分离变量法求解两端固定的弦的自由振动

$$
\begin{cases}
u_{tt} = a^2 u_{xx}, \quad 0 < x < l, t > 0 \\
u\big|_{x=0} = u\big|_{x=l} = 0, \quad t > 0 \\
u\big|_{t=0} = \phi(x), \quad u_t\big|_{t=0} = \psi(x), \quad 0 \le x \le l
\end{cases}
$$

驻波法

基本思想：有界弦形成驻波,相位不随时间变化。

因此猜测 $$u(x,t)=X(x)T(t)$$

分离变量

分离变量

将 $u(x,t)=X(x)T(t)$ 代入到 $u_{tt}=a^2u_{xx}$

$$\begin{cases} X(x)T''(t)=a^2X''(x)T(t), & 0<x<l, t>0 \\ X(0)T(t)=0, X(l)T(t)=0, & t>0 \end{cases}$$

$$\frac{T''(t)}{a^2T(t)}=\frac{X''(x)}{X(x)}$$

偏微分方程化为两个常微分方程问题

$$\frac{T''(t)}{a^2T(t)}=\frac{X''(x)}{X(x)}=-\lambda$$

$$\begin{cases} T''(t)+\lambda a^2T(t)=0, \quad t>0 \\ X''(x)+\lambda X(x)=0, \quad 0<x<l \\ X(0)=X(l)=0 \end{cases}$$

分离变量

$$\begin{cases} X''(x)+\lambda X(x)=0, & 0<x<l \\ X(0)=X(l)=0 \end{cases}$$

特征方程 $\eta^2+\lambda=0$ $\Longrightarrow$ $\begin{cases} \eta_1 = i\sqrt{\lambda} \\ \eta_2 = -i\sqrt{\lambda} \end{cases}$

通解 $X = Ae^{\eta_1 x} + Be^{i\eta_2 x}$

$$\begin{cases} X = A\cos\sqrt{\lambda}x + B\sin\sqrt{\lambda}x & \lambda>0 \\ X = A+Bx & \lambda=0 \end{cases}$$

代入边界条件

$$\begin{cases} A=0,\quad B\sin\sqrt{\lambda}l=0, & \lambda\neq 0 \\ A=B=0, & \lambda=0 \end{cases}$$

$$\sin\sqrt{\lambda}l=0\Rightarrow\sqrt{\lambda}l=n\pi, n=+1,+2,\cdots$$

$$\therefore\lambda=\frac{n^2\pi^2}{l^2},\qquad X_n(x)=B\sin\frac{n\pi}{l}x$$

$$X_n(x)=B\sin\frac{n\pi}{l}x,\quad \lambda=\frac{n^2\pi^2}{l^2},\qquad n\in N^+$$

时间的常微分方程

$$T''(t)+\lambda_n a^2 T(t)=0, \quad t>0 \qquad \lambda_n=\frac{n^2\pi^2}{l^2}, \quad n\in N^+$$

$$T_n(t)=C_{1n}\cos\frac{n\pi a}{l}t+C_{2n}(n)\sin\frac{n\pi a}{l}t$$

有界波动方程的通解

$$u_n(x,t) = X_n(x)T_n(t) = B\sin\frac{n\pi}{l}x\left[C_{1n}\cos\frac{n\pi a}{l}t + C_{2n}\sin\frac{n\pi a}{l}t\right]$$

$$u_n(x,t) = X_n(x)T_n(t) = \sin\frac{n\pi}{l}x\left[C_{1n}\cos\frac{n\pi a}{l}t + C_{2n}\sin\frac{n\pi a}{l}t\right]$$

有界波动方程的通解

$$u(x,t)=\sum_{n=1,2,\cdots} u_n(x,t)=\sum_{n=1,2,\cdots} X_n(x)T_n(t)=\sum_{n=1,2,\cdots} \sin\frac{n\pi}{l}x\left[C_{1n}\cos\frac{n\pi a}{l}t+C_{2n}\sin\frac{n\pi a}{l}t\right]$$

代入初始条件

$$\begin{cases} u\big|_{t=0} = \sum_{n=1}^{\infty} C_{1n} \sin\frac{n\pi}{l}x = \phi(x) \\ u_t\big|_{t=0} = \sum_{n=1}^{\infty} C_{2n} \frac{n\pi a}{l} \sin\frac{n\pi}{l}x = \psi(x) \end{cases}$$

