QFT - Lecture 8

11.9.2023

Contents

Lorentz Transformation:

$$\Lambda = e^{\frac{1}{2}\Omega_{\rho\sigma}M^{\rho\sigma}}$$

Spinor Representation:

$$S[\Lambda] = e^{\frac{1}{2}\Omega_{\rho\sigma}S^{\rho\sigma}}$$

Some things:

- $S[\Lambda]$ is not unitary (for boosts)
- $S[\Lambda] \neq \Lambda$

In the chiral representation,

$$(\gamma^0)^\dagger = \gamma^0 \tag{1}$$
$$(\gamma^i)^\dagger = -\gamma^i \tag{2}$$
$$(\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0 \tag{3}$$

1 Dirac Spinor ψ^α

The Dirac spinor are four complex numbers ψ^α that transform as

$$\psi_\alpha(x) \mapsto S[\Lambda]_{\alpha\beta}\psi_\beta(\Lambda^{-1}x) \tag{4}$$

This is similar to the four vector transformation

$$A^\mu(x) \mapsto \Lambda^\mu_\nu A^\nu(\Lambda^{-1}x)$$

- ψ^α is not a four-vector.
- $\psi^\dagger\psi \mapsto \psi^\dagger S^\dagger S\psi \neq \psi^\dagger\psi$ in general, as S is not unitary. $\psi^\dagger\psi$ is then not a scalar.
- Define $\bar{\psi} = \psi^\dagger\gamma^0$.

$$\bar{\psi} \mapsto \psi^\dagger S^\dagger\gamma^0, S[\Lambda]^\dagger = e^{\frac{1}{2}\Omega_{\rho\sigma}(S^{\rho\sigma})^\dagger} \tag{5}$$
$$(S^{\rho\sigma})^\dagger = \frac{1}{4}[\gamma^\rho\gamma^\sigma - \gamma^\sigma\gamma^\rho]^\dagger = \frac{1}{4}(\gamma^0\gamma^\sigma\gamma^0\gamma^0\gamma^\rho\gamma^\sigma - \gamma^0\gamma^\rho\gamma^0\gamma^0\gamma^\sigma\gamma^0) \tag{6}$$
$$= \frac{1}{4}\gamma^0[\gamma^\sigma, \gamma^\rho]\gamma^0 = -\gamma^0 S^{\rho\sigma}\gamma^0 \tag{7}$$
$$\bar{\psi} \mapsto \psi^\dagger S[\Lambda]^\dagger\gamma^0 \tag{8}$$
$$= \psi^\dagger(e^{-\frac{1}{2}\Omega_{\rho\sigma}S^{\rho\sigma}}\gamma^0) = \psi^\dagger\gamma^0 e^{-\frac{1}{2}\Omega_{\rho\sigma}\gamma^0 S^{\rho\sigma}\gamma^0}\gamma^0 \tag{9}$$
$$= \bar{\psi}S[\Lambda]^{-1} \tag{10}$$

- $\bar{\psi}\psi$ is a scalar.

- $\bar{\psi}\gamma^\mu\psi$ is a four vector!

$$\bar{\psi}\gamma^\mu\psi \mapsto \bar{\psi}S[\Lambda]^{-1}\gamma^\mu S[\Lambda]\psi \stackrel{?}{=} \Lambda^\mu_\nu\bar{\psi}\gamma^\nu\psi \tag{11}$$

  This is the case if $S^{-1}\gamma^\mu S = \Lambda^\mu_\nu\gamma^\nu$. We use Taylor series:

$$\left(1 - \frac{1}{2}\Omega_{\rho\sigma}S^{\rho\sigma}\right)\gamma^\mu\left(1 + \frac{1}{2}\Omega_{\rho\sigma}S^{\rho\sigma}\right) \stackrel{?}{=} \left(\delta^\mu_\nu + \frac{1}{2}\Omega_{\rho\sigma}(M^{\rho\sigma})^\mu_\nu\right)\gamma^\nu \tag{12}$$

$$-\frac{1}{2}\Omega_{\rho\sigma}[S^{\rho\sigma}, \gamma^\mu] \stackrel{?}{=} \frac{1}{2}\Omega_{\rho\sigma}(M^{\rho\sigma})^\mu_\nu\gamma^\nu \tag{13}$$

$$-[S^{\rho\sigma}, \gamma^\mu] \stackrel{?}{=} (M^{\rho\sigma})^\mu_\nu\gamma^\nu \tag{14}$$

$$= g^{\rho\mu}\gamma^\sigma - g^{\sigma\mu}\gamma^\rho \tag{15}$$

  This is in fact true, proving the equality, proving that $\bar{\psi}\gamma^\mu\psi$ is a four vector.

