Lectures in GR

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xx.xx.2025

Contents

Lecture 1 - 7.4

0.1 Literature

Mainly using Sean M Carol.
-Lecture Notes on General Relativity, arXiv: gt-qc/97/20/9
-Book: Spacetime and Geometry, Camebridge University Press
Also used is Heinz Stephani
-Book: General Relativity, Camebridge University Press
Also used is S Weinberg
-Book: Gravitation and Cosmology
For mathematical topics, M Nakahara's Geometry, Topology and Physics is used. For Recreational reading there is Kip Thorne's Black Holes and Timewarps and Brian Cox & Jeff Forshaw's Black Holes.

Outline of the Lecture:

- Introduction
- Special Relativity
- Relativistic Field Theory
- Principle of Equivalence
- Manifolds and Curvature (Math block)
- Tests for GR, Schwarschild Solution
- Gravitational Waves
- Black Holes
- Cosmology

0.2 Introduction

General Relativity unifies Special Relativity and Newtonian gravity.
Newtonian Gravity:

$$\Delta V_N = 4\pi G\rho$$

For Point masses m_N at positions x_N we have

$$m_N \frac{\mathrm{d}^2 x_N}{\mathrm{d}t^2} = G \sum_{M \neq N} \frac{m_N m_M (x_M - x_N)}{|x_M - x_N|^3}$$

with G the Gravitational constant
$G = 6.6 \cdot 10^{-11} m^3 kg^{-1} s^{-2}$.
$G/c^2 = 7.41 \cdot 10^{-29} cm/g$
For two masses

$$F = -\frac{G m_1 m_2 r}{r^3}, \quad V \propto 1/r$$

Keplers law follows from this.
Uranus' Orbit predicted Neptune

precession of Mercury's perihelion was off by 35 arcseconds per century. Vulcan predicted, but not observed. The explanation for this was given by GR.

0.3 Special Relativity

Newton: The principle of relativity states that the laws of physics are invariant under Gallilei transformations.
Maxwell: Electrodynamics is not invariant under Galilei transformations. Proposed solution: Ether (not observed), Or the principle of relativity doesn't apply.

0.3.1 Einstein 1905

Galilei transformations $\longrightarrow$ Lorentz transformations. Then, he called this principle special relativity. In Newton's Mechanics there is an absolute space, so every observer sees the same distances, this is replaced by absolute spacetime. Special Relativity is leaves spacetime distances invariant for all observers.
If $v << c$ then special relativity becomes newtonian mechanics. This also implies weak gravitational fields.

0.3.2 Einstein 1916

Generalized Principle of Relativity. In GR there is no absolute space or absolute spacetime. Newtonian Mehcanics obtained in the weak gravity limit. There also exists a Geometric Interpretation. Observations confirmed Einsteins Theory, Mercury was discovered, the deflection of light by the sun, red shift of spectral lines, Lense-Thirtring effect, Gravitational Waves

0.3.3 Gravity is Special

Gravity is the weakest force. In Order, we have The strong force, electromagnetic force, weak force, gravitational force. Assuming the strong force has a strength of 1, electromagnetism has a strenght of 10^{-2}, weak force a strength of 10^{-5} and Gravity of 10^{-35}.
Gravity is the only really long force. Weak and Strong forces have short ranges and Electromagnetism tends to cancel out through opposite charges.
GR is relevant in the Universe; deviations from Newton, large masses (black holes), cosmic singularities, cosmology. Gravity is transmitted by spin 2 particles called gravitons. This produces a fundamental difference to spin half or spin one particles.

0.4 Special Relativity (for real)

0.4.1 Principle of Relativity in Newton Mechanics

There exists an absolute space inertial frame with constant speed relative to absolute space. All related by Galilei transformations

$$t' = t + a \tag{0.1}$$
$$x' = x + vt \tag{0.2}$$
$$x' = Ax + b, \quad A^T = A^{-1} \tag{0.3}$$

Electrodynamics and Experiments confirmed that the speed of light is constant. Maxwell Equations are not covariant under Galilei Transformations
A Way out: modify the notion of inertial frames.

0.4.2 Principle of Relativity in Special Relativity

consider

$$(\Delta S)^2 = -c^2(\Delta t)^2 + (\Delta x)^2$$

impose c is the same in all inertial frames. If $(\Delta S)^2 = 0$ in one Inertial frame, then $(\Delta S')^2 = 0$ in another. (Argument following will be from Landau, Lifshitz)

So it must be that $(\Delta S)^2 = a(\Delta S')^2$
Consider three inertial frames K, K_1, K_2. v_1 is the velocity between K, K_1, v_2 between K, K_2 and v_{12} between K_1 and K_2.

$$(\Delta S_2)^2 = a(v_2)(\Delta S)^2 = a(v_{12})(\Delta S_1)^2 = a(v_{12})a(v_1)(\Delta S)^2 \tag{0.4}$$
$$\Rightarrow a(v_{12}) = \frac{a(v_2)}{a(v_2)} \tag{0.5}$$

Then v_{12} depends on the angle between v_1, v_2. v_1, v_2 do not. Therefore, $a(v) = \text{const}$, then $a = 1$. For Intertial frames

$$(\Delta S)^2 = (\Delta S')^2 \tag{0.6}$$

and ΔS is the Minkowski distance. Transformations leaving this distance invariant are called Lorentz Transformations

