# 波动、扩散方程的一般表达式

波动方程

$$\begin{cases} u_{tt} = a^2 u_{xx} + f\left(x,t\right), & 0 < x < l, t > 0 \\ u\,|_{x=0} = u_1(t), u\,|_{x=l} = u_2(t), & t > 0 \\ u\,|_{t=0} = \varphi(x), u_t\,|_{t=0} = \psi(x), & \end{cases}$$

扩散方程

$$\begin{cases} u_{t} = a^2 u_{xx} + f\left(x,t\right), & 0 < x < l, t > 0 \\ u\,|_{x=0} = u_1(t), u\,|_{x=l} = u_2(t), & t > 0 \\ u\,|_{t=0} = \varphi(x) & \end{cases}$$

# 第一步 将非齐次边界化为齐次边界问题

$$\begin{cases} u_{tt} = a^2 u_{xx} + f(x,t), & 0 < x < l, t > 0 \\ u|_{x=0} = u_1(t), u|_{x=l} = u_2(t), & t > 0 \\ u|_{t=0} = \varphi(x), u_t|_{t=0} = \psi(x), & \end{cases}$$

$$\begin{cases} u_t = a^2 u_{xx} + f(x,t), & 0 < x < l, t > 0 \\ u|_{x=0} = u_1(t), u|_{x=l} = u_2(t), & t > 0 \\ u|_{t=0} = \varphi(x) & \end{cases}$$

选取一个特殊函数将非齐次边界化为齐次边界问题

$$u(x,t) = v(x,t) + w(x,t)$$

$$w = \alpha_1(x)\mu_1(t) + \alpha_2(x)\mu_2(t) \qquad w_x\big|_{x=0} = \mu_1(t), \quad w\big|_{x=l} = \mu_2(t)$$

$$\begin{cases} v_{tt} = a^2 v_{xx} + a^2 w_{xx} - a^2 w_{tt} + f(x,t), & 0 < x < l, t > 0 \\ v_x\big|_{x=0} = \mu_1(t) - w_x\big|_{x=0}, \quad v\big|_{x=l} = \mu_2(t) - w\big|_{x=l}, & t > 0 \\ v\big|_{t=0} = \phi(x) - w\big|_{t=0}, \quad v_t\big|_{t=0} = \psi(x) - w_t\big|_{t=0}, & 0 \le x \le l \end{cases}$$

# 第二步 将使用叠加原理简化非齐次方程（级数法可省略这一步）

$$\begin{cases} u_{tt} = a^2 u_{xx} + f(x,t), & 0<x<l, t>0 \\ u|_{x=0} = 0, u|_{x=l} = 0, & t>0 \\ u|_{t=0} = \varphi(x), u_t|_{t=0} = \psi(x), \end{cases}$$

$$u = w + v$$

$$\begin{cases} w_{tt} = a^2 w_{xx}, & 0<x<l, t>0 \\ w|_{x=0} = 0, w|_{x=l} = 0, & t>0 \\ w|_{t=0} = \varphi(x), w_t|_{t=0} = \psi(x), \end{cases}$$

$$\begin{cases} v_{tt} = a^2 v_{xx} + f(x,t), & 0<x<l, t>0 \\ v|_{x=0} = 0, v|_{x=l} = 0, & t>0 \\ v|_{t=0} = 0, v_t|_{t=0} = 0, \end{cases}$$

# 第三步 将非齐次方程转化为齐次方程

$$\begin{cases} v_{tt} = a^2 v_{xx} + f(x,t), & 0 < x < l, t > 0 \\ v|_{x=0} = 0, v|_{x=l} = 0, & t > 0 \\ v|_{t=0} = 0, v_t|_{t=0} = 0, & \end{cases}$$

