



A critical analysis of the Weighted Least Squares Monte Carlo method for pricing American options[☆]

R. Mark Reesor^a, Lars Stentoft^b, Xiaotian Zhu^{c,*}

^a Department of Mathematics, Wilfrid Laurier University, Waterloo, ON N2L 3C7, Canada

^b Department of Economics and Department of Statistical and Actuarial Sciences, University of Western Ontario, London, ON N6A 5C2, Canada

^c Department of Statistical and Actuarial Sciences, University of Western Ontario, London, ON N6A 5C2, Canada



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ABSTRACT

Least-squares Monte Carlo generates regression-based continuation value estimators that are heteroscedastic. Fabozzi et al. (2017) propose weighted least-squares regression to correct for this. We show that heteroscedastic-corrected estimators are more accurate than uncorrected estimators far from the exercise boundary and where the exercise decision is obvious. However, the corrected estimators do not translate into improved exercise decisions and hence correcting has little effect on option price estimates. This holds when using alternative specifications for the correction and when implementing an iterative method. We conclude that correcting for heteroscedasticity does not result in more efficient prices and generally should be avoided.

1. Introduction

The least-squares Monte Carlo (LSMC) method of Longstaff and Schwartz (2001) has become a standard method for pricing American options. The method has been shown to converge to the true price, cf. Clément et al. (2002) and Stentoft (2004b), and the method is shown to perform exceptionally well when compared to other regression and simulation based methods like those of Carrière (1996) and Tsitsiklis and Van Roy (2001), cf. Stentoft (2014). However, simulation methods in general, and the LSMC method in particular, are computationally intensive and may result in numerically inefficient price estimates. Therefore much research has gone into improving its performance. A first strand of literature has sought to apply standard variance reduction techniques in this setting. For an early implementation of importance sampling and control variates see Moreni (2003) and Rasmussen (2005), respectively. In Boire et al. (2022) the effect of combining various such methods in the LSMC method is analysed carefully.

The LSMC method uses a regression at each time step to estimate the continuation values of each path and a second strand of literature attempts to improve directly on this regression. Kan and Reesor (2012) and Boire et al. (2023) construct a bias-corrected estimator by subtracting estimated bias from the uncorrected estimator at each exercise opportunity. Choi et al. (2018) and Cheng and Joshi (2017) propose to use leave-one-out type cross-validation techniques to remove overfitting from the estimation of continuation values in the LSMC method. Rasmussen (2005) and Boire et al. (2022) demonstrate that variance reduction techniques can be used when performing the cross sectional regressions and this leads to improved estimates. Another example is Fabozzi et al.

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* Corresponding author.

E-mail addresses: mreesor@wlu.ca (R.M. Reesor), lars.stentoft@uwo.ca (L. Stentoft), xzhu444@uwo.ca (X. Zhu).

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(2017), who argue that the OLS regression is not the best linear unbiased estimator because of heteroscedasticity. They propose to use a weighted-least squares (WLSMC) method to correct it and report results showing that this improves on pricing American basket put options on multiple stocks.

In this paper, we provide further insights on the type of method that seeks to correct for heteroscedasticity in the cross-sectional regressions in the LSMC method. To do so we implement two robust methods, from Park (1966) and Harvey (1976), respectively, for correcting for heteroscedasticity, and we propose to combine each of these with an iterative correction method as suggested in, e.g., Greene (2012). However, our results demonstrate limited potential benefits from this type of approach. First, in a simple example we show that while correcting for heteroscedasticity may improve on the estimated continuation value, this does not necessarily translate into better exercise decisions. The reason is that most of the improvement in the fit from using a weighting scheme occurs for extreme values of the regressors and not around values of importance for making the right exercise decision (i.e., values near the exercise boundary). Thus, this type of correction may have little or no effect on exercise decisions (relative to not correcting) and hence no effect on option price estimates.

Second, we show that the improvements achieved with this method are generally also very limited in a realistic option pricing setting with multiple early exercise times. This holds in particular when standard metrics such as root mean squared errors (RMSE) are used.¹ It also holds when pricing American basket put options as in Fabozzi et al. (2017) or when considering a model with empirically relevant time-varying volatility. Again, in all settings considered here this holds irrespective of the heteroscedasticity correction method used, and for some of these cases using a one-step correction can lead to much larger bias, although this is corrected with our iterative method. Because of the added computational complexity of implementing this type of correction, we conclude that heteroscedasticity corrections do not result in more efficient prices in general and should therefore be avoided.

This rest of the paper proceeds as follows: Section 2 introduces the LSMC method and explains how WLSMC and iteratively re-weighted least-squares (IRLSMC) algorithms are implemented. Section 3 provides simulation results for options with two exercise opportunities where the true continuation values are known. Section 4 compares the performance across various more realistic settings. Finally, Section 5 discusses some limitations and future research areas and Section 6 concludes. The Appendix contains some additional figures.

2. Simulation and regression based option pricing

Consider the price process $V(t, S_t)$ of an American style option under risk-neutral evaluation, given by the formula

$$V(t, S_t) = \sup_{\tau \in [t, T]} \mathbb{E}_t^{\mathbb{Q}}[e^{-r\tau} f(\tau, S_\tau)], \quad (1)$$

where τ is a stopping time, $f(\tau, S_\tau)$ is the payoff function, and $\mathbb{E}_t^{\mathbb{Q}}$ denotes the risk-neutral expectation conditional on information known at time t . The standard method for solving the problem in Eq. (1) is to assume that time can be discretized and to use backward dynamic programming.

There are several numerical ways of estimating the price of the American option. In this paper we consider methods based on simulation and regression and we first detail how the least-squares Monte Carlo method of Longstaff and Schwartz (2001) is implemented. Next, we explain how different heteroscedasticity corrections can be implemented and we propose an iterative method that can be applied to improve the efficiency of this correction.

2.1. The least squares Monte Carlo method

The LSMC method solves the backward dynamic programming problem by approximating the continuation value at each time step using ordinary least squares (OLS) linear regression. The LSMC method proceeds as follows:

1. Discretize time into J time steps and generate a set of N stock paths denoted as S_n^j , where n and j denote simulated path and time, respectively. For example, S_{10}^2 denotes the stock price on the 10th path at time index 2.
2. Calculate the payoffs at maturity T , i.e., P_n^J .
3. Discount P_n^J back one time step as P_n^{J-1} . Let \mathcal{L} denote the set of in-the-money (ITM) paths and $\bar{\mathcal{L}}$ the complement of this.² For an ITM path $\ell \in \mathcal{L}$, construct a regression with polynomials of degree M , where³

$$P_\ell^{J-1} = \sum_{i=0}^M \beta_i^{J-1} (S_\ell^{J-1})^i + \epsilon_\ell. \quad (2)$$

4. Estimate $\hat{\beta}_i^{J-1}$, and calculate the fitted continuation values \hat{P}_ℓ^{J-1} and their corresponding payoff Z_ℓ^{J-1} from immediate exercise. The estimated optimal decision then is to exercise the option if $Z_\ell^{J-1} \geq \hat{P}_\ell^{J-1}$.

¹ Fabozzi et al. (2017), on the other hand, considers only the mean relative error (MRE) which neglects the potentially important effect the heteroscedasticity correction has on the variance of the estimated price.

² ITM paths are paths for which exercising the option immediately would lead to a positive payoff.

³ In our implementation, we divide the stock price by the strike price (i.e., use powers of (S/K) as regressors), a commonly-used method to help ensure numerical stability.

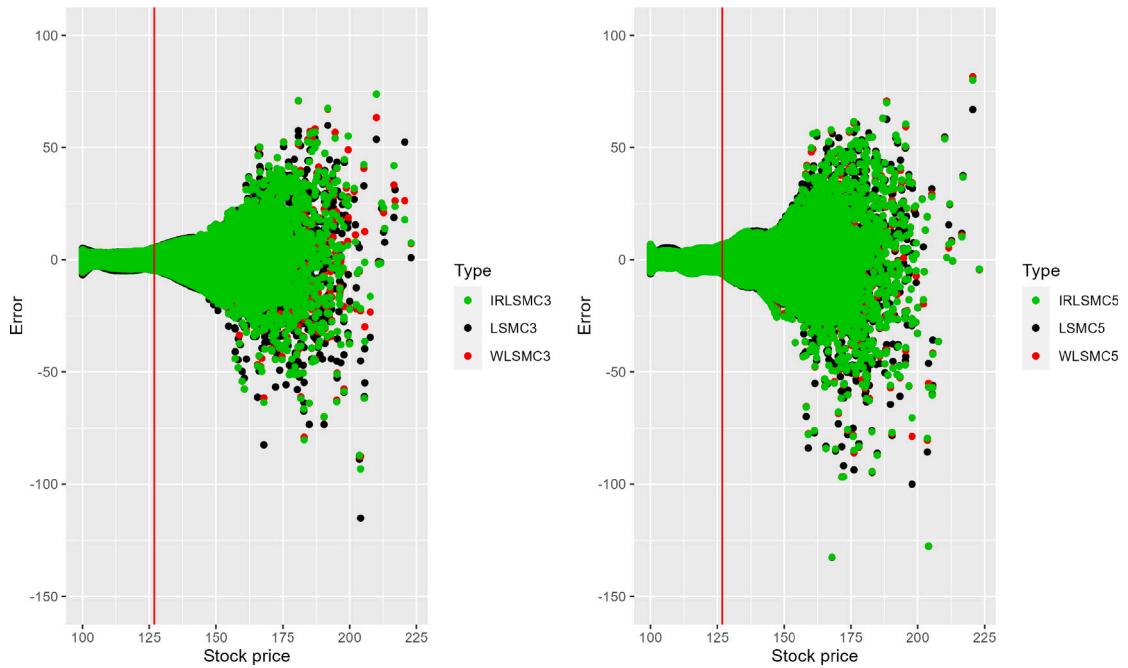


Fig. 1. This figure plots the estimated continuation value errors against stock price using LSMC (in black), Method 1 WLSMC (in red), and Method 1 IRLSMC (in green) with $N = 1000$ simulated paths and $I = 1000$ repetitions. The option characteristics are those from Table 1. Left and right panels correspond to 3rd and 5th order polynomials. The red line is the true exercise boundary with the exercise region lying to the right of the boundary as this is a call option.

