Cutting Planes Width
&
The Complexity of Graph Isomorphism Refutations

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SAT 2023, Alghero

Cutting Planes as Propositional Proof System

In CP a CNF formula F is translated into a system of linear inequalities:

- Clause gets translated to linear inequality:

$$(x \vee \bar{y} \vee \bar{z}) \quad \leftrightsquigarrow \quad x + (1 - y) + (1 - z) \geq 1.$$

- Relax Boolean variables: axioms $x \geq 0$, $-x \geq -1$ for every variable.

The system has **integer solutions** if and only if the formula is **satisfiable**.

Cutting Planes Rules: Linear Combination + Rounding

Linear combination: For $\alpha_1, \ldots, \alpha_k \in \mathbb{N}_0$:

$$\frac{\langle \mathbf{a^{(1)}}, \mathbf{x} \rangle \geq \gamma_1 \quad \ldots \quad \langle \mathbf{a^{(k)}}, \mathbf{x} \rangle \geq \gamma_k}{\sum_{i=1}^{k} \alpha_i \langle \mathbf{a^{(i)}}, \mathbf{x} \rangle \geq \sum_{i=1}^{k} \alpha_i \gamma_i}.$$

Rounding: If all the coefficients in the vector **a** are divisible by a $c \in \mathbb{N}$:

$$\frac{\langle \mathbf{a}, \mathbf{x} \rangle \geq \gamma}{\langle \frac{\mathbf{a}}{c}, \mathbf{x} \rangle \geq \lceil \frac{\gamma}{c} \rceil}.$$

Both rules together: **GC-cut rule**

CP Refutations

A **CP refutation** for a set $f = \{f_1, \ldots, f_m\}$ of linear inequalities is a sequence $(g_1, \ldots, g_t)$ of inequalities satisfying:

- each g_i is either an *axiom* or obtained from previous inequalities by a GC-cut,
- and g_t is the inequality $0 \geq 1$.

Sound and complete system for integer solutions.

Complexity Measures for CP

Length: number of vertices in the refutation graph.

(Chvátal) rank: max. number of roundings in a path from an axiom to the $0 \geq 1$ inequality.

(Cut)width: max. number of variables after rounding.

CP-width introduced in [Dantchev, Martin 11];
Supercritical trade-offs between width and rank in [Razborov 17]

Very natural measures:
Rank similar to depth measure in resolution
Cutwidth measure similar to width in resolution

$$\mathcal{C}(f \vdash) := \min_{\pi : f \vdash} \mathcal{C}(\pi).$$

Formulas for Graph Isomorphism

$G = (V_G, E_G)$ and $H = (V_H, E_H)$ graphs with $V_G = V_H = \{1, \ldots, n\}$.
Variables $x_{i,j}$ with $i, j \in [n]$.
If $x_{i,j} > 0$, this indicates that vertex i in G is mapped to vertex j in H.

A and **B** adjacency matrices of graphs G and H. The graphs are isomorphic if and only if there is a permutation matrix **X** satisfying

$$\mathbf{AX} = \mathbf{XB}.$$

Iso(G, H) axioms:

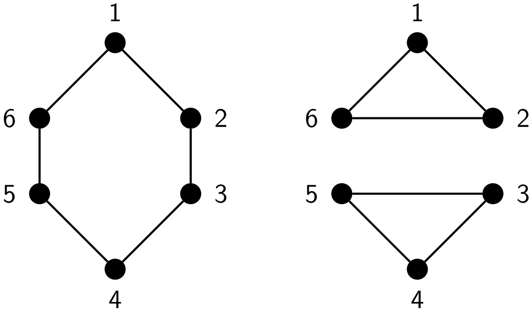
Type 1 axioms: Matrix **X** is doubly stochastic.
$\forall v \in V_G$: the equality $\sum_{w \in V_H} x_{v,w} = 1$,
$\forall w \in V_H$: the equality $\sum_{v \in V_G} x_{v,w} = 1$.

