be another morphism in Fin and $k\colon \langle \ell \rangle \to \langle \ell' \rangle$ another inert arrow. If a lift $\varphi\colon i \to k$ of $\beta \circ \alpha$ in $\mathrm{Ar}^{\mathrm{int}}(\mathrm{Fin})$ exists, we can find an inert arrow j and lifts $i \to j$, $j \to k$ of α, β in such a way that φ is the composition $i \to j \to k$.*

For ($\boxtimes_1$), any such lift is a commutative diagram of the form

$$
\begin{array}{ccc}
\langle n \rangle & \xrightarrow{\ \alpha\ } & \langle m \rangle \\
\ \downarrow{\scriptstyle i} & /\!/\!/ & \ \downarrow{\scriptstyle j} \\
\langle n' \rangle & \dashrightarrow & \langle m' \rangle
\end{array}
$$

in Fin. Let $S \subseteq \langle n \rangle$ be the subset where i is defined, so that $i\colon S \to \langle n' \rangle$ is a bijection. If $j \circ \alpha$ is defined anywhere outside S, then no dashed arrow can exist. If $j \circ \alpha$ is undefined outside S, then the dashed arrow exists and is uniquely determined as $j \circ \alpha|_S \circ i^{-1}$. This shows ($\boxtimes_1$). In the situation of ($\boxtimes_2$), we deduce that $k \circ \beta \circ \alpha$ must be undefined outside S or φ wouldn't exist. We can now choose $j\colon \langle m \rangle \to \alpha(S)$ to be defined precisely on $\alpha(S)$. Then $j \circ \alpha$ is undefined outside S, hence $i \to j$ exists, and $k \circ \beta$ is undefined outside $\alpha(S)$, hence $j \to k$ exists. By uniqueness, it follows that φ must be the composition $i \to j \to k$. This shows ($\boxtimes_2$).

Since the category Δ^2 is a partially ordered set, ($\boxtimes_1$) implies that for all $\Delta^2 \to \mathrm{Fin}$ the pullback $\Delta^2 \times_{\mathrm{Fin}} \mathrm{Ar}^{\mathrm{int}}(\mathrm{Fin})$ is a partially ordered set as well. Furthermore, ($\boxtimes_2$) implies that the partially ordered set $\Delta^2 \times_{\mathrm{Fin}} \mathrm{Ar}^{\mathrm{int}}(\mathrm{Fin})$ is the concatenation of the partially ordered sets $\Delta^{\{0,1\}} \times_{\mathrm{Fin}} \mathrm{Ar}^{\mathrm{int}}(\mathrm{Fin})$ and $\Delta^{\{1,2\}} \times_{\mathrm{Fin}} \mathrm{Ar}^{\mathrm{int}}(\mathrm{Fin})$ along their common partially ordered subset $\{1\} \times_{\mathrm{Fin}} \mathrm{Ar}^{\mathrm{int}}(\mathrm{Fin})$. By a straightforward inspection[(8.8)], this shows that $- \times_{\mathrm{Fin},s} \mathrm{Ar}^{\mathrm{int}}(\mathrm{Fin})$ preserves pushouts of the form $\Delta^{\{0,1\}} \sqcup_{\{1\}} \Delta^{\{1,2\}} \simeq \Delta^2$, as desired. $\square$

8.18. Construction. — Let $\mathcal{C}$ and $\mathcal{D}$ be symmetric monoidal ∞-categories and consider the functor

$$- \times_{\mathrm{Fin},s} \mathrm{Ar}^{\mathrm{int}}(\mathrm{Fin}) \times_{t,\mathrm{Fin}} \mathcal{C}^{\otimes}\colon \mathrm{Cat}_{\infty/\mathrm{Fin}} \longrightarrow \mathrm{Cat}_{\infty/\mathrm{Fin}}$$

(there are two ways to make the values of this functor into objects of $\mathrm{Cat}_{\infty/\mathrm{Fin}}$; we choose the one coming from $\mathcal{C}^{\otimes} \to \mathrm{Fin}$). Using a combination of Lemmas 8.8 and 8.17 and the fact that

---

[(8.7)]This is not the same as $\mathrm{Ar}(\mathrm{Fin}^{\mathrm{int}})$, as there can be non-inert morphisms between inert arrows.

[(8.8)]For example, such pushouts of partially ordered sets are easily understood via model category fact 6.13.

