University of Salzburg | Lecturer: Roland Kwitt

Machine Learning (911.236)

Exercise sheet C (April 21, 2018) | Deadline: **April 29, 2018 (11pm)**

Uniform convergence

Exercise 1. 5 P.

In the lecture, we showed that finite hypothesis classes have the uniform convergence property. This then led to agnostic PAC learnability.

However, we have always assumed that the range of the loss function is $[0, 1]$, i.e., $l : \mathcal{H} \times Z \to [0, 1]$ (remember that $Z = \mathcal{X} \times \mathcal{Y}$). Show that, in case the loss function has range $[a, b]$, the inequality chain

$$m_{\mathcal{H}}(\epsilon, \delta) \leq m_{\mathcal{H}}^{UC}(\epsilon/2, \delta) \leq \left\lceil \frac{2\log(2|\mathcal{H}|/\delta)(b-a)^2}{\epsilon^2} \right\rceil$$

holds.

VC-Dimension

Exercise 2. 5 P.

Show the following monotonicity property of the VC-Dimension: For every two hypothesis classes $\mathcal{H}', \mathcal{H}$:

$$\text{if } \mathcal{H}' \subseteq \mathcal{H} \Rightarrow \text{VCdim}(\mathcal{H}') \leq \text{VCdim}(\mathcal{H})$$

Exercise 3. 10 P.

Let $\mathcal{H}_{si}$ be the class of **signed intervals**, i.e.,

$$\mathcal{H}_{si} = \{h_{a,b,s} : a \leq b, s \in \{+1, -1\}\}$$

where

$$h_{a,b,s}(x) = \begin{cases} +s & \text{if } x \in [a, b] \\ -s & \text{if } x \notin [a, b] \end{cases}$$

What is the VC-dimension $\text{VCdim}(\mathcal{H}_{si})$?

Exercise 4. 10 P.

Let $\mathcal{H}$ be a hypothesis class of functions from X to $\{0, 1\}$. Proof that if $\text{VCdim}(\mathcal{H}) = \infty$, then $\mathcal{H}$ is not PAC learnable.

Exercise 5. 10 P.

We have shown that for a **finite** hypothesis class $\mathcal{H}$, $\text{VCdim}(\mathcal{H}) \leq \lfloor \log(|\mathcal{H}|) \rfloor$. However, this is just an upper bound. The VC-dimension of a class can be much lower than that.

1. Find an example of a class $\mathcal{H}$ of functions over the real interval $\mathcal{X} = [0, 1]$ such that $\mathcal{H}$ is <u>infinite</u> while $\text{VCdim}(\mathcal{H}) = 1$. In other words, you have an infinite number of hypothesis, but you can only shatter sets of size ≤ 1 (Note: the empty set is always shattered).

2. Give an example of a finite hypothesis class $\mathcal{H}$ over the domain $\mathcal{X} = [0, 1]$, where

$$\text{VCdim}(\mathcal{H}) = \lfloor \log_2(|\mathcal{H}|) \rfloor$$

Exercise 6. 10 P.

Let $\mathcal{H}_1, \ldots, \mathcal{H}_r$ be hypothesis classes over some fixed domain $\mathcal{X}$. Let

$$d = \max_i \mathrm{VCdim}(\mathcal{H}_i)$$

and assume, for simplicity, that $d \geq 3$. Prove that

$$\mathrm{VCdim}(\cup_i \mathcal{H}_i) \leq 4d \log(2d) + 2 \log(r)$$

Hint(s): For any two sets A and B of functions from $\mathcal{X} \to \{0, 1\}$ (with finite VC dimension), you need to argue that for the growth-function it holds that $\tau_{A \cup B}(m) \leq \tau_A(m) + \tau_B(m)$.

Make use of Sauer's lemma and the Lemma 1 given below.

Lemma 1. *Let $a \geq 1$ and $b > 0$. Then*

$$x \geq 4a \log(2a) + 2b \Rightarrow x \geq a \log(x) + b$$