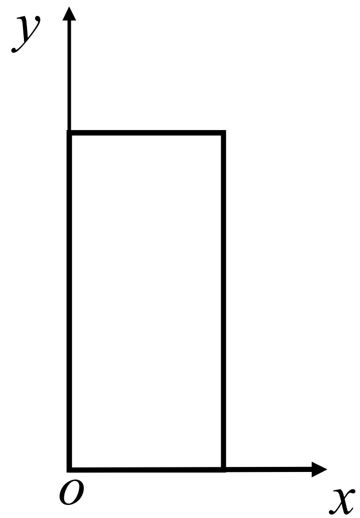
稳恒方程求解

$$\begin{cases} u_{xx}+u_{yy}=0; \quad 0<x<a,\ 0<y<b \\ u\big|_{x=0}=u_0, \quad u\big|_{x=a}=u_0; \quad 0<y<b \\ u\big|_{y=0}=u_0, \quad u\big|_{y=b}=U; \quad 0<x<a \end{cases}$$



二阶常系数齐次常微分方程

$$X''(x)+\lambda X(x)=0$$

特征方程 $\eta^2+\lambda=0$

通解 $X=Ae^{\eta_1 x}+Be^{\eta_2 x}$

傅里叶变换求解

特征方程 $$\left(\lambda - k^2\right)F\left[X\left(x\right)\right] = 0$$

$$F[X(x)] = \begin{cases} A_{\pm} & \lambda = \pm k_0 \\ 0 & \lambda \neq \pm k_0 \end{cases} = A_+\delta(k + k_0) + A_-\delta(k - k_0)$$

$$X(x) = \int e^{ikx}\left[A_+\delta(k + k_0) + A_-\delta(k - k_0)\right]dx$$

$$= A_+ e^{-ik_0 x} + A_- e^{+ik_0 x} = \begin{pmatrix} A_+ & A_- \end{pmatrix}\begin{pmatrix} e^{-ik_0 x} \\ e^{+ik_0 x} \end{pmatrix}$$

通解是以 exp(ik0x)和exp(-k0x)为基矢的二维矢量。

傅里叶变换角度理解分离变量法

傅里叶级数变换

$$U(x,\omega_n) \sim \int u(x,t)\sin\omega_n t dt \qquad or \qquad \int u(x,t)\cos\omega_n t dt$$

傅里叶级数展开

$$u(x,t) = \sum_n \left[U_1(x,\omega_n)\sin\omega_n t + U_2(x,\omega_n)\cos\omega_n t\right]\sin(\omega_n x/a)$$

$$= \sum_n \left[C_1(\omega_n)\sin\omega_n t + C_2(\omega_n)\cos\omega_n t\right]\sin(\omega_n x/a)$$

非齐次偏微分方程的级数解法

$$
\begin{cases}
u_{tt} - a^2 u_{xx} = A\cos\dfrac{\pi x}{l}\sin\omega_0 t, & 0 < x < l, t > 0 \\
u_x\big|_{x=0} = u_x\big|_{x=l} = 0, & t > 0 \\
u\big|_{t=0} = \phi(x), \quad u_t\big|_{t=0} = \psi(x), & 0 \le x \le l
\end{cases}
$$

P162，分离变量思路

条件放宽

$$\begin{cases} V_{tt} - a^2 V_{xx} = 0, \quad 0 < x < l, t > 0 \\ V_x\big|_{x=0} = V_x\big|_{x=l} = 0, \quad t > 0 \end{cases}$$

$$V(x,t) = \sum_{n=0} T_n(t) \cos\frac{n\pi x}{l}$$

寻找 $T_n(t)$ 使之满足

$$V_{tt} - a^2 V_{xx} = A\cos\frac{\pi x}{l}\sin\omega_0 t$$

$$V\big|_{t=0} = \phi(x), \quad V_t\big|_{t=0} = \psi(x), \quad 0 \le x \le l$$

P162，分离变量思路

$$\begin{cases} T_n''(t)+\left(\dfrac{n\pi a}{l}\right)^2 T_n = A\sin\omega_0 t & n=1 \\ T_n''(t)+\left(\dfrac{n\pi a}{l}\right)^2 T_n = 0 & n\neq 1 \end{cases}$$