1. 级数法
2. 冲量定理法
3. 特解法

# 1.非齐次偏微分方程的级数解法

$$\begin{cases} u_{tt} - a^2 u_{xx} = A\cos\frac{\pi x}{l}\sin\omega_0 t, \quad 0<x<l, t>0 \\ u_x\big|_{x=0} = u_x\big|_{x=l} = 0, \quad t>0 \\ u\big|_{t=0} = \phi(x), \quad u_t\big|_{t=0} = \psi(x), \quad 0\le x\le l \end{cases}$$

$$V(x,t) = \sum_{n=0} T_n(t)\cos\frac{n\pi x}{l}$$

$$V_{tt} - a^2 V_{xx} = A\cos\frac{\pi x}{l}\sin\omega_0 t$$

$$V\big|_{t=0} = \phi(x), \quad V_t\big|_{t=0} = \psi(x), \quad 0\le x\le l$$

$$T_1''(t) + \left(\frac{\pi a}{l}\right)^2 T_1 = A\sin\omega_0 t$$

$$T_1(0) = \varphi_1, \quad T_1'(0) = \psi_1$$

1.傅里叶变换法求解常微分方程
2.猜特解sinwt

$$T_1(t) = C_1\sin(\omega_0 t) + C_3\sin\left(\frac{\pi a}{l}t\right) + C_4\cos\left(\frac{\pi a}{l}t\right)$$

## 2冲量定理法求解非齐次方程

作用力分解为瞬时力作用 $f(x,t)=\int_0^{\infty} f(x,t)\delta(t-\tau)\mathrm{d}\tau=\int_0^{\infty} f(x,\tau)\delta(t-\tau)\mathrm{d}\tau$

作用力分解为瞬时力作用，瞬时力作用相当于初始速度引起的振动

$$f(x,\tau)\delta(t-\tau)\to V$$

然后将瞬时力引起的振动线性叠加。 $\therefore u(x,t)=\int_0^{\infty} V(x,t;\tau)\mathrm{d}\tau=\int_0^{t} V(x,t;\tau)\mathrm{d}\tau$

# 冲量定理法求解非齐次方程

作用力分解为瞬时力作用，瞬时力作用相当于初始速度引起的振动；然后将瞬时力引起的振动线性叠。

$$\begin{cases} V_{tt} = a^2 V_{xx} + f(x,t)\delta(t-\tau), & 0 < x < l, t > 0 \\ V|_{x=0} = V|_{x=l} = 0, & t > 0 \\ V|_{t=0} = V_t|_{t=0} = 0, & \end{cases}$$

$$\therefore u(x,t) = \int_0^t V(x,t;\tau)\,\mathrm{d}\tau$$

# 冲量定理法求解非齐次方程

$$\begin{cases} u_{tt}=a^2u_{xx}+f(x,t), \quad 0<x<l,t>0 \\ u|_{x=0}=u|_{x=l}=0, \qquad t>0 \\ u|_{t=0}=u_t|_{t=0}=0, \end{cases}$$

$$\begin{cases} V_{tt}=a^2V_{xx}+f(x,t)\delta(t-\tau), \quad 0<x<l,t>0 \\ V|_{x=0}=V|_{x=l}=0, \qquad t>0 \\ V|_{t=0}=V_t|_{t=0}=0, \end{cases}$$

## 冲量定理

$$\begin{cases} V_{tt}=a^2V_{xx}, \quad 0<x<l \\ V|_{x=0}=V|_{x=l}=0, \\ V|_{t=\tau}=0, V_t|_{t=\tau}=f(x,\tau), \end{cases}$$

$$\therefore u(x,t)=\int_0^{\infty}V(x,t;\tau)\,\mathrm{d}\tau=\int_0^{t}V(x,t;\tau)\,\mathrm{d}\tau$$

