Watersheds, mosaics, and the emergence paradigm

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Abstract

In this paper, we investigate the links between the flooding paradigm and the topological watershed. Guided by the analysis of a classical flooding algorithm, we present several notions that lead us to a better understanding of the watershed: minima extension, mosaic, pass value and separation. We first make a detailed examination of the effectiveness of the divide set produced by watershed algorithms. We introduce the mosaic to retrieve the altitude of points along the divide set. A desirable property is that, when two minima are separated by a crest in the original image, they are still separated by a crest of the same altitude in the mosaic. Our main result states that this is the case if and only if the mosaic is obtained through a topological thinning. We investigate the possibility for a flooding to produce a topological watershed, and conclude that this is not feasible. This leads us to reverse the flooding paradigm, and to propose a notion of *emergence*. An emergence process is a transformation based on a topological criterion, in which points are processed in decreasing altitude order while preserving the number of connected components of lower cross-sections. Our main result states that any emergence watershed is a topological watershed, and more remarkably, that any topological watershed of a given image can be obtained as an emergence watershed of the image.
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1. Introduction

The watershed has been extensively studied during the 19th century by Maxwell [17] and Jordan [15] among others. One hundred years later, the watershed transform was introduced by Beucher and Lantuéjoul [4] for image segmentation, and is now used as a fundamental step in many powerful segmentation procedures [20,5]. Image segmentation usually requires several processing steps. For example, a typical morphological segmentation procedure includes a filtering step, a gradient, a marker extraction or a reduction of the number of minima, a watershed step and some post-processing. Most of these steps are often very dependent on the application, only the watershed step is application independent. In this paper, we focus exclusively on watersheds and we study some mathematical properties of several discrete watershed operators.

A popular presentation of the watershed in the morphological community [32,14,12] is based on a flooding paradigm. Let us consider the greyscale image as a topographical relief: the grey level of a pixel becomes the elevation of a point, the basins and valleys of the relief correspond to the dark areas, whereas the mountains and crest lines correspond to the light areas. Let us suppose the surface being immersed in a lake, with holes pierced in local minima. Water fills up basins starting at these local minima, and, at points where waters coming from different basins would meet, dams are built. As a result, the surface is partitioned into regions or basins separated by dams, called watershed divides.

Efficient watershed algorithms based on immersion simulation were proposed by Vincent, Soille [36] and Meyer [18] in the early 90s. Those algorithms build a partition of the space by associating an influence zone to each minimum of the image, and by producing (in their "dividing" variant) a divide set which separates those influence zones; that is to say, they "extend" the minima. The building of the influence zones is based on a flooding paradigm which consists in processing points of the image in increasing grey level order. We can find a presentation of most of the existing morphological watershed algorithms in a paper by Roerdink and Meijster [27]. Nevertheless, to our best knowledge, no attempt has been made to propose comparison criteria. Let us note that a mathematical approach for regular continuous functions has been proposed by Najman and Schmitt [23,24], introducing in particular the equivalence for regular functions between the flooding approach and a distance-based approach to the watershed. Algorithms for computing distance-based watersheds have been proposed in [19]. Such distance-based or cost-based [16] watersheds will not be studied in this paper.

An original approach to the watershed transform, called the topological watershed, has been proposed in [7]. The idea is to define a "topological thinning" that transforms the image while preserving some topological properties, namely the number of connected components of each lower cross-section. Let F be a greyscale image and λ be a grey level, the lower cross-section $\overline{F_\lambda}$ is the set composed of all the points having an altitude strictly lower than λ. A point x is said to be W-destructible for F (where W stands for Watershed) if its altitude can be lowered by one without changing the number of connected components of $\overline{F_k}$, with $k = F(x)$. A map G is called a W-thinning of F if it may be obtained from F by iteratively selecting a W-destructible point and lowering it by one. A topological watershed of F is a W-thinning of F which contains no W-destructible point (see Fig. 1a,c). A major feature of this transform is to produce a greyscale image. A divide set of the original image can
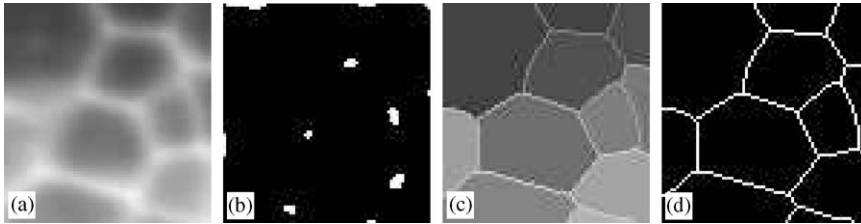
Fig. 1. (a) Original image; (b) regional minima of (a) (in white), (c) a topological watershed of (a), (d) a divide set of (a), obtained by taking the complement of the regional minima of (c).

easily be computed on the transformed image, by taking the complement of the minima of the transformed image (see Fig. 1d). Recently, Bertrand [1] proposed a framework in which fundamental properties of the topological watershed have been derived. Quasi-linear algorithms for computing the topological watershed transform have been obtained and proved using this framework [10].

In this framework, a notion of contrast plays an important role. We will say informally that a transformation "preserves the contrast" if the transformation preserves the altitude of the minima of the image and if, when two minima are separated by a crest in the original image, they are still separated by a crest of the same altitude in the transform. The formal definition relies on the altitude of the lowest pass which separates two minima, named pass value. One of the main results obtained in [1] states that any topological thinning preserves the contrast (in this sense), and that any transformation that preserves the contrast is a topological thinning.

One of the goals of this paper is to examine the links between the flooding paradigm and the topological watershed. In the first part of this paper, guided by the analysis of a classical flooding algorithm, we present some notions that lead us to a better understanding of the watershed: minima extension, mosaic, pass values and separation (see also [21]). A mosaic image is obtained from an image F and a divide set D of F by valuating the points of D with the corresponding values of these points for F. We prove in particular that a mosaic "preserves the contrast" *if and only if* the mosaic is obtained through a topological thinning. We investigate the possibility for a flooding to produce a topological watershed, and we propose a monotone flooding transformation that preserves the number of connected components of each lower cross-section. We show that this monotone flooding does not always produce a topological watershed.

This leads us to the paradigm of emergence: reversing the flooding paradigm, we start with the highest level first. We call emergence watershed a transformation that lowers points in decreasing altitude order while preserving the number of connected components of lower cross-sections. Our main result states that an emergence watershed is a topological watershed, and more remarkably, that any topological watershed of a given image can be obtained as an emergence watershed of the image.

2. Basic notions and notations

Many fundamental notions related to watersheds in discrete spaces can be expressed in the framework of graphs.

Let E be a finite set of vertices (or points), and let $\mathscr{P}(E)$ denote the set of all subsets of E. Throughout this paper, Γ denotes a binary relation on E, which is reflexive $((x, x) \in \Gamma)$ and symmetric $((x, y) \in \Gamma \Leftrightarrow (y, x) \in \Gamma)$. We say that the pair (E, Γ) is a *graph*. We also denote by Γ the map from E to $\mathscr{P}(E)$ such that, for all $x \in E$, $\Gamma(x) = \{y \in E | (x, y) \in \Gamma\}$. For any point x, the set $\Gamma(x)$ is called the *neighborhood of x*. If $y \in \Gamma(x)$ then we say that x and y are *adjacent*.

Let $X \subseteq E$ We denote by $\overline{X}$ the complement of X in E. Let $x_0, x_n \in X$. A *path from x_0 to x_n in X* is a sequence $\pi = (x_0, x_1, \ldots, x_n)$ of points of X such that $x_{i+1} \in \Gamma(x_i)$, with $i = 0 \ldots n - 1$. Let $x, y \in X$, we say that x and y are *linked for X* if there exists a path from x to y in X. We say that X is *connected* if any x and y in X are linked for X. We say that $Y \subseteq E$ is a *connected component of X* if $Y \subseteq X, Y$ is connected, and Y is maximal for these two properties (i.e., $Y = Z$ whenever $Y \subseteq Z \subseteq X$ and Z is connected). In the following, we assume that the graph (E, Γ) is connected, that is, E is made of exactly one connected component.

We denote by $\mathscr{F}(E)$ the set composed of all maps from E to $\mathbb{Z}$. A map $F \in \mathscr{F}(E)$ is also called an *image*, and if $x \in E$, $F(x)$ is called the *altitude of x (for F)*. Let $F \in \mathscr{F}(E)$. We write $F_k = \{x \in E | F(x) \geqslant k\}$ with $k \in \mathbb{Z}$; F_k is called an *upper (cross-) section* of F, and $\overline{F_k}$ is called a *lower (cross-) section* of F. A non-empty connected component of a lower section $\overline{F_k}$ is called a *(level k) lower-component of F*. A level k lower-component of F that does not contain a level $(k - 1)$ lower-component of F is called a *(regional) minimum of F*.

A subset X of E is *flat for F* if any two points x, y of X are such that $F(x) = F(y)$. If X is flat for F, we denote by $F(X)$ the altitude of any point of X for F.

3. The flooding paradigm

The flooding paradigm corresponds to the intuitive idea of immersion described in the second paragraph of the introduction. In mathematical morphology, it was first proposed by Digabel and Lantuéjoul [11] and used for image segmentation by Beucher and Lantuéjoul [4]. Among the numerous morphological algorithms that were developed following this idea, Meyer's algorithm [18] (called *flooding algorithm* in the sequel) is probably the simplest to describe and understand. We are going to use it as a guide that will help us to introduce the questions we are studying in this paper.

