傅里叶变换求解扩散方程

$$\begin{cases} u_t = a^2 u_{xx} + f(x,t), & |x| < \infty, t > 0 \\ u|_{t=0} = 0, & |x| < \infty \end{cases}$$

1.泛定方程两边同时对空间做傅里叶变换变成常微分方程

$$U' + k^2 a^2 U = F(k,t) \qquad V' e^{-k^2 a^2} - k^2 a^2 V e^{-k^2 a^2} + k^2 a^2 V e^{-k^2 a^2} = F(k,t)$$

$$U = 0, \quad t = 0 \qquad U = V e^{-k^2 a^2} \qquad V' = F(k,t) e^{k^2 a^2}$$

2. 直接求解常微分方程，代入初值求出系数。

$$U(k,t) = e^{-k^2 a^2 t} \int_0^t F(k,\xi) e^{k^2 a^2 \xi} d\xi = e^{-k^2 a^2 t} \int_0^t \int_{-\infty}^{\infty} f(x,\xi) e^{-ikx} dx e^{k^2 a^2 \xi} d\xi$$

3. 对空间做傅里叶逆变换

冲量法求解非齐次方程

作用力分解为瞬时力作用 $f(x,t)=\int_0^{\infty} f(x,t)\delta(t-\tau)\mathrm{d}\tau=\int_0^{\infty} f(x,\tau)\delta(t-\tau)\mathrm{d}\tau$

作用力分解为瞬时力作用，瞬时力作用相当于初始速度引起的振动

$$\bar{f} \rightarrow V$$

然后将瞬时力引起的振动线性叠。 $\therefore u(x,t)=\int_0^{\infty} V(x,t;\tau)\mathrm{d}\tau=\int_0^{t} V(x,t;\tau)\mathrm{d}\tau$

冲量法求解非齐次方程

作用力分解为瞬时力作用，瞬时力作用相当于初始速度引起的振动；然后将瞬时力引起的振动线性叠。

$$\begin{cases} V_{tt} = a^2 V_{xx} + f(x,t)\delta(t-\tau), \quad 0<x<l, t>0 \\ V|_{x=0} = V|_{x=l} = 0, \qquad t>0 \\ V|_{t=0} = V_t|_{t=0} = 0, \end{cases}$$

$$\therefore u(x,t) = \int_0^t V(x,t;\tau)\,\mathrm{d}\tau$$

冲量法求解非齐次方程

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冲量定理

$$V'(\tau+\Delta\tau) - V'(\tau-\Delta\tau) = \int_{\tau-\Delta\tau}^{\tau+\Delta\tau} f(x,t)\delta(t-\tau)dt$$

$$V'(\tau+\Delta\tau) = f(x,\tau)$$

$$\Delta\tau \to 0 \quad V'(\tau) = f(x,\tau)$$

冲量法求解非齐次方程

$$\begin{cases} V_{tt} = a^2 V_{xx}, \quad 0 < x < l \\ V|_{x=0} = V|_{x=l} = 0, \\ V|_{t=\tau} = 0, V_t|_{t=\tau} = f(x,\tau), \end{cases}$$

$$\therefore u(x,t) = \int_0^{\infty} V(x,t;\tau)\,\mathrm{d}\tau = \int_0^{t} V(x,t;\tau)\,\mathrm{d}\tau$$

非齐次方程的时空分解P318,例一

一维无界空间中波的传播

$$\begin{cases} u_{tt} = a^2 u_{xx} + f\left(x,t\right), t > 0 \\ u\big|_{t=0} = 0, u_t\big|_{t=0} = 0 \end{cases}$$

把外力分解为一系列瞬时脉冲力

$$f\left(x,t\right) = \int_0^t \mathrm{d}\tau \int_{-\infty}^{\infty} \mathrm{d}\xi f\left(\xi,\tau\right)\delta\left(x-\xi\right)\delta\left(t-\tau\right)$$

瞬时脉冲力 $f\left(\xi,\tau\right)\delta\left(x-\xi\right)\delta\left(t-\tau\right)$

对弦的影响 $G(t,x;\tau,\xi)$

时空分解

$$\begin{cases} G_{tt} = a^2 G_{xx} + \delta(x-\xi)\delta(t-\tau) \\ G|_{t=0} = 0, G_t|_{t=0} = 0 \end{cases}$$

