QFT - Lecture 10

September 18, 2024

Contents

Dirac Equation:

$$(i\partial\!\!\!/ - m)\psi = 0$$

Solutions:

$$\psi = u(p)e^{-ipx}, \quad u^s(p) = \begin{pmatrix} \sqrt{p\sigma}\xi^s \\ \sqrt{p\bar{\sigma}}\xi^s \end{pmatrix} \tag{1}$$

$$\psi = v(p)e^{ipx}, \quad v^s(p) = \begin{pmatrix} \sqrt{p\sigma}\eta^s \\ -\sqrt{p\bar{\sigma}}\eta^s \end{pmatrix} \tag{2}$$

we usually take $\eta^s = \xi^s$, and $\xi^1 = |+\rangle\,, \quad \xi^2 = |-\rangle$.
$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

0.1 Special case of Solution

Let $p = (E, 0, 0, p^3)$.

$$p\sigma = \begin{pmatrix} E - p^3 & 0 \\ 0 & E + p^3 \end{pmatrix}, \quad p\bar{\sigma} = \begin{pmatrix} E + p^3 & 0 \\ 0 & E - p^3 \end{pmatrix} \tag{3}$$

Then

$$\xi^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} : \quad u^1(p) = \begin{pmatrix} \sqrt{E - p^3}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \sqrt{E + p^3}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{pmatrix} \to_{\text{ultra-rel}} \begin{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ \sqrt{2E}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{pmatrix} \tag{4}$$

$$\xi^2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} : \quad u^2(p) = \begin{pmatrix} \sqrt{E - p^3}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ \sqrt{E - p^3}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{pmatrix} \to \begin{pmatrix} \sqrt{2E}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{pmatrix} \tag{5}$$

from Quantum Mechanics:

$$S = \frac{\sigma}{2} \tag{6}$$

And we will take

$$S = \frac{1}{2}\begin{pmatrix} \sigma & 0 \\ 0 & \sigma \end{pmatrix} \tag{7}$$

Which is our spin operator.
We will define the helicity, which is the spin in the direction of motion:

$$h = S \cdot p$$

for p along the 3-Axis we have

$$h = \frac{1}{2}\begin{pmatrix} \sigma^3 & 0 \\ 0 & \sigma^3 \end{pmatrix}$$

the helicity is not Lorentz Invariant, but it is a constant of motion.

To correct last lecture:

$$\gamma^{5\dagger} = -i\gamma^{3\dagger}\gamma^{2\dagger}\gamma^{1\dagger}\gamma^{0\dagger} \tag{8}$$
$$= -i\gamma^0\gamma^3\gamma^0\gamma^0\gamma^2\gamma^0\gamma^0\gamma^1\gamma^0\gamma^0\gamma^0\gamma^0 \tag{9}$$
$$= -i\gamma^0\gamma^3\gamma^2\gamma^1 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \gamma^5 \tag{10}$$

From Exercise:

$$j^\mu = \bar{\psi}\gamma^\mu\psi \tag{11}$$
$$\partial_\mu j^\mu = 0 \tag{12}$$

is a conserved quantity. We define

$$(j^\mu)^5 = \bar{\psi}\gamma^\mu\gamma^5\psi$$

$(j^\mu)^5$ is a vector:

$$(j^\mu)^5 \mapsto^\Lambda \bar{\psi}S[\Lambda]^{-1}\gamma^\mu\gamma^5 S[\Lambda]\psi \tag{13}$$
$$S[\Lambda] = e^{\frac{1}{2}\Omega_{\rho\sigma}S^{\rho\sigma}}, \quad S^{\rho\sigma} = \frac{1}{4}(\gamma^\rho\gamma^\sigma - \gamma^\sigma\gamma\rho) \tag{14}$$
$$[\gamma^5, \gamma^\mu\gamma^\nu] = 0 \tag{15}$$
$$(13) = \bar{\psi}\Lambda^\mu_\nu\gamma^\nu\gamma^5\psi = \Lambda^\mu_\nu\bar{\psi}\gamma^\nu\gamma^5\psi \tag{16}$$

so it is a vector. It is conserved (for $m = 0$):

$$\partial_\mu(j^\mu)^5 = \partial_\mu\bar{\psi}\gamma^\mu\gamma^5\psi + \bar{\psi}\gamma^\mu\gamma^5\partial_\mu\psi \tag{17}$$
$$= 0 - \bar{\gamma}^5\gamma^\mu\partial_\mu\psi = 0 \text{ (From Dirac Equation)} \tag{18}$$

The transformation for the current is $\psi \mapsto e^{i\alpha\gamma^5}\psi$.

j^μ is the electric current.

Consider $\frac{1-\gamma^5}{2} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $\frac{1+\gamma^5}{2} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$.

