数学物理方程 第一次作业（共计 20 分）

$$\delta(x-x_0)=\begin{cases}\infty & x=x_0\\ 0 & x\neq x_0\end{cases}$$

$$\int \delta(x-x_0)\varphi(x)dx=\varphi(x_0)$$

广义函数的积分复合运算与普通函数完全一致

$$\delta\left[T(x)\right]=\sum_n\frac{\delta(x-x_n)}{\left|T'(x_n)\right|}$$

第一题，计算下列式子（**4 分**）。

（1）$\int_0^3(5x-2)\delta(2-x)dx=8$ （2）$\int_{-1}^1\cos x\delta(-2x)dx=1/2$

（3）$\int_{-\pi/2}^{\pi/2}\cos x\delta(\sin x)dx=1$

（4）$\int_0^\infty\phi(x)\delta(x^2-a^2)dx=\frac{1}{2|a|}\phi(|a|)$

答案：(1) $\int_0^3(5x-2)\delta(2-x)dx$ 令 $2-x=y$ 所以 $x=2-y$，

$$\int_2^{-1}[5(2-y)-2]\delta(y)d(2\text{-y})=-\int_2^{-1}(8-5y)\delta(y)\mathrm{dy}=\int_{-1}^2(8-5y)\delta(y)dy$$

由 $\int \delta(x-x_0)\varphi(x)dx=\varphi(x_0)$

所以 $\int_{-1}^2(8-5y)\delta(y)\mathrm{dy}=(8-5*0)=8$

(2)

$$\int_{-1}^{1}\cos x\delta(-2x)dx=\cos 0/|-2|=1/2$$

$$\int_{-1}^{1}\cos x\delta(-2x)dx=\int_{-2}^{2}\cos(-\frac{1}{2})\delta(y)d(-\frac{1}{2}y)=\int_{2}^{-2}-\frac{1}{2}\cos(-\frac{1}{2})\delta(y)dy=\frac{1}{2}$$

(3) $$\int_{-\pi/2}^{\pi/2}\cos x\delta(\sin x)dx=\cos 0/|\cos 0|=1$$

$$\int_{-\pi/2}^{\pi/2}\cos x\delta(\sin x)dx=\int_{-1}^{1}\cos(\arcsin y)\delta(y)d\arcsin y$$

$$=\int_{-1}^{1}\cos(\arcsin y)\delta(y)\frac{1}{\sqrt{1-y^2}}dy=\cos(\arcsin 0)\frac{1}{\sqrt{1-0^2}}=\cos 0=1$$

(4) $$\int_{0}^{\infty}\phi(x)\delta(x^2-a^2)dx=\begin{cases}\phi(a)/2a & a>0\\ \phi(-a)/-2a & a<0\end{cases}$$

第二题，具有第一类间断点，分段连续可导函数 $f(x)=\begin{cases}f_1(x) & x<x_0\\ f_2(x) & x>x_0\end{cases}$，

$h=f_2(x_0)-f_1(x_0)$，证明

$$\frac{\mathrm{d}f(x)}{\mathrm{d}x}=\begin{cases}\dfrac{\mathrm{d}f_1(x)}{\mathrm{d}x} & x<x_0\\ \dfrac{\mathrm{d}f_2(x)}{\mathrm{d}x} & x>x_0\end{cases}+h\delta(x-x_0)$$ （2 分）

