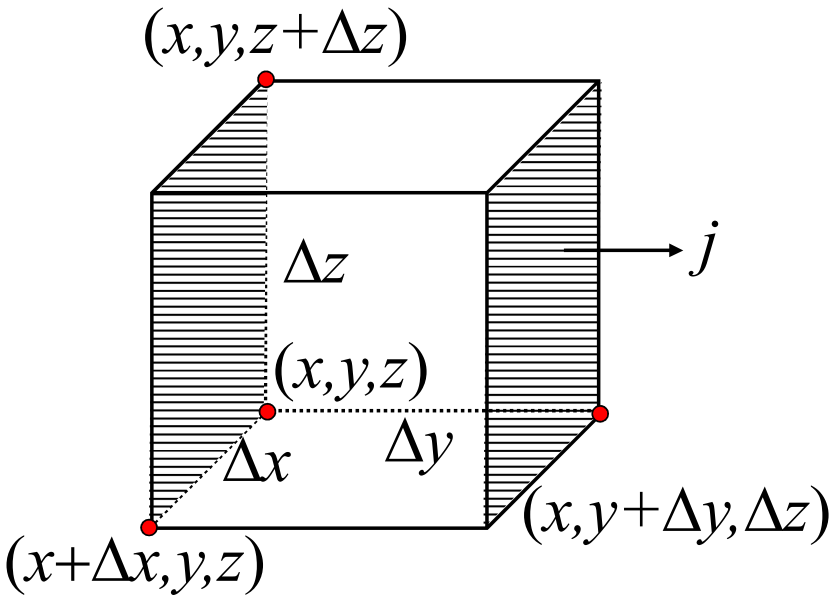
3. 扩散运动

物理图像：粒子在空间的运动



空间离散化，每一个粒子都在一个正方体中。不同时间从一个盒子转移到另一个盒子。

$$\rho_i(t) = N_i / \Delta V$$

第i个盒子内粒子的密度

质量能量守恒---粒子不能凭空产生也不能凭空消灭，只能从一个格子转移到另一个格子

$$\rho_i(t+\Delta t)-\rho_i(t)=(n_{x\uparrow}+n_{x\downarrow}+n_{y\uparrow}+n_{y\downarrow}+n_{z\uparrow}+n_{z\downarrow})/\Delta V$$

第i个正方体格子内，粒子数量的变化等于6个方向进来的粒子数减去出去的粒子数

$$\frac{\rho_i(t+\Delta t)-\rho_i(t)}{\Delta t}=\frac{n_{x\uparrow}}{\Delta t}\frac{1}{\Delta V}+\cdots$$

$$\mathrm{J}=\frac{n}{\Delta S\Delta t}$$

单位时间**正向**流过单位面积的粒子数

$$\frac{\rho_i(t+\Delta t)-\rho_i(t)}{\Delta t}=\frac{1}{\Delta t}\frac{n_{x\uparrow}}{\Delta S}\frac{\Delta S}{\Delta V}+\cdots$$

$$\frac{\rho_i(t+\Delta t)-\rho_i(t)}{\Delta t}=\frac{(J_{x\uparrow}-J_{x\downarrow})}{\Delta x}+\cdots$$

扩散运动

连续性方程 $\rho_t(x,y,z,t)=\frac{\partial J(x,y,z,t)}{\partial x}+\cdots=\nabla\bullet J(x,y,z,t)$

Fick's law $J=-k\nabla\bullet\rho$

浓度高的区域向浓度低的区域流动

$$\rho_t=-k\nabla\nabla\bullet\rho$$

数学物理方程的分类

波动方程wave equation：　（双曲型方程）

描述现象：声波、电磁波等波动

$$u_{tt} - a^2\nabla^2 u = 0$$

输运方程Diffusion Equation：　（抛物型方程）

描述现象：热扩散、物质扩散等扩散过程

$$u_t - a^2\nabla^2 u = 0$$

稳恒状态方程：　（椭圆型方程）

描述现象：电势、稳定温度场分布等与时间无关的稳定场。

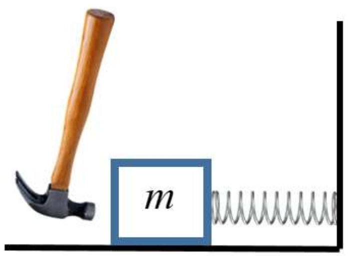
$$\nabla^2 u = 0$$

$$\rho u_{tt} = Tu_{xx} - Ru_t \Longrightarrow Tu_{xx} - Ru_t = 0$$

定解条件

- 泛定方程和定解条件:
  - 泛定方程:
    描述了系统内部具有代表性（一般性）的点处的运动规律的偏微分方程
  - 定解条件:
    - 边界条件：描述外界影响
    - 衔接条件：描述内部特殊点的运动规律
    - 初始条件：描述历史的作用