3.1. The flooding algorithm

Starting from an image $F \in \mathscr{F}(E)$ and the set M composed of all points belonging to the minima of F, the flooding algorithm expands as much as possible the set M, while preserving the connected components of M. It can be described as follows:

1. Attribute to each minimum a label, two distinct minima having distinct labels; mark each point belonging to a minimum with the label of the corresponding minimum. Initialize two sets Q and V to the empty set.
2. Insert every non-marked neighbor of every marked point in the set Q;

3. Extract from the set Q a point x which has the minimal altitude, that is, a point x such that $F(x) = \min\{F(y)|y \in Q\}$. Insert x in V. If all marked points in $\Gamma(x)$ have the same label, then

    – Mark x with this label; and
       Insert in Q every $y \in \Gamma(x)$ such that $y \notin Q \cup V$;

4. Repeat step 3 until the set Q is empty.

The divide set is the complement of the set of marked points.

3.2. Illustration of the algorithm

In all the examples of the paper, we assume that the graph (E, Γ) corresponds to the 4-adjacency relation on a subset $E \subset \mathbb{Z}^2$, i.e., for all $x = (x_1, x_2) \in E$, $\Gamma(x) = \{(x_1, x_2), (x_1 + 1, x_2), (x_1 - 1, x_2), (x_1, x_2 + 1), (x_1, x_2 - 1)\} \cap E$.

Let us illustrate the behaviour of the algorithm on the example of Fig. 2a which presents an image with three minima at altitudes 0, 1 and 2.

- The minima at altitudes 2, 1, 0 are marked with the labels A, B, C respectively (Fig. 2b). All the non-marked neighbors of the marked points are put into the set Q.
- The first point which is extracted from the set Q is the point x at altitude 10, which has points marked B and C among its neighbors (Fig. 2b). This point cannot be marked.
- The next point to process is one of the points at altitude 20, for instance y (Fig. 2b). The only marked points in the neighborhood of such a point are marked with the label A, and thus y is marked with the label A (Fig. 2c), and the points at altitude 10 which are neighbors of y are put into the set Q.
- The next points to process are points at altitude 10. A few steps latter, all points at altitude 10 but x are processed, and marked with the label A (Fig. 2d).
- Then the other points at altitude 20 are processed. They are marked with the label A (Fig. 2e).
- The next points to process are those at altitude 30, and we finally obtain the set of labeled points shown in Fig. 2f. The divide set is circled in the figure.

Remark 1. We observe that the algorithm is not "monotone", in the following sense: if a point y of altitude $F(y) = k$ is extracted from the set Q, it is sometimes possible to find in the neighborhood of y a point z not already labeled such that $F(z) < k$. This point z will be the next point processed by the algorithm. Thus this algorithm does not always process points according to increasing altitude.

Remark 2. A second observation is related to the contrast of the original image: in the original image, to go from e.g., the minimum at altitude 0 to the minimum at altitude 2, one has to climb to at least an altitude of 20: indeed, there exists a contour at altitude 20 that we have to overcome. We observe that this contour is not present in the divide set produced by
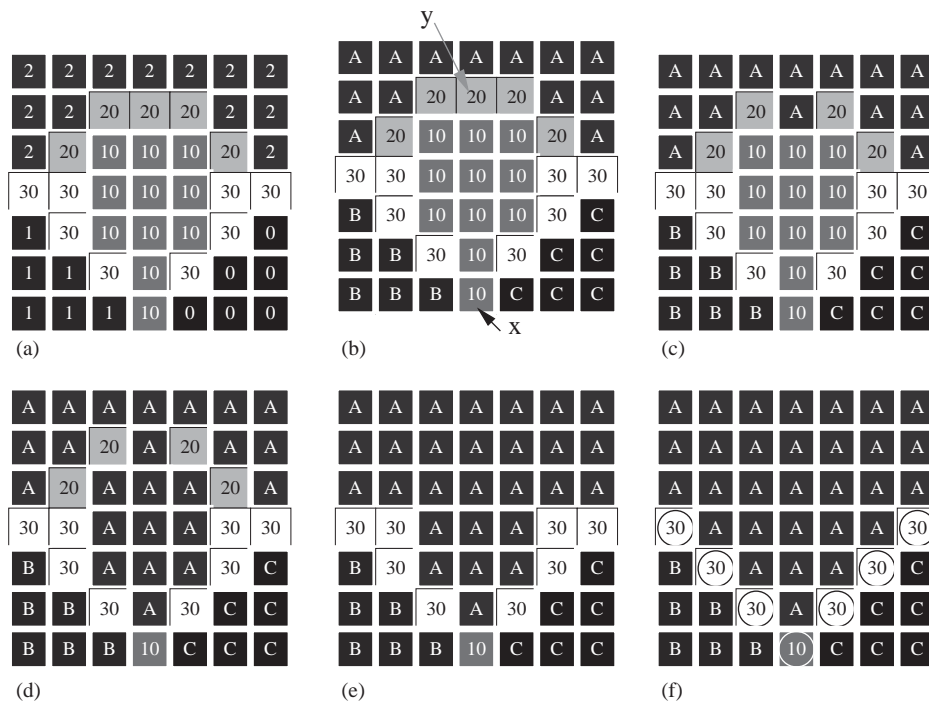
Fig. 2. (a) Original image, (b–f) several steps of the flooding algorithm. One can see that this algorithm is not "monotone": some points at altitude 10 are processed after one of the points at altitude 20. One can also note that the contour at altitude 20 in the original image (a) is not present in the result (f).

the algorithm. Let us emphasize that similar configurations can be found for other adjacency relations, and in particular for the 6- and the 8-adjacency relation. Configurations similar to the examples presented in this paper are found in real-world images.

The following section introduces the formal framework that leads to a better understanding of the previous observations.

4. Minima extensions, mosaics, and pass values

A result of the previous algorithm is to associate an influence zone to each minimum of the image. We formalize this through the definition of a minima extension.

Definition 1. Let $F \in \mathscr{F}(E)$. A *minima extension of F* is a subset X of E such that:

- each connected component of X contains one and only one minimum of F, and
- each minimum of F is included in a connected component of X.
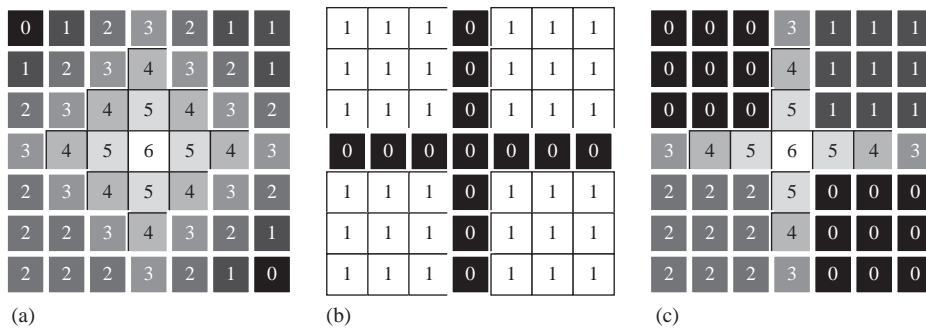 The complementary of a minima extension of F is called a *divide set* (of F).

Fig. 3. (a) An image, (b) a minima extension of (a), and (c) the associated mosaic.

It is easy to prove the following result: let $F \in \mathscr{F}(E)$, and let X be the set composed of all the points labeled by the flooding algorithm applied on F; the set X is indeed a minima extension of F. We call any such set X produced by the flooding algorithm a *flooding extension* (of F). Note that, in general, there may exists several flooding extensions of a given map F.

Intuitively, for application to image analysis, the divide set represents the location of points which best separate the dark objects (regional minima), in terms of grey level difference (contrast). In order to evaluate the effectiveness of this separation, we have to consider the values of points along the divide set. This motivates the following definition.

Definition 2. Let $F \in \mathscr{F}(E)$ and let X be a minima extension of F. The *mosaic* of F associated with X is the map $F_X \in \mathscr{F}(E)$ such that

- for any $x \notin X$, $F_X(x) = F(x)$; and
- for any $x \in X$, $F_X(x) = \min\{F(y) | y \in C_x\}$, where C_x denotes the connected component of X that contains x.

The term 'mosaic' for this kind of construction, was coined by Beucher [3].

Fig. 3 shows a simple example of a minima extension and its associated mosaic. The flooding extension of Fig. 3a is the minima extension 3b, and the associated mosaic is Fig. 3c.

Fig. 4 is another illustration of the definitions of minima extension and mosaic, using the flooding algorithm on the image of Fig. 2a.

Let F be a map and let F_X be the mosaic of F associated with a minima extension X of F. It is natural to try to associate any minimum of F_X to a connected component of X and conversely, and to compare the altitude of each minimum of F_X to the altitude of the corresponding minimum of F. We will see with forthcoming properties and examples, that both problems are in fact closely linked.