0.4.3 Notation, Conventions, Terminology

Coordinates in spacetime $x^0 = ct$, $x^m = x_m$: x^μ.
Putting $c = 1$ means that $1s = 3 \times 10^8 m$
Sum convention: $(\Delta S)^2 = \sum_\mu \sum_\nu \eta_{\mu\nu}\Delta x^\mu \Delta x^\nu = \eta_{\mu\nu}\Delta x^\mu \Delta x^\nu$
Minkowski Metric $(- + ++)$

Lecture 2 - 9.4

0.4.1 twin Paradox

proper time of

$$\text{twin 1: } \Delta\tau_{ABC} = \Delta t \tag{0.7}$$

$$\text{twin 2: } \Delta\tau_{AB'C} = \Delta\tau_{AB'} + \Delta\tau_{B'C} \tag{0.8}$$

$$\Delta\tau_{AB'} = \sqrt{1 - v^2}\frac{\Delta t}{2} = \Delta\tau_{B'C} \tag{0.9}$$

Therefore

$$\Delta\tau_{AB'C} = \sqrt{1 - v^2}\Delta\tau_{ABC} \tag{0.10}$$

travelling twin is younger

0.5 Lorentz Transformations

$$\tilde{x}^\mu = \Lambda^\mu_\nu x^\nu \tag{0.11}$$

impose that the Minkowski Distance from the origin is invariant, such that

$$\tilde{x}^\mu \tilde{x}^\nu \eta_{\mu\nu} = x^\mu x^\nu \eta_{\mu\nu} \tag{0.12}$$

Then

$$\eta_{\mu\nu}\Lambda^\mu_\kappa \Lambda^\nu_\lambda = \eta_{\kappa\lambda} \tag{0.13}$$

This property describes the Lorentz Group 0(3, 1). In this Group there exist 4 branches.

$$\kappa = \lambda = 0 \quad \Rightarrow \quad \eta_{00} = -1 = -(\Lambda^0_0)^2 + \underbrace{(\Lambda^i_0)^2}_{\geq 0} \tag{0.14}$$

$$\Rightarrow \quad \Lambda^0_0 \geq 0 \text{ or } \Lambda^0_0 \leq -1 \text{from Matrix equation } \Lambda^T \eta \Lambda = \eta \tag{0.15}$$

$$\Rightarrow \quad \det(\Lambda^T \eta \Lambda) = \det(\eta) \quad \Rightarrow \quad \det(\Lambda)^2 = 1 \tag{0.16}$$

then we get the four branches

- proper Lorentz group (contains the identity) $\Lambda^0_0, \det\Lambda = 1$
- combine with time reversal $\Lambda^0_0 \leq -1, \det\Lambda = -1$
- combine with parity (e.g. $x \to -x$) $\Lambda^0_0 \geq 1, \det\Lambda = -1$
- time reversal and parity $\Lambda^0_0 \leq -1, \det\Lambda = +1$

Raising and lowering an index with the (Minkowski) metric tensor

$$a_\mu = \eta_{\mu\nu} a^\nu \tag{0.17}$$

and we have that $\eta_{\mu\nu} = \eta^{\mu\nu}$

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0.6 last time

$$\Lambda^{\rho}_{\mu}\Lambda^{\kappa}_{\nu}\eta_{\rho\kappa} = \eta_{\mu\nu} \tag{0.18}$$
$$x^{\mu} \mapsto \Lambda^{\mu}_{\nu}x^{\nu} \tag{0.19}$$
$$a_{\mu} = \eta_{\nu\mu}a^{\nu} \tag{0.20}$$
$$(\mathrm{d}\tau)^2 = -(\mathrm{d}s)^2 \tag{0.21}$$

0.6.1 Examples

4-vector: $x^{\mu} = (ct, \vec{x})$.
4-velocity: $\frac{\mathrm{d}x^{\mu}}{\mathrm{d}\tau} = u^{\mu}$

$$\vec{x} = \vec{v}t,\ u^{\mu} = (\frac{\mathrm{d}t}{\mathrm{d}\tau}), \vec{u} = \frac{\mathrm{d}t}{\mathrm{d}\tau}) \tag{0.22}$$
$$\mathrm{d}\tau^2 = \mathrm{d}t^2 - \mathrm{d}x^2,\quad \mathrm{d}\tau = \frac{1}{\gamma}\mathrm{d}t \tag{0.23}$$
$$u^{\mu} = \gamma(1, \vec{v}),\quad u_{\mu}u^{\mu} = -1 \tag{0.24}$$

4-acceleration: $a^{\mu} = \frac{\mathrm{d}^2x^{\mu}}{\mathrm{d}\tau^2},\quad a_{\mu}u^{\mu} = 0$

Tensors

$$\tilde{T}^{\mu_1,\cdots,\mu_k}_{\nu_1,\cdots,\nu_m} = \Lambda^{\mu_1}_{\kappa_1}\cdots\Lambda^{\mu_k}_{\kappa_k}\Lambda^{\lambda_1}_{\nu_1}\cdots\Lambda^{\lambda_m}_{\nu_m}T^{\kappa_1,\cdots,\kappa_k}_{\lambda_1,\cdots,\lambda_m} \tag{0.25}$$