# 3特解法

$$\nabla^2 u = a + b(x^2 - y^2)$$

$$(x, y) \in 圆域\rho \langle \rho_0$$

$$u|_{\rho=\rho_0} = c$$

寻找特解函数v

$$\nabla^2 v = a + b(x^2 - y^2)$$

$$v = \frac{ax^2}{2} + \frac{ay^2}{2} + \frac{bx^4}{12} - \frac{by^4}{12}$$

$$\nabla^2 u = a + b(x^2 - y^2)$$

$$\nabla^2 (u - v) = \nabla^2 \omega = 0$$

$$\nabla^2 \omega = 0,\quad \omega|_{\rho=\rho_0} = c$$

# 第四步 分离变量法求解

$$\begin{cases} u_{tt} = a^2 u_{xx}, & 0 < x < l, t > 0 \\ u|_{x=0} = u|_{x=l} = 0, & t > 0 \\ u|_{t=0} = \phi(x), \quad u_t|_{t=0} = \psi(x), & 0 \le x \le l \end{cases}$$

(1) $u(x,t) = X(x)T(t)$ 得到两个常微分方程

(2) 利用齐次边界条件获得常微分方程的解

(3) 利用初始条件确定叠加系数

$$\frac{T''(t)}{a^2T(t)} = \frac{X''(x)}{X(x)} = -\lambda \qquad \begin{cases} X''(x) + \lambda X(x) = 0, & 0 < x < l \\ X'(0) = X'(l) = 0 \end{cases}$$

$$X = Ae^{\eta_1 x} + Be^{\eta_2 x}$$

$$X_n(x) = A\cos\frac{n\pi}{l}x, \quad \lambda_n = \frac{n^2\pi^2}{l^2}, \qquad n \in N$$

$$\begin{cases} u|_{t=0} = \sum_{n=1}^{\infty} C_{1n}\cos\frac{n\pi}{l}x + C_{10} = \phi(x) \\ u_t|_{t=0} = \sum_{n=1}^{\infty} C_{2n}\frac{n\pi a}{l}\cos\frac{n\pi}{l}x + C_{20} = \psi(x) \end{cases}$$

$$T''(t) + \lambda_n a^2 T(t) = 0, \quad t > 0 \qquad T_n(t) = C_{1n}\cos\frac{n\pi a}{l}t + C_{2n}(n)\sin\frac{n\pi a}{l}t + C_{10} + C_{20}t$$

$$u_n(x,t) = X_n(x)T_n(t) = \sin\frac{n\pi}{l}x\left[C_{1n}\cos\frac{n\pi a}{l}t + C_{2n}\sin\frac{n\pi a}{l}t + C_{10} + C_{20}t\right]$$

# 分离变量法扩散方程求解

$$\begin{cases} u_t = a^2 u_{xx}, & 0 < x < l, t > 0 \\ u\big|_{t=0} = u_0 x / l, & 0 \le x \le l \end{cases}$$

$$u(x,t) = \sum_{n=1,2,\cdots}^{\infty} T_n(t) X_n(x) = \sum_{n=1,2,\cdots}^{\infty} C_{1n} e^{-\lambda_n a^2 t} \cos\sqrt{\lambda_n} x + C_{2n} e^{-\lambda_n a^2 t} \sin\sqrt{\lambda_n} x$$

# 圆形边界分离变量

$$\begin{cases} \Delta u = 0，\quad a < \rho < \infty, 0 \le \varphi \le 2\pi \\ u\big|_{\rho=a} = 0 \\ u\big|_{\rho\to\infty} \sim u_0 + \dfrac{q_0}{2\pi\varepsilon_0}\ln\rho - E_0\rho\cos\varphi (\sim \text{表示量级相当}) \end{cases}$$

$$\Delta = \nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial}{\partial\theta^2} + \frac{\partial^2}{\partial z^2}$$

$$\frac{\partial^2 u}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial u}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2 u}{\partial\varphi^2} = 0$$

$$\frac{\rho^2 R''(\rho)}{R(\rho)} + \frac{\rho R'(\rho)}{R(\rho)} + \frac{\Phi''(\varphi)}{\Phi(\varphi)} = 0 \qquad u(\rho,\varphi) = R(\rho)\Phi(\varphi)$$

$$\begin{cases} \Phi''(\varphi) + \lambda\Phi(\varphi) = 0 \\ \Phi(\varphi + 2\pi) = \Phi(\varphi) \end{cases}$$

