QFT - Lecture 13

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Contents

1 Interacting fields

1)

$$\varphi^4\text{-Theory: } \mathcal{L} = \frac{1}{2}\partial_\mu \varphi \partial^\mu \varphi - \frac{1}{2} m^2 \varphi^2 - \frac{\lambda}{4!}\varphi^4$$

- Self interaction theory
- Simplest interaction theory (Interaction between Fourier components)
- Contains all essential features of an interaction theory
- Not possible to solve analytically
- Relevance to the Standard model and in statistical mechanics and solid state physics

2)

$$\text{Yukawa-Theory: } \mathcal{L}_{\text{Yukawa}} = L_{\text{Dirac}} + L_{\text{KG}} - g\bar{\psi}\psi\varphi$$

- Very similar to QED, but simpler

3)

$$\begin{aligned}\text{QED: } \mathcal{L}_{\text{QED}} &= \mathcal{L}_{\text{Maxwell}} + \mathcal{L}_{\text{Dirac}} - \underbrace{e\bar{\psi}\gamma^\mu\psi A_\mu}_{j^\mu A_\mu} \\ &= \mathcal{L}_{\text{Maxwell}} + \bar{\psi}(i\partial\!\!\!/ - m)\psi - \underbrace{e\bar{\psi}\gamma^\mu\psi A_\mu}_{e\bar{\psi}A\!\!\!/\psi} \\ &= \mathcal{L}_{\text{Maxwell}} + \bar{\psi}(iD\!\!\!\!/ - m)\psi, \quad D\!\!\!\!/ = \partial\!\!\!/ + ieA\!\!\!/, \quad \text{Gauge covariant Derivative}\end{aligned}$$

- explains almost all phenomena down to $10^{-15}m$
- The QED Lagrangian is invariant under gauge transformation $A_\mu \mapsto A_\mu - \frac{1}{e}\partial_\mu \alpha(x)$ and $\psi \mapsto e^{i\alpha(x)}\psi$ together

2 The Interaction Picture

- Define a t_0 where all pictures coincide.
- Schrödinger Picture:
  - State kets evolve in time $|\psi(t)\rangle = e^{iH(t-t_0)}|\psi(t_0)\rangle$
  - Operators $A(t_0)$ independent of time
- Heisenberg Picture:
  - State kets $|\psi(t_0)\rangle$ are indepdentent of time
  - Operators evolve in time $A(t) = e^{iH(t-t_0)}A(t_0)e^{-iH(t-t_0)}$

- Interaction Picture:
  - $A(t) := e^{iH_0(t-t_0)}A(t_0)e^{-iH_0(t-t_0)}$ where we have $H = H_0 + H_{\text{interacting}}$
  - $|\psi(t)\rangle := e^{iH_0(t-t_0)}e^{-iH(t-t_0)}|\psi(t_0)\rangle$

Calculating the expectation values:

$$\langle A\rangle = \langle\psi(t)|\, A(t)\,|\psi(t)\rangle = \cdots \tag{1}$$

$$= \langle\psi(t_0)|\, e^{iH(t-t_0)}A(t_0)e^{-iH(t-t_0)}\,|\psi(t_0)\rangle \tag{2}$$

Showing that the definitions are equivalent physically.
We define the Operator:

$$U(t,t_0) = e^{iH_0(t-t_0)}e^{-iH(t-t_0)}$$

Some properties we want:

- $U(t_3,t_2)U(t_2,t_1) = U(t_3,t_1)$
- $U(t_2,t_1) = U^\dagger(t_1,t_2)$
- $U(t,t')$ General

We want an expression for $U(t,t') = U(t,t_0)U(t_0,t')$

$$U(t,t_0)U^\dagger(t',t_0) = e^{iH_0(t-t_0)}e^{-iH(t-t_0)}e^{iH(t'-t_0)}e^{-iH_0(t'-t_0)} \tag{3}$$

$$= e^{iH_0(t-t_0)}e^{iH(t'-t)}e^{-iH_0(t'-t_0)} \tag{4}$$

3 Perturbation Theory

Goal: Calculating $\langle\Omega|\, T\phi(x)\phi(y)\,|\Omega\rangle$ in ϕ^4-Theory

$$H = H_0 + \int \mathrm{d}^3x \frac{\lambda}{4!}\phi^4$$

with H_0 being the KG Hamiltonian.
Hamiltonian in the interaction picture:

$$e^{iH_0(t-t_0)}(H_0 + H_{\text{int}})e^{-iH_0(t-t_0)} \tag{5}$$

$$= H_0 + H_I(t), \quad H_I = e^{iH_0(t-t_0)}H_{\text{int}}e^{-iH_0(t-t_0)} \tag{6}$$

$$\phi_I(t) = e^{iH_0(t-t_0)}\phi(t_0)e^{-iH_0(t-t_0)} \tag{7}$$

$$= e^{iH_0(t-t_0)}e^{-iH(t-t_0)}\phi(t)e^{iH(t-t_0)}e^{-iH_0(t-t_0)} \tag{8}$$

$$= U(t,t_0)\phi(t)U^\dagger(t,t_0) \tag{9}$$

$$\Rightarrow \phi(t) = U^\dagger(t,t_0)\phi_I(t)U(t,t_0) \tag{10}$$

Find a useful expression for $U(t,t')$.

$$i\frac{\partial U(t,t')}{\partial t} = e^{iH_0(t-t_0)}\underbrace{(H-H_0)}_{H_{\text{int}}}\left(e^{-iH_0(t-t_0)}e^{iH_0(t-t_0)}\right)e^{iH(t'-t)}e^{-iH_0(t-t_0)} \tag{11}$$

$$= H_I(t)U(t,t') \tag{12}$$

Solution through Dyson series:

$$U(t,t') = 1 + (-i)\int_{t'}^{t} \mathrm{d}t_1 H_I(t_1) + (-i)^2 \int_{t'}^{t}\int_{t'}^{t_1} \mathrm{d}t_1 \mathrm{d}t_2 H_I(t_1)H_I(t_2) + \cdots \tag{13}$$

$$\frac{\partial U}{\partial t_1} = (-i)H_I(t_1) + (-i)^2 H_I(t_1)\int_{t'}^{t} \mathrm{d}t_2 H_I(t_2) + \cdots \tag{14}$$

Useful rewrite:

$$U(t,t') = 1 + (-1)\int_{t'}^{t} \mathrm{d}t_1 H_I(t_1) + \frac{(-i)^2}{2}\int_{t'}^{t}\int_{t'}^{t_1} \mathrm{d}t_1 \mathrm{d}t_2 T\{H_I(t_1)H_I(t_2)\} + \cdots \tag{15}$$

$$= T\left\{\exp\left[-i\int_{t'}^{t} \mathrm{d}t'' H_I(t'')\right]\right\} \tag{16}$$

Why is this more convenient?

$$H_I(t) = \int \mathrm{d}^3 x \frac{\lambda}{4!}\varphi_I^4 \tag{17}$$

We now have $\phi(t)$, ϕ_I and $U(t,t')$.

Next time: what is $|\Omega\rangle$?