5. Update the pathwise payoffs as follows:

$$P_{\ell}^{J-1} = \begin{cases} Z_{\ell}^{J-1}, & Z_{\ell}^{J-1} \geq \hat{P}_{\ell}^{J-1} \\ P_{\ell}^{J-1}, & Z_{\ell}^{J-1} < \hat{P}_{\ell}^{J-1}. \end{cases} \quad (3)$$

That is, for paths that are optimally exercised the exercise value is used whereas for paths that are not optimally exercised the discounted payoffs continue to be used.

6. Work backwards in time repeating steps 3-5 until time step $j = 1$.

7. The final estimated price \hat{V} is given by

$$\hat{V} = \frac{e^{-r\Delta t}}{N} \sum_{n=1}^N P_n^1, \quad (4)$$

where $\Delta t = \frac{T}{J}$. That is, the price is estimated by the average discounted pathwise payoffs.

Key to the implementation of this method is the cross-sectional regression in Eq. (2). If this is to yield an optimal fit certain assumptions need to be satisfied, including homoscedasticity of the error term. Fig. 1 plots the errors that are obtained for a particular option in black and clearly demonstrates that the size of the errors depends on the stock price. In other words, the assumption of homoscedasticity is clearly questionable in this setting. In line with this finding, Fabozzi et al. (2017) showed that the assumption of homoscedastic errors is generally not satisfied, and to counter this they proposed that instead of using OLS for the regression, weighted least squares (WLS) should be used. They name this method the WLSMC.

2.2. Heteroscedasticity corrections to the LSMC method

In this paper, we consider two approaches to correct for heteroscedasticity originating in Park (1966) and Harvey (1976), respectively, in a WLSMC method. Similar to what is done in Fabozzi et al. (2017), we use these two methods in the two step algorithm outlined in Greene (2012) to correct for heteroscedasticity in each of the cross-sectional regressions. While both applications have the same second step they differ in terms of the first step which details how the heteroscedasticity in the fitted residuals, e_{ℓ} , is modelled.

For Method 1, which is based on Park (1966), in the first step we assume the following structure for the errors:

$$(e_{\ell})^2 = \sigma^2 \exp^{v_{\ell}} (S_{\ell}/K)^{\lambda}, \quad (5)$$

where e_ℓ represent the fitted residual from Eq. (2) and v_ℓ is an error term. As noted above, to prevent possible numerical issues caused by large values from the predictors we use powers of the stock price divided by the strike (e.g., powers of (S_ℓ/K)) as regressors in Eq. (2) for both LSMC and WLSMC. Equivalently we may write Eq. (5) as

$$\ln(e_\ell)^2 = \ln\sigma^2 + \lambda \ln(S_\ell/K) + v_\ell, \quad (6)$$

and we can estimate the unknown parameters, i.e., $\ln\sigma^2$ and λ , using OLS.

For Method 2, which is based on Harvey (1976), in the first step we assume the following structure for the errors:

$$\ln(e_\ell)^2 = \alpha_0 + \alpha_1(S_\ell/K) + v_\ell. \quad (7)$$

Again we can estimate the unknown parameters, in this case α_0 and α_1 , using OLS. As stated in Harvey (1976), to provide a consistent estimate of $\alpha = (\alpha_0, \alpha_1)$, $\hat{\alpha}$ should be corrected with $\hat{\alpha} = (\hat{\alpha}_0 + 1.2704, \hat{\alpha}_1)$.

The second step is the same for both methods. Let \hat{W} represent a diagonal matrix with elements equal to the fitted values from the regressions in Eq. (6) for Method 1 and Eq. (7) for Method 2, respectively. Then the coefficient estimates from the WLS regression, $\hat{\beta}^W$, can be calculated as

$$\hat{\beta}^W = (X^T \hat{W} X)^{-1} X^T \hat{W} Y. \quad (8)$$

Using these coefficients, heteroscedasticity corrected estimated continuation values are calculated as $\hat{P}_\ell = \sum_{i=0}^M \hat{\beta}_i^W \left(\frac{S_\ell}{K}\right)^i$, and in the WLSMC these are used in place of \hat{P}_ℓ in Step 4 of the LSMC method.

The WLSMC method considered in Fabozzi et al. (2017) is similar to our Method 2 but differs in two important aspects. First, while we consider only a first order approximation Fabozzi et al. (2017) use a quadratic function. We find little indication that a second order term is needed, cf. Fig. 1, and our results do not change in any material way when a second order term is used. Second, and most importantly, while we model $\ln(e_\ell)^2$, Fabozzi et al. (2017) choose to model $(e_\ell)^2$. In our tests we have found that their approach often results in negative weights in the WLS regression, causing WLS to break down.

Besides using these two alternative and more robust methods for implementing WLS, we also consider a straightforward generalization of these termed iterative re-weighted least squares (IRLS) resulting in a pricing method we call IRLSMC. Following Greene (2012), the procedure for conducting IRLS involves, simply described, re-computing the residual and re-weighting in the regression. More specifically IRLS proceeds with the following steps:

1. Let $\hat{\beta}_0^W$ and e_0^W represent the coefficients and residuals from the OLS regression.
2. Estimate the heteroscedasticity regression in Eq. (6) for Method 1 or Eq. (7) for Method 2, respectively, use these to construct \hat{W} , and calculate the WLS estimates from Eq. (8).
3. Let $\hat{\beta}_k^W$ and e_k^W represent the coefficients and residuals from this k th iteration.
4. Continue step 2 and 3 until $\|\hat{\beta}_k^W - \hat{\beta}_{k-1}^W\|$ is less than a specified tolerance level or a maximum iteration number is reached.⁴

The IRLS method has the same asymptotic properties as the WLS estimators. And although Greene (2012) states that IRLS may provide little additional benefit, we will consider to what extent the IRLS improves on the results from WLS.⁵

3. Results for a two period option

In this section, we investigate the effect the proposed heteroscedasticity corrections with WLSMC and IRLSMC have on (i) the estimated continuation values; and (ii) the resulting estimated option prices when compared with the standard LSMC. The model we consider is a univariate GBM for the underlying assets and we assume that the option has only one early exercise time. When there are only two exercise opportunities (including one at maturity), the true continuation values at the intermediate time step can be calculated by the Black–Scholes formula, cf. Black and Scholes (1973). This true value can now be compared with the estimated continuation values from either LSMC or the modified WLSMC/IRLSMC and gives us an error E_n defined as

$$E_n = P_n - \hat{P}_n, \quad (9)$$

for the n th path in a simulation. Note that the time superscript is suppressed here, as these are all quantities at the intermediate time step.

Throughout this section, we perform I repetitions each of sample size N and for the i th repetition we let \mathcal{L}_i and $\ell_i = |\mathcal{L}_i|$ denote the set of and number of simulated ITM paths at the time of early exercise, respectively, for $1 \leq i \leq I$. Additionally, let n_i denote the n th path from repetition i and $\mathcal{L} = \bigcup_{i=1}^I \mathcal{L}_i$ denote the set of all ITM paths across all repetitions, giving $\sum_{i=1}^I \ell_i = |\mathcal{L}|$ as the total number of ITM paths. Moreover, we fit all LSMC, WLSMC and IRLSMC with 3rd and 5th order polynomials. In the tables and figures that follow, we denote them with the polynomial order attached to LSMC, WLSMC or IRLSMC, e.g., WLSMC3 means that a third order polynomial is used in the WLSMC method. We also set the stopping criteria for IRLSMC to be a maximum number of 10 iterations and set a tolerance value of 0.01. Note that the analysis presented in this section and the resulting insights are absent from Fabozzi et al. (2017).

⁴ The second criterion ensures that the loop stops in the rare occasions that the method does not converge.

⁵ Note that WLS is, of course, a special case of the IRLS where the maximum iteration number is 1.

Table 1
Continuation value errors.

Type	Met	$N = 1000$		$N = 10,000$	
		MRE	RMSE	MRE	RMSE
LSMC3	–	0.072 (0.001)	1.878 (0.024)	0.024 (0.001)	0.629 (0.023)
WLSMC3	1	0.069 (0.001)	1.834 (0.024)	0.022 (0.001)	0.607 (0.023)
IRLSMC3	1	0.069 (0.001)	1.838 (0.024)	0.022 (0.001)	0.607 (0.023)
WLSMC3	2	0.069 (0.001)	1.852 (0.024)	0.022 (0.001)	0.621 (0.024)
IRLSMC3	2	0.069 (0.001)	1.861 (0.024)	0.022 (0.001)	0.621 (0.024)
LSMC5	–	0.083 (0.001)	2.361 (0.026)	0.025 (0.001)	0.800 (0.026)
WLSMC5	1	0.081 (0.001)	2.342 (0.026)	0.024 (0.001)	0.788 (0.025)
IRLSMC5	1	0.081 (0.001)	2.343 (0.026)	0.024 (0.001)	0.788 (0.025)
WLSMC5	2	0.081 (0.001)	2.349 (0.026)	0.024 (0.001)	0.794 (0.025)
IRLSMC5	2	0.081 (0.001)	2.353 (0.026)	0.024 (0.001)	0.795 (0.025)

This table compares the continuation value errors across different types of polynomial regressions using different specifications of the heteroscedasticity method (Met) and using different number of paths in the simulation (N). The call option characteristics are as follows: $S_0 = 100$, $K = 100$, $r = q = 0.06$, $T = 1$, $\sigma = 0.25$, and we consider $J = 2$ exercise opportunities. The total computational cost is set to be 1,000,000 and we report averages across the $I = 1,000,000/N$ repetitions with standard errors in brackets.

