Theorem 1. *The VC dimension of the class of homogeneous halfspaces in* $\mathbb{R}^d$*, i.e.,*

$$\mathcal{H} = \{x \mapsto \operatorname{sign}(\langle w, x\rangle) \,:\, w \in \mathbb{R}^d\}$$

is $\operatorname{VCdim}(\mathcal{H}) = d$.

Proof. The proof is done in two parts, first we show $\operatorname{VCdim}(\mathcal{H}) \geq d$, second we show $\operatorname{VCdim}(\mathcal{H}) < d + 1$.

$\underline{\operatorname{VCdim}(\mathcal{H}) \geq d}$: We need to show that there exists a set C of size d that is shattered by $\mathcal{H}$. Choose $\mathbf{e}_1, \ldots, \mathbf{e}_d$, where $\mathbf{e}_i$ is the i-th unit vector in $\mathbb{R}^d$, i.e., all zeros except at position i. For any labeling $y_1, \ldots, y_d$, we can choose $\mathbf{w} = [y_1, \ldots, y_d]$ and obtain

$$\forall i \in [d] \,:\, \langle \mathbf{e}_i, \mathbf{w}\rangle = y_i$$

which establishes our lower bound.

$\underline{\operatorname{VCdim}(\mathcal{H}) < d + 1}$: We need to show that *no* set of size $d + 1$ is shattered by $\mathcal{H}$. We show this by *contradiction*. Assume that a set $\mathbf{x}_1, \ldots, \mathbf{x}_{d+1}$ of size $d + 1$ is shattered by $\mathcal{H}$, hence, all 2^{d+1} labelings can be obtained. In other words, for every possible labeling, there is a $\mathbf{w}$ such that this labeling can be obtained. Lets denote these weight vectors as $\mathbf{w}_1, \ldots, \mathbf{w}_{2^{d+1}}$ and write down the full matrix of all classifier outputs (before the sign), denoted by $\mathbf{H}$:

$$\mathbf{H} = \begin{pmatrix} \langle \mathbf{x}_1, \mathbf{w}_1\rangle & \cdots & \langle \mathbf{x}_1, \mathbf{w}_{2^{d+1}}\rangle \\ \vdots & \ddots & \vdots \\ \langle \mathbf{x}_{d+1}, \mathbf{w}_1\rangle & \cdots & \langle \mathbf{x}_{d+1}, \mathbf{w}_{2^{d+1}}\rangle \end{pmatrix} = \begin{pmatrix} \mathbf{x}_1 \\ \vdots \\ \mathbf{x}_{d+1} \end{pmatrix} (\mathbf{w}_1 \cdots \mathbf{w}_{2^{d+1}})$$

$\mathbf{H}$ is a $(d + 1) \times (2^{d+1})$ matrix, and each column of $\operatorname{sign}(\mathbf{H})$ identifies one labeling of $\mathbf{x}_1, \ldots, \mathbf{x}_{d+1}$. Importantly, the rows are *linearly independent*, i.e., there is no non-trivial $\mathbf{a}$, such that

$$\mathbf{a}^\top \mathbf{H} = \mathbf{0}^\top \ .$$

Just consider the k-th entry, u_k, in $\mathbf{u} = \mathbf{a}^\top \mathbf{H}$ which is given by

$$\mathbf{a}^\top \mathbf{X} \mathbf{w}_k = u_k$$

Due to our shattering assumption, we know that there is some k for which

$$\operatorname{sign}(\mathbf{X}\mathbf{w}_k) = \operatorname{sign}(\mathbf{a})$$

implying that $u_k > 0$ (as we sum over non-negative numbers). Hence, no non-trivial $\mathbf{a}$ exists and the linear independency claim holds.

Finally, all $d+1$ rows being linearly independent then implies that

$$\operatorname{rank}(\mathbf{H}) = d + 1$$

However, as $\mathbf{H} = \mathbf{X}\mathbf{W}$, we also have that

$$\begin{aligned} d + 1 = \operatorname{rank}(\mathbf{H}) &\leq \min\{\operatorname{rank}(\mathbf{X}), \operatorname{rank}(\mathbf{W})\} \\ &\leq d \end{aligned} \tag{1}$$

which is a contradiction. The last inequality holds as the rank of $\mathbf{X}$ is bounded by the smallest dimension, i.e., d (same for $\mathbf{W}$). Hence, no set of size $d+1$ can be shattered by $\mathcal{H}$, i.e., $\operatorname{VCdim}(\mathcal{H}) < d+1$ and we conclude $\operatorname{VCdim}(\mathcal{H}) = d$. □

Intervals in $\mathbb{R}$. Let

$$\mathcal{H} = \{x \mapsto h_{a,b}(x) = 1_{x \in (a,b)} \, : \, a < b\}$$

be the class of intervals in $\mathbb{R}$. For the lower bound, $\operatorname{VCdim}(\mathcal{H}) \geq 1$ is obvious, for $\operatorname{VCdim}(\mathcal{H}) \geq 2$, see the illustration below with interval boundaries (a, b) in red. For sets of size 3, assume w.l.o.g., $x_1 \leq x_2 \leq x_3$. In fact, using $\mathcal{H}$, the labeling $(1, 0, 1)$ cannot be achieved and we have $\operatorname{VCdim}(\mathcal{H}) < 3$; consequently $\operatorname{VCdim}(\mathcal{H}) = 2$.