波动方程的特殊解（d'Alembert公式）

无界

$$\begin{cases} u_{tt} = a^2 u_{xx}, & |x| < \infty, t > 0 \\ u\big|_{t=0} = \phi(x), \quad u_t\big|_{t=0} = \psi(x), & |x| < \infty \end{cases}$$

半有界

$$\begin{cases} u_{tt} = a^2 u_{xx}, & 0 < x < \infty, t > 0 \\ u\big|_{t=0} = \phi(x), \quad u_t\big|_{t=0} = \psi(x), & 0 < x < \infty \\ u\big|_{x=0} = 0 \end{cases}$$

d'Alembert公式

$$u(x,t) = \frac{\phi(x+at)+\phi(x-at)}{2} + \frac{1}{2a}\int_{x-at}^{x+at} \psi(\xi)\,\mathrm{d}\xi$$

分离变量法总结

（1）假定 $u(x,t)=X(x)T(t)$ 得到两个常微分方程

（2）利用齐次边界条件获得常微分方程的通解

（3）利用初始条件确定叠加系数

分离变量法求解两端固定的弦的自由振动

$$\begin{cases} u_{tt} = a^2 u_{xx}, & 0 < x < l, t > 0 \\ u\big|_{x=0} = u\big|_{x=l} = 0, & t > 0 \\ u\big|_{t=0} = \phi(x), \quad u_t\big|_{t=0} = \psi(x), & 0 \le x \le l \end{cases}$$

$$u(x,t) = \sum_{n=1,2,\cdots} u_n(x,t) = \sum_{n=1,2,\cdots} X_n(x) T_n(t) = \sum_{n=1,2,\cdots} \sin\frac{n\pi}{l}x \left[C_{1n} \cos\frac{n\pi a}{l}t + C_{2n} \sin\frac{n\pi a}{l}t \right]$$

第二类边界条件

$$\begin{cases} u_{tt} = a^2 u_{xx}, & 0 < x < l, t > 0 \\ u_x\big|_{x=0} = u_x\big|_{x=l} = 0, & t > 0 \\ u\big|_{t=0} = \phi(x), \quad u_t\big|_{t=0} = \psi(x), & 0 \le x \le l \end{cases}$$

$$u(x,t) = \sum_{n=1,2,\cdots} u_n(x,t) = \sum_{n=1,2,\cdots} X_n(x) T_n(t) = \sum_{n=1,2,\cdots} \sin\frac{n\pi}{l}x \left[C_{1n}\cos\frac{n\pi a}{l}t + C_{2n}\sin\frac{n\pi a}{l}t + C_{10} + C_{20}t \right]$$

扩散方程求解

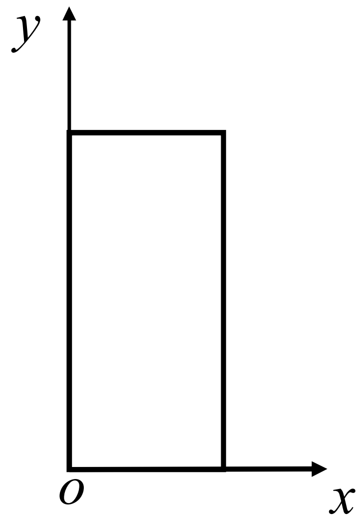
$$\begin{cases} u_t = a^2 u_{xx}, & 0 < x < l, t > 0 \\ u\big|_{x=0} = u_x\big|_{x=l} = 0, & t > 0 \\ u\big|_{t=0} = u_0 x / l, & 0 \le x \le l \end{cases}$$

$$u(x,t) = \sum_{n=1,2,\cdots}^{\infty} T_n(t) X_n(x) = \sum_{n=1,2,\cdots}^{\infty} C_n e^{-\lambda_n a^2 t} \sin\sqrt{\lambda_n}\, x$$

$$\lambda_n = \left[\pi(n+\frac{1}{2})\right]^2$$

稳恒方程求解

$$\begin{cases} u_{xx} + u_{yy} = 0; \quad 0 < x < a,\ 0 < y < b \\ u\,|_{x=0} = u_0, \quad u\,|_{x=a} = u_0; \quad 0 < y < b \\ u\,|_{y=0} = u_0, \quad u\,|_{y=b} = U; \quad 0 < x < a \end{cases}$$



化为齐次边界问题求解

作变换 $u(x,y)=u_0+v(x,y)$，定解问题化为齐次边界问题

$$\begin{cases} v_{xx}+v_{yy}=0; \quad 0<x<a,\ 0<y<b \\ v|_{x=0}=0, \quad v|_{x=a}=0; \quad 0<y<b \\ v|_{y=0}=0, \quad v|_{y=b}=U-u_0; \quad 0<x<a \end{cases}$$

