数学物理方程

波动方程Wave equation：

描述现象：声波、电磁波等波动

$$u_{tt} - a^2\nabla^2 u = 0$$

双曲型方程

输运、扩散方程

Diffusion Equation：

描述现象：热扩散、物质扩散等扩散过程

$$u_t - a^2\nabla^2 u = 0$$

抛物型方程

不随时间改变的特例：稳恒状态方程：

描述现象：电势、稳定温度场分布等与时间无关的稳定场。

$$\nabla^2 u = 0$$

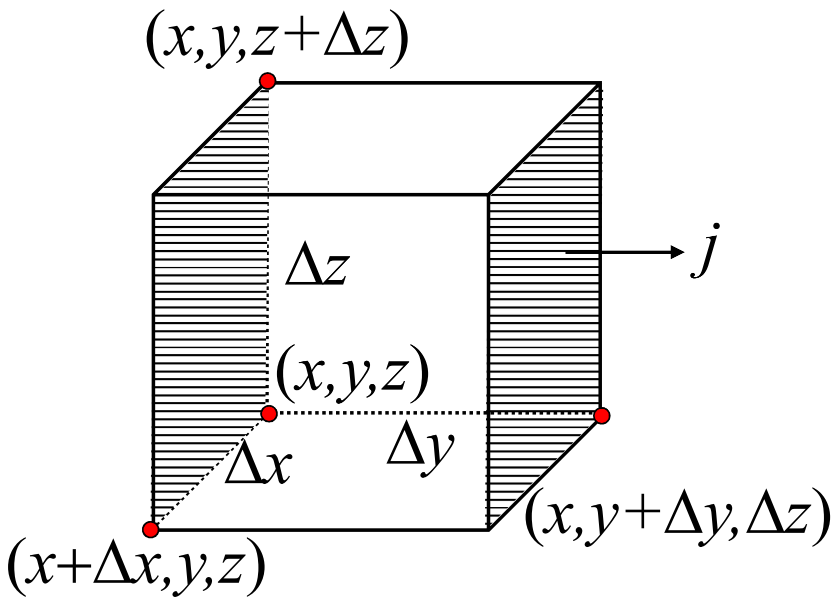
椭圆型方程

扩散方程

连续性方程 $$\rho_t(x,y,z,t) = \frac{\partial J(x,y,z,t)}{\partial x} + \cdots = \nabla \bullet J(x,y,z,t)$$

质量能量守恒

粒子不能凭空产生也不能凭空消灭，只能从一个格子转移到另一个格子



$$J = -k\nabla \bullet \rho$$

浓度高的区域向浓度低的区域流动

Fick's law $$\rho_t = -k\nabla\nabla \bullet \rho$$

定解条件

- 泛定方程和定解条件:
  - 泛定方程:
    描述了系统内部具有代表性（一般性）的点处的运动规律的偏微分方程 $u_{tt} - a^2\nabla^2 u = 0$
  - 定解条件:
    - 边界条件：描述外界影响
    - 衔接条件：描述内部特殊点的运动规律
    - 初始条件：描述历史的作用

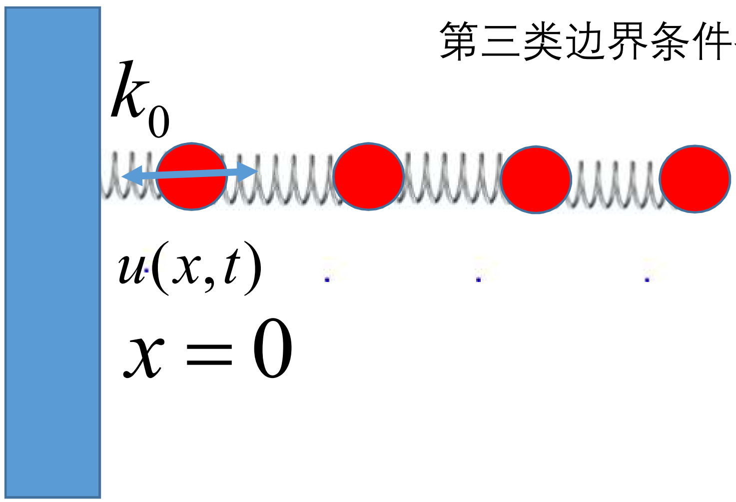
边界条件

第一类：表征量在边界处的值 $u(x,t)|_{x\in S}$ Dirichlet boundary condition

第二类：其法向导数在边界处的值 $u_x(x,t)|_{x\in S}$ Neumann boundary condition

第三类：前两者的线性组合 $u(x,t)+Au_x(x,t)|_{x\in S}$

第三类边界条件举例



$$-k_0u(0,t)+k[u(\Delta x,t)-u(0,t)]=0$$

行波法求解无界弦波动方程

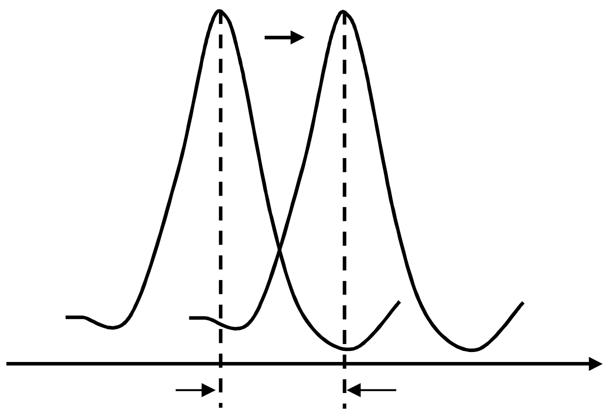
$$\begin{cases} u_{tt} = a^2 u_{xx}, & |x| < \infty, t > 0 \\ u|_{t=0} = \phi(x), \quad u_t|_{t=0} = \psi(x), & |x| < \infty \end{cases}$$