Examples for Tensors

Electrodynamics $\boldsymbol{E}$, $\boldsymbol{B}$ not part of 4 vectors.

$$F_{\mu\nu}\partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu},\quad B_i = \epsilon_{ijk}F_{jk},\quad E_i = F_{i0} \tag{0.26}$$

$\eta_{\mu\nu}$ invariant.
$\epsilon_{\mu\nu\rho\lambda}$ the totally anti-symmetric pseudo-tensor, invariant.

$$\tilde{\epsilon}_{\mu\nu\rho\lambda} = \det(\Lambda)\underbrace{\Lambda^{\kappa}_{\mu}\Lambda^{\phi}_{\nu}\Lambda^{\psi}_{\rho}\Lambda^{\xi}_{\lambda}\epsilon_{\kappa\phi\psi\xi}}_{\text{totally anti-symmetric}} \tag{0.27}$$
$$= \det(\Lambda)\epsilon_{\mu\nu\rho\lambda}\Lambda^{\kappa}_{0}\Lambda^{\phi}_{1}\Lambda^{\psi}_{2}\Lambda^{\xi}_{3}\epsilon_{\kappa\phi\psi\xi} = \det(\Lambda)^2\epsilon_{\mu\nu\rho\lambda} \tag{0.28}$$

the determinant is the cause for the pseudo-tensor property.

1 Relativistic Field Theory

-Relativistic Quantum Field Theory → Particle Physics
-Here, Classical Field Theories, only two relevant examples. That is, Electrodynamics and General Relativity.

1.1 Euler Lagrange Equation

For simplicity, one scalar field $\tilde{\phi}(\tilde{x}^\mu) = \phi(x^\mu)$.

$$S = \int \mathrm{d}t L = \int \mathrm{d}^4 x \mathcal{L}\left(\phi(x^\mu), \partial_\mu \phi(x^\mu)\right) \tag{1.1}$$

(The derivative is also a 4-vector with a lower index.)
Least Action Principle; the action should be extremal, so the variation should be 0.

$$0 = \delta S = \int \mathrm{d}^4 x \frac{\partial \mathcal{L}}{\partial \phi} \delta\phi + \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} \delta \partial_\mu \phi \tag{1.2}$$

Now integrate by parts and choose boundary conditions so that the contributions vanish.

$$0 = \int \mathrm{d}^4 x \left(\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} \right) \delta\phi \tag{1.3}$$

$$\Rightarrow \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} - \frac{\partial \mathcal{L}}{\partial \phi} = 0 \tag{1.4}$$

1.1.1 Examples

- Massive real scalar field (e.g. Higgs)

$$\mathcal{L} = \frac{-1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{m^2}{2} \phi^2$$

- Electromagnetic field

$$S = \int \mathrm{d}^4 x \left(-\frac{1}{4} F_{\kappa\lambda} F^{\kappa\lambda} + j_\lambda A^\lambda \right)$$

1.2 Noether Theorem

For every continuous global symmetry there exists one conserved current.
Proof: A continuous symmetry is parametrized by a continuous parameter (e.g. rotation angle).

Global means that the symmetry does not depend on space or time.
Consider one parameter symmetry, call parameter a. Consider$a(x^\mu)$ non-constant, then

$$\delta_a S \int \mathrm{d}^4 x (\partial_\mu a) j^\mu .$$

Integration by parts, $a_{\partial r} = 0$.

$$\delta_a S = -\int \mathrm{d}^4 x a \partial_\mu j^\mu .$$

If the Equations of Motions are satisfied then $\delta S = 0$ for any a.

$$0 = \int \mathrm{d}^4 x a \partial_\mu j^\mu$$

for any $a \to \partial_\mu j^\mu = 0$.

1.2.1 Noether Charge

$$Q = \int \mathrm{d}^3 x j^0 \tag{1.5}$$

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = \int \mathrm{d}^3 x \partial_0 \partial^0 = \mathrm{d}^3 x \div j = 0 \tag{1.6}$$

1.2.2 examples

Complex Scalar Field

$$\mathcal{L} = -\frac{1}{2}\partial_\mu \phi \partial^\mu \phi^\dagger - m^2 \phi \phi^\dagger \tag{1.7}$$

invariant under complex phase change.

$$e^{i\alpha}\phi \approx (1 + i\alpha)\phi \tag{1.8}$$

$$e^{-i\alpha}\phi \approx (1 - i\alpha)\phi \tag{1.9}$$

$$\delta_\alpha S = -\partial_\mu (i\alpha\phi)\partial^\mu \phi^\dagger - \partial_\mu \phi \partial^\mu (-i\alpha\phi^\dagger) - m^2 i\alpha\phi\phi^\dagger - m^2\phi(-i\alpha\phi^\dagger) \tag{1.10}$$

$$= \partial_\mu \alpha \left(-i\phi\partial^\mu\phi^\dagger + i(\partial^\mu\phi)\phi^\dagger\right) \tag{1.11}$$

So then we can see that

$$j^\mu = -i\phi\partial^\mu\phi^\dagger + i(\partial_\mu\phi)\phi^\dagger \tag{1.12}$$

This would be the electromagnetic current.

