前情提要:

我们将正则型广义函数的运算推广到非正则型广义函数的运算

（a）**线性运算** $\langle \alpha_1 f_1 + \alpha_2 f_2, \phi \rangle = \alpha_1 \langle f_1, \phi \rangle + \alpha_2 \langle f_2, \phi \rangle$

（b）**乘法运算** $\langle \alpha(x) f(x), \phi(x) \rangle \equiv \langle f(x), \alpha(x)\phi(x) \rangle$,其中 $\alpha(x) \in C^\infty$ **是无穷可微函数。** $x\delta(x) = 0$

（c）**复合函数运算**

$$\langle f[T(x)], \phi(x) \rangle = \left\langle f(x), \left| \frac{\mathrm{d}T^{-1}(x)}{\mathrm{d}x} \right| \phi[T^{-1}(x)] \right\rangle = \left\langle f(x), \frac{1}{|T'(x)|} \phi[T^{-1}(x)] \right\rangle$$

$$\delta(-x) = \delta(x)$$

(d) **若** $y=T(x)$**连续，且只有单根** x_n**，并在** x_n **处可导，则定义** $\delta[T(x)] = \sum_n \frac{\delta(x - x_n)}{|T'(x_n)|}$

$$\delta(|x| - 1) = \delta(x+1) + \delta(x-1)$$

$$\delta(\sin x) = \sum_{n=-\infty}^{\infty} \delta(x - n\pi)$$

(e) **广义函数的微商和积分** $\left\langle \frac{\mathrm{d}f}{\mathrm{d}x}, \phi \right\rangle \equiv -\left\langle f, \frac{\mathrm{d}\phi}{\mathrm{d}x} \right\rangle$

若 $\left\langle \frac{\mathrm{d}g}{\mathrm{d}x}, \varphi \right\rangle = \langle f, \varphi \rangle$ 则 $\int f(x)\,\mathrm{d}x \equiv g(x) + C$

$$H(x) = \int_{-\infty}^{x} \delta(y)dy$$ **（非常重要）**

（f）**卷积** $< f * g, \varphi > = < f(x), < g(y), \varphi(x+y) >>$

$$\begin{aligned}
&\int dx \int dy [f(y) g(x-y)]\, \varphi(x) \\
&= \int dy f(y) \int dx g(x-y) \varphi(x) \\
&= < f(y), < g(x-y), \varphi(x) >> \\
&= < f(y), < g(x), \varphi(x+y) >> \\
&= < f(y), \varphi * g(y) >
\end{aligned}$$

(g) **多元广义函数**

δ/微元、球坐标$\dfrac{1}{r^2\sin\vartheta}$，柱坐标$\dfrac{1}{r}$，极坐标$\dfrac{1}{r}$

电场的散度可以得到电荷的密度分布$\nabla\cdot\dfrac{\mathbf{e_r}}{\mathbf{r}^2}=4\pi\delta(\mathbf{r})$

第三课时 广义函数的傅里叶变换

$$\widehat{f}(\mathbf{q})=F[f(\mathbf{x})]=\int_{\mathbf{R}^n}\frac{1}{(\sqrt{2\pi})^n}\mathbf{dx}\,f(\mathbf{x})e^{-i\mathbf{qx}}$$

$$f(\mathbf{x})=F^{-1}[\widehat{f}(\mathbf{q})]=\int_{\mathbf{R}^n}\frac{1}{(\sqrt{2\pi})^n}\mathbf{dq}\,\widehat{f}(\mathbf{q})e^{i\mathbf{qx}}$$

$$<F[f],\varphi>=<f,F[\varphi]>$$

$$<F^{-1}[\widehat{f}],\varphi>=<\widehat{f},F^{-1}[\varphi]>$$

我们上节课展示了，当广义函数为正则型，上述两个定义等价。

求广义函数$\delta(x-x_0)$的傅里叶变换

$$\begin{aligned}&<F[\delta(x-x_0)],\varphi>=<\delta(x-x_0),F[\varphi]>\\&=\widehat{\varphi}(x_0)\\&=\frac{1}{\sqrt{2\pi}}\int dye^{-ix_0y}\varphi(y)\\&=<\frac{1}{\sqrt{2\pi}}e^{-ix_0y},\varphi(y)>\end{aligned}$$

$$F[\delta(x-x_0)](y)=\frac{1}{\sqrt{2\pi}}e^{-ix_0y}$$

因此我们得到，$\dfrac{1}{\sqrt{2\pi}}\displaystyle\int_{-\infty}^{\infty}dx\delta(x-x_0)e^{-ixy}=\dfrac{1}{\sqrt{2\pi}}e^{-ix_0y}$