Type 2 axioms: These encode the matrix product $\mathbf{AX} = \mathbf{XB}$.
$\forall i, j \in [n]$: the equality $(\mathbf{AX})_{i,j} = (\mathbf{XB})_{i,j}$.

Type 3 axioms: For every variable x: the CP axioms $x \leq 1$ and $x \geq 0$.

$G \ncong H \iff \mathrm{Iso}(G, H)$ has no integer solution.

Non-isomorphic Graphs Can Have Fractional Isomorphisms

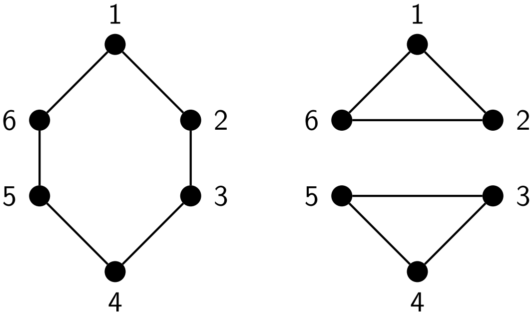


$$\begin{pmatrix} 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 \\ 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 \\ 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 \\ 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 \\ 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 \\ 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 \end{pmatrix}$$

The k-pebble Game on G and H

- Duplicator tries to maintain a partial isomorphism $p = \{(v_1, w_1), \ldots, (v_\ell, w_\ell)\}$ of size $\leq k$ between the graphs.
- Spoiler tries to show that they are non-isomorphic.
- In each round:
  1. Spoiler chooses $p' \subseteq p$ with $|p'| < k$.
  2. Duplicator extends this partial isomorphism to a bijection $\varphi\colon V_G \to V_H$.
  3. Spoiler picks a vertex $a \in V_G$. New position is $p' \cup \{(a, \varphi(a))\}$.
- Spoiler wins if p is no local isomorphism on induced subgraphs.
- Duplicator wins if she can make the game last indefinitely.

Example: 3 Pebbles



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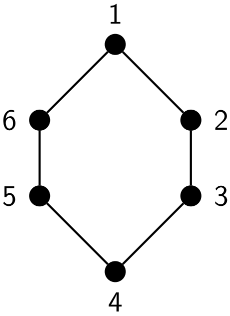

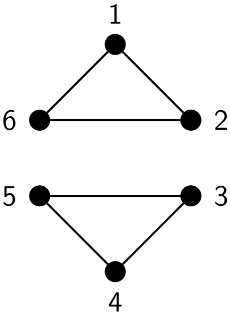

Bijection:

$$1 \longmapsto 1$$
$$2 \longmapsto 2$$
$$3 \longmapsto 3$$
$$4 \longmapsto 4$$
$$5 \longmapsto 5$$
$$6 \longmapsto 6$$

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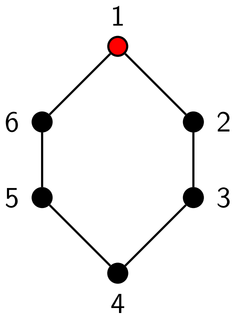

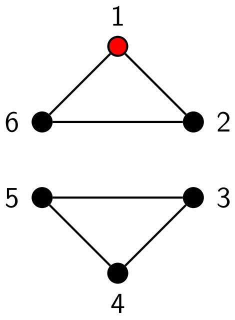

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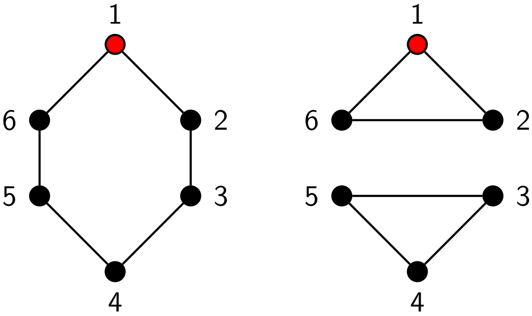