分离变量

$$X''(x)Y(y)+X(x)Y''(y)=0$$

$$\frac{X''(x)}{X(x)}=-\frac{Y''(y)}{Y(y)}=-\lambda$$

分离变量

$$\begin{cases} X''(x) + \lambda X(x) = 0 \\ X(0) = X(a) = 0 \end{cases}$$

$$X_n(x) = \sin\frac{n\pi}{a}x, \quad \lambda_n = \frac{n^2\pi^2}{a^2}, \quad n = 1, 2, \cdots$$

分离变量

$$Y''(y) - \lambda_n Y(y) = 0$$

$$Y_n(y) = A_n \mathrm{e}^{\frac{n\pi}{a}y} + B_n \mathrm{e}^{-\frac{n\pi}{a}y}$$

$$v(x,y) = \sum_{n=1}^{\infty}\left(A_n \mathrm{e}^{\frac{n\pi}{a}y} + B_n \mathrm{e}^{-\frac{n\pi}{a}y}\right)\sin\frac{n\pi}{a}x,$$

边界条件

$$\begin{cases} \displaystyle\sum_{n=1}^{\infty}\left(A_n + B_n\right)\sin\frac{n\pi}{a}x = 0 \\ \displaystyle\sum_{n=1}^{\infty}\left(A_n \mathrm{e}^{\frac{n\pi b}{a}} + B_n \mathrm{e}^{-\frac{n\pi b}{a}}\right)\sin\frac{n\pi}{a}x = U - u_0 \end{cases}$$

傅里叶级数

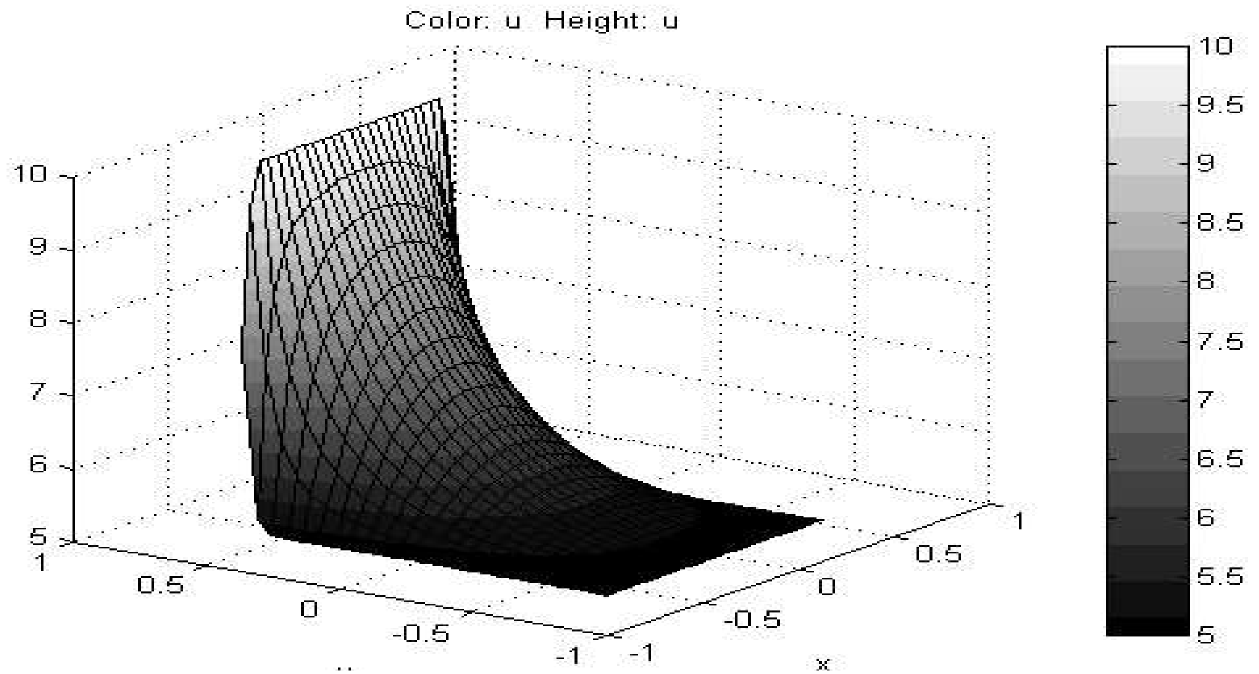
$$A_n + B_n = 0\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$A_n \mathrm{e}^{\frac{n\pi b}{a}} + B_n \mathrm{e}^{-\frac{n\pi b}{a}} = \frac{2(U - u_0)}{a}\int_0^a \sin\frac{n\pi}{a}x\,\mathrm{d}x$$