$\mathcal{C}^{\otimes} \to \mathrm{Fin}$ is a cocartesian fibration, we see that this functor preserves colimits. By the adjoint functor theorem (Theorem 6.66(a); presentability is ensured by Corollaries 6.64 and 6.65), it therefore admits a right adjoint. Let $\overline{\mathrm{Fun}}(\mathcal{C}, \mathcal{D})^{\otimes\mathrm{Day}}$ be the value of the right adjoint on $\mathcal{D}^{\otimes}$. Then for all $\mathcal{X} \to \mathrm{Fin}$ we obtain a functorial equivalence[(8.9)]

$$\mathrm{Fun}_{\mathrm{Fin}}\big(\mathcal{X}, \overline{\mathrm{Fun}}(\mathcal{C}, \mathcal{D})^{\otimes\mathrm{Day}}\big) \simeq \mathrm{Fun}_{\mathrm{Fin}}\big(\mathcal{X} \times_{\mathrm{Fin}, s} \mathrm{Ar}^{\mathrm{int}}(\mathrm{Fin}) \times_{t, \mathrm{Fin}} \mathcal{C}^{\otimes}, \mathcal{D}^{\otimes}\big).$$

In particular, we can describe the fibers of $\overline{\mathrm{Fun}}(\mathcal{C}, \mathcal{D})^{\otimes\mathrm{Day}} \to \mathrm{Fin}$: First observe that even though $\mathrm{Ar}(\mathrm{Fin}^{\mathrm{int}}) \to \mathrm{Ar}^{\mathrm{int}}(\mathrm{Fin})$ is not fully faithful, we still have

$$\big\{\langle n \rangle\big\} \times_{\mathrm{Fin}, s} \mathrm{Ar}^{\mathrm{int}}(\mathrm{Fin}) \simeq \mathrm{Fin}^{\mathrm{int}}_{\langle n \rangle /};$$

this is straightforward using our characterisation of morphisms in $\mathrm{Ar}^{\mathrm{int}}(\mathrm{Fin})$ in the proof of Lemma 8.17. Thus, the fibre over $\langle n \rangle \in \mathrm{Fin}$ is given by

$$\mathrm{Fun}_{\mathrm{Fin}}\big(\{\langle n \rangle\}, \overline{\mathrm{Fun}}(\mathcal{C}, \mathcal{D})^{\otimes\mathrm{Day}}\big) \simeq \mathrm{Fun}_{\mathrm{Fin}}\big(\mathrm{Fin}^{\mathrm{int}}_{\langle n \rangle /} \times_{t, \mathrm{Fin}} \mathcal{C}^{\otimes}, \mathcal{D}^{\otimes}\big).$$

We now let $\mathrm{Fun}(\mathcal{C}, \mathcal{D})^{\otimes\mathrm{Day}} \subseteq \overline{\mathrm{Fun}}(\mathcal{C}, \mathcal{D})^{\otimes\mathrm{Day}}$ be the full sub-∞-category spanned fibrewise by those functors $T\colon \mathrm{Fin}^{\mathrm{int}}_{\langle n \rangle /} \times_{t, \mathrm{Fin}} \mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$ that *preserve inert lifts* in the sense that T sends any morphism whose projection to $\mathcal{C}^{\otimes}$ is an inert lift to an inert lift in $\mathcal{D}^{\otimes}$.

Our goal is now to show that $\mathrm{Fun}(\mathcal{C}, \mathcal{D})^{\otimes\mathrm{Day}} \to \mathrm{Fin}$ satisfies all properties claimed in Theorem 8.16. We start with describing its fibres.

8.19. Lemma. — *We have $\mathrm{Fun}(\mathcal{C}, \mathcal{D})^{\otimes\mathrm{Day}}_n \simeq \mathrm{Fun}(\mathcal{C}, \mathcal{D})^n$ for all $n \geqslant 0$.*

Proof sketch. Throughout the proof, we abusingly denote by $\mathcal{C}$ and $\mathcal{D}$ both the underlying ∞-categories of $\mathcal{C}^{\otimes}$ and $\mathcal{D}^{\otimes}$, but also the associated functors $\mathcal{C}, \mathcal{D}\colon \mathrm{Fin} \to \mathrm{Cat}_{\infty}$.

