Selected Exercises

Complex Analysis

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Here are selected problems I solved while self-studying *Complex Analysis* by Stein and Sharkarchi.

1.9. We first deduce the Cauchy-Riemann equations in polar coordinates. Notice that:

$$\begin{aligned}
\frac{\partial u}{\partial r} &= \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} \\
&= \frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta \\
&= \frac{1}{r}\frac{\partial v}{\partial y} r\cos\theta - \frac{1}{r}\frac{\partial v}{\partial x} r\sin\theta \\
&= \frac{1}{r}\frac{\partial v}{\partial \theta} \\
\frac{1}{r}\frac{\partial u}{\partial \theta} &= \frac{1}{r}\left(\frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta}\right) \\
&= \frac{1}{r}\left(-\frac{\partial u}{\partial x} r\sin\theta + \frac{\partial u}{\partial y} r\cos\theta\right) \\
&= -\frac{\partial v}{\partial y}\sin\theta - \frac{\partial v}{\partial x}\cos\theta \\
&= -\frac{\partial v}{\partial r}.
\end{aligned}$$

Write $\log z = \log r + i\theta = u(r,\theta) + iv(r,\theta)$. By Theorem 2.4, it suffices to show u and v satisfy the Cauchy-Riemann equations. Clearly, for the given domain, we have

$$\frac{\partial u}{\partial r} = \frac{1}{r} \quad \frac{\partial u}{\partial \theta} = 0 \quad \frac{\partial v}{\partial r} = 0 \quad \frac{\partial v}{\partial \theta} = 1.$$

Thus, we get that

$$\frac{\partial u}{\partial r} = \frac{1}{r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$$

and

$$\frac{1}{r}\frac{\partial u}{\partial \theta} = 0 = -\frac{\partial v}{\partial r}.$$

Thus, $\log z$ is holomorphic for $r > 0$ and $\theta \in (-\pi, \pi)$.

1.10. Applying the differentiation operators as usual, we compute

$$\begin{aligned}
4\frac{\partial}{\partial z}\frac{\partial}{\partial \overline{z}} &= 2\frac{\partial}{\partial z}\frac{\partial}{\partial x} - \frac{2}{i}\frac{\partial}{\partial z}\frac{\partial}{\partial y} \\
&= \frac{\partial^2}{\partial x^2} + \frac{1}{i}\frac{\partial^2}{\partial y \partial x} - \frac{1}{i}\frac{\partial^2}{\partial x \partial y} + \frac{\partial^2}{\partial y^2} \\
&= \Delta
\end{aligned}$$

$$\begin{aligned}
4\frac{\partial}{\partial \overline{z}}\frac{\partial}{\partial z} &= 2\frac{\partial}{\partial \overline{z}}\frac{\partial}{\partial x} + \frac{2}{i}\frac{\partial}{\partial \overline{z}}\frac{\partial}{\partial y} \\
&= \frac{\partial^2}{\partial x^2} - \frac{1}{i}\frac{\partial^2}{\partial y \partial x} + \frac{1}{i}\frac{\partial^2}{\partial x \partial y} + \frac{\partial^2}{\partial y^2} \\
&= \Delta.
\end{aligned}$$

1.24. Using the reverse parameterization defined in the book, we observe

$$\begin{aligned}
\int_{\gamma^-} f(x)\, dz &= \int_a^b f(z^-(t))(z^-)'(t)\, dt \\
&= -\int_a^b f(z(a+b-t))z'(b+a-t)\, dt \\
&= \int_b^a f(z(u))z'(u)\, du \\
&= -\int_a^b f(z(u))z'(u)\, du
\end{aligned}$$

$$= -\int_\gamma f(z)\, dz.$$

2.1. Denote the flat arc at the bottom by γ_1, the curved arc by γ_2, and the segment of the line $\theta = \pi/4$ by γ_3. Next, notice that $z \mapsto e^{-z^2}$ is holomorphic as a complex Gaussian. Thus, it follows that $\int_{\gamma_1} e^{-z^2}\, dz + \int_{\gamma_2} e^{-z^2}\, dz + \int_{\gamma_3} e^{-z^2}\, dz = 0$. We explicitly compute the first two integrals in order to solve for the third. For the first one, notice that this is just the usual Gaussian integral over the positive part of the real line. Thus, we get that, as R approaches infinity,

$$\int_{\gamma_1} e^{-z^2}\, dz \to \frac{\sqrt{\pi}}{2}.$$

For the second term, notice that

$$\begin{aligned}
\left| \int_{\gamma_2} e^{-z^2}\, dz \right| &= \left| \int_0^{\pi/4} e^{-R^2 e^{4i\theta}} 2Rie^{i\theta}\, d\theta \right| \\
&\leq \int_0^{\pi/4} \left| e^{-R^2 e^{4i\theta}} 2Rie^{i\theta} \right|\, d\theta \\
&\leq \frac{CR}{e^{R^2}} \xrightarrow{R\to\infty} 0.
\end{aligned}$$

Thus, we see that

$$\int_{\gamma_3} e^{-z^2}\, dz = -\frac{\sqrt{\pi}}{2}.$$

Next, notice that

$$\begin{aligned}
-\int_{\gamma_3} e^{-z^2}\, dz &= \int_0^\infty e^{-x^2 e^{i\pi/2}} e^{i\pi/4}\, dx \\
&= e^{i\pi/4} \int_0^\infty e^{-ix^2}\, dx \\
&= e^{i\pi/4} \int_0^\infty \cos(-x^2) + i\sin(-x^2)\, dx \\
&= e^{i\pi/4} \int_0^\infty \cos(x^2) - i\sin(x^2)\, dx \\
&= \frac{\sqrt{\pi}}{2}.
\end{aligned}$$

Therefore, moving the $(1+i)/\sqrt{2}$ to the other side of the equation and taking real and imaginary parts, we obtain the equalities

$$\begin{aligned}
\mathrm{Re}\left(\int_0^\infty \cos(x^2) - i\sin(x^2)\ dx\right) &= \int_0^\infty \cos(x^2)\ dx \\
&= \frac{\sqrt{2\pi}}{4} \\
-\mathrm{Im}\left(\int_0^\infty \cos(x^2) - i\sin(x^2)\ dx\right) &= \int_0^\infty \sin(x^2)\ dx \\
&= \frac{\sqrt{2\pi}}{4},
\end{aligned}$$

which completes the computation.