求出瞬时脉冲力对弦振动的影响再叠加起来，可以得到总的影响。

$$u(x,t) = \int_0^t \mathrm{d}\tau \int_{-\infty}^{\infty} \mathrm{d}\xi f(\xi,\tau) G(x,t;\xi,\tau)$$

格林函数 Green’s Function

In mathematics, a Green's function of an inhomogeneous linear differential operator defined on a domain with specified initial conditions or boundary conditions is its impulse response.

格林函数代表一个点源在一定边界条件和初始条件下产生的场。

冲量定理法求解格林函数

$$\begin{cases} G_{tt} = a^2 G_{xx} + \delta(x-\xi)\delta(t-\tau) \\ G|_{t=0} = 0, G_t|_{t=0} = 0 \end{cases}$$

化为齐次方程

$$\begin{cases} G_{tt} = a^2 G_{xx} \\ G\big|_{t=\tau} = 0, G_t\big|_{t=\tau} = \delta(x-\xi) \end{cases}$$

达朗贝尔公式

$$\begin{cases} G_{tt} = a^2 G_{xx} \\ G|_{t=\tau} = 0, G_t|_{t=\tau} = \delta(x-\xi) \end{cases}$$

$$u(x,t) = \frac{\phi(x+at)+\phi(x-at)}{2} + \frac{1}{2a}\int_{x-at}^{x+at} \psi(\xi)\,\mathrm{d}\xi$$

$$G(x,t;\xi,\tau) = \frac{1}{2a}\int_{x-a(t-\tau)}^{x+a(t-\tau)} \delta(\xi-\xi_0)\,\mathrm{d}\xi_0 = \begin{cases} 0 & otherwise \\ \dfrac{1}{2a} & \xi \in [x-a(t-\tau), x+a(t-\tau)] \end{cases}$$

叠加求解

$$u(x,t)=\int_{-\infty}^{\infty}\mathrm{d}\tau\int_{-\infty}^{\infty}\mathrm{d}\xi f(\xi,\tau)G(x,t;\xi,\tau)$$

$$G(x,t;\xi,\tau)=\frac{1}{2a}\int_{x-a(t-\tau)}^{x+a(t-\tau)}\delta(\xi-\xi_0)\,\mathrm{d}\xi_0=\begin{cases}0 & otherwise\\ \dfrac{1}{2a} & \xi\in[x-a(t-\tau),x+a(t-\tau)]\end{cases}$$

叠加求解

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$$u(x,t)=\int_0^t\mathrm{d}\tau\int_{x-a(t-\tau)}^{x+a(t-\tau)}\mathrm{d}\xi f(\xi,\tau)\frac{1}{2a}$$

由叠加原理

$$\begin{cases} u_{tt} = a^2 u_{xx} + f(x,t), \quad t > 0 \\ u|_{t=0} = \phi(x), \quad u_t|_{t=0} = \psi(x) \end{cases}$$

$$\begin{cases} v_{tt} = a^2 u_{xx}, \quad t > 0 \\ v|_{t=0} = \phi(x), \quad \mathrm{v}_t|_{t=0} = \psi(x) \end{cases}$$

$$\begin{cases} \omega_{tt} = a^2 \omega_{xx} + f(x,t), \quad t > 0 \\ \omega|_{t=0} = 0, \quad \omega_t|_{t=0} = 0 \end{cases}$$

$$u(x,t) = \frac{1}{2}\left[\phi(x-at) + \phi(x+at)\right] + \frac{1}{2a}\int_{x-at}^{x+at} \mathrm{d}\xi\, \psi(\xi)$$

$$+ \frac{1}{2a}\int_0^t \mathrm{d}\tau \int_{x-a(t-\tau)}^{x+a(t-\tau)} \mathrm{d}\xi\, f(\xi,\tau)$$