Remark

if the action is invariant under a local symmetry then the Noether Theorem does not apply. However, conserved currents can also come from the Equations of motion.
e.g.

$$\partial_\mu F^{\mu\nu} = j^\nu, \quad \partial_\nu j^\nu = \partial_\nu \partial_\mu F^{\mu\nu} = 0, \quad \text{Noether's 2nd Theorem} \tag{1.13}$$

conservation laws appears as integrability condition on equation of motion

$$A_\mu \to A_\mu + \partial_\mu \alpha \tag{1.14}$$

1.2.3 Energy Momentum

symmetry $x\mu \to x^\mu + a^\mu$

Lecture 4 - 16.4

1.3 Energy Momentum Tensor

the symmetry $x^\mu \to x^\mu + a^\mu$ leads to the Energy momentum tensor $T^{\mu\nu}$.
Energy conservation in classical mechanics, start with L

$$\frac{\mathrm{d}L}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial q}q\right)$$

which leads to

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial q}\dot{q} - L\right) = 0$$

which is energy conservation. Now with $\mathcal{L}$

$$\partial_\mu \mathcal{L} = \frac{\partial \mathcal{L}}{\partial \phi}\partial_\mu \phi + \frac{\partial \mathcal{L}}{\partial \partial_\nu \phi}\partial_\mu \partial_\nu \phi = \left(\partial_\nu \frac{\partial L}{\partial \partial_\nu \phi}\right)\partial_\mu \phi + \frac{\partial \mathcal{L}}{\partial \partial_\nu \phi}\partial_\nu \partial_\mu \phi \tag{1.15}$$

$$= \partial_\nu \left(\frac{\partial \mathcal{L}}{\partial \partial_\mu \phi}\partial_\mu \phi\right) \tag{1.16}$$

$$T_\mu^\nu = \delta_\mu^\nu \mathcal{L} - \frac{\partial L}{\partial \partial_\nu \phi}\partial_\mu \phi \tag{1.17}$$

1.3.1 conserved quantities

$$Q_\mu = \int \mathrm{d}^3 x T_\mu^0 \tag{1.18}$$

$$\Rightarrow T_0^0 \text{ Energy Density}, \quad T_i^0 = \pi_i = \text{ momentum density} \tag{1.19}$$

remark: we could have obtained the above result with Noether's method

$$x^\mu \to x^\mu + a^\mu(x) \Rightarrow \delta_{a^\mu} S = \int \mathrm{d}^4 x \partial_\nu a^\mu (T_\mu^\nu)$$

Infinitesimal rotations:

$$x^0 \to x^0$$
$$x \to x + \omega \times x$$

so therefore

$$\delta_\omega S = \int \mathrm{d}^4 x \partial_\mu \left(\epsilon^{ijk}\omega_j x_k\right) T_i^\mu \tag{1.20}$$

For a rotational symmetry $\partial_\omega S = 0$ for constant ω.

$$\delta_\omega S = \int \mathrm{d}^4 x \left(\partial_\mu \omega_j\right) \underbrace{\epsilon^{ijk} x_k T_i^\mu}_{J_k^\mu} \tag{1.21}$$

charge density

$$\epsilon^{ijk} x_k T_i^0$$
$$\epsilon^{jki} x_k \pi_i = (x \times \Pi)^j \tag{1.22}$$

1.3.2 Example of energy momentum tensor

Take the Maxwell Theory without a source

$$S = \int \mathrm{d}^4x(-\frac{1}{4}F^{\mu\nu}F_{\mu\nu}) \tag{1.23}$$

$$T^{\mu\nu} = \eta^{\mu\nu}\mathcal{L} - \frac{\partial L}{\partial\partial_\mu A_\lambda}\partial^\nu A_\lambda \tag{1.24}$$

$$= -\frac{1}{4}\eta^{\mu\nu}F^2 - F^{\mu\lambda}\partial^\mu A_\lambda \tag{1.25}$$

not "nice": $T^{\mu\nu} \neq T^{\nu\mu}$, not gauge invariant

1.3.3 "improvement"

add something that is conserved by itself.

$$\partial_\mu\left(-F^{\mu\lambda}\partial_\lambda A^\nu\right) = \partial_\mu F^{\mu\lambda}\partial_\lambda A^\nu + F^{\mu\lambda}\partial_\mu\partial_\lambda A^\nu \tag{1.26}$$

So we can add tis to the previous energy momentum tensor without affecting the action

$$T^{\mu\nu} = F^{\mu\nu}F^\nu_\lambda - \frac{1}{4}\eta^{\mu\nu}F^2 \tag{1.27}$$

Lecture 5 - 23.4

1.3.1 Last Time

$$x^\mu \to x^\mu + \alpha^\mu, \quad \alpha^\mu = \alpha^\mu(x)$$

Noether Theorem

$$\delta_a S = \int \mathrm{d}^4 x \partial_{\mu a} j^\mu = 0 \to \partial_\mu j^\mu = 0 \tag{1.28}$$

Energy Momentum tensor

$$\mathcal{L}(\phi, \partial_\mu \phi) \to \partial_\mu T^{\mu\nu} = 0, \;\; T^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} \partial^\nu \phi - \eta^{\mu\nu} \mathcal{L} \tag{1.29}$$