初始条件



$u(t=0)$ t=0,小木块静止在平面上

$u'(t=0)$ t=0,小木块速度为0

$$\frac{\partial^2 u(t)}{\partial t^2} = -\omega^2 u(t) + f_n(t)$$

初始条件应当给出 $u\left(x, y, z, t_0\right)$

整个系统在t0时刻的状态

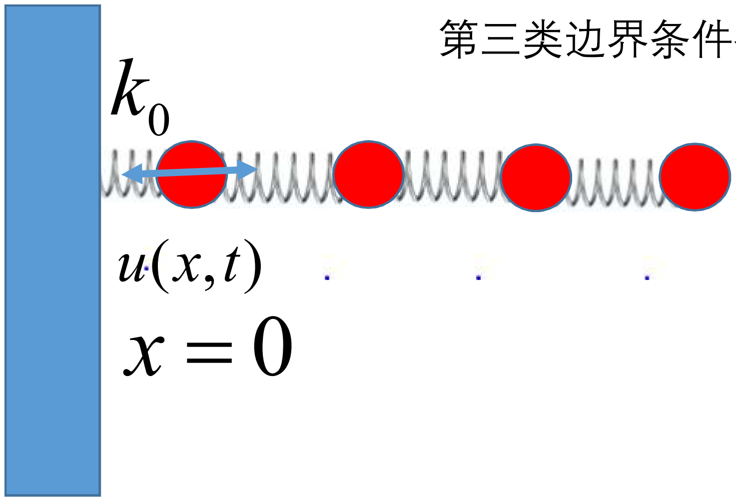
边界条件

第一类：表征量在边界处的值 $u(x,t)|_{x\in S}$ Dirichlet boundary condition

第二类：其法向导数在边界处的值 $u_x(x,t)|_{x\in S}$ Neumann boundary condition

第三类：前两者的线性组合 $u(x,t)+Au_x(x,t)|_{x\in S}$

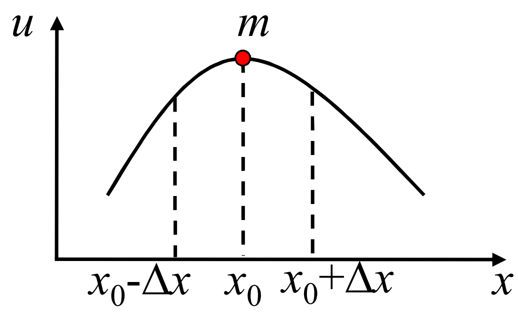
第三类边界条件举例



$$-k_0u(0,t)+k[u(\Delta x,t)-u(0,t)]=0$$

衔接条件

外界原因，体系在研究区域内出现越变点，使得该点处不满足泛定方程。



$$\left(x_0-\Delta x, x_0+\Delta x\right)$$

弦的横向运动,x0处受外力f(t)，使得该点静止。

$$u\left(x_0+0,t\right)=u\left(x_0-0,t\right)$$

$$mu_{tt}\left(x_0,t\right)=T\left[u_x\left(x_0+0,t\right)-u_x\left(x_0-0,t\right)\right]-f(t)=0$$

弦的自由横振动---行波法

$$\begin{cases} u_{tt} = a^2 u_{xx}, & |x| < \infty, t > 0 \\ u\big|_{t=0} = \phi(x), \quad u_t\big|_{t=0} = \psi(x), & |x| < \infty \end{cases}$$