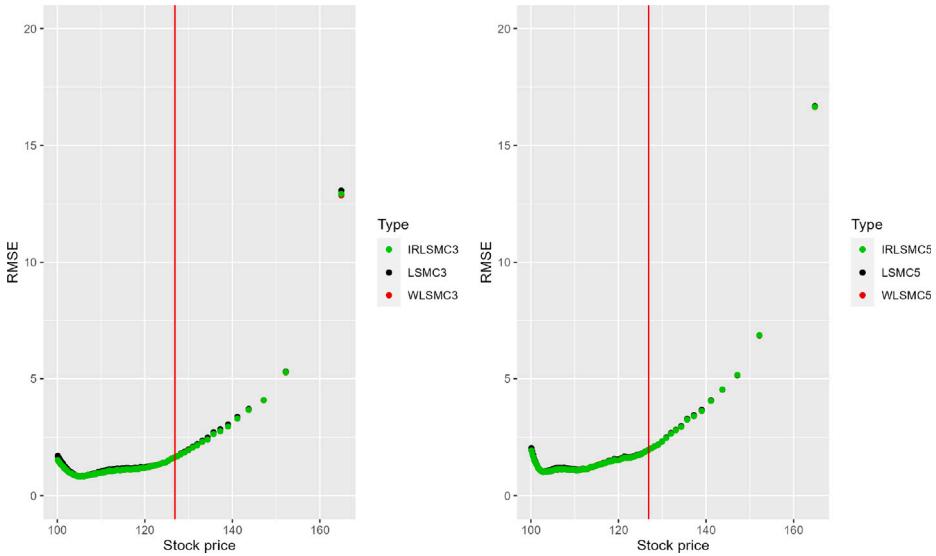


Fig. 2. This figure plots the RMSE of the estimated continuation values against stock price percentile using LSM (in black), Method 1 WLSMC (in red), and Method 1 IRLSMC (in green) with $N = 1000$ simulated paths and $I = 1000$ repetitions. The option characteristics are those from Table 1. Left and right panels correspond to 3rd and 5th order polynomials. The red line is the true exercise boundary with the exercise region to the right of the boundary.

3.1. Continuation value errors

The first criteria we consider are the mean relative error (MRE) and the root mean squared error (RMSE) between the true continuation value and the estimated continuation value calculated as

$$MRE = \frac{\sum_{i=1}^I \sum_{n_i \in \mathcal{L}_i} |E_{n_i} - P_{n_i}|}{|\mathcal{L}|} \text{ and } RMSE = \sqrt{\frac{\sum_{i=1}^I \sum_{n_i \in \mathcal{L}_i} (E_{n_i} - P_{n_i})^2}{|\mathcal{L}|}}, \quad (10)$$

respectively, where P_{n_i} is the true continuation value and E_{n_i} is the error from Eq. (9) for path n_i .

The two error metrics from Eq. (10) are reported in Table 1 for different methods of the polynomial regression and number of paths used in the simulation. The table shows that compared to LSMC, WLSMC using Methods 1 and 2 yields lower MRE and RMSE and hence provides a more accurate estimation of the continuation value. WLSMC using Method 2 performs slightly worse than when using Method 1 in terms of RMSE. The proposed iterative algorithm IRLSMC, however, does not give additional benefits in any of the cases and in fact may lead to worse performance when the number of paths (N) used is low and when Method 2 is used.

Table 2
 2×2 contingency tables of exercise decisions.

Type	Met	True exercise	$N = 1000$			$N = 10,000$		
			Estim. exercise			Estim. exercise		
			Yes	No	Cor.	Yes	No	Cor.
LSMC3	-	Yes	0.0454	0.0292	0.9387	0.0602	0.0151	0.9715
		No	0.0321	0.8933	0.0613	0.0135	0.9112	0.0285
WLSMC3	1	Yes	0.0451	0.0295	0.9388	0.0590	0.0163	0.9704
		No	0.0317	0.8937	0.0612	0.0133	0.9114	0.0296
IRLSMC3	1	Yes	0.0451	0.0294	0.9388	0.0590	0.0163	0.9704
		No	0.0318	0.8937	0.0612	0.0134	0.9114	0.0296
WLSMC3	2	Yes	0.0450	0.0296	0.9387	0.0587	0.0166	0.9702
		No	0.0316	0.8937	0.0612	0.0133	0.9115	0.0298
IRLSMC3	2	Yes	0.0450	0.0296	0.9388	0.0587	0.0166	0.9702
		No	0.0316	0.8938	0.0612	0.0133	0.9115	0.0298
LSMC5	-	Yes	0.0433	0.0312	0.9321	0.0553	0.0200	0.9648
		No	0.0367	0.8888	0.0679	0.0151	0.9096	0.0352
WLSMC5	1	Yes	0.0434	0.0312	0.9329	0.0557	0.0196	0.9665
		No	0.0359	0.8895	0.0671	0.0139	0.9108	0.0335
IRLSMC5	1	Yes	0.0434	0.0312	0.9328	0.0557	0.0196	0.9665
		No	0.0360	0.8894	0.0672	0.0139	0.9108	0.0335
WLSMC5	2	Yes	0.0433	0.0312	0.9328	0.0558	0.0195	0.9665
		No	0.0360	0.8895	0.0672	0.0140	0.9108	0.0335
IRLSMC5	2	Yes	0.0433	0.0313	0.9328	0.0558	0.0195	0.9666
		No	0.0359	0.8895	0.0672	0.0140	0.9107	0.0335

This table shows the 2×2 contingency tables of exercise decisions using the true continuation value and the estimated continuation value at time $t = 1$. The column headed “Cor.” reports the proportion of time the exercise decision is correct, i.e, the sum of the diagonal elements, in rows labelled “Yes” and the proportion of time the exercise decision is wrong, i.e., the sum of the off-diagonal elements, in rows labelled “No”. The option characteristics are those from Table 1.

Next, we split all ITM paths in \mathcal{L} into 100 bins and plot the RMSE of the estimated continuation values compared to the true values against the mean stock price in each bin. Fig. 2 displays the results of RMSE in each bin with Method 1.⁶ The figure shows that smaller RMSE are obtained for lower stock prices (ITM but not in the exercise region), consistent with the error plot in Fig. 1. However, the figure indicates that the heteroscedasticity corrected estimators do not systematically outperform the non-corrected one for most of the stock price range of values. In particular there is little difference between corrected and uncorrected estimators near the exercise boundary where this would be most important. Small differences between corrected and non-corrected estimators are evident far from the exercise boundary, where the exercise decision is not in doubt. Therefore, there is no clear indication that the corrected estimators yield more optimal exercise decisions and improved price estimates even though they may fit the continuation value better.

3.2. Early exercise decision errors

As a second criteria, we examine whether the estimated continuation value leads to the correct exercise/hold decision at time step 1. To examine this we construct 2×2 frequency tables detailing the frequency of correct and incorrect decisions. In terms of option pricing, this criterion is more important than the previous ones, since it measures directly the improvement in terms of optimal decision making. A perfect method will have all the mass on the diagonals in these relative frequency tables and provide the best estimate of the option price for a given sample of simulated paths.

The resulting contingency tables are presented in Table 2, which also reports the proportion of correct decisions made when estimating the continuation value. The first thing to notice is that whereas the previous section showed that WLSMC may lead to more accurate estimated continuation values, Table 2 shows that there is little improvement in terms of the decisions made from the estimated continuation values of the LSMC, WLSMC and IRLSMC methods, respectively. Again this is important when it comes to pricing options since option prices are averages of optimally exercised discounted cash-flows as shown in Eq. (4) and methods that yield similar exercise decisions result in the same price.

In fact, when it comes to making the correct decisions the proportions differ at most on the third decimal.⁷ The largest differences are found when using $N = 10,000$ paths. However, in this case Table 2 shows that when using a polynomial of order $M = 3$ the LSMC actually leads to the highest number of correct decisions. With a polynomial of order $M = 5$ WLSMC or IRLSMC improves on the LSMC, though the frequency of correct decisions is lower here than with a polynomial of order $M = 3$. It is also noteworthy

⁶ The corresponding results for Method 2 are shown in Fig. 8 in the Appendix and are virtually identical.

⁷ With $M = 3$ and $N = 1000$ the frequencies only differ on the fourth decimal.

