QFT - Lecture 7

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Contents

1 Lorentz Transformations

$$x^\mu \mapsto x'^\mu = \Lambda^\mu_\nu x^\nu \quad (1)$$

$$x' = \Lambda x \quad (2)$$

Scalar Field:

$$\varphi'(x) = \varphi(\Lambda^{-1}x) \quad (3)$$

Vector field:

$$A'^\mu(x) = \Lambda^\mu_\nu A^\nu(\Lambda^{-1}x) \quad (4)$$

$$\Lambda = e^{\frac{1}{2}\Omega_{\rho\sigma}M^{\rho\sigma}} \quad (5)$$

$$S[\Lambda] = e^{\frac{1}{2}\Omega_{\rho\sigma}S^{\rho\sigma}}(NotationfromTom'sLecturenotes) \quad (6)$$

Here, Ω is a set of coefficients, and S, M are bases.

$$[M^{\rho\sigma}M^{\tau\nu}] = g^{\sigma\tau}M^{\rho\nu} - g^{\rho\tau}M^{\sigma\nu} + g^{\rho\nu}M^{\sigma\tau} - g^{\sigma\nu}M^{\rho\tau} \quad (7)$$

$$[S^{\rho\sigma}, S^{\tau\nu}] = \text{same as before just with S.} \quad (8)$$

The Dirac Spinor:

$$\psi'^\alpha = S[\Lambda]^\alpha_\beta \psi(\Lambda^{-1}x) \quad (9)$$

$S[\Lambda]$ must be a representation of the Lorentz Group.

$$\Lambda = \Lambda_2\Lambda_1 \quad (10)$$

$$\underbrace{S[\Lambda_2]S[\Lambda_1]}_{S[\Lambda]}\psi \quad (11)$$

Baker-Campbell-Hausdorff Forumla:

$$e^X e^Y = e^Z \quad (12)$$

$$Z = X + Y + \frac{1}{2}[X, Y] + \frac{1}{12}\cdots \quad (13)$$

If we know the commutator of a Group, we can construct the whole Group.

We will work on finding a representation for $S[\Lambda]$.

2 Spinor Representation

Clifford Algebra

- 4 matrices γ^μ
- $\{\gamma^\mu, \gamma^\nu\} = 2g^\mu\nu I.$
- $(\gamma^0)^2 = 1, (\gamma^i)^2 = -1$

Chiral representation

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$$

with the Pauli matrices

$$\sigma^1 = 1, 0, 0, 1 \quad \sigma^2 = 0, -i, i, 0 \quad \sigma^3 = 1, 0, 0, -1 \tag{14}$$

anti- and commutator

$$\{\sigma^i, \sigma^j\} = 2\delta^{ij}, \quad [\sigma^i, \sigma^j] = 2i\epsilon^{ijk}\sigma^k \tag{15}$$

$$\sigma^i\sigma^j = \delta^{ij} + i\epsilon^{ijk}\sigma^k \tag{16}$$

..

$$S^{\rho\sigma} = \frac{1}{4}[\gamma^\rho, \gamma^\sigma] == (\frac{1}{2}\gamma^\rho\gamma^\sigma - g^{\rho\sigma}) \tag{17}$$

$$\begin{aligned}
[S^{\mu\nu}, \gamma^\rho] &=_{\mu\neq\nu} \frac{1}{2}[\gamma^\mu\gamma^\nu, \gamma^\rho] = \frac{1}{2}(\gamma^\mu\gamma^\nu\gamma^\rho - \gamma^\rho\gamma^\mu\gamma^\nu) \\
&= \frac{1}{2}(\gamma^\mu\{\gamma^\nu, \gamma^\rho\} - \gamma^\nu\gamma^\rho\gamma^\nu - \{\gamma^\rho, \gamma^\mu\}\gamma^\nu + \gamma^\mu\gamma^\rho\gamma^\nu \\
&= \frac{1}{2}(\gamma^\mu 2g^{\mu\rho} - 2g^{\rho\mu}\gamma^\nu) = \gamma^\nu g^{\mu\rho} - g^{\rho\mu}\gamma^\mu
\end{aligned} \tag{18}$$

$$[S^{\rho\sigma}, S^{\tau\nu}] = \cdots = g^{\sigma\tau}S^{\rho\nu} - g^{\rho\tau}S^{\sigma\nu} + S^{\rho\nu}S^{\sigma\tau} - g^{\sigma\nu}S^{\rho\tau} \tag{19}$$

Is $S[\Lambda] \neq \Lambda$?

