QFT - Lecture9

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Contents

1 Functions of matrices

1)
A is the square matrix.
$f x) = \sum_n c_n x^n$. Then

$$f(A) = \sum_n c_n A^n.$$

assuming that the Taylor expansion exists, and converges.

2)
Now, assuming that A is normal, meaning that $A^\dagger A = AA^\dagger$. Then

$$A = U\Lambda U^\dagger$$

then we can define

$$f(A) = Uf(\Lambda)U^\dagger, \quad f(\Lambda) = \begin{pmatrix} f(\lambda_1) & 0 & \cdots \\ 0 & f(\lambda_2) & \cdots \\ \vdots & \cdots & 0 \end{pmatrix}$$

When 1) and 2) both exist:

$$f_1 = \sum c_n A^n = \sum c_n (U\Lambda U^\dagger)^n = \sum_n c_n U\Lambda^n U^\dagger \tag{1}$$

$$= U \sum c_n A^n U^\dagger = Uf(\Lambda)U^\dagger = f_2 \tag{2}$$

If normal A and B commute, then $\sqrt{AB} = \sqrt{A}\sqrt{B}$.

Proof:

$$[A, B] = 0 \Rightarrow A = U\Lambda_A U^\dagger, B = U\Lambda_B U^\dagger. \tag{3}$$

$$\sqrt{AB} = sqrt U\Lambda_A U^\dagger U\Lambda_B U^\dagger = \sqrt{U\Lambda_A\Lambda_B U^\dagger} = U\sqrt{\Lambda_A\Lambda_B}U^\dagger \tag{4}$$

$$= U\sqrt{\Lambda_A}U^\dagger U\sqrt{\Lambda_B}U^\dagger = \sqrt{A}\sqrt{B}. \tag{5}$$

2 Solutions to the Dirac Equation

$$(i\partial\!\!\!/ - m)\psi = 0 \tag{6}$$

Let $\psi(x) = u(p) \cdot e^{ipx}$

Psi must also be a solution to the Klein Gordon equation.

$$(\partial_\mu \partial^\mu + m^2)\psi = 0 \tag{7}$$

Starting:

$$(i(-ip\!\!\!/) - m)u(p) = 0 = (p\!\!\!/ - m)u(p) \tag{8}$$

$$(-p^2 + m^2)u(p) \stackrel{!}{=} 0 \tag{9}$$

Which means that we require the On-Shell condition, so that $p^2 = m^2$.

Assume that we are in the rest frame, such that $p = (m, 0)$. Thus

$$\not{p} = \gamma^0 p_0 = m\gamma^0 = m \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \tag{10}$$

So (8) yields

$$m \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} u = 0, \; u = \begin{pmatrix} \xi \\ \xi \end{pmatrix}, \; \xi = (1, 0)^T \text{ or } \xi = (0, 1)^T \tag{11}$$

Rotation: $u \mapsto S(\Lambda)u = \begin{pmatrix} e^{-i\varphi\sigma/2} & 0 \\ 0 & e^{-i\varphi\sigma/2} \end{pmatrix} \begin{pmatrix} \xi \\ \xi \end{pmatrix}$

Boost: $u \mapsto S(\Lambda)u = \begin{pmatrix} e^{-\eta\sigma/2} & 0 \\ 0 & e^{+\eta\sigma/2} \end{pmatrix} \begin{pmatrix} \xi \\ \xi \end{pmatrix}$

...

$$u(p) = \begin{pmatrix} \sqrt{p\sigma}\xi \\ \sqrt{p\bar{\sigma}}\xi \end{pmatrix}$$

Claim: $\sqrt{(p\sigma)(p\bar{\sigma})} = m$.

Proof:

$$(p\sigma)(p\bar{\sigma}) = (E - \vec{p}\vec{\sigma})(E + \vec{p}\vec{\sigma}) = E^2 - (\vec{p}\vec{\sigma})^2 \tag{12}$$

$$= E^2 - p^i\sigma^i p^j \sigma^j = E^2 - p^i p^j \cdot \frac{\sigma^i\sigma^j + \sigma^j\sigma^i}{2} = E^2 - p^i p^j = E^2 - \vec{p}^2 = p^2 = m \tag{13}$$

Verify that $u(p)$ satisfies the Dirac equation:

$$\begin{pmatrix} -m & i\sigma\partial \\ i\bar{\sigma}\partial & -m \end{pmatrix} \begin{pmatrix} \sqrt{p\sigma}\xi \\ \sqrt{p\bar{\sigma}}\xi \end{pmatrix} e^{-ipx} = \begin{pmatrix} -m\sqrt{p\sigma}\xi + i\sigma(-ip)\sqrt{p\bar{\sigma}}\xi \\ i\bar{\sigma}(-ip)\sqrt{p\sigma}\xi - m\sqrt{p\sigma}\xi \end{pmatrix} \tag{14}$$

$$(-m\sqrt{p\sigma} + \sqrt{p\sigma}\sqrt{p\sigma}\sqrt{p\bar{\sigma}})\xi = 0 \tag{15}$$

So, it is zero and $u(p)$ fulfills the Dirac equation.

