QFT - Lecture14

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Contents

1 ϕ^4 self interacting theory

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu \phi \partial^\mu \phi\right) - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4$$

$$H = \underbrace{H_0}_{\text{free KG}} + \underbrace{H_{\text{interacting}}}_{\int \mathrm{d}^3x \frac{\lambda}{4!}\phi^4}$$

2 Interaction picture

$$\phi_I(t) = \int \frac{\mathrm{d}^3 p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\left(a_p e^{-ipx} + a_p^\dagger e^{ipx}\right)$$

And related to the Heisenberg picture

$$\phi(x) = U^\dagger(t, t_0)\phi_I U(t, t_0)$$

defining the operator U with the dyson series

$$U(t, t_0) = T \exp\left[-i\int_{t'}^{t} \mathrm{d}t'' H_I(t'')\right] \tag{1}$$

$$H_I(t) = e^{iH_0(t-t_0)} H_{\text{int}} e^{-iH_0(t-t_0)}$$
$$U(t_0, t') = e^{iH(t'-t_0)} e^{-iH_0(t'-t_0)}$$

3 Correlation functions

Correlation functions are of the form

$$\langle\Omega|\, T\phi(x)\phi(y)\, |\Omega\rangle$$

the vacuum state $|0\rangle$ has changed to $|\Omega\rangle$ because of the interaction.

we claim that this is equal to

$$\frac{\langle 0|\, T\phi_I(x)\phi_I(y)U(\infty, -\infty)\, |0\rangle}{\langle 0|\, U(\infty, -\infty)\, |0\rangle}$$

which is from Tong's lecture notes p76 (around about)
we will prove this and then derive Wick's theorem.

Assume a mass gap. Meaning that you should assume that between the Energy E_0 of the state $|\Omega\rangle$ and the next possible Energy $E_0 + m$ there is a gap. Meaning the energy spectrum is not continuous down to the base Energy.

Assume that $x^0 > y^0$. Start with the numerator.

$$\langle 0| U(\infty, x^0)\phi_I(x)U(x^0, y^0)\phi_I(y)U(y^0, -\infty) |0\rangle \tag{2}$$

$$= \langle 0| U(\infty, x^0)U(x^0, t_0)\phi(x)U^\dagger(x^0, t_0)U(x^0, y^0)U(y^0, t_0)\phi(y)U^\dagger(y^0, t_0)U(y^0, -\infty) |0\rangle \tag{3}$$

$$= \underbrace{\langle 0| U(\infty, t_0)\phi(x)U(t_0, t_0)\phi(y)}_{\langle\psi|} U(t_0, -\infty) |0\rangle \tag{4}$$

Because of the differences in vacuums we now consider:

$$\langle\psi| U(t_0, t') |0\rangle$$

There is a Mass gap, s.t.

$$H = E_0 |\Omega\rangle \langle\Omega| + \int \mathrm{d}\lambda \int \frac{\mathrm{d}^3 p}{(2\pi)^3} |\lambda_p\rangle \langle\lambda_p| E_{\lambda_p}$$

From Quantum mechanics, we have the form $H = \int \mathrm{d}n E_n |n\rangle \langle n|$ for a continuous spectrum n. When ommitting the Energies you get the Identity operator

$$1 = |\Omega\rangle \langle\Omega| + \int \mathrm{d}\lambda \int \frac{\mathrm{d}^3 p}{(2\pi)^3} |\lambda_p\rangle \langle\lambda_p|$$

And we return to

$$\langle\psi| e^{iH(t'-t_0)}e^{-iH_0(t'-t_0)} |0\rangle \tag{5}$$

Only the 0th part of H_0 survives, as the annihilator kills the vacuum.

$$\langle\psi| e^{iH(t'-t_0)} |0\rangle \tag{6}$$

$$\langle\psi| e^{iE_0(t'-t_0)} |\Omega\rangle \langle\Omega| |0\rangle + \int \mathrm{d}\lambda \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \langle\psi|\lambda_p\rangle \langle\lambda_p|0\rangle e^{iE_{p\lambda}(t'-t_0)} \tag{7}$$

$$\Rightarrow e^{iE_0(t't_0)} \langle\psi|\Omega\rangle \langle\Omega|0\rangle + 0, \text{ by Riemann-Lebesgue Lemma} \tag{8}$$

