Definition of ordinal regression

The dependent random variable Y has ordinal level of measurement. The number of possible values that Y can take is $m+1$. A realization of Y is denoted by y and $y \in \{0, 1, \ldots, m\}$. The conditional probability distribution of Y given a set of interval predictors $\mathbf{x} = (x_1, \ldots, x_k)'$ is $P(Y = y \,|\, \mathbf{x})$. The regression of Y on $\mathbf{x}$ is ordinal if and only if

$$P(Y \leq y \,|\, \mathbf{x}) = \sum_{t=0}^{y} P(Y = t \,|\, \mathbf{x}), \quad \text{for all } y, \tag{1}$$

is a monotonic function of $\mathbf{x}$. If higher values of y are associated with higher values of $\mathbf{x}$, then this function should be non-increasing, for all y. If $P(Y \leq y \,|\, \mathbf{x})$ is a monotonic function of x_i, then its derivative with respect to x_i is either non-negative or non-positive for all x_i.

Multinomial logistic regression

Let $z_s = 1$ if $y = s$ and $z_s = 0$ otherwise, for $s = 1, \ldots, m$, then the conditional probability distribution of Y given $\mathbf{x}$ can be written as

$$P(Y = y \,|\, \mathbf{x}) = P(Y = 0 \,|\, \mathbf{x}) \prod_{s=1}^{m} \left\{ \frac{P(Y = s \,|\, \mathbf{x})}{P(Y = 0 \,|\, \mathbf{x})} \right\}^{z_s}, \tag{2}$$

where

$$P(Y = 0 \,|\, \mathbf{x}) = \left\{ 1 + \sum_{s=1}^{m} \frac{P(Y = s \,|\, \mathbf{x})}{P(Y = 0 \,|\, \mathbf{x})} \right\}^{-1}. \tag{3}$$

In the multinomial logistic regression model,

$$\frac{P(Y = s \,|\, \mathbf{x})}{P(Y = 0 \,|\, \mathbf{x})} = \exp(\alpha_{0s} + \boldsymbol{\alpha}_s'\mathbf{x}), \text{ for } s = 1, \ldots, m, \tag{4}$$

where $\boldsymbol{\alpha}_s = (\alpha_{1s}, \ldots, \alpha_{ks})'$, so that

$$P(Y = y \,|\, \mathbf{x}) = \frac{\exp\left\{ \sum_{s=1}^{m} z_s(\alpha_{0s} + \boldsymbol{\alpha}_s'\mathbf{x}) \right\}}{1 + \sum_{s=1}^{m} \exp(\alpha_{0s} + \boldsymbol{\alpha}_s'\mathbf{x})}. \tag{5}$$

It follows that

$$P(Y \le y \,|\, \mathbf{x}) = \begin{cases} \dfrac{1}{1+\sum_{s=1}^{m} \exp(\alpha_{0s}+\boldsymbol{\alpha}_s'\mathbf{x})}, & \text{for } y = 0, \\[2ex] \dfrac{1+\sum_{s=1}^{y} \exp(\alpha_{0s}+\boldsymbol{\alpha}_s'\mathbf{x})}{1+\sum_{s=1}^{m} \exp(\alpha_{0s}+\boldsymbol{\alpha}_s'\mathbf{x})}, & \text{for } y > 0. \end{cases} \tag{6}$$

The multinomial logistic regression model has $(k+1)m$ parameters.

Adjacent categories model

In general, we have

$$\frac{P(Y = s \,|\, \mathbf{x})}{P(Y = 0 \,|\, \mathbf{x})} = \prod_{u=1}^{s} \frac{P(Y = u \,|\, \mathbf{x})}{P(Y = u-1 \,|\, \mathbf{x})}, \text{ for } s = 1, \ldots, m. \tag{7}$$

In an adjacent categories model,

$$\frac{P(Y = u \,|\, \mathbf{x})}{P(Y = u-1 \,|\, \mathbf{x})} = \exp(\beta_{0u} + \boldsymbol{\beta}_u'\mathbf{x}), \tag{8}$$

where $\boldsymbol{\beta}_u = (\beta_{1u}, \ldots, \beta_{ku})'$, so that $\alpha_{0s} = \sum_{u=1}^{s} \beta_{0u}$ and $\boldsymbol{\alpha}_s = \sum_{u=1}^{s} \boldsymbol{\beta}_u$. This general adjacent categories model is just a reparameterization of the multinomial logistic regression model. Under the adjacent categories model, it follows that

$$\frac{P(Y = u \,|\, \mathbf{x})}{P(Y = u-1 \,|\, \mathbf{x}) + P(Y = u \,|\, \mathbf{x})} = \frac{\exp(\beta_{0u} + \boldsymbol{\beta}_u'\mathbf{x})}{1 + \exp(\beta_{0u} + \boldsymbol{\beta}_u'\mathbf{x})}. \tag{9}$$

Ordinal logistic regression

Adjacent categories

In the ordinal logistic adjacent categories model, $\beta_{is} = \beta_i$, for $i = 1, \ldots, k$ and all s, so that $\alpha_{is} = s\beta_i$ and

$$P(Y = y \,|\, x_1, \ldots, x_k) = \frac{\exp\left(\sum_{s=1}^{m} z_s \alpha_{0s} + y\boldsymbol{\beta}'\mathbf{x}\right)}{1 + \sum_{s=1}^{m} \exp(\alpha_{0s} + s\boldsymbol{\beta}'\mathbf{x})}, \tag{10}$$

where $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_k)'$. The ordinal logistic adjacent categories model has $k + m$ parameters.

Cumulative probabilities

In general, we have

$$P(Y = y\,|\,\mathbf{x}) = \begin{cases} P(Y \le 0\,|\,\mathbf{x}), & \text{for } y = 0, \\ P(Y \le y\,|\,\mathbf{x}) - P(Y \le y-1\,|\,\mathbf{x}), & \text{for } y = 1, \ldots, m-1, \\ 1 - P(Y \le m-1\,|\,\mathbf{x}), & \text{for } y = m. \end{cases}$$

In the ordinal logistic cumulative probabilities model,

$$P(Y \le y\,|\,\mathbf{x}) = \frac{1}{1 + \exp\{\alpha_{0y} + \boldsymbol{\beta}'\mathbf{x}\}}, \quad \text{for } y = 0, 1, \ldots, m-1, \tag{11}$$

where $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_k)' > 0$. This model has also $k + m$ parameters.