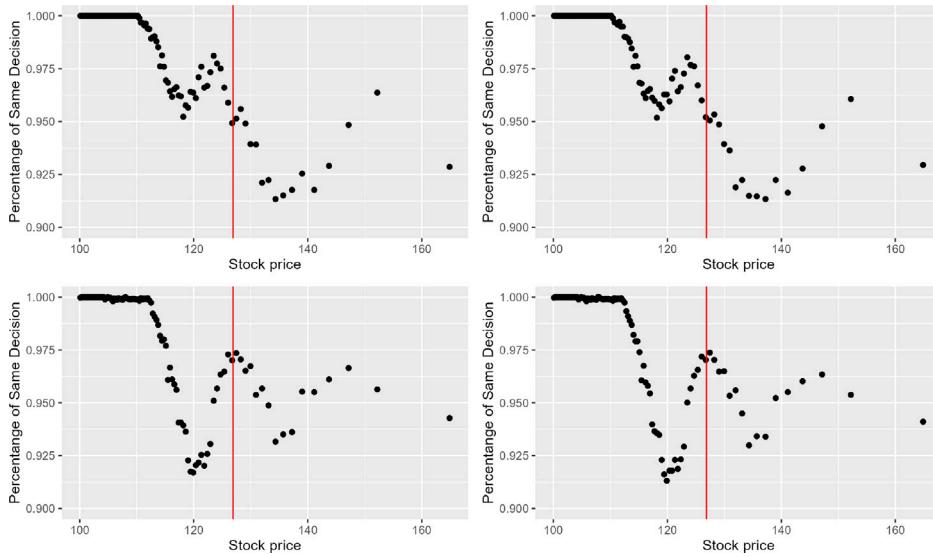


Fig. 3. This figure plots the proportions for which the same decision is made against stock price percentile for LSMC3 vs. WLSMC3 (top left), LSMC3 vs. IRLSMC3 (top right), LSMC5 vs. WLSMC5 (bottom left), and LSMC5 vs. IRLSMC5 (bottom right) across the same bins as in Fig. 2 using Method 1. $N = 1000$ simulated paths with $I = 1000$ repetitions were used in pricing ATM American Call option. The option characteristics are those from Table 1. The red line is the true exercise boundary with the exercise region to the right of the boundary.

that, for a given polynomial order and number of paths, the proportions of correct decisions differ only on the fourth decimal for the heteroscedasticity corrected methods.

Next, we compare the exercise decisions made by LSMC, WLSMC, and IRLSMC across the 100 stock-price bins used in Fig. 2 and we plot the percentages of paths where the same decision is made by two methods, e.g., LSMC vs. WLSMC or LSMC vs. IRLSMC, against the mean stock price in each bin. The results are shown in Fig. 3 for Method 1.⁸ By plotting the proportion of times the same decision is made by two methods in each bin, we find that at any stock price, over 90% of the time LSMC and the heteroskedastic corrected WLSMC/IRLSMC make the same decision. This percentage increases to 95% when the stock price approaches the early exercise boundary and therefore the methods also result in almost identical price estimates.

4. General results

The results from the previous section indicate that while WLSMC/IRLSMC may improve on the fit of the estimated continuation values this does not appear to translate into improved early exercise decisions. In this section, we consider the impact this has on the estimated option price. The criteria we use to compare the performance of LSMC, WLSMC and IRLSMC are the bias of the estimated option values, the standard deviation of the price estimate, and the MRE and RMSE of the estimates calculated from

$$MRE = \frac{\sum_{i=1}^I \left| \frac{\hat{V}_i - V}{V} \right|}{I} \text{ and } RMSE = \sqrt{\frac{\sum_{i=1}^I (\hat{V}_i - V)^2}{I}}, \quad (11)$$

respectively, where \hat{V}_i represents the estimated price from the i th repetition of sample size N , $1 \leq i \leq I$ where I denotes the number of repetitions, and where V represents the benchmark price.

We first consider the case in which the underlying dynamics are those from a simple univariate GBM where we extend the exercise opportunities from two to many and investigate the pricing performance of WLSMC/IRLSMC compared to LSMC. Next, we consider extensions of this model to the case with a multivariate model with basket options and the case with time-varying volatility for the underlying dynamics.

4.1. Results for single asset options

We start out by considering the option price estimates from the previous section with two exercise opportunities, the results for which are reported in Table 3. The table reports results for polynomials of order $M = 3$ and $M = 5$ and when using $N = 1000$ and $N = 10,000$ simulated paths. For each (M, N) combination the best performing model, i.e., the model with the smallest errors is shown in *italics* whereas the minimum error overall is shown in **bold**. The first thing to notice from the table is that the bias of the

⁸ The corresponding results for Method 2 are shown in Fig. 9 in the Appendix and are virtually identical.

Table 3
American call option with $J = 2$ exercise opportunities.

Type	Met	$N = 1000$				$N = 10,000$			
		Bias	StDev	MRE	RMSE	Bias	StDev	MRE	RMSE
LSMC3	–	0.1061	0.4829	0.0416	0.4941	0.0127	0.1348	0.0117	0.1347
WLSMC3	1	0.1052	0.4791	0.0415	0.4903	0.0123	0.1329	0.0114	0.1328
IRLSMC3	1	0.1056	0.4791	0.0414	0.4904	0.0125	0.1326	0.0114	0.1327
WLSMC3	2	0.1067	0.4783	0.0414	0.4898	0.0118	0.1325	0.0114	0.1324
IRLSMC3	2	0.1068	0.4774	<i>0.0413</i>	<i>0.4890</i>	0.0120	0.1328	0.0114	0.1327
LSMC5	–	0.1381	0.4782	0.0419	0.4975	0.0204	0.1326	0.0115	0.1335
WLSMC5	1	0.1386	0.4763	<i>0.0419</i>	<i>0.4958</i>	0.0165	0.1322	0.0115	0.1326
IRLSMC5	1	0.1391	0.4764	0.0419	0.4960	0.0166	0.1323	0.0115	0.1327
WLSMC5	2	0.1386	0.4768	0.0420	0.4963	0.0170	0.1321	<i>0.0114</i>	<i>0.1326</i>
IRLSMC5	2	0.1388	0.4766	0.0419	0.4962	0.0174	0.1325	0.0115	0.1329

This table shows the bias and standard deviation, along with the resulting mean relative error (MRE) and root mean squared error (RMSE), of $I = 1,000,000/N$ price estimates, for the American call option from Table 1. The true price calculated with the explicit finite difference method from Brennan and Schwartz (1977) is 9.422. The minimum error for each combination of polynomial order and simulated sample size is shown in *italics* whereas the minimum error overall is shown in **bold**.

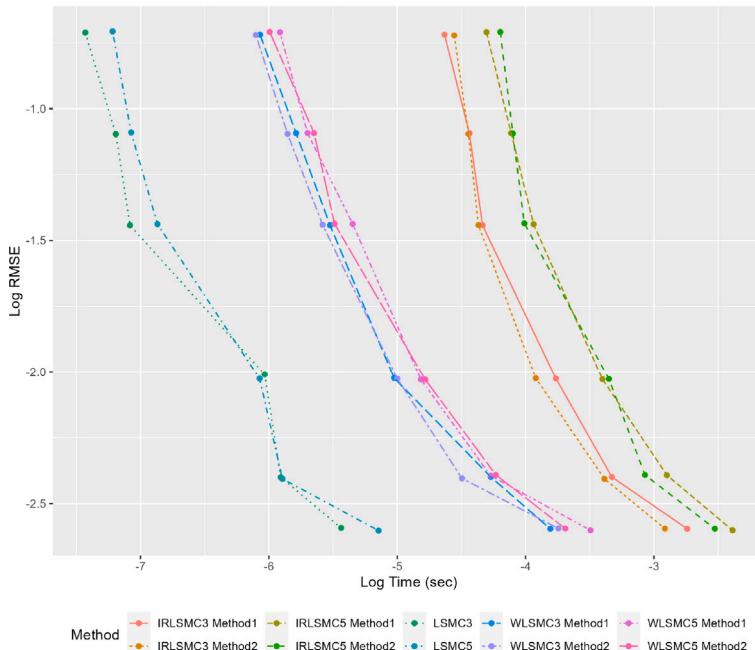


Fig. 4. This figure plots the log RMSE against the log mean computational time for each of the methods considered. $I = 1000$ repetitions were used with a number of simulated paths $N \in \{1000, 2000, 4000\}$, and $I = 100$ repetitions were used with a number of simulated paths $N \in \{10,000, 20,000, 40,000\}$. The option considered has the same parameters as in Table 3.

estimated prices are surprisingly close and so are the standard deviations. In general, the bias differs only at the third decimal place and the largest differences are found when using a low order polynomial and few simulated paths. Because of this the resulting mean relative error (MRE) and root mean squared error (RMSE) are also very close for similar polynomial orders and simulated sample sizes.

Table 3 also shows that there is much more value to increasing the sample size than to attempt to correct for heteroscedasticity. For example, when increasing the sample size from $N = 1000$ to $N = 10,000$ for the LSMC with a third order polynomial the RMSE drops from 0.5650 to 0.1396 whereas implementing the IRLSMC method only reduces the RMSE to 0.5642. Computational time is essentially linear in the sample size, and it is thus interesting to examine the trade-off between RMSE and computational time to see if one method dominates the other. To do so, Fig. 4 plots log RMSE against log mean computational time for different numbers of simulated paths for each of the methods considered. This figure clearly shows that the simple LSMC methods dominate any of the weighted regression methods.

To complement the results presented in Table 3 for 2 early exercise opportunities, we report results corresponding to 25, 50, and 100 exercise opportunities (keeping the option maturity time fixed) in Tables 4–6, respectively. These tables show that the conclusions from above extend to situations with more exercise opportunities. In particular, there is often very little difference

Table 4
American call option with $J = 25$ exercise opportunities.