傅里叶级数展开求系数

等式两边做积分 $\frac{2}{l}\int_0^l \cdots \sin\frac{n\pi}{l}x\,\mathrm{d}x$

$$\int_0^l \sin\frac{n\pi}{l}x\sin\frac{m\pi}{l}x\,\mathrm{d}x=\frac{l}{2}\delta_{mn}$$

$$C_{1n}=\frac{2}{l}\int_0^l \phi(x)\sin\frac{n\pi}{l}x\,\mathrm{d}x=\phi_n$$

$$C_{2n}=\frac{2}{n\pi a}\int_0^l \psi(x)\sin\frac{n\pi}{l}x\,\mathrm{d}x=\frac{l}{n\pi a}\psi_n$$

有界弦振动定解

$$u(x,t)=\sum_{n=1}^{\infty}\left(\phi_n\cos\frac{n\pi a}{l}t+\frac{l}{n\pi a}\psi_n\sin\frac{n\pi a}{l}t\right)\sin\frac{n\pi}{l}x$$

分离变量法总结

（1）假定 $u(x,t)=X(x)T(t)$ 得到两个常微分方程

（2）利用齐次边界条件获得常微分方程的解

（3）利用初始条件确定叠加系数

第二类边界条件

$$\begin{cases} u_{tt} = a^2 u_{xx}, \quad 0 < x < l, t > 0 \\ u_x\big|_{x=0} = u_x\big|_{x=l} = 0, \quad t > 0 \\ u\big|_{t=0} = \phi(x), \quad u_t\big|_{t=0} = \psi(x), \quad 0 \le x \le l \end{cases}$$

分离变量

将 $u(x,t)=X(x)T(t)$ 代入到 $u_{tt}=a^2u_{xx}$

$$\begin{cases} X(x)T''(t)=a^2X''(x)T(t), & 0<x<l,t>0 \\ X(0)T(t)=0,X(l)T(t)=0, & t>0 \end{cases}$$

$$\frac{T''(t)}{a^2T(t)}=\frac{X''(x)}{X(x)}$$

分离变量

$$\begin{cases} X''(x)+\lambda X(x)=0, & 0<x<l \\ X'(0)=X'(l)=0 \end{cases}$$

特征方程 $\eta^2+\lambda=0$ $\quad \begin{cases} \eta_1 = i\sqrt{\lambda} \\ \eta_2 = -i\sqrt{\lambda} \end{cases}$

通解 $\quad X = Ae^{\eta_1 x} + Be^{\eta_2 x}$

$$\begin{cases} X = A\cos\sqrt{\lambda}x + B\sin\sqrt{\lambda}x & \lambda>0 \\ X = A+Bx & \lambda=0 \end{cases}$$

代入边界条件

$$\begin{cases} -\sqrt{\lambda} A \sin\sqrt{\lambda} l = 0, \quad B = 0, \quad \lambda > 0 \\ B = 0, \qquad\qquad\qquad \lambda = 0 \end{cases}$$

$$\sin\sqrt{\lambda} l = 0 \Rightarrow \sqrt{\lambda} l = n\pi, n = 0, +1, +2, \cdots$$

$$\therefore \lambda = \frac{n^2\pi^2}{l^2}, \qquad X_n(x) = A\cos\frac{n\pi}{l}x$$

$$X_n(x) = A\cos\frac{n\pi}{l}x, \quad \lambda = \frac{n^2\pi^2}{l^2}, \qquad n \in N$$

时间的常微分方程

$$T''(t)+\lambda_n a^2 T(t)=0, \quad t>0 \qquad\qquad \lambda_n=\frac{n^2\pi^2}{l^2}, \quad n\in N$$

$$T_n(t)=C_{1n}\cos\frac{n\pi a}{l}t+C_{2n}(n)\sin\frac{n\pi a}{l}t+C_{10}+C_{20}t$$