The following definition extends to maps the minima extension previously defined for sets.
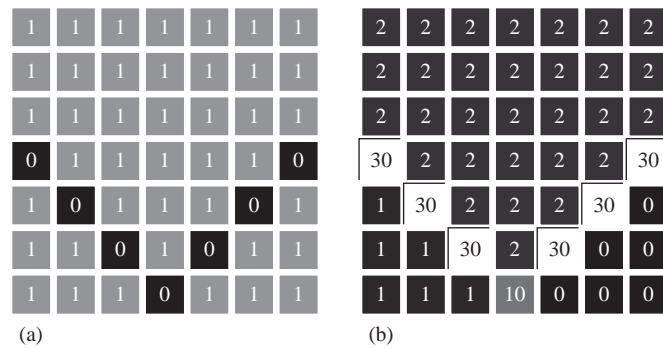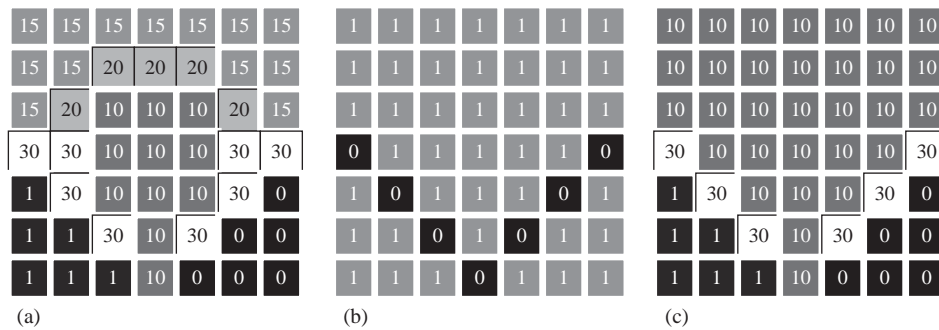
Fig. 4. (a) The flooding extension of Fig. 2a, and (b) the associated mosaic.

Fig. 5. (a) An image F, and (b) the flooding extension of F, and (c) the associated mosaic.

Definition 3. Let F and G in $\mathcal{F}(E)$ such that $G \leqslant F$. We say that G is a *minima extension* (*of F*) if:

(i) the set composed by the union of all the minima of G is a minima extension of F.
(ii) for any $X \in \mathcal{M}(F)$ and $Y \in \mathcal{M}(G)$ such that $X \subseteq Y$, we have $F(X) = G(Y)$.

The image of Fig. 3c (resp. 4b) is an example of a minima extension of the image of Fig. 3a (resp. 2a).

On the other hand, Fig. 5a shows an image F and Fig. 5c shows the mosaic F_X associated with the flooding extension X (Fig. 5b) of the image F. One can notice that the connected component of X which corresponds to the minimum of altitude 15 for F has an altitude of 10 for F_X, and is not a minimum of F_X. Thus, this mosaic F_X is not a minima extension of F. In other words, Fig. 5 shows that mosaics produced by the flooding algorithm are not always minima extensions of the original map.

We can now turn back to a more precise analysis of Remark 2. To this aim, we present the pass value and the separation. Intuitively, the pass value between two points corresponds to the lowest altitude to which one has to climb to go from one of these points to the other.
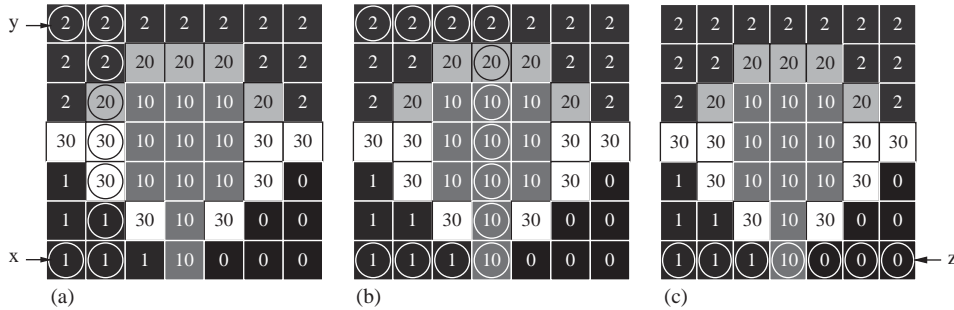
Fig. 6. Illustration of paths and pass values on the image F of Fig. 2a. (a) A path π_1 from the point x to the point y such that $F(\pi_1) = 30$. (b) A path π_2 from the point x to the point y such that $F(\pi_2) = 20$. It is not possible to find a path from x to y with a lower maximal altitude, hence $F(x, y) = 20$. (c) A path π_3 from the point x to the point z such that $F(\pi_3) = 10$, and we can easily check that $F(x, z) = 10$.

Definition 4. Let $F \in \mathscr{F}(E)$. Let $\pi = (x_0, \ldots, x_n)$ be a path in the graph (E, Γ), we set $F(\pi) = \max\{F(x_i) | i = 0, \ldots, n\}$.

Let x, y be two points of E, the *pass value for F between x and y* is defined as $F(x, y) = \min\{F(\pi) | \pi \in \Pi(x, y)\}$, where $\Pi(x, y)$ is the set of all paths from x to y.

Let X, Y be two subsets of E, the pass value for F between X and Y is defined by $F(X, Y) = \min\{F(x, y) | \text{for any } x \in X \text{ and any } y \in Y\}$.

A notion equivalent to the pass value up to an inversion of F (that is, replacing F by $-F$), has been introduced by Rosenfeld [28–30] under the name of *degree of connectivity* for studying connectivity in the framework of fuzzy sets. Fig. 6 illustrates the pass value on the image F of Fig. 2a.

Informally, a transformation "preserves the separation" if, when two points are separated by a crest in the original map, they are still separated by a crest of the same "height" in the transform.

Definition 5 (*Bertrand [1]*). Let $F \in \mathscr{F}(E)$, let $x, y \in E$. We say that x and y are *separated* (*for F*) if $F(x, y) > \max\{F(x), F(y)\}$.

We say that x and y are *linked* (*for F*) if $F(x, y) = \max\{F(x), F(y)\}$.

We say that x and y are *k-separated* (*for F*) if they are separated for F and if $k = F(x, y)$.

Let $G \in \mathscr{F}(E)$, with $G \leqslant F$. We say that G is a *separation of F* if, for all x and y in E, whenever x and y are k-separated for F, x and y are k-separated for G.

We say that G is a *strong separation of F* is G is both a separation of F and a minima extension of F.

Remark 3. We can now restate Remark 2 using the notions we have introduced in this section. Fig. 5 shows that a mosaic produced by the flooding algorithm is not always a minima extension of the original map. Fig. 4 shows that a mosaic produced by the flooding algorithm, even in the case where it is a minima extension, is not necessarily a separation of the original map.

5. Topological watershed

A different approach to the watershed was presented by Couprie and Bertrand [7]. The idea is to transform the image F into an image G while preserving some topological properties of F, namely the number of connected components of the lower cross-sections of F. A minima extension of F can then be obtained easily from G, by extracting the minima of G.

5.1. Definitions

We begin by defining a "simple" point (in a graph), in a sense which is adapted to the watershed, then we extend this notion to weighted graphs through the use of lower sections [7].

Definition 6. Let $X \subseteq E$. The point $x \in X$ is W-*simple* (for X) if x is adjacent to one and only one connected component of $\overline{X}$.

In other words, x is W-simple (for X) if the number of connected components of $\overline{X} \cup \{x\}$ equals the number of connected components of $\overline{X}$.

We can now define the notions of W-destructible point, W-thinning, and topological watershed:

Definition 7. Let $F \in \mathscr{F}(E)$, $x \in E$, and $k = F(x)$.

The point x is *W-destructible* (*for F*) if x is W-simple for F_k.

We say that $G \in \mathscr{F}(E)$ is a *W-thinning of F* if $G = F$ or if G may be derived from F by iteratively lowering W-destructible points by one.

We say that $G \in \mathscr{F}(E)$ is a *topological watershed of F* if G is a W-thinning of F and if there is no W-destructible point for G.

The differences between topological watershed and the notion of *homotopic greyscale skeleton* are discussed in Appendix A.

As a consequence of the definition, a topological watershed G of a map F is a map which has the same number of minima as F. Furthermore, the number of connected components of any lower cross-section is preserved during this transformation.

By the very definition of a W-destructible point, it may easily be proved that, if G is a W-thinning of F, then the union of all minima of G is a minima extension of F (this result is also a consequence of Theorem 10). This allows us to propose the following definition.

Definition 8. Let $F \in \mathscr{F}(E)$ and let G be a W-thinning of F. The *mosaic of F associated with G* is the mosaic of F associated with the union of all minima of G.

Notice that in general, there exist different topological watersheds for a given map F. Fig. 7a presents one of the possible topological watersheds of Fig. 2a, and Fig. 7b shows the associated mosaic. One can note that both Fig. 7a and b are separations of Fig. 2a.

An extensive algorithmic study of the topological watershed is made in [10], which proposes in particular a quasi-linear algorithm.
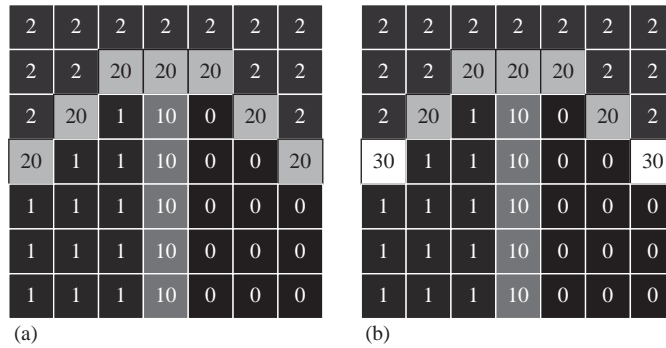
Fig. 7. Example of topological watershed. (a) a topological watershed of Fig. 2a (b) the associated mosaic.