Type	Met	$N = 1000$				$N = 10,000$			
		Bias	StDev	MRE	RMSE	Bias	StDev	MRE	RMSE
LSMC3	–	0.3522	0.4428	0.0484	0.5650	0.0436	0.1337	0.0123	0.1396
WLSMC3	1	0.3497	0.4454	0.0485	0.5655	0.0413	0.1378	0.0128	0.1430
IRLSMC3	1	0.3464	0.4499	0.0485	0.5670	0.0420	0.1385	0.0127	0.1438
WLSMC3	2	0.3435	0.4522	0.0484	0.5671	0.0372	0.1344	0.0124	0.1386
IRLSMC3	2	0.3416	0.4500	<i>0.0482</i>	<i>0.5642</i>	0.0387	0.1349	0.0124	0.1394
LSMC5	–	0.4927	0.4432	0.0582	0.6618	0.0813	0.1381	<i>0.0137</i>	0.1592
WLSMC5	1	0.4901	0.4435	<i>0.0578</i>	0.6601	0.0784	0.1386	0.0139	0.1581
IRLSMC5	1	0.4910	0.4390	0.0579	<i>0.6578</i>	0.0773	0.1392	0.0137	0.1582
WLSMC5	2	0.4932	0.4414	0.0580	0.6609	0.0761	0.1399	0.0139	0.1580
IRLSMC5	2	0.4925	0.4407	0.0582	0.6600	0.0779	0.1386	0.0137	<i>0.1579</i>

This table shows the bias and standard deviation, along with the resulting mean relative error (MRE) and root mean squared error (RMSE), of $I = 1,000,000/N$ price estimates, for the American call option from Table 1 with $J = 25$ exercise opportunities. The benchmark price calculated with the explicit finite difference method from Brennan and Schwartz (1977) is 9.493. The minimum error for each combination of polynomial order and simulated sample size is shown in *italics* whereas the minimum error overall is shown in **bold**.

Table 5
American call option with $J = 50$ exercise opportunities.

Type	Met	$N = 1000$				$N = 10,000$			
		Bias	StDev	MRE	RMSE	Bias	StDev	MRE	RMSE
LSMC3	–	0.3871	0.4527	0.0504	0.5955	0.0416	0.1322	0.0114	0.1380
WLSMC3	1	0.3644	0.4667	0.0503	0.5920	0.0329	0.1370	0.0113	0.1402
IRLSMC3	1	0.3693	0.4560	0.0595	0.5866	0.0339	0.1375	0.0115	0.1409
WLSMC3	2	0.3561	0.4768	0.0505	0.5949	0.0198	0.1494	0.0122	0.1500
IRLSMC3	2	0.3604	0.4624	<i>0.0497</i>	<i>0.5861</i>	0.0205	0.1452	0.0121	0.1459
LSMC5	–	0.5584	0.4452	0.0626	0.7140	0.0847	0.1378	0.0134	0.1612
WLSMC5	1	0.5380	0.4517	<i>0.0613</i>	<i>0.7024</i>	0.0793	0.1407	0.0135	0.1609
IRLSMC5	1	0.5451	0.4529	0.0618	0.7085	0.0762	0.1408	0.0131	0.1595
WLSMC5	2	0.5398	0.4530	0.0616	0.7046	0.0716	0.1372	<i>0.0127</i>	<i>0.1542</i>
IRLSMC5	2	0.5396	0.4564	0.0618	0.7066	0.0709	0.1448	0.0134	0.1605

This table shows the bias and standard deviation, along with the resulting mean relative error (MRE) and root mean squared error (RMSE), of $I = 1,000,000/N$ price estimates, for the American call option from Table 1 with $J = 50$ exercise opportunities. The benchmark price calculated with the explicit finite difference method from Brennan and Schwartz (1977) is 9.497. The minimum error for each combination of polynomial order and simulated sample size is shown in *italics* whereas the minimum error overall is shown in **bold**.

Table 6
American call option with $J = 100$ exercise opportunities.

Type	Met	$N = 1000$				$N = 10,000$			
		Bias	StDev	MRE	RMSE	Bias	StDev	MRE	RMSE
LSMC3	–	0.4087	0.4428	0.0516	0.6025	0.0360	0.1451	0.0132	0.1488
WLSMC3	1	0.2512	0.6432	<i>0.0561</i>	0.6903	0.0216	0.1552	0.0134	0.1560
IRLSMC3	1	0.3352	0.4836	0.0498	0.5883	0.0223	0.1494	0.0131	0.1503
WLSMC3	2	0.1937	0.6947	0.0580	0.7209	-0.0947	0.4376	0.0230	0.4455
IRLSMC3	2	0.3182	0.4883	<i>0.0491</i>	<i>0.5827</i>	-0.0239	0.1813	0.0149	0.1820
LSMC5	–	0.5914	0.4337	0.0656	0.7334	0.1004	0.1352	0.0139	<i>0.1680</i>
WLSMC5	1	0.5443	0.4806	0.0641	0.7261	0.0322	0.2149	0.0160	0.2163
IRLSMC5	1	0.5589	0.4517	0.0633	0.7169	0.0543	0.1607	<i>0.0137</i>	0.1688
WLSMC5	2	0.5259	0.4940	0.0635	0.7215	-0.0176	0.3133	0.0191	0.3122
IRLSMC5	2	0.5393	0.4613	<i>0.0624</i>	<i>0.7096</i>	0.0312	0.1765	0.0144	0.1784

This table shows the bias and standard deviation, along with the resulting mean relative error (MRE) and root mean squared error (RMSE), of $I = 1,000,000/N$ price estimates, for the American call option from Table 1 with $J = 100$ exercise opportunities. The benchmark price calculated with the explicit finite difference method from Brennan and Schwartz (1977) is 9.499. The minimum error for each combination of polynomial order and simulated sample size is shown in *italics* whereas the minimum error overall is shown in **bold**.

between the bias of estimated prices obtained with the five methods for a given choice of polynomial order and simulated sample size. The only exception to this is when there are 100 exercise points for which Table 6 shows that in several cases the bias and standard deviation with the WLSM for both methods differ a lot. This is not the case when using an iterative method and the results

Table 7
Two asset American basket put option with $J = 13$ exercise opportunities.

Type	Met	$N = 1000$				$N = 10,000$			
		Bias	StDev	MRE	RMSE	Bias	StDev	MRE	RMSE
LSMC3	–	0.1717	0.1324	0.0582	0.2169	0.0554	0.0403	0.0185	0.0685
WLSMC3	1	0.1689	0.1344	0.0577	0.2159	0.0546	0.0398	0.0185	0.0676
IRLSMC3	1	0.1696	0.1334	0.0578	0.2159	0.0546	0.0401	0.0185	0.0677
WLSMC3	2	0.1706	0.1346	0.0580	0.2174	0.0558	0.0400	0.0188	0.0687
IRLSMC3	2	0.1698	0.1332	<i>0.0576</i>	0.2159	0.0552	0.0403	0.0188	0.0683
LSMC5	–	0.2578	0.1321	0.0828	0.2898	0.0770	0.0382	0.0247	0.0859
WLSMC5	1	0.2567	0.1330	<i>0.0825</i>	0.2891	0.0772	0.0378	0.0247	0.0859
IRLSMC5	1	0.2577	0.1330	0.0827	0.2901	0.0768	0.0382	<i>0.0246</i>	<i>0.0857</i>
WLSMC5	2	0.2577	0.1341	0.0828	0.2906	0.0778	0.0383	0.0249	0.0867
IRLSMC5	2	0.2585	0.1335	0.0830	0.2910	0.0779	0.0387	0.0250	0.0870

This table shows the average bias and standard deviation, along with the resulting mean relative error (MRE) and root mean squared error (RMSE) for an American basket put option on two assets with characteristics similar to those used in [Fabozzi et al. \(2017\)](#). The benchmark price from [Kovalov et al. \(2007\)](#) is 3.137. The minimum error for each combination of polynomial order and simulated sample size is shown in *italics* whereas the minimum error overall is shown in **bold**.

Table 8
Three asset American basket put option with $J = 13$ exercise opportunities.

Type	Met	$N = 1000$				$N = 10,000$			
		Bias	StDev	MRE	RMSE	Bias	StDev	MRE	RMSE
LSMC3	–	0.2346	0.1194	0.0804	0.2632	0.0621	0.0375	0.0215	0.0724
WLSMC3	1	<i>0.2327</i>	0.1204	0.0799	0.2620	0.0615	0.0385	0.0214	0.0724
IRLSMC3	1	0.2322	0.1210	0.0798	0.2618	0.0627	0.0385	0.0218	0.0735
WLSMC3	2	0.2325	0.1193	0.0797	0.2613	0.0615	0.0386	0.0214	0.0725
IRLSMC3	2	0.2320	0.1197	<i>0.0797</i>	<i>0.2610</i>	0.0621	0.0382	0.0216	0.0728
LSMC5	–	0.4297	0.1159	0.1460	0.4450	0.1034	0.0388	<i>0.0351</i>	<i>0.1104</i>
WLSMC5	1	0.4297	0.1166	0.1460	0.4450	0.1059	0.0389	0.0360	0.1127
IRLSMC5	1	0.4268	0.1177	<i>0.1450</i>	<i>0.4427</i>	0.1044	0.0397	0.0355	0.1116
WLSMC5	2	0.4299	0.1176	0.1460	0.4457	0.1049	0.0387	0.0356	0.1118
IRLSMC5	2	0.4276	0.1170	0.1452	0.4433	0.1054	0.0387	0.0358	0.1122

This table shows the average bias and standard deviation, along with the resulting mean relative error (MRE) and root mean squared error (RMSE) for an American basket put option on three assets with characteristics similar to those used in [Fabozzi et al. \(2017\)](#). The benchmark price from [Kovalov et al. \(2007\)](#) is 2.944. The minimum error for each combination of polynomial order and simulated sample size is shown in *italics* whereas the minimum error overall is shown in **bold**.