$$T_n(t)=A_n\cos\frac{n\pi a}{l}t+B_n\sin\frac{n\pi a}{l}t \qquad n\neq 1$$

非齐次常系数常微分方程

$$T_1''(t)+\left(\frac{\pi a}{l}\right)^2 T_1 = A\sin\omega_0 t$$

$$T_1(0)=\varphi_1$$

$$T_1'(0)=\psi_1$$

P162，分离变量思路

$$T_1''(t)+\left(\frac{\pi a}{l}\right)^2 T_1 = A\sin\omega_0 t$$

$$T_1(t)=C_1\sin\omega_0 t+C_2\cos\omega_0 t+C_3\sin\frac{\pi a}{l}t+C_4\cos\frac{\pi a}{l}t$$

$$-C_1\omega_0{}^2\sin\omega_0 t+C_1\left(\frac{\pi a}{l}\right)^2\sin\omega_0 t-C_2\omega_0{}^2\cos\omega_0 t+C_2\left(\frac{\pi a}{l}\right)^2\cos\omega_0 t$$

$$-C_3\left(\frac{\pi a}{l}\right)^2\sin\frac{\pi a}{l}t+C_3\left(\frac{\pi a}{l}\right)^2\sin\frac{\pi a}{l}t-C_4\left(\frac{\pi a}{l}\right)^2\cos\frac{\pi a}{l}t+C_4\left(\frac{\pi a}{l}\right)^2\cos\frac{\pi a}{l}t$$

$$=A\sin\omega_0 t$$

P162，分离变量思路

$$-C_1\omega_0^{\ 2}\sin\omega_0 t + C_1\left(\frac{\pi a}{l}\right)^2\sin\omega_0 t - C_2\omega_0^{\ 2}\cos\omega_0 t + C_2\left(\frac{\pi a}{l}\right)^2\cos\omega_0 t$$

$$-C_3\left(\frac{\pi a}{l}\right)^2\sin\frac{\pi a}{l}t + C_3\left(\frac{\pi a}{l}\right)^2\sin\frac{\pi a}{l}t - C_4\left(\frac{\pi a}{l}\right)^2\cos\frac{\pi a}{l}t + C_4\left(\frac{\pi a}{l}\right)^2\cos\frac{\pi a}{l}t$$

$$= A\sin\omega_0 t$$

$$\omega_0 \neq \frac{\pi a}{l}$$

$$T_1(t) = \begin{cases} C_1 = -\dfrac{A}{\omega_0^2 - \left(\dfrac{\pi a}{l}\right)^2} \\ C_2 = 0 \\ C_3 \\ C_4 \end{cases}$$

P162，分离变量思路

$$T_1(t) = C_1 \sin(\omega_0 t) + C_3 \sin\left(\frac{\pi a}{l} t\right) + C_4 \cos\left(\frac{\pi a}{l} t\right)$$

$$T_1(0) = \varphi_1$$

$$T_1'(0) = \psi_1$$

$$C_4 = \varphi_1$$

$$\omega_0 C_1 + \frac{\pi a}{l} C_3 = \psi_1$$

$$C_1 = -\frac{A}{\omega_0^2 - \left(\frac{\pi a}{l}\right)^2}$$

$$C_2 = 0$$

傅里叶变换思路

$$\omega \neq \pm \frac{\pi a}{l}$$

$$\left[-\omega^2 + \left(\frac{\pi a}{l}\right)^2\right] F[T_1] = \frac{A}{2}\left\{\delta(\omega+\omega_0) - \delta(\omega-\omega_0)\right\}$$

