Homework Sheet 02

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Contents

1 Canonical Transformations and Classical Trajectories

1.1

We require that (q_i, p_i) is a valid trajectory, so we require that $\frac{\mathrm{d}q_i}{\mathrm{d}t} = \frac{\partial H}{\partial p_i}$, $\frac{\mathrm{d}p_i}{\mathrm{d}t} = -\frac{\partial H}{\partial q_i}$. We want to see that

$$\frac{\mathrm{d}\bar{q}_i}{\mathrm{d}t}, \quad \frac{\mathrm{d}\bar{p}_i}{\mathrm{d}t}$$

Are valid trajectories.

We will use the mappings

$$q_i \mapsto \bar{q}_i = q_i + \epsilon \frac{\partial g}{\partial p_i}, \quad p_i \mapsto \bar{p}_i = p_i - \epsilon \frac{\partial g}{\partial q_i} \tag{1}$$

and that

$$\{g, H\} = 0 = \frac{\partial g}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial g}{\partial p_i}\frac{\partial H}{\partial q_i} = \frac{\partial g}{\partial q_i}\frac{\partial q_i}{\partial t} + \frac{\partial g}{\partial p_i}\frac{\partial p_i}{\partial t} = \frac{\partial g}{\partial t}$$

We calculate explicitly:

$$\frac{\mathrm{d}\bar{q}}{\mathrm{d}t} = \frac{\mathrm{d}q}{\mathrm{d}t} + \epsilon \frac{\partial^2 g}{\partial t \partial p} = \frac{\mathrm{d}q}{\mathrm{d}t} \tag{2}$$

$$\frac{\mathrm{d}\bar{p}}{\mathrm{d}t} = \frac{\mathrm{d}p}{dt} - \epsilon \frac{\partial^2 g}{\partial t \partial q} = \frac{\mathrm{d}p}{\mathrm{d}t} \tag{3}$$

So, as we know that (q, p) describes a valid trajectory, and the time derivatives of $(\bar{q}, \bar{p})$ are equivalent, they must also describe a valid trajectory.

1.2

we have a transformation

$$x_k \mapsto x'_k = x_k + \delta$$

We will take a look at the generating function. We know that for transformation of the coordinate $\delta = \delta \frac{\partial g}{\partial p_k}$. We also know that we do not transform our conjugate momentum, so $0 = \delta \frac{\partial g}{\partial q_k}$. Therefore we know that

$$g = p$$

WE want to check if this transformation gives us a valid trajectory if (x_k, p_k) is a valid trajectory, hence we will use the method that we have used in the previous task. We will calculate the

Poisson bracket:

$$\{p, H\} = \underbrace{\frac{\partial p}{\partial q_i}\frac{\partial H}{\partial p_i}}_{=0} - \frac{\partial p}{\partial p_i}\frac{\partial H}{\partial q_i}$$
$$= -\frac{\partial p}{\partial p_i}\frac{\partial H}{\partial q_i} = -e_i\frac{\partial H}{\partial q_i}$$
$$= e_i\dot{p}_i = \dot{p}$$
$$\{p, H\} = \frac{\partial p}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial p}{\partial p_i}\frac{\partial H}{\partial q_i} = \frac{\partial p}{\partial q_i}\frac{\partial q_i}{\partial t} + \frac{\partial p}{\partial p_i}\frac{\partial p_i}{\partial t}$$
$$\frac{\partial p}{\partial t} + \frac{\partial p}{\partial t} = 2\dot{p}$$
$$\Rightarrow \{p, H\} = \dot{p} = 2\dot{p} = 0$$

Now that we know that the Poisson bracket is 0 we can apply what we have found out from the previous task, and infer that $(x'(t), p(t))$ is a valid trajectory, seeing as $(x(t), p(t))$ is a valid trajectory, and the Poisson bracket $\{g, H\} = 0$.

2 Canonical Transformation in Quantum Mechanics

2.1

We apply the operator $U(\xi) = e^{-i\xi g/\hbar}$ on the wave function

$$e^{-i\xi g/\hbar}\psi(q_i, t) = \sum_n e^{-i\xi g/\hbar}c_n(t)\psi_n(q_i)$$
$$= \sum_n c_n(t)e^{-i\xi g_n/\hbar}\psi_n(q_i)$$
$$= \sum_n e^{-i\xi g_n/\hbar}c_n(t)\psi_n(q_i) = \sum_n \bar{c}_n(t)\psi_n(q_i)$$

we from this we infer $\bar{c}_n = e^{-i\xi g_n/\hbar}c_n$.

