QFT - Lecture 3

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Contents

1 Classical Klein-Gordon field

$$L = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}m^2\phi^2 \tag{1}$$

$$\pi = \dot{\phi}, \;\; \mathcal{H} = \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2 \tag{2}$$

$$\text{Klein-Gordon Equation: } (\partial^2 + m^2)\phi = 0 \tag{3}$$

$$\text{Solution: } \phi(x) = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} \left(a_p e^{-ipx} + \bar{a_p} e^{ipx}\right) \tag{4}$$

2 Quantized Klein-Gordon field

Fields promoted to Operators

$$[\phi(x), \pi(y)] = i\delta(x-y) \tag{5}$$

$$[\phi(x), \phi(y)] = [\pi(x), \pi(y)] = 0 \tag{6}$$

We consider only equal-time commutators.

$$\phi(x)\text{satisfies the Klein-Gordon equation} \tag{7}$$

$$\phi(x) = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} \left[a_p e^{-ipx} + a_p^\dagger e^{ipx}\right] \tag{8}$$

($\phi(x)$ is hermitian)

$$= \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} [a_p e^{-i\omega_p t + ipx} + a^\dagger e^{i\omega_p t - ipx}] \tag{9}$$

$$\pi(x) = \dot{\phi}(x) = (-i) \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \sqrt{\frac{\omega}{2}} [a_p e^{-i\omega_p t + ipx} - a^\dagger e^{i\omega_p t - ipx}] \tag{10}$$

we want the commutator relations of the a_p.

$$[a_p, a_q^\dagger] = (2\pi)^3\delta(p-q)[a_p, a_q] = [a_p^\dagger, a_q^\dagger] = 0 \tag{11}$$

Assume this to be true and evaluate the previous commutator relations:

$$\begin{aligned}
[\phi(x), \pi(y)] &= \frac{-i}{2} \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \int \frac{\mathrm{d}^3 q}{(2\pi)^3} \frac{\omega_p}{\omega_q} \left(- \underbrace{[a_p, a_q^\dagger]}_{(2\pi)^3\delta(p-q)} e^{i(px-qy)} + \underbrace{[a_p^\dagger, a_q]}_{-(2\pi)^3\delta(p-q)} e^{i(qy-px)} \right) \\
&= \frac{-i}{2} \int \frac{\mathrm{d}^3 p}{(2\pi)^3} (-e^{ip(x-y)} - e^{-ip(x-y)}) \\
&= \frac{-i}{2} \int \frac{\mathrm{d}^3 p}{(2\pi)^3} (-2e^{ip(x-y)}) \\
&= (-i)(-1)\delta(x-y)
\end{aligned}$$

Next task: calculating the hamiltonian.

$$\mathcal{H} = \frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2\text{: Hamiltonian Density} \tag{12}$$

$$H = \int \mathrm{d}^3x\mathcal{H} = \int \mathrm{d}^3x\mathcal{H}\big|_{t=0}\text{: no explicit t dependency in } L. \tag{13}$$

$$\phi(x) = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_p}}[a_p + a^\dagger_{-p}]e^{ipx} \tag{14}$$

$$\pi(x) = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\sqrt{\frac{\omega_p}{2}}[a_q - a^\dagger_{-q}]e^{iqx} \tag{15}$$

$$\begin{aligned} H = \frac{1}{4}\int \mathrm{d}^3x \int \frac{\mathrm{d}^3p}{(2\pi)^3}\int \frac{\mathrm{d}^3q}{(2\pi)^3}\Big[-\sqrt{\omega_p\omega_q}(a_p - a^\dagger_{-p})(a_q - a^\dagger_{-q})e^{i(p+q)x} \\ +\frac{1}{\sqrt{\omega_p\omega_q}}(ip)(iq)(a_p + a^\dagger_{-p})(a_q + a^\dagger_{-q})e^{i(p+q)x} \\ +\frac{m^2}{\sqrt{\omega_p\omega_q}}(a_p + a^\dagger_{-p})(a_q + a^\dagger_{-q})e^{i(p+q)x}\Big] \end{aligned} \tag{16}$$

Now use Fubini to interchange the spatial and momentum integrals, which leads to delta functions $\delta(p+q)$.

