QFT - Lecture 23

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Contents

1 Vertex correction: Example

Fig1.

Static B field

$$iM = -ie(2m)\xi'^{\dagger}\left(\frac{-1}{2m}\sigma^k\left[F_1(0)+F_2(0)\right]\xi\tilde{B}^k(q)\right) \tag{1}$$

$$iM = -i\tilde{V}(q) \text{ ignoring } (2m) \tag{2}$$

$$\Rightarrow V(x) = -\langle\mu\rangle B(x), \tag{3}$$

$$\langle\mu\rangle = \frac{e}{m}\left[F_1(0)+F_2(0)\right]\xi'^{\dagger}\frac{\sigma}{2}\xi \tag{4}$$

$$\Rightarrow = g\frac{e}{2m}S, \quad S = \chi'^{\dagger}\frac{\sigma}{2}\xi \tag{5}$$

$$\text{Landé g-factor: } g = 2\left[F_1(0)+F_2(0)\right] \approx 2 \tag{6}$$

2 Evaluaton of vertex correction

Fig2.

By the Feynman rules:

$$\bar{u}(p')\underbrace{\delta T^{\mu}(p',p)}_{=T^{\mu}-\gamma^{\mu}}u(p) = \int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{-ig_{\nu\rho}}{(k-p)^2+i\epsilon} \tag{7}$$

$$\cdot\bar{u}(p')(-ie\gamma^{\nu})\frac{i(k'+m)}{k'^2-m^2+i\epsilon}\gamma^{\mu}\frac{i(k+m}{k^2-m^2+i\epsilon}(-ie\gamma^{\rho})u(p) \tag{8}$$

$$= \int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{-i}{(k-p)^2+i\epsilon}\cdot\bar{u}(p')(-ie\gamma_{\rho})\frac{i(k'+m)}{k'^2-m^2+i\epsilon}\gamma^{\mu}\frac{i(k+m}{k^2-m^2+i\epsilon}(-ie\gamma^{\rho})u(p) \tag{9}$$

$$\text{Numerator: } \gamma_{\rho}(\not{k}'+m)\gamma^{\mu}(\not{k}+m)\gamma^{\rho} \tag{10}$$

Using the Gamma identities

$$\gamma_{\rho}\gamma^{\mu}\gamma^{\rho} = -2\gamma^{\mu}, \ \gamma_{\rho}\gamma^{\mu}\gamma^{\nu}\gamma^{\rho} = 4g^{\mu\nu} \tag{11}$$

$$\gamma_{\rho}\gamma^{\mu}\gamma^{\nu}\gamma^{\sigma}\gamma^{\rho} = -2\gamma^{\sigma}\gamma^{\nu}\gamma^{\mu} \tag{12}$$

and we then obtain

$$\bar{u}(p')\delta T^{\mu}u(p) = 2ie^2\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{\bar{u}(p')\left[\not{k}\gamma^{\mu}\not{k}'+m^2\gamma^{\mu}-2m(k+k')^{\mu}\right]u(p)}{\left((k-p)^2+i\epsilon\right)(k'^2-m^2+i\epsilon)(k^2-m^2+i\epsilon)} \tag{13}$$

3 Feynman Parameters:

$$\frac{1}{AB} = \int_0^1 \mathrm{d}x \frac{1}{(xA+(1-x)B)^2} = \int_0^2 \mathrm{d}x \int_0^1 \mathrm{d}y \delta(x+y-1) \frac{1}{(xA+(1-x)B)^2} \tag{14}$$

$$\frac{1}{B-A}\int_A^B \frac{\mathrm{d}t}{t^2} = \frac{AB}{AB(B-A)}\left(\frac{1}{A}-\frac{1}{B}\right) = \frac{1}{AB(B-A)}(B-A) = \frac{1}{AB} \tag{15}$$

$$\text{letting } t = B + (A-B)x \text{ and } \mathrm{d}t = (A-B)\mathrm{d}t \tag{16}$$

Generalizing it by Induction leads to

$$\frac{1}{A_1 A_2 \cdots A_n} = \int_0^1 \mathrm{d}x_1 \mathrm{d}x_2 \cdots \mathrm{d}x_n \delta\left(\sum_i x_i - 1\right) \frac{(n-1)!}{\left(x_1 A_1 + x_2 A_2 + \cdots\right)^n} \tag{17}$$

denominator of (13):

$$\frac{1}{\left((k-p)^2+i\epsilon\right)(k'^2-m^2+i\epsilon)(k^2-m^2+i\epsilon} \tag{18}$$

$$= \int_0^1 \mathrm{d}x\mathrm{d}y\mathrm{d}z\delta(x+y+z-1)\frac{2}{D^3} \tag{19}$$

$$D = x(k^2-m^2+i\epsilon) + y\left(\underbrace{(k+q)^2}_{k'=k+q} - m^2 + i\epsilon\right) + z\left((k-p)^2+i\right) = k^2 + 2k(\cdots) + \cdots \tag{20}$$

and then through completing the square and defining $l = k + yq - zp$

$$\Rightarrow \cdots \Rightarrow D = l^2 - \Delta + i\epsilon,\ \Delta = -xyq^2 + (1-z)^2 m^2 \tag{21}$$

$$\text{needed was also: } p^2 = m^2 = p'^2 = (p+q)^2 \Rightarrow 2pq + q^2 = 0 \tag{22}$$