$j^\mu_L = \bar{\psi}\gamma^\mu\frac{1-\gamma^5}{2}\psi, j^\mu_R = \bar{\psi}\gamma^\mu\frac{1+\gamma^5}{2}\psi$ Then, j^μ_L only depends on ψ_L as does j^μ_R on ψ_R. They are the respective electrical currents.

0.2 Dirac field bilinears:

- $\bar{\psi}\psi$
- $\bar{\psi}\gamma^\mu\psi$
- $\bar{\psi}\gamma^\mu\gamma^\nu\psi$
- $\bar{\psi}\gamma^\mu\gamma^5\psi$
- $\bar{\psi}T\psi$

where T is a 4×4 matrix

1 Quantization of the Dirac field

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi \tag{19}$$

and the canonical momentums

$$\pi = \frac{\partial L}{\partial \dot{\psi}} = i\bar{\psi}\gamma^0 = i\psi^\dagger \tag{20}$$

$$\pi_{\bar{\psi}} = 0 \tag{21}$$

The Hamiltonian is given by the integral of the Hamiltonian density

$$\int \mathrm{d}^3x(\pi\dot{\psi} - L) = \int \mathrm{d}^3x(i\psi^\dagger\dot{\psi} - i\psi^\dagger\dot{\psi} - \bar{\psi}i\gamma^i\partial_i\psi + m\bar{\psi}\psi) \tag{22}$$

Which results in a Hamiltonian

$$H = \int \mathrm{d}^3x(-i\bar{\psi}\gamma\nabla\psi + m\bar{\psi}\psi) \tag{23}$$

General Solution:

$$\psi(x) = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\sum_s \left(a_p^s u^s(p)e^{-ipx} + b_p^{s\dagger}v^s(p)e^{ipx}\right) \tag{24}$$

$$\bar{\psi}(x) = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\sum_s \left(a_p^{s\dagger}\bar{u}^s(p)e^{ipx} + b_p^s\bar{v}^s(p)e^{-ipx}\right) \tag{25}$$

1.1 Canonical quantization

$$\left[\psi_\alpha(x), \psi_\beta^\dagger(y)\right] = \delta(x-y)\delta_{\alpha\beta} \tag{26}$$

This would lead to ladder operator properties of a_p^s and b_p^s

$$\Rightarrow \cdots \Rightarrow H = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\sum_s \left(E_p a_p^{s\dagger}a_p^s - E_p b_s^{s\dagger}b_p^s\right) \tag{27}$$

where the b particles have negative Energy. But this does not lead to the correct theory.

Instead, impose

$$\left\{\psi_\alpha(x), \psi_\beta^\dagger(y)\right\} = \delta(x-y)\delta_{\alpha\beta} \tag{28}$$

$$\left\{\psi_\alpha(x), \psi_\beta(y)\right\} = 0 \tag{29}$$

This leads to:

$$\left\{a_p^r, a_q^{s\dagger}\right\} = (2\pi)^3\delta(p-q)\delta^{rs} \tag{30}$$

$$\left\{b_p^r, b_q^{s\dagger}\right\} = (2\pi)^3\delta(p-q)\delta^{rs} \tag{31}$$

all others vanish. Note:

$$(a\dagger)^2 = a^\dagger a^\dagger = \frac{\left\{a^\dagger, a^\dagger\right\}}{2} = 0 \tag{32}$$

We will prove the opposite implication, proving that (30, 31) lead to (28,29).

$$\{\psi_\alpha(x), \psi^\dagger_\beta(y)\} = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{2E_p} \sum_s \left(u^s_\alpha(p) v^{s\dagger}_\beta(p) e^{ip(x-y)} + v^s_\alpha(p) v^{s\dagger}_\beta(p) e^{-ip(x-y)} \right) \tag{33}$$

$$\int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{2E_p} \left[(\not{p} + m)\gamma^0 + (\underbrace{\not{p}}_{\text{flip } \vec{p}} - m)\gamma^0 \right]_{\alpha\beta} e^{ip(x-y)} \tag{34}$$

$$\int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{2E_p} \left[(\gamma^0 p_0 + \gamma^i p_i + m)\gamma^0 + (\gamma^0 p_0 - \gamma^i p_i - m)\gamma^0 \right]_{\alpha\beta} e^{ip(x-y)} \tag{35}$$

$$= \int \frac{\mathrm{d}^3 p}{(2\pi)^3} e^{ip(x-y)} \delta_{\alpha\beta} = \delta(x-y)\delta_{\alpha\beta} \tag{36}$$

Fermionic Property important here:

$$\sum_s u^s(p)\bar{u}^s(p) = \not{p} + m, \qquad \sum_s u^s(p) u^{s\dagger}(p) = (\not{p} + m)\gamma^0 \tag{37}$$