弦的自由横振动---行波法

$$(\partial_t - a\partial_x)(\partial_t + a\partial_x)u = 0$$

做变换: $\partial_\xi = \partial_t + a\partial_x$ $\quad \begin{cases} \xi = x + at \\ \eta = x - at \end{cases}$

$\partial_\eta = \partial_t - a\partial_x$



$$\partial_\eta \partial_\xi u(\xi, \eta) = 0$$

弦振动通解 $\quad u = \int f_1(\xi)\mathrm{d}\xi + g(\eta) = f(\xi) + g(\eta) = f(x+at) + g(x-at)$

利用初始条件求出定解

代入初始条件 $\begin{cases} f(x)+g(x)=\phi(x) \\ \left[f_t(x+at)+g_t(x-at)\right]\Big|_{t=0}=\psi(x) \end{cases}$

$$\begin{cases} f(x)+g(x)=\phi(x) \\ af(x)-af(x_0)-ag(x)+ag(x_0)=\int_{x_0}^{x} dx\psi(x) \end{cases}$$

$$\begin{cases} f(x)=\dfrac{1}{2}\phi(x)+\dfrac{1}{2}[f(x_0)-g(x_0)]+\dfrac{1}{2a}\displaystyle\int_{x_0}^{x}\psi(x)dx \\ g(x)=\dfrac{1}{2}\phi(x)-\dfrac{1}{2}[f(x_0)-g(x_0)]-\dfrac{1}{2a}\displaystyle\int_{x_0}^{x}\psi(x)dx \end{cases}$$

求出定解

$$\begin{cases} f(x)=\dfrac{1}{2}\phi(x)+\dfrac{1}{2}[f(x_0)-g(x_0)]+\dfrac{1}{2a}\displaystyle\int_{x_0}^{x}\psi(x)dx \\ g(x)=\dfrac{1}{2}\phi(x)-\dfrac{1}{2}[f(x_0)-g(x_0)]-\dfrac{1}{2a}\displaystyle\int_{x_0}^{x}\psi(x)dx \end{cases}$$

$$u=f(x+at)+g(x-at)$$

$$=\frac{1}{2}\left[\phi(x+at)+\phi(x-at)\right]+\frac{1}{2a}\int_{x_0}^{x+at}\psi(x)dx-\frac{1}{2a}\int_{x_0}^{x-at}\psi(x)dx$$

$$=\frac{1}{2}\left[\phi(x+at)+\phi(x-at)\right]+\frac{1}{2a}\int_{x-at}^{x+at}\psi(x)dx$$

定解问题的解（d'Alembert公式）

$$u(x,t)=\frac{\phi(x+at)+\phi(x-at)}{2}+\frac{1}{2a}\int_{x-at}^{x+at}\psi(\xi)\,\mathrm{d}\xi$$

端点反射

$$\begin{cases} u_{tt} = a^2 u_{xx}, & 0 < \mathrm{x} < \infty, t > 0 \\ u\big|_{t=0} = \phi(x), \quad u_t\big|_{t=0} = \psi(x), & 0 < \mathrm{x} < \infty \\ u\big|_{x=0} = 0 & \end{cases}$$

端点为零---延拓为奇函数

$$u(x,t) = -u(-x,t)$$

$$\phi(x+at)+\phi(x-at) = -\phi(-x+at)-\phi(-x-at)$$

$$\frac{1}{2a}\int_{x-at}^{x+at}\psi(\xi)\mathrm{d}\xi = -\frac{1}{2a}\int_{-x-at}^{-x+at}\psi(\xi)\mathrm{d}\xi$$

$$=\frac{1}{2a}\int_{x+at}^{x\text{-}at}\psi(-\xi)\mathrm{d}\xi=\frac{1}{2a}\int_{x\text{-}at}^{x+at}-\psi(-\xi)\mathrm{d}\xi$$

$$\phi(x) = -\phi(-x) \qquad \psi(x) = -\psi(-x)$$

端点为零---延拓为奇函数

$$u(x,t) = -u(-x,t) \qquad \phi(x) = -\phi(-x) \quad \psi(x) = -\psi(-x)$$

重新定义初始条件

$$\Phi(x) = \begin{cases} \phi(x) & x > 0 \\ -\phi(-x) & x < 0 \\ 0 & x = 0 \end{cases} \qquad \Psi(x) = \begin{cases} \psi(x) & x > 0 \\ -\psi(-x) & x < 0 \\ 0 & x = 0 \end{cases}$$

端点为零---延拓为奇函数

$$u(x,t)=\begin{cases}\dfrac{\phi(x+at)+\phi(x-at)}{2}+\dfrac{1}{2a}\displaystyle\int_{x-at}^{x+at}\psi(\xi)\,\mathrm{d}\xi & x>at\\[2ex] \dfrac{\phi(x+at)-\phi(x-at)}{2}+\dfrac{1}{2a}\displaystyle\int_{-x+at}^{x+at}\psi(\xi)\,\mathrm{d}\xi & x<at\end{cases}$$

球面波---三维波动方程

$$\begin{cases} u_{tt} = a^2 \nabla^2 u, \\ u\big|_{t=0} = \phi(r), \quad u_t\big|_{t=0} = \psi(r) \end{cases}$$

$$\begin{cases} u_{tt} = a^2 \dfrac{1}{r^2} \dfrac{\partial}{\partial r}\left(r^2 \dfrac{\partial u}{\partial r} \right), \qquad r \geq 0, t > 0 \\ u\big|_{t=0} = \phi(r), \quad u_t\big|_{t=0} = \psi(r) \end{cases}$$