3 Normalization

$$\bar{\psi} = \psi^\dagger\gamma^0, \quad \bar{u} = u^\dagger\gamma^0 \tag{16}$$

So,

$$(\xi^\dagger\sqrt{p\sigma}, \xi^\dagger\sqrt{p\bar{\sigma}}) \begin{pmatrix} \sqrt{p\sigma}\xi \\ \sqrt{p\bar{\sigma}}\xi \end{pmatrix} = \xi^\dagger(p\sigma + p\bar{\sigma})\xi = 2E\xi^\dagger\xi \tag{17}$$

$$\bar{u}u = \xi^\dagger(\sqrt{p\sigma}\sqrt{p\bar{\sigma}} + \sqrt{p\bar{\sigma}}\sqrt{p\sigma})\xi = 2m\xi^\dagger\xi \tag{18}$$

We have 2 linearly independent solutions $\psi(x) = u^s(p)e^{-ipx}, u^s(p) = \begin{pmatrix} \sqrt{p\sigma}\xi^s \\ \sqrt{p\bar{\sigma}}\xi^s \end{pmatrix}, \quad s = 1, 2.$
and $\psi(x) = v^s(p)e^{ipx}, v^s(p) = \begin{pmatrix} \sqrt{p\sigma}\eta^s \\ -\sqrt{p\bar{\sigma}}\eta^s \end{pmatrix}$

Orthogonality:

$$\bar{u}^r(p)v^s(p) = (\xi^{r\dagger}\sqrt{p\sigma}, \xi^{r\dagger}\sqrt{p\bar{\sigma}}) \begin{pmatrix} -\sqrt{p\bar{\sigma}}\eta^s \\ \sqrt{p\sigma}\eta^s \end{pmatrix} = \xi^{r\dagger}(-m+m)\eta^s = 0 \tag{19}$$

$$u^{r\dagger}(\vec{p})v^s(-\vec{p}) = (\xi^{r\dagger}\sqrt{p\sigma}, \xi^{r\dagger}\sqrt{p\bar{\sigma}}) \begin{pmatrix} -\sqrt{p\bar{\sigma}}\eta^s \\ \sqrt{p\sigma}\eta^s \end{pmatrix} \tag{20}$$

$$E + \vec{p}\vec{\sigma} = p\bar{\sigma}, \quad E - \vec{p}\vec{\sigma} = p\sigma \tag{21}$$

$\bar{u}^r(p)u^s(p) = 2m\delta^{rs}$

4 Spin sums

When calculating cross sections, we will need:

$$\sum_{s=1,2} u^s(p)\bar{u}^s(p) = \sum_{s=1,2} \begin{pmatrix} \sqrt{p\sigma}\xi^s \\ \sqrt{p\bar{\sigma}}\xi \end{pmatrix} (\xi^{s\dagger}\sqrt{p\sigma},\ \xi^{s\dagger}\sqrt{p\bar{\sigma}})\gamma^0 \tag{22}$$

$$\left[\sum_s \xi^s\xi^{s\dagger} = 1, \quad \sum_s |s\rangle\langle s| = 1\right] \tag{23}$$

$$= \begin{pmatrix} p\sigma & \sqrt{p\sigma}\sqrt{p\bar{\sigma}} \\ \sqrt{p\bar{\sigma}}\sqrt{p\sigma} & p\bar{\sigma} \end{pmatrix}\gamma^0 = \begin{pmatrix} m & p\sigma \\ p\bar{\sigma} & m \end{pmatrix} = m + p^0\gamma^0 - p^i\gamma^i = m + \not{p} \tag{24}$$

$$\sum_s v^s(p)v^{-s}(p) = \cdots = \not{p} - m \tag{25}$$

Addition: $m = \sqrt{(p\sigma)(p\bar{\sigma})} = \sqrt{p\sigma}\sqrt{p\bar{\sigma}}$

$$p\sigma = E - \vec{p}\vec{\sigma}, \quad p\bar{\sigma} = E + \vec{p}\vec{\sigma} \tag{26}$$

$$[p\sigma, p\bar{\sigma}] = 0. \tag{27}$$

5 γ^5

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 \tag{28}$$

$$(\gamma^5)^\dagger = -i\gamma^0\gamma^0\gamma^0\gamma^0\gamma^1\gamma^0\gamma^0\gamma^2\gamma^0\gamma^0\gamma^3\gamma^0 = -i\gamma^1\gamma^2\gamma^3\gamma^0 = i\gamma^0\gamma^1\gamma^2\gamma^3 \tag{29}$$

$$(\gamma^5)^2 = 1, \quad \gamma^5 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \tag{30}$$

$$\{\gamma^5, \gamma^\mu\} = 0, \ \gamma^5\gamma^\mu = i\gamma^0\gamma^1\gamma^2\gamma^3\gamma^\mu = -\gamma^\mu\gamma^5 \tag{31}$$