5.2. Topological watershed and separation

Recently, Bertrand [1] showed that a mathematical key underlying the topological watershed is the *separation*. The following theorem asserts that it is sufficient to consider the minima of F for testing if G is a separation of F.

Theorem 9 (*Bertrand [1]*). *Let F and G be two elements of $\mathcal{F}(E)$ such that $G \leqslant F$. The map G is a separation of F if and only if, for all distinct minima X, Y of F, $F(X, Y) = G(X, Y)$.*

The following theorem states the equivalence between the notions of W-thinning and strong separation. The "if" part implies in particular that a topological watershed of an image F preserves the pass values between the minima of F. Furthermore, the "only if" part of the theorem mainly states that if one needs a transformation which is guaranteed to preserve the pass values between the minima of the original map, then this transformation is necessarily a W-thinning.

Theorem 10 (*Bertrand [1]*). *Let F and G be two elements of $\mathcal{F}(E)$. The map G is a W-thinning of F if and only if G is a strong separation of F.*

Let $F \in \mathcal{F}(E)$ and $p \in E$. We denote by $\Gamma^{-}(p, F)$ the set of (strictly) lower neighbors of p, that is, $\Gamma^{-}(p, F) = \{q \in \Gamma(p) | F(q) < F(p)\}$. In the sequel, we will need the following characterization of W-destructible points:

Property 11 (*Couprie et al. [10]*). *Let $F \in \mathcal{F}(E)$ and $p \in E$. The point p is W-destructible for F if and only if $\Gamma^{-}(p, F) \neq \emptyset$ and, for all x and y in $\Gamma^{-}(p, F)$ with $x \neq y$, we have $F(x, y) < F(p)$.*

5.3. Mosaic and separation

We can now prove that the mosaic associated with any W-thinning of a map F is also a W-thinning of F (and thus, it is a separation of F). Furthermore, we prove that an arbitrary

mosaic F_X of a map F is a separation of F if and only if F_X is a W-thinning of F. These strong results can be obtained thanks to the three following properties.

Property 12. *Let $F \in \mathscr{F}(E)$, let X be a minima extension of F, and let F_X be the mosaic of F associated with X. Then, any minimum M of F_X is a connected component of X; furthermore $F_X(M) = F(m)$ where m denotes the unique minimum of F such that $m \subseteq M$.*

A proof of Proposition 12 can be found in Appendix B. The following property follows straightforwardly.

Property 13. *Let $F \in \mathscr{F}(E)$, let X be a minima extension of F, and let F_X be the mosaic of F associated with X. If any connected component of X is a minimum of F_X, then F_X is a minima extension of F.*

Property 14. *Let $F \in \mathscr{F}(E)$, let X be a minima extension of F, and let F_X be the mosaic of F associated with X. If F_X is a separation of F, then F_X is a minima extension of F.*

Proof. As X is a minima extension of F, by Proposition 12, we know that any minimum of F_X is a connected component of X. We have to prove that any connected component of X is a minimum of F_X.

Let M be a connected component of X, and let m be the minimum of F that is included in M. Suppose that M is not a minimum of F_X. Let $k = F_X(M) + 1$, and let C be the connected component of $\overline{(F_X)_k}$ that contains M. Let N be a minimum of F_X that is included in C. By Proposition 12, $N \subseteq X$. Let n the minimum of F that is included in N. We see easily that n and m are such that $F_X(n, m) = F_X(m)$. But F_X is a separation of F, and by Theorem 9, $F_X(n, m) = F(n, m)$. As n and m are minima of F, we have $F(n, m) > \max\{F(n), F(m)\}$, a contradiction. Thus, any connected component of X is a minimum of F_X.

By Proposition 13, F_X is thus a minima extension of F. $\quad\square$

Property 15. *Let $F \in \mathscr{F}(E)$, let G be a W-thinning of F, and let H be the mosaic of F associated with G. Then H is a separation of F.*

Proof. Let M and M' be two distinct minima of F and let $k = F(M, M')$. There exists a path π from a point of M to a point of M' such that $F(\pi) = k$. Since $G \leqslant F$, we have $G(\pi) \leqslant k$, but, by Theorem 9, we must have $G(\pi) \geqslant k$ (otherwise we would have $G(M, M') < k$). Hence $G(\pi) = k$. Since $G \leqslant H$, we have $H(\pi) \geqslant k$. But since $H \leqslant F$, $H(\pi) \leqslant k$. It follows that $H(\pi) = k$ and we may affirm that $H(M, M') \leqslant k$. Now suppose that $H(M, M') < k$. It means that there exists a path π' from a point of M to a point of M' such that $H(\pi') < k$. Since $G \leqslant H$, we would have $G(\pi') < k$ which contradicts $G(M, M') = k$. So $H(M, M') = k$, and, from Theorem 9, we deduce that H is a separation of F. $\quad\square$

Property 16. *Let $F \in \mathscr{F}(E)$, let G be a W-thinning of F, and H be the mosaic of F associated with G. Then H is necessarily a W-thinning of F.*

Proof. By Proposition 15, H is a separation of F. By Proposition 14, H is a minima extension of F. In consequence H is a strong separation of F which, by Theorem 10, implies that H is a W-thinning of F. $\quad\square$

The following theorem is a straightforward consequence of Theorem 10 and Proposition 14.

Theorem 17. *Let $F \in \mathcal{F}(E)$, let X be a minima extension of F, and let F_X be the mosaic of F associated with X. Then F_X is a separation of F if and only if F_X is a W-thinning of F.*

6. Emergence watershed

In this section, we first design a monotone algorithm based on both the flooding paradigm and W-destructible points. We show that such an algorithm does not always produce a topological watershed, more precisely, there may exist points of the divide set that are still W-destructible. This will lead us, in the second part of the section, to reverse the flooding paradigm and to propose the notion of emergence.

To produce a W-thinning, we sequentially lower the altitude of W-destructible points by one. A particular case of this process is obtained if, when a point has been lowered, we immediately check whether the same point is W-destructible or not, and continue until the point is no more W-destructible.

Let $F \in \mathcal{F}(E)$, and let x be a W-destructible point for F.

- We call *W-lowering of x* the action of lowering the altitude of x by one.
- We call *W⋆-lowering of x* the action of successively W-lowering the altitude of x until it is no more W-destructible for the result.

Let us denote by $\mathcal{F}_0(E)$ the set of all maps $F \in \mathcal{F}(E)$ such that $\min\{F(x)|x \in E\} = 0$. In the sequel, for the sake of simplicity and without loss of generality, we will often restrict ourselves to maps belonging to $\mathcal{F}_0(E)$.

6.1. A monotone W-flooding

Let us design a "monotone" flooding-like algorithm based on the lowering of W-destructible points by increasing order of altitude. By Theorem 10, such an algorithm will always produce separation.

Let $F \in \mathcal{F}(E)$. We say that

- G is a *W⋆-thinning of F for level k* if $G = F$ or if we can obtain G from F by iteratively W⋆-lowering some W-destructible points p such that $F(p) = k$.
- G is an *ultimate W⋆-thinning of F for level k* if G is a W⋆-thinning of F for level k and if G contains no W-destructible point p such that $F(p) = k$.

The following algorithm builds a W-thinning that is called a *monotone W-flooding of F*.
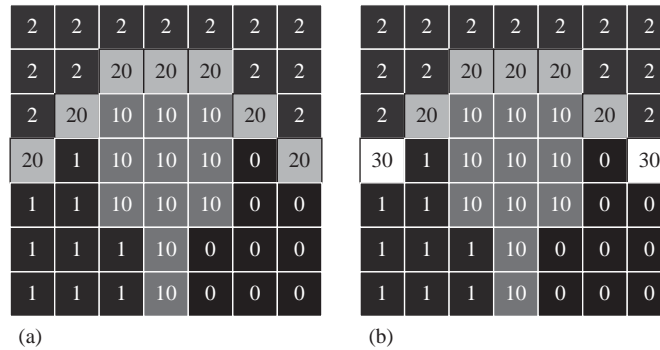
Fig. 8. Example of monotone W-flooding. (a) a monotone W-flooding of Fig. 2a (b) the associated mosaic.

Definition 18. Let $F \in \mathscr{F}_0(E)$, and let $m = \max\{F(x) | x \in E\}$. Let $G^{(0)} = F$, and for any $k = 0 \ldots m - 1$, let $G^{(k+1)}$ be an ultimate W*-thinning of $G^{(k)}$ for level $k + 1$. The sequence $(G^{(0)}, \ldots, G^{(m)})$ is called a *monotone W-flooding sequence for F*, and $G^{(m)}$ is called a *monotone W-flooding of F*.

Let $F \in \mathscr{F}_0(E)$. It is obvious that any monotone W-flooding of F is a W-thinning of F.

Nevertheless, a monotone W-flooding process does not necessarily produces a topological watershed. A monotone W-flooding of the image 2a is depicted in Fig. 8a. It may be seen that, while the monotone W-flooding 8a is a W-thinning of 2a, several points in 8a are W-destructible.