Table 9
Four asset American basket put option with $J = 13$ exercise opportunities.

Type	Met	$N = 1000$				$N = 10,000$			
		Bias	StDev	MRE	RMSE	Bias	StDev	MRE	RMSE
LSMC3	–	0.3179	0.1143	0.1120	0.3378	0.0815	0.0363	0.0287	0.0891
WLSMC3	1	0.3161	0.1170	0.1113	0.3370	0.0839	0.0376	0.0295	0.0919
IRLSMC3	1	0.3128	0.1168	<i>0.1102</i>	<i>0.3338</i>	0.0836	0.0373	0.0294	0.0915
WLSMC3	2	0.3177	0.1169	0.1119	0.3385	0.0833	0.0377	0.0293	0.0914
IRLSMC3	2	0.3148	0.1164	0.1109	0.3356	0.0841	0.0365	0.0296	0.0916
LSMC5	–	0.6827	0.1146	0.2404	0.6922	0.1594	0.0378	<i>0.0561</i>	<i>0.1637</i>
WLSMC5	1	0.6780	0.1143	0.2387	0.6875	0.1602	0.0373	0.0564	0.1644
IRLSMC5	1	0.6341	0.1183	<i>0.2233</i>	<i>0.6450</i>	0.1601	0.0371	0.0564	0.1643
WLSMC5	2	0.6791	0.1148	0.2391	0.6887	0.1609	0.0387	0.0567	0.1654
IRLSMC5	2	0.6431	0.1171	0.2265	0.6537	0.1606	0.0381	0.0566	0.1650

This table shows the average bias and standard deviation, along with the resulting mean relative error (MRE) and root mean squared error (RMSE) for an American basket put option on four assets with characteristics similar to those used in [Fabozzi et al. \(2017\)](#). The benchmark price from [Kovalov et al. \(2007\)](#) is 2.840. The minimum error for each combination of polynomial order and simulated sample size is shown in *italics* whereas the minimum error overall is shown in **bold**.

seem to indicate that the iterative method can correct some of the cases where simple weighting leads to unfortunate results when there are many exercise opportunities.

4.2. Results with basket options or with alternative dynamics

In [Fabozzi et al. \(2017\)](#) the focus is primarily on basket options on multiple underlying assets. Therefore, as a first extension of the results above [Tables 7, 8, and 9](#) report the corresponding results for basket options on 2, 3, and 4 assets, respectively. Together,

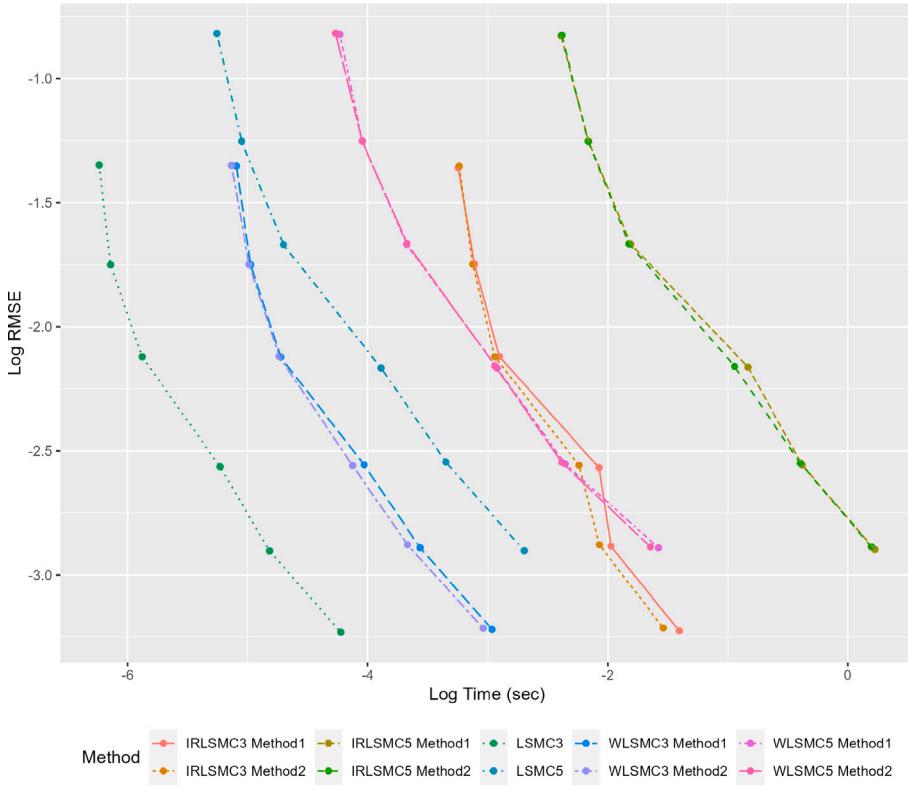


Fig. 5. This figure plots the log RMSE against the log mean computational time for each of the methods considered for three asset basket option. $I = 1000$ repetitions were used with a number of simulated paths $N \in \{1000, 2000, 4000\}$, and $I = 100$ repetitions were used with a number of simulated paths $N \in \{10,000, 20,000, 40,000\}$. The option considered has the same parameters as in Table 8.

these tables confirm our findings from above and show that there is only marginal improvement from using weighted regression techniques if any at all. The largest improvements are found when a low order of the polynomial and a small number of paths are used. In the two asset case for example, the WLSMC with Method 1 improves on the LSMC but only marginally with the MRE decreasing from 0.0582 to 0.0577.

Moreover, as it was the case with the plain vanilla option above, the best results are obtained with a polynomial of order $M = 3$ and with $N = 10,000$ simulated paths and this holds across the dimensionality of the problem in Tables 7, 8, and 9. These tables also show that though WLSMC with Method 1 performs slightly better in the two dimensional case with an RMSE of 0.0676 compared to 0.0686 for the LSMC method, in three dimensions the two methods perform equally well, and when the dimension is four the LSMC actually performs the best with a RMSE of 0.0891 compared to 0.0919 for WLSMC with Method 1.

The results for basket options thus shows that, similar to the one dimensional case, one is much better off increasing the number of simulated paths than considering correcting for heteroscedasticity. This also means that when factoring in computational time a weighting scheme is likely suboptimal. This is particularly so since in multiple dimensions the number of regressors used in the cross-sectional regressions potentially grows fast.⁹ Fig. 5 plots log RMSE against log computational time for each of the methods considered for a three asset American basket put option and clearly shows that the simple LSMC method with a polynomial order of $M = 3$ dominates any of the weighted regression methods.¹⁰

We thus conclude that using weighted regressions is not worthwhile in realistic settings when pricing options with a reasonable number of early exercise points in high dimensions. So why do our conclusions differ from those in Fabozzi et al. (2017)? First of all, Fabozzi et al. (2017) only report MRE and therefore essentially neglect the possible increase in the standard error of the estimate. Second, in drawing conclusions the increase in computational time from using any weighting scheme is neglected. When using a more reasonable metric such as the RMSE and when considering computational time as a constraint, we find that it is not worth using heteroscedasticity corrected estimators in the cross-sectional regressions.

The above results are rather negative in terms of the value that one might extract from using heteroscedasticity corrected estimators and a reasonable question might be if there are any cases in which this idea can be salvaged? We now show that there

⁹ This is indeed the case if one uses the complete polynomial with all cross products as a basis.

¹⁰ Using a polynomial of order $M = 5$ now clearly performs worse than using an order of $M = 3$.

Table 10
American put option with $J = 21$ exercise opportunities in a GARCH model.

Type	Met	$N = 1000$				$N = 10,000$			
		Bias	StDev	MRE	RMSE	Bias	StDev	MRE	RMSE
LSMC3	–	0.1471	0.1125	0.0597	0.1852	0.0392	0.0310	0.0161	0.0499
WLSMC3	1	0.1509	0.1120	0.0611	0.1879	0.0377	0.0312	0.0156	0.0489
IRLSMC3	1	0.1506	0.1120	0.0611	0.1877	0.0378	0.0304	0.0154	0.0484
WLSMC3	2	0.1505	0.1115	0.0608	0.1873	0.0391	0.0306	0.0157	0.0496
IRLSMC3	2	0.1498	0.1124	0.0608	0.1873	0.0400	0.0309	0.0160	0.0504
LSMC5	–	0.2406	0.1110	0.0924	0.2650	0.0603	0.0318	0.0232	0.0681
WLSMC5	1	0.2429	0.1115	0.0933	0.2673	0.0610	0.0310	0.0234	0.0684
IRLSMC5	1	0.2419	0.1120	0.0929	0.2673	0.0612	0.0313	0.0235	0.0687
WLSMC5	2	0.2431	0.1129	0.0934	0.2680	0.0623	0.0316	0.0239	0.0698
IRLSMC5	2	0.2422	0.1131	0.0932	0.2673	0.0630	0.0306	0.0242	0.0699

This table shows the average bias and standard deviation, along with the resulting mean relative error (MRE) and root mean squared error (RMSE) for an American put option with characteristics similar to those used in Stentoft (2011). The benchmark price from Stentoft (2011) is 2.610. The minimum error for each combination of polynomial order and simulated sample size is shown in *italics* whereas the minimum error overall is shown in **bold**.

Table 11
American put option with $J = 63$ exercise opportunities in a GARCH model.

