AQT Homework Sheet 1

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Contents

I will be using the dagger to signify complex conjugates because the asterisk looks ugly with my compilation setup. For comparison: $\psi *= \psi^\dagger$.

1 Hermitean Operators

1.1

Let $\psi_1 = \psi_2 = \psi$ be Eigenstates of Q with the Eigenvalue λ. So

$$Q\psi = \lambda\psi$$

We then calculate by Eq(1) in the Sheet

$$\int \mathrm{d}x\psi^\dagger Q\psi = \int \mathrm{d}x(Q\psi)^\dagger\psi \tag{1}$$

$$\int \mathrm{d}x\psi^\dagger \lambda\psi = \int \mathrm{d}x\lambda^\dagger\psi^\dagger\psi \tag{2}$$

$$\lambda\int \mathrm{d}x\psi^\dagger\psi = \lambda^\dagger\int \mathrm{d}x\psi^\dagger\psi \tag{3}$$

$$\Rightarrow \lambda = \lambda^\dagger \tag{4}$$

and through this we see that λ must be real, as $Im(\lambda) = -Im(\lambda) \Rightarrow Im(\lambda) = 0$

1.2

Let ψ_1 and ψ_2 be Eigenstates of Q with respective Eigenvalues λ_1, λ_2 and $\psi_1 \neq \psi_2$.

Using Eq(1) off the Sheet again:

$$\int \mathrm{d}x\psi_1^\dagger Q\psi_2 = \int \mathrm{d}x(Q\psi_1)^\dagger\psi_2 \tag{5}$$

$$\int \mathrm{d}x\psi_1^\dagger \lambda_2\psi_2 = \int \mathrm{d}x\lambda_1^\dagger\psi_1^\dagger\psi_2 \tag{6}$$

$$\lambda_2\int \mathrm{d}x\psi_1^\dagger\psi_2 = \lambda_1\int \mathrm{d}x\psi_1^\dagger\psi_2 \tag{7}$$

we now see why non-degenerative eigenstates must be Orthonormal (Because the equality must hold). This proof doesn't hold for degenerative Eigenstates, as λ_1 could equal λ_2.

1.3

The dagger notation could be confusing here so I will swap to using bars for complex conjugates $\bar{z}$

We want to show that a hermitean operator Q is represented by a hermitean Matrix Q, where hermitean matrices follow the relation $(A^\dagger)_{ij} = \bar{A}_{ji}$
We use the relation in the hint

$$Q_{ij} = \int \mathrm{d}x\bar{\psi}_i Q\psi_j \tag{8}$$

Take the complex conjugate

$$\bar{Q}_{ij} = \int \mathrm{d}x (\bar{Q\psi_j})\psi_i \tag{9}$$

Where we recognize Q as the hermitean matrix, as in

$$\bar{Q}_{ij} = \int \mathrm{d}x \bar{\psi}_j Q^\dagger \psi_i = (Q^\dagger)_{ji} \tag{10}$$

because the hermitean matrix acts to the left like the matrix itself would act to the right.

2 Decomposition of a wave function

2.1

We take $\psi(x,t) = \psi$ as the wave function. We decompose it as

$$\psi = \sum_n u_n(t)\psi_n(x) = u_n\psi^n \tag{11}$$

Rewriting the Integral Eq(4) off the sheet we get:

$$u_n = \int \mathrm{d}x \psi_n^\dagger \psi \tag{12}$$

$$\rightarrow u_n = \int \mathrm{d}x \psi_n^\dagger u_m \psi^m \tag{13}$$

$$\rightarrow u_n = u_m \int \mathrm{d}x \psi_n^\dagger \psi^m \tag{14}$$

$$\text{using the orthonormality condition of Eq(3)} \tag{15}$$

$$\rightarrow u_n = u_m \delta_n^m \tag{16}$$

$$\text{using the implied summation we then receive } u_n \text{ as expected} \tag{17}$$

basically we use the ability to swap integration and summation to establish a kronecker delta over which we sum with the coefficients, then receiving only the desired coefficient.