First recall that the computation of the fibres of $\overline{\mathrm{Fun}}(\mathcal{C}, \mathcal{D})^{\otimes\mathrm{Day}}$ in Construction 8.18. The Segal maps $e_i\colon \langle n \rangle \to \langle 1 \rangle$ are objects of $\mathrm{Fin}^{\mathrm{int}}_{\langle n \rangle /}$ and induce a functor $\coprod_{i=1}^{n}\{e_i\} \to \mathrm{Fin}^{\mathrm{int}}_{\langle n \rangle /}$. Restriction along this functor yields a map

$$\mathrm{Fun}_{\mathrm{Fin}}\big(\mathrm{Fin}^{\mathrm{int}}_{\langle n \rangle /} \times_{t, \mathrm{Fin}} \mathcal{C}^{\otimes}, \mathcal{D}^{\otimes}\big) \longrightarrow \mathrm{Fun}\left(\coprod_{i=1}^{n}\{e_i\} \times \mathcal{C}_1^{\otimes}, \mathcal{D}_1^{\otimes}\right) \simeq \mathrm{Fun}(\mathcal{C}, \mathcal{D})^n$$

and thus a map $\mathrm{Fun}(\mathcal{C}, \mathcal{D})^{\otimes\mathrm{Day}}_n \to \mathrm{Fun}(\mathcal{C}, \mathcal{D})^n$. To check whether this map is an equivalence, it's enough to do so after $\mathrm{core}(-)$ and $\mathrm{core}\,\mathrm{Ar}(-)$. We'll start with $\mathrm{core}(-)$. The target projection $t\colon \mathrm{Fin}^{\mathrm{int}}_{\langle n \rangle /} \to \mathrm{Fin}$ factors through $\mathrm{Fin}^{\mathrm{int}}$ and so a functor $T\colon \mathrm{Fin}^{\mathrm{int}}_{\langle n \rangle /} \times_{t, \mathrm{Fin}} \mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$ over Fin is equivalently given by a functor

$$T^{\mathrm{int}}\colon \mathrm{Fin}^{\mathrm{int}}_{\langle n \rangle /} \times_{t, \mathrm{Fin}} \mathcal{C}^{\otimes} \longrightarrow \mathrm{Fin}^{\mathrm{int}} \times_{\mathrm{Fin}} \mathcal{D}^{\otimes}$$

[(8.9)]A priori, the adjunction only yields this equivalence for $\mathrm{Hom}_{\mathrm{Cat}_{\infty}/\mathrm{Fin}}$ instead of $\mathrm{Fun}_{\mathrm{Fin}}$. But we can upgrade it to $\mathrm{Fun}_{\mathrm{Fin}}$ via a standard argument. Indeed, the counit

$$c_{\mathcal{D}^{\otimes}}\colon \overline{\mathrm{Fun}}(\mathcal{C}, \mathcal{D})^{\otimes\mathrm{Day}} \times_{\mathrm{Fin}, s} \mathrm{Ar}^{\mathrm{int}}(\mathrm{Fin}) \times_{t, \mathrm{Fin}} \mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$$

certainly induces a natural transformation for $\mathrm{Fun}_{\mathrm{Fin}}$ as well. Checking whether this is an equivalence can be done pointwise and after applying $\mathrm{core}(-)$ and $\mathrm{core}\,\mathrm{Ar}(-)$, since these functors are jointly conservative (see the proof of Corollary 6.64). After $\mathrm{core}(-)$, we get the assertion for $\mathrm{Hom}_{\mathrm{Cat}_{\infty}/\mathrm{Fin}}$. After $\mathrm{core}\,\mathrm{Ar}(-)$, we get the same assertion, except that the argument $\mathcal{X}$ has been replaced by $\Delta^1 \times \mathcal{X}$

over $\mathrm{Fin}^{\mathrm{int}}$ (see Lemma 6.23(a)). Since $t\colon \mathrm{Fin}^{\mathrm{int}}_{\langle n\rangle/} \to \mathrm{Fin}$ is a left fibration, every morphism is cocartesian. Thus, T preserves inert lifts if and only if T^{int} is a morphism of cocartesian fibrations over $\mathrm{Fin}^{\mathrm{int}}$. Via straightening/unstraightening, it follows that

$$\mathrm{core}\,\mathrm{Fun}(\mathcal{C},\mathcal{D})^{\otimes\mathrm{Day}}_n \simeq \mathrm{Hom}_{\mathrm{Fun}(\mathrm{Fin}^{\mathrm{int}},\mathrm{Cat}_\infty)}\big(\mathrm{Hom}_{\mathrm{Fin}^{\mathrm{int}}}(\langle n\rangle,-) \times \mathcal{C}|_{\mathrm{Fin}^{\mathrm{int}}}, \mathcal{D}|_{\mathrm{Fin}^{\mathrm{int}}}\big)$$