New Bijection:

$$1 \longmapsto 1$$
$$2 \longmapsto 2$$
$$3 \longmapsto 3$$
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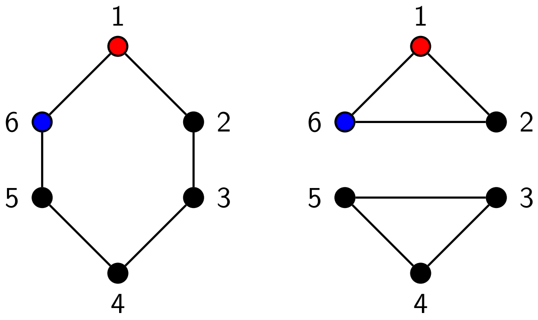
$$6 \longmapsto 6$$

Example: 3 Pebbles



New Bijection:

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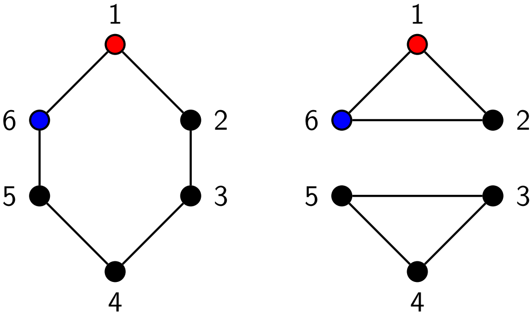

Yet Another Bijection:

$$1 \longmapsto 1$$
$$2 \longmapsto ??$$
$$3 \longmapsto \ldots$$
$$4 \longmapsto \ldots$$
$$5 \longmapsto \ldots$$
$$6 \longmapsto 6$$

Weisfeiler–Leman

Write $G \not\equiv_k H$ if G and H can be distinguished in k-game.

If G and H are non-isomorphic, their differentiation number is:

$$\mathsf{WL}(G, H) := \min\left\{k \in \mathbb{N} \,\middle|\, G \not\equiv_k H\right\}.$$

$\mathsf{WL}(G, H) = k \iff (k-1)$-dimensional Weisfeiler–Leman (coloring) algorithm is needed to distinguish the graphs [Cai, Fürer, Immerman 92]

Previously Known Tight Connections

[Atserias Maneva 13], [Malkin 14].
$G \not\equiv_k H \Rightarrow \mathrm{Iso}(G, H)$ refutable in the k-th level of Sherali–Adams.
$\mathrm{Iso}(G, H)$ refutable in k-th level of Sherali-Adams $\Rightarrow G \not\equiv_{k+1} H$.

[Grohe Otto 15]
The result of Atserias and Maneva is optimal.

[Berkholz Grohe 12], [Atserias Ochremiak 18]
$\mathrm{Iso}(G, H)$ refutable in ~~monomial~~ PC of rank $k \Longleftrightarrow G \not\equiv_k H$.

[Torán W. 22]
$\mathrm{Iso}(G, H)$ refutable in (narrow) Resolution of width $k \Longleftrightarrow G \not\equiv_{\mathscr{L}^k} H$.

Our Result

[Atserias Maneva 13], [Malkin 14].
$G \not\equiv_k H \Rightarrow \mathrm{Iso}(G,H)$ refutable in the k-th level of Sherali-Adams.
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[Berkholz Grohe 12]
$\mathrm{Iso}(G,H)$ refutable in monomial PC of rank $k \iff G \not\equiv_k H$.

[Torán W. 22]
$\mathrm{Iso}(G,H)$ refutable in (narrow) Resolution of width $k \iff G \not\equiv_{\mathscr{L}^k} H$.

[Grohe Otto 15]
The result of Atserias and Maneva is optimal.