球面波---三维波动方程

$$v = ru \qquad \begin{cases} v_{tt} = a^2 v_{rr}, & r > 0, t > 0 \\ v\big|_{r=0} = 0, & t \geq 0 \\ v\big|_{t=0} = r\phi(r), \quad v_t\big|_{t=0} = r\psi(r) \end{cases}$$

通解为 $$v = f(r+at) + g(r-at)$$

$$u=\begin{cases}\dfrac{(r+at)\phi(r+at)+(r-at)\phi(r-at)}{2r} \\ \qquad +\dfrac{1}{2ar}\displaystyle\int_{r-at}^{r+at}\xi\psi(\xi)\,\mathrm{d}\xi, \quad r-at\geq 0 \\ \dfrac{(r+at)\phi(r+at)-(at-r)\phi(at-r)}{2r} \\ \qquad +\dfrac{1}{2ar}\displaystyle\int_{at-r}^{r+at}\xi\psi(\xi)\,\mathrm{d}\xi, \quad r-at<0\end{cases}$$

傅里叶变换求解波动方程

$$\begin{cases} u_{tt} = a^2 u_{xx}, & |x| < \infty, t > 0 \\ u|_{t=0} = \phi(x), \quad u_t|_{t=0} = \psi(x), & |x| < \infty \end{cases}$$

泛定方程两边同时对空间做傅里叶变换

$$U'' + k^2 a^2 U = 0$$

$$U = \Phi(k), \quad U' = \Psi(k), \quad t = 0$$

带初值的齐次常微分方程求解

$$U''(k,t)+k^2a^2U(k,t)=0$$

通解

$$U(k,t)=C_1e^{\lambda_1 t}+C_2e^{\lambda_2 t}$$

$$U(k,t)=C_1(k)e^{\lambda_1 t}+C_2(k)e^{\lambda_2 t}$$

带初值的齐次常微分方程求解

$$\begin{cases} U=\Phi(k),\quad U'=\Psi(k),\quad t=0 \\ U=C_1(k)e^{\lambda_1 t}+C_2(k)e^{\lambda_2 t} \end{cases}$$

$$\begin{cases} C_1(k)=\dfrac{1}{2}\Phi(k)+\dfrac{1}{2a}\dfrac{1}{ik}\Psi(k) \\ C_2(k)=\dfrac{1}{2}\Phi(k)-\dfrac{1}{2a}\dfrac{1}{ik}\Psi(k) \end{cases}$$

$$U(k,t)=\Phi(k)\cos(kat)-\frac{1}{ak}\Psi(k)\sin(kat)$$

达朗贝尔公式

$$U(k,t)=\Phi(k)\cos(kat)-\frac{1}{ak}\Psi(k)\sin(kat)$$

$$u(x,t)=\frac{\phi(x+at)+\phi(x-at)}{2}+\frac{1}{2a}\int_{x-at}^{x+at}\psi(\xi)\,\mathrm{d}\xi$$

