University of Salzburg | Lecturer: Roland Kwitt

Machine Learning (911.236)

Exercise sheet **D**

VC-Dimension & Dudley classes

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a real-valued function and define

$$\mathrm{POS}(f)(x) = \mathbf{1}_{f(x)>0}$$

i.e., a classifier built from f. For a *class* of real-valued functions, denoted as $\mathcal{F}$, we define the corresponding class of classifiers via

$$\mathrm{POS}(\mathcal{F}) = \{\mathrm{POS}(f) : f \in \mathcal{F}\}$$

Also, for $g : \mathbb{R}^n \to \mathbb{R}$ and a class $\mathcal{F}$, we define

$$\mathcal{F} + g = \{f + g : f \in \mathcal{F}\} \enspace ,$$

where $f+g$ is to be understood point wise, i.e., $(f+g)(\mathbf{x}) = f(\mathbf{x})+g(\mathbf{x})$. Hypothesis classes that have a representation as $\mathrm{POS}(\mathcal{F} + g)$ for some *vector space of functions*, $\mathcal{F}$, and some function g (as above) are called *Dudley classes*.

Exercise 1. 15 P.

Part A (5 points). Show that for every $g : \mathbb{R}^n \to \mathbb{R}$ and every vector space of functions, $\mathcal{F}$, we have

$$\mathrm{VC}(\mathrm{POS}(\mathcal{F} + g)) = \mathrm{VC}(\mathrm{POS}(\mathcal{F})) \enspace .$$

In particular, show that a set $C = \{\mathbf{x}_1, \ldots, \mathbf{x}_m\}$ is shattered by $\mathrm{POS}(\mathcal{F})$ if and only if ($\Leftrightarrow$) the set C is shattered by $\mathrm{POS}(\mathcal{F} + g)$, i.e., show the "$\Rightarrow$" and the "$\Leftarrow$" part.

Part B (5 points). Consider the following theorem from Wenocour & Dudley [WD81, Theorem 3.1]

Theorem 1. *If $\mathcal{F}$ is a n-dimensional real vector space of real-valued functions defined on $\mathcal{X}$ and $g : \mathcal{X} \to \mathbb{R}$, then VC(POS($\mathcal{F} + g$)) = n.*

Using this theorem, show (1) that the hypothesis class of open balls in $\mathbb{R}^n$, i.e.,

$$\mathcal{B}_n = \{h_{\mathbf{a},r} : \mathbf{a} \in \mathbb{R}^n, r \in \mathbb{R}_+\}$$

with

$$h_{\mathbf{a},r} : \mathbb{R}^b \to \{0,1\}, \quad h_{\mathbf{a},r}(\mathbf{x}) = 1 \Leftrightarrow \|\mathbf{x} - \mathbf{a}\| < r$$

has a representation as a Dudley class and (2) then establish the VC dimension of $\mathrm{VC}(\mathcal{B}_n)$ using Theorem 1. Hint: Rephrase $h_{\mathbf{a},r}$ such that it fits into the Dudley class formalism, i.e., $s(\mathbf{x}) + g(\mathbf{x})$ with $s \in \mathcal{F}$ and $g : \mathbb{R}^n \to \mathbb{R}$ chosen appropriately, then think about the functions that span $\mathcal{F}$ and infer the dimensionality of that vector space.

Part C (5 points). Similar in spirit to Part B, let

$$\mathcal{P}_n^d = \{h_p : p \text{ is a polynomial of degree} \le d \text{ in variables } x_1, \ldots, x_n\}$$

be a hypothesis class with $h_p(\mathbf{x}) = \mathbf{1}_{p(\mathbf{x})>0}$. Find the VC dimension of $\mathcal{P}_1^d$, i.e., all degree d polynomials over $\mathbb{R}$.

References

[WD81] R.S. Wenocour and P.M. Dudley. Some special Vapnik-Chervonenkis classes. *Discrete Mathematics*, 33:313–318, 1981.

Total #points: 15 P.