格林函数形式

$$G(x,t;\xi,\tau)=\frac{1}{2a}\int_{x-a(t-\tau)}^{x+a(t-\tau)}\delta(\xi-\xi_0)\,\mathrm{d}\xi_0=\begin{cases}0 & otherwise\\ \dfrac{1}{2a} & \xi\in[x-a(t-\tau),x+a(t-\tau)]\end{cases}$$

$$u(x,t)=\frac{1}{2}[\phi(x-at)+\phi(x+at)]+\frac{1}{2a}\int_{x-at}^{x+at}\mathrm{d}\xi\,\psi(\xi)+\frac{1}{2a}\int_0^t\mathrm{d}\tau\int_{x-a(t-\tau)}^{x+a(t-\tau)}\mathrm{d}\xi\,f(\xi,\tau)$$

$$=\int_{-\infty}^{\infty}\mathrm{d}\xi\,\phi(\xi)\,\delta(|\xi-x|+at)+\frac{1}{2a}\int_{x-at}^{x+at}\mathrm{d}\xi\,\psi(\xi)+\frac{1}{2a}\int_0^t\mathrm{d}\tau\int_{x-a(t-\tau)}^{x+a(t-\tau)}\mathrm{d}\xi\,f(\xi,\tau)+\frac{1}{2a}\int_0^t\mathrm{d}\tau\int_{x-a(t-\tau)}^{x+a(t-\tau)}\mathrm{d}\xi\,f(\xi,\tau)$$

$$-\frac{d}{dt}G(x,t;\xi,0)=\frac{1}{2a}\delta(x-\xi-at)+\frac{1}{2a}\delta(-x+\xi-at)$$

$$u(x,t)=-\int_{-\infty}^{\infty}\mathrm{d}\xi\,\phi(\xi)\frac{\partial}{\partial t}G(x,t;\xi,0)+\int_{-\infty}^{\infty}\mathrm{d}\xi\,\psi(\xi)\,G(x,t;\xi,0)+\int_0^t\mathrm{d}\tau\int_{-\infty}^{\infty}\mathrm{d}\xi\,f(\xi,\tau)\,G(x,t;\xi,\tau$$

格林函数 Green’s Function

In mathematics, a Green's function of an inhomogeneous linear differential operator defined on a domain with specified initial conditions or boundary conditions is its impulse response.

格林函数代表一个点源在一定边界条件和初始条件下产生的场。

格林函数是构造一种方法，把连续的场做“点源分解”。

核心就是点源和叠加原理。

从物理上讲，就是我们知道了点源的场，那么对于给定的源的分布，我们就可以从叠加原理写出这个源的场。

Green's function (GF) is a fundamental solution to a linear differential equation, a building block that can be used to construct many useful solutions.

一维无界有源热传导问题时空分解P318，例三

一维无界空间中有源热传导问题

$$\begin{cases} u_t = a^2 u_{xx} + f(x,t), \quad t>0 \\ u|_{t=0} = 0 \end{cases}$$

把热源分解为一系列瞬时点热源

$$f(x,t) = \int_0^t \mathrm{d}\tau \int_{-\infty}^{\infty} \mathrm{d}\xi f(\xi,\tau)\delta(x-\xi)\delta(t-\tau)$$

求出点热源引起的温度变化

$$\begin{cases} G_t = a^2 G_{xx} + \delta(x-\xi)\delta(t-\tau), \quad t>0 \\ G|_{t=0} = 0 \end{cases}$$

将每一个点热源引起的温度变化叠加

$$u(x,t) = \int_0^t \mathrm{d}\tau \int_{-\infty}^{\infty} \mathrm{d}\xi f(\xi,\tau) G(x,t;\xi,\tau)$$

时空分解

$$\begin{cases} G_t = a^2\nabla^2 G + \delta\left(\mathbf{r}-\mathbf{r_0}\right)\delta\left(t-\tau\right), \quad t,\tau > 0 \\ G\big|_{t=\tau} = 0 \end{cases}$$

$$\begin{cases} G_t = a^2\nabla^2 G, \quad t,\tau > 0 \\ G\big|_{t=\tau} = \delta\left(\mathbf{r}-\mathbf{r_0}\right) \end{cases}$$