Lorentz Transformation Matrix

$$(\Lambda^{-1})^\mu_{\ \nu} = \Lambda_\nu^{\ \mu} \neq (\Lambda^T)^\mu_{\ \nu} = \Lambda^\nu_{\ \mu} \tag{1.30}$$

1.4 Energy Momentum Tensor of Ideal Fluid

remark: not a relativistic field theory.
non-interacting particles: positions $\vec{x}_n(t)$, $x_n^0 = t$. All particles have the same mass m. Particles have a momentum

$$p_n^\mu = m \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} = \frac{m}{\sqrt{1 - v_n^2}} \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} = E_n \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} \tag{1.31}$$

Noether Charge: total 4-momentum.

$$T^{0\mu} = \sum_n p_n^\mu(t) \delta(x - x_n(t)) \qquad \text{Charge Density} \tag{1.32}$$

What is the Noether current? Recall continuity equation from fluid dynamics

$$\dot{\rho} + \nabla j = 0 \quad j = \rho v. \tag{1.33}$$

for us $\rho \to T^{0\mu}$ and $j \to T^{i\mu}$

$$T^{i\mu} = \sum_n p_n^\mu(t) \frac{\mathrm{d}x_n(t)}{\mathrm{d}t} \delta\left(x - x_n(t)\right) \tag{1.34}$$

Together this is

$$T^{\mu\nu} = \sum_n p_n^\nu \frac{\mathrm{d}x^\mu}{\mathrm{d}t} \delta\left(x - x_n(t)\right) \tag{1.35}$$

Then we can put in eq 1.31

$$\sum_n \frac{p_n^\mu p_n^\nu}{E_n} \delta\left(x - x_n(t)\right) \tag{1.36}$$

average over ensemble of fluids with given temperature (volume = ∞) and particle number fixed (canonical ensemble) and the fluid is at rest with respect to the observer.

$$\langle p_n^\mu \rangle = \langle \frac{\mathrm{d}x^\mu}{\mathrm{d}t} \rangle = 0 \tag{1.37}$$

isotropy leads to rotational invariance $\langle p_n^\mu p_n^\nu \rangle = 0$ for $\mu \neq \nu$.

$$T^{00} = \sum_n \langle p_n^0 \delta(x - x_n(t)) \rangle = \rho \ \text{ Energy Density} \tag{1.38}$$

$$T^{ii} = \sum_n \langle p_n^i \sigma_n^i \delta(x - x_n(t)) \rangle \tag{1.39}$$

consider a classical ideal gas $v << 1$, then $\gamma \approx 1$.

$$T^{ii} = \sum_k \langle m(v_n^i)^2 \delta(x - x_n(t)) \rangle = \cdots \tag{1.40}$$

$$\text{Isotropy: } \langle (v_n^i)^2 \rangle = 1/3 \langle v^2 \rangle$$

$$E_{n,kin} = \frac{m}{2}(v_n)^2$$

$$\cdots = \frac{2}{3}\mathcal{E}_{kin} \tag{1.41}$$

equation of state $\mathcal{E}_{kin} = \frac{3}{2}\frac{NkT}{V} = \frac{3}{2}P$ pressure.

$$T^{ii} = P \tag{1.42}$$

So then

$$T^{\mu\nu} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & P \end{pmatrix} \tag{1.43}$$

true for all ideal fluids (relativistic, quantum)

For example, non-relativistic: $T^{\mu\nu} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ and relativistic: $\begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & -1/3\rho & 0 & 0 \\ 0 & 0 & -1/3\rho & 0 \\ 0 & 0 & 0 & -1/3 \end{pmatrix}$

Lecture 6 - 28.4

2 Principle of Equivalence

2.1 Special Relativity Pros and Cons

Pros

- composition of velocity
- $E = \gamma mc^2$
- time dilation
- unify with QM for QFT

Cons

- notion of absolute spacetime
- What happens in accelerated frames?

2.2 Principle of Equivalence

inertial mass: Resistance against acceleration

$$F = m_i a = m_i \frac{\mathrm{d}v}{\mathrm{d}t}, \quad m_i \text{ : Intertial Mass} \tag{2.1}$$

Gravitational mass: factor of proportionality in gravitational force

2.2.1 Galilei (Newton, Huggels)

forces fall at same rate independent of their mass. In a free Fall:

$$m_i a = m_g g \to a = \frac{m_g}{m_i} g \tag{2.2}$$

Observation: $\frac{m_g}{m_i}$ is universal, g can be defined such that $m_i = m_g$

2.2.2 Eötvös Experiment (1889)

(Weinberg) Simplified picture:
revolving heavy disk with a scale that balances vertical forces and at the same time radial forces. The torque pendulum: There are two masses on a scale held up by a string that can freely rotate. Let F_g denote the respective gravity force on the mass, and let F_C be the centrifugal forces on the masses. The gravitational balance is

$$l_A m_{gA} = l_B m_{g_B} \tag{2.3}$$

The resulting torque through centripetal force

$$T = (l_A m_{iA} - l_B m_{iB}) g_i \tag{2.4}$$

Then g_i is thhe acceleration due to the centripetal force. Observation: $T = 0$, so we take that $m_i = m_g$

2.2.3 Einstein Thought Experiment

freely falling elevator of mass m.
The external observer sees

$$m\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = mg$$

The internal observer sees

$$x' = x - \frac{1}{2}gt^2$$

Then we compute x' and we find

$$m\frac{\mathrm{d}^2 x'}{\mathrm{d}t^2} = 0$$

The principle of equivalence tells us that the gravitational and inertial forces can be cancelled for all freely falling objects.

