冲量法求解非齐次方程

$$\begin{cases} u_{tt} = a^2 u_{xx} + f\left(x,t\right), & 0 < x < l, t > 0 \\ u\big|_{x=0} = u\big|_{x=l} = 0, & t > 0 \\ u\big|_{t=0} = u_t\big|_{t=0} = 0, & \end{cases}$$

冲量法求解非齐次方程

作用力分解为瞬时力作用 $f(x,t)=\int_0^{\infty} f(x,t)\delta(t-\tau)\mathrm{d}\tau=\int_0^{\infty} f(x,\tau)\delta(t-\tau)\mathrm{d}\tau$

作用力分解为瞬时力作用，瞬时力作用相当于初始速度引起的振动

$$f(x,\tau)\delta(t-\tau)\to V$$

然后将瞬时力引起的振动线性叠。 $\therefore u(x,t)=\int_0^{\infty} V(x,t;\tau)\mathrm{d}\tau=\int_0^{t} V(x,t;\tau)\mathrm{d}\tau$

冲量法求解非齐次方程

作用力分解为瞬时力作用，瞬时力作用相当于初始速度引起的振动；然后将瞬时力引起的振动线性叠。

$$\begin{cases} V_{tt} = a^2 V_{xx} + f(x,t)\delta(t-\tau), & 0 < x < l, t > 0 \\ V|_{x=0} = V|_{x=l} = 0, & t > 0 \\ V|_{t=0} = V_t|_{t=0} = 0, & \end{cases}$$

$$\therefore u(x,t) = \int_0^t V(x,t;\tau)\,\mathrm{d}\tau$$

冲量法求解非齐次方程

$$\begin{cases} V_{tt} = a^2 V_{xx} + f(x,\tau)\delta(t-\tau), & 0<x<l, t>0 \\ V|_{x=0} = V|_{x=l} = 0, & t>0 \\ V|_{t=0} = V_t|_{t=0} = 0, & \end{cases}$$

冲量定理 $V'(\tau+\Delta\tau) - V'(\tau-\Delta\tau) = \int_{\tau-\Delta\tau}^{\tau+\Delta\tau} f(x,t)\delta(t-\tau)dt$

$$V'(\tau+\Delta\tau) = f(x,\tau)$$

$$\Delta\tau \to 0 \quad V'(\tau) = f(x,\tau)$$

冲量法求解非齐次方程

$$\begin{cases} u_{tt} = a^2 u_{xx} + f(x,t), & 0 < x < l, t > 0 \\ u|_{x=0} = u|_{x=l} = 0, & t > 0 \\ u|_{t=0} = u_t|_{t=0} = 0, & \end{cases}$$

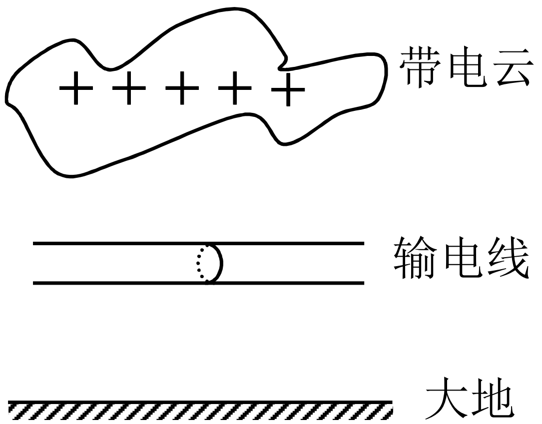
$$\begin{cases} V_{tt} = a^2 V_{xx}, & 0 < x < l \\ V|_{x=0} = V|_{x=l} = 0, & \\ V|_{t=\tau} = 0, V_t|_{t=\tau} = f(x,\tau), & \end{cases}$$