$$A_n = -B_n = \begin{cases} 0 & (n\text{为偶数}) \\ \dfrac{4}{n\pi}\dfrac{U - u_0}{\mathrm{e}^{n\pi b/a} - \mathrm{e}^{-n\pi b/a}} & (n\text{为奇数}) \end{cases}$$

$$u(x,y)=u_0$$

$$+\frac{4\left(U-u_0\right)}{\pi}\sum_{k=0}^{\infty}\frac{1}{(2k+1)}\frac{\operatorname{sh}\frac{(2k+1)\pi y}{a}}{\operatorname{sh}\frac{(2k+1)\pi b}{a}}\sin\frac{(2k+1)\pi x}{a}$$



二阶常系数齐次常微分方程

$$X''(x)+\lambda X(x)=0$$

特征方程 $\eta^2+\lambda=0$

通解 $X=Ae^{\eta_1 x}+Be^{\eta_2 x}$

傅里叶变换

$$F\left[X''(x)+\lambda X(x)\right]=0$$

$$F\left[X''(x)\right]+F\left[\lambda X(x)\right]=0$$

$$(-ik)^2 F\left[X(x)\right]+\lambda F\left[X(x)\right]=0$$

$$\left(\lambda-k^2\right)F\left[X(x)\right]=0$$

傅里叶变换求解

特征方程 $\left(\lambda - k^2\right) F\left[X\left(x\right)\right] = 0$

$$F[X(x)] = \begin{cases} A_{\pm} & \lambda = \pm k \\ 0 & \lambda \neq \pm k \end{cases}$$

傅里叶变换求解

特征方程 $\left(\lambda-k^2\right)F\left[X\left(x\right)\right]=0$

$$F[X(x)]=\begin{cases} A_{\pm} & \lambda=\pm k_0 \\ 0 & \lambda\neq\pm k_0 \end{cases}=A_+\delta(k+k_0)+A_-\delta(k-k_0)$$

$$X(x)=\int e^{ikx}\left[A_+\delta(k+k_0)+A_-\delta(k-k_0)\right]dx$$

$$=A_+e^{-ik_0x}+A_-e^{+ik_0x}=\left(A_+ \quad A_-\right)\begin{pmatrix} e^{-ik_0x} \\ e^{+ik_0x} \end{pmatrix}$$

通解是以 exp(ik0x)和exp(-ik0x)为基矢的二维矢量。

为什么扩散方程、拉普拉斯方程也可以分离变量?

$$\begin{cases} u_{tt} = a^2 u_{xx}, & 0 < x < l, t > 0 \\ u\big|_{x=0} = u\big|_{x=l} = 0, & t > 0 \\ u\big|_{t=0} = \phi(x), \quad u_t\big|_{t=0} = \psi(x), & 0 \le x \le l \end{cases}$$

$$F[u_{tt} - a^2 u_{xx}] = (-i\omega)^2 U - a^2 U_{xx} = 0$$

$$U_{xx} + \frac{\omega^2}{a^2} U = 0 \qquad U = A(\omega) e^{i\omega x/a} + B(\omega) e^{-i\omega x/a}$$

为什么扩散方程、拉普拉斯方程也可以分离变量?

$$U = A(\omega)e^{i\omega x/a} + B(\omega)e^{-i\omega x/a}$$

由于边界条件 $u\big|_{x=0} = u\big|_{x=l} = 0$

$$U_n(x,\omega_n) = C(\omega_n)\sin(\omega_n x / a)$$

从而时间和空间函数分开

傅里叶级数连续得到傅里叶变换

傅里叶级数

$$f(x)=\sum a(k)\sin kx$$

$$\Delta k\to 0$$

傅里叶变换

$$f(x)=\sum a(k)\sin kx$$

$$k=\frac{n\pi}{l}$$

傅里叶级数中，任意给定f(x), 如何求sinkx的系数?

