Selected Exercises

Complex Analysis
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Here are selected problems I solved while self-studying *Complex Analysis* by Stein and Sharkarchi. I tended to write up computational questions because I have seen the theory come up in my other classes.

1.9. We first deduce the Cauchy-Riemann equations in polar coordinates. Notice that:

$$\begin{aligned}
\frac{\partial u}{\partial r} &= \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} \\
&= \frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta \\
&= \frac{1}{r}\frac{\partial v}{\partial y} r\cos\theta - \frac{1}{r}\frac{\partial v}{\partial x} r\sin\theta \\
&= \frac{1}{r}\frac{\partial v}{\partial \theta} \\
\frac{1}{r}\frac{\partial u}{\partial \theta} &= \frac{1}{r}\left(\frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta}\right) \\
&= \frac{1}{r}\left(-\frac{\partial u}{\partial x} r\sin\theta + \frac{\partial u}{\partial y} r\cos\theta\right) \\
&= -\frac{\partial v}{\partial y}\sin\theta - \frac{\partial v}{\partial x}\cos\theta \\
&= -\frac{\partial v}{\partial r}.
\end{aligned}$$

Write $\log z = \log r + i\theta = u(r,\theta) + iv(r,\theta)$. By Theorem 2.4, it suffices to show u and v satisfy the Cauchy-Riemann equations. Clearly, for the given domain, we have

$$\frac{\partial u}{\partial r} = \frac{1}{r} \quad \frac{\partial u}{\partial \theta} = 0 \quad \frac{\partial v}{\partial r} = 0 \quad \frac{\partial v}{\partial \theta} = 1.$$

Thus, we get that

$$\frac{\partial u}{\partial r} = \frac{1}{r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$$

and

$$\frac{1}{r}\frac{\partial u}{\partial \theta} = 0 = -\frac{\partial v}{\partial r}.$$

Thus, $\log z$ is holomorphic for $r > 0$ and $\theta \in (-\pi, \pi)$.

1.10. Applying the differentiation operators as usual, we compute

$$\begin{aligned} 4\frac{\partial}{\partial z}\frac{\partial}{\partial \bar{z}} &= 2\frac{\partial}{\partial z}\frac{\partial}{\partial x} - \frac{2}{i}\frac{\partial}{\partial z}\frac{\partial}{\partial y} \\ &= \frac{\partial^2}{\partial x^2} + \frac{1}{i}\frac{\partial^2}{\partial y \partial x} - \frac{1}{i}\frac{\partial^2}{\partial x \partial y} + \frac{\partial^2}{\partial y^2} \\ &= \Delta \end{aligned}$$

$$\begin{aligned} 4\frac{\partial}{\partial \bar{z}}\frac{\partial}{\partial z} &= 2\frac{\partial}{\partial \bar{z}}\frac{\partial}{\partial x} + \frac{2}{i}\frac{\partial}{\partial \bar{z}}\frac{\partial}{\partial y} \\ &= \frac{\partial^2}{\partial x^2} - \frac{1}{i}\frac{\partial^2}{\partial y \partial x} + \frac{1}{i}\frac{\partial^2}{\partial x \partial y} + \frac{\partial^2}{\partial y^2} \\ &= \Delta. \end{aligned}$$

1.24. Using the reverse parameterization defined in the book, we observe

$$\begin{aligned} \int_{\gamma^-} f(x)\, dz &= \int_a^b f(z^-(t))(z^-)'(t)\, dt \\ &= -\int_a^b f(z(a+b-t))z'(a+b-t)\, dt \\ &= \int_b^a f(z(u))z'(u)\, du \end{aligned}$$

$$= -\int_a^b f(z(u))z'(u)\, du$$

$$= -\int_\gamma f(z)\, dz.$$

2.1. Denote the flat arc at the bottom by γ_1, the curved arc by γ_2, and the segment of the line $\theta = \pi/4$ by γ_3. Next, notice that $z \mapsto e^{-z^2}$ is holomorphic as a complex Gaussian. Thus, it follows that $\int_{\gamma_1} e^{-z^2}\, dz + \int_{\gamma_2} e^{-z^2}\, dz + \int_{\gamma_3} e^{-z^2}\, dz = 0$. We explicitly compute the first two integrals in order to solve for the third. For the first one, notice that this is just the usual Gaussian integral over the non-negative portion of the real line. Thus, we get that, as R approaches infinity,

$$\int_{\gamma_1} e^{-z^2}\, dz \to \frac{\sqrt{\pi}}{2}.$$

For the second term, notice that

$$\left|\int_{\gamma_2} e^{-z^2}\, dz\right| = \left|\int_0^{\pi/4} e^{-R^2 e^{4i\theta}} 2Rie^{i\theta}\, d\theta\right|$$

$$\leq \int_0^{\pi/4} \left|e^{-R^2 e^{4i\theta}} 2Rie^{i\theta}\right|\, d\theta$$

$$\leq \frac{CR}{e^{R^2}} \xrightarrow{R\to\infty} 0.$$

Thus, we see that

$$\int_{\gamma_3} e^{-z^2}\, dz = -\frac{\sqrt{\pi}}{2}.$$

Next, notice that

$$-\int_{\gamma_3} e^{-z^2}\, dz = \int_0^\infty e^{-x^2 e^{i\pi/2}} e^{i\pi/4}\, dx$$

$$= e^{i\pi/4} \int_0^\infty e^{-ix^2}\, dx$$

$$= e^{i\pi/4} \int_0^\infty \cos(-x^2) + i\sin(-x^2)\, dx$$

$$= e^{i\pi/4} \int_0^\infty \cos(x^2) - i\sin(x^2)\ dx$$

$$= \frac{\sqrt{\pi}}{2}.$$

Therefore, moving the $(1+i)/\sqrt{2}$ to the other side of the equation and taking real and imaginary parts, we obtain the equalities

$$\text{Re}\left(\int_0^\infty \cos(x^2) - i\sin(x^2)\ dx\right) = \int_0^\infty \cos(x^2)\ dx$$

$$= \frac{\sqrt{2\pi}}{4}$$

$$-\text{Im}\left(\int_0^\infty \cos(x^2) - i\sin(x^2)\ dx\right) = \int_0^\infty \sin(x^2)\ dx$$

$$= \frac{\sqrt{2\pi}}{4},$$

which completes the computation.