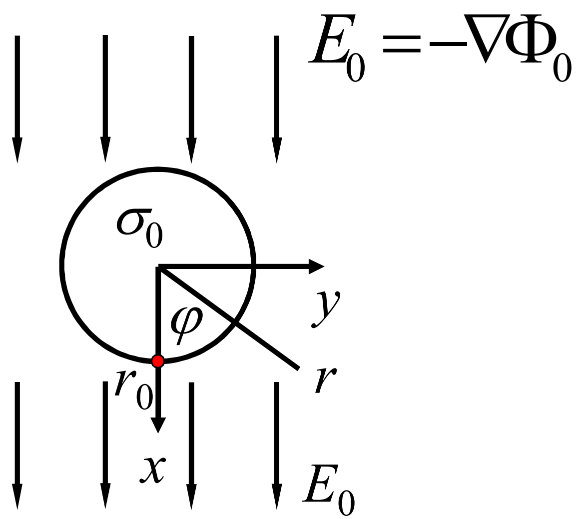
$$\therefore u(x,t) = \int_0^{\infty} V(x,t;\tau)\,\mathrm{d}\tau = \int_0^{t} V(x,t;\tau)\,\mathrm{d}\tau$$

求解柱型边界P154,例4

$$\begin{cases} \Delta u = 0,\quad a < \rho < \infty, 0 \le \varphi \le 2\pi \\ u\big|_{\rho=a} = 0 \\ u\big|_{\rho\to\infty} \sim u_0 + \dfrac{q_0}{2\pi\varepsilon_0}\ln\rho - E_0\rho\cos\varphi \end{cases}$$

（~表示量级相当）





场算符

$$\nabla = \mathbf{e}_x \frac{\partial}{\partial x} + \mathbf{e}_y \frac{\partial}{\partial y} + \mathbf{e}_z \frac{\partial}{\partial z}, \quad \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

$$\nabla = \mathbf{e}_\mathbf{r} \frac{\partial}{\partial r} + \frac{\mathbf{e}_\theta}{r} \frac{\partial}{\partial \theta} + \mathbf{e}_\mathbf{z} \frac{\partial}{\partial z} \qquad \nabla^2 = \frac{1}{r} \frac{\partial}{\partial r}\left(r \frac{\partial}{\partial r} \right) + \frac{1}{r^2} \frac{\partial}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}$$

化为极标分离变量

$$\frac{\partial^2 u}{\partial \rho^2}+\frac{1}{\rho}\frac{\partial u}{\partial \rho}+\frac{1}{\rho^2}\frac{\partial^2 u}{\partial \varphi^2}=0$$

$$u(\rho,\varphi)=R(\rho)\Phi(\varphi)$$

化为极标分离变量

$$R''(\rho)\Phi(\varphi)+\frac{1}{\rho}R'(\rho)\Phi(\varphi)+\frac{1}{\rho^2}R(\rho)\Phi''(\varphi)=0$$

$$\frac{\rho^2 R''(\rho)}{R(\rho)}+\frac{\rho R'(\rho)}{R(\rho)}+\frac{\Phi''(\varphi)}{\Phi(\varphi)}=0$$

化为两个常微分方程，优先求解齐次边界的方程

$$\frac{\rho^2 R''(\rho)}{R(\rho)} + \frac{\rho R'(\rho)}{R(\rho)} = -\frac{\Phi''(\varphi)}{\Phi(\varphi)} = \lambda$$

$$\begin{cases} \Phi''(\varphi) + \lambda\Phi(\varphi) = 0 \\ \Phi(\varphi + 2\pi) = \Phi(\varphi) \end{cases}$$

$$\Phi(\varphi) = \begin{cases} A\cos\sqrt{\lambda}\varphi + B\sin\sqrt{\lambda}\varphi & \lambda > 0 \\ A + B\varphi & \lambda = 0 \\ Ae^{\sqrt{\lambda}\varphi} + Be^{-\sqrt{\lambda}\varphi} & \lambda < 0 \end{cases}$$

由周期性边界条件，确定特解

$$\Phi(\varphi)=\begin{cases} A\cos\sqrt{\lambda}\varphi+B\sin\sqrt{\lambda}\varphi & \lambda>0 \\ A+B\varphi & \lambda=0 \\ Ae^{\sqrt{\lambda}\varphi}+Be^{-\sqrt{\lambda}\varphi} & \lambda<0 \end{cases}$$

$$\Phi(\varphi+2\pi)=\Phi(\varphi)$$

$$\Rightarrow \begin{cases} \lambda>0 \\ \sqrt{\lambda}=n \\ n=1,2,3,\cdots \end{cases}$$