Let us note that a monotone algorithm based on flooding has been proposed by Vincent and Soille [31,34–36]. The application of the dividing variant of this algorithm on an image $F \in \mathscr{F}(E)$ produces a minima extension X of F, but the mosaic F_X of F associated with X is not always a W-thinning of F (see [21]). An illustration is provided in Fig. 11.

6.2. Emergence watershed

We have seen that the flooding paradigm does not lead to a satisfying result, even when we proceed by lowering exclusively W-destructible points. Surprisingly, we will see that reversing the level scanning order leads to an algorithm which possesses good properties. We introduce in this section the emergence watershed, which is based on a process where points are considered in decreasing altitude order, and prove one of the main results of the paper: for any map F, any emergence watershed of F is a topological watershed of F (and thus a separation of F), and more remarkably, any topological watershed of F is an emergence watershed of F. Let us note that a process similar to the emergence has been proposed in [8] in the framework of orders, but no property of this emergence process had been studied in this latter work.

Let $F \in \mathscr{F}(E)$. We say that

- G is a *W-thinning* (*of F*) for level k if $G = F$ or if we can obtain G from F by iteratively W-lowering some W-destructible points p such that $F(p) = k$.
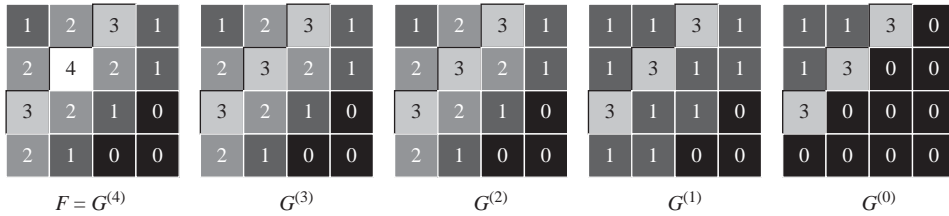
Fig. 9. An image F, and an emergence sequence for F: $(G^{(4)}, G^{(3)}, G^{(2)}, G^{(1)}, G^{(0)})$.

- G is a *ultimate W-thinning* (*of F*) *for level k* if G is W-thinning for level k of F and if G contains no W-destructible point p such that $F(p) = k$.

Definition 19. Let $F \in \mathscr{F}_0(E)$, and let $m = \max\{F(x)|x \in E\}$.

Let $G^{(m)} = F$ and, and for any $k = 1 \ldots m$, let $G^{(k-1)}$ be an ultimate W-thinning of $G^{(k)}$ for level k.

The sequence $(G^{(m)} \ldots G^{(0)})$ is called a *emergence sequence for F*, and $G^{(0)}$ is called an *emergence watershed of F*.

Fig. 9 illustrates the emergence process.

Before stating and proving our results, we introduce some notations, definitions and intermediate properties.

Let $F \in \mathscr{F}(E)$. If $x \in E$, we denote by $F \backslash x$ the element of $\mathscr{F}(E)$ such that $(F \backslash x)(y) = F(y)$ for any $y \neq x$ and $(F \backslash x)(x) = F(x) - 1$.

The following two lemmas arise immediately from Property 11 and from the definition of a W-destructible point.

We recall that $\Gamma^-(p, F) = \{q \in \Gamma(p)|F(q) < F(p)\}$.

Lemma 20. *A point p is not W-destructible for F if and only if either $\Gamma^-(p, F) = \emptyset$ or there exist x and y in $\Gamma^-(p, F)$ with $x \neq y$ such that $F(x, y) = F(p)$.*

Proof. It follows from Property 11 and from the fact that the path $\pi = (x, p, y)$ is such that $F(\pi) = F(p)$. $\square$

Lemma 21. *Let $F \in \mathscr{F}(E)$, let p be a point such that $F(p) = k$, and let q be a point such that $F(q) < k$. If p is W-destructible for F, then p is W-destructible for $F \backslash q$.*

Proof. Since the lower cross-section $\overline{F_k}$ is equal to the lower cross-section $\overline{(F \backslash q)_k}$, the property follows from the very definition of a W-destructible point. $\square$

The following notion of stable point is essential for the understanding of the emergence properties.

Definition 22. Let $F \in \mathscr{F}(E)$ and $p \in E$. We say that p is a *stable point* (*for F*) if p is not W-destructible for any W-thinning of F.

We say that G is a *topological watershed* (*of F*) *above level k* if G is a W-thinning of F and if any p such that $G(p) > k$ is a stable point for G.

Notice that, since any map $F \in \mathscr{F}(E)$ is by definition a W-thinning of F itself, any point which is a stable point for F is not W-destructible for F.

6.3. Emergence and topological watershed

We shall prove that all the points "emerging" from the emergence process (that is, points above the current altitude) are stable points. The proof relies on the following property.

Property 23. *Let $F \in \mathscr{F}(E)$. Let $p \in E$ be a point which is not W-destructible for F and let $q \in E$ be a point W-destructible for F. If p is W-destructible for $F \backslash q$, then $F(q) = F(p)$.*

Proof. Suppose that there exist x and $y \in \Gamma^-(p, F)$ such that $F(x, y) = F(p) = k$. Since q is W-destructible for F, we know that $q \neq p$. Furthermore, since $F \backslash q$ is a W-thinning of F, we know from Theorem 10 that x and y are k-separated for $F \backslash q$, thus p is not W-destructible for $F \backslash q$, a contradiction.

Thus by Lemma 20, we deduce that $\Gamma^-(p, F) = \emptyset$. Since p has no lower neighbor for F and has a lower neighbor for $F \backslash q$, this lower neighbor is q and $F(q) = F(p)$. $\square$

We can now prove that in an emergence sequence, all the points above the current altitude are stable points.

Property 24. *Let $F \in \mathscr{F}_0(E)$. Let $(G^{(m)} \ldots G^{(0)})$ be an emergence sequence for F. Let $k \in [0 \ldots m]$. Then $G^{(k)}$ is a topological watershed of F above level k.*

Proof. Obviously, $G^{(k)}$ is a W-thinning of F. Thus, in order to prove the property, it is sufficient to show that (1) any point p such that $G^{(k)}(p) > k$ is a stable point for $G^{(k)}$.

The property (1) is true for $k = m$ since there is no point $p \in E$ such that $G^{(m)}(p) > m$. Suppose that the property is true for all $i > k$. We set $h = G^{(k)}(p)$, we have $h > k$.

- Suppose that $h > k + 1$. By the recurrence hypothesis, p is a stable point for $G^{(k+1)}$, thus p is obviously a stable point for $G^{(k)}$ which is a W-thinning of $G^{(k+1)}$.
- Suppose now that $h = k + 1$. Suppose that p is not stable for $G^{(k)}$, i.e., p is *W*-destructible for a W-thinning G of $G^{(k)}$. By construction of the emergence, the point p is not W-destructible for $G^{(k)}$.

Let us write $G = G^{(k)} \backslash x_0 \backslash \ldots \backslash x_n$ where for all $i \in [0 \ldots n]$, x_i is W-destructible for $G^{(k)} \backslash x_0 \backslash \ldots \backslash x_{i-1}$. Without loss of generality, we can assume that for any $G^{(k)} \backslash x_0 \backslash \ldots \backslash x_i$, $i < n$, no point of level h has been lowered (otherwise, we choose the first one among such points instead of p). We can also assume that p is not W-destructible for $G^{(k)} \backslash x_0 \backslash \ldots \backslash x_{n-1}$ (otherwise we choose n such that it is the case).

By recurrence hypothesis and by construction, no point of level greater or equal to h has been lowered by this sequence. Thus all the points $x_0, \ldots, x_n$ are such that $G^{(k)}(x_i) < h$. On

the other hand, by Property 23, we may affirm that $(G^{(k)}\backslash x_0\backslash\ldots\backslash x_{n-1})(x_n)=G^{(k)}(p)=h$, hence $G^{(k)}(x_n)=h$, a contradiction. $\quad\square$

We shall now prove that any topological watershed of a map can be obtained by an emergence sequence.

Property 25. *Let $F \in \mathscr{F}_0(E)$ and G a topological watershed of F. Then G is an emergence watershed of F.*

Proof. Let us write $G=F\backslash x_1\backslash x_2\backslash\ldots\backslash x_n$, meaning that G is obtained from F by iteratively W-lowering the points $x_1,\ldots,x_n$. For the sake of brevity, we will denote this sequence of W-lowerings by $(x_1,x_2,\ldots x_n)$.

Let $i \in [1\ldots(n-1)]$. Suppose that, at step i, we have $(F\backslash x_1\backslash\ldots\backslash x_{i-1})(x_i) < (F\backslash x_1\backslash\ldots\backslash x_i)(x_{i+1})$.

Let us show that in this case, we can "exchange" the lowerings of points x_i and x_{i+1}, while still proceeding by W-lowerings.

Let us write $F' = F\backslash x_1\backslash\ldots\backslash x_{i-1}$, the hypothesis becomes $F'(x_i) < (F'\backslash x_i)(x_{i+1})$. Notice that we have necessarily $x_{i+1} \neq x_i$, and thus $F'(x_i) < F'(x_{i+1})$.

We need to prove that (a): x_{i+1} is W-destructible for F'; and that (b): x_i is W-destructible for $F'\backslash x_{i+1}$.

