QFT - Lecture 12

25.09.24

Contents

1 Quantized Dirac field 3

2 Dirac propagator 4
2.1 Define time ordering 4

3 Feynmann propagator 4

4 Discrete symmetries of Dirac theory 5
4.1 Parity 5

1 Quantized Dirac field

$$\psi(x) = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{\sqrt{E_p}} \sum_s \left(a_p^s u^s(p) e^{-ipx} + b_p^{s\dagger} v^s(p) e^{ipx} \right) \tag{1}$$

$$\bar{\psi}(x) = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_s \left(b_p^s \bar{v}^s(p) e^{-ipx} + a_p^{s\dagger} \bar{u}^s(p) e^{ipx} \right) \tag{2}$$

Single fermion: Energy, Momentum, Charge, Spin

$$\sqrt{2E_p} a_p^{s\dagger} \left|0\right\rangle, \quad E_p, \quad p, \quad +1, \tag{3}$$

Single anti-fermion:

$$\sqrt{2E_p} b_p^{s\dagger} \left|0\right\rangle, \quad E_p, \quad p, \quad -1, \tag{4}$$

Noether, Rotation:

$$\int \mathrm{d}^3 x \psi^\dagger (x \times (-i\nabla) + \frac{1}{2}\Sigma)\psi \tag{5}$$

$$\text{is conserved. } \Sigma^i = \begin{pmatrix} \sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix} \tag{6}$$

Spin:

$$\vec{J} = \int \mathrm{d}^3 x \psi^\dagger \frac{1}{2} \Sigma \psi \tag{7}$$

Now,

$$\xi^1 = (1,\, 0)^T, \quad \xi^2 = (0,\, 1)^T, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{8}$$

$$J_z a_p^{s\dagger} \left|0\right\rangle = \frac{1}{2} \sum_r \xi^{r\dagger} \sigma^3 \xi^s a_p^{s\dagger} \left|0\right\rangle = \begin{cases} \frac{1}{2} & s = 1 \\ -\frac{1}{2} & s = 2 \end{cases} a_p^{s\dagger} \left|0\right\rangle \tag{9}$$

$$= \pm \frac{1}{2} a_p^{s\dagger} \left|0\right\rangle \tag{10}$$

Therefore the Eigenvalues of J_z are $\pm 1/2$ for fermions and $\mp 1/2$ for anti-fermions.

2 Dirac propagator

$$\langle 0| \psi_\alpha(x)\bar{\psi}_b(y) |0\rangle \tag{11}$$

$$= \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \int \frac{\mathrm{d}^3 q}{(2\pi)^3} \frac{1}{\sqrt{2E_p 2E_q}} \sum_{r,s} \langle 0| a_p^r u^r(p) e^{-ipx} a_q^{s\dagger} \bar{u}^s(q) e^{iqx} |0\rangle \tag{12}$$

$$\text{anti commute with } \{a_p^r, a_q^{s\dagger}\} = \delta^{rs}\delta(p-q)(2\pi)^3$$

$$\text{and also use that } \sum_s u^s(p)\bar{u}^s(q) = \not{p} + m$$

$$= \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{2E_p} (\not{p} + m)_{\alpha\beta} e^{-ip(x-y)} \tag{13}$$

$$= (i\not{\partial} + m) \underbrace{\int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{2E_p} e^{-ip(x-y)}}_{D(x-y)} = (i\not{\partial} + m) D(x-y) \tag{14}$$

Klein Gordon:

$$D_F(x-y) = \langle 0| T\phi(x)\phi(y) |0\rangle = \begin{cases} D(x-y) & x^0 > y^0 \\ D(y-x) & x^0 < y^0 \end{cases} = \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{i e^{-ip(x-y)}}{p^2 - m^2 + i\epsilon} \tag{15}$$

$$\langle 0| \bar{\psi}_\beta(y)\psi_\alpha(x) |0\rangle = \cdots = -(i\not{\partial} + m) D(y-x) \tag{16}$$

2.1 Define time ordering

Define time ordering for fermionic fields.

$$T\psi_A(x)\psi_B(y) = \begin{cases} \psi_A(x)\psi_B(y) & x^0 > y^0 \\ -\psi_B(y)\psi_A(x) & x^0 < y^0 \end{cases} \tag{17}$$

3 Feynmann propagator

Defining the Feynmann propagator

$$S_F(x-y) = \langle 0| T\psi(x)\bar{\psi}(y) |0\rangle = \begin{cases} (i\not{\partial} + m) D(x-y) & x^0 > y^0 \\ (i\not{\partial} + m) D(y-x) & x^0 < y^0 \end{cases} \tag{18}$$

So,

$$S_F(x) = (i\not{\partial} + m) D_F(x-y) \tag{19}$$

$$= (i\not{\partial} + m) \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{i e^{-ip(x-y)}}{p^2 - m^2 + i\epsilon} = \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{i(\not{p} + m) e^{-ip(x-y)}}{p^2 - m^2 + i\epsilon} \tag{20}$$

$$= F^{(4)} \left\{ \frac{i(\not{p} + m)}{p^2 - m^2 + i\epsilon} \right\} \tag{21}$$

4 Discrete symmetries of Dirac theory

Parity (Space flip)

$$P : (t, x) \mapsto (t, -x)$$

Time reversal

$$T : (t, x) \mapsto (-t, x)$$

$$C : \text{interchange fermions to anti-fermions}$$

P, T are Lorentz transformations, C is not.