或 $\lambda=0$，$n=0$

Cauchy-Euler equation

$$\rho^2 R''(\rho) + \rho R'(\rho) - n^2 R(\rho) = 0$$

求解方法 $t = \ln \rho$

Cauchy-Euler equation

$$\rho^2 R''(\rho) + \rho R'(\rho) - n^2 R(\rho) = 0 \qquad t = \ln \rho$$

$$\frac{dR}{d\rho} = \frac{dR}{dt}\frac{dt}{d\rho} = \frac{1}{\rho}\frac{dR}{dt}$$

$$\frac{d^2R}{d\rho^2} = \frac{d}{dt}\left(\frac{1}{\rho}\frac{dR}{dt}\right)\frac{dt}{d\rho} = \frac{1}{\rho}\left(-\frac{1}{e^t}\frac{dR}{dt} + \frac{1}{\rho}\frac{d^2R}{dt^2}\right) = -\frac{1}{\rho^2}\frac{dR}{dt} + \frac{1}{\rho^2}\frac{d^2R}{dt^2}$$

$$R''(t) - n^2 R(t) = 0$$

通解

$$R''(t)-n^2R(t)=0$$

$$R(t)=C_{1n}e^{-nt}+C_{2n}e^{nt}=\frac{C_{1n}}{\rho^n}+C_{2n}\rho^n \qquad n=1,2,3,\cdots$$

$$R(t)=C_{10}+C_{20}\text{t}=C_{10}+C_{20}\ln\rho \qquad n=0$$

$$\Phi(\varphi)R(\rho)=C_{10}+C_{20}\ln\rho+\sum_{n=1,2,3,\cdots}(A_n\cos n\varphi+B_n\sin n\varphi)\left(\frac{C_{1n}}{\rho^n}+C_{2n}\rho^n\right)$$

利用边界条件确定特解

$$\Phi(\varphi)R(\rho)=C_{10}+C_{20}\ln\rho+\sum_{n=1,2,3,\cdots}(A_n\cos n\varphi+B_n\sin n\varphi)\left(\frac{C_{1n}}{\rho^n}+C_{2n}\rho^n\right)$$

$$\Phi(\varphi)R(\rho)=C_{10}+C_{20}\ln\rho+\sum_{n=1,2,3,\cdots}A_n\cos n\varphi\frac{1}{\rho^n}+B_n\cos n\varphi\rho^n+C_n\sin n\varphi\frac{1}{\rho^n}+D_n\sin n\varphi\rho^n$$

$$\begin{cases}\Delta u=0,\quad a<\rho<\infty,0\le\varphi\le 2\pi\\ u\big|_{\rho=a}=0\\ u\big|_{\rho\to\infty}\sim -E_0\rho\cos\varphi(\sim \text{表示量级相当})\end{cases}$$

利用边界条件确定特解

$$\begin{cases} u\big|_{\rho=a} = 0 \\ u\big|_{\rho\to\infty} \sim\sim u_0 + \dfrac{q_0}{2\pi\varepsilon_0}\ln\rho - E_0\rho\cos\varphi (\sim \text{表示量级相当}) \end{cases}$$

$$\Phi(\varphi)R(\rho) = C_{10} + C_{20}\ln\rho + \sum_{n=1,2,3,\cdots} A_n\cos n\varphi\frac{1}{\rho^n} + B_n\cos n\varphi\rho^n + C_n\sin n\varphi\frac{1}{\rho^n} + D_n\sin n\varphi\rho^n$$

$\rho = a$

$$\Phi(\varphi)R(\rho) = C_{10} + C_{20}\ln a + \sum_{n=1,2,3,\cdots}\left(\frac{A_n}{a^n} + B_n a^n\right)\cos n\varphi + \left(\frac{C_n}{a^n} + D_n a^n\right)\sin n\varphi = 0$$

$$C_{10} + C_{20}\ln a = 0 \qquad \frac{A_n}{a^n} + B_n a^n = 0 \qquad \frac{C_n}{a^n} + D_n a^n = 0$$

