Claim: The Fourier transform of $f(x-a) = e^{-iya}\hat{f}(y)$ for $a > 0$ (i.e., the translation property).

Proof of the Fourier **translation property**: Let $g(x) = f(x-a)$.

$$\hat{g}(y) = \int_{-\infty}^{\infty} f(x-a)e^{-ixy}\mathrm{d}x \tag{1}$$

Set $z = x - a \Rightarrow \mathrm{d}z = \mathrm{d}x$. We obtain

$$\begin{aligned}
\hat{g}(y) &= \int_{-\infty}^{\infty} f(x-a)e^{-ixy}\mathrm{d}x \\
&= \int_{-\infty}^{\infty} f(z)e^{-i(z+a)y}\mathrm{d}z \\
&= \int_{-\infty}^{\infty} f(z)e^{-iyz}e^{-iya}\mathrm{d}z \\
&= e^{-iya}\int_{-\infty}^{\infty} f(z)e^{-iyz}\mathrm{d}z \\
&= e^{-iya}\hat{f}(y)
\end{aligned} \tag{2}$$

which shows our claim.

Claim: The Fourier transform of $f(ax) = 1/a\hat{f}(y/a)$ for $a > 0$ (i.e., the scaling property).

Proof of the Fourier **scaling property**:

$$\hat{f}(y) = \int_{-\infty}^{\infty} f(ax)e^{-ixy}\mathrm{d}x \tag{3}$$

Set $z = ax \Rightarrow \mathrm{d}z = a\mathrm{d}x$ and we get

$$\begin{aligned}
\hat{f}(y) &= \int_{-\infty}^{\infty} f(ax)e^{-ixy}\mathrm{d}x \\
&= \frac{1}{a}\int_{-\infty}^{\infty} f(z)e^{-iyz/a}\mathrm{d}z \\
&= \frac{1}{a}\hat{f}(y/a)
\end{aligned} \tag{4}$$

which confirms our claim. *Note*: If you allow a to be negative as well, you need to be more careful, but essentially you get $1/|a|\hat{f}(y/a)$ as the corresponding Fourier transform.

The convolution between two functions f and g is defined as

$$(f * g)(x) = \int_{-\infty}^{\infty} f(x-t)g(t)\mathrm{d}t$$

Claim (convolution property): $(\widehat{f * g}) = \hat{f}\hat{g}$.

Proof of the Fourier **convolution property**:

$$(\widehat{f * g})(y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x-t)g(t)e^{-iyt}\mathrm{d}t\mathrm{d}x \tag{5}$$

We first write e^{-iyt} as $e^{-iyt} = e^{-iy(x-t)}e^{-iyt}$ and substitute $z = x - t \Rightarrow \mathrm{d}z = \mathrm{d}x$. We obtain

$$\begin{aligned}
(\widehat{f * g})(y) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x-t)g(t)e^{-iy(x-t)}e^{-iyt}\mathrm{d}t\mathrm{d}x \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(z)g(t)e^{-iyz}e^{-iyt}\mathrm{d}t\mathrm{d}z \\
&= \int_{-\infty}^{\infty} f(z)e^{-iyz}\mathrm{d}z + \int_{-\infty}^{\infty} g(t)e^{-iyt}\mathrm{d}z \\
&= \hat{f}(y)\hat{g}(y)
\end{aligned} \tag{6}$$

which concludes the proof.

Fourier transform example

Say $f(x) = e^{-|x|}$ and we want to know its Fourier transform $\hat{f}(y)$. First, we note that $f(x)$ is an *even* function, as $f(x) = f(-x)$. This is nice, as for even functions, we know that

$$\int_{-\infty}^{\infty} f(x) sin(xy)\mathrm{d}x = 0$$

You can easily verify this using `Mathematica`

```
Integrate[Exp[-Abs[x]]*Sin[x*y], {x, -Infinity, Infinity},
    Assumptions -> y \[Element] Reals]
```

Now, the Fourier transform boils down to

$$\begin{aligned}
\hat{f}(y) &= \int_{-\infty}^{\infty} f(x) cos(xy)\mathrm{d}x \\
&= 2\int_{0}^{\infty} e^{-x} cos(xy)\mathrm{d}x
\end{aligned} \tag{7}$$

as we can split $e^{-iyx} = \cos(xy) - i * \sin(xy)$. We are not going to solve the integral in Eq. (7) by hand, but use `Mathematica` again,

```
Integrate[Exp[-x]*Cos[y*x], {x, 0, Infinity},
    Assumptions -> y \[Element] Reals]
```

from which we get

$$\int_0^\infty e^{-x} cos(xy) \mathrm{d}x = \frac{1}{1+y^2}$$

and we finally obtain

$$\hat{f}(y) = \frac{2}{1+y^2} \quad .$$