The strong equivalence principle (SEP)(Popular in exam): At every spacetime point in an arbitrary gravitational field it is possible to choose a locally inertial coordinate system such that, within a sufficiently small region around the point in question the laws of nature take the same form as in an unaccelerated cartesian coordinate system in the absence of gravity

Effects of gravitational field can be detected by non-local experiment. Applied to measure earth gravitational field GOCE. Gravitational wave detection is non local.

The weak equivalence principle: Laws of nature replaced by laws of motion ($m_i = m_g$).

Einstein Equivalence Principle: Laws of nature replaced by non gravitational laws of nature

Gravitational binding energy affects m_g differently from m_i

2.2.4 gravitational red shift (from SEP or EEP)

Let 1 and 2 be spaceships with a constant distance z, while accelerating with a. 1 sends a photon to 2, time between being sent and being detected is $\Delta t = z/c$. Gained speed during Δt is $\Delta v = a\Delta t = \frac{az}{2}$. Incremental Doppler Shift $\frac{\Delta v}{c} = \frac{\Delta \lambda}{\lambda_0} = \frac{az}{c}$

Equivalent setup with a=g

let 1 and 2 be observers, where 2 is on top of a tower. 1 sends a photon to 2. Applying the previous relation we see that

$$\frac{\Delta\lambda}{\lambda_0} = \frac{gz}{c^2}$$

is the gravitational redshift.
Intuition: The energy of a photon is given by $h\nu$, the change is $\Delta(h\nu) = -\frac{gz}{c^2}h\nu$. The change of energy in a photon is the change of potential energy ($mgz = \frac{h\nu}{c^2}hz$ if $m \to \frac{E}{c^2}$.

Observation: Pound & Rebka (1960)

let γ ray from Fe^{57} fall 22.6 meters. Resonant absorption showed

$$\frac{\Delta\nu}{\nu} = (2.57 \pm 0.26) \cdot 10^{-15} \tag{2.5}$$

$$\text{With the theory: } \frac{\Delta\nu}{\nu} = 2.46 \cdot 10^{-15} \tag{2.6}$$

3 Manifolds

3.1 Definitions and Examples

Topologies generalize the notion of open intervalls, balls, etc

3.1.1 Def:

Let there be a collection of subsects $\tau = \left\{U_i | i \in I\right\}$ in a set X such that it possesses the following properties:

- $\emptyset, X \in \tau$
- for any $J \subset I$ we have that $\cup_J U_j \in \tau$.
- for any finite $J \subset I$ we have that $\cap_J U_j \in \tau$.

Then I, J are index sets, τ is a topology in X, and X is a topological space. The elements of τ are called open subsets. For example

- trivial topology $\tau = \{\emptyset, X\}$
- discrete topology $\tau = \{\text{all subsets in } X\}$
- Euclidean topology $X = \mathbb{R}^n, I = \mathbb{R}^n, U_y$ is the open ball with radius r and all unions and intersections

Lecture 7 - 30.4

3.1.1 Continuous Map

let X, Y be topological spaces. Let f be a map from X to Y. Then f is continuous if and only if the inverse of an open set Y is open in X.

example

let $f(x) = \begin{cases} x+1, & x \geq 0 \\ x, & x < 0 \end{cases}$
on $y \in (0, 1/2)$ this is open, as $f^{-1}(0, 1/2) = \emptyset$.
for $y \in (1/2, 3/2)$ this is not open, as $f^{-1}(1/2, 3/2) = [0, 1/2]$ therefore f is not continuous.

3.1.2 Def: Homeomorphism

Let X_1 and X_2 be topological spaces, then $f : X_1 \to X_2$ is a homeomorphism if

- f is bijective
- f is continuous
- f^{-1} is continuous

If a homeomorphism exists between X_1 and X_2 then the spaces are called homeomorphic.

3.1.3 Def: Manifolds

M is an $n-$dimensional differentiable manifold if and only if

- M is a topological space
- M is provided with a family of pairs $\left\{(u_i, \phi_i)\right\}$
- u_i is a family of open sets which covers M
- ϕ_i is a homeomorphism from u_i to an open subset in $\mathbb{R}^n$.
- given u_i, u_f such that $u_i \cap u_f \neq \emptyset$ and the map $\psi_{if} = \phi_i \cdot \phi_f^{-1} : \quad \phi_f(u_1 \cap u_2) \to \phi_i(u_i \cap u_f)$ is infinitely differentiable from $\mathbb{R}^n$ to $\mathbb{R}^n$.
- $\phi_i : p \in \to \underbrace{(x^1(p), \cdots, x^n(p))}_{\text{coordinates of } p} \in \mathbb{R}^n$
- the family $\{(u_i, \phi_i\}$ is called the atlas consisting of charts (u_i, ϕ_i).