- $\bar{\psi}\gamma^\mu\gamma^\nu\psi$ is a tensor.

- $\bar{\psi}\gamma^\mu\partial_\mu\psi$ is a scalar.

$$\bar{\psi}\gamma^\nu\partial_\mu\psi \mapsto \bar{\psi}S^{-1}\gamma^\mu\Lambda^{-1\nu}_{\mu}\partial_\nu S\psi \tag{16}$$

  S may be moved after γ^μ.

$$\bar{\psi}\Lambda^\mu_\kappa\gamma^\kappa\Lambda^{-1}{}_\mu^{\ \nu}\partial_\nu\psi = \bar{\psi}\gamma^\kappa\delta^\nu_\kappa\partial_\nu\psi \tag{17}$$

$$= \bar{\psi}\gamma^\nu\partial_\nu\psi \tag{18}$$

2 Dirac field

$$L = \bar{\psi}(x)(i\gamma^\mu\gamma_\mu - m)\psi(x) \tag{19}$$

$$= \bar{\psi}_\alpha(i\gamma^\mu_{\alpha\beta}\gamma_\mu - m\delta_{\alpha\beta})\psi_\beta \tag{20}$$

2.1 Euler-Lagrange equations

considering ψ and $\bar{\psi}$ as independent. For $\bar{\psi}_\alpha$

$$\frac{\partial L}{\partial\bar{\psi}_\alpha} - \partial_\mu\frac{\partial L}{\partial(\partial_\mu\bar{\psi}_\alpha)} = 0 \tag{21}$$

$$i\gamma^\mu_{\alpha\beta}\partial_\mu\psi_\beta - m\delta_{\alpha\beta}\psi_\beta = 0 \tag{22}$$

$$(i\gamma^\mu\partial_\mu - m)\psi = 0, \quad \text{Dirac equation} \tag{23}$$

$$(i\not{\partial} - m)\psi \tag{24}$$

- $\not{A}\not{A} = \gamma^\mu A_\mu\gamma^\nu A_\nu = \gamma^\mu\gamma^\nu A_\mu A_\nu = \frac{1}{2}(\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu)A_\mu A_\nu = g^{\mu\nu}A_\mu A_\nu = A^2$, It is always implied that A is multiplied by the 4×4 identity.

Each spinor component satisfies the Klein-Gordon equation.

$$(i\partial\!\!\!/ + m)\underbrace{(i\partial\!\!\!/ - m)\psi = 0}_{\text{Dirac}} \tag{25}$$

$$\Rightarrow (\partial^2 + m^2)\psi = 0, \quad \text{Klein Gordon} \tag{26}$$

2.2 Forms of the Dirac Equation

- $(i\gamma^\mu \partial_\mu - m)\psi = 0$
- $(i\partial\!\!\!/ - m)\psi = 0$
- write $\psi = (\psi_L, \psi_R)^T$. Then

$$\begin{pmatrix} -m & i(\partial_0 + \sigma\nabla) \\ i(\partial_0 - \sigma\nabla) & -m \end{pmatrix} \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} = 0$$

  This is useful, for example, for the massless case (Photons).

Define that

$$\sigma^\mu = (1, \sigma), \quad \bar{\sigma}^\mu = (1, -\sigma)$$

then

- $$\begin{pmatrix} -m & i\sigma\partial \\ i\bar{\sigma}\partial & -m \end{pmatrix} \begin{pmatrix} \psi_R \\ \psi_L \end{pmatrix} = 0$$

We had

$$S[\Lambda] = \begin{pmatrix} e^{\frac{i}{2}\phi\sigma} & 0 \\ 0 & e^{-\frac{i}{2}\phi\sigma} \end{pmatrix}$$

for rotations, and

$$S[\Lambda] = \begin{pmatrix} e^{-\eta\sigma/2} & 0 \\ 0 & e^{\eta\sigma/2} \end{pmatrix}$$

for boosts.

Thus, ψ_R and ψ_L transform separately. ψ_R, ψ_L are called Weyl spinors.

When $m = 0$ then the dirac Equation can be written as

$$i(\partial_0 + \sigma\nabla)\psi_R = 0 \tag{27}$$

$$i(\partial_0 - \sigma\nabla)\psi_L = 0 \tag{28}$$