The Segal condition implies that $\mathcal{D}|_{\mathrm{Fin}^{\mathrm{int}}}\colon \mathrm{Fin}^{\mathrm{int}} \to \mathrm{Cat}_\infty$ is the right Kan extension of the functor $\{\langle 1\rangle\} \to \mathrm{Cat}_\infty$ that sends $\langle 1\rangle \mapsto \mathcal{D}$. Therefore, the right-hand side above can be simplified to $\mathrm{Hom}_{\mathrm{Cat}_\infty}(\mathrm{Hom}_{\mathrm{Fin}^{\mathrm{int}}}(\langle n\rangle,\langle 1\rangle) \times \mathcal{C},\mathcal{D})$. Since $\mathrm{Hom}_{\mathrm{Fin}^{\mathrm{int}}}(\langle n\rangle,\langle 1\rangle) \simeq \coprod_{i=1}^{n}\{e_i\}$, this finishes the proof that $\mathrm{core}\,\mathrm{Fun}(\mathcal{C},\mathcal{D})^{\otimes\mathrm{Day}}_n \to \mathrm{core}\,\mathrm{Fun}(\mathcal{C},\mathcal{D})^n$ is an equivalence. The argument for $\mathrm{core}\,\mathrm{Ar}(-)$ is entirely analogous, one just has to adapt the calculation from Construction 8.18 to compute $\mathrm{Fun}_{\mathrm{Fin}}(\Delta^1 \times \{\langle n\rangle\}, \overline{\mathrm{Fun}}(\mathcal{C},\mathcal{D})^{\otimes\mathrm{Day}})$ instead. $\qquad\square$

8.20. Objects and morphisms in $\mathbf{Fun}(\mathcal{C},\mathcal{D})^{\otimes\mathrm{Day}}$. — As a consequence of Lemma 8.19, objects of $\mathrm{Fun}(\mathcal{C},\mathcal{D})^{\otimes\mathrm{Day}}$ can be thought of as n-tuples $F := (F_1,\dots,F_n)$ of functors $F_i\colon \mathcal{C} \to \mathcal{D}$. Our next goal is to describe morphisms. To this end, fix a morphism $\alpha\colon \langle n\rangle \to \langle m\rangle$ in Fin as well as objects $F := (F_1,\dots,F_n)$ and $G := (G_1,\dots,G_m)$ in the fibres over $\langle n\rangle$ and $\langle m\rangle$, respectively. Let us denote by

$$\mathrm{Hom}^\alpha_{\mathrm{Fun}(\mathcal{C},\mathcal{D})^{\otimes\mathrm{Day}}}(F,G) \subseteq \mathrm{Hom}_{\mathrm{Fun}(\mathcal{C},\mathcal{D})^{\otimes\mathrm{Day}}}(F,G)$$

the collection of path components spanned by those morphisms that map to α (this agrees with Hom in the fibre product $\mathrm{Fun}(\mathcal{C},\mathcal{D})^{\otimes\mathrm{Day}}_\alpha := \Delta^1 \times_{\alpha,\mathrm{Fin}} \mathrm{Fun}(\mathcal{C},\mathcal{D})^{\otimes\mathrm{Day}}$). Let us also denote $\mathrm{Ar}^{\mathrm{int}}_\alpha(\mathrm{Fin}) := \Delta^1 \times_{\alpha,\mathrm{Fin},s} \mathrm{Ar}^{\mathrm{int}}(\mathrm{Fin})$ and recall from 8.3 that $\alpha_{\mathcal{C}\otimes}\colon \mathcal{C}^n \to \mathcal{C}^m$ and $\alpha_{\mathcal{D}\otimes}\colon \mathcal{D}^n \to \mathcal{D}^m$ denote the functors obtained via unstraightening of the cocartesian fibrations $\mathcal{C}^\otimes_\alpha \to \Delta^1$ and $\mathcal{D}^\otimes_\alpha \to \Delta^1$. Unravelling the definition of $\overline{\mathrm{Fun}}(\mathcal{C},\mathcal{D})^{\otimes\mathrm{Day}}$, we then get a pullback square as follows:

