Group Theory - Sheet 1

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10.10.2024

Contents

1 Rotation Symmetries of a cube

1.1 a)

The identity is the 1st invariance.
Then there is an Invariance for every $\pi/2$ turn around any axis. That is, taking the example of the z axis, we can turn the cube $\pi/2$, π, $3\pi/2$ and get an invariance. 2π is not counted as it is equivalent to the Identity.
These are 3 Invariances for 3 unique axes, meaning that there are 9 Invariances.
There is an Invariance for every $2\pi/3$ turn around any vertex. That is for an axis of the form (b_x, b_y, b_z) where b_i is a basis vector. We can turn the cube $2\pi/3$ and $4\pi/3$ and get an Invariance.
These are 2 Invariances for 4 unique axes, meaning there are 8 Invariances.
There is an invariance for every π turn around any edge. That is for an axis of the form $(b_x, b_y, 0)$.
There is 1 Invariance for 6 unique axes, meaning there are 6 Invariances.

In total we have $6 + 8 + 9 + 1 = 24$ Rotational invariances.

2 Chiral Group Axioms

2.1 a)

In the lecture:

- Associativity: $(a_i * a_j) * a_k = a_i * (a_j * a_k)$
- Neutral element: $\exists e \in G \forall a \in G(e * a = a * e = a)$
- Inverse element: $\forall a \exists a^{-1}(a * a^{-1} = a^{-1} * a = e)$

We want to prove that these are equivalent to the Definitions on the Homework sheet. Associativity is trivially equivalent.

2.1.1 Neutral Element

$$\exists e \forall a \in G(e * a = a * e = a) \Leftrightarrow \exists e \in G \forall a \in G(a * e = a)$$

So, we want to see that $a * e = a$ somehow implies $e * a = a$ for this to be equivalent.

Proof: Assume that $\exists e \in G \forall a \in G(a * e = a)$. Assume that $\forall a \in G \exists a^{-1} \in G(a * a^{-1} = e)$. We now want to see that $a * e = e * a$. We know that two elements are the same if they are equal if connected with another element $a * e * d = e * a * d$. We will show this by picking $d = a^{-1}$.

$$a * e * a^{-1} = e * a * a^{-1} \tag{1}$$

$$a * a^{-1} = e * e \tag{2}$$

$$e = e \tag{3}$$

So, $a * e = e$ and $a * a^{-1} = e$ implies $e * a = a * e$. And as equality is transitive, $e = a * e = e * a$. Meaning that the axioms are equivalent, if swapped.

2.1.2 Inverse Element

$$\forall a \exists a^{-1}(a * a^{-1} = a^{-1} * a = e) \Leftrightarrow \forall a \exists a^{-1}(a * a^{-1} = e)$$

We then want to see if $a * a^{-1} = e$ implies $a^{-1} * a = e$. We do the same as we did previously, showing that $a * a^{-1} = a^{-1} * a$

Proof: Assume that $\forall a \exists a^{-1}(a * a^{-1} = e)$ and $\exists e \forall a(a * e = a)$. Through the previously proven statement we can also Assume that $\exists e \forall a(e * a = a)$.

$$a * a^{-1} = a^{-1} * a \tag{4}$$
$$a * a * a^{-1} = a * a^{-1} * a \tag{5}$$
$$a * e = e * a \tag{6}$$
$$a = a \tag{7}$$

So $a * a^{-1} = e$ and $e * a = a * e = a$ imply that $a * a^{-1} = a^{-1} * a$, and through the transitive property of equality we have $e = a * a^{-1} = a^{-1} * a$.

2.1.3 Why only one direction

I only proved the "$\Leftarrow$" direction in both cases, that is because the "$\Rightarrow$" direction is trivial, as the right hand side is included in the left hand side.

2.1.4 Alternative direction

These axioms would still be equivalent with $e * a = a$ or $a^{-1} * a = e$, however the proofs would have to be adjusted slightly.

2.2 b)

We want to show that $e^{-1} = e$, so that e^{-1} acts on elements as e would.

Proof: Assume $\exists e \forall a(e * a = a)$. As this holds for all a, it also holds for e. So we know that $e * e = e$. Then, from the definition of inverses we know that $a * a^{-1} = e$, particularly we know that $e * e^{-1} = e$, setting them equal we get $e * e = e * e^{-1}$, from this we find that $e = e^{-1}$.

3 Groups, Yes or No?

3.1 a)

It does not obey associativity:

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \times \left(\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \times \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \right) = \left(\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \times \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \right) \times \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \tag{8}$$

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \times \begin{pmatrix} b_2c_3 - b_3c_2 \\ b_3c_1 - b_1c_3 \\ b_1c_2 - b_2c_1 \end{pmatrix} = \begin{pmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{pmatrix} \times \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \tag{9}$$

With the first entry on both sides being:

$$(a_3b_1 - a_1b_3)c_3 - (a_1b_2 - a_2b_1)c_2 = a_2(b_1c_2 - b_2c_1) - a_3(b_3c_1 - b_1c_3) \tag{10}$$

Which is not equal as the left side contains a_1 while the right side does not.