第二类边界弦振动通解

$$u_n(x,t)=X_n(x)T_n(t)=\sin\frac{n\pi}{l}x\left[C_{1n}\cos\frac{n\pi a}{l}t+C_{2n}\sin\frac{n\pi a}{l}t+C_{10}+C_{20}t\right]$$

$$n=+1,+2,\cdots$$

代入初始条件

$$\begin{cases} u\big|_{t=0} = \sum_{n=1}^{\infty} C_{1n} \cos\dfrac{n\pi}{l}x + C_{10} = \phi(x) \\ u_t\big|_{t=0} = \sum_{n=1}^{\infty} C_{2n} \dfrac{n\pi a}{l} \cos\dfrac{n\pi}{l}x + C_{20} = \psi(x) \end{cases}$$

利用傅里叶级数求系数

等式两边做积分 $\frac{2}{l}\int_0^l \cdots \cos\frac{n\pi}{l}x\,\mathrm{d}x$

$$\int_0^l \cos\frac{n\pi}{l}x\cos\frac{m\pi}{l}x\,\mathrm{d}x=\frac{l}{2}\delta_{mn}$$

$$C_{1n}=\frac{1}{l}\int_0^l \phi(x)\cos\frac{n\pi}{l}x\,\mathrm{d}x=\phi_n \qquad C_{10}=\frac{2}{l}\int_0^l \phi(x)\,\mathrm{d}x$$

$$C_{2n}=\frac{1}{n\pi a}\int_0^l \psi(x)\cos\frac{n\pi}{l}x\,\mathrm{d}x=\frac{l}{n\pi a}\psi_n \qquad C_{20}=\frac{2}{n\pi a}\int_0^l \psi(x)\,\mathrm{d}x$$

$$n=+1,+2,\cdots$$

扩散方程求解

$$\begin{cases} u_t = a^2 u_{xx}, \quad 0 < x < l, t > 0 \\ u\big|_{x=0} = u_x\big|_{x=l} = 0, \quad t > 0 \\ u\big|_{t=0} = u_0 x / l, \quad 0 \le x \le l \end{cases}$$

分离变量

将 $u(x,t)=X(x)T(t)$ 代入到 $u_t=a^2u_{xx}$

$$\begin{cases} X(x)T'(t)=a^2X''(x)T(t), & 0<x<l,t>0 \\ X(0)T(t)=0, X(l)T(t)=0, & t>0 \end{cases}$$

$$\frac{T'(t)}{a^2T(t)}=\frac{X''(x)}{X(x)}$$

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通解 $X=Ae^{\eta_1 x}+Be^{\eta_2 x}$

$$\begin{cases} X=A\cos\sqrt{\lambda}x+B\sin\sqrt{\lambda}x & \lambda>0 \\ X=A+Bx & \lambda=0 \end{cases}$$

代入边界条件

$$\begin{cases} \sqrt{\lambda} B \cos \sqrt{\lambda} l = 0, \quad \mathrm{A} = 0, \quad \lambda > 0 \\ B = 0, A = 0 \qquad\qquad \lambda = 0 \end{cases}$$

$$\cos \sqrt{\lambda} l = 0 \Rightarrow \sqrt{\lambda} l = \pi (n + \frac{1}{2}), n = 0, +1, +2, \cdots$$

$$\therefore \lambda_n = \left[\pi (n + \frac{1}{2}) \right]^2, \quad X_n(x) = B \sin \frac{\pi}{l} \frac{(2n+1)}{2} x$$

关于时间的微分方程

$$T'(t) + \lambda_n a^2 T(t) = 0$$

$$T(t) = \sum_{n=1,2,\cdots}^{\infty} T_n(t) = \sum_{n=1,2,\cdots}^{\infty} C_n e^{-\lambda_n a^2 t} \quad \lambda_n = \left[\pi(n+\frac{1}{2})\right]^2$$

扩散方程通解

$$u(x,t)=\sum_{n=1,2,\cdots}^{\infty} T_n(t)X_n(x)=\sum_{n=1,2,\cdots}^{\infty} C_n e^{-\lambda_n a^2 t}\sin\sqrt{\lambda_n}\,x$$

$$\lambda_n=\left[\pi(n+\frac{1}{2})\right]^2$$

代入初始条件并用傅里叶级数展开求解

$$\sum_{n=1,2,\cdots}^{\infty} C_n \sin\sqrt{\lambda_n}\, x = \frac{u_0}{l} x \qquad \lambda_n = \left[\pi\left(n+\frac{1}{2}\right)\right]^2$$