2.2

$$1 = \int \mathrm{d}x \psi^\dagger \psi = \int \mathrm{d}x u_n^\dagger \psi^{n\dagger} u_m \psi^m \tag{18}$$

$$= u_n^\dagger u^m \int \psi^{n\dagger} \psi^m = u_n^\dagger u^m \int \psi^{n\dagger} \psi_m \tag{19}$$

$$= u_n^\dagger u^m \delta_m^n = u_n^\dagger u^n = 1 \tag{20}$$

Proven

2.3

$$\langle Q \rangle := \int \mathrm{d}x \psi^\dagger Q \psi = \int \mathrm{d}x u_n^\dagger \psi^{n\dagger} Q u_m \psi^m \tag{21}$$

$$= u_n^\dagger u_m \int \mathrm{d}x \psi^{n\dagger} Q \psi^m = u_n^\dagger u_m \int \mathrm{d}x \psi^{n\dagger} q_m \psi^m \tag{22}$$

Now you will notice that there will be 3 m indeces. That's a bummer but whatever.

$$= u_n^\dagger u_m q_m \int \mathrm{d}x \psi^{n\dagger} \psi^m = u_n^\dagger u_m q_m \delta^{nm} \tag{23}$$

$$= \sum_m u_m^\dagger u_m q_m = \sum_m |u_m|^2 q_m \tag{24}$$

3 Angular Momentum Operator

3.1

We apply the Operator to the function.

$$L_z \psi = -i\hbar \frac{\partial}{\partial \phi} \frac{1}{\sqrt{2\pi}} e^{im\phi} \tag{25}$$

$$= -i\hbar i m \frac{1}{\sqrt{2\pi}} e^{im\phi} = -(i)^2 m\hbar\psi = m\hbar\psi \tag{26}$$

Which, according to the definition of Eigenfunctions and Eigenvalues, ψ is an Eigenfunction with the Eigenvalue $m\hbar$.

3.2

we set $\psi(\phi) = \psi(\phi + 2\pi)$.

$$\frac{1}{\sqrt{2\pi}} e^{im\phi} = \frac{1}{\sqrt{2\pi}} e^{im\phi + im2\pi} \tag{27}$$

$$\rightarrow \psi = \psi \cdot e^{im2\pi} \tag{28}$$

for this equality to hold, $e^{im2\pi} = 1$, this happens when $m2\pi$ is a multiple of 2π, meaning that m must be an Integer.

3.3

calculate the integral explicitly.

$$\frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\phi e^{-il\phi}e^{im\phi} \tag{29}$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\phi e^{i(m-l)\phi} \tag{30}$$

$$\text{for } l = m \text{ we have: } \frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\phi = \frac{1}{2\pi}2\pi = 1 \tag{31}$$

$$\text{for } l \neq m \text{ we have: } \frac{1}{2\pi}\left[\frac{1}{m-l}e^{i(m-l)\phi}\right]_{\phi=0}^{2\pi} \tag{32}$$

$$= \frac{1}{2\pi(m-l)}\left(e^{i(m-l)2\pi} - e^0\right) = \frac{1}{2\pi(m-l)}(1-1) = 0 \tag{33}$$

Therefore we come to the conclusion that

$$\int_0^{2\pi} \mathrm{d}\phi \psi_l^\dagger \psi_m = \delta_m^l$$

4 Canonical Transformation

4.1

$$\dot{\bar{q}}_i = \frac{\partial H}{\partial \bar{p}_i} = \frac{\partial H}{\partial p_i}\frac{\partial p_i}{\partial \bar{p}_i} + \frac{\partial H}{\partial q_i}\frac{\partial q_i}{\partial \bar{p}_i} \tag{34}$$