2.1 Example Rotations

Let $\Omega_{0\nu} = \Omega\mu 0 = 0$ for all μ, ν.

$$S^{ij} =_{i\neq j} \frac{1}{2}\gamma^i\gamma^j = \frac{1}{2}\begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^j \\ -\sigma^j & 0 \end{pmatrix} \tag{20}$$

$$= \frac{-1}{2}\begin{pmatrix} \sigma^i\sigma^j & 0 \\ 0 & \sigma^i\sigma^j \end{pmatrix} = \begin{pmatrix} -\frac{i}{2}\epsilon^{ijk}\sigma^k & 0 \\ 0 & -\frac{i}{2}\epsilon^{ijk}\sigma^k \end{pmatrix} \quad \Omega_{ij} = \epsilon_{ijk}\varphi^k \tag{21}$$

$$S[\Lambda] = e^{\frac{1}{2}\Omega_{ij}S^{ij}} = exp\begin{pmatrix} -\frac{i}{2}\Omega_{ij}\epsilon^{ijk}\sigma^k & 0 \\ 0 & -\frac{i}{2}\Sigma_{ij}\epsilon^{ijk}\sigma^k \end{pmatrix} \tag{22}$$

$$\tag{23}$$

use that $\epsilon_{ijk}\epsilon^{ij\ell} = \delta_k^\ell$

$$= exp\begin{pmatrix} -\frac{i}{2}\varphi^k\sigma^k & 0 \\ 0 & -\frac{i}{2}\varphi^k\sigma^k \end{pmatrix} = \begin{pmatrix} e^{-\frac{i}{2}\varphi\sigma} & 0 \\ 0 & e^{-\frac{i}{2}\varphi\sigma} \end{pmatrix} \tag{24}$$

$$\Lambda = e^{\frac{1}{2}\Omega_{ij}M^{ij}} =_{\Omega_{12}=-\Omega_{21}=1} e^{\Omega_{12}M^{12}} \tag{25}$$

$$= exp\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\varphi^3 & 0 \\ 0 & \varphi^3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{26}$$

Only the middle part is nontrivial. Calculate

$$\Lambda_{2\times2} = exp\begin{pmatrix} 0 & -\varphi^3 \\ \varphi^3 & 0 \end{pmatrix} = exp\left(-i\begin{vmatrix} 0 & -i \\ i & 0 \end{vmatrix}\varphi^3\right) = \exp(-i\sigma^2\varphi^3) \tag{27}$$

$$= \cos(\varphi^3) - i\sigma^2\sin(\varphi^3) \tag{28}$$

This is a rotation about the 3− axis

For $\varphi^3 = 2\pi$ we expect the Identity. So $\Lambda = 1$. But what about $S[\Lambda]$? We will get

$$\begin{pmatrix} e^{-\frac{i}{2}2\pi\sigma^3} & 0 \\ 0 & e^{-\frac{i}{2}2\pi\sigma^3} \end{pmatrix} = \begin{pmatrix} e^{-i\pi\sigma^3} & 0 \\ 0 & e^{-i\pi\sigma^3} \end{pmatrix} = -1 \tag{29}$$

That means that doing nothing results in a -1 change. This means that the Dirac Spinor is not a physical object, not an observable.

2.2 Example Boosts

$\Omega_{ij} = 0$, $\Omega_{i0} = -\Omega_{0i} = -\eta_i$. $-\eta$ is a real quantitiy called rapidity. In Tom's notes this is χ. We start

$$S^{0i} = \frac{1}{2}\gamma^0\gamma^i == \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -\sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix} \tag{30}$$

$$S[\Lambda] = \exp\left[\frac{1}{2}\Omega_{0i}S^{0i} + \frac{1}{2}\Omega_{i0}S^{i0}\right] \tag{31}$$

$$= \exp\left[\Omega_{0i}S^{0i}\right] = \exp\left[-\frac{\eta_i}{2}\begin{pmatrix} -\sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix}\right] = \begin{pmatrix} e^{\frac{\eta_i\sigma^i}{2}} & 0 \\ 0 & e^{\frac{\eta_i\sigma^i}{2}} \end{pmatrix} \tag{32}$$

Boost along the 1−direction:

$$\eta_1 = 1, \eta_2 = \eta_3 = 0. \tag{33}$$

$$\Lambda = \exp[\Omega_{0i}M^{0i}] = \exp[\eta_1 M^{01}].\Lambda_{2\times2} = \exp\left(\eta_1\sigma^1\right) = \cosh\eta_1 + \sigma^1\sinh\eta_1 \tag{34}$$

We saw that $S[\Lambda] \neq \Lambda$ and we see that $S[\Lambda]$ is not unitary.