Type	Met	$N = 1000$				$N = 10,000$			
		Bias	StDev	MRE	RMSE	Bias	StDev	MRE	RMSE
LSMC3	–	0.2793	0.1776	0.0670	0.3310	0.0556	0.0544	0.0157	0.0777
WLSMC3	1	0.2523	0.2027	0.0648	0.3236	0.0137	0.0756	0.0143	0.0764
IRLSMC3	1	0.2447	0.2013	0.0627	0.3168	0.0199	0.0660	0.0133	0.0686
WLSMC3	2	0.2504	0.1957	0.0632	0.3178	0.0364	0.0617	0.0139	0.0714
IRLSMC3	2	0.2581	0.1934	0.0644	0.3224	0.0372	0.0611	0.0139	0.0713
LSMC5	–	0.4670	0.1784	0.1094	0.4999	0.0953	0.0552	0.0230	0.1100
WLSMC5	1	0.4596	0.1846	0.1077	0.4952	0.0753	0.0639	0.0200	0.0986
IRLSMC5	1	0.4510	0.1877	0.1057	0.4885	0.0732	0.0617	0.0193	0.0955
WLSMC5	2	0.4715	0.1857	0.1102	0.5056	0.0844	0.0593	0.0216	0.1030
IRLSMC5	2	0.4447	0.1995	0.1052	0.4874	0.0831	0.0594	0.0214	0.1019

This table shows the average bias and standard deviation, along with the resulting mean relative error (MRE) and root mean squared error (RMSE) for an American put option with characteristics similar to those used in Stentoft (2011). The benchmark price from Stentoft (2011) is 4.268. The minimum error for each combination of polynomial order and simulated sample size is shown in *italics* whereas the minimum error overall is shown in **bold**.

Table 12
American put option with $J = 126$ exercise opportunities in a GARCH model.

Type	Met	$N = 1000$				$N = 10,000$			
		Bias	StDev	MRE	RMSE	Bias	StDev	MRE	RMSE
LSMC3	–	0.3852	0.2306	0.0686	0.4489	0.0691	0.0682	0.0140	0.0968
WLSMC3	1	0.2031	0.3747	0.0599	0.4261	-0.1944	0.1699	0.0358	0.2577
IRLSMC3	1	0.1958	0.3490	0.0576	0.4000	-0.1456	0.1388	0.0286	0.2006
WLSMC3	2	0.2451	0.3487	0.0614	0.4261	-0.0258	0.0972	0.0140	0.1001
IRLSMC3	2	0.2456	0.3211	0.0583	0.4042	-0.0255	0.0932	0.0137	0.0962
LSMC5	–	0.6532	0.2354	0.1140	0.6943	0.1236	0.0712	0.0221	0.1424
WLSMC5	1	0.6041	0.2596	0.1059	0.6571	-0.0296	0.1365	0.0195	0.1390
IRLSMC5	1	0.5565	0.2813	0.0988	0.6235	-0.0174	0.1131	0.0160	0.1139
WLSMC5	2	0.6212	0.2588	0.1089	0.6729	0.0577	0.0965	0.0160	0.1120
IRLSMC5	2	0.5801	0.2659	0.1019	0.6381	0.0526	0.0988	0.0160	0.1115

This table shows the average bias and standard deviation, along with the resulting mean relative error (MRE) and root mean squared error (RMSE) for an American put option with characteristics similar to those used in Stentoft (2011). The benchmark price from Stentoft (2011) is 5.730. The minimum error for each combination of polynomial order and simulated sample size is shown in *italics* whereas the minimum error overall is shown in **bold**.

might be, but again only if computational time is neglected as a constraint. Since heteroscedasticity is key to the method proposed by Fabozzi et al. (2017) one suggestion would be to consider a setting in which this is potentially even more of an issue. We choose here to work with option pricing in the GARCH model of Duan (1995). GARCH models, short for Generalized Autoregressive Conditional Heteroskedasticity, naturally, contain even more exposure to heteroscedasticity. In Tables 10–12 we compare the pricing performance for our 5 methods for GARCH options with 1 month, 3 months and 6 months maturity, respectively. We assume daily exercise with 252 trading days per year.

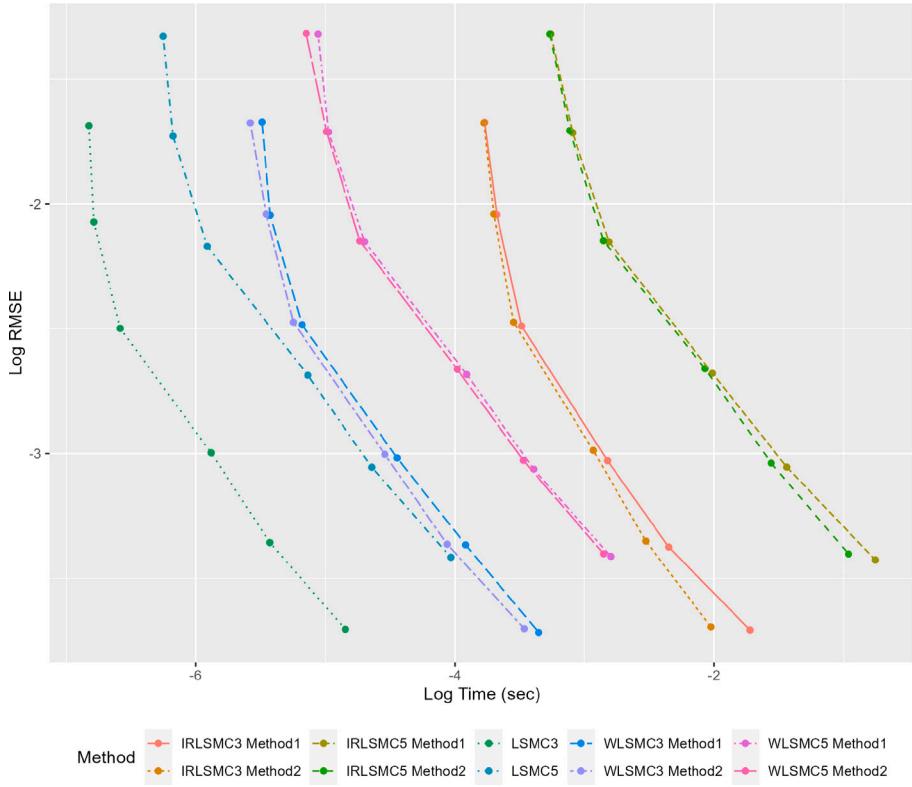


Fig. 6. This figure plots the log RMSE against the log mean computational time for each of the methods considered for a GARCH option. $I = 1000$ repetitions were used with a number of simulated paths $N \in \{1000, 2000, 4000\}$, and $I = 100$ repetitions were used with a number of simulated paths $N \in \{10,000, 20,000, 40,000\}$. The option considered has the same parameters as in Table 10.

Across the three tables, a model that uses a weighting scheme is always the best performing model overall and the improvements may be quite large in fact.¹¹ For example, for the 3 month option with $J = 63$ exercise points the iterative IRLSMC method has a RMSE of 0.0686 compared to 0.0777 for the LSMC, an improvement of almost 12%. The corresponding improvement in the MRE is 15.2% in this case.

Although these results may seem to salvage at least in part the idea of correcting for heteroscedasticity in the cross-sectional regressions in the LSMC method, this conclusion still strongly depends on abstracting from the increased computational time that such methods carry with them. This is demonstrated in Fig. 6, which plots the log RMSE against the log computational time for the GARCH option with $J = 21$ exercise opportunities and shows that again all other methods are dominated by the LSMC method. In particular, using a polynomial order of $M = 3$ in the LSMC is the optimal choice in this setting and we conclude that using heteroscedasticity corrected estimators is not recommended.

5. Discussion and open research questions

Our results have some limitations and, as a result, open up some future areas of research. First, our application of the LSMC method considered only simply polynomial regressions. In the literature on the use of regression and simulation-based pricing approaches several alternatives have been proposed. For example, Stentoft (2004a) considers using various different (orthogonal) families in the cross-sectional regressions and finds that since these are generally more complicated to compute and lead to very small differences in estimated prices using a polynomial regression is the best (see also Moreno and Navas (2003)). However, it is possible that heteroskedasticity correction would improve significantly on these alternative regressions. Alternatively, these regressions, essentially parametric approximations of the conditional continuation value, can potentially be improved upon by using more general statistical or machine learning (ML) techniques. See, e.g. Ludkovski (2023) for a review of some of the literally hundreds of potential

¹¹ Interestingly, for the 1 month option with $J = 21$ exercise points the LSMC method performs the best for all combinations except when using a polynomial of order $M = 3$ and $N = 10,000$ simulated paths.

methods.¹² We leave the interesting work of examining the effect of corrections in all these alternative cross-sectional approximation settings for future research.

Second, in the LSMC method a new regression is performed at each time where early exercise is possible, which allows determining if exercise is optimal conditional on the state variables. Once this has been done for all possible exercise times, one has essentially (implicitly) obtained the early exercise strategy for the option at time $t = 0$. An alternative to doing this is to approximate the early exercise strategy with one “regression”. Such a method was first proposed in this type of classical analytical approach by [Tsitsiklis and Van Roy \(2001\)](#). More recently, new developments in ML have been used to solve this option pricing problem as well as more general types of financial decision-making problems. One example is the Least-Squares Policy Iteration (LSPI) algorithm of [Lagoudakis and Parr \(2003\)](#), which was applied to the American put pricing problem in [Li et al. \(2009\)](#) and extended to pricing convertible bonds in [Dubrov \(2015\)](#).¹³ These papers show that ML methods can improve on the LSMC method, and an interesting problem for future research is to examine if heteroscedasticity (or any other type of) corrections can be implemented with benefit in these alternative algorithms for determining optimal policies.

