QFT - Lecture 4

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Contents

1 Lorentz Transformation

$$x^\mu \mapsto x'^\mu = \Lambda^\mu_\nu x^\nu \tag{1}$$

$$g_{\mu\nu} = g_{\rho\sigma}\Lambda^\rho_\mu \Lambda^\sigma_\nu \tag{2}$$

$$g = \Lambda^T g \Lambda \det(g) = \det(\Lambda)^2 \det(g) \quad \Rightarrow \quad \det(\Lambda) = \pm 1 \tag{3}$$

We take these as defining properties for the Lorentz Transformation matrix.
Discrete transformation:

$$\Lambda = P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{4}$$

Then there is also

$$\Lambda = T = -P \tag{5}$$

and then

$$PT = -1 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{6}$$

Proper Lorentz Transformations:

$$\Lambda^0_0 > 1 \tag{7}$$

$$\det(\Lambda) = +1 \tag{8}$$

The proper Lorentz Transformation can be continuously connected to the identity. The previous 3 Transformations cannot be transformed.

Rotation:

$$\Lambda = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & R_{xx} & R_{xy} & R_{xz} \\ 0 & R_{yx} & R_{yy} & R_{yz} \\ 0 & R_{zx} & R_{zy} & R_{zz} \end{pmatrix} \tag{9}$$

Boost:

$$\Lambda = \begin{pmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \beta = v/c, \quad \gamma = \frac{1}{\sqrt{1-\beta^2}} \tag{10}$$

$$\det(\Lambda) = \gamma^2 - \gamma^2\beta^2 = \gamma^2(1-\beta^2) = 1 \tag{11}$$

2 Quantized Klein-Gordon field

the ket $a_p^\dagger \left|0\right\rangle$ is a single particle state. The ket has Energy E_p with momentum p

We will now consider the normalization.

$$\left\langle 0\right| a_q a_p^\dagger \left|0\right\rangle = (2\pi)^3 \delta(p-q) \tag{12}$$

We now need to make this Lorentz-Invariant.

2.1 Lorentz Transformation of $\delta(p-q)$

$$\delta(p-q) = \delta(p_1-q_1)\delta(p_2-q_2)\delta(p_3-q_3) \tag{13}$$

Consider a boost in the 3-direction:

$$p_3' = \gamma(p_3+\beta E) \qquad p = (E, p_1, p_2, p_3),\ p' = (E', p_1, p_2, p_3')E' = \gamma(E+\beta p_3) \tag{14}$$

$$\delta(p-q) \mapsto \delta(p_1-q_1)\delta(p_2-q_2)\delta(p_3'-q_3') \cdot \left|\frac{\mathrm{d}p_3'}{\mathrm{d}p_3}\right| \tag{15}$$

$$\delta(f(x)-f(x_0)) = \frac{1}{|f'(x_0)|}\delta(x-x_0) \tag{16}$$

$$E^2 = \sum_i (p^i)^2 + m^2, \quad 2E\mathrm{d}E = 2p_3\mathrm{d}p_3. \tag{17}$$

$$\frac{\mathrm{d}p_3'}{\mathrm{d}p_3} = \gamma\left(1+\beta\frac{\mathrm{d}E}{\mathrm{d}p_3}\right) = \gamma\left(1+\beta\frac{p_3}{E}\right) = \frac{\gamma}{E}(E+\beta p_3) = \frac{E'}{E} \tag{18}$$

$$\Rightarrow \delta(p-q) = \delta(p'-q')\frac{E'}{E} \longrightarrow E\delta(p-q) = E'\delta(p'-q') : \text{ Lorentz-Invariant} \tag{19}$$

2.2 Normalization

Choose normalization as follows.

$$\left|p\right\rangle = \sqrt{2E_p} a_p^\dagger \left|0\right\rangle \tag{20}$$

$$\left\langle q\right| \left|p\right\rangle = 2\sqrt{E_pE_q}\left\langle 0\right| a_q a_p^\dagger \left|0\right\rangle = 2\sqrt{E_pE_q}(2\pi)^3\delta(p-q) = 2E_p(2\pi)^3\delta(p-q) \tag{21}$$

3 Lorentz Invariance of $\int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}f(p)$

Invariant under proper Lorentz transformations:

$$\int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2E_p} = \int \frac{\mathrm{d}^4p}{(2\pi)^4}\frac{1}{2E_p}2\pi\delta(p^0-E_p) \tag{22}$$

$$= \int_{p^0>0} \frac{\mathrm{d}^4p}{(2\pi)^4} 2\pi\delta(p^2-m^2) \tag{23}$$

to see that these delta functions are the same, consider:

$$p^2 - m^2 = (p^0)^2 - p^2 - m^2 = (p^0)^2 - E_p^2 = (p^0 - E_p)\underbrace{(p^0 + E_p)}_{=2E_p} \tag{24}$$

$$\delta(ax) = \frac{1}{|a|}\delta(x) \tag{25}$$

Lorentz Invariant, because:

- $\mathrm{d}^4p' = J\mathrm{d}^4p = |\det(\Lambda)|\mathrm{d}^4p = \mathrm{d}^4p$
- a proper Lorentz Transformation transforms $p^0 > 0$ into $p^{0\prime} > 0$, Since it can be continuously connected to the identity.

4 .

$$D(x-y) = \langle 0|\,\varphi(x)\varphi(y)\,|0\rangle \tag{26}$$

in Peskin and Schröder: "$\varphi(x)\,|0\rangle$ is a particle at position x."

$$\varphi(\vec{x})\,|0\rangle = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p)}a_p^\dagger e^{-ipx}}\,|0\rangle \tag{27}$$

$$= \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2E_p}e^{-ipx}\,|p\rangle\text{"} \approx |x\rangle\text{"} \tag{28}$$

if we ignore the factor $1/2E_p$ then we have a normal Fourier-Transform as in non relativistic QM, which would make our approxmiation equal.

non-relativistic case: $E_p^2 = p^2 + m^2 \approx m^2$

$$\Rightarrow D(x-y) = \langle 0|\,\varphi(x)\varphi(y)\,|0\rangle \tag{29}$$

"is the probability amplitude that a particle at x is detected at position y"

Next Lecture: We will prove $D(x-y) \neq 0$ even if x-y is spacelike.