弦的自由横振动---行波法

$$(\partial_t - a\partial_x)(\partial_t + a\partial_x)u = 0$$

做变换得到:

$$\partial_\xi = \partial_t + a\partial_x$$
$$\partial_\eta = \partial_t - a\partial_x$$

$$\partial_\eta \partial_\xi u(\xi,\eta) = 0$$

$$t = ?$$

$$x = ?$$

弦的自由横振动---行波法

$$\begin{bmatrix} \xi \\ \eta \end{bmatrix} = T \begin{bmatrix} t \\ x \end{bmatrix} = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} \begin{bmatrix} t \\ x \end{bmatrix}$$

$$\left.\begin{aligned}
\partial_t &= \frac{\partial}{\partial \xi}\frac{\partial \xi}{\partial t} + \frac{\partial}{\partial \eta}\frac{\partial \eta}{\partial t} = T_{11}\frac{\partial}{\partial \xi} + T_{12}\frac{\partial}{\partial \eta} \\
\partial_x &= \frac{\partial}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial}{\partial \eta}\frac{\partial \eta}{\partial x} = T_{12}\frac{\partial}{\partial \xi} + T_{22}\frac{\partial}{\partial \eta}
\end{aligned}\right\} \quad (\partial_t - a\partial_x)(\partial_t + a\partial_x)u = 0$$

弦的自由横振动---行波法

$$
\begin{aligned}
&(\partial_t - a\partial_x)(\partial_t + a\partial_x)u \\
&= T_{11}^2\left(\frac{\partial}{\partial\xi}\right)^2 + 2T_{12}T_{11}\frac{\partial}{\partial\eta}\frac{\partial}{\partial\xi} + T_{12}^2\left(\frac{\partial}{\partial\eta}\right)^2 - a^2T_{21}^2\left(\frac{\partial}{\partial\xi}\right)^2 - 2a^2T_{21}T_{22}\frac{\partial}{\partial\eta}\frac{\partial}{\partial\xi} - a^2T_{22}^2\left(\frac{\partial}{\partial\eta}\right)^2 \\
&= (T_{11}^2 - a^2T_{21}^2)\left(\frac{\partial}{\partial\xi}\right)^2 + (2T_{12}T_{11} - 2a^2T_{21}T_{22})\frac{\partial}{\partial\eta}\frac{\partial}{\partial\xi} + (T_{12}^2 - a^2T_{22}^2)\left(\frac{\partial}{\partial\eta}\right)^2
\end{aligned}
$$

$$
\frac{\partial}{\partial\eta}\frac{\partial}{\partial\xi}u = 0
$$

弦的自由横振动---行波法

$$\partial_\xi = \partial_t + a\partial_x$$

$$\partial_\eta = \partial_t - a\partial_x$$

$$\partial_\xi = \frac{\partial}{\partial t}\frac{\partial t}{\partial \xi} + \frac{\partial}{\partial x}\frac{\partial x}{\partial \xi} = \partial_t + a\partial_x$$

$$\partial_\eta = \frac{\partial}{\partial t}\frac{\partial t}{\partial \eta} + \frac{\partial}{\partial x}\frac{\partial x}{\partial \eta} = \partial_t - a\partial_x$$

$$\frac{\partial t}{\partial \xi} = 1 \quad \frac{\partial x}{\partial \xi} = a \quad \frac{\partial t}{\partial \eta} = 1 \quad \frac{\partial x}{\partial \eta} = -a$$

$$\begin{cases} t = \xi + \eta \\ x = a(\xi - \eta) \end{cases}$$

弦的自由横振动---行波法

采用书上的变换：

$$\begin{cases} \xi = x + at \\ \eta = x - at \end{cases}$$

$$\frac{\partial}{\partial x} = \frac{\partial \xi}{\partial x}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial x}\frac{\partial}{\partial \eta} = \frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}$$

$$\frac{\partial^2}{\partial x^2} = \left(\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}\right)^2 = \frac{\partial^2}{\partial \xi^2} + 2\frac{\partial^2}{\partial \xi \partial \eta} + \frac{\partial^2}{\partial \eta^2}$$

$$\frac{\partial^2}{\partial t^2} = a^2\left(\frac{\partial^2}{\partial \xi^2} - 2\frac{\partial^2}{\partial \xi \partial \eta} + \frac{\partial^2}{\partial \eta^2}\right)$$