数学物理方程求解的线索

明线

行波法
分离变量法
格林函数法
冲量定理法

暗线

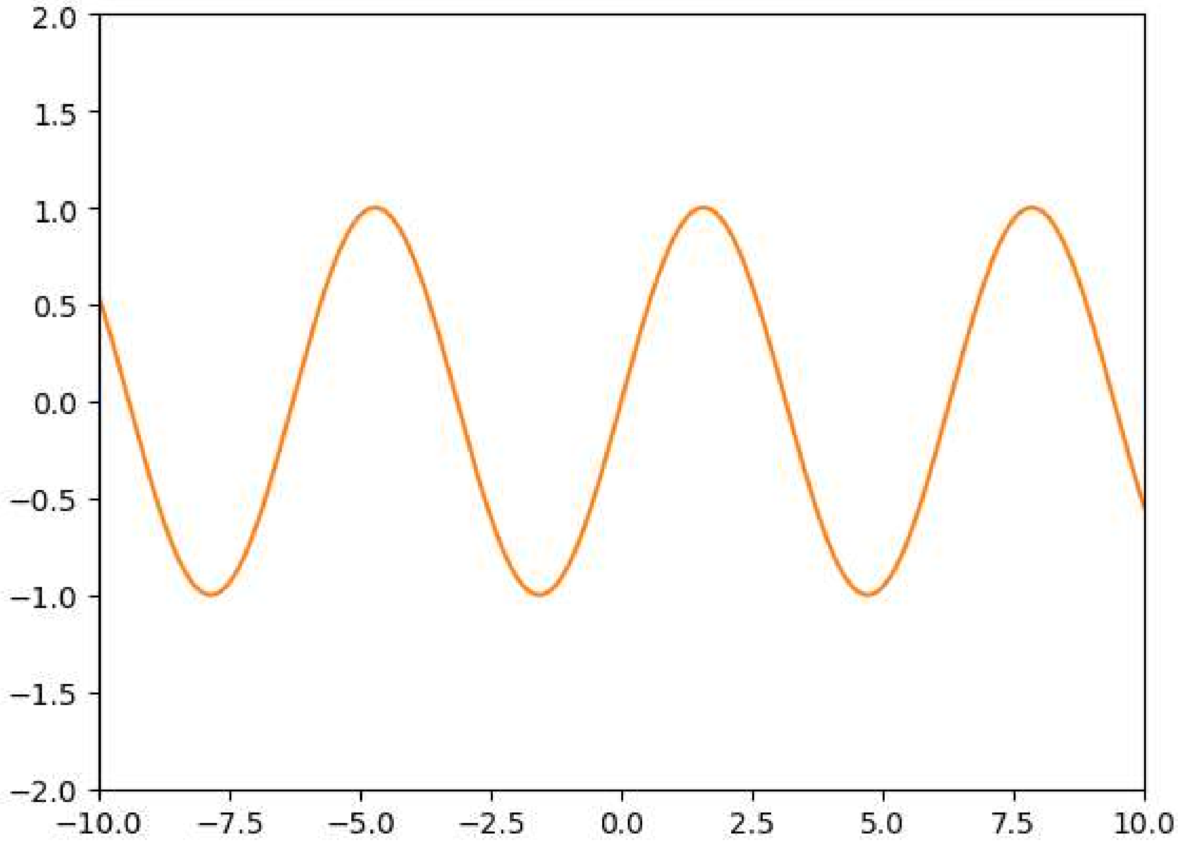
线性叠加，求解偏微分方程。

$$f(x)=\sum_k a_k g_k(x)$$

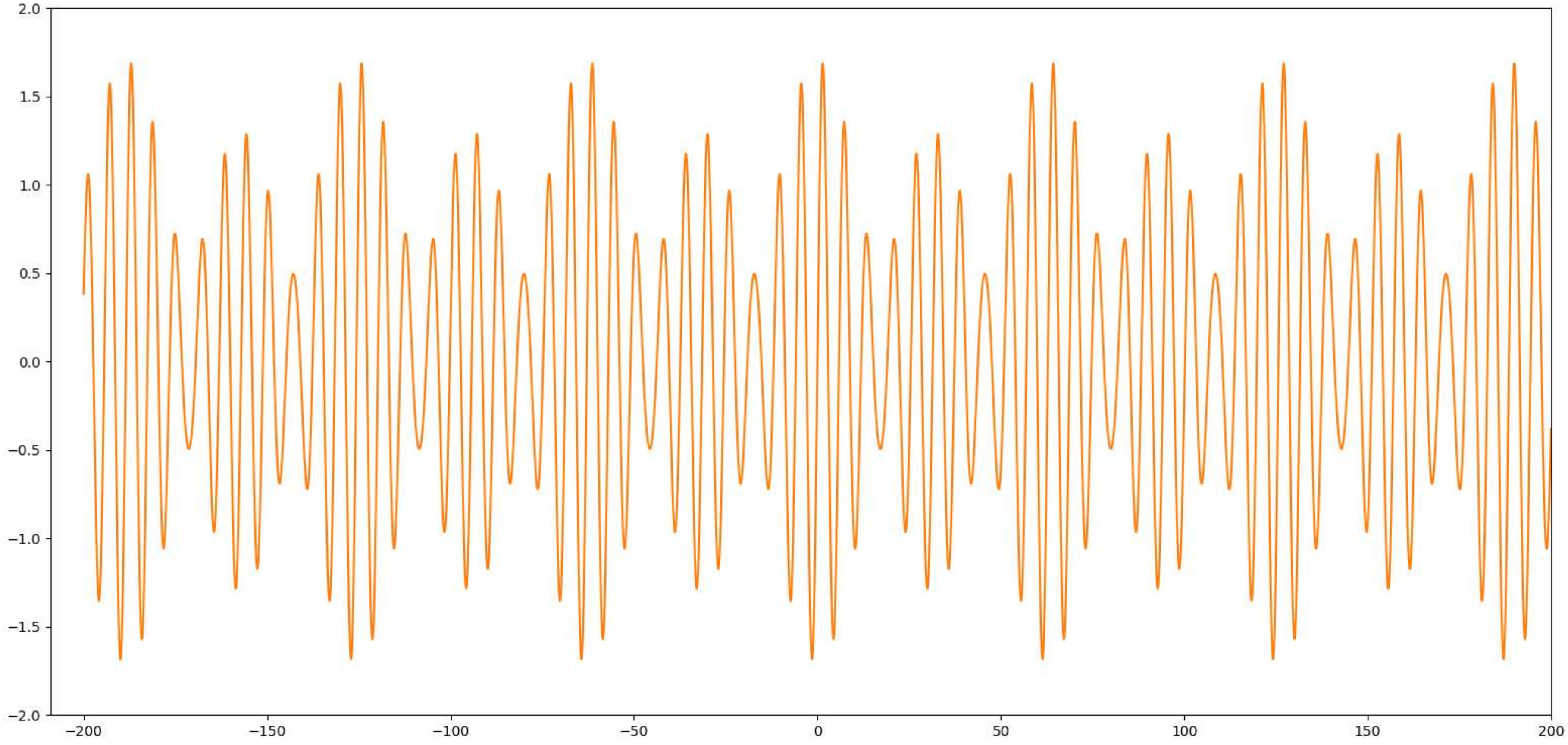
傅里叶级数展开
傅里叶变换
特殊函数

函数空间

$$f(x) = \sin(x)$$



$$f(x) = \sin(x) + 0.2\sin(1.1x) + 0.5\sin(1.2x)$$



$$f(x)=\sin(x)$$

周期	振幅
2pi	1

$$f(x)=\sin(x)+0.2\sin(1.1x)+0.5\sin(1.2x)$$

周期	振幅
2pi	1
2pi/1.1	0.2
2pi/1.2	0.5

描述具有周期性规律的信号，我们用一个新的函数描述。这个函数以sin、cos函数的周期为自变量，振幅为函数值。这就是傅里叶级数展开（或傅里叶变换）

傅里叶级数、傅里叶变换

傅里叶级数

$$f(x)=\sum a(k)\sin kx$$

$$\Delta k\to 0$$

傅里叶变换

$$f(x)=\sum a(k)\sin kx$$

$$k=\frac{n\pi}{l}$$

傅里叶级数中，任意给定f(x), 如何求sinkx的系数?

