

Lecture 3

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1 Classical Klein-Gordon field

$$L = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}m^2 \phi^2 \quad (1)$$

$$\pi = \dot{\phi}, \quad \mathcal{H} = \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}(\nabla \phi)^2 + \frac{1}{2}m^2 \phi^2 \quad (2)$$

$$\text{Klein-Gordon Equation: } (\partial^2 + m^2)\phi = 0 \quad (3)$$

$$\text{Solution: } \phi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} (a_p e^{-ipx} + \bar{a}_p e^{ipx}) \quad (4)$$

2 Quantized Klein-Gordon field

Fields promoted to Operators

$$[\phi(x), \pi(y)] = i\delta(x - y) \quad (5)$$

$$[\phi(x), \phi(y)] = [\pi(x), \pi(y)] = 0 \quad (6)$$

We consider only equal-time commutators.

$$\phi(x) \text{ satisfies the Klein-Gordon equation} \quad (7)$$

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} [a_p e^{-ipx} + a_p^\dagger e^{ipx}] \quad (8)$$

($\phi(x)$ is hermitian)

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} [a_p e^{-i\omega_p t + ipx} + a_p^\dagger e^{i\omega_p t - ipx}] \quad (9)$$

$$\pi(x) = \dot{\phi}(x) = (-i) \int \frac{d^3p}{(2\pi)^3} \sqrt{\omega_p} [a_p e^{-i\omega_p t + ipx} - a_p^\dagger e^{i\omega_p t - ipx}] \quad (10)$$

we want the commutator relations of the a_p .

$$[a_p, a_q^\dagger] = (2\pi)^3 \delta(p - q) [a_p, a_q] = [a_p^\dagger, a_q^\dagger] = 0 \quad (11)$$

Assume this to be true and evaluate the previous commutator relations:

$$\begin{aligned} [\phi(x), \pi(y)] &= \frac{-i}{2} \int \frac{d^3p}{(2\pi)^3} \int \frac{d^3q}{(2\pi)^3} \frac{\omega_p}{\omega_q} \left(- \underbrace{[a_p, a_q^\dagger]}_{(2\pi)^3 \delta(p-q)} e^{i(px-xy)} + \underbrace{[a_p^\dagger, a_q]}_{-(2\pi)^3 \delta(p-q)} e^{i(qy-px)} \right) \\ &= \frac{-i}{2} \int \frac{d^3p}{(2\pi)^3} (-e^{ip(x-y)} - e^{-ip(x-y)}) \\ &= \frac{-i}{2} \int \frac{d^3p}{(2\pi)^3} (-2e^{ip(x-y)}) \\ &= (-i)(-1)\delta(x - y) \end{aligned}$$

Next task: calculating the hamiltonian.

$$\mathcal{H} = \frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2: \text{ Hamiltonian Density} \quad (12)$$

$$H = \int d^3x \mathcal{H} = \int d^3x \mathcal{H}|_{t=0}: \text{ no explicit } t \text{ dependency in } L. \quad (13)$$

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} [a_p + a_{-p}^\dagger] e^{ipx} \quad (14)$$

$$\pi(x) = \int \frac{d^3p}{(2\pi)^3} \sqrt{\frac{\omega_p}{2}} [a_q - a_{-q}^\dagger] e^{iqx} \quad (15)$$

$$\begin{aligned} H = \frac{1}{4} \int d^3x \int \frac{d^3p}{(2\pi)^3} \int \frac{d^3q}{(2\pi)^3} & \left[-\sqrt{\omega_p \omega_q} (a_p - a_{-p}^\dagger)(a_q - a_{-q}^\dagger) e^{i(p+q)x} \right. \\ & + \frac{1}{\sqrt{\omega_p \omega_q}} (ip)(iq)(a_p + a_{-p}^\dagger)(a_q + a_{-q}^\dagger) e^{i(p+q)x} \\ & \left. + \frac{m^2}{\sqrt{\omega_p \omega_q}} (a_p + a_{-p}^\dagger)(a_q + a_{-q}^\dagger) e^{i(p+q)x} \right] \end{aligned} \quad (16)$$

Now use Fubini to interchange the spatial and momentum integrals, which leads to delta functions $\delta(p+q)$.