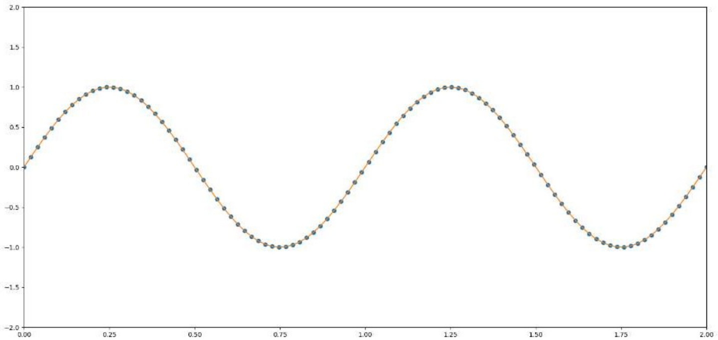
$$\int f(x)\sin k'xdx=\sum a(k)\delta(k-k')=a(k')$$

正交性 $\int \sin k'x\sin kxdx=\begin{cases} c & k=k' \\ 0 & k\neq k' \end{cases}$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}dqe^{-i(x-x')q}=\delta(x-x')$$

傅里叶变换离散退化为傅里叶级数

函数离散化 $f(k) \approx \tilde{f}(k) = \sum_{n} f(k_n)\delta(k - k_n) \quad k_n = \frac{n\pi}{l}$



$$f(x) = \int e^{ikx} f(k)dk \approx \int e^{ikx} \sum_{n} f(k_n)\delta(k - k_n)dk$$

$$= \sum_{n} f(k_n) \int e^{ikx} \delta(k - k_n)dk = \sum_{n} f(k_n)e^{ik_n x}$$

$$= \sum_{n} f(k_n)\cos k_n x + if(k_n)\sin k_n x$$

为什么扩散方程、拉普拉斯方程也可以分离变量?

$$U(x,\omega_n) = C(\omega_n)\sin(\omega_n x / a)$$

$$U(x,\omega) \sim \int u(x,t) e^{-i\omega t} dt$$

Omega只能取离散的值，傅里叶变换退化为傅里叶级数变换。

$$U(x,\omega_n) \sim \int u(x,t)\sin\omega_n t dt \qquad or \qquad \int u(x,t)\cos\omega_n t dt$$

为什么扩散方程、拉普拉斯方程也可以分离变量?

傅里叶级数变换

$$U(x,\omega_n) \sim \int u(x,t)\sin\omega_n t dt \qquad or \qquad \int u(x,t)\cos\omega_n t dt$$

傅里叶级数展开

$$u(x,t) = \sum_n \left[U_1(x,\omega_n)\sin\omega_n t + U_2(x,\omega_n)\cos\omega_n t\right]\sin(\omega_n x/a)$$

$$= \sum_n \left[C_1(\omega_n)\sin\omega_n t + C_2(\omega_n)\cos\omega_n t\right]\sin(\omega_n x/a)$$

非齐次偏微分方程的级数解法

$$
\begin{cases}
u_{tt} - a^2 u_{xx} = A\cos\dfrac{\pi x}{l}\sin\omega_0 t, & 0<x<l, t>0 \\
u_x\big|_{x=0} = u_x\big|_{x=l} = 0, & t>0 \\
u\big|_{t=0} = \phi(x), \quad u_t\big|_{t=0} = \psi(x), & 0\le x\le l
\end{cases}
$$

P162，分离变量思路

条件放宽

$$\begin{cases} V_{tt} - a^2 V_{xx} = 0, \quad 0 < x < l, t > 0 \\ V_x\big|_{x=0} = V_x\big|_{x=l} = 0, \quad t > 0 \end{cases}$$

$$V(x,t) = \sum_{n=0} T_n(t) \cos\frac{n\pi x}{l}$$

寻找 $T_n(t)$ 使之满足

$$V_{tt} - a^2 V_{xx} = A\cos\frac{\pi x}{l}\sin\omega_0 t$$

$$V\big|_{t=0} = \phi(x), \quad V_t\big|_{t=0} = \psi(x), \quad 0 \le x \le l$$

P162，分离变量思路

$$\begin{cases} T_n''(t) + \left(\dfrac{n\pi a}{l}\right)^2 T_n = A\sin\omega_0 t & n=1 \\ T_n''(t) + \left(\dfrac{n\pi a}{l}\right)^2 T_n = 0 & n\neq 1 \end{cases}$$

$$T_n(t) = A_n\cos\frac{n\pi a}{l}t + B_n\sin\frac{n\pi a}{l}t \qquad n\neq 1$$

P162，分离变量思路

$$T_1''(t) + \left(\frac{\pi a}{l}\right)^2 T_1 = A\sin\omega_0 t$$

$$T_1(0) = \varphi_1$$

$$T_1'(0) = \psi_1$$

P162，分离变量思路

$$T_1''(t)+\left(\frac{\pi a}{l}\right)^2 T_1 = A\sin\omega_0 t$$

$$T_1(t)=C_1\sin\omega_0 t+C_2\cos\omega_0 t+C_3\sin\frac{\pi a}{l}t+C_4\cos\frac{\pi a}{l}t$$