$$\int f(x)\sin k'xdx=\sum a(k)\delta(k-k')=a(k')$$

$$\int\sin k'x\sin kxdx=\begin{cases}c & k=k'\\0 & k\neq k'\end{cases}$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}dqe^{-i(x-x')q}=\delta(x-x')$$

傅里叶变换把一个函数翻译成了另一个函数

$$f(\mathbf{x}) = F^{-1}[F[f(\mathbf{x})]]$$

$$\hat{f}(\mathbf{q}) = F[f(\mathbf{x})] = \int_{\mathbf{R^n}} \frac{1}{(\sqrt{2\pi})^n} \mathbf{dx}\, f(\mathbf{x}) e^{-i\mathbf{qx}}$$

$$f(\mathbf{x}) = F^{-1}[\hat{f}(\mathbf{q})] = \int_{\mathbf{R^n}} \frac{1}{(\sqrt{2\pi})^n} \mathbf{dq}\, \hat{f}(\mathbf{q}) e^{i\mathbf{qx}}$$

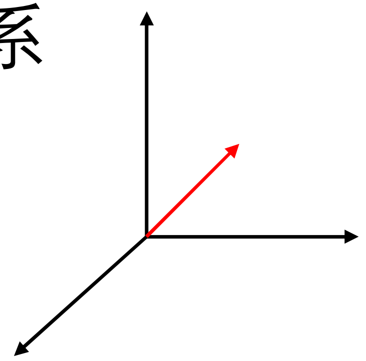
因为是翻译，所以正变换要过去逆变换要回来。
傅里叶变换最核心的性质

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} dy e^{-ixy} = \delta(x)$$

$$\begin{aligned}
&F[F^{-1}[\hat{f}]] \\
&= \int_{\mathbf{R^n}} \frac{1}{(\sqrt{2\pi})^n} \mathbf{dx} \int_{\mathbf{R^n}} \frac{1}{(\sqrt{2\pi})^n} \mathbf{dq}'\, \hat{f}(\mathbf{q}') e^{i\mathbf{q}'x} e^{-i\mathbf{qx}} \\
&= \frac{1}{2\pi^n} \int_{\mathbf{R^n}} \mathbf{dq}' \hat{f}(\mathbf{q}') \int \mathbf{dx} e^{i(\mathbf{q}'-\mathbf{q})\mathbf{x}} \\
&= \int_{\mathbf{R^n}} \mathbf{dq}' \hat{f}(\mathbf{q}') \delta(\mathbf{q}'-\mathbf{q}) \\
&= \hat{f}(\mathbf{q})
\end{aligned}$$

欧氏空间

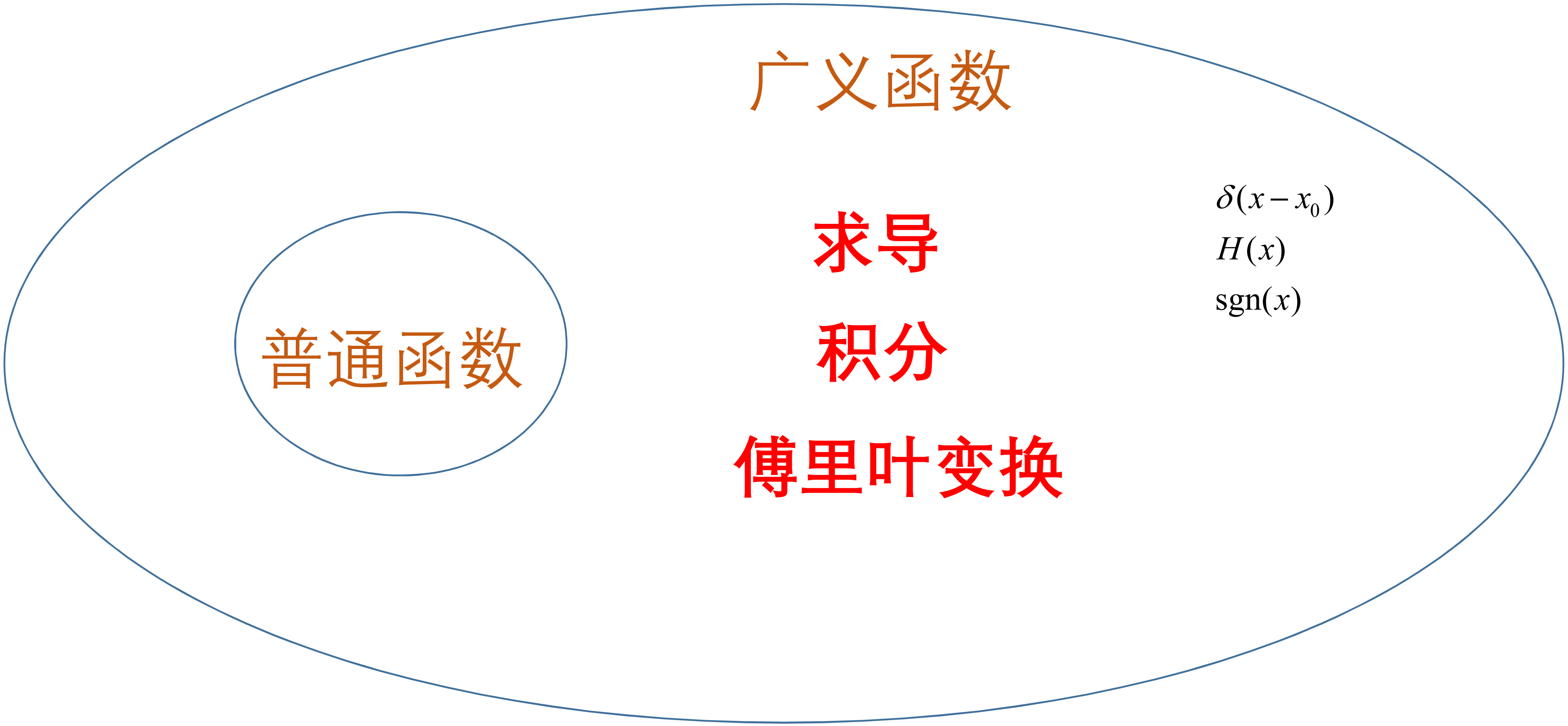
- 线性矢量空间
  - 三维欧氏空间 $\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3$ $\boxed{\mathbf{r}=\sum_{i=1}^{3}x_i\mathbf{e}_i}$ $\mathbf{e}_i\cdot\mathbf{e}_j=\delta_{ij}$
  - n维欧氏空间 $\mathbf{r}=\sum_{i=1}^{n}x_i\mathbf{e}_i$ $x_i=\dfrac{\mathbf{r}\cdot\mathbf{e}_i}{\mathbf{e}_i\cdot\mathbf{e}_i}$
  - 内积 $\mathbf{r}\cdot\mathbf{e}_i=\langle\mathbf{r},\mathbf{e}_i\rangle$ $x_i=\langle\mathbf{r},\mathbf{e}_i\rangle$
  - 子空间：个数小于n的正交矢量系
  - 完备