答案：

$$f_c'=\begin{cases}\dfrac{\mathrm{d}f_1(x)}{\mathrm{d}x} & x<x_0\\ \dfrac{\mathrm{d}f_2(x)}{\mathrm{d}x} & x>x_0\end{cases}$$

$$\left\langle \frac{\mathrm{d} f(x)}{\mathrm{d} x}, \phi(x) \right\rangle = -\langle f(x), \phi'(x) \rangle$$

$$= -\int_{-\infty}^{x_0} f_1 \phi' \mathrm{d} x - \int_{x_0}^{\infty} f_2 \phi' \mathrm{d} x$$

$$= -f_1 \phi \Big|_{-\infty}^{x_0} + \int_{-\infty}^{x_0} f_1' \phi \mathrm{d} x - f_2 \phi \Big|_{x_0}^{\infty} + \int_{x_0}^{\infty} f_2' \phi \mathrm{d} x$$

$$= -f_1(x_0)\phi(x_0) + \int_{-\infty}^{x_0} f_1' \phi \mathrm{d} x + f_2(x_0)\phi(x_0) + \int_{x_0}^{\infty} f_2' \phi \mathrm{d} x$$

$$= [f_2(x_0) - f_1(x_0)]\phi(x_0) + \int_{-\infty}^{x_0} f_1' \phi \mathrm{d} x + \int_{x_0}^{\infty} f_2' \phi \mathrm{d} x$$

$$= h\phi(x_0) + \langle f_c', \phi \rangle = \langle f_c' + h\delta(x - x_0), \phi \rangle$$

或者 $f(x) \equiv f_1(x)H(x) + f_2(x)H(-x)$

$$H(x) \equiv \begin{cases} 1 & x > 0 \\ 0 & x < 0 \end{cases}$$

$H'(x) = \delta(x)$

$$f'(x) \equiv f_1'(x)H(x) + f_1(x)\delta(x) + f_2'(x)H(-x) - f_2'(x)\delta(x)$$
$$= f_1'(x)H(x) + f_2'(x)H(-x) + (f_1(x) - f_2(x))\delta(x)$$

因为 $\int a(x)\delta(x)\varphi(x)dx = a(0)\varphi(0) = \int a(0)\delta(x)\varphi(x)dx$

所以 $a(x)\delta(x) = a(0)\delta(x)$

$$f'(x) = f_1'(x)H(x) + f_2'(x)H(-x) + (f_1(0) - f_2(0))\delta(x)$$

第三题， 设 $f(x) = \begin{cases} x^2 & x \geq 0 \\ x & x < 0 \end{cases}$， 求解 $\frac{df}{dx}, \frac{df^2}{dx^2}, \frac{df^3}{dx^3}$ 。（3 分）

直接带入第二题的公式， $\frac{\mathrm{d} f(x)}{\mathrm{d} x} = \begin{cases} \frac{\mathrm{d} f_1(x)}{\mathrm{d} x} & x < x_0 \\ \frac{\mathrm{d} f_2(x)}{\mathrm{d} x} & x > x_0 \end{cases} + h\delta(x - x_0)$

$$\frac{df}{dx} = \begin{cases} 2x & x \geq 0 \\ 1 & x < 0 \end{cases}$$

$$\frac{df^2}{dx^2}=\begin{cases}2 & x\ge 0\\ 0 & x<0\end{cases}-\delta(x)$$

$$\frac{df^3}{dx^3}=2\delta(x)-\delta'(x)=2\delta(x)-\frac{1}{2\pi}\int_{-\infty}^{\infty} i\omega e^{-i\omega t}d\omega$$

第四题，证明在球坐标下，

$$\delta(\mathbf{r}-\mathbf{r}_0)=\delta(r-r_0)\delta(\cos\theta-\cos\theta_0)\delta(\phi-\phi_0)/r^2\text{。}$$

（1 分）

与公式 $\delta(x-x_0)\delta(y-y_0)\delta(z-z_0)=\frac{1}{r^2\sin\vartheta}\delta(r-r_0)\delta(\phi-\phi_0)\delta(\vartheta-\vartheta_0)$ 比较，只需证明， $\delta(\cos\theta-\cos\theta_0)=\frac{1}{\sin\vartheta}\delta(\vartheta-\vartheta_0)$

利用公式 $\delta\left[T(x)\right]=\sum_n\frac{\delta(x-x_n)}{\left|T'(x_n)\right|}$， 得到 $\delta(\cos\theta-\cos\theta_0)=\frac{1}{|\sin\vartheta_0|}\delta(\vartheta-\vartheta_0)$

