QFT - Lecture 11

23.9.2024

Contents

1 Quantized Dirac Field

$$\psi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_s \left(a_p^s u^s(p) e^{-ipx} + b_p^{s\dagger} v^s(p) e^{ipx} \right) \tag{1}$$

$$\pi(x) = \frac{\partial L}{\partial(\partial_0 \psi)} = i\psi^\dagger(x), \quad H = \int d^3x \bar{\psi}(-i\gamma\nabla + m)\psi \tag{2}$$

$$\text{anti-commutators: } \{a_p^r, a_q^{s\dagger}\} = (2\pi)^3 \delta(p-q)\delta^{rs}, \text{ all others are zero} \tag{3}$$

$$H = \dots = \int \frac{d^3p}{(2\pi)^3} \sum_s E_p (a_p^{s\dagger} a_p^s + b_p^{s\dagger} b_p^s) \tag{4}$$

$$Q = \int d^3x \bar{\psi}\gamma^0\psi = \int d^3x \psi^\dagger \psi = \dots = \int \frac{d^3p}{(2\pi)^3} \sum_s (a_p^{s\dagger} a_p^s + b_p^{s\dagger} b_p^s) \tag{5}$$

1.1 Are a and b ladder operators?

Quantum Mechanics:
Assume: $[H, a] = -\omega a$ (harmonic Oscillator)

$$(Ha - aH) |\psi\rangle = -\omega a |\psi\rangle$$

where $|\psi\rangle$ is an energy Eigenstate with Energy E.

$$(Ha - aE) |\psi\rangle = -\omega a |\psi\rangle \tag{6}$$

$$Ha |\psi\rangle = (E - \omega) a |\psi\rangle \tag{7}$$

$$\text{there is a vacuum state s.t. } a|0\rangle = 0 \tag{8}$$

QFT:

$$[H, a_p^s] =_{[BC,A]=B\{A,C\}-\{A,B\}C} -a_p^s \cdot E_p \tag{9}$$

This ladder only has 2 steps.

$$a_p^s |0\rangle = 0$$

for all p and s defines the vacuum state $|0\rangle$

$$a_p^{s\dagger} |0\rangle$$

$$\left(a_p^{s\dagger}\right)^2 |0\rangle = 0$$

because the anticommutator is 0.
Let $|p, s\rangle = \sqrt{2E_p} a_p^{s\dagger} |0\rangle$ be a single fermion.

$$\langle p, r | q, s\rangle = \sqrt{2E_p 2E_q} \langle 0| a_p^r a_q^{s\dagger} |0\rangle = 2E_p (2\pi)^3 \delta(p-q)\delta^{rs}$$

Similarly, $\sqrt{2E_p} b_p^{s\dagger} |0\rangle$ is an anti-fermion.

1.2 Angular Momentum, Spin

Consider an infinitesimal rotation Θ around the 3-axis.

$$\Lambda = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -\theta & 0 \\ 0 & \theta & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\psi(x) = S[\Lambda]\psi(\Lambda^{-1}x), \quad S[\Lambda] = \begin{pmatrix} e^{-i\theta\sigma^3/2} & 0 \\ 0 & e^{-i\theta\sigma^3/2} \end{pmatrix} \tag{10}$$

$$S[\Lambda] = 1 - \frac{i\theta}{2}\Sigma^3, \quad \Sigma^3 = \begin{pmatrix} \sigma^3 & 0 \\ 0 & \sigma^3 \end{pmatrix} \tag{11}$$

$$\Lambda^{-1}x = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \theta & 0 \\ 0 & -\theta & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} t \\ x + \theta y \\ -\theta x + y \\ z \end{pmatrix} \tag{12}$$

$$\psi \mapsto \psi + \theta\Delta\psi, \quad L \mapsto L + \theta\partial_\mu J^\mu \tag{13}$$