如果我们把函数比作欧式空间中的矢量，我们可以用积分定义函数的内积。

欧式空间	函数空间
$\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$	$\{\sin kx\}$
$\langle \mathbf{e}_j, \mathbf{e}_i \rangle = \delta_{ij}$	$\frac{1}{c}\int \sin k' x \sin kx dx = \delta_{ij}$
$\mathbf{r} = \sum_{i=1}^{n} x_i \mathbf{e}_i$	$f(x) = \sum f(k) \sin kx$
$x_i = \langle \mathbf{r}, \mathbf{e}_i \rangle$	$f(k) = \int f(x) \sin k' x dx$

普通函数满足广义函数的全部运算规则



广义函数的积分复合运算与普通函数完全一致

求解 $\int_0^3 (5x-2)\delta(2-x)dx$

$$y = 2 - x \quad x = 2 - y$$

$$\int_2^{-1} [5(2-y)-2]\delta(y)d(2\text{-y})$$

$$= -\int_2^{-1} (8-5y)\delta(y)d\text{y}$$

$$= \int_{-1}^{2} (8-5y)\delta(y)d\text{y} = 8$$

$$\delta(x-x_0) = \begin{cases} \infty & x = x_0 \\ 0 & x \neq x_0 \end{cases}$$

$$\int \delta(x-x_0)\varphi(x)dx = \varphi(x_0)$$

$$\delta\left[T(x)\right] = \sum_n \frac{\delta(x-x_n)}{\left|T'(x_n)\right|}$$

广义函数求导

$$\left\langle \frac{\mathrm{d}\,f}{\mathrm{d}\,x},\phi \right\rangle \equiv -\left\langle f,\frac{\mathrm{d}\,\phi}{\mathrm{d}\,x} \right\rangle$$

$$\int \frac{\mathrm{d}\,f}{\mathrm{d}\,x}\phi dx = -\int f\,\frac{\mathrm{d}\,\phi}{\mathrm{d}\,x}dx$$

$$\left\langle \delta'(x),\phi(x) \right\rangle = -\left\langle \delta(x),\phi'(x) \right\rangle = -\phi'(0)$$

$$f\,'(x) = \begin{cases} \dfrac{\mathrm{d}\,f_1(x)}{\mathrm{d}\,x} & x < x_0 \\ \dfrac{\mathrm{d}\,f_2(x)}{\mathrm{d}\,x} & x > x_0 \end{cases} + h\delta(x - x_0)$$

作业第二题

$$\left\langle \frac{\mathrm{d} f(x)}{\mathrm{d} x}, \phi(x) \right\rangle = -\langle f(x), \phi'(x) \rangle$$

$$f_c' = \begin{cases} \dfrac{\mathrm{d} f_1(x)}{\mathrm{d} x} & x < x_0 \\ \dfrac{\mathrm{d} f_2(x)}{\mathrm{d} x} & x > x_0 \end{cases}$$

$$= -\int_{-\infty}^{x_0} f_1 \phi' \,\mathrm{d} x - \int_{x_0}^{\infty} f_2 \phi' \,\mathrm{d} x$$

$$= -f_1 \phi \Big|_{-\infty}^{x_0} + \int_{-\infty}^{x_0} f_1' \phi \,\mathrm{d} x - f_2 \phi \Big|_{x_0}^{\infty} + \int_{x_0}^{\infty} f_2' \phi \,\mathrm{d} x$$

$$= -f_1(x_0)\phi(x_0) + \int_{-\infty}^{x_0} f_1' \phi \,\mathrm{d} x + f_2(x_0)\phi(x_0) + \int_{x_0}^{\infty} f_2' \phi \,\mathrm{d} x$$

$$= \left[f_2(x_0) - f_1(x_0) \right] \phi(x_0) + \int_{-\infty}^{x_0} f_1' \phi \,\mathrm{d} x + \int_{x_0}^{\infty} f_2' \phi \,\mathrm{d} x$$

$$= h\phi(x_0) + \langle f_c', \phi \rangle = \langle f_c' + h\delta(x - x_0), \phi \rangle$$