$\mathrm{WL}(G,H) \leq k \implies \mathrm{Iso}(G,H)$ refutable in CP width k.
$\mathrm{WL}(G,H) > k \implies \mathrm{Iso}(G,H)$ **not** refutable in CP width $k-2$.

$G \not\equiv_k H \Longrightarrow \mathrm{Iso}(G, H)$ Refutable in Width k CP

For a game position $q = \{(v_1, w_1), \ldots, (v_\ell, w_\ell)\} \subseteq V_G \times V_H$ we let

$$S_q := \sum_{i=1}^{\ell} x_{v_i, w_i}.$$

In particular, $S_\emptyset = 0$ and Spoiler can win from the empty position.

Theorem: If Spoiler has a winning strategy for the r-round k-pebble game played on with initial position q_0, then there is a CP derivation of the inequality $S_{q_0} \leq |q_0| - 1$ from $\mathrm{Iso}(G, H)$ having width k and rank r simultaneously.

Induction in the number of rounds r.
Base $r = 0$.

$G \not\equiv_k H \Longrightarrow \mathrm{Iso}(G, H)$ Refutable in Width k CP

Def: Let $q \subseteq V(G) \times V(H)$ be an initial position of the k-pebble game. The bipartite graph $B := B_r^k(q)$ is defined to have vertices $V_B = V_G \uplus V_H$ and edges

$$E_B = \Big\{\{v, w\} \;\Big|\; \text{Spoiler cannot win } k\text{-game in } r \text{ rounds from } q \cup (v, w)\Big\}.$$

Lemma: [Berkholz Grohe 15]
If Spoiler has a winning position for the k-pebble game in $r + 1$ rounds starting from position q. Then, in the graph B there are two sets $S \subseteq V_G$ and $T \subseteq V_H$ satisfying:

- $N(S) = T$, $N(T) = S$, and $|S| > |T|$;
- Spoiler can win the game in r rounds from the starting position $q \cup (v, w)$ for every pair $(v, w) \in V_G \times V_H$ with the property $v \in S \leftrightarrow w \notin T$.

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Lemma: [Berkholz Grohe 15]
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Idea: For fixed S and T, there are $\gamma := |S||\overline{T}| + |\overline{S}||T|$ such positions. By induction we can derive the lines $S_{q\cup(v,w)} \leq |q|$ for all of them.

$G \not\equiv_k H \implies \mathrm{Iso}(G,H)$ Refutable in Width k CP

Idea: There are $\gamma := |S||\overline{T}| + |\overline{S}||T|$ such positions.
By induction we can derive the lines $S_{q\cup(v,w)} \leq |q|$ for all of them.
Let $\ell := |q|$. By a **linear combination** of the axioms we obtain

$$\sum_{v\in S, w\in T} x_{v,w} + \sum_{v\in\overline{S}, w\in\overline{T}} x_{v,w} \leq n-1.$$

Linearly combine induction hypotheses:

$$\sum_{v\in S}\sum_{w\in\overline{T}}(S_q + x_{v,w}) + \sum_{w\in T}\sum_{v\in\overline{S}}(S_q + x_{v,w}) \leq (|S||\overline{T}| + |\overline{S}||T|)\ell.$$

Finally, we get

$$\sum_{v\in S, w\in T} x_{v,w} + \sum_{v\in\overline{S}, w\in\overline{T}} x_{v,w} - \gamma S_q \geq n - \gamma\ell$$

and

$$-\gamma S_q \geq 1 - \gamma\ell.$$

$G \not\equiv_k H \implies \mathrm{Iso}(G,H)$ Refutable in Width k CP

$\gamma := |S||\overline{T}| + |\overline{S}||T|$, $\ell := |q|$.

$$-\gamma S_q \geq 1 - \gamma\ell.$$

Using the **rounding rule** dividing by γ, we get

$$-S_q \geq \left\lceil \frac{1-\gamma\ell}{\gamma} \right\rceil = 1 - \ell,$$

which is equivalent to $S_q \leq \ell - 1$.