时空分解

$$\begin{cases} G_t = a^2\nabla^2 G + \delta(\mathbf{r}-\mathbf{r_0})\delta(t-\tau), \quad t,\tau>0 \\ G|_{t=\tau} = 0 \end{cases}$$

$$\begin{cases} G_t = a^2\nabla^2 G, \quad t,\tau>0 \\ G|_{t=\tau} = \delta(\mathbf{r}-\mathbf{r_0}) \end{cases}$$

$$U(k,t) = e^{-k^2a^2t}\int_0^t F(k,\xi)e^{k^2a^2\xi}d\xi = e^{-k^2a^2t}\int_0^t\int_{-\infty}^{\infty} f(x,\xi)e^{-ikx}dxe^{k^2a^2\xi}d\xi$$

考虑一维问题 $G(x,t;\xi,\tau) = \int_{-\infty}^{\infty} \frac{\varphi(x_0)}{2a\sqrt{\pi t}} e^{-(x-x_0)^2/4a^2t} dx_0$ P331,公式13.1.2

格林函数的解

$$G(x,t;\xi,\tau)=\int_{-\infty}^{\infty}\frac{\varphi(x')}{2a\sqrt{\pi t}}e^{-(x-x')^2/4a^2t}dx'$$

$$G(x,t;\xi,\tau)=\int_{-\infty}^{\infty}\mathrm{d}x'\delta(x'-\xi)\frac{1}{2a\sqrt{\pi(t-\tau)}}\mathrm{e}^{-\frac{(x-x')^2}{4a^2(t-\tau)}}=\frac{1}{2a\sqrt{\pi(t-\tau)}}\mathrm{e}^{-\frac{(x-\xi)^2}{4a^2(t-\tau)}}$$

广义函数弱收敛

$$G(x,t;\xi,\tau)=\int_{-\infty}^{\infty}\frac{\varphi(x')}{2a\sqrt{\pi t}}e^{-(x-x')^2/4a^2t}dx'$$

$$G(x,t;\xi,\tau)=\int_{-\infty}^{\infty}\mathrm{d}x'\delta(x'-\xi)\frac{1}{2a\sqrt{\pi(t-\tau)}}\mathrm{e}^{-\frac{(x-x')^2}{4a^2(t-\tau)}}=\frac{1}{2a\sqrt{\pi(t-\tau)}}\mathrm{e}^{-\frac{(x-\xi)^2}{4a^2(t-\tau)}}$$

若有一列广义函数 $\{f_n\}$，对于基本函数空间内任一给定的元素 $\varphi(x)$，当 $n\to\infty$，有

$$\lim_{n\to\infty}<f_n-f,\varphi>=0$$

则称 $\{f_n\}$ 的极限是 f（或弱收敛于 f）。

这个定义可以帮助我们用正则型广义函数逼近得到非正则型广义函数。

对于正则型广义函数列，如果满足

$\lim\limits_{\varepsilon\to\varepsilon_0}f_\varepsilon(x)=\begin{cases}\infty & x=0\\ 0 & x\neq 0\end{cases}$ 且任意给定的区域 $(-a,a)$ 内积分，有 $\int_{-a}^{a}f_\varepsilon(x)dx=1$。

我们有 $\lim\limits_{\varepsilon\to\varepsilon_0}f_\varepsilon(x)=\delta(x)$

$$t>\tau\quad \int_{-\infty}^{\infty}G(x,t;\xi,\tau)=\int_{-\infty}^{\infty}\frac{1}{2a\sqrt{\pi(t-\tau)}}\mathrm{e}^{-\frac{(x-\xi)^2}{4a^2(t-\tau)}}d(x-\xi)=1$$

$$x\neq\xi\quad \lim_{t\to\tau}G(x,t;\xi,\tau)=\lim_{t\to\tau}\frac{1}{2a\sqrt{\pi(t-\tau)}}\mathrm{e}^{-\frac{(x-\xi)^2}{4a^2(t-\tau)}}=0$$