$\rho \to \infty$

$$\Phi(\varphi)R(\rho) = C_{10} + C_{20}\ln\rho + \sum_{n=1,2,3,\cdots} B_n\cos n\varphi\rho^n + D_n\sin n\varphi\rho^n \sim u_0 + \frac{q_0}{2\pi\varepsilon_0}\ln\rho - E_0\rho\cos\varphi$$

$$C_{10} = u_0 \qquad C_{20} = \frac{q_0}{2\pi\varepsilon_0} \qquad \mathrm{B}_1 = -E_0$$

非齐次拉普拉斯方程

$$\nabla^2 u = a + b(x^2 - y^2)$$

$$(x, y) \in \text{圆域} \rho \langle \rho_0$$

$$u \big|_{\rho=\rho_0} = c$$

寻找特解v

$$\nabla^2 v = a + b(x^2 - y^2)$$

$$\nabla^2 (u - v) = 0$$

$$\nabla^2 \omega = 0$$

非齐次拉普拉斯方程

$$\nabla^2 u = a + b(x^2 - y^2)$$

$$v = \frac{ax^2}{2} + \frac{ay^2}{2} + \frac{bx^4}{12} - \frac{by^4}{12}$$

非齐次拉普拉斯方程通解

$$\nabla^2 \omega = 0$$

$$\omega = \Phi(\varphi)R(\rho) = C_{10} + C_{20}\ln\rho + \sum_{n=1,2,3,\cdots} A_n \cos n\varphi \frac{1}{\rho^n} + B_n \cos n\varphi \rho^n + C_n \sin n\varphi \frac{1}{\rho^n} + D_n \sin n\varphi \rho^n$$

$$v = \frac{a(x^2+\mathrm{y}^2)}{2} + \frac{bx^4}{12} - \frac{by^4}{12} = \frac{a\rho^2}{2} + \frac{b\rho^4}{12}\cos 2\varphi$$

代入边界中求解

$$\omega = \Phi(\varphi)R(\rho) = C_{10} + C_{20}\ln\rho + \sum_{n=1,2,3,\cdots} A_n\cos n\varphi\frac{1}{\rho^n} + B_n\cos n\varphi\rho^n + C_n\sin n\varphi\frac{1}{\rho^n} + D_n\sin n\varphi\rho^n$$

$$v = \frac{a(x^2+y^2)}{2} + \frac{bx^4}{12} - \frac{by^4}{12} = \frac{a\rho^2}{2} + \frac{b\rho^4}{12}\cos 2\varphi$$

$$\omega = \Phi(\varphi)R(\rho) = C_{10} + \sum_{n=1,2,3,\cdots} B_n\cos n\varphi\rho_0{}^n + D_n\sin n\varphi\rho_0{}^n + \frac{a\rho_0{}^2}{2} + \frac{b\rho_0{}^4}{12}\cos 2\varphi = c$$

特解

$$\omega=\Phi(\varphi)R(\rho)=C_{10}+\sum_{n=1,2,3,\cdots}B_n\cos n\varphi\rho_0{}^n+D_n\sin n\varphi\rho_0{}^n+\frac{a\rho_0{}^2}{2}+\frac{b\rho_0{}^4}{12}\cos 2\varphi=c$$

$$\sum_{n=1,2,3,\cdots}B_n\cos n\varphi\rho_0{}^n+D_n\sin n\varphi\rho_0{}^n+\frac{b\rho_0{}^4}{12}\cos 2\varphi+\frac{a\rho_0{}^2}{2}+C_{10}-c=0$$

$$C_{10}-c+\frac{a\rho_0{}^2}{2}=0$$

$$\left(B_2\rho_0{}^2+\frac{b\rho_0{}^4}{12}\right)\cos 2\varphi=0$$

非齐次边界处理

$$\begin{cases} u_{tt} = a^2 u_{xx} + f(x,t) & 0 < x < l, t > 0 \\ u_x\big|_{x=0} = \mu_1(t), \quad u\big|_{x=l} = \mu_2(t), & t > 0 \\ u\big|_{t=0} = \phi(x), \quad u_t\big|_{t=0} = \psi(x), & 0 \le x \le l \end{cases}$$

$$u(x,t) = v(x,t) + w(x,t)$$

$$\begin{cases} \\ w_x\big|_{x=0} = \mu_1(t), \quad w\big|_{x=l} = \mu_2(t), \qquad t > 0 \\ \\ \end{cases}$$

$$\begin{cases} \\ v_x\big|_{x=0} = 0, \quad v\big|_{x=l} = 0, \qquad t > 0 \\ \\ \end{cases}$$