Lecture 8 - 5.5

3.2 Vectors

remark about chain rule:
consider two functions $f : \mathbb{R}^m \to \mathbb{R}^m$ and $g : \mathbb{R}^n \to \mathbb{R}^l$. Then

$$g \cdot f = \mathbb{R}^m \to \mathbb{R}^l$$

then with $a \in \{1, \cdots, m\}$ and $c \in \{1, \cdots, l\}$

$$\frac{\partial}{\partial x^a}(g \cdot f)^c = \sum_{b=1} \frac{\partial f^b}{\partial x^a}\frac{\partial g^c}{\partial f^b}$$

More formally, renaming $f \to y$ (we use the Einstein sum convention).

$$\frac{\partial}{\partial x^a} = pdvy^b x^a \frac{\partial}{y^b}$$

applying this to a coordinate transformation $y^\mu = \tilde{x}^\mu = \tilde{x}^\mu(x)$.

$$\frac{\partial}{\partial x^\mu} = \frac{\partial \tilde{x}^\nu}{\partial x^\mu}\frac{\partial}{\partial \tilde{x}^\nu}$$

We also have that

$$\partial_{\nu'} = \frac{\partial x^\mu}{\partial x^{\nu'}}\partial_\mu$$

so, in a matrix sense we have

$$\left(\frac{\partial x^\mu}{\partial x^{\nu'}}\right)^{-1} = \frac{\partial^{\nu'}}{\partial x^\mu}$$

Taking a look at vectors again, we want to look at a general Vector $V = V^\mu e_\mu = V^\mu \partial_\mu$. Under a coordinate change:

$$\partial_\mu = \frac{\partial x^{\mu'}}{\partial x^\mu}\partial_{\mu'}, \quad V^\mu = \frac{\partial x^\mu}{\partial^{\mu'}}V^{\mu'} \quad \implies V \text{ invariant}$$

Let $\gamma : \mathbb{R} \to \mathbb{R}^n$ be a curve in Minkowski space. Then we have the tangent vector

$$V^\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda} = \lim_{\epsilon\to 0}\frac{x^\mu(\lambda+\epsilon) - x^\mu(\lambda)}{\epsilon}$$

$$V = \frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\partial_\mu = \frac{\mathrm{d}}{\mathrm{d}\lambda} \quad \text{coordinate invariant}$$

Now we replace $\mathbb{R}^n$ with M, and the curve $\lambda \in \mathbb{R} \mapsto \gamma(\lambda) \in M$.

$$\frac{\mathrm{d}\gamma}{\mathrm{d}\lambda} = \lim_{\epsilon\to 0}\frac{\gamma(\lambda+\epsilon) - \gamma(\lambda)}{\epsilon}$$

which cannot be done as M is not a vector space and so we have no definition for subtraction.
Now consider a function $f : M \to \mathbb{R}$. Then

$$\frac{\mathrm{d}f(\gamma(\lambda))}{\mathrm{d}\lambda} = \frac{\mathrm{d}}{\mathrm{d}\lambda}f \cdot \gamma$$

where $\mathrm{d}/\mathrm{d}\lambda$ is the tangent vector, the directional derivative.

3.2.1 Def: Tangent Space

We now define the Tangent Space T_p, which is the space of all directional derivative operators along curves through $p \in M$.

3.2.2 Def: Tangent Bundle

the Tangent Bundle is

$$T(M) = \bigcup_{p \in M} T_p$$

Now we claim that T_p is a vector space.

$$\frac{\mathrm{d}f}{\mathrm{d}\lambda} = \frac{\mathrm{d}f \cdot \gamma}{\mathrm{d}\lambda} = \frac{\mathrm{d}}{\mathrm{d}\lambda}\left(f \cdot \phi^{-1} \cdot \underbrace{\phi \cdot \gamma}_{\lambda \mapsto x^\mu(\lambda)} \right) = \frac{\partial f \cdot \phi^{-1}}{\mathrm{d}x^\mu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda} = \partial_\mu \frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda} \tag{3.1}$$

$$\Rightarrow \frac{\mathrm{d}}{\mathrm{d}\lambda} = \frac{\partial x^\mu}{\partial \lambda}\partial_\mu = V^\mu \partial_\mu \tag{3.2}$$

the Right hand side is a vector in $\mathbb{R}^n$, so it's an isomorphism $T_p \cong R^n$
We look at a coordinate transformation $x^\mu \to x^{\mu'}(x)$. We impose invariance of $\mathrm{d}/\mathrm{d}\lambda$.

$$V^\mu = \frac{\partial \lambda^\mu}{\partial \lambda^{\mu'}} V^{\mu'}$$

3.2.3 Remarks:

- coordinates $\neq$ 4 vector
- Lie bracket maps two vectors to one vector

$$[X, Y] = XY - YX, [X, Y]^\mu \partial_\mu = (X^\lambda \partial_\lambda Y^\mu - Y\lambda \partial_\lambda X^\mu)\partial_\mu$$

3.3 Dual Vectors - One Forms

T_p* is the cotangent space. It is defined through the linear forms on T_p.
Let $w \in T_p*$. w is a linear map $T_p \to \mathbb{R}$.
now consider $f : M \to \mathbb{R}$ as before. Define the gradient of f, $\mathrm{d}f \in T_p*$.