$$F[T_1] = \frac{\frac{A}{2}\left\{\delta(\omega+\omega_0) - \delta(\omega-\omega_0)\right\}}{\left[-\omega_0{}^2 + \left(\frac{\pi a}{l}\right)^2\right]}$$

$$\omega = \pm \frac{\pi a}{l}$$

等式左边为0，因此右边在w点一定也为零。

因此右边为delta函数要满足 $\omega_0 \neq \pm \pi a / l$

傅里叶变换思路

$$F[T_1]=\frac{\frac{A}{2}\{\delta(\omega+\omega_0)-\delta(\omega-\omega_0)\}}{\left[-\omega_0^{\ 2}+\left(\frac{\pi a}{l}\right)^2\right]}+C_3\delta(\omega+\frac{\pi a}{l})+C_4\delta(\omega-\frac{\pi a}{l})$$

$$\omega\neq\pm\frac{\pi a}{l}\quad \frac{A\sin\omega_0 t}{\left[-\omega_0^{\ 2}+\left(\frac{\pi a}{l}\right)^2\right]}$$

$$\omega=\pm\frac{\pi a}{l}\quad C_3\sin\left(\frac{\pi a}{l}t\right)+C_4\cos\left(\frac{\pi a}{l}t\right)$$

$$T_1(x)=\frac{A\sin\omega_0 t}{\left[-\omega_0^{\ 2}+\left(\frac{\pi a}{l}\right)^2\right]}+C_3\sin\left(\frac{\pi a}{l}t\right)+C_4\cos\left(\frac{\pi a}{l}t\right)$$

$$T_1(0)=\varphi_1$$

$$T_1{}'(0)=\psi_1$$

冲量法求解非齐次方程

$$\begin{cases} u_{tt} = a^2 u_{xx} + f\left(x,t\right), & 0 < x < l, t > 0 \\ u\big|_{x=0} = u\big|_{x=l} = 0, & t > 0 \\ u\big|_{t=0} = u_t\big|_{t=0} = 0, & \end{cases}$$

冲量法求解非齐次方程

作用力分解为瞬时力作用 $f(x,t)=\int_0^{\infty} f(x,t)\delta(t-\tau)\,\mathrm{d}\tau=\int_0^{\infty} f(x,\tau)\delta(t-\tau)\,\mathrm{d}\tau$

作用力分解为瞬时力作用，瞬时力作用相当于初始速度引起的振动

$$\bar{f}\to V$$

然后将瞬时力引起的振动线性叠加。 $\therefore u(x,t)=\int_0^{\infty} V(x,t;\tau)\,\mathrm{d}\tau=\int_0^{t} V(x,t;\tau)\,\mathrm{d}\tau$

冲量法求解非齐次方程

作用力分解为瞬时力作用，瞬时力作用相当于初始速度引起的振动；然后将瞬时力引起的振动线性叠。

$$\begin{cases} V_{tt} = a^2 V_{xx} + f(x,t)\delta(t-\tau), \quad 0 < x < l, t > 0 \\ V|_{x=0} = V|_{x=l} = 0, \qquad t > 0 \\ V|_{t=0} = V_t|_{t=0} = 0, \end{cases}$$

$$\therefore u(x,t) = \int_0^t V(x,t;\tau)\,\mathrm{d}\tau$$

冲量法求解非齐次方程

$$\begin{cases} V_{tt} = a^2 V_{xx} + f(x,\tau)\delta(t-\tau), & 0<x<l, t>0 \\ V|_{x=0} = V|_{x=l} = 0, & t>0 \\ V|_{t=0} = V_t|_{t=0} = 0, & \end{cases}$$