非齐次边界处理

$$w=\alpha_1(x)\mu_1(t)+\alpha_2(x)\mu_2(t) \qquad w_x\big|_{x=0}=\mu_1(t),\quad w\big|_{x=l}=\mu_2(t)$$

$$\begin{cases} v_{tt}=a^2v_{xx}+a^2w_{xx}-a^2w_{tt}+f(x,t), & 0<x<l,t>0 \\ v_x\big|_{x=0}=\mu_1(t)-w_x\big|_{x=0},\quad v\big|_{x=l}=\mu_2(t)-w\big|_{x=l}, & t>0 \\ v\big|_{t=0}=\phi(x)-w\big|_{t=0},\quad v_t\big|_{t=0}=\psi(x)-w_t\big|_{t=0}, & 0\le x\le l \end{cases}$$

非齐次边界处理

$$w=\alpha_1(x)\mu_1(t)+\alpha_2(x)\mu_2(t)$$

$$w_x\big|_{x=0}=\mu_1(t),\quad w\big|_{x=l}=\mu_2(t)$$

$$\alpha_1'(0)\mu_1(t)+\alpha_2'(0)\mu_2(t)=\mu_1(t)$$

$$\alpha_1(l)\mu_1(t)\ +\ \alpha_2(l)\mu_2(t)=\mu_2(t)$$

非齐次边界处理

$$\alpha_1'(0)\mu_1(t)+\alpha_2'(0)\mu_2(t)=\mu_1(t)$$

$$\alpha_1(l)\mu_1(t) \quad + \quad \alpha_2(l)\mu_2(t)=\mu_2(t)$$

$$w=(x-l)\mu_1(t)+\mu_2(t)$$

非齐次边界处理

$$\alpha_1'(0)\mu_1(t)+\alpha_2'(0)\mu_2(t)=\mu_1(t)$$

$$\alpha_1(l)\mu_1(t)\ +\ \alpha_2(l)\mu_2(t)=\mu_2(t)$$

$$\begin{cases}\alpha_1'(0)=1, & \alpha_2'(0)=0\\ \alpha_1(l)=0, & \alpha_2(l)=1\end{cases}$$

$$\begin{cases}\alpha_1(x)=Ax+B\\ \alpha_2(x)=Cx+D\end{cases} \Longrightarrow \begin{cases}A=1, & B=-l\\ C=0, & D=1\end{cases}$$

$$w=(x-l)\mu_1(t)+\mu_2(t)$$

特解法

$$
\begin{cases}
u_{tt} = a^2 u_{xx} & 0 < x < l, t > 0 \\
u\big|_{x=0} = 0, \quad u\big|_{x=l} = A\sin\omega t, & t > 0 \\
u\big|_{t=0} = 0, \quad u_t\big|_{t=0} = 0, \quad 0 \le x \le l &
\end{cases}
$$

特解法

$$\begin{cases} u_{tt} = a^2 u_{xx} & 0 < x < l, t > 0 \\ u\big|_{x=0} = 0, \quad u\big|_{x=l} = A\sin\omega t, & t > 0 \\ u\big|_{t=0} = 0, \quad u_t\big|_{t=0} = 0, \quad 0 \le x \le l \end{cases}$$

