Lecture 2 - General Relativity

9.4.2025

Contents

0.1 twin Paradox

proper time of

$$\text{twin 1: } \Delta\tau_{ABC} = \Delta t \tag{1}$$

$$\text{twin 2: } \Delta\tau_{AB'C} = \Delta\tau_{AB'} + \Delta\tau_{B'C} \tag{2}$$

$$\Delta\tau_{AB'} = \sqrt{1-v^2}\frac{\Delta t}{2} = \Delta\tau_{B'C} \tag{3}$$

Therefore

$$\Delta\tau_{AB'C} = \sqrt{1-v^2}\Delta\tau_{ABC} \tag{4}$$

travelling twin is younger

1 Lorentz Transformations

$$\tilde{x}^\mu = \Lambda^\mu_\nu x^\nu \tag{5}$$

impose that the Minkowski Distance from the origin is invariant, such that

$$\tilde{x}^\mu \tilde{x}^\nu \eta_{\mu\nu} = x^\mu x^\nu \eta_{\mu\nu} \tag{6}$$

Then

$$\eta_{\mu\nu}\Lambda^\mu_\kappa \Lambda^\nu_\lambda = \eta_{\kappa\lambda} \tag{7}$$

This property describes the Lorentz Group $0(3,1)$. In this Group there exist 4 branches.

$$\kappa = \lambda = 0 \quad \Rightarrow \quad \eta_{00} = -1 = -(\Lambda^0_0)^2 + \underbrace{(\Lambda^i_0)^2}_{\geq 0} \tag{8}$$

$$\Rightarrow \quad \Lambda^0_0 \geq 0 \text{ or } \Lambda^0_0 \leq -1 \text{from Matrix equation } \Lambda^T\eta\Lambda = \eta \tag{9}$$

$$\Rightarrow \quad \det(\Lambda^T\eta\Lambda) = \det(\eta) \quad \Rightarrow \quad \det(\Lambda)^2 = 1 \tag{10}$$

then we get the four branches

- proper Lorentz group (contains the identity) $\Lambda^0_0, \det\Lambda = 1$
- combine with time reversal $\Lambda^0_0 \leq -1, \det\Lambda = -1$
- combine with parity (e.g. $x \to -x$) $\Lambda^0_0 \geq 1, \det\Lambda = -1$
- time reversal and parity $\Lambda^0_0 \leq -1, \det\Lambda = +1$

Raising and lowering an index with the (Minkowski) metric tensor

$$a_\mu = \eta_{\mu\nu}a^\nu \tag{11}$$

and we have that $\eta_{\mu\nu} = \eta^{\mu\nu}$