6. Conclusion

The least squares Monte Carlo (LSMC) method of [Longstaff and Schwartz \(2001\)](#) is the state-of-the-art method for pricing American style options. However, because the method uses simulation and regression methods it may be numerically inefficient and therefore much research has gone into improving its performance. One suggestion put forth by [Fabozzi et al. \(2017\)](#) is to improve upon the regressions used to determine optimal early exercise since the standard assumptions underlying these regressions may not be satisfied due to heteroscedasticity. Based on this finding, we consider two methods for heteroscedasticity correction in the cross-sectional regression part of this algorithm and we propose an iterative re-weighting method.

To provide some intuition, we first consider a simple case with one early exercise time for which the continuation value is known and show that there is indeed evidence of heteroscedasticity in the errors when approximating this with an ordinary linear regression. However, our results also show that though weighted linear regressions, which attempt to correct for this heteroscedasticity, improve on the fit of the continuation value function this does not lead to significantly improved early exercise decisions on average. As a result, the improvements on the price estimate are limited.

This finding is confirmed in several realistic settings with multiple early exercise times where we find that, though heteroscedasticity corrected price estimates may have slightly lower bias, they very often have higher variance than their uncorrected counterparts. As a result, using more relevant error metrics like root mean squared errors, the LSMC method often performs better than the corrected versions. Moreover, since it is computationally more complicated to correct for heteroscedasticity, we conclude that such methods are not necessarily successful at improving price estimator efficiency in realistic settings.

CRediT authorship contribution statement

R. Mark Reesor: Conceptualization, Formal analysis, Methodology, Writing – original draft, Writing – review & editing, Supervision, Project administration, Funding acquisition. **Lars Stentoft:** Conceptualization, Software, Formal analysis, Methodology, Writing – original draft, Writing – review & editing, Supervision, Project administration, Funding acquisition. **Xiaotian Zhu:** Conceptualization, Software, Formal analysis, Methodology, Validation, Writing – original draft, Writing – review & editing, Visualization.

Data availability

Data will be made available on request.

Appendix. Additional results

This appendix contains additional results complementing what is found in the main text. In particular, [Fig. 7](#), [Fig. 8](#), and [Fig. 9](#) report results for Method 2 that correspond to and are essentially equivalent to that reported for Method 1 in [Fig. 1](#), [Fig. 2](#), and [Fig. 3](#), respectively.

¹² Interestingly, one of the first applications of ML in option pricing was to learn the option pricing function, corresponding to the conditional continuation value at time zero, see, e.g., [Hutchinson et al. \(1994\)](#).

¹³ [Dubrov \(2015\)](#) also proposes to use a random forest type method, where each tree learns the early exercise strategy from a LSMC algorithm. This method has similarities with the bootstrapping approach proposed by [Létourneau and Stentoft \(2019\)](#) in which an average of several LSMC algorithms is used at each time step. These methods are shown to determine the optimal stopping time more precisely and to improve on the price estimates. However, since our research has shown that the strategy determined with each of the simulations is unaffected by correcting for heteroskedasticity, we conjecture that such corrections would have at best only a marginal effect on these algorithms.

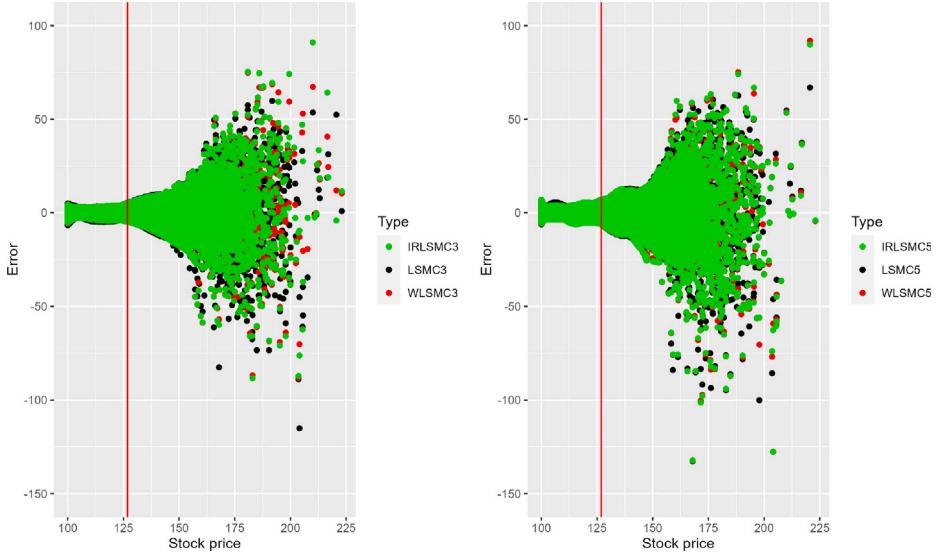


Fig. 7. This figure plots the estimated continuation value errors against stock price using LSMC (in black), Method 2 WLSMC (in red), and Method 2 IRLSMC (in green) with $N = 1000$ simulated paths and $I = 1000$ repetitions. The option characteristics are those from Table 1. Left and right panels correspond to 3rd and 5th order polynomials. The red line is the true exercise boundary.

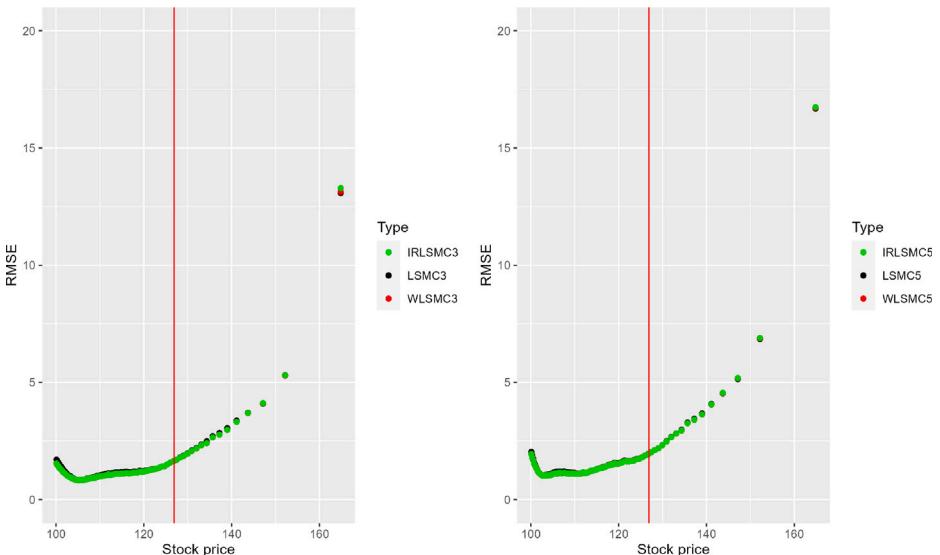


Fig. 8. This figure plots the RMSE of the estimated continuation values against stock price percentile using LSMC (in black), Method 2 WLSMC (in red), and Method 2 IRLSMC (in green) with $N = 1000$ simulated paths and $I = 1000$ repetitions. The option characteristics are those from Table 1. Left and right panels correspond to 3rd and 5th order polynomials. The red line is the true exercise boundary.

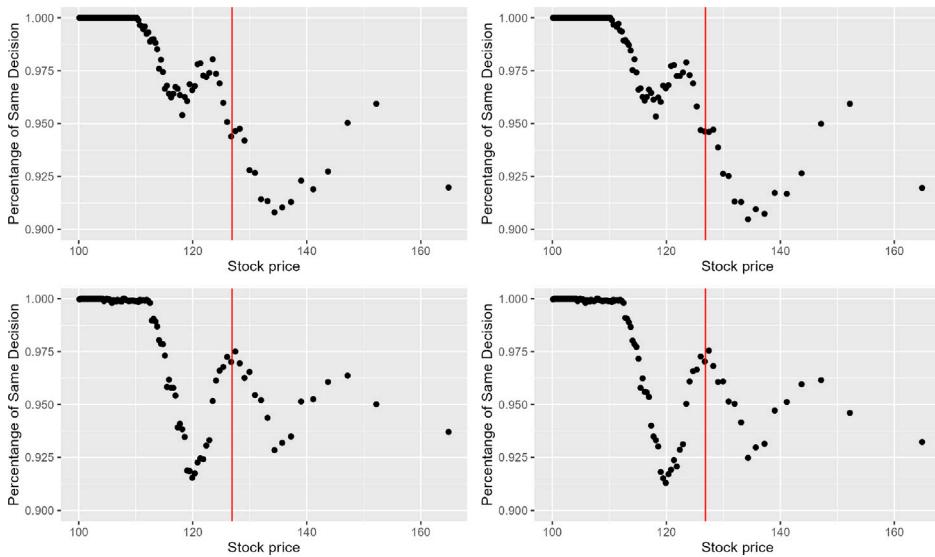


Fig. 9. This figure plots the frequencies for which the same decision is made against stock price percentile for LSMC3 vs. WLSMC3 (top left), LSMC3 vs. IRLSMC3 (top right), LSMC5 vs. WLSMC5 (bottom left), and LSMC5 vs. IRLSMC5 (bottom right) across the same bins as in Fig. 2 using Method 2. $N = 1000$ simulated paths with $I = 1000$ repetitions were used in pricing ATM American Call option. The option characteristics are those from Table 1. The red line is the true exercise boundary.

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