$$\lim_{t\to\tau}G(x,t;\xi,\tau)=\delta(\mathbf{r}-\mathbf{r}_0)$$

$$G\big|_{t=\tau}=\delta(\mathbf{r}-\mathbf{r}_0)$$

一般形式的无解有热源输运问题

$$\begin{cases} u_t = a^2 u_{xx} + f\left(x,t\right), \quad t>0 \\ u\big|_{t=0} = \varphi(x) \end{cases}$$

$$\begin{cases} v_t = a^2 v_{xx}, \quad t>0 \\ u\big|_{t=0} = \varphi(x) \end{cases} \qquad \begin{cases} \omega_t = a^2 \omega_{xx} + f\left(x,t\right), \quad t>0 \\ \omega\big|_{t=0} = 0 \end{cases}$$

$$G\left(x,t;\xi,\tau\right) = \frac{1}{2a\sqrt{\pi\left(t-\tau\right)}} \mathrm{e}^{-\frac{(x-\xi)^2}{4a^2(t-\tau)}} \qquad \omega\left(x,t\right) = \int_0^t \mathrm{d}\tau \int_{-\infty}^{\infty} \mathrm{d}\xi f\left(\xi,\tau\right) G\left(x,t;\xi,\tau\right)$$

$$v(x,t) = \int_{-\infty}^{\infty} \frac{\varphi\left(\xi\right)}{2a\sqrt{\pi t}} e^{-(x-\xi)^2/4a^2 t} d\xi = \int_{-\infty}^{\infty} G(x,t;\xi,0)\varphi\left(\xi\right) d\xi$$

$$u(x,t) = v + \omega = \int_{-\infty}^{\infty} G(x,t;\xi,0)\varphi\left(\xi\right) d\xi + \int_0^t \mathrm{d}\tau \int_{-\infty}^{\infty} \mathrm{d}\xi f\left(\xi,\tau\right) G\left(x,t;\xi,\tau\right)$$

半无界输运问题P320,例四

$$\begin{cases} u_t = a^2 u_{xx} + f\left(x,t\right), \quad x > 0 \\ u\big|_{t=0} = 0 \\ u\big|_{x=0} = 0 \end{cases}$$

半无界输运问题

$$
\begin{cases}
u_t = a^2 u_{xx} + f(x,t), \quad x > 0 \\
u\big|_{t=0} = 0 \\
u\big|_{x=0} = 0
\end{cases}
$$

$$
\begin{cases}
u_t = a^2 u_{xx} + \begin{cases} f(x,t) & x > 0 \\ -f(-x,t) & x < 0 \end{cases} \\
u\big|_{t=0} = 0
\end{cases}
$$

半无界输运问题

$$\begin{cases} u_t = a^2 u_{xx} + \begin{cases} f(x,t) & x>0 \\ -f(-x,t) & x<0 \end{cases} \\ u|_{t=0} = 0 \end{cases}$$

$$G(x,t;\xi,\tau) = \frac{1}{2a\sqrt{\pi(t-\tau)}} \mathrm{e}^{-\frac{(x-\xi)^2}{4a^2(t-\tau)}}$$

$$u(x,t) = v + \omega = \int_{-\infty}^{\infty} G(x,t;\xi,0)\varphi(\xi)d\xi + \int_0^t \mathrm{d}\tau \int_{-\infty}^{\infty} \mathrm{d}\xi f(\xi,\tau) G(x,t;\xi,\tau)$$

半无界输运问题P320,例五

$$\begin{cases} u_t = a^2 u_{xx} + f(x,t), \quad x>0 \\ u\,|_{t=0} = 0 \\ u_x\,|_{x=0} = 0 \end{cases}$$