特解法

$$\begin{cases} v_{tt} = a^2 v_{xx} & 0 < x < l, t > 0 \\ v\big|_{x=0} = 0, \quad v\big|_{x=l} = A\sin\omega t, & t > 0 \\ \end{cases}$$

$$v = X(x)\sin\omega t$$

$$-\omega^2 X(x)\sin\omega t - a^2 X''(x)\sin\omega t = 0$$

$$X''(x) + \frac{\omega^2}{a^2} X(x) = 0 \quad X(0) = 0, X(l) = A$$

特解法

$$\begin{cases} v_{tt} = a^2 v_{xx} & 0 < x < l, t > 0 \\ v_{x=0} = 0, \quad v\big|_{x=l} = A\sin\omega t, & t > 0 \\ \end{cases}$$

$$v = X(x)\sin\omega t$$

$$-\omega^2 X(x)\sin\omega t - a^2 X''(x)\sin\omega t = 0$$

$$X''(x) + \frac{\omega^2}{a^2}X(x) = 0 \quad X(0) = 0, X(l) = A$$

特解法

$$X''(x)+\frac{\omega^2}{a^2}X(x)=0$$

$$X(x)=A_1\cos\omega x/a+B_1\sin\omega x/a$$

$$X(0)=0,X(l)=A \qquad B_1=\frac{A}{\sin\frac{\omega l}{a}}$$

$$X(x)=B_1\sin\omega x/a$$

$$v=X(x)\sin\omega t=B_1\sin\omega x/a\sin\omega t$$

特解法

$$X''(x)+\frac{\omega^2}{a^2}X(x)=0$$

$$X(x)=A_1\cos\omega x/a+B_1\sin\omega x/a$$

$$X(0)=0, X(l)=A \qquad B_1=\frac{A}{\sin\frac{\omega l}{a}}$$

$$X(x)=B_1\sin\omega x/a$$

$$u=X(x)\sin\omega t=B_1\sin\omega x/a\sin\omega t$$

特解法

$$u = v + w = w + X(x)\sin\omega t = B_1 \sin\omega x / a \sin\omega t$$

$$\begin{cases} u_{tt} = a^2 u_{xx} & 0 < x < l, t > 0 \\ u\big|_{x=0} = 0, \quad u\big|_{x=l} = A\sin\omega t, & t > 0 \\ u\big|_{t=0} = 0, \quad u_t\big|_{t=0} = 0, \quad 0 \le x \le l \end{cases}$$

$$\begin{cases} w_{tt} = a^2 w_{xx} & 0 < x < l, t > 0 \\ w\big|_{x=0} = 0, \quad w\big|_{x=l} = 0, & t > 0 \\ w\big|_{t=0} = 0, \quad w_t\big|_{t=0} = -\omega B_1 \sin\omega x / a, \quad 0 \le x \le l \end{cases}$$

特解法

$$u_t - a^2 u_{xx} = 0$$

$$u\Big|_{x=0} = Ae^{i\omega t}$$

月的平均值可认为作简谐
变化，T=2π/ω，取单位为日、月或年；真实变化可视为谐
变化的叠加）

$$u(x,t) = X(x)e^{i\omega t}$$

无初值问题，地下温度的变化规律

$$u_t - a^2 u_{xx} = 0 \qquad 0 < x < \infty$$

$$u\Big|_{x=0} = Ae^{i\omega t}$$

$$|u|\Big|_{x\to\infty} < \infty$$

月的平均值可认为作简谐变化，T=2π/ω，取单位为日、月或年；真实变化可视为谐变化的叠加）

$$u(x,t) = X(x)e^{i\omega t}$$

无初值问题，地下温度的变化规律

$$u_t - a^2 u_{xx} = 0$$

$$u(x,t) = X(x)e^{i\omega t}$$

$$a^2 X''(x) - i\omega X = 0, X(0) = A$$

$$X(x) = C_1 e^{x\sqrt{\omega/a^2}\sqrt{i}} + C_2 e^{-x\sqrt{\omega/a^2}\sqrt{i}}$$

无初值问题，地下温度的变化规律

$$X(x) = C_1 e^{x\sqrt{\omega/2a^2}} e^{ix\sqrt{\omega/2a^2}} + C_2 e^{-x\sqrt{\omega/2a^2}} e^{-ix\sqrt{\omega/2a^2}}$$

$$u(x,t) = Ae^{-x\sqrt{\omega/2a^2}} e^{-ix\sqrt{\omega/2a^2}+i\omega t}$$

$$u(x,t) = Ae^{-x\sqrt{\omega/2a^2}} \cos\omega(t - x\sqrt{\omega/2a^2})$$