冲量定理 $$V'(\tau+\Delta\tau) - V'(\tau-\Delta\tau) = \int_{\tau-\Delta\tau}^{\tau+\Delta\tau} f(x,t)\delta(t-\tau)\,dt$$

$$V'(\tau+\Delta\tau) = f(x,\tau)$$

$$\Delta\tau \to 0 \quad V'(\tau) = f(x,\tau)$$

冲量法求解非齐次方程

$$\begin{cases} V_{tt} = a^2 V_{xx}, \quad 0 < x < l \\ V\big|_{x=0} = V\big|_{x=l} = 0, \\ V\big|_{t=\tau} = 0, V_t\big|_{t=\tau} = f(x,\tau), \end{cases}$$

$$\therefore u(x,t) = \int_0^{\infty} V(x,t;\tau)\,\mathrm{d}\tau = \int_0^{t} V(x,t;\tau)\,\mathrm{d}\tau$$

P168 例2

$$\begin{cases} u_{tt} = a^2 u_{xx} + A\cos\dfrac{\pi x}{l}\sin\omega_0 t, \quad 0 < x < l, t > 0 \\ u_x\big|_{x=0} = u_x\big|_{x=l} = 0, \qquad\qquad t > 0 \\ u\big|_{t=0} = u_t\big|_{t=0} = 0, \end{cases}$$

例2 将力化为叠加脉冲力叠加

$$A\cos\frac{\pi x}{l}\sin\omega_0 t = \int \sin\omega_0 t\delta(t-\tau)d\tau$$

$$\begin{cases} V_{tt} = a^2 V_{xx} + A\cos\frac{\pi x}{l}\sin\omega_0 t\delta(t-\tau), \quad 0<x<l, t>0 \\ V_x|_{x=0} = V_x|_{x=l} = 0, \qquad t>0 \\ V|_{t=0} = V_t|_{t=0} = 0, \end{cases}$$

例2 将力化为叠加脉冲力叠加

$$A\cos\frac{\pi x}{l}\sin\omega_0 t=\int\sin\omega_0 t\delta(t-\tau)d\tau$$

$$\begin{cases} V_{tt}=a^2V_{xx}, \quad 0<x<l, t>0 \\ V_x|_{x=0}=V_x|_{x=l}=0, \qquad t>0 \\ V|_{t=\tau}=0, V_t|_{t=\tau}=A\cos\frac{\pi x}{l}\sin\omega_0\tau, \end{cases}$$

冲量法求解非齐次方程

$$\begin{cases} V_{tt} = a^2 V_{xx}, \quad 0 < x < l \\ V_x \big|_{x=0} = V_x \big|_{x=l} = 0, \end{cases}$$

$$V(x,t;\tau) = \sum_{n=0} T_n(t;\tau) \cos\frac{n\pi x}{l} \qquad n = 0,1,2,\cdots$$

冲量法求解非齐次方程

$$V(x,t;\tau)=\sum_{n=0} T_n(t;\tau)\cos\frac{n\pi x}{l} \qquad n=0,1,2,\cdots$$

$$V_{tt}=a^2V_{xx}$$

冲量法求解非齐次方程

$$\left[\sum_{n=0} T_n''(t;\tau)+\left(\frac{n\pi a}{l}\right)^2 T_n(t;\tau)\right]\cos\frac{n\pi x}{l}=0$$

$$n=0,1,2,\cdots$$

冲量法求解非齐次方程

$$T_n''(t;\tau)+\left(\frac{n\pi a}{l}\right)^2 T_n(t;\tau)=0 \qquad n=0,1,2,\cdots$$

$$n=0 \qquad T_0(t;\tau)=A_0+B_0(t-\tau)$$

$$n=1,2,\cdots \quad T_n(t;\tau)=A_n\cos\frac{n\pi a(t-\tau)}{l}+B_n\cos\frac{n\pi a(t-\tau)}{l}$$