2.2

We express our Quantum Operator with $g = L_z$, such that $U(\xi) = \exp(-i\xi L_z/\hbar)$

We know the form of L_z, but more importantly we know the Eigenvalues of L_z. We will take ψ_n to be Eigenfunctions of L_z, that means that $\psi_n(\rho, \phi, \theta) = f(\rho, \theta)e^{-in\phi}$. Then

$$e^{-i\xi L_z/\hbar}\psi = e^{-i\xi L_z/\hbar}\sum_n c_n f(\rho, \theta)e^{-in\phi}$$
$$= \sum_n c_n e^{-i\xi n}f(\rho, \theta)e^{-in\phi}$$
$$= \sum_n c_n f(\rho, \theta)e^{-in(\phi+\xi)}$$

From this we can see that $g = L_z$ generates a shift in ϕ, which we define as the rotation around the z axis, so L_z generates a rotation around the z axis.

2.3

3 Gauge Invariance in Classical Electrodynamics

3.1

We will show this

$$\begin{aligned}\nabla \times A &\mapsto \nabla \times (A + \nabla\lambda) \\ &= \nabla \times A + \underbrace{\nabla \times \nabla\lambda}_{=0} = \nabla \times A\end{aligned}$$

$$\begin{aligned}E = -\nabla U - \frac{\partial A}{\partial t} &\mapsto -\nabla\left(U - \frac{\partial\lambda}{\partial t}\right) - \frac{\partial A + \nabla\lambda}{\partial t} \\ &= -\nabla U + \nabla\frac{\partial\lambda}{\partial t} - \frac{\partial A}{\partial t} - \nabla\frac{\partial\lambda}{\partial t} \\ &= -\nabla U - \frac{\partial A}{\partial t}\end{aligned}$$

Therefore, the $\boldsymbol{E}$ and $\boldsymbol{B}$ fields stay invariant.

3.2

for the $\boldsymbol{B}$ field

$$\nabla \boldsymbol{B} = \nabla(\nabla \times A) = 0 \tag{4}$$

for the $\boldsymbol{E}$ field

$$\nabla \times \boldsymbol{E} = -\nabla \times (\nabla U) - \nabla \times \frac{\partial A}{\partial t} \tag{5}$$

$$= -\frac{\partial}{\partial t}\nabla \times A = -\frac{\partial B}{\partial t} \tag{6}$$

3.3

We plug in the expressions for our fields from the potentials:

$$\begin{aligned}\nabla\left(-\nabla U - \frac{\partial A}{\partial t}\right) &= \frac{\rho}{\epsilon_0} \\ -\Delta U - \frac{\partial \nabla A}{\partial t} &= \frac{\rho}{\epsilon_0} \\ -\Delta U + \mu_0\epsilon_0\frac{\partial^2 U}{\partial t^2} &= \frac{\rho}{\epsilon_0}\end{aligned}$$

and

$$
\begin{aligned}
\nabla \times (\nabla \times A) &= \mu_0 j + \mu_0 \epsilon_0 \frac{\partial - \nabla U - \frac{\partial A}{\partial t}}{\partial t} \\
\nabla \times (\nabla \times A) &= \mu_0 j + \mu_0 \epsilon_0 \left(-\nabla \frac{\partial U}{\partial t} - \frac{\partial^2 A}{\partial t^2} \right) \\
\nabla(\nabla A) - \nabla^2 A &= \mu_0 j - \mu_0 \epsilon_0 \frac{\partial^2 A}{\partial t^2} - \mu_0 \epsilon_0 \nabla \frac{\partial U}{\partial t} \\
-\nabla \left(\mu_0 \epsilon_0 \frac{\partial U}{\partial t} \right) - \nabla^2 A &= \mu_0 j - \mu_0 \epsilon_0 \frac{\partial^2 A}{\partial t^2} - \mu_0 \epsilon_0 \nabla \frac{\partial U}{\partial t} \\
-\Delta A + \mu_0 \epsilon_0 \frac{\partial^2 A}{\partial t^2} &= \mu_0 j
\end{aligned}
$$

Both of these equations could be made much prettier with the co and contravariant derivative.

3.4

The fields stay invariant, so I will only consider the changes in the Potentials themselves:

$$
-\rho U + jA \mapsto -\rho U + \rho \frac{\partial \lambda}{\partial t} + jA + j\nabla \lambda
$$

Charge is conserved, this means that

$$
\begin{aligned}
\partial_t \rho + \nabla j &= 0 \\
\int \mathrm{d}^4 x \partial_t \rho + \nabla j &= 0
\end{aligned}
$$

We will consider the Spacetime Integral over the additional terms of the Lagrangian Density.

$$
\delta S = \int \mathrm{d}^4 x \rho \frac{\partial \lambda}{\partial t} + j \nabla \lambda
$$

integration by parts, and the surface terms of λ are ignored.

$$
= -\lambda \int \mathrm{d}^4 x \partial_t \rho + \nabla j = 0
$$

hence, there is no change in the Action, so it stays Invariant.