半无界输运问题

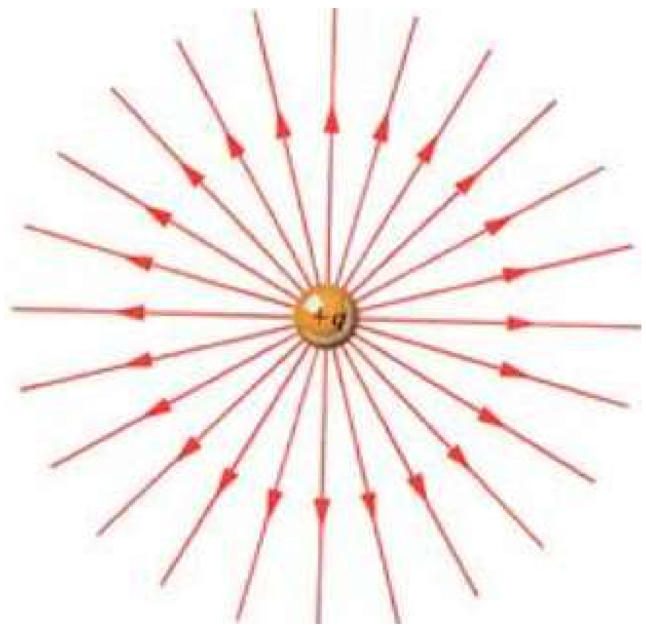
$$\begin{cases} u_t = a^2 u_{xx} + f(x,t), \quad x > 0 \\ u|_{t=0} = 0 \\ u|_{x=0} = 0 \end{cases}$$

$$\begin{cases} u_t = a^2 u_{xx} + \begin{cases} f(x,t) & x > 0 \\ f(-x,t) & x < 0 \end{cases} \\ u|_{t=0} = 0 \end{cases}$$

泊松方程的格林函数

空间中放置一个电量为+q的点电荷 $\rho(r)=q\delta\left(\mathbf{r}-\mathbf{r}_0\right)$

在空间中产生的电场强度为



三维空间

$$\mathbf{E}=-\frac{q(\mathbf{r}-\mathbf{r}_0)}{4\pi\varepsilon|\mathbf{r}-\mathbf{r}_0|^3}$$

二维空间

$$\mathbf{E}=-\frac{q(\mathbf{r}-\mathbf{r}_0)}{2\pi\varepsilon|\mathbf{r}-\mathbf{r}_0|}$$

$$\mathbf{E}=-\nabla\Phi$$

$$\Phi=\frac{q}{4\pi\varepsilon|\mathbf{r}-\mathbf{r}_0|}$$

$$\Phi=-\frac{q}{2\pi\varepsilon}\ln|\mathbf{r}-\mathbf{r}_0|$$

点源格林函数与点电荷

$$\mathbf{E}=-\frac{q(\mathbf{r}-\mathbf{r}_0)}{4\pi\varepsilon|\mathbf{r}-\mathbf{r}_0|^3}$$

$$\nabla\cdot\frac{\mathbf{e}_\mathbf{r}}{\mathbf{r}^2}=4\pi\delta(\mathbf{r})$$

$$\nabla\frac{\mathbf{r}-\mathbf{r}_0}{|\mathbf{r}-\mathbf{r}_0|^3}=4\pi\delta(\mathbf{r}-\mathbf{r}_0)$$

$$\nabla\bullet\mathbf{E}=\nabla\bullet\frac{q(\mathbf{r}-\mathbf{r}_0)}{4\pi\varepsilon|\mathbf{r}-\mathbf{r}_0|^3}=\frac{q}{\varepsilon}\delta(\mathbf{r}-\mathbf{r}_0)$$

$$\mathbf{E}=-\nabla\Phi$$

$$\left.\begin{array}{l}\end{array}\right\} \quad \nabla^2\Phi=-\frac{q}{\varepsilon}\delta(\mathbf{r}-\mathbf{r}_0)$$

$$\nabla^2 G_0(\vec{r},\vec{\xi})=\delta(\vec{r}-\vec{\xi})$$

三维空间格林函数

$$\Phi=\frac{q}{4\pi\varepsilon|\mathbf{r}-\mathbf{r}_0|} \quad G_0(\vec{r},\vec{\xi})=-\frac{1}{4\pi|\vec{r}-\vec{\xi}|}$$