$$
\begin{array}{ccc}
\mathrm{Hom}^\alpha_{\mathrm{Fun}(\mathcal{C},\mathcal{D})^{\otimes\mathrm{Day}}}(F,G) & \longrightarrow & \mathrm{Fun}_{\mathrm{Fin}}\big(\mathrm{Ar}^{\mathrm{int}}_\alpha(\mathrm{Fin}) \times_{t,\mathrm{Fin}} \mathcal{C}^\otimes, \mathcal{D}^\otimes\big) \\
\downarrow & \lrcorner & \downarrow \\
\{F \sqcup G\} & \longrightarrow & \mathrm{Fun}_{\mathrm{Fin}}\Big(\big(\mathrm{Fin}^{\mathrm{int}}_{\langle n\rangle/} \sqcup \mathrm{Fin}^{\mathrm{int}}_{\langle m\rangle/}\big) \times_{t,\mathrm{Fin}} \mathcal{C}^\otimes, \mathcal{D}^\otimes\Big)
\end{array}
$$

8.21. Lemma. — *We have an equivalence of animae*

$$\mathrm{Hom}^\alpha_{\mathrm{Fun}(\mathcal{C},\mathcal{D})^{\otimes\mathrm{Day}}}(F,G) \simeq \mathrm{Hom}_{\mathrm{Fun}(\mathcal{C}^n,\mathcal{D}^m)}(\alpha_{\mathcal{D}\otimes} \circ F, G \circ \alpha_{\mathcal{C}\otimes}).$$

In particular, a morphism $\varphi\colon F \to G$ in $\mathrm{Fun}(\mathcal{C},\mathcal{D})^{\otimes\mathrm{Day}}$ is locally cocartesian if and only if under the equivalence above the following is a left Kan extension diagram:

$$
\begin{array}{ccc}
\mathcal{C}^n & \xrightarrow{\ F\ } \mathcal{D}^n \xrightarrow{\ \alpha_{\mathcal{D}\otimes}\ } & \mathcal{D}^m \\
{\scriptstyle \alpha_{\mathcal{C}\otimes}}\downarrow & {\scriptstyle \varphi}\!\!\Nearrow & \nearrow \\
\mathcal{C}^m & \underset{G}{\cdots\cdots\cdots\cdots} &
\end{array}
$$

Proof sketch (special case $\alpha = f_n\colon \langle n\rangle \to \langle 1\rangle$). Let us first consider the special case where α equals the unique everywhere defined morphism $f_n\colon \langle n\rangle \to \langle 1\rangle$. In light of the pullback square

from 8.20, we need to understand the category $\mathrm{Ar}^{\mathrm{int}}_{f_n}(\mathrm{Fin})$: It's objects are inert morphisms $(i\colon \langle m\rangle \to \langle m'\rangle)$, where $\langle m\rangle \in \{\langle n\rangle, \langle 1\rangle\}$. These two possibilities for $\langle m\rangle$ give rise to a functor

$$\mathrm{Fin}^{\mathrm{int}}_{\langle n\rangle/} \sqcup \mathrm{Fin}^{\mathrm{int}}_{\langle 1\rangle/} \longrightarrow \mathrm{Ar}^{\mathrm{int}}_{f_n}(\mathrm{Fin}),$$

which is faithful and essentially surjective. It is not fully faithful, but not many morphisms are missing: The missing ones are $f_n\colon (\mathrm{id}_{\langle n\rangle}\colon \langle n\rangle \to \langle n\rangle) \to (\mathrm{id}_{\langle 1\rangle}\colon \langle 1\rangle \to \langle 1\rangle)$ as well as a unique morphism $(i\colon \langle n\rangle \to \langle n'\rangle) \to (\langle 1\rangle \to \langle 0\rangle)$ for any inert i. In particular, we see that $(\langle 1\rangle \to \langle 0\rangle)$ is a terminal object and we obtain the following picture of $\mathrm{Ar}^{\mathrm{int}}_{f_n}(\mathrm{Fin})$:
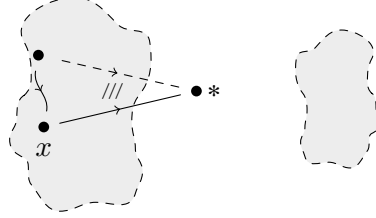
[TODO] $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