$$\theta\Delta\psi = S[\Lambda]\psi(\Lambda^{-1}x) - \psi(x) = \left(1 - \frac{i\theta}{2}\Sigma^3\right)\psi(t, x + \theta y, -\theta x + y, z) - \psi(t, x, y, z) \tag{14}$$

$$= \left(1 - \frac{i\theta}{2}\Sigma^3\right)(\psi + \theta y\partial_x\psi - \theta x\partial_y\psi) - \psi \tag{15}$$

$$= \theta\underbrace{\left(\frac{-i}{2}\sigma^3\psi + y\partial_x\psi - x\partial_y\psi\right)}_{\Delta\psi} \tag{16}$$

$$\theta\partial_\mu j^\mu = L(\Lambda^{-1}) - L(x) = \theta\left(y\partial_x L - x\partial_y L\right) \tag{17}$$

$$\Rightarrow j^\mu = (0, yL, -xL, 0). \tag{18}$$

$$\text{Conserved Noether Current: } j^\mu = \frac{\partial L}{\partial(\partial_\mu\psi)}\Delta\psi + J^\mu \tag{19}$$

We consider the timelike component

$$j^0 = \pi\Delta\psi + J^0 \tag{20}$$

$$= \pi\left(\frac{-i}{2}\Sigma^3\psi + y\partial_x\psi - x\partial_y\psi\right) \tag{21}$$

$$= -i\psi^\dagger\left(\frac{i}{2}\Sigma^3 + x\partial_y - y\partial_x\right)\psi \quad \text{conserved} \tag{22}$$

$$\vec{J} = \int \mathrm{d}^3x - i\psi^\dagger\left(\frac{i}{2}\Sigma + (x \times (-i\nabla))\right)\psi \tag{23}$$

For nonrelative fermions: Spin: $J_z = \int \mathrm{d}^3x - i\psi^\dagger\frac{i}{2}\Sigma^3\psi$ To find the spin we will want to find the Eigenvalue of J_z in a one-particle state $a_p^{s\dagger}|0\rangle$.

$$J_z a_0^{s\dagger}|0\rangle =_{J_z|0\rangle=0} [J_z, a_0^{s\dagger}]|0\rangle$$

Through the Poincaré symmetry we know that $[J_z, H] = 0$. That means that the Vacuum state is an Eigenvalue of J_z, meaning that $\langle 0| J_z |0\rangle = 0$ implies $J_z |0\rangle = 0$.

$$J_z = \frac{1}{2} \int \mathrm{d}^3 x \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \int \frac{\mathrm{d}^3 p'}{(2\pi)^3} \frac{1}{\sqrt{2E_p 2E_{p'}}} e^{-ip'x+ipx} \tag{24}$$

$$\sum_{r,r'} \left(a_{p'}^{r'\dagger} u^{r'\dagger}(p') + b_{-p'}^{r'} v^{r'\dagger}(-p') \right) \Sigma^3 \left(a_p^r u^r(p) + b_{-p}^{r\dagger} v^r(-p) \right) \tag{25}$$

having put $x^0 = 0$. Now use: $\left[a_p^{r'\dagger} a_p^r, a_0^{s\dagger} \right] = a_p^{r'\dagger} (2\pi)^3 \delta^{rs} \delta(p - 0)$

$$\int \mathrm{d}^3 x \cdots \Rightarrow \delta(p - p') \tag{26}$$

$$\left[J_z, a_0^{s\dagger} \right] |0\rangle = \frac{1}{4m} \sum_{r'} u^{r'\dagger}(0) \Sigma^3 u^s(0) a_0^{r'\dagger} |0\rangle \tag{27}$$

$$= \frac{1}{4} \sum_r (\xi^{r\dagger}, \xi^{r\dagger}) \begin{pmatrix} \sigma^3 & 0 \\ 0 & \sigma^3 \end{pmatrix} \begin{pmatrix} \xi^s \\ \xi^s \end{pmatrix} a_0^{r\dagger} |0\rangle \tag{28}$$

$$= \frac{1}{2} \xi^{s\dagger} \sigma^3 \xi^s \tag{29}$$