And we have the numerator

$$\langle 0| U(t'', t_0)\phi(x)\phi(y) |\Omega\rangle \langle\Omega| |0\rangle e^{iE_0(t'-t_0)} \tag{9}$$

$$e^{iE_0(t'-t_0)}e^{-iE_0(t''-t_0)} \langle 0| |\Omega\rangle \langle\Omega| \phi(x)\phi(y) |\Omega\rangle \langle\Omega| |0\rangle \tag{10}$$

The denominator is the exact same except without the fields.

$$\Rightarrow e^{iE_0(t',t_0)}e^{-iE_0(t''-t_0)} \langle 0| |\Omega\rangle \langle\Omega| |\Omega\rangle \langle\Omega| |0\rangle \tag{11}$$

And we calculate the nominator and the denominator together:

$$\frac{\langle\Omega| \phi(x)\phi(y) |\Omega\rangle}{\langle\Omega| |\Omega\rangle} = \langle\Omega| \phi(x)\phi(y) |\Omega\rangle \tag{12}$$

proving the claim.

And the infinite time evolution operator:

$$U(\infty, -\infty) = T \exp\left[-i \int_{-\infty}^{\infty} \mathrm{d}t'' H_I(t'')\right],\ H_I(t) = \int \mathrm{d}^3 x \frac{\lambda}{4!}\phi_I^4 \tag{13}$$

So everything is expressed in terms of ϕ_I which we already know.

4 Wick's Theorem

$$\langle 0|\, T \underbrace{\phi(x_1)\phi(x_2)\cdots\phi(x_n)}_{\text{interaction picture fields}} |0\rangle \tag{14}$$

in the case of $n = 2$ we can calculate the expectation value with the Feynmann Propagator. With $n > 2$ we can calculate it with multiple Feynmann Propagators.

4.1 calculating Eq.14

Write $\phi(x) = \phi^+(x) + \phi^-(x)$, here the positive frequency is defined as

$$\phi^+(x) = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} a_p e^{-ipx}, \quad \phi^{+\dagger} = \phi^-$$

Consider $T\phi(x)\phi(y)$. Assume $x^0 > y^0$. Then

$$T\phi(x)\phi(y) = \phi^+(x)\phi^+(y) + \phi^+(x)\phi^-(y) + \phi^-(x)\phi^+(y) + \phi^-(x)\phi^-(y) \tag{15}$$

$$= \phi^+(x)\phi^+(y) + \phi^-(x)\phi^+(y) + \phi^-(y)\phi^+(x) + \left[\phi^+(x), \phi^-(y)\right] + \phi^-(x)\phi^-(y). \tag{16}$$

Assuming $x^0 < y^0$

$$\phi^+(x)\phi^+(y) + \phi^-(y)\phi^+(x) + \phi^-(x)\phi^+(y) + \left[\phi^+(y), \phi^-(x)\right] + \phi^-(x)\phi^-(y) \tag{17}$$

For both cases,

$$T\phi(x)\phi(y) = N\phi(x)\phi(y) + \begin{cases} \left[\phi^+(x), \phi^-(y)\right], & x^0 > y^0 \\ \left[\phi^+(y), \phi^-(x)\right], & y^0 < x^0 \end{cases}$$

Normal ordering is defined as putting the annihilation operators to the right.

4.2 Wick's Theorem for 2 Fields

$$T\phi(x)\phi(y) = N\phi(x)\phi(y) + contraction\phi(x)\phi(y) \tag{18}$$

with the contraction defined as

$$\overbracket{\phi(x)\phi(y)} := \begin{cases} \left[\phi^+(x), \phi^-(y)\right], & x^0 > y^0 \\ \cdots \end{cases} = \langle 0|\, T\phi(x)\phi(y)\, |0\rangle = D_F(x-y) \tag{19}$$

4.3 Wick's Theorem

$$T\phi(x_1)\phi(x_2)\cdots\phi(x_n) = N\phi(x_1)\cdots\phi(x_n) + \text{all contractions} \tag{20}$$

for four fields: $\phi_1\phi_2\phi_3\phi_4$

$$\overbracket{\phi_1\phi_2}\phi_3\phi_4 + \overbracket{\phi_1\phi_2\phi_3}\phi_4 + \cdots + \overbracket{\phi_1\phi_2}\overbracket{\phi_3\phi_4} + \phi_1\phi_2\phi_3\phi_4 + \cdots \tag{21}$$

4.3.1 Think about this but not too much because it's stupid

$1 = aa^\dagger - a^\dagger a$
$N1 = Naa^\dagger - a^\dagger a = a^\dagger a - a^\dagger a = 0$