Construction of Stiefel-Whitney Classes

In these notes, we will construct important topological invariants: Stiefel-Whitney classes. These cohomology classes classify vector bundles to some extent. Let $\pi : E \to B$ be a vector bundle. A straightforward way to define the Stiefel-Whitney classes is by the following axioms:

1. The i-th Stiefel-Whitney class, $w_i(E)$, is an element of $H^i(B; \mathbb{Z}/2)$, $w_0(E) = 1$, and $w_j(E) = 0$ for $j > \mathrm{rank}(E)$.
2. The Stiefel-Whitney classes are natural in the sense that they commute with pullbacks.
3. The total Stiefel-Whitney class, $w(E) = \sum w_j(E)$, of a direct sum splits as a cup product: $w(E_1 \oplus E_2) = w(E_1) \smile w(E_2)$.
4. The tautological line bundle over $\mathbb{RP}^1$ has non-zero w_1.

This gives us some nice immediate properties, such as $w_i(E_1 \oplus E_2) = w_i(E_1)$, if E_2 is trivial. Another interesting property is that $w_j(E) = 0$ for $j > n - k$ if there are k fiber-wise linearly independent non-vanishing sections of E. One can show that, in fact, real vector bundles have Stiefel-Whitney classes described by a pullback of the class on the tautological bundle over the infinite Grassmannian, $\mathrm{G}_n(\mathbb{R}^\infty)$ of the corresponding dimension. Using these ideas, we can show that Stiefel-Whitney classes are unique.

Now, there exists a unique cohomology class, $u \in H^n(E, E_0)$ that restricts to the nonzero element of $H^n(F, F_0)$ in each fiber, where a zero subscript denotes a subtraction of the image of the zero section. Cup product with u then yields an isomorphism, and additionally, $B \hookrightarrow E$ as the zero section, which means that E deformation retracts onto B and yields an isomorphism $\pi^* : H^k(B) \to H^k(E)$. Hence, the map $\phi : H^k(B) \to H^{n+k}(E, E_0)$ is an isomorphism called the Thom isomorphism.

Another important element of this construction is the construction of the Steenrod squaring operations. These are written with respect to $\mathbb{Z}/2$ coefficients and are characterized by the axioms:

1. For spaces, $A \subset X$, and integers i and n, there is a homomorphism $\mathrm{Sq}^i : H^n(X, A) \to H^{n+i}(X, A)$.
2. These homomorphisms are natural in the same sense as above.
3. $\mathrm{Sq}^0 = \mathrm{id}$, Sq^n is the cup square, and $\mathrm{Sq}^i = 0$ if $i > n$ by definition.
4. If $\mathrm{Sq} = \sum \mathrm{Sq}^i$, then $\mathrm{Sq}(a \smile b) = \mathrm{Sq}(a) \smile \mathrm{Sq}(b)$.

We can then define $w_i(E) = \phi^{-1}\mathrm{Sq}^i\phi(1)$, which can be shown to coincide with the original definition. Finally, some important geometric notes about Stiefel-Whitney classes are that $w_1(E) = 0$ if and only if E is orientable, and $w_2(E) = 0$ if and only if E admits a spin structure.