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QFT - Lecture 17

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Contents

1 LSZ Reduction

$$a_p^\psi = i\int \mathrm{d}^3x \bar{\Psi}_p(x)\partial_0\phi(x) \tag{1}$$

$$\Psi_p(x) = \int \frac{\mathrm{d}^3 p}{(2\pi)^3}\frac{\psi(k-p)}{\sqrt{2E_p}}e^{-ikx} \tag{2}$$

$$k^0 = E_k = \sqrt{k^2+m^2} \tag{3}$$

Wavepacket in and out states

$$|k_1,k_2\rangle_{\text{in}} := |\text{int}\rangle = \prod_i \sqrt{\frac{2E_{k_i}}{Z}} a_{k_1}^{\psi\dagger}(-\infty)a_{k_2}^{\psi\dagger}(-\infty)\cdots|\Omega\rangle \tag{4}$$

$$|p_1,p_2\rangle_{\text{out}} := |\text{out}\rangle = \prod_j \sqrt{\frac{2E_{p_j}}{Z}} a_{p_1}^{\psi\dagger}(\infty)a_{p_2}^{\psi\dagger}(\infty)\cdots|\Omega\rangle \tag{5}$$

with renormalization constants

$$Z_\lambda = |\langle\Omega|\,\phi(0)\,|\lambda_{p=0}\rangle|^2 \tag{6}$$

$$Z = |\langle\Omega|\,\phi(0)\,|p=0\rangle|^2 \tag{7}$$

we consider the S matrix, which looks like

$$\langle\text{out}|\text{int}\rangle = \langle p_1,p_2,\cdots|\,S\,|k_1,k_2,\cdots\rangle \tag{8}$$

We want to get an expression that uses the fields, instead of the ladder operators, as we know how to do calculations with them

$$\langle\Omega|\,T\phi(x)\phi(y)\,|\Omega\rangle \tag{9}$$

We will define from now on that $\prod_i \sqrt{2E_{k_i}/Z} = A$.

$$\langle\text{int}|\text{out}\rangle = A\,\langle\text{out}|\,a_{k_1}^{\psi\dagger}(-\infty)a_{k_2}^{\psi\dagger}(-\infty)\,|\Omega\rangle \tag{10}$$

$$= A\,\langle\text{out}|\left((a_{k_1}^{\psi\dagger}(-\infty) - a_{k_1}^{\psi\dagger}(\infty)\right)a_{k_2}^{\psi\dagger}(-\infty)\,|\Omega\rangle \tag{11}$$

$$\text{Holding as far as } k_1 \neq p_1,p_2,.. \tag{12}$$

if $k_1 \neq p_1, p_2$, then they go in different directions. No matter how close they are at the beginning, they will be spacially separated at ∞. This means that you can commute the ladder operators which kills the vacuum, resulting in a zero, which is why we can add the term. We will now insert the definitions of the ladder operators with limits.

$$iA\left[\lim_{t_1\to\infty}-\lim_{t_1\to-\infty}\right]\int \mathrm{d}^3x\Psi_{k_1}(x_1)\overleftrightarrow{\partial}_0^{x_1}\langle\text{out}|\,\phi(x_1)a_{k_2}^{\psi\dagger}(-\infty)\,|\Omega\rangle \tag{13}$$

we use the identity

$$\left[\lim_{t\to\infty}-\lim_{t\to-\infty}\right]\int \mathrm{d}^3x f(x)\overleftrightarrow{\partial}_0 g(x)=\int \mathrm{d}^4x\partial_0\left(f(x)\overleftrightarrow{\partial}_0 g(x)\right) \tag{14}$$

$$=\int \mathrm{d}^4x\left[f(x)\partial_0^2 g(x)-\left(\partial_0^2 f(x)\right)g(x)\right],\ f(x)\text{ statisfies KG-eq} \tag{15}$$

$$=\int \mathrm{d}^4x\left[f(x)\partial_0^2 g(x)-\left((m^2-\nabla^2)f(x)\right)g(x)\right] \tag{16}$$

and we use integration by parts, surface term goes to 0 if $f(x)\to 0$.

$$\int \mathrm{d}^4x f(x)\left(\partial_\mu\partial^\mu+m^2\right)g(x) \tag{17}$$

And we use this property on the previous Integral

$$(13)=iA\int \mathrm{d}^4x\Psi_1(x_1)\left((\partial_\mu\partial^\mu)_{x_1}+m^2\right)\langle\text{out}|\,\phi(x_1)a_{k_2}^{\psi\dagger}(-\infty)\,|\Omega\rangle \tag{18}$$

Next step. Result:

$$\langle\text{out}|\text{in}\rangle=\prod_i i\sqrt{\frac{2E_{k_i}}{Z}}\int \mathrm{d}^4x_i\Psi_{k_i}(x_i)\left((\partial_\mu\partial^\mu)_{x_i}+m^2\right)$$

$$\prod_j i\sqrt{\frac{2E_{p_j}}{Z}}\int \mathrm{d}^4y_j\bar{\Psi}_{p_j}(y_j)\left((\partial_\mu\partial^\mu)_{y_j}+m^2\right)\langle\Omega|\,T\phi(x_1)\cdots\phi(y_1)\cdots|\Omega\rangle \tag{19}$$

Take plane-wave limit, $\psi(k)\to(2\pi)^3\delta(k)$ which then results in

$$\Psi_p(x)\to\frac{1}{\sqrt{2E_p}}e^{-ipx}$$

And then we get

$$\langle\text{out}|\text{in}\rangle=\prod_i\frac{1}{i\sqrt{Z}}\left(k_i^2-m^2\right)\int \mathrm{d}^4x_i e^{-ik_ix_i}\prod_j\frac{1}{i\sqrt{Z}}\left(p_j^2-m^2\right)\int \mathrm{d}^4y_j e^{ip_jy_j} \tag{20}$$

$$\underbrace{\langle\Omega|\,T\phi(x_1)\cdots\phi(y_1)\cdots|\Omega\rangle}_{\text{sum of connected diagrams}} \tag{21}$$

Fourier transforming the diagrams can be done by omitting the exponentials and Integrals of external points.

Consider Fig1.

Fourier Transforming:

$$-i\lambda(2\pi)^4\delta(p_1+p_2-k_1-k_2)D_F(k_1)D_F(k_2)D_F(p_1)D_F(p_2) \tag{22}$$

it is then convenient to define an S Matrix

$$\text{Define: } i\mathcal{M}: \quad \langle\text{out}|\text{in}\rangle = (2\pi)^4\delta\left(\sum p_j - \sum k_i\right) i\mathcal{M}(k_1, k_2, \cdots p_1, \cdots) \tag{23}$$

$$i\mathcal{M} = \prod_i \frac{1}{i\sqrt{Z}}\left(k_i^2 - m^2\right) \prod_j \frac{1}{\sqrt{Z}}\left(p_j^2 - m^2\right) \sum \text{connected diagrams} \tag{24}$$

where the connected diagrams are with the Feynman rules in momentum space, but not with exponentials and momentum integrals for external lines.