又 $\pi>\vartheta_0\ge 0$， $\sin\vartheta_0\ge 0$，所以 $\delta(\cos\theta-\cos\theta_0)=\frac{1}{\sin\vartheta_0}\delta(\vartheta-\vartheta_0)=\frac{1}{\sin\vartheta}\delta(\vartheta-\vartheta_0)$

第五题，计算下列函数的广义傅里叶变换（10 分），

$$f(\mathbf{q})=F[f(\mathbf{x})]=\int_{\mathbf{R}^n}\frac{1}{(\sqrt{2\pi})^n}\mathbf{dx}\ f(\mathbf{x})e^{-i\mathbf{qx}}$$

$$f(\mathbf{x})=F^{-1}[f(\mathbf{q})]=\int_{\mathbf{R}^n}\frac{1}{(\sqrt{2\pi})^n}\mathbf{dq}\ f(\mathbf{q})e^{i\mathbf{qx}}$$

$$F[\delta(x-x_0)]=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}dx\delta(x-x_0)e^{-ixy}=\frac{1}{\sqrt{2\pi}}e^{-ix_0y}$$

$$F[1]=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}dye^{-ixy}=\sqrt{2\pi}\delta(x)$$

$$F[H(x)]=\frac{1}{\sqrt{2\pi}}\int_{0}^{\infty}dxe^{-iqx}=\frac{1}{\sqrt{2\pi}}\frac{1}{iq}+\frac{1}{\sqrt{2\pi}}\pi\delta(q)$$

(1) $\sin x \to i\sqrt{\frac{\pi}{2}}[\delta(q+1)-\delta(q-1)]$

代入傅里叶变换公式 $f(\mathbf{q})=F[f(\mathbf{x})]=\int_{\mathbf{R}^n}\frac{1}{(\sqrt{2\pi})^n}\mathbf{dx}\, f(\mathbf{x})e^{-i\mathbf{qx}}$

$$F[\sin x]=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-iqx}\sin x dx=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-iqx}\frac{e^{ix}-e^{-ix}}{2i}dx$$

$$=\frac{1}{2i\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-i(q-1)x}-e^{-i(q+1)x}dx$$

利用公式 $F[1]=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}dye^{-ixy}=\sqrt{2\pi}\delta(x)$

$$F[\sin x]=\frac{1}{2i\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-i(q-1)x}-e^{-i(q+1)x}dx=\frac{\sqrt{2\pi}}{2i}[\delta(q-1)-\delta(q+1)]$$

$$i\sqrt{\frac{\pi}{2}}[\delta(q+1)-\delta(q-1)]$$

$$f(x)=\begin{cases}\sin x & x>0\\ 0 & x<0\end{cases}\to$$

(2)

$$\frac{1}{2i\sqrt{2\pi}}\left\{\frac{1}{i(q-1)}+\pi\delta(q-1)-\frac{1}{i(q+1)}-\pi\delta(q+1)\right\}$$

代入傅里叶变换公式 $f(\mathbf{q})=F[f(\mathbf{x})]=\int_{\mathbf{R}^n}\frac{1}{(\sqrt{2\pi})^n}\mathbf{dx}\, f(\mathbf{x})e^{-i\mathbf{qx}}$

$$=\frac{1}{\sqrt{2\pi}}\int_{0}^{\infty}e^{-iqx}\sin x dx=\frac{1}{\sqrt{2\pi}}\int_{0}^{\infty}e^{-iqx}\frac{e^{ix}-e^{ix}}{2i}dx$$

$$=\frac{1}{2i\sqrt{2\pi}}\int_{0}^{\infty}e^{-i(q-1)x}-e^{-i(q+1)x}dx$$

利用公式 $F[H(x)]=\frac{1}{\sqrt{2\pi}}\int_{0}^{\infty}dxe^{-iqx}=\frac{1}{\sqrt{2\pi}}\frac{1}{iq}+\frac{1}{\sqrt{2\pi}}\pi\delta(q)$

$$\frac{1}{2i\sqrt{2\pi}}\int_{0}^{\infty}e^{-i(q-1)x}-e^{-i(q+1)x}dx=\frac{1}{2i\sqrt{2\pi}}[\frac{1}{i(q-1)}+\pi\delta(q-1)-\frac{1}{i(q+1)}-\pi\delta(q+1)]$$