$$\Phi(\varphi) = \begin{cases} A\cos\sqrt{\lambda}\varphi + B\sin\sqrt{\lambda}\varphi & \lambda > 0 \\ A + B\varphi & \lambda = 0 \\ Ae^{\sqrt{\lambda}\varphi} + Be^{-\sqrt{\lambda}\varphi} & \lambda < 0 \end{cases}$$

$$\rho^2 R''(\rho) + \rho R'(\rho) - n^2 R(\rho) = 0$$

$$t = \ln\rho$$

$$R(t) = C_{1n}e^{-nt} + C_{2n}e^{nt} = \frac{C_{1n}}{\rho^n} + C_{2n}\rho^n$$

$$u = \Phi(\varphi)R(\rho) = C_{10} + C_{20}\ln\rho + \sum_{n=1,2,3,\cdots} A_n\cos n\varphi\frac{1}{\rho^n} + B_n\cos n\varphi\rho^n + C_n\sin n\varphi\frac{1}{\rho^n} + D_n\sin n\varphi\rho^n$$

# 总结

## 去掉非齐次边界，处理非齐次项，最终化为以下形式的方程

$$\begin{cases} u_{tt} = a^2 u_{xx}, \quad 0 < x < l, t > 0 \\ u\big|_{x=0} = u\big|_{x=l} = 0, \quad t > 0 \\ u\big|_{t=0} = \phi(x), \quad u_t\big|_{t=0} = \psi(x), \quad 0 \le x \le l \end{cases}$$

$$u_n(x,t) = X_n(x)T_n(t) = \sin\frac{n\pi}{l}x\left[C_{1n}\cos\frac{n\pi a}{l}t + C_{2n}\sin\frac{n\pi a}{l}t + C_{10} + C_{20}t\right]$$

$$\begin{cases} u_t = a^2 u_{xx}, \quad 0 < x < l, t > 0 \\ u\big|_{t=0} = u_0 x / l, \quad 0 \le x \le l \end{cases}$$

$$u(x,t) = \sum_{n=1,2,\cdots}^{\infty} T_n(t)X_n(x) = \sum_{n=1,2,\cdots}^{\infty} C_{1n}e^{-\lambda_n a^2 t}\cos\sqrt{\lambda_n}x + C_{2n}e^{-\lambda_n a^2 t}\sin\sqrt{\lambda_n}x$$

$$\begin{cases} \Delta u = 0, \quad a < \rho < \infty, 0 \le \varphi \le 2\pi \\ u\big|_{\rho=a} = 0 \end{cases}$$

$$u = \Phi(\varphi)R(\rho) = C_{10} + C_{20}\ln\rho + \sum_{n=1,2,3,\cdots} A_n\cos n\varphi\frac{1}{\rho^n} + B_n\cos n\varphi\rho^n + C_n\sin n\varphi\frac{1}{\rho^n} + D_n\sin n\varphi\rho^n$$

# 傅里叶变换求解有界波动方程

$$\begin{cases} u_{tt} = a^2 u_{xx}, & 0 < x < l, t > 0 \\ u|_{x=0} = u|_{x=l} = 0, & t > 0 \\ u|_{t=0} = \phi(x), \quad u_t|_{t=0} = \psi(x), & 0 \le x \le l \end{cases}$$

$$f(x) = \int e^{ikx} f(k)dk \approx \int e^{ikx} \sum_n f(k_n)\delta(k-k_n)dk$$

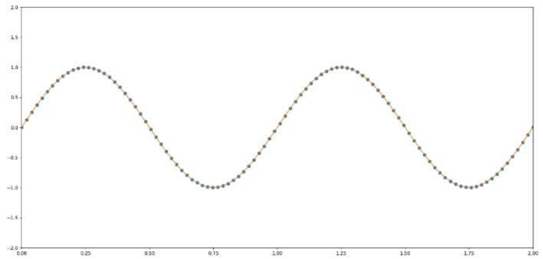
$$= \sum_n f(k_n) \int e^{ikx}\delta(k-k_n)dk = \sum_n f(k_n) e^{ik_n x}$$

$$= \sum_n f(k_n)\cos k_n x + if(k_n)\sin k_n x$$

1.泛定方程两边同时对时间做傅里叶变换变成常微分方程

$$F[u_{tt} - a^2 u_{xx}] = (-i\omega)^2 U - a^2 U_{xx} = 0$$

$$U_{xx} + \frac{\omega^2}{a^2} U = 0 \quad U = A(\omega)e^{i\omega x/a} + B(\omega)e^{-i\omega x/a}$$