4.1 Parity

$3D$ space: P can be implemented as a reflection followed by a rotation. In particular, a reflection about the yz plane, and a π rotation about the x axis.

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

consider the reflection. A vector will be reflected, such that it flips the sign under parity. There are also vectors such as torque or spin, which represent a rotation, which do not flip sign under parity.

$$\text{flipped sign: } r,\ \dot{r},\ \ddot{r},\ F,\ E \tag{22}$$

$$\text{invariant sign: } T = r \times F,\ L = r \times p,\ B \tag{23}$$

Vectors which flip sign are called polar vectors, vectors. Vectors which do not flip sign are called axial-vectors, or pseudovectors.
On a quantum state P is implemented as a unitary transformation $U(p)$, but call it P. We expect

$$a_p^{s\dagger} |0\rangle = \bar{\eta}_\alpha a_{-p}^{s\dagger} |0\rangle, \quad |\eta_\alpha| = 1$$

$$P a_p^{s\dagger} |0\rangle = P a_p^{s\dagger} P P |0\rangle \tag{24}$$

$$P a_p^{s\dagger} P = \bar{\eta}_\alpha a_{-p}^{s\dagger} \tag{25}$$

$$P a_p^{s} P = \eta_\alpha a_{-p}^{s} \tag{26}$$

$$P b_p^{s\dagger} P = \bar{\eta}_\beta b_{-p}^{s\dagger} \tag{27}$$

What happens to ψ under parity?

$$P\psi(x)P = \int \frac{\mathrm{d}^3 p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\sum_s \left(\eta_\alpha a^s_{-p} u(p) e^{-ipx} + \bar{\eta}_\beta b^{s\dagger}_{-p} v(p) e^{ipx}\right) \tag{28}$$

Change the variable to $p' = (p^0, -p)$. Define $x' = (x^0, -x)$.

$$px = p'x'.\ p'\sigma = p\bar{\sigma},\ p'\bar{\sigma} = p\sigma$$

$$u(p) = \begin{pmatrix} \sqrt{p\sigma}\xi \\ \sqrt{p\bar{\sigma}}\xi \end{pmatrix} = \begin{pmatrix} \sqrt{p'\bar{\sigma}}\xi \\ \sqrt{p'\sigma}\xi \end{pmatrix} = \gamma^0 u(p').\ v(p) = -\gamma^0 v(p')$$

$$\int \frac{\mathrm{d}^3 p'}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\sum_s \left(\eta_\alpha a^s_{p'} \gamma^0 u(p') e^{-ip'x'} - \bar{\eta}_\beta b^{s\dagger}_{p'} \gamma^0 v(p') e^{ip'x'}\right) \tag{29}$$

$$= \gamma^0 \psi(x),\ \text{if } n_\alpha = 1 \text{ and } \eta_\beta = -1. \tag{30}$$

We conclude that the Parity operation produces a Gamma naught.

$$P\psi(x)P = \gamma^0\psi(x) \tag{31}$$

$$P\bar{\psi}(x)P = P\psi^\dagger\gamma^0 P = P\psi^\dagger PP\gamma^0 P = \psi^\dagger\gamma^0\gamma^0 = \bar{\psi}\gamma^0 \tag{32}$$

$$P\bar{\psi}\psi P = P\bar{\psi}PP\psi P = \bar{\psi}\gamma^0\gamma^0\psi = \bar{\psi}\psi \tag{33}$$

$$P\bar{\psi}\gamma^\mu\psi P = P\bar{\psi}PP\gamma^\mu PP\psi P = \bar{\psi}\gamma^0\gamma^\mu\gamma^0\psi \begin{cases} \bar{\psi}\gamma^\mu\psi & \mu = 0 \\ -\psi\gamma^\mu\psi & \mu = i \end{cases} \tag{34}$$

$$P\bar{\psi}\gamma^\mu\gamma^5\psi P = \cdots = \bar{\psi}\gamma^0\gamma^\mu\gamma^5\gamma^0\psi = \begin{cases} -\bar{\psi}\gamma^\mu\gamma^5\psi & \mu = 0 \\ \bar{\psi}\gamma^\mu\gamma^5\psi & \mu = i \end{cases} \tag{35}$$

There is a Table in Peskin and Schröder about Parity operations.