弦的自由横振动---行波法

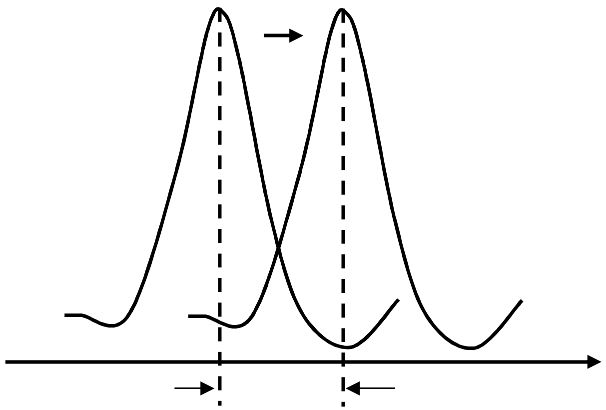
则泛定方程变为

$$u_{tt}-a^2u_{xx}=-4a^2u_{\xi\eta}=0$$

求积分，得 $u=\int f_1(\xi)\mathrm{d}\xi+g(\eta)=f(\xi)+g(\eta)$

弦振动方程的通解

$$u=f(x+at)+g(x-at)$$



无界弦的自由横振动---行波法

$$\begin{cases} f(x)+g(x)=\phi(x) \\ \left[f_t(x+at)+g_t(x-at)\right]\Big|_{t=0}=\psi(x) \end{cases}$$

代入初始条件

$$\begin{cases} f(x)+g(x)=\phi(x) \\ af'(x)-ag'(x)=\psi(x) \end{cases}$$

定解问题的解（d’Alembert公式）

$$u(x,t)=f(x+at)+g(x-at)=\frac{\phi(x+at)+\phi(x-at)}{2}+\frac{\Psi(x+at)+\Psi(x-at)}{2}$$

$$=\frac{\phi(x+at)+\phi(x-at)}{2}+\frac{1}{2a}\int_{x-at}^{x+at}\psi(\xi)\,\mathrm{d}\xi$$

多元积分坐标变换

$$\begin{cases} x = r\cos\theta\sin\varphi \\ y = r\sin\theta\sin\varphi \\ z = r\cos\varphi \end{cases}$$

$$\boxed{\begin{array}{l} r \geq 0 \\ 0 \leq \theta \leq 2\pi \\ 0 \leq \varphi \leq \pi \end{array}}$$

$$\int f(x, y, z)dxdydz$$

$$= \int f(r\cos\theta\cos\varphi, r\sin\theta\cos\varphi, r\sin\varphi)J(r, \theta, \varphi)drd\theta d\varphi$$

球柱积分元

$$\begin{cases} x = r\cos\theta\sin\varphi \\ y = r\sin\theta\sin\varphi \\ z = r\cos\varphi \end{cases}$$

$$J = \begin{vmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \theta} \\ \dfrac{\partial x}{\partial r} & \dfrac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r$$

$$J = \begin{vmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \theta} & \dfrac{\partial x}{\partial \varphi} \\ \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial \theta} & \dfrac{\partial y}{\partial \varphi} \\ \dfrac{\partial z}{\partial r} & \dfrac{\partial z}{\partial \theta} & \dfrac{\partial z}{\partial \varphi} \end{vmatrix} = \begin{vmatrix} \cos\theta\sin\varphi & -r\sin\theta\sin\varphi & r\cos\theta\cos\varphi \\ \sin\theta\sin\varphi & r\cos\theta\sin\varphi & r\sin\theta\cos\varphi \\ \cos\varphi & 0 & -r\sin\varphi \end{vmatrix} = r^2\sin\varphi$$

场算符

$$\nabla = \mathbf{e}_x \frac{\partial}{\partial x} + \mathbf{e}_y \frac{\partial}{\partial y} + \mathbf{e}_z \frac{\partial}{\partial z}, \quad \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

$$\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}$$

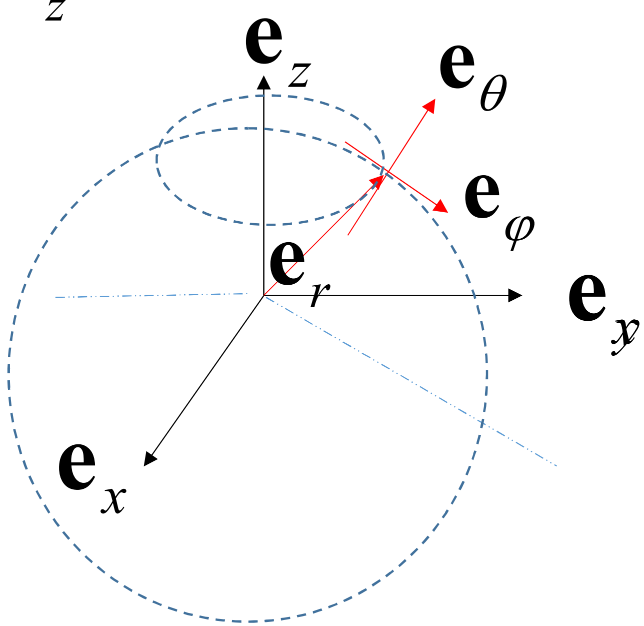
$$\begin{cases} x = r\cos\theta\sin\varphi \\ y = r\sin\theta\sin\varphi \\ z = r\cos\varphi \end{cases}$$