4.2

$$\bar{q} = \ln(q^{-1}\sin p), \quad \bar{p} = q\cot p \tag{35}$$

which then comes to be

$$\bar{q}_i = \ln(q_j^{-1}\sin p_j), \quad \bar{p}_i = q_j \cot p_j \tag{36}$$

We check the Poisson brackets

$$\left\{\ln(q_i^{-1}\sin p_i), \ln(q_j^{-1}\sin p_j)\right\} \tag{37}$$

$$\frac{\partial \ln(q_i^{-1}\sin p_i)}{\partial q_k}\frac{\partial \ln(q_j^{-1}\sin p_j)}{\partial p_k} - \frac{\partial \ln(q_i^{-1}\sin p_i)}{\partial p_k}\frac{\partial \ln(q_j^{-1}\sin p_j)}{\partial q_k} \tag{38}$$

$$= \frac{\partial q_i^{-1}\sin p_i}{\partial q_k}\frac{\partial q_j^{-1}\sin p_j}{\partial p_k}\frac{q_i q_j}{\sin p_i \sin p_j} - \frac{\partial q_i^{-1}\sin p_i}{\partial p_k}\frac{\partial q_j^{-1}\sin p_j}{\partial q_k}\frac{q_i q_j}{\sin p_i \sin p_j} \tag{39}$$

$$= \left(-\delta^{ik}q_i^{-2}\sin p_i \delta^{jk}q_j^{-1}\cos p_j + \delta^{ik}q_i^{-1}\cos p_i \delta^{jk}q_j^{-2}\sin p_j\right)\frac{q_i q_j}{\sin p_i \sin p_j} \tag{40}$$

$$= 0 \tag{41}$$

$$\left\{q_i \cot p_i, q_j \cot p_j\right\} \tag{42}$$

$$\frac{\partial q_i \cot p_i}{\partial q_k}\frac{\partial q_j \cot p_j}{\partial p_k} - \frac{\partial q_i \cot p_i}{\partial p_k}\frac{\partial q_j \cot p_j}{\partial q_k} \tag{43}$$

$$-\delta^{ik} \cot p_i q_j \delta^{jk}\frac{1}{\sin^2 p_j} + q_i \delta^{ik}\frac{1}{\sin^2 p_i}\delta^{jk} \cot p_j \tag{44}$$

$$j = i = k$$

$$\frac{q_k \cot p_k}{\sin^2 p_k} - \frac{q_k \cot p_k}{\sin^2 p_k} = 0 \tag{45}$$

$$\left\{\ln(q_i \sin p_i), q_j \cot p_j\right\} \tag{46}$$

$$\frac{\partial \ln(q_i^{-1} \sin p_i)}{\partial q_k}\frac{\partial (q_j \cot p_j)}{\partial p_k} - \frac{\partial \ln(q_i^{-1} \sin p_i)}{\partial p_k}\frac{\partial (q_j \cot p_j)}{\partial q_k} \tag{47}$$

$$-\frac{\partial q_i^{-1} \sin p_i}{\partial q_k}\frac{q_i}{\sin p_i}\delta^{jk}\frac{q_j}{sin^2(p_j)} - \frac{\partial q_i^{-1} \sin p_i}{\partial p_k}\frac{q_i}{\sin p_i}\delta^{jk} \cot p_j \tag{48}$$

$$\delta^{ik} q_i^{-2} q_i \delta^{jk}\frac{q_j}{\sin^2 p_j} - \delta^{ik} \cos p_i q_i^{-1}\frac{q_i}{\sin p_i}\delta^{jk} \cot p_j \tag{49}$$

$\delta^{ik}\delta^{jk} = \delta^{ij}$ because it only equals one if $i = j$.

and we are eliminating k because we're summing over it

$$\delta^{ij}\frac{1}{\sin^2 p_j} - \delta^{ij} \cot^2 p_j \tag{50}$$

$$= \delta^{ij} \tag{51}$$

4.3

I also wanted to say, that I (Florian Bierlage) am currently not in Bonn, and will only be able to attend tutorials coming next year, as I will come back to Germany on the 22nd of December.