冲量法求解非齐次方程

$$V(x,t;\tau)=\sum_{n=0} T_n(t;\tau)\cos\frac{n\pi x}{l}$$

$$V(x,t;\tau)=\sum_{n=1}\left[A_n\cos\frac{n\pi a(t-\tau)}{l}+B_n\sin\frac{n\pi a(t-\tau)}{l}\right]\cos\frac{n\pi x}{l}+A_0+B_0(t-\tau)$$

$$V|_{t=\tau}=0, V_t|_{t=\tau}=A\cos\frac{\pi x}{l}\sin\omega_0\tau$$

冲量法求解非齐次方程

$$V(x,t;\tau)=\sum_{n=1}\left[A_n\cos\frac{n\pi a(t-\tau)}{l}+B_n\sin\frac{n\pi a(t-\tau)}{l}\right]\cos\frac{n\pi x}{l}+A_0+B_0(t-\tau)$$

$$V\big|_{t=\tau}=0,\quad V_t\big|_{t=\tau}=A\cos\frac{\pi x}{l}\sin\omega_0\tau$$

$$\sum_{n=1}A_n\cos\frac{n\pi x}{l}+A_0=0$$

$$\sum_{n=1}B_n\frac{n\pi a}{l}\cos\frac{n\pi x}{l}+B_0=A\cos\frac{\pi x}{l}\sin\omega_0\tau$$

冲量法求解非齐次方程

$$\sum_{n=1} A_n \cos\frac{n\pi x}{l} + A_0 = 0$$

$$\sum_{n=1} B_n \frac{n\pi a}{l}\cos\frac{n\pi x}{l} + B_0 = A\cos\frac{\pi x}{l}\sin\omega_0\tau$$

$$B_1\frac{n\pi a}{l} = A\sin\omega_0\tau$$

$$V(x,t;\tau) = B_1\sin\frac{\pi a(t-\tau)}{l}\sin\frac{n\pi a(t-\tau)}{l}\cos\frac{n\pi x}{l} = \frac{Al}{\pi a}\sin\omega\tau\sin\frac{\pi a(t-\tau)}{l}\cos\frac{n\pi x}{l}$$

冲量法求解非齐次方程

$$u(x,t)=\int_0^t V(x,t;\tau)\,\mathrm{d}\tau=\int_0^t \frac{Al}{\pi a}\sin\omega\tau\sin\frac{\pi a(t-\tau)}{l}\cos\frac{n\pi x}{l}d\tau$$

$$=\cos\frac{n\pi x}{l}\int_0^t \frac{Al}{\pi a}\sin\omega\tau\sin\frac{\pi a(t-\tau)}{l}d\tau$$

$$=\frac{Al}{\pi a}\cos\frac{n\pi x}{l}\int_0^t \sin\omega\tau\sin\frac{\pi a(t-\tau)}{l}d\tau$$

拉普拉斯变换法求积分

$$\int_0^t \sin\omega\tau \sin\frac{\pi a(t-\tau)}{l}d\tau$$

$$=\frac{1}{2}\int_0^t \cos\left[\omega\tau-\frac{\pi a(t-\tau)}{l}\right]-\cos\left[\omega\tau+\frac{\pi a(t-\tau)}{l}\right]d\tau$$

$$=\frac{1}{2}\int_0^t \cos\left[\left(\omega+\frac{\pi a}{l}\right)\tau-\frac{\pi a}{l}t\right]+\cos\left[\left(\omega-\frac{\pi a}{l}\right)\tau+\frac{\pi a}{l}t\right]d\tau$$

$$=\frac{1}{2}\int_0^t \cos\left[\left(\omega+\frac{\pi a}{l}\right)\tau\right]\cos\frac{\pi a}{l}t+\sin\left[\left(\omega+\frac{\pi a}{l}\right)\tau\right]\sin\frac{\pi a}{l}t+\cos\left[\left(\omega-\frac{\pi a}{l}\right)\tau\right]\cos\frac{\pi a}{l}t-\sin\left[\left(\omega-\frac{\pi a}{l}\right)\tau\right]\sin\frac{\pi a}{l}td\tau$$