(a) Let us write $k = F'(x_{i+1})$. Since $F'(x_i) < k$, we have $\overline{(F'\backslash x_i)_k} = \overline{F'_k}$. Since x_{i+1} is W-destructible for $(F'\backslash x_i)$, x_{i+1} is W-destructible for F'.
(b) Let us write $h = F'(x_i)$. Since $h < F'(x_{i+1})$, we have $\overline{F'_h} = \overline{(F'\backslash x_{i+1})_h}$. Since x_i is W-destructible for F', x_i is W-destructible for $(F'\backslash x_{i+1})$.

Obviously, $F'\backslash x_i\backslash x_{i+1} = F'\backslash x_{i+1}\backslash x_i$. Thus, the new sequence $(x_1,\ldots,x_{i-1},x_{i+1},x_i, x_{i+2},\ldots,x_n)$ of lowerings is indeed composed of W-lowerings and also produces the map G.

By repeating such exchanges until stability, we see that we can obtain a sequence $S = (x_1',\ldots,x_n')$ of W-lowerings such that $G = F\backslash x_1'\backslash\ldots\backslash x_n'$ and such that for all $1\leqslant i < n$, $(F\backslash x_1'\backslash\ldots\backslash x_{i-1}')(x_i')\geqslant(F\backslash x_1'\backslash\ldots\backslash x_i')(x_{i+1}')$.

We write $F^{(m)}=F$. For any $k \in [1\ldots m]$, we define $F^{(k-1)}=F\backslash x_1'\backslash\ldots\backslash x_i'$, such that x_i' is the last point in the sequence S for which $(F\backslash x_1'\backslash\ldots\backslash x_{i-1}')(x_i')\geqslant k$. We have $F^{(0)}=G$.

The sequence $(F^{(m)}\ldots F^{(0)})$ is an emergence sequence for F. Indeed, suppose that there exists a point p W-destructible for $F^{(k-1)}$ such that $F(p)=k$. By construction and Lemma 21, this point would be W-destructible for $F^{(0)}$. This is not possible since $F^{(0)}$ is a topological watershed. $\quad\square$

We can now state the main result of this section.

Theorem 26. *Let $F \in \mathscr{F}_0(E)$. A map $G \in \mathscr{F}_0(E)$ is a topological watershed of F if and only if it is an emergence watershed of F.*

Proof. Suppose that $(G^{(m)},\ldots,G^{(0)})$ with $G^{(0)} = G$ is an emergence sequence for F. Obviously, the map $G^{(0)}$ is a W-thinning of F. Property 24 states that for all $k \in [0\ldots m]$,

0	10	10	1
0	20	10	1
0	10	10	1

0	0	10	1
0	0	10	1
0	0	10	1

0	10	1	1
0	10	1	1
0	10	1	1

(a) (b) (c)
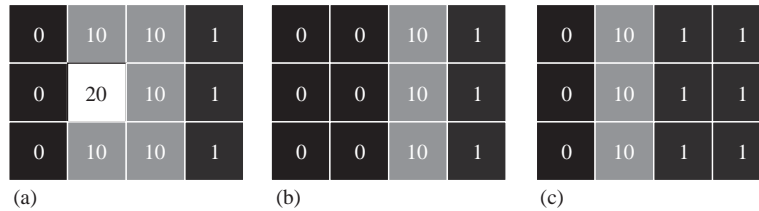
Fig. 10. An image (a), and two associated topological watersheds (b) and (c). Note that, contrary to the topological watershed (b), the topological watershed (c) cannot be obtained through a reverse W-flooding process.

$G^{(k)}$ is a topological watershed above level k of F. In particular $G^{(0)}$ has no W-destructible point such that $G^{(0)}(p) > 0$. Any point p such that $G^{(0)}(p) = 0$ is in a minimum of $G^{(0)}$, hence p is not W-destructible. Thus $G^{(0)}$ is a topological watershed of F. The converse is proved by Property 25. □

6.4. Emergence and reverse W-flooding

We may wonder if we can propose a variant of the emergence process where, instead of lowering the value of points by one (W-lowerings), we lower the value of points until those points are no more W-destructible (W*-lowerings). We are going to see that, although such a process always produces a topological watershed, there exist topological watersheds that cannot be obtained in this way.

The following algorithm, called *reverse W-flooding* is a direct inversion of the monotone W-flooding.

Definition 27. Let $F \in \mathscr{F}_0(E)$, let $m = \max\{F(x)|x \in E\}$.

Let $G^{(m)} = F$ and, for any $k = 1 \dots m$, let $G^{(k-1)}$ be an ultimate $W^\star$-thinning of $G^{(k)}$ for level k.

The sequence $(G^{(m)}, \dots, G^{(0)})$ is called a *reverse W-flooding sequence for F*, and $G^{(0)}$ is called a *reverse W-flooding* of F.

A major feature of the reverse W-flooding is that, in opposition to the monotone W-flooding, the result is guaranteed to be a topological watershed. The proof of the following property is very similar to the one of Property 24, and will thus be omitted.

Property 28. Let $F \in \mathscr{F}_0(E)$, and let G be a reverse W-flooding of F. Then G is a topological watershed of F.

Fig. 10 shows an image and two associated topological watersheds. It can be easily seen that the topological watershed 10c cannot be obtained through a reverse W-flooding process. The point at altitude 20 is necessarily lowered to 0 by any reverse W-flooding.

7. Conclusion

The watershed transform is more and more used as a low-level operator in complex segmentation chains. Among those segmentation procedures, we can cite *hierarchical seg-*

mentation [6] and *geodesic saliency of watershed contours* [23,25]. Such approaches need to compare several divides, or are based on neighborhood relationship between extended minima. It is thus important to be able to characterize some properties of the divides produced by watershed algorithms. This paper is a step in this direction. We introduced several notions that helped us to understand the watershed: minima extension, mosaic, and we also consider the pass values and separation.

The topic of this paper is to examine the links between the flooding paradigm and the topological watershed. We prove in particular that a mosaic is a separation *if and only if* it is a W-thinning. Inspired by the analysis of the flooding algorithm, we present the monotone W-flooding. A monotone W-flooding does not necessarily produce a topological watershed. This leads us to propose the emergence paradigm. A major result of this paper is that any emergence of a given image is a topological watershed of this image, and more remarkably, that any topological watershed of a given image can be obtained as an emergence of the image.

Future work will build up on those results to revisit the saliency of contours. We also aim at exploring definitions and properties of "watersheds without divides" (Fig. 11).

Appendix A. Topological watershed versus homotopic greyscale skeleton

There exists in the literature an approach called *homotopic greyscale skeleton* [13,2,9,26,33] that can be used for thinning a greyscale image. It can be easily proved that the pass values between the minima of a homotopic greyscale skeleton G of an image $F \in \mathscr{F}(E)$ are the same than the pass values between the minima of F.

Fig. 12 presents a 2D image (Fig. 12a), and both a topological watershed (Fig. 12b) and a homotopic greyscale skeleton (Fig. 12c) of this image.

Let us emphasize the essential difference between the topological watershed and the homotopic greyscale skeleton. With the topological watershed, only the number of connected components of the lower cross-sections of the map are preserved, while the homotopic greyscale skeleton preserves both these components and the components of the upper cross-sections. As a consequence, a homotopic greyscale skeleton may be computed by using a purely local criterion for testing whether a point may be lowered or not, while computing a topological watershed requires the reiteration of global algorithms for computing connected components, or the use of a global data structure called *component tree* [7,22]. Notice that a topological watershed only produces closed contours around the regions of interest (Fig. 12b). One can see in Fig. 12c that this is not the case for a homotopic greyscale skeleton: there is a "skeleton branch" at level 11 which does not separate different minima.

Appendix B. Proof of Property 12

Let $F \in \mathscr{F}(E)$, let x, y be two points of E, recall that "x and y are linked for F" means that $F(x, y) = \max\{F(x), F(y)\}$. Let X, Y be two subsets of E which are flat for F, we say that *x and Y are linked for F* if for any $y \in Y$, x and y are linked for F; and we say that *X and Y are linked for F* if for any $x \in X$, for any $y \in Y$, x and y are linked for F. In the same way, we say that *x and Y are separated for F* if for any $y \in Y$, x and y are separated for F.