We now write down the result of (13)

$$= 2ie^2 \int \frac{\mathrm{d}^4 l}{(2\pi)^4} \int_0^1 \mathrm{d}x\mathrm{d}y\mathrm{d}z\delta(x+y+z-1)\frac{1}{D^3} \tag{23}$$

$$\cdot \bar{u}(p')\left[\gamma^\mu \underbrace{\left(\frac{-l^2}{2} + (1-x)(1-y)q^2 + (1-4z+z^2)m^2\right)}_{\text{leads to } F_1(q^2)} + \frac{i\sigma^{\mu\nu}q_\nu}{2m}\underbrace{\left(2m^2 z(1-z)\right)}_{\text{leads to } F_2(q^2)}\right]u(p) \tag{24}$$

We now have a total factor of $1/l$ in the integral for $F_1(q^2)$, which leads to a logarithmic divergence. First, evaluate the $F_2(q^2)$ part, as it is unproblematic.

4 Wick Rotation

$$\frac{\mathrm{d}^4 l}{(2\pi)^4}\frac{1}{(l^2-\Delta+i\epsilon)^3} \tag{25}$$

with our square being the Minkowski metric. We want to integrate in the Euclydian metric though. We define l_E such that

$$l_E: \quad l^0 = i l_E, \quad l = l_E \tag{26}$$
$$\Rightarrow l^2 = l^{02} - l^2 = -(l_E^0)^2 - l^2 = -l_E^2 \tag{27}$$
$$\mathrm{d}^4 l = i\mathrm{d}^4 l_E \tag{28}$$

so we can express our integral as

$$i\int \frac{\mathrm{d}^4 l_E}{(2\pi)^4}\frac{1}{\left(-l_E^2-\Delta+i\epsilon\right)} \tag{29}$$

We originally integrated l^0 along the Real axis from $-\infty$ to ∞.

We rotate our previous Integral path before our variable transformation, such that we integrate from $-i\infty$ to $i\infty$. We need to rotate the Integral path so that we do not go through the poles, as they would bring a contribution to the integral which we don't want.

Then

$$= \frac{-i}{(2\pi)^4}\int_0^\infty \mathrm{d}l_E \underbrace{2\pi^2 l_E^3}_{\text{surface of 4D sphere}} \frac{1}{\left(l_E^2+\Delta-i\epsilon\right)^3} \tag{30}$$
$$= \frac{-i}{(2\pi)^2}\frac{1}{2\Delta} \tag{31}$$

4.1 $F_2(q^2)$

$$F_2(q^2) = 2ie^2\int\frac{\mathrm{d}^4 l}{(2\pi)^4}\int_0^1 \mathrm{d}x\mathrm{d}y\mathrm{d}z\delta(x+y+z-1)\frac{2}{D^3}(2m^2 z(1-z)) \tag{32}$$
$$= \frac{\alpha}{2\pi}\int_0^1 \mathrm{d}x\mathrm{d}y\mathrm{d}z\delta(x+y+z-1)\frac{2m^2 z(1-z)}{(1-z)^2 m^2 - xyq^2} \tag{33}$$
$$\Rightarrow F_2(0) = \frac{\alpha}{2\pi}\int_0^1 \mathrm{d}x\mathrm{d}y\mathrm{d}z\delta(x+y+z-1)\frac{2z}{1-z} \tag{34}$$

For a non-zero result from the x-integration, $y+z-1$ mus be between -1 and 0. From this, we get $0 \le y \le 1-z$.

$$F(0) = \frac{\alpha}{2\pi}\int \mathrm{d}z\int_0^{1-z}\mathrm{d}y\frac{2z}{1-z} = \frac{\alpha}{2\pi}\int_0^1 \mathrm{d}z 2z = \frac{\alpha}{2\pi} \tag{35}$$

so then we have the result that

$$
\begin{aligned}
g &= 2 + F_2(0) && (36) \\
&= 2 + \frac{\alpha}{\pi} + O(\alpha^2) && (37)
\end{aligned}
$$

And it is conventional to define

$$
a_e = \frac{g-2}{2} = \frac{\alpha}{2\pi} \approx 0.00161 \tag{38}
$$

$$
\text{experimental value: } a_e \approx 0.001160 \tag{39}
$$