分步求积分

$$\int_0^t \sin\omega\tau \sin\frac{\pi a(t-\tau)}{l}d\tau$$

$$=-\frac{1}{\omega}\int_0^t \sin\frac{\pi a(t-\tau)}{l}d\cos\omega\tau=-\frac{1}{\omega}\sin\frac{\pi a(t-\tau)}{l}\cos\omega\tau\,|_0^t+\frac{1}{\omega}\frac{\pi a}{l}\int_0^t \cos\omega\tau\cos\frac{\pi a(t-\tau)}{l}d\tau$$

$$=\frac{1}{\omega}\sin\frac{\pi at}{l}+\frac{1}{\omega^2}\frac{\pi a}{l}\int_0^t \cos\frac{\pi a(t-\tau)}{l}d\sin\omega\tau$$

$$=\frac{1}{\omega}\sin\frac{\pi at}{l}+\frac{1}{\omega^2}\frac{\pi a}{l}\sin\omega\tau\cos\frac{\pi a(t-\tau)}{l}\,|_0^t-\left(\frac{\pi a}{l\omega}\right)^2\int_0^t \sin\frac{\pi a(t-\tau)}{l}\sin\omega\tau d\tau$$

$$=\frac{1}{\omega}\sin\frac{\pi at}{l}+\frac{1}{\omega^2}\frac{\pi a}{l}\sin\omega t-\left(\frac{\pi a}{l\omega}\right)^2\int_0^t \sin\frac{\pi a(t-\tau)}{l}\sin\omega\tau d\tau$$

拉普拉斯变换法求积分

$$\int_0^t \sin\omega\tau \sin\frac{\pi a(t-\tau)}{l}d\tau$$

$$=\int_0^t \sin\omega\tau \sin\frac{\pi a(t-\tau)}{l}H(t-\tau)d\tau=\int_0^\infty \sin\omega\tau \sin\frac{\pi a(t-\tau)}{l}H(t-\tau)d\tau$$

$$\text{Laplacian transform}\rightarrow\int_0^\infty dte^{-pt}\int_0^\infty \sin\omega\tau \sin\frac{\pi a(t-\tau)}{l}H(t-\tau)d\tau$$

$$=\int_0^\infty \sin\omega\tau e^{-p\tau}\int_0^\infty e^{-p(t-\tau)}\sin\frac{\pi a(t-\tau)}{l}H(t-\tau)dtd\tau$$

$$=\int_0^\infty \sin\omega\tau e^{-p\tau}\int_{-\tau}^\infty e^{-pt}\sin\frac{\pi at}{l}H(t)dtd\tau=\int_0^\infty \sin\omega\tau e^{-p\tau}d\tau\int_0^\infty e^{-pt}\sin\frac{\pi at}{l}dt$$

$$=\frac{\omega}{p^2+\omega^2}\frac{\left(\frac{\pi a}{l}\right)}{p^2+\left(\frac{\pi a}{l}\right)^2}$$

$$L[\sin\omega x]=\frac{\omega}{p^2+\omega^2}$$

拉普拉斯变换法求积分

$$\frac{\omega}{p^2+\omega^2}\frac{\left(\frac{\pi a}{l}\right)}{p^2+\left(\frac{\pi a}{l}\right)^2}=\frac{A}{p^2+\omega^2}-\frac{B}{p^2+\left(\frac{\pi a}{l}\right)^2}$$

$$L[\sin\omega x]=\frac{\omega}{p^2+\omega^2}$$

$$L^{-1}=\frac{A}{\omega}\sin\omega t-\frac{B}{\pi a/l}\sin\frac{\pi a}{l}t$$

拉普拉斯法求积分

$$\frac{\omega}{p^2+\omega^2}\frac{\left(\frac{\pi a}{l}\right)}{p^2+\left(\frac{\pi a}{l}\right)^2}=\frac{A}{p^2+\omega^2}-\frac{B}{p^2+\left(\frac{\pi a}{l}\right)^2}$$

$$A\left[p^2+\left(\frac{\pi a}{l}\right)^2\right]-B\left[p^2+\omega^2\right]=\omega\frac{\pi a}{l}$$

$$A=B$$

$$A\left[\left(\frac{\pi a}{l}\right)^2-\omega^2\right]=\omega\frac{\pi a}{l}$$