(a) F^1

2	2	2	2	2	2	2
2	2	20	20	20	2	2
2	20	10	10	10	20	2
30	30	10	10	10	30	30
1	30	10	10	10	30	0
1	1	30	10	30	0	0
1	1	1	10	0	0	0

(b) F^2

2	3	50	2
3	50	3	50
255	128	50	4
2	255	128	50
1	2	255	5

(c) F^3

0	255	1
0	255	128
0	255	2

(d) EF^1

2	2	2	2	2	2	2
2	2	2	2	2	2	2
2	2	2	2	2	2	2
30	2	2	2	2	2	30
1	30	2	2	2	30	0
1	1	30	2	30	0	0
1	1	1	10	0	0	0

(e) EF^2

2	2	50	2
2	50	3	50
255	1	50	4
1	1	1	50
1	1	255	5

(f) EF^3

0	255	1
0	0	128
0	255	2

(g) VS^1

2	2	2	2	2	2	2
2	2	2	2	2	2	2
2	2	10	10	10	2	2
30	2	10	10	10	2	30
1	30	10	10	10	30	0
1	1	1	10	0	0	0
1	1	1	10	0	0	0

(h) VS^2

2	2	50	2
2	50	3	50
255	128	50	4
1	1	128	50
1	1	255	5

(i) VS^3

0	255	1
0	0	128
0	255	2

(j) MW^1

2	2	2	2	2	2	2
2	2	20	20	20	2	2
2	20	10	10	10	20	2
20	1	10	10	10	0	20
1	1	10	10	10	0	0
1	1	1	10	0	0	0
1	1	1	10	0	0	0

(k) MW^2

2	2	50	2
2	50	3	50
255	50	50	4
1	255	50	50
1	1	255	5

(l) MW^3

0	255	1
0	255	128
0	255	2

(m) TW^1

2	2	2	2	2	2	2
2	2	20	20	20	2	2
2	20	1	10	0	20	2
20	1	1	10	0	0	20
1	1	1	10	0	0	0
1	1	1	10	0	0	0
1	1	1	10	0	0	0

(n) TW^1

2	2	50	2
2	50	3	50
255	50	50	4
1	255	50	50
1	1	255	5

(o) TW^3

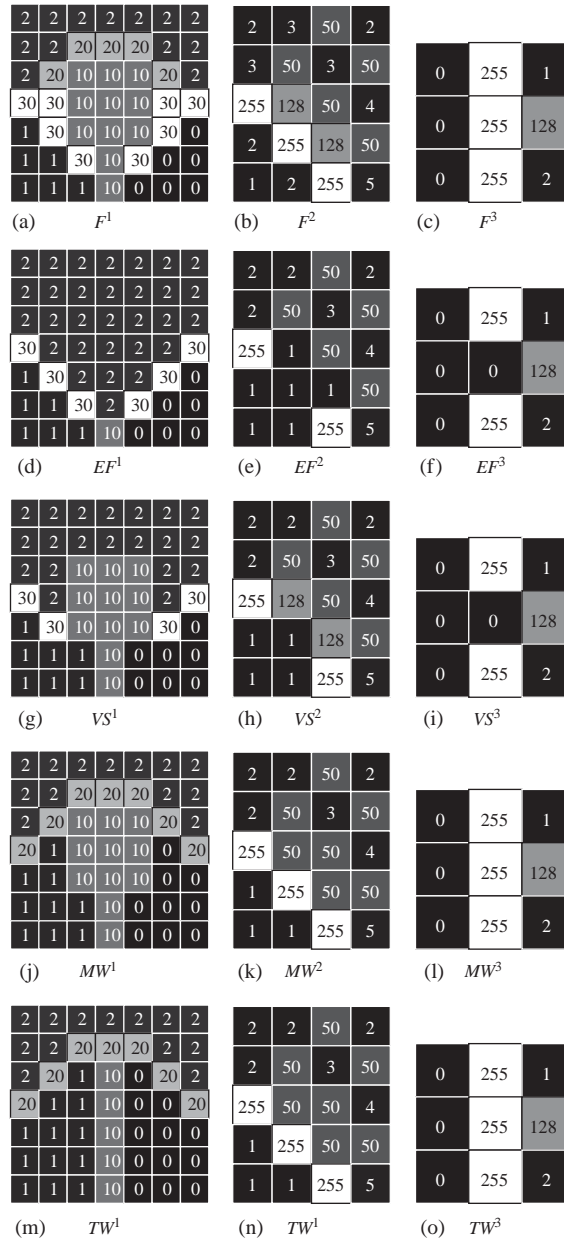| 0	255	1
0	255	128
0	255	2

Fig. 11. Examples of the application of the flooding algorithm, the Vincent–Soille algorithm, the monotone W-flooding and the topological watershed on several images F^i ($i = 1, 2$ and 3). The mosaics produced by the flooding algorithm and by Vincent–Soille's algorithm are denoted by EF^i and VS^i, respectively, the monotone W-floodings are denoted by MW^i and the topological watersheds are denoted by TW^i. One can observe that the pass value between the minima (at altitude) 1 and the minima 2 is 20 for F^1, 10 for EF^1 and VS^1, and 20 for MW^1 and TW^1; the pass value between the minima (at altitude) 1 and any other minima is 255 for F^2, 50 for EF^2, 128 for VS^2 and 255 for MW^2 and TW^2; the pass value between the minima (at altitude) 0 and any other minima is 255 for F^3, 128 for EF^3 and VS^3, and 255 for MW^3 and TW^3.

0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	1	2	11	11	11	4	4	3	3	0	5	11	11	11	3	2	0
0	3	12	6	7	5	11	7	6	5	7	11	5	7	7	12	4	0
0	15	7	4	2	3	5	11	11	11	11	5	3	1	7	7	15	0
0	3	14	7	3	5	11	8	7	8	8	11	5	3	7	14	5	0
0	1	2	13	11	11	3	2	2	1	2	4	11	11	13	4	3	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

(a)

0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	11	11	11	0	0	0	0	0	0	11	11	11	0	0	0
0	0	11	2	2	2	11	0	0	0	0	11	1	1	1	11	0	0
0	11	2	2	2	2	2	11	0	0	11	1	1	1	1	1	11	0
0	0	11	2	2	2	11	0	0	0	0	11	1	1	1	11	0	0
0	0	0	11	11	11	0	0	0	0	0	0	11	11	11	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

(b)

0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	11	11	11	0	0	0	0	0	0	11	11	11	0	0	0
0	0	11	2	2	2	11	0	0	0	0	11	1	1	1	11	0	0
0	15	2	2	2	2	2	11	11	11	11	1	1	1	1	1	15	0
0	0	11	2	2	2	11	0	0	0	0	11	1	1	1	11	0	0
0	0	0	11	11	11	0	0	0	0	0	0	11	11	11	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

(c)

Fig. 12. An image (a), a topological watershed (b) of the image (a) and a homotopic greyscale skeleton (c) of the image (a).

Let us state two basic properties which are fundamental to understand subsequent proofs, and can easily be verified.

Property 29. *Let $F \in \mathscr{F}(E)$, let m be a minimum of F, and let $x \in E$. If x and m are linked for F, then we have*:

$F(x) = F(m)$ *if and only if $x \in m$, and*
$F(x) > F(m)$ *if and only if $x \notin m$*

Property 30. *Let $F \in \mathscr{F}(E)$. For any $x \in E$, there exists a minimum m of F such that x and m are linked. Furthermore, $F(x) \geqslant F(m)$.*

Property 31. *Let $F \in \mathscr{F}(E)$, let X be a minima extension of F, let F_X be the mosaic of F associated with X. Let M be a connected component of X, and let m be the unique minimum of F such that $m \subseteq M$.*

If M is a minimum of F_X, then we have $F_X(M) = F(m)$.

Proof. Since $F_X \leqslant F$, we have $F_X(M) \leqslant F(m)$. Suppose that $F_X(M) < F(m)$. By definition of F_X, there exists a point $x \in M$ such that $F(x) = F_X(M) < F(m)$, furthermore x and m must be separated (Proposition 29). By Proposition 30, there exists a minimum m' of F, $m' \neq m$, such that x and m' are linked for F and $F(x) \geqslant F(m')$. Let $\delta(M)$ denote the set of points of $\overline{M}$ which are adjacent to M. Since M is a minimum of F_X we know that for any $y \in \delta(M)$, $F_X(y) > F_X(M) = F(x)$ and thus for any $y \in \delta(M)$, $F(y) > F(x)$ since $y \in \overline{X}$ and thus $F_X(y) = F(y)$. The fact that x and m' are linked for F thus implies that m' is included in M as well as m, a contradiction with the definition of a minima extension. $\square$

Property 32. *Let $F \in \mathscr{F}(E)$, let X be a minima extension of F, let F_X be the mosaic of F associated with X, let $x \in E$ and let m be a minimum of F. If x is linked to m for F and if $F_X(x) = F(x)$, then x is linked to m for F_X.*

Proof. Since m is a minimum for F and x is linked to m for F, by Proposition 29 we have $F(x) \geqslant F(m)$, thus for any point y of m we have $F(x, y) = F(x)$. Thus, there exists a path $\pi = (x_0, \ldots, x_n)$ from x to m, with $x_0 = x$ and $x_n \in m$, such that $F(\pi) \leqslant F(x)$. For any $i = 1 \ldots n$ we have $F_X(x_i) \leqslant F(x_i)$, thus since $F_X(x) = F(x)$ we have $F_X(\pi) = F_X(x)$. $\square$

Proof of Proposition 12. Let m be any minimum of F, we denote by C_m the connected component of X such that $m \subseteq C_m$. We are going to prove that either (a) C_m is a minimum of F_X, and in this case $F_X(C_m) = F(m)$, or (b) C_m is disjoint with any minimum of F_X. We will also prove that (c) no minimum of F_X is included in $\overline{X}$. It may be seen that the property follows from (a), (b), (c).

(a) Let $\delta(C_m)$ denote the set of points $x \in \overline{C_m}$ which are adjacent to C_m. If all the points x of $\delta(C_m)$ are such that $F(x) > F(m)$, then for any x of $\delta(C_m)$ we have $F_X(x) > F(m)$ (since $x \in \overline{X}$, $F_X(x) = F(x)$). Furthermore, from the very definition of F_X, $\forall z \in C_m$, $F_X(z) \leqslant F(m)$; thus C_m is a minimum for F_X. By Proposition 31, we deduce that $F_X(C_m) = F(m)$.