2. 直接求解常微分方程，代入边界条件求出系数。

$$U_n(x, \omega_n) = C(\omega_n)\sin(\omega_n x / a)$$

3. 对时间做傅里叶逆变换

$$u(x,t) = \sum_n [U_1(x,\omega_n)\sin\omega_n t + U_2(x,\omega_n)\cos\omega_n t]\sin(\omega_n x / a) = \sum_n [C_1(\omega_n)\sin\omega_n t + C_2(\omega_n)\cos\omega_n t]\sin(\omega_n x / a)$$

# 傅里叶变换求解无界波动方程

$$\begin{cases} u_{tt} = a^2 u_{xx}, & |x| < \infty, t > 0 \\ u\big|_{t=0} = \phi(x), \quad u_t\big|_{t=0} = \psi(x), & |x| < \infty \end{cases}$$

1.泛定方程两边同时对空间做傅里叶变换变成常微分方程

$$U'' + k^2 a^2 U = 0$$

$$U = \Phi(k), \quad U' = \Psi(k), \quad t = 0$$

2. 直接求解常微分方程，代入初值求出系数。

$$C_1(k) = \frac{1}{2}\Phi(k) + \frac{1}{2a}\frac{1}{ik}\Psi(k)$$

$$C_2(k) = \frac{1}{2}\Phi(k) - \frac{1}{2a}\frac{1}{ik}\Psi(k)$$

$$U(k,t) = C_1(k)e^{\lambda_1 t} + C_2(k)e^{\lambda_2 t}$$

$$U(k,t) = \Phi(k)\cos(kat) - \frac{1}{ak}\Psi(k)\sin(kat)$$

3. 对空间做傅里叶逆变换

$$u(x,t) = \frac{\phi(x+at) + \phi(x-at)}{2} + \frac{1}{2a}\int_{x-at}^{x+at} \psi(\xi)\,\mathrm{d}\xi$$

# 傅里叶变换求解扩散方程P330 例2

$$\begin{cases} u_t = a^2 u_{xx}, & |x| < \infty, t > 0 \\ u|_{t=0} = \phi(x), & |x| < \infty \end{cases}$$

1.泛定方程两边同时对空间做傅里叶变换变成常微分方程

2. 直接求解常微分方程，代入初值求出系数。

3.泛定方程两边同时对空间做傅里叶逆变换

# 傅里叶变换求解扩散方程

$$\begin{cases} u_t = a^2 u_{xx}, & |x| < \infty, t > 0 \\ u|_{t=0} = \phi(x), & |x| < \infty \end{cases}$$

1.泛定方程两边同时对空间做傅里叶变换变成常微分方程

$$U' + k^2 a^2 U = 0$$

$$U = \Phi(k), \quad t = 0$$

2. 直接求解常微分方程，代入初值求出系数。

$$U(k,t) = \Phi(k) e^{-k^2 a^2 t}$$

3.泛定方程两边同时对空间做傅里叶逆变换

$$u(x,t) = F^{-1}[U(k,x)]$$

# 傅里叶变换求解扩散方程

$$u(x,t)=F^{-1}[U(k,x)]=\int_{-\infty}^{\infty}dke^{ikx}\Phi(k)e^{-k^2a^2t}$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}dke^{ikx}e^{-k^2a^2t}\int_{-\infty}^{\infty}\varphi(\xi)e^{-ik\xi}d\xi$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}d\xi\varphi(\xi)\left[\int_{-\infty}^{\infty}e^{ik(x-\xi)}e^{-k^2a^2t}dk\right]$$