$$H = \frac{1}{4} \int \frac{d^3p}{(2\pi)^3} \left[-\omega_p (a_p - a_{-p}^\dagger)(a_{-p} - a_p^\dagger) \right. \quad (17)$$

$$\left. + \frac{p^2}{\omega_p} (a_p + a_{-p}^\dagger)(a_{-p} + a_p^\dagger) - \frac{m^2}{\omega_p} (\dots)(\dots) \right] \quad \omega_p = p^2 + m^2$$

$$H = \frac{1}{4} \int \frac{d^3p}{(2\pi)^3} -\omega_p (a_p - a_{-p}^\dagger)(a_{-p} - a_p^\dagger) + \omega_p (a_p + a_{-p}^\dagger)(a_{-p} + a_p^\dagger) \quad (18)$$

$$= \frac{1}{4} \int \frac{d^3p}{(2\pi)^3} \omega_p (-a_p a_{-p} + a_p a_p^\dagger + a_{-p}^\dagger a_{-p} - a_{-p}^\dagger a_p^\dagger + a_p a_{-p} + a_p a_p^\dagger + a_{-p}^\dagger a_{-p} + a_{-p}^\dagger a_p^\dagger) \quad (19)$$

$$= \frac{1}{2} \int \frac{d^3p}{(2\pi)^3} \omega_p (a_p a_p^\dagger + a_p^\dagger a_p) \quad [a_p, a_q^\dagger] = (2\pi)^3 \delta(p-q) \quad (20)$$

$$= \int \frac{d^3p}{(2\pi)^3} \omega_p \left(a_p^\dagger a_p + \frac{1}{2} (2\pi)^3 \delta(0) \right) \quad (21)$$

We get an infinity in the Hamiltonian, by subtracting the infinity. Then

$$H = \int \frac{d^3p}{(2\pi)^3} \omega_p (a_p^\dagger a_p) \quad (22)$$

For the Quantum Mechanical Harmonic Oscillator we have the Hamiltonian

$$H = \frac{p^2}{2} + 1/2 \omega^2 q^2 = \dots = a^\dagger a + \frac{1}{2} \quad (23)$$

We could have just as well written the Hamiltonian in the form

$$\frac{1}{2}(\omega q - ip)(\omega q + ip) = a^\dagger a \quad (24)$$

So we essentially picked a different point of view to write down the Hamiltonian. This also goes for our Infinite Hamiltonian.

We have: $[H, a_p^\dagger] = \int \frac{d^3 p}{(2\pi)^3} \omega_q (a_q^\dagger a_q, a_p)$

$$\int \frac{d^3 p}{(2\pi)^3} \omega_q a_q^\dagger (2\pi)^3 (q - p) = \omega_p a_p^\dagger \quad (25)$$

$$\text{Similarly, } [H, a_p] = -\omega_p a_p \quad (26)$$

Which are the same commutators as the ones in Quantum Mechanics. Then, we know that a_p^\dagger and a_p are ladder operators. And, every p -mode is a harmonic oscillator.

What "is" the state $a_p^\dagger |0\rangle$? We will calculate the Eigenvalue of the Hamiltonian of the state.

$$H a_p^\dagger |0\rangle = (a_p^\dagger H + \omega_p a_p^\dagger) |0\rangle \quad (27)$$

$$= \omega_p a_p^\dagger |0\rangle \quad (28)$$

which means that our ket $a_p^\dagger |0\rangle$ has an Eigenvalue ω_p ($\hbar\omega_p$).

The total momentum operator looks like

$$P = - \int d^3 x \pi \nabla \phi = \dots = \int \frac{d^3 p}{(2\pi)^3} p a_p^\dagger a_p \quad (29)$$

so then, similarly $P a_p^\dagger |0\rangle = p a_p^\dagger |0\rangle$.

Then, a_p^\dagger is a single excitation of the harmonic oscillator associated with mode p , with Energy ω_p , and the momentum p . Called a **particle**.

Then, a particle is an excitation of a quantum field.

We have a two particle system $a_p^\dagger a_q^\dagger |0\rangle$. And an n -particle state $(a_p^\dagger)^n |0\rangle$, these are Bosons. Later on, for fermions $(a_p^\dagger)^2 = 0$.

To recall, we have the Klein Gordon field, which is the solution to the KG Equation. We define the commutator relations that we want. It turns out that a_p and a_p^\dagger are ladder operators. The excitation of the empty space is a particle. A particle is an infinite plane wave, which means that we need to overlap several particles to get a localized particle.

next Lecture: Lorentz Invariance.

3 Lorentz-Transformation

$$x^\mu \mapsto x'^\mu = \Lambda^\mu_\nu x^\nu \quad (30)$$

we want a transformation such that

$$ds^2 = d(x^0)^2 - dx^2 = g_{\mu\nu} dx^\mu dx^\nu \quad (31)$$

is invariant.

$$g_{\mu\nu} = dx^\mu dx^\nu = g_{\rho\sigma} dx'^\rho dx'^\sigma \quad (32)$$

$$= g_{\rho\sigma} \Lambda_\mu^\rho dx^\mu \Lambda_\nu^\sigma dx^\nu \quad (33)$$

$$\Rightarrow g_{\mu\nu} = g_{\rho\sigma} \Lambda_\mu^\rho \Lambda_\nu^\sigma \quad (34)$$