对于特殊的广义函数 f 不必采用定义式计算，可以直接采用

当$x_0=0$，得到$F[\delta(x)]=\dfrac{1}{\sqrt{2\pi}}$

由逆变换的定义，$F^{-1}[\dfrac{1}{\sqrt{2\pi}}]=\delta(x)$

得到$\dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty}dye^{ixy}=\delta(x)$（非常重要）。

用广义函数弱收敛的性质证明该公式，见教材 P86

该公式可以验证傅里叶变换和拉普拉斯变换可逆。也就是证明

$\widehat{f}=F[F^{-1}[\widehat{f}]]$。

$$F[F^{-1}[\hat{f}]]$$
$$=\int_{\mathbf{R}^n}\frac{1}{(\sqrt{2\pi})^n}\mathbf{dx}\int_{\mathbf{R}^n}\frac{1}{(\sqrt{2\pi})^n}\mathbf{dq}'\hat{f}(\mathbf{q}')e^{i\mathbf{q}'x}e^{-i\mathbf{qx}}$$
$$=\frac{1}{2\pi^n}\int_{\mathbf{R}^n}\mathbf{dq}'\hat{f}(\mathbf{q}')\int\mathbf{dx}e^{i(\mathbf{q}'-\mathbf{q})\mathbf{x}}$$
$$=\int_{\mathbf{R}^n}\mathbf{dq}'\hat{f}(\mathbf{q}')\delta(\mathbf{q}'-\mathbf{q})$$
$$=\hat{f}(\mathbf{q})$$

这个证明告诉我们为什么一定有系数$\frac{1}{2\pi}$。

同理也可验证拉普拉斯变换，拉普拉斯变换公式

$$\overline{f}(p=\sigma+i\omega)=L[f]=\int_0^\infty dtf(t)e^{-pt}$$

$$f(t)=L^{-1}[f]=\frac{1}{2\pi}\int_{-\infty}^{\infty}d\omega\overline{f}(p)e^{pt}$$

$$f(t)=L[L^{-1}[f]]=\frac{1}{2\pi}\int_{-\infty}^{\infty}d\omega\int_0^\infty dt'f(t')e^{-(\sigma+i\omega)t'}e^{(\sigma+i\omega)t}$$
$$=\frac{1}{2\pi}\int_0^\infty dt'f(t')e^{-\sigma t'}e^{\sigma t}\int_{-\infty}^{\infty}d\omega e^{i\omega(t-t')}$$
$$=\int_0^\infty dt'f(t')e^{-\sigma(t'-t)}\delta(t-t')$$

拉普拉斯变换关心的是初值问题，所以$t<0$， $f(t)=0$

$$=\int_{-\infty}^{\infty}dt'f(t')e^{-\sigma(t'-t)}\delta(t-t')$$
$$=f(t)$$

积分变换求解，初值问题

$$\frac{\partial^2u(t)}{\partial t^2}=-\omega^2u(t)+f_n(t)，\quad f_n(t)=\begin{cases}ne^{-nt} & t>0\\ 0 & t\le 0\end{cases}，\quad \omega=\sqrt{k/m}。$$

采用拉普拉斯变换求解该方程，

$$
\begin{aligned}
&\int_0^\infty e^{-pt}u''dt \\
&= \int_0^\infty e^{-pt}du' = e^{-pt}u'|_0^\infty + \int_0^\infty pe^{-pt}u'dt \\
&= \int_0^\infty pe^{-pt}du = pe^{-pt}u\,|_0^\infty + \int_0^\infty p^2e^{-pt}udt \\
&= p^2\overline{u}(p)
\end{aligned}
$$

$$
\begin{aligned}
&\int_0^\infty e^{-pt}ne^{-nt}dt = n\int_0^\infty e^{-(p+n)t}dt \\
&= -n\frac{1}{p+n}e^{-(p+n)t}\,|_0^\infty = \frac{n}{p+n}
\end{aligned}
$$