All linear combinations can be done in one step.

One use of the GC-rule.

Observations and Consequences

All the lines in the proof are either axioms or have the form $S_q \leq |q| - 1$ for some game position q, i. e.,

$$x_{i_1,j_1} + x_{i_2,j_2} + \ldots x_{i_\ell,j_\ell} \leq \ell - 1.$$

There are only $n^{O(k)}$ such positions $\Rightarrow$ The CP proof has size $n^{O(k)}$.

Following [Grohe 10] isomorphism testing for planar and minor-free graphs can be done with polynomial size CP.

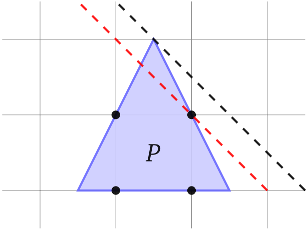
$G \equiv_k H \implies \mathit{Iso}(G, H)$ Is Not Refutable in Width $k-2$ CP

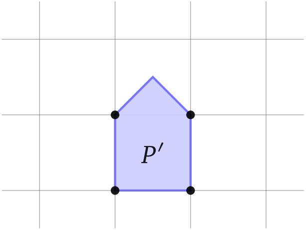
Let $P_{G,H}$ be the polytope in $[0,1]^{n \times n}$ defined by the $\mathrm{Iso}(G, H)$ inequalities.

For a matrix $\mathbf{X} \in \mathbb{R}^{n \times n}$ and $J \subseteq [n]$, let $\mathbf{X}|_J$ be the projection of $\mathbf{X}$ to the rows with indices in J.

For $k \in \mathbb{N}$, we define $P'_{G,H}$ as the following set of survival points in $P_{G,H}$:

$$P'_{G,H}(k) := \left\{ \mathbf{X} \in P_{G,H} \;\middle|\; \begin{array}{l} \forall \mathbf{A} \in \mathbb{Z}^{n \times n},\ \forall b \in \mathbb{R},\ \forall J \subseteq [n],\ |J| = k : \\ \langle \mathbf{A}, \mathbf{X}|_J \rangle \geq b \implies \langle \mathbf{A}, \mathbf{X}|_J \rangle \geq \lceil b \rceil \end{array} \right\}.$$





$G \equiv_k H \implies Iso(G, H)$ Is Not Refutable in Width $k-2$ CP

Protection Lemma for Graph Isomorphism:

Let $k \in \mathbb{N}$. Further, let $\mathbf{X}$ be a fractional point in $P_{G,H}$ and suppose that for any $J \subseteq [n]$, $|J| \leq k$, there exists a set of matrices $\mathbf{Y^{(1)}}, \ldots, \mathbf{Y^{(s)}}$ satisfying:

- For all $t \in [s]$, $\mathbf{Y^{(t)}}$ is $0, 1$ on the rows with indices in J;
- for all $t \in [s]$, $\mathbf{Y^{(t)}}$ is a fractional solution of $P_{G,H}$; and
- $\mathbf{X}|_J$ is a convex combination of $\mathbf{Y^{(1)}}|_J, \ldots, \mathbf{Y^{(s)}}|_J$.

Then, $\mathbf{X} \in P'_{G,H}(k)$.

$G \equiv_k H \implies Iso(G, H)$ Is Not Refutable in Width $k - 2$ CP

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Then, $\mathbf{X} \in P'_{G,H}(k)$.

Idea: Translate winning positions of Duplicator to protection matrices.

CP Length Lower Bounds for GI Formulas?

Interpolation ☹

Lifting ☹

Possible: Exponential lower bounds for tree-like CP with polynomially bounded coefficients.

Idea:

1. Use known results of block sensitivity for Tseitin formulas [Impagliazzo Pitassi Urquhart 94], [Huynh Nordström 12], [Göös Pitassi 13]
2. Tseitin formulas $\cong$ CFI graphs