(b) Suppose now that there exists a point $x \in \delta(C_m)$ such that $F(x) \leqslant F(m)$. Then, x and m are separated for F, otherwise if $F(x) = F(m)$ we would have $x \in m$ and thus $x \in C_m$, and if $F(x) < F(m)$, m would not be a minimum of F (Proposition 29).

Thus, there exists a minimum m' of F, $m' \neq m$, such that x is linked to m' for F and $F(x) \geqslant F(m')$ (Proposition 30). Suppose that $F(x) = F(m')$, since x is linked for F with the minimum m' of F, it would imply that $x \in m'$ (Proposition 29), thus C_m and $C_{m'}$ are adjacent, a contradiction with the definition of the minima extension X. Thus we have

$F(x) > F(m')$. On the other hand, since $x \in \overline{X}$ we have $F_X(x) = F(x)$, thus by Proposition 32, x is linked to m' for F_X. Now two cases must be distinguished.

- If $F_X(C_m) = F(m)$, then we have $F_X(C_m) = F(m) \geqslant F(x) > F(m') \geqslant F_X(m')$, thus C_m is linked to m' for F_X with $F_X(C_m) > F_X(m')$. Now suppose that C_m has a non-empty intersection with a minimum M of F_X. Thus both C_m and M are flat for F_X with the same altitude and since M is a minimum, we have $C_m \subseteq M$. The fact that C_m is linked to m', with $F_X(C_m) > F_X(m')$, raises a contradiction with the fact that M is a minimum of F_X.
- If $F_X(C_m) < F(m)$, then there exists a point $y \in C_m$ such that $F(m) > F(y) = F_X(C_m)$, thus $F(y) = F_X(y)$. Since $F(y) < F(m)$ and m is a minimum of F, we know that y does not belong to m, and with the same arguments as above we see that y and m are separated for F. Thus, there exists a minimum m' of F, $m' \neq m$, such that y is linked to m' for F and $F(y) \geqslant F(m')$. As above, we can see that indeed $F(y) > F(m')$, that y is linked to m' for F_X, that $F_X(C_m) = F(y) > F(m') \geqslant F_X(m')$, and finally that C_m cannot have a non-empty intersection with a minimum of F_X.

(c) Let M be any subset of $\overline{X}$ which is flat for F_X (thus M is also flat for F), and let k denote $F_X(M)$ (which is equal to $F(M)$). Since X is a minima extension for F, we know that M is not a minimum of F, thus there exists a point y of $\overline{M}$ adjacent to M such that $F(y) \leqslant k$. Hence, $F_X(y) \leqslant k$ and M is not a minimum of F_X. $\quad\square$

References

[1] G. Bertrand, Some properties of topological greyscale watersheds, in: L.J. Latecki, D.M. Mount, A.Y. Wu (Eds.), Vision Geometry XII. Number 5300 in IS&T/SPIE Symposium on Electronic Imaging, San-Jose, CA, USA, SPIE, 2004, pp. 127–137.

[2] G. Bertrand, J. Everat, M. Couprie, Image segmentation through operators based upon topology, J. Electronic Imaging 6 (1997) 395–405.

[3] S. Beucher, Segmentation d'images et morphologie mathématique, Ph.D. Thesis, École des Mines de Paris, France, 1990.

[4] S. Beucher, C. Lantuéjoul, Use of watersheds in contour detection, in: Proceedings of International Workshop on Image Processing, Real-Time Edge and Motion Detection/Estimation, Rennes, France, 1979.

[5] S. Beucher, F. Meyer, The morphological approach to segmentation: the watershed transformation, in: E. Dougherty (Ed.), Mathematical Morphology in Image Processing, Marcel Decker, 1993, pp. 433–481.

[6] S. Beucher, Watershed, hierarchical segmentation and waterfall algorithm, in: J. Serra, P. Soille (Eds.), Proceedings of Mathematical Morphology and its Applications to Image Processing, Fontainebleau, France, Kluwer, 1994, pp. 69–76.

[7] M. Couprie, G. Bertrand, Topological grayscale watershed transform, in: SPIE Vision Geometry V Proceedings, vol. 3168, 1997, pp. 136–146.

[8] M. Couprie, G. Bertrand, Tesselations by connection in orders, in: Discrete geometry for computer imagery, Lecture Notes in Computer Science, vol. 1953, Springer, Berlin, 2000, pp. 15–26.

[9] M. Couprie, F.N. Bezerra, G. Bertrand, Topological operators for grayscale image processing, J. Electronic Imaging 10 (2001) 1003–1015.

[10] M. Couprie, L. Najman, G. Bertrand, Quasi-linear algorithms for topological watershed, Journal of Mathematical Imaging and Vision (2005), Special issue on Mathematical Morphology, to appear.

[11] H. Digabel, C. Lantuéjoul, Iterative algorithms, in: J.L. Chermant (Ed.), Second European Symposium on Qualitative Analysis of Microstructures in Material Science, Biology and Medicine, Stuttgart FRG, Riederer Verlag, 1978, pp. 85–99.

[12] E. Dougherty, R. Lotufo, Hands-on Morphological Image Processing, SPIE Press, 2003.

[13] V. Goetcherian, From binary to grey tone image processing using fuzzy logic concepts, Pattern Recognition 12 (1980) 7–15.

[14] H. Heijmans, Morphological Image Operators, Academic Press, New York, 1994.

[15] C. Jordan, Nouvelles observations sur les lignes de faîtes et de thalweg, Comptes Rendus des Séances de l'Académie des Sciences 75 (1872) 1023–1025.

[16] R.A. Lotufo, A.X. Falcao, F.A. Zampirolli, Ift-watershed from gray-scale marker, in: SIBGRAPI'02, Fortaleza-CE, Brazil, 2002, pp. 146–152.

[17] J. Maxwell, On hills and dales, Philosophical Magazine 4/40 (1870) 421–427.

[18] F. Meyer, Un algorithme optimal de ligne de partage des eaux, in: Actes du 8ème Congrès AFCET, Lyon-Villeurbanne, France, 1991, pp. 847–859.

[19] F. Meyer, Topographic distance and watershed lines, Signal Processing 38 (1994) 113–126 (special issue on Mathematical Morphology).

[20] F. Meyer, S. Beucher, Morphological segmentation, Journal of Visual Communication and Image Representation 1 (1990) 21–46.

[21] L. Najman, M. Couprie, Watershed algorithms and contrast preservation, in: DGCI'03. Lecture Notes in Computer Sciences, vol. 2886, Springer, Berlin, 2003, pp. 62–71.

[22] L. Najman, M. Couprie, Quasi-linear algorithm for the component tree, in: L.J. Latecki, D.M. Mount, A.Y. Wu (Eds.), Vision Geometry XII. Number 5300 in IS&T/SPIE Symposium on Electronic Imaging, San-Jose, CA, USA, SPIE, 2004, pp. 98–107.

[23] L. Najman, Morphologie Mathématique: de la Segmentation d'Images à l'Analyse Multivoque, Ph.D. Thesis, Université Paris-Dauphine, 1994.

[24] L. Najman, M. Schmitt, Watershed of a continuous function, Signal Processing 38 (1994) 99–112 (special issue on Mathematical Morphology).

[25] L. Najman, M. Schmitt, Geodesic saliency of watershed contours and hierarchical segmentation, IEEE Trans. on PAMI 18 (1996) 1163–1173.

[26] V. Ranwez, P. Soille, Order independent homotopic thinning for binary and grey tone anchored skeletons, Pattern Recognition Lett. 23 (2002) 687–702.

[27] J. Roerdink, A. Meijster, The watershed transform: definitions, algorithms and parallelization strategies, Fundamenta Informaticae 41 (2000) 187–228.

[28] A. Rosenfeld, Fuzzy digital topology, Information and Control 40 (1979) 76–87.

[29] A. Rosenfeld, On connectivity properties of grayscale pictures, Pattern Recognition 16 (1983) 47–50.

[30] A. Rosenfeld, The fuzzy geometry of image subsets, Pattern Recognition Lett. 2 (1984) 311–317.

[31] P. Soille, Morphologie mathématique: du relief à la dimensionalité–Algorithmes et méthodes, Ph.D. Thesis, Université catholique de Louvain, Partnership with the Paris's School of Mines, 1992.

[32] P. Soille, Morphological Image Analysis, Springer, Berlin, 1999.

[33] P. Soille, M. Ansoult, Automated basin delineation from digital elevation models using mathematical morphology, Signal Processing 20 (1990) 171–182.

[34] P. Soille, L. Vincent, Determining watersheds in digital pictures via flooding simulations, in: M. Kunt (Ed.), Visual Communications and Image Processing'90, vol. 1360, Bellingham, Society of Photo-Instrumentation Engineers, 1990, pp. 240–250.

[35] L. Vincent, Algorithmes morphologiques à base de files d'attente et de lacets, Extension aux graphes (Morphological Algorithms Based on Queues and Loops, with Extension to Graphs), Ph.D. Thesis, School of Mines, Paris, 1990.

[36] L. Vincent, P. Soille, Watersheds in digital spaces: an efficient algorithm based on immersion simulations, IEEE Transactions on PAMI 13 (1991) 583–598.