作业第二题 第二种证法

Heaviside step function

$$H(x) \equiv \begin{cases} 1 & x > 0 \\ 0 & x < 0 \end{cases}$$

$$f(x) \equiv f_1(x)H(x) + f_2(x)H(-x)$$

$$f'(x) \equiv f_1'(x)H(x) + f_1(x)\delta(x) + f_2'(x)H(-x) - f_2'(x)\delta(x)$$

$$= f_1'(x)H(x) + f_2'(x)H(-x) + (f_1(x) - f_2(x))\delta(x)$$

因为 $$\int a(x)\delta(x)\varphi(x)dx = a(0)\varphi(0) = \int a(0)\delta(x)\varphi(x)dx$$

所以 $$a(x)\delta(x) = a(0)\delta(x)$$

$$f_1'(x)H(x) + f_2'(x)H(-x) + (f_1(0) - f_2(0))\delta(x)$$

作业第三题

$$f(x)=\begin{cases} x^2 & x\geq 0 \\ x & x<0 \end{cases}$$

$$\frac{df}{dx}=\begin{cases} 2x & x\geq 0 \\ 1 & x<0 \end{cases}$$

$$\frac{df^2}{dx^2}=\begin{cases} 2 & x\geq 0 \\ 0 & x<0 \end{cases}-\delta(x)$$

$$\frac{df^3}{dx^3}=2\delta(x)-\delta'(x)=2\delta(x)-\frac{1}{2\pi}\int ike^{ikx}dk$$

$$\left\langle \delta'(x),\phi(x)\right\rangle=-\left\langle \delta(x),\phi'(x)\right\rangle=-\phi'(0)$$

作业第四题

$$\delta(x-x_0)\delta(y-y_0)\delta(z-z_0)=\frac{1}{r^2\sin\vartheta}\delta(r-r_0)\delta(\phi-\phi_0)\delta(\vartheta-\vartheta_0)$$

$$\delta(\mathbf{r}-\mathbf{r}_0)=\delta(r-r_0)\delta(\cos\theta-\cos\theta_0)\delta(\phi-\phi_0)/r^2$$

只需证 $\delta(\cos\theta-\cos\theta_0)=\dfrac{1}{\sin\theta}\delta(\theta-\theta_0)$

作业第四题

与公式$\delta(x-x_0)\delta(y-y_0)\delta(z-z_0)=\frac{1}{r^2\sin\vartheta}\delta(r-r_0)\delta(\phi-\phi_0)\delta(\vartheta-\vartheta_0)$比较，只需证明，$\delta(\cos\theta-\cos\theta_0)=\frac{1}{\sin\vartheta}\delta(\vartheta-\vartheta_0)$

利用公式$\delta[T(x)]=\sum_n\frac{\delta(x-x_n)}{|T'(x_n)|}$，得到$\delta(\cos\theta-\cos\theta_0)=\frac{1}{|\sin\vartheta_0|}\delta(\vartheta-\vartheta_0)$

又$\pi>\vartheta_0\geq 0$，$\sin\vartheta_0\geq 0$，所以$\delta(\cos\theta-\cos\theta_0)=\frac{1}{\sin\vartheta_0}\delta(\vartheta-\vartheta_0)=\frac{1}{\sin\vartheta}\delta(\vartheta-\vartheta_0)$

$$\delta[T(x)]=\sum_n\frac{\delta(x-x_n)}{|T'(x_n)|}\qquad a(x)\delta(x)=a(0)\delta(x)$$

以后任何函数的傅里叶变换，直接套5个公式

$$\hat{f}(\mathbf{q}) = F[f(\mathbf{x})] = \int_{\mathbf{R}^n} \frac{1}{(\sqrt{2\pi})^n} \mathbf{dx}\, f(\mathbf{x}) e^{-i\mathbf{qx}}$$

$$f(\mathbf{x}) = F^{-1}[\hat{f}(\mathbf{q})] = \int_{\mathbf{R}^n} \frac{1}{(\sqrt{2\pi})^n} \mathbf{dq}\, \hat{f}(\mathbf{q}) e^{i\mathbf{qx}}$$

满足普通函数傅里叶变换的函数一定满足广义傅里叶变换

根据广义函数定义

$$F[\delta(x-x_0)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx \delta(x-x_0) e^{-ixy} = \frac{1}{\sqrt{2\pi}} e^{-ix_0 y}$$

$$F[1] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dy e^{-ixy} = \sqrt{2\pi}\delta(x)$$

$$F[H(x)] = \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} dx e^{-iqx} = \frac{1}{\sqrt{2\pi}} \frac{1}{iq} + \frac{1}{\sqrt{2\pi}} \pi\delta(q)$$

作业第五题（1）

$$(1)\ \sin x \to i\sqrt{\frac{\pi}{2}}[\delta(q+1)-\delta(q-1)]$$

代入傅里叶变换公式 $\hat{f}(\mathbf{q}) = F[f(\mathbf{x})] = \int_{\mathbf{R}^n} \frac{1}{(\sqrt{2\pi})^n} \mathbf{dx}\, f(\mathbf{x}) e^{-i\mathbf{qx}}$

$$F[\sin x] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-iqx} \sin x dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-iqx} \frac{e^{ix}-e^{ix}}{2i} dx$$

$$= \frac{1}{2i\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i(q-1)x} - e^{-i(q+1)x} dx$$

