QFT - Lecture 16

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Contents

1 The scattering (S) matrix

Interaction can involve:

- self-interaction
- scattering
- production of new particles

2 Eigenstates of interacting theories (ϕ^4)

Hamiltonian H.
Momentum Operator P.
From the poincaré symmetry: $[H, P] = 0$.
Assumptions:

- there exists a unique, translationally invariant and Lorentz invariant state $|\Omega\rangle$. The Energy zero is chose such that $H|\Omega\rangle = 0$
- $\langle\Omega|\,\phi(x)\,|\Omega\rangle = 0$.

The common Eigenstates of P and H, $|\lambda_p\rangle$ with p the total momentum and λ the degrees of freedom of the state.

$$H|\lambda_p\rangle = E_p^\lambda |\lambda_p\rangle, \quad P|\lambda_p\rangle = p|\lambda_p\rangle \tag{1}$$

let $m_\lambda := E_p^\lambda I_{p=0}$ be the rest Energy.
Consider a Lorentz transformation from $(m_\lambda, 0) \mapsto (p^0, p)$

$$U^\dagger P^\mu U = \Lambda_\nu^\mu P^\nu$$

$$P^\mu U|\lambda_0\rangle = UU^\dagger P^\mu U|\lambda_0\rangle \tag{2}$$
$$= U\Lambda_\nu^\mu P^\mu |\lambda_0\rangle = U\Lambda_0^\mu m_\lambda |\lambda_0\rangle \tag{3}$$
$$= p^\mu U|\lambda_0\rangle \tag{4}$$

And therefore we conclude that $|\lambda_p\rangle = U|\lambda_0\rangle$

As the Lorentz transformation leaves Four-Vector length invariant, we have

$$(p^0)^2 - p^2 = m_\lambda^2 \tag{5}$$
$$E_p^{\lambda^2} = m_\lambda^2 + p^2 m = m_\lambda \text{ for a single particle} \tag{6}$$

A single particle starts at the Energy m, and is only on the shell. This is the mass gap. A second particle is at the Energy $2m$, and through internal motion can have any Energy above the shell.

$$\phi(x) = e^{iPx}\phi(0)e^{-iPx} \tag{7}$$
$$\langle\Omega|\,\phi(x)\,|\lambda_p\rangle = \langle\Omega|\,e^{iPx}\phi(0)e^{-iPx}\,|\lambda_p\rangle \tag{8}$$
$$= \langle\Omega|\,\phi(0)\,|\lambda_p\rangle\,e^{-ipx} \tag{9}$$
$$= \langle\Omega|\,UU^\dagger\phi(0)U\,|\lambda_0\rangle\,e^{-ipx} \tag{10}$$
$$= \langle\Omega|\,\phi(0)\,|\lambda_0\rangle\,e^{-ipx} \tag{11}$$

when λ is a single particle, $|\lambda\rangle_0 = |p=0\rangle$.

$$Z := \langle\Omega|\,\phi(0)\,|p=0\rangle^2 \tag{12}$$

3 Defining in and out states

$$|\lambda_p^\psi\rangle = \int \frac{\mathrm{d}^3k}{(2\pi)^3}\psi(k-p)\,|\lambda_k\rangle \tag{13}$$
$$|p^\psi\rangle = \int \frac{\mathrm{d}^3k}{(2\pi)^3}\psi(k-p)\,|k\rangle \tag{14}$$

Now Define Operators: $a_p^\psi(t) = i\int \bar{\Psi}_p(x)\overleftrightarrow{\partial}_0\phi(x)$
$a_p^{\psi\dagger}(t) = -i\int \Psi(x)\overleftrightarrow{\partial}_0\phi(x)$

Where we have define $\Psi(x) = \int \frac{\mathrm{d}^3k}{(2\pi)^3}\frac{\psi(k-p)}{\sqrt{2E_k}}e^{-ikx}$ with $k^0 = E_k = \sqrt{k^2+m^2}$.

Limits:

- $t \to \pm\infty$
- $\psi(k) \to (2\pi)^3\delta(k)$

Result:

$$a_p^\psi(\pm\infty), a_p^{\psi\dagger}(\pm\infty) \tag{15}$$

Work as ladder operators for vacuum and single particles. (First two steps of the ladder)

$$|p\rangle = \frac{\sqrt{2E_p}}{\sqrt{Z}}a_p^{\psi\dagger}(\pm\infty)\,|\Omega\rangle \tag{16}$$
$$\langle p|q\rangle = 2E_p(2\pi)^3\delta(p-q) \tag{17}$$
$$a_p^\psi(\pm\infty)\,|\Omega\rangle = 0 \tag{18}$$

Asymptotic in and out states:

$$|k_1, k_2, \cdots\rangle_{\text{in}} = \prod_i \sqrt{\frac{2E_{k_i}}{Z}} a^{\psi\dagger}_{k_i}(-\infty)\,|\Omega\rangle \tag{19}$$

$$|p_1, p_2, \cdots\rangle_{\text{out}} = \prod_j \sqrt{\frac{2E_{p_j}}{Z}} a^{\psi\dagger}_{p_j}(+\infty)\,|\Omega\rangle \tag{20}$$

Both of these states are in the Heisenberg picture.

4 Scattering in Schrödinger Picture

let the reference time be $t = -\infty$. Consider Evolution from $t = -T$ to $t = T$: $S = e^{-iH2T}$.
For $t = -T$ in Schrödinger picture: $|k_1, k_2, \cdots\rangle = |k_1, k_2, \cdots\rangle_{\text{in}}$ from the Heisenberg picture.
For $t = T$ in Schrödinger picture: $|p_1, p_2, \cdots\rangle = S\,|p_1, p_2, \cdots\rangle_{\text{out}}$.
We define the S-Matrix as the overlap of the in and out states

$$S_m = \langle p_1, p_2, \cdots|_{\text{out}} k_1, k_2, \cdots\rangle_{\text{in}} = \langle p_1, p_2, \cdots|_{\text{out}} S^\dagger S|k_1, k_2, \cdots\rangle_{\text{in}} \tag{21}$$

$$= \langle p_1, p_2, \cdots|S|k_1, k_1, \cdots\rangle \tag{22}$$