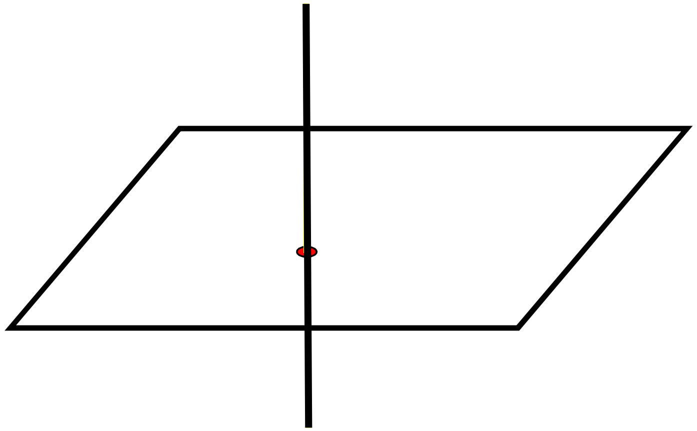
二维空间格林函数

$$\Phi=\frac{q}{4\pi\varepsilon}\ln|\mathbf{r}-\mathbf{r}_0| \quad G(\vec{r},\vec{\xi})=-\frac{1}{2\pi}\ln|\vec{r}-\vec{\xi}|$$

三维无界空间中的长导线产生的电势分布

$$u = \iiint \rho(\mathbf{r}_0) G(\mathbf{r};\mathbf{r}_0) \mathrm{d}^3 \mathbf{r}_0$$

$$\rho(x, y, z) = \sigma\delta(x)\delta(y)$$



$$G_0(\vec{r}, \vec{\xi}) = -\frac{1}{4\pi\left|\vec{r} - \vec{\xi}\right|}$$

$$\Phi(x,y,z) = \frac{\sigma}{4\pi\varepsilon}\int_{-\infty}^{\infty} dx'\int_{-\infty}^{\infty} dy'\int_{-\infty}^{\infty} dz' \frac{\delta(x')\delta(y')}{\left[(x-x')^2+(y-y')^2+(z-z')^2\right]^{1/2}}$$

$$= \frac{\sigma}{4\pi\varepsilon}\int_{-\infty}^{\infty} dz' \frac{1}{\left[x^2+y^2+(z-z')^2\right]^{1/2}}$$

$$\nabla^2\Phi = -\frac{1}{\varepsilon}\rho(x, y, z)$$

三维无界空间中的长导线产生的电势分布

$$\Phi(x,y,z)=-\frac{1}{4\pi}\int_{-\infty}^{\infty}dz'\frac{1}{\left[x^2+y^2+(z-z')^2\right]^{1/2}}$$

$$\Phi(x,y,z)-\Phi(x_0,y_0,z_0)=\frac{\sigma}{4\pi\varepsilon}\int_{-\infty}^{\infty}dz'\frac{1}{\left[x^2+y^2+(z-z')^2\right]^{1/2}}-\frac{\sigma}{4\pi\varepsilon}\int_{-\infty}^{\infty}dz'\frac{1}{\left[{x_0}^2+{y_0}^2+(z_0-z')^2\right]^{1/2}}$$

$$=\frac{\sigma}{4\pi\varepsilon}\int_{-\infty}^{\infty}dz'\frac{1}{\left[x^2+y^2+z'^2\right]^{1/2}}-\frac{\sigma}{4\pi\varepsilon}\int_{-\infty}^{\infty}dz'\frac{1}{\left[{x_0}^2+{y_0}^2+z'^2\right]^{1/2}}$$

$$=\lim_{M\to\infty}\ln 2[z'+\sqrt{{x_0}^2+{y_0}^2+z'^2}]-\ln 2[z'+\sqrt{x^2+y^2+z'^2}]\Big|_0^M\frac{2\sigma}{4\pi\varepsilon}$$

$$=\lim_{M\to\infty}\ln\frac{z'+\sqrt{{x_0}^2+{y_0}^2+z'^2}}{z'+\sqrt{x^2+y^2+z'^2}}\Big|_0^M\frac{2\sigma}{4\pi\varepsilon}$$

$$=\frac{\sigma}{2\pi\varepsilon}\ln\frac{\sqrt{{x_0}^2+{y_0}^2}}{\sqrt{x^2+y^2}}=-\frac{\sigma}{2\pi\varepsilon}\ln R+\frac{\sigma}{2\pi\varepsilon}\ln R_0$$

$$\Phi=-\frac{q}{2\pi\varepsilon}\ln\left|\mathbf{r}-\mathbf{r_0}\right|$$