(3) $\operatorname{sgn}(x)=\begin{cases}1 & x>0\\ 0 & x=0\\ -1 & x<0\end{cases} \to \sqrt{\frac{2}{\pi}}\frac{1}{iq}$

代入傅里叶变换公式 $f(\mathbf{q})=F[f(\mathbf{x})]=\int_{\mathbf{R}^n}\frac{1}{(\sqrt{2\pi})^n}\mathbf{dx}\, f(\mathbf{x})e^{-i\mathbf{qx}}$

$$=\frac{1}{\sqrt{2\pi}}\int_0^{\infty}e^{-iqx}dx-\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0}e^{-iqx}dx$$

$$=\frac{1}{\sqrt{2\pi}}\int_0^{\infty}e^{-iqx}dx+\frac{1}{\sqrt{2\pi}}\int_{\infty}^{0}e^{iqx}dx=\frac{1}{\sqrt{2\pi}}\int_0^{\infty}e^{-iqx}dx-\frac{1}{\sqrt{2\pi}}\int_0^{\infty}e^{iqx}dx$$

利用公式 $F[H(x)]=\frac{1}{\sqrt{2\pi}}\int_0^{\infty}dxe^{-iqx}=\frac{1}{\sqrt{2\pi}}\frac{1}{iq}+\frac{1}{\sqrt{2\pi}}\pi\delta(q)$

$$=\frac{1}{\sqrt{2\pi}}\left(\frac{1}{iq}+\frac{1}{\sqrt{2\pi}}\pi\delta(q)\right)-\frac{1}{\sqrt{2\pi}}\left(-\frac{1}{iq}+\frac{1}{\sqrt{2\pi}}\pi\delta(q)\right)=\frac{2}{\sqrt{2\pi}}\frac{1}{iq}$$

(4) $f(x)=1/x\to -i\sqrt{\frac{2}{\pi}}\operatorname{sgn}(q)$

注意题（3）可知，$\operatorname{sgn}(x)$ 的傅里叶变换是 $\frac{2}{\sqrt{2\pi}}\frac{1}{iq}$，所以 $\frac{2}{\sqrt{2\pi}}\frac{1}{iq}$ 的傅里叶逆变换

是 $\operatorname{sgn}(x)$。利用傅里叶逆变换的公式

$f(\mathbf{x})=F^{-1}[f(\mathbf{q})]=\int_{\mathbf{R}^n}\frac{1}{(\sqrt{2\pi})^n}\mathbf{dq}\, f(\mathbf{q})e^{i\mathbf{qx}}$ 得到，

$$\operatorname{sgn}(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{iqx}\frac{1}{iq}\frac{2}{\sqrt{2\pi}}dq$$

所以

$$\operatorname{sgn}(x)=\frac{1}{\sqrt{2\pi}}\int_{\infty}^{-\infty}e^{-iqx}\frac{1}{-iq}\frac{2}{\sqrt{2\pi}}d(-q)$$

$$=-\frac{1}{i\pi}\int_{-\infty}^{\infty}e^{-iqx}\frac{1}{q}dq=-\frac{\sqrt{2\pi}}{i\pi}F[\frac{1}{q}]$$

也就是 $\operatorname{sgn}(x)=-\frac{\sqrt{2\pi}}{i\pi}F[\frac{1}{q}]$

即 $F[\frac{1}{q}]=-i\sqrt{\frac{2}{\pi}}\operatorname{sgn}(x)$