8.22. Graded and filtered tensor products. — content...

8.23. The smash product of pointed animae. — content...

8.24. The tensor product of spectra. — content...

§8.3. Homology and cohomology

8.25. The derived tensor product — Let R be a ring. Recall from 6.36 that the category of chain complexes $\mathrm{Ch}(R)$ can be equipped with a Kan enrichment. Let $\mathcal{K}(R) \coloneqq \mathrm{N}^\Delta(\mathrm{Ch}(R))$ denote its simplicial nerve. Also recall the full subcategory $K\text{-}\mathrm{Proj}_R \subseteq \mathrm{Ch}(R)$ of *K-projective complexes.* After taking simplicial nerves, we obtain an inclusion $\mathcal{D}(R) \subseteq \mathcal{K}(R)$.

If R is commutative, then the tensor product of chain complexes $- \otimes_R -$ turns $\mathrm{Ch}(R)$ into a symmetric monoidal Kan-enriched category. Hence $\mathcal{K}(R)$ has a canonical symmetric monoidal structure by Example 8.11(b). It is now straightforward to check that the condition of Lemma 8.9(a) applies to $\mathcal{D}(R) \subseteq \mathcal{K}(R)$: Indeed, the tensor unit $R[0]$ is contained in $\mathcal{D}(R)$, since it is bounded below and degree-wise projective, and if P_* and Q_* are K-projective complexes, then so is $P_* \otimes_R Q_*$ since $\underline{\mathrm{Hom}}_R(P_* \otimes_R Q_*, -) \cong \underline{\mathrm{Hom}}_R(P_*, \underline{\mathrm{Hom}}_R(Q_*, -))$ still preserves quasi-isomorphisms. Thus, we obtain a symmetric monoidal structure on $\mathcal{D}(R)$, which we call the *derived tensor product* $- \otimes_R^{\mathrm{L}} -$. The same argument works for $\mathcal{D}_{\geqslant 0}(R)$; in particular, this shows that the inclusion $\mathcal{D}_{\geqslant 0}(R) \subseteq \mathcal{D}(R)$ can be equipped with a symmetric monoidal structure.

8.26. Theorem. — *The Eilenberg–MacLane functor $\mathcal{D}(\mathbb{Z}) \to \mathrm{Sp}$ from 7.47 upgrades to an equivalence of stable ∞-categories*

$$\mathcal{D}(R) \xrightarrow{\simeq} \mathrm{LMod}_R(\mathrm{Sp})$$

for every ordinary ring R. If R is commutative, then $\mathrm{LMod}_R(\mathrm{Sp}) \simeq \mathrm{Mod}_R(\mathrm{Sp})$ admits a canonical symmetric monoidal structure and the above equivalence can be made strictly monoidal if we equip $\mathcal{D}(R)$ with the derived tensor product $- \otimes_R^{\mathrm{L}} -$ from 8.25. ∎

8.27. Corollary. — *If $X \in \mathrm{An}$ is an anima, then the unreduced and reduced homology and cohomology of X with coefficients in an abelian group A are given by*

$$\mathrm{H}_*(X, A) \cong \pi_*\big(\mathbb{S}[X] \otimes A\big), \qquad \widetilde{\mathrm{H}}_*(X, A) \cong \pi_*\Big(\mathrm{fib}\big(\mathbb{S}[X] \to \mathbb{S}[*]\big) \otimes A\Big),$$

$$\mathrm{H}^*(X, A) \cong \pi_{-*}\mathrm{hom}_{\mathbb{S}}\big(\mathbb{S}[X], A\big), \qquad \widetilde{\mathrm{H}}^*(X, A) \cong \pi_{-*}\mathrm{hom}_{\mathbb{S}}\Big(\mathrm{fib}\big(\mathbb{S}[X] \to \mathbb{S}[*]\big), A\Big).$$

Appendix to §8. A glimpse of higher algebra

§8.4. ∞-Operads

§8.5. $\mathbb{E}_n$-Algebras and iterated loop animae

§8.6. The Lurie tensor product

§9. Cool topology applications 😎

§9. Cool topology applications 😎

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