$$-C_1\omega_0{}^2\sin\omega_0 t+C_1\left(\frac{\pi a}{l}\right)^2\sin\omega_0 t-C_2\omega_0{}^2\cos\omega_0 t+C_2\left(\frac{\pi a}{l}\right)^2\cos\omega_0 t$$

$$-C_3\left(\frac{\pi a}{l}\right)^2\sin\frac{\pi a}{l}t+C_3\left(\frac{\pi a}{l}\right)^2\sin\frac{\pi a}{l}t-C_4\left(\frac{\pi a}{l}\right)^2\cos\frac{\pi a}{l}t+C_4\left(\frac{\pi a}{l}\right)^2\cos\frac{\pi a}{l}t$$

$$=A\sin\omega_0 t$$

P162，分离变量思路

$$-C_1\omega_0^{\ 2}\sin\omega_0 t + C_1\left(\frac{\pi a}{l}\right)^2\sin\omega_0 t - C_2\omega_0^{\ 2}\cos\omega_0 t + C_2\left(\frac{\pi a}{l}\right)^2\cos\omega_0 t$$

$$-C_3\left(\frac{\pi a}{l}\right)^2\sin\frac{\pi a}{l}t + C_3\left(\frac{\pi a}{l}\right)^2\sin\frac{\pi a}{l}t - C_4\left(\frac{\pi a}{l}\right)^2\cos\frac{\pi a}{l}t + C_4\left(\frac{\pi a}{l}\right)^2\cos\frac{\pi a}{l}t$$

$$= A\sin\omega_0 t$$

$$\omega_0 \neq \frac{\pi a}{l} \qquad T_1(t) = \begin{cases} C_1 = -\dfrac{A}{\omega_0^2 - \left(\dfrac{\pi a}{l}\right)^2} \\ C_2 = 0 \\ C_3 \\ C_4 \end{cases}$$

$$\omega_0 = \frac{\pi a}{l} \qquad T_1(t) = \begin{cases} C_1 = 0 \\ C_2 = 0 \\ C_3 \\ C_4 \end{cases}$$

P162，分离变量思路

$$T_1(t) = C_1 \sin(\omega_0 t) + C_3 \sin\left(\frac{\pi a}{l} t\right) + C_4 \cos\left(\frac{\pi a}{l} t\right)$$

$$T_1(0) = \varphi_1$$

$$T_1'(0) = \psi_1$$

$$C_4 = \varphi_1$$

$$\omega_0 C_1 + \frac{\pi a}{l} C_3 = \psi_1$$

$$C_1 = -\frac{A}{\omega_0^2 - \left(\frac{\pi a}{l}\right)^2}$$

$$C_2 = 0$$

傅里叶变换思路

$$\omega_0 \neq \frac{\pi a}{l} \qquad \left[-\omega^2+\left(\frac{\pi a}{l}\right)^2\right]F[T_1]=\frac{A}{2}\left\{\delta(\omega+\omega_0)-\delta(\omega-\omega_0)\right\}$$

$$F[T_1]=\frac{\frac{A}{2}\left\{\delta(\omega+\omega_0)-\delta(\omega-\omega_0)\right\}}{\left[-\omega_0{}^2+\left(\frac{\pi a}{l}\right)^2\right]}$$

傅里叶变换思路

$$F[T_1]=\frac{\frac{A}{2}\{\delta(\omega+\omega_0)-\delta(\omega-\omega_0)\}}{\left[-\omega_0{}^2+\left(\frac{\pi a}{l}\right)^2\right]}$$

$$\omega_0\neq\frac{\pi a}{l}\qquad\frac{A\sin\omega_0 t}{\left[-\omega_0{}^2+\left(\frac{\pi a}{l}\right)^2\right]}$$

$$T_1(x)=\frac{A\sin\omega_0 t}{\left[-\omega_0{}^2+\left(\frac{\pi a}{l}\right)^2\right]}+C_3\sin\left(\frac{\pi a}{l}t\right)+C_4\cos\left(\frac{\pi a}{l}t\right)$$

$$\omega_0=\frac{\pi a}{l}\qquad C_3\sin\left(\frac{\pi a}{l}t\right)+C_4\cos\left(\frac{\pi a}{l}t\right)$$

$$T_1(0)=\varphi_1$$

$$T_1{}'(0)=\psi_1$$