3.2 b)

The set lacks an inverse for 3. (The neutral element is 1)

$$3 \cdot 1(\text{mod}3) = 3(\text{mod}3) \tag{11}$$

$$3 \cdot 2(\text{mod}3) = 6(\text{mod}3) = 3(\text{mod}3) \tag{12}$$

$$3 \cdot 3(\text{mod}3) = 9(\text{mod}3) = 3(\text{mod}3) \tag{13}$$

3.3 c)

3.3.1 Neutral Element

The neutral element is f_1.

$$f_1(f_1(x)) = x, \quad f_1(f_2(x)) = -x \tag{14}$$

$$f_1(f_3(x)) = \frac{1}{x}, \quad f_1(f_4(x)) = \frac{-1}{x} \tag{15}$$

3.3.2 Inverse Element

The inverse element of every element is itself.

$$f_1(f_1(x)) = x \tag{16}$$

$$f_2(f_2(x)) = -(-x) = x \tag{17}$$

$$f_3(f_3(x)) = \frac{1}{1/x} = x \tag{18}$$

$$f_4(f_4(x)) = \frac{-1}{-1/x} = x \tag{19}$$

3.3.3 Associativity

I will prove associativity of composition which is more general than the composition of these 4 elements.

Proof: let $f : X \to X, g : X \to X, h : X \to X$ be functions. Then

$$f \cdot (g \cdot h) = (f \cdot g) \cdot h \tag{20}$$
$$f \cdot (g(h(x))) = (f(g(x)) \cdot h \tag{21}$$
$$f(g(h(x))) = f(g(h(x))) \tag{22}$$

Both sides are equivalent, proving associativity.

Hence, The functions form a Group. More generally, bijective functions form a Group.

3.4 Simple Statements

3.4.1 a)

$$\forall a \in G(aG = G), \quad aG := \{ab | b \in G\}$$

Proof: As a Group underlies the closure property, $\forall a, b \in G(ab \in G)$. So, all elements of aG are elements of G, meaning $aG \subset G$. Every element b can be represented as $b = aa^{-1}b$. As $a^{-1}b \in G$, $aa^{-1}b = b \in aG$. So, $aG \subset G$.

As $G \subset aG$ and $aG \subset G$, $G = aG$.

3.4.2 b)

$$\forall a \in G(a^2 = e) \Rightarrow \forall a, b \in G(ab = ba)$$

Proof: Assume $\forall x \in G(x^2 = e)$. Let $a, b \in G$.

$$ab = ba \tag{23}$$

Multiplying ab from the left

$$abab = abba \tag{24}$$
$$(ab)^2 = aea \tag{25}$$

using $x^2 = e$

$$e = aa \tag{26}$$
$$e = e \tag{27}$$

Which proves the statement.

3.5 c)

$Z_2 \times Z_4 = \{a_1b_1, a_1b_2, a_1b_3, a_1b_4, a_2b_1, a_2b_2, a_2b_3, a_2b_4\}$
$Z_8 = \{c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8\}$

	a_1b_1	a_1b_2	a_1b_3	a_1b_4	a_2b_1	a_2b_2	a_2b_3	a_2b_4
a_1b_1	a_1b_1	a_1b_2	a_1b_3	a_1b_4	a_2b_1	a_2b_2	a_2b_3	a_2b_4
a_1b_2	a_1b_2	a_1b_3	a_1b_4	a_1b_1	a_2b_2	a_2b_3	a_2b_4	a_2b_1
a_1b_3	a_1b_3	a_1b_4	a_1b_1	a_1b_2	a_2b_3	a_2b_4	a_2b_1	a_2b_2
a_1b_4	a_1b_4	a_1b_1	a_1b_2	a_1b_3	a_2b_4	a_2b_1	a_2b_2	a_2b_3
a_2b_1	a_2b_1	a_2b_2	a_2b_3	a_2b_4	a_1b_1	a_1b_2	a_1b_3	a_1b_4
a_2b_2	a_2b_2	a_2b_3	a_2b_4	a_2b_1	a_1b_2	a_1b_3	a_1b_4	a_1b_1
a_2b_3	a_2b_3	a_2b_4	a_2b_1	a_2b_2	a_1b_3	a_1b_4	a_1b_1	a_1b_2
a_2b_4	a_2b_4	a_2b_1	a_2b_2	a_2b_3	a_1b_4	a_1b_1	a_1b_2	a_1b_3

	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8
c_1	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8
c_2	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_1
c_3	c_3	c_4	c_5	c_6	c_7	c_8	c_1	c_2
c_4	c_4	c_5	c_6	c_7	c_8	c_1	c_2	c_3
c_5	c_5	c_6	c_7	c_8	c_1	c_2	c_3	c_4
c_6	c_6	c_7	c_8	c_1	c_2	c_3	c_4	c_5
c_7	c_7	c_8	c_1	c_2	c_3	c_4	c_5	c_6
c_8	c_8	c_1	c_2	c_3	c_4	c_5	c_6	c_7

As we can see, the neutral elements a_1b_1, c_1 are in completely different places. Meaning that if we tried to establish an isomorphic mapping, where for example $c_1 = a_1b_1, c_2 = a_1b_2, \cdots, c_8 = a_2b_2$ then we would have $c_1c_1 = e = a_1b_1a_1b_1$, however we can see from the first table that $a_1b_2a_1b_4 = e$, meaning that $c_2c_4 = e$, but as you can see from the second table this isn't true. So there is no isomorphic mapping that preserves the Group structure.

This could also be proven by comparing the Order of each element of each Group, which is the smallest n such that $g^n = e$ for an element $g \in G$.