利用公式 $F[1] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dy e^{-ixy} = \sqrt{2\pi}\delta(x)$

$$F[\sin x] = \frac{1}{2i\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i(q-1)x} - e^{-i(q+1)x} dx = \frac{\sqrt{2\pi}}{2i}[\delta(q-1)-\delta(q+1)]$$

作业第五题（2）

$$f(x)=\begin{cases}\sin x & x>0\\ 0 & x<0\end{cases}\rightarrow$$

$$\frac{1}{2i\sqrt{2\pi}}\left\{\frac{1}{i(q-1)}+\pi\delta(q-1)-\frac{1}{i(q+1)}-\pi\delta(q+1)\right\}$$

(2) $$\frac{1}{2i\sqrt{2\pi}}\left\{\frac{1}{i(q-1)}+\pi\delta(q-1)-\frac{1}{i(q+1)}-\pi\delta(q+1)\right\}$$

代入傅里叶变换公式 $\hat{f}(\mathbf{q})=F[f(\mathbf{x})]=\int_{\mathbf{R}^n}\frac{1}{(\sqrt{2\pi})^n}\mathbf{dx}\,f(\mathbf{x})e^{-i\mathbf{qx}}$

$$=\frac{1}{\sqrt{2\pi}}\int_0^\infty e^{-iqx}\sin x dx=\frac{1}{\sqrt{2\pi}}\int_0^\infty e^{-iqx}\frac{e^{ix}-e^{ix}}{2i}dx$$

$$=\frac{1}{2i\sqrt{2\pi}}\int_0^\infty e^{-i(q-1)x}-e^{-i(q+1)x}dx$$

利用公式 $F[H(x)]=\frac{1}{\sqrt{2\pi}}\int_0^\infty dxe^{-iqx}=\frac{1}{\sqrt{2\pi}}\frac{1}{iq}+\frac{1}{\sqrt{2\pi}}\pi\delta(q)$

$$=\frac{1}{2i\sqrt{2\pi}}\int_0^\infty e^{-i(q-1)x}-e^{-i(q+1)x}dx=\frac{1}{2i\sqrt{2\pi}}\left[\frac{1}{i(q-1)}+\pi\delta(q-1)-\frac{1}{i(q+1)}-\pi\delta(q+1)\right]$$

作业第五题（3）

$$\operatorname{sgn}(x)=\begin{cases}1 & x>0\\ 0 & x=0\\ -1 & x<0\end{cases}\rightarrow\sqrt{\frac{2}{\pi}}\frac{1}{iq}$$

代入傅里叶变换公式 $\hat{f}(\mathbf{q})=F[f(\mathbf{x})]=\int_{\mathbf{R}^n}\frac{1}{(\sqrt{2\pi})^n}\mathbf{dx}\,f(\mathbf{x})e^{-iq\mathbf{x}}$

$$=\frac{1}{\sqrt{2\pi}}\int_0^{\infty}e^{-iqx}dx-\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0}e^{-iqx}dx$$

$$=\frac{1}{\sqrt{2\pi}}\int_0^{\infty}e^{-iqx}dx+\frac{1}{\sqrt{2\pi}}\int_{\infty}^{0}e^{iqx}dx=\frac{1}{\sqrt{2\pi}}\int_0^{\infty}e^{-iqx}dx-\frac{1}{\sqrt{2\pi}}\int_0^{\infty}e^{iqx}dx$$

利用公式 $F[H(x)]=\frac{1}{\sqrt{2\pi}}\int_0^{\infty}dxe^{-iqx}=\frac{1}{\sqrt{2\pi}}\frac{1}{iq}+\frac{1}{\sqrt{2\pi}}\pi\delta(q)$

$$=\frac{1}{\sqrt{2\pi}}\left(\frac{1}{iq}+\frac{1}{\sqrt{2\pi}}\pi\delta(q)\right)-\frac{1}{\sqrt{2\pi}}\left(-\frac{1}{iq}+\frac{1}{\sqrt{2\pi}}\pi\delta(q)\right)=\frac{2}{\sqrt{2\pi}}\frac{1}{iq}$$

作业第五题（4）

$$f(x) = 1/x \to -i\sqrt{\frac{2}{\pi}}\,\mathrm{sgn}(q)$$

由上题可知$\mathrm{sgn}(x)$的傅里叶变换是$\frac{2}{\sqrt{2\pi}}\frac{1}{iq}$，所以$\frac{2}{\sqrt{2\pi}}\frac{1}{iq}$的傅里叶逆变换是$\mathrm{sgn}(x)$。利用傅里叶逆变换的公式

$f(\mathbf{x}) = F^{-1}[\hat{f}(\mathbf{q})] = \int_{\mathbf{R}^n} \frac{1}{(\sqrt{2\pi})^n} \mathbf{dq}\, \hat{f}(\mathbf{q}) e^{i\mathbf{qx}}$ 得到，

$$\mathrm{sgn}(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{iqx} \frac{1}{iq} \frac{2}{\sqrt{2\pi}} dq$$

所以

$$\mathrm{sgn}(x) = \frac{1}{\sqrt{2\pi}} \int_{\infty}^{-\infty} e^{-iqx} \frac{1}{-iq} \frac{2}{\sqrt{2\pi}} d(-q)$$

$$= -\frac{1}{i\pi} \int_{-\infty}^{\infty} e^{-iqx} \frac{1}{q} dq = -\frac{\sqrt{2\pi}}{i\pi} F[\frac{1}{q}]$$

也就是$\mathrm{sgn}(x) = -\frac{\sqrt{2\pi}}{i\pi} F[\frac{1}{q}]$