坐标变换

$$\mathbf{e}_r = \frac{\mathbf{r}}{r} = \cos\theta\sin\varphi\mathbf{e}_x + \sin\theta\sin\varphi\mathbf{e}_y + \cos\varphi\mathbf{e}_z$$

$$\mathbf{e}_\theta = -\sin\theta\mathbf{e}_x + \cos\theta\mathbf{e}_y$$

$$\mathbf{e}_\varphi = \cos\theta\cos\varphi\mathbf{e}_x + \sin\theta\cos\varphi\mathbf{e}_y - \sin\varphi\mathbf{e}_z$$



$$\frac{\partial}{\partial x}=\frac{\partial r}{\partial x}\frac{\partial}{\partial r}+\frac{\partial \theta}{\partial x}\frac{\partial}{\partial \theta}+\frac{\partial \varphi}{\partial x}\frac{\partial}{\partial \varphi}=\cos\theta\sin\varphi\frac{\partial}{\partial r}-r\sin\theta\sin\varphi\frac{\partial}{\partial \theta}+r\cos\varphi\frac{\partial}{\partial \varphi}$$

$$\frac{\partial}{\partial y}=\sin\theta\sin\varphi\frac{\partial}{\partial r}+r\cos\theta\sin\varphi\frac{\partial}{\partial \theta}+r\cos\varphi\frac{\partial}{\partial \varphi}$$

$$\frac{\partial}{\partial z}=\cos\varphi\frac{\partial}{\partial r}-r\sin\varphi\frac{\partial}{\partial \varphi}$$

$$\mathbf{e}_x=\cos\theta\sin\varphi\mathbf{e}_r-\sin\theta\mathbf{e}_\theta+\cos\theta\cos\varphi\mathbf{e}_\varphi$$

$$\mathbf{e}_y=\sin\theta\sin\varphi\mathbf{e}_r+\cos\theta\mathbf{e}_\theta+\sin\theta\cos\varphi\mathbf{e}_\varphi$$

$$\mathbf{e}_z=\cos\varphi\mathbf{e}_r-\sin\varphi\mathbf{e}_\varphi$$

http://mathworld.wolfram.com/SphericalCoordinates.html

$$\nabla=\mathbf{e}_x\frac{\partial}{\partial x}+\mathbf{e}_y\frac{\partial}{\partial y}+\mathbf{e}_z\frac{\partial}{\partial z}$$

场算符

$$\nabla = \mathbf{e}_\mathbf{r}\frac{\partial}{\partial r} + \frac{\mathbf{e}_\theta}{r}\frac{\partial}{\partial \theta} + \mathbf{e}_\mathbf{z}\frac{\partial}{\partial z}$$

$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}$$

$$\nabla = \mathbf{e}_\mathbf{r}\frac{\partial}{\partial r} + \frac{\mathbf{e}_\theta}{r\sin\varphi}\frac{\partial}{\partial \theta} + \frac{\mathbf{e}_\varphi}{r}\frac{\partial}{\partial \varphi}$$

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\varphi}\frac{\partial}{\partial \varphi}\left(\sin\varphi\frac{\partial}{\partial \varphi}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial \theta^2}$$

球面波

$$\begin{cases} u_{tt} = a^2 \nabla^2 u, \\ u\big|_{t=0} = \phi(r), \quad u_t\big|_{t=0} = \psi(r) \end{cases}$$

$$\begin{cases} u_{tt} = a^2 \dfrac{1}{r^2} \dfrac{\partial}{\partial r}\left(r^2 \dfrac{\partial u}{\partial r} \right), \qquad r \geq 0, t > 0 \\ u\big|_{t=0} = \phi(r), \quad u_t\big|_{t=0} = \psi(r) \end{cases}$$

球面波

$$v = ru \qquad \begin{cases} v_{tt} = a^2 v_{rr}, & r > 0, t > 0 \\ v\big|_{r=0} = 0, & t \ge 0 \\ v\big|_{t=0} = r\phi(r), \quad v_t\big|_{t=0} = r\psi(r) \end{cases}$$

通解为 $$v = f(r + at) + g(r - at)$$