$$\mathrm{d}f\left(\frac{\mathrm{d}}{\mathrm{d}\lambda}\right) = \frac{\mathrm{d}f}{\mathrm{d}\lambda} \tag{3.3}$$

which refers to the curve $\gamma(\lambda)$. Pick a basis in T_p which would be $\{\partial_\mu\}$ for us. The natural basis in T_p* is then $\mathrm{d}x^\mu$. We have the property

$$\mathrm{d}^\mu \partial_\nu = \delta^\mu_\nu$$

The left hand side of 3.3

$$= (\mathrm{d}f)_\mu \mathrm{d}x^\mu \left(\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda}\partial_\mu\right) = (\mathrm{d}f)_\mu \frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda}\mathrm{d}x^\mu(\partial_\nu) = (\mathrm{d}f)_\mu \frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda} \tag{3.4}$$

right hand side of 3.3

$$\frac{\mathrm{d}f}{\mathrm{d}\lambda} = \partial_\mu f \frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda} \tag{3.5}$$

$$\Rightarrow (\mathrm{d}f)_\mu = \partial_\mu f \to \mathrm{d}f = \partial_\mu f \mathrm{d}x^\mu \tag{3.6}$$

Doing a coordinate transformation $\partial_{\mu'} = \cdots$ impose that $\mathrm{d}x^{\mu'}(\partial_{\nu'} = \delta^{\mu'}_{\nu'}$ Then

$$\to \mathrm{d}x^{\mu'} = \frac{\partial x^{\mu'}}{\partial x^\mu}\mathrm{d}x^\mu$$

a general element of T_p*. Then impose indepence in choice of coordinate

$$w_{\mu'} = \frac{\partial x^\mu}{\partial^{\mu'}} w_\mu \tag{3.7}$$

3.4 Tensors

Define the direct product of vector spaces. V_1, V_2 are vector spaces, then

$$V_1 \times V_2 = \{(v_1, v_2) | v_1 \in V_1, v_2 \in V_2\}$$

Tensors are a multi linear form

$$T : T_p * \cdots T_p * \times T_p \cdots T_p \to \mathbb{R}$$

3.4.1 Tensor Product

$T = (k, l)$ tensor
$S = (m, n)$ tensor
$T \otimes S = (k + m, l + n)$ tensor.

$$T \otimes S(\omega^1, \cdots, \omega^{(k+m)}, V^1, \cdots V^{l+n}) \tag{3.8}$$

$$= T(\omega^1, \cdots, \omega^k, V^1, \cdots, V^l)S(\omega^{k+1}, \cdots, \omega^{(k+n)}, V^{l+1}, \cdots, V^{(l+n)} \tag{3.9}$$

elements of T_p can be viewed as linear on T_p*

Lecture 9 - 7.5

- basis in (k,l) tensor space = tensor product of $(1,0)$ and

$$T = T^{\mu_1\cdots\mu_k}_{\nu_1\cdots\nu_l}\partial_{\mu_1}\otimes\cdots\otimes\partial_{\mu_k}\otimes \mathrm{d}x^{\nu_1}\otimes\cdots\otimes\mathrm{d}x^{\nu_l}$$

example $(2,0)$ tensor

$$T(\omega^1,\omega^2) = T^{\mu_1\mu_2}\partial_{\mu_1}\otimes\partial_{\mu_2}\left(\omega^1_{\nu_1}\mathrm{d}x^{\nu_1},\omega^2_{\nu_2}\mathrm{d}x^{\nu_2}\right)$$

3.5 Manipulating Tensors

3.5.1 Tensor product

$$T\otimes S(\omega^1,\cdots,\omega^{k+m},V^1,\cdots,V^{l+n}) = \tag{3.10}$$
$$T(\omega^1,\cdots,\omega^k,V^1,\cdots,V^l)S(\omega^{k+1},\cdots,\omega^{k+l},V^{l+1},V^{l+n}) \tag{3.11}$$

in components

$$(T\otimes S)^{\mu_1\cdots\mu_{k+m}}_{\nu_1\cdots\nu_{l+n}} = T^{\mu_1\cdots\mu_k}_{\nu_1\cdots\nu_l}S^{\mu_{k+1}\cdots\mu_{k+m}}_{\nu_{l+1}\cdots\nu_{l+n}}$$

3.5.2 contraction

first for $(1,1)$ tensor. $T : T_p*\times T_p \to \mathbb{R}$
we can view $T\in \mathrm{Span}(T_p\otimes T_p*)$
and the contraction $C : T_p\otimes T_p* \to \mathbb{R}$. $C(V,\omega)=\omega(V)$.

$$T = T^\mu_\nu\partial_\mu\otimes\mathrm{d}x^\nu$$

and

$$C(T) = T^\mu_\nu C(\partial_\mu,\mathrm{d}x^\nu) = T^\mu_\nu\mathrm{d}x^\nu(\partial_\mu) = T^\mu_\mu$$

Generalization to $T \in \mathrm{Span}(T_p\otimes\cdots\otimes T_p\otimes T_p*\otimes\cdots\otimes T_p*)$ with k vector spaces and l covector spaces. Pick a pair T_p, T_p* and treat that as in the $(1,1)$ case $\Rightarrow (k-1,l-1)$ tensor.