# 傅里叶变换求解扩散方程

$$\int_{-\infty}^{\infty} e^{-ax^2}dx = \int_{-\infty}^{\infty} e^{-ax^2}dx = \left(\int_{-\infty}^{\infty} e^{-ax^2}dx \int_{-\infty}^{\infty} e^{-ay^2}dy\right)^{1/2} = \left(\int_{-\infty}^{\infty} e^{-a(x^2+y^2)}dxdy\right)^{1/2} = \left(2\pi\int_{0}^{\infty} e^{-ar^2}rdr\right)^{1/2}$$

$$= \left(2\pi\int_{0}^{\infty} e^{-ar^2}rdr\right)^{1/2} = \sqrt{\frac{\pi}{a}}$$

$$\int_{-\infty}^{\infty} e^{ik(x-\xi)}e^{-k^2a^2t}dk = \int_{-\infty}^{\infty} e^{-a^2t\left(k-\frac{x-\xi}{2}i\right)^2-\frac{(x-\xi)^2}{4a^2t}}dk = e^{-\frac{(x-\xi)^2}{4a^2t}}\int_{-\infty}^{\infty} e^{-a^2t\left(k-\frac{x-\xi}{2}i\right)^2}dk$$

$$= \sqrt{\frac{\pi}{a^2t}}e^{-\frac{(x-\xi)^2}{4a^2t}}$$

# 傅里叶变换求解扩散方程

$$u(x,t) = F^{-1}[U(k,x)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\xi \varphi(\xi)\left[\int_{-\infty}^{\infty} e^{ik(x-\xi)} e^{-k^2a^2t} dk\right]$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} d\xi \varphi(\xi)\sqrt{\frac{\pi}{a^2t}} e^{-\frac{(x-\xi)^2}{4a^2t}}$$

# 傅里叶变换求解有热源扩散方程P331 例3

$$\begin{cases} u_t = a^2 u_{xx} + f(x,t), & |x| < \infty, t > 0 \\ u|_{t=0} = \phi(x), & |x| < \infty \end{cases}$$

1.泛定方程两边同时对空间做傅里叶变换变成常微分方程

2. 直接求解常微分方程，代入初值求出系数。

3. 对空间做傅里叶逆变换

# 傅里叶变换求解扩散方程

$$\begin{cases} u_t = a^2 u_{xx} + f(x,t), & |x| < \infty, t > 0 \\ u|_{t=0} = 0, & |x| < \infty \end{cases}$$

1.泛定方程两边同时对空间做傅里叶变换变成常微分方程

$$U' + k^2 a^2 U = F(k,t) \qquad V' e^{-k^2 a^2} - k^2 a^2 V e^{-k^2 a^2} + k^2 a^2 V e^{-k^2 a^2} = F(k,t)$$

$$U = 0, \quad t = 0 \qquad U = V e^{-k^2 a^2} \qquad V' = F(k,t) e^{k^2 a^2}$$

2. 直接求解常微分方程，代入初值求出系数。

$$U(k,t) = e^{-k^2 a^2 t} \int_0^t F(k,\xi) e^{k^2 a^2 \xi} d\xi = e^{-k^2 a^2 t} \int_0^t \int_{-\infty}^{\infty} f(x,\xi) e^{-ikx} dx e^{k^2 a^2 \xi} d\xi$$

3. 对空间做傅里叶逆变换

# 傅里叶变换求解扩散方程

$$\begin{cases} u_t = a^2 u_{xx} + f(x,t), & |x| < \infty, t > 0 \\ u|_{t=0} = 0, \quad |x| < \infty \end{cases}$$