$$(p^2+\omega^2)\overline{u} = \frac{n}{p+n}$$

$$\frac{\partial^2 u(t)}{\partial t^2} = -\omega^2 u(t) + \delta(t)$$

注意$\int_{x_0}^{a} \delta(x-x_0)\varphi(x)dx = \varphi(x_0)$

对于黎曼积分，在某一点处的积分，值恒为零，因此可以随意拆分。

广义函数不可以随意拆分

$$\int_{-a}^{a} \delta(x-x_0)\varphi(x)dx \neq \int_{x_0}^{a} \delta(x-x_0)\varphi(x)dx + \int_{-a}^{x_0} \delta(x-x_0)\varphi(x)dx$$

$$
\begin{aligned}
&(p^2+\omega^2)\overline{u}(p) = 1 \\
&\overline{u}(p) = \frac{\omega}{(p^2+\omega^2)}\frac{1}{\omega}
\end{aligned}
$$

因此δ的确是可以认为是$f_n(t)$在$n \to \infty$的极限，可以反映出物理量的瞬时变化

广义函数傅里叶变换的性质

1. 线性$F[a_1f_1 + a_2f_2] = a_1F[f_1] + a_2F[f_2]$

2. 微分$F[f'(x)](q) = iqF[f(x)](q) = iq\widehat{f}(q)$

$$< F[f'],\varphi > = < f',F[\varphi] >$$
$$= - < f,F'[\varphi] >$$
$$= -\int dx\, f(x)\frac{1}{\sqrt{2\pi}}\int dq \text{ -}iqe^{-iqx}\varphi(q)$$
$$= -\int dq \text{ -}iq\varphi(q)\int dx\, f(x)e^{-iqx}$$
$$= < \mathrm{iq}F(f),\varphi >$$

3．平移 $F[f(x-a)] = e^{-iqa}F[f(x)]$

$$< F[f(x-a)],\varphi > = < f(x-a),F[\varphi] >$$
$$= < f(x),F[\varphi](x+a) >$$
$$= \int dx\, f(x)\frac{1}{\sqrt{2\pi}}\int dq \text{ -}iqe^{-iq(x+a)}\varphi(q)$$
$$= \int dq \text{ -}iq\varphi(q)e^{-iqa}\frac{1}{\sqrt{2\pi}}\int dx\, f(x)e^{-iqx}$$
$$= < e^{-iqa}f,\varphi >$$

4．卷积 F[f*g]=F[f]F[g]

$$< F[f*g],\varphi > = < f*g,F[\varphi] >$$
$$= < f(x), < g(y),F[\varphi](x+y) >>$$
$$< f(x), < g(y),\frac{1}{\sqrt{2\pi}}\int dqe^{-iq(x+y)}\varphi(q) >>$$
$$= < f(x),\frac{1}{\sqrt{2\pi}}\int dq\varphi(q)e^{-iqx}\int dye^{-iqy}g(y) >$$
$$= < f(x),\int dq\hat{g}(q)e^{-iqx}\varphi(q) >$$
$$= \int dq\hat{g}(q)\varphi(q)\int dxe^{-iqx}f(x)$$
$$= \sqrt{2\pi}\int dq\hat{g}(q)\hat{f}(q)\varphi(q)$$
$$= \sqrt{2\pi} < \hat{g}(q)\hat{f}(q),\varphi(q) >$$

5．定义积分 $\mathrm{F}[\int_{-\infty}^{x} f(y)dy] = \frac{\hat{f}(q)}{iq} + \pi\hat{f}(0)\delta(q)$

例．求广义函数 $H(x) = \begin{cases} 0 & x < 0 \\ 1 & x > 0 \end{cases}$ 的广义傅里叶变换

$$H(x) = \int_{-\infty}^{x} \delta(y)dy$$

由定义 5，得到

$F[H(x)]=\frac{1}{\sqrt{2\pi}}[\frac{1}{iq}+\pi\delta(q)]$，从而$\int_0^{\infty}dxe^{-iqx}=\frac{1}{iq}+\pi\delta(q)$ **（非常重要）**

参见书 P88，用广义函数弱收敛解释$H(x)$傅里叶变换为$\frac{1}{\sqrt{2\pi}}[\frac{1}{iq}+\pi\delta(q)]$。