$$\begin{aligned} H = \frac{1}{4}\int \frac{\mathrm{d}^3p}{(2\pi)^3}\Big[-\omega_p(a_p - a^\dagger_{-p})(a_{-p} - a^\dagger_p) \\ +\frac{p^2}{\omega_p}(a_p + a^\dagger_{-p})(a_{-p} + a^\dagger_p) - \frac{m^2}{\omega_p}(\cdots)(\cdots)\Big] \quad \omega_p = p^2 + m^2 \end{aligned} \tag{17}$$

$$H = \frac{1}{4}\int \frac{\mathrm{d}^3p}{(2\pi)^3} - \omega_p(a_p - a^\dagger_{-p})(a_{-p} - a^\dagger_p) + \omega_p(a_p + a^\dagger_{-p})(a_{-p} + a^\dagger_p) \tag{18}$$

$$= \frac{1}{4}\int \frac{\mathrm{d}^3p}{(2\pi)^3}\omega_p(-a_pa_{-p} + a_pa^\dagger_p + a^\dagger_{-p}a_{-p} - a^\dagger_{-p}a^\dagger_p + a_pa_{-p} + a_pa^\dagger_p + a^\dagger_{-p}a_{-p} + a^\dagger_{-p}a^\dagger_p \tag{19}$$

$$= \frac{1}{2}\int \frac{\mathrm{d}^3p}{(2\pi)^3}\omega_p(a_pa^\dagger_p + a^\dagger_pa_p) \quad [a_p, a^\dagger_q] = (2\pi)^3\delta(p-q) \tag{20}$$

$$= \int \frac{\mathrm{d}^3p}{(2\pi)^3}\omega_p\left(a^\dagger_pa_p + \frac{1}{2}(2\pi)^3\delta(0)\right) \tag{21}$$

We get an infinity in the Hamiltonian, by subtracting the infinity. Then

$$H = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\omega_p(a^\dagger_pa_p) \tag{22}$$

For the Quantum Mechanical Harmonic Oscillator we have the Hamiltonian

$$H = \frac{p^2}{2} + 1/2\omega^2q^2 = \cdots = a^\dagger a + \frac{1}{2} \tag{23}$$

We could have just as well written the Hamiltonian in the form

$$\frac{1}{2}(\omega q - ip)(\omega q + ip) = a^\dagger a \tag{24}$$

So we essentially picked a different point of view to write down the Hamiltonian. This also goes for our Infinite Hamiltonian.

We have: $[H, a_p^\dagger] = \int \frac{\mathrm{d}^3 p}{(2\pi)^3}\omega_q(a_q^\dagger a_q, a_p)$

$$\int \frac{\mathrm{d}^3 p}{(2\pi)^3}\omega_q a_q^\dagger (2\pi)^3 (q-p) = \omega_p a_p^\dagger \tag{25}$$

$$\text{Similarly, } [H, a_p] = -\omega_p a_p \tag{26}$$

Which are the same commutators as the ones in Quantum Mechanics. Then, we know that $a_p^\dagger$ and a_p are ladder operators. And, every p-mode is a harmonic oscillator.

What "is" the state $a_p^\dagger |0\rangle$? We will calculate the Eigenvalue of the Hamiltonian of the state.

$$Ha_p^\dagger |0\rangle = (a_p^\dagger H + \omega_p a_p^\dagger) |0\rangle \tag{27}$$

$$= \omega_p a_p^\dagger |0\rangle \tag{28}$$

which means that our ket $a_p^\dagger |0\rangle$ has an Eigenvalue ω_p ($\hbar\omega_p$).

The total momentum operator looks like

$$P = -\int \mathrm{d}^3 x \pi \nabla\phi = \cdots = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} p a_p^\dagger a_p \tag{29}$$

so then, similarly $Pa_p^\dagger |0\rangle = pa_p^\dagger |0\rangle$.
Then, $a_p^\dagger$ is a single excitation of the harmonic oscillator associated with mode p, with Energy ω_p, and the momentum p. Called a **particle**.

Then, a particle is an excitation of a quantum field.
We have a two particle system $a_p^\dagger a_q^\dagger |0\rangle$. And an n-particle state $(a_p^\dagger)^n |0\rangle$, these are Bosons. Later on, for fermions $(a_p^\dagger)^2 = 0$.

To recall, we have the Klein Gordon field, which is the solution to the KG Equation. We define the commutator relations that we want. It turns out that a_p and $a_p^\dagger$ are ladder operators. The excitation of the empty space is a particle. A particle is an infinite plane wave, which means that we need to overlap several particles to get a localized particle.

next Lecture: Lorentz Invariance.

3 Lorentz-Transformation

$$x^\mu \mapsto x'^\mu = \Lambda^\mu_\nu x^\nu \tag{30}$$

we want a transformation such that

$$\mathrm{d}s^2 = \mathrm{d}(x^0)^2 - \mathrm{d}x^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu \tag{31}$$

is invariant.

$$g_{\mu\nu} = \mathrm{d}x^{\mu}\mathrm{d}x^{\nu} = g_{\rho\sigma}\mathrm{d}x'^{\rho}\mathrm{d}x'^{\sigma} \tag{32}$$

$$= g_{\rho\sigma}\Lambda^{\rho}_{\mu}\mathrm{d}x^{\mu}\Lambda^{\sigma}_{\nu}\mathrm{d}x^{\nu} \tag{33}$$

$$\Rightarrow g_{\mu\nu} = g_{\rho\sigma}\Lambda^{\rho}_{\mu}\Lambda^{\sigma}_{\nu} \tag{34}$$