3. 做傅里叶逆变换

$$U(k,t) = e^{-k^2a^2t}\int_0^t F(k,\xi)e^{k^2a^2\xi}d\xi = e^{-k^2a^2t}\int_0^t\int_{-\infty}^{\infty} f(x,\xi)e^{-ikx}dxe^{k^2a^2\xi}d\xi$$

$$F^{-1}[U(k,t)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}e^{-k^2a^2t}\int_0^t F(k,\xi)e^{k^2a^2\xi}d\xi dk$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}e^{-k^2a^2t}\int_0^t\int_{-\infty}^{\infty} f(x',\xi)e^{-ikx'}dx'e^{k^2a^2\xi}d\xi dk$$

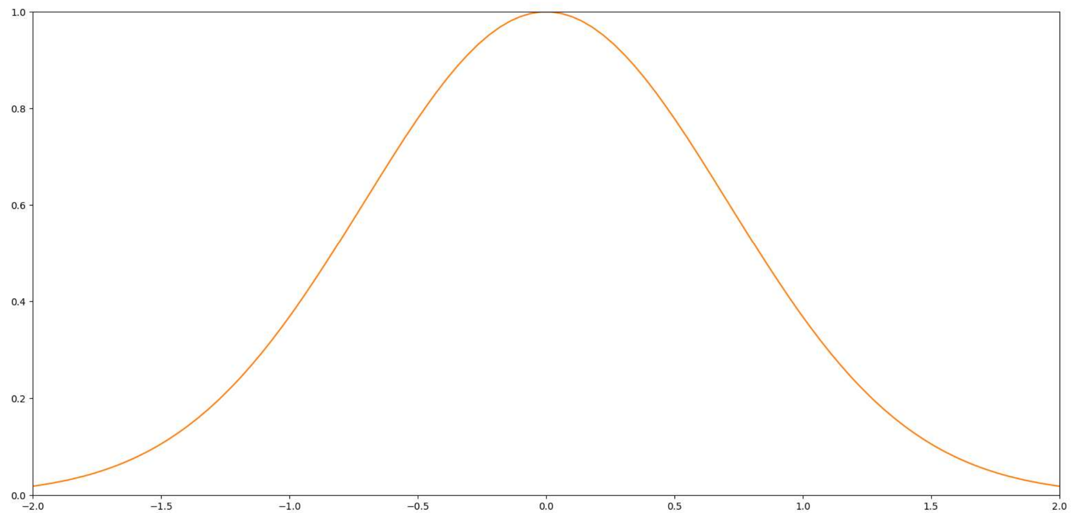
$$= \frac{1}{2\pi}\int_0^t\int_{-\infty}^{\infty} f(x',\xi)dx'd\xi\int_{-\infty}^{\infty} e^{ikx}e^{-ikx'}e^{k^2a^2\xi}dk$$

# 傅里叶变换求解有热源扩散方程P331 例4

$$\begin{cases} u_t = a^2 u_{xx}, \\ u_x \big|_{x=0} = 0 \\ u\big|_{t=0} = 2\Phi_0 \delta(x), \quad x > 0 \end{cases}$$

高斯函数

$$u = \frac{\Phi}{2a\sqrt{t}} \frac{2}{\sqrt{\pi}} e^{-x^2/4a^2 t}$$



# 傅里叶变换求解有热源扩散方程P332 例5

$$\begin{cases} u_t = a^2 u_{xx}, \\ u|_{x=0} = N_0 \\ u|_{t=0} = 0, \quad x > 0 \end{cases}$$

$$u = w + N_0$$

$$\begin{cases} w_t = a^2 w_{xx}, \\ w|_{x=0} = 0 \\ w|_{t=0} = -N_0, \quad x > 0 \end{cases}$$

$$\begin{cases} w_t = a^2 w_{xx}, \\ w|_{x=0} = 0 \end{cases}$$

$$w|_{t=0} = \begin{cases} -N_0 & x > 0 \\ 0 & x = 0 \\ N_0 & x < 0 \end{cases}$$

1.泛定方程两边同时对空间做傅里叶变换变成常微分方程

2. 直接求解常微分方程，代入初值求出系数。

3. 对空间做傅里叶逆变换

$$u = -N_0 erf\left(\frac{x}{2a\sqrt{t}}\right)$$