QFT - Lecture 1

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Contents

Why QFT?

- nonrelativistic Quantum Mechanics is Noncausal
- Quantum Mechanics cannot describe generation/annihilation of particles
- Even in low-energy Physics, QFT is used to describe the Photon field

1 Classical field theory

In classical mechanics:

$$s = \int L(q_i, \dot{q}_i)\mathrm{d}t \tag{1}$$

It is required that the action is stationary, $\delta s = 0$, from which the Euler Lagrange equations are derived.

1.1 Example: String

Abbildung 1.2

we consider longitudal oscillation. And use the discretization:

Abbildung 1.2

We then have the Lagrangian

$$L = T - V = \sum_i \left(\frac{1}{2} m \dot{q_i}^2 - \frac{1}{2} k (q_{i+1} - q_i)^2 \right) \tag{2}$$

For field theory, q will become the field, and x will be the discretization i.

$$L = \sum_i a \left(\frac{1}{2} \frac{m}{a} \dot{q_i}^2 - \frac{1}{2} ka \left(\frac{q_{i+1} - q_i}{a} \right)^2 \right) \tag{3}$$

$a \to 0$.

$$L = \int \mathrm{d}x \underbrace{\left[\frac{1}{2} \mu \dot{q_i}^2 - \frac{1}{2} Y \left(\frac{\partial q}{\partial x} \right)^2 \right]}_{\text{Lagrange Density}} \tag{4}$$

$i \mapsto x$, and $q_i \mapsto \varphi$.

Abbildung 2

2 Lagrangian field theory

$$s = \int \mathrm{d}t L = \int \mathrm{d}^3 x \int \mathrm{d}t L = \int \mathrm{d}^4 x L \tag{5}$$

where we have

$$L = L(\varphi, \partial_\mu \varphi) \tag{6}$$

and we use the principle of least action, such that $\delta s = 0$.

$$\delta s = \int \mathrm{d}^4 x \left(\frac{\partial L}{\partial \varphi} \delta\varphi + \frac{\partial L}{\partial(\partial_\mu \varphi)} \delta(\partial_\mu \varphi) \right) \tag{7}$$

$$= \int \mathrm{d}^4 x \, \partial_\mu \underbrace{\left(\frac{\partial L}{\partial(\partial_\mu \varphi)} \delta\varphi \right)}_{K^\mu} + \int \mathrm{d}^4 x \left(\frac{\partial L}{\partial \varphi} - \partial_\mu \left(\frac{\partial L}{\partial(\partial_\mu \varphi)} \right) \right) \delta\varphi \tag{8}$$

$$\int \mathrm{d}^4 x \partial_\mu K^\mu = \int_{\text{bounding surface, } B} K^\mu \mathrm{d}S_\mu \tag{9}$$

$$\text{Remember: } \int_V \mathrm{d}^3 x \nabla A = \oint_S dSA \tag{10}$$

and we require the action to be zero at the bounding surface, so for all points $x \in B$, $\delta\phi(x) = 0$.

$$\text{Then: } \int \mathrm{d}S_\mu K^\mu = 0. \tag{11}$$

And then

$$\frac{\partial L}{\partial \varphi} - \partial_\mu \left(\frac{\partial L}{\partial(\partial_\mu \varphi)} \right) = 0 \tag{12}$$

Which is the Euler-Lagrange equation, as desired. This follows from the fundamental theorem of variational calculus and $Eq.8$.

2.1 Example: String

$q = \varphi$.

$$\frac{\partial L}{\partial q} = 0 \ , \ \frac{\partial L}{\partial \dot{q}} = \mu \dot{q} \ , \ \frac{\partial L}{\partial(\frac{\partial q}{\partial x})} = -Y \frac{\partial q}{\partial x} \tag{13}$$

$$\frac{\partial L}{\partial q} - \partial_0 \frac{\partial L}{\partial \partial_0 q} - \partial_i \left(\frac{\partial L}{\partial(\partial_i q)} \right) = 0 \tag{14}$$

$$-\mu \ddot{q} + Y \frac{\partial^2 q}{\partial x^2} = 0 \quad \text{Wave Equation} \tag{15}$$

2.1.1 Four-Vector

A^μ is a four-vector if it transforms as x^μ under Lorentz Transformations.

3 Hamiltonian Field Theory

Classical mechanics:

$$p_i = \frac{\partial L}{\partial \dot{q}_i} \text{: conjugated momentum} \tag{16}$$

$$H = \sum_i p_i \dot{q}_i - L = E \tag{17}$$

For the string, this means that

$$p_i = m\dot{q}_i \ , \ H = \sum_i \left(\frac{1}{2} m\dot{q}_i{}^2 + \frac{1}{2} k (q_{i+1} - q_i)^2 \right) \tag{18}$$

and with $a \to 0$

$$H = \int \mathrm{d}x \left(\frac{1}{2} \mu \dot{q}_i{}^2 + \frac{1}{2} Y \left(\frac{\partial L}{\partial x} \right)^2 \right) \tag{19}$$

Classical field theory:

$$\pi = \frac{\partial L}{\partial \dot{\varphi}} \text{: conjugated momentum density} \tag{20}$$

$$\mathcal{H} = \pi \dot{\varphi} - L \ , \ H = \int \mathrm{d}x \mathcal{H} \tag{21}$$

4 Klein-Gordon field

$$\frac{1}{2} \underbrace{(\partial_\mu \varphi)(\partial^\mu \varphi)}_{\text{P\&S: } (\partial_\mu \varphi)^2} - \frac{1}{2} m^2 \phi^2 \tag{22}$$

Euler-Equation:

$$-m^2 \varphi - \partial_\mu \partial^\mu \varphi = (m^2 + \partial_\mu \partial^\mu) \varphi = 0 \tag{23}$$

Then we have

$$\frac{\partial^2 \varphi}{\partial t^2} - \nabla^2 \varphi + m^2 \varphi = 0 \tag{24}$$

and for the Hamiltonian field

$$\pi = \frac{\partial L}{\partial \dot{\varphi}} = \dot{\varphi} \tag{25}$$

$$\mathcal{H} = \dot{\varphi}^2 - L = \frac{1}{2} (\dot{\varphi}^2 + (\nabla \varphi)^2) + \frac{1}{2} m^2 \varphi^2 \tag{26}$$

Which is not Lorentz invariant, as you give Energy into the system.

5 Noether Theorem

If you have a continuous symmetry, then you get a conservation law.

- Translational Symmetry: conservation of momentum
- Rotational Symmetry: conservation of angular momentum
- Time Symmetry: conservation of energy

5.0.1 Symmetry Definition

A symmetry is a transformation such that the equation of motion is unchanged.

$$L \mapsto L + \alpha \partial_\mu J^\mu \tag{27}$$

We start with transforming L to L with a four-divergence. This does not change the action, as a four-divergence is 0 at the boundary.