

# Selected Exercises

Complex Analysis

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Here are selected problems I solved while self-studying *Complex Analysis* by Stein and Sharkarchi.

**1.9.** We first deduce the Cauchy-Riemann equations in polar coordinates. Notice that:

$$\begin{aligned}\frac{\partial u}{\partial r} &= \frac{\partial u}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial r} \\&= \frac{\partial u}{\partial x} \cos \theta + \frac{\partial u}{\partial y} \sin \theta \\&= \frac{1}{r} \frac{\partial v}{\partial y} r \cos \theta - \frac{1}{r} \frac{\partial v}{\partial x} r \sin \theta \\&= \frac{1}{r} \frac{\partial v}{\partial \theta} \\ \frac{1}{r} \frac{\partial u}{\partial \theta} &= \frac{1}{r} \left( \frac{\partial u}{\partial x} \frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial \theta} \right) \\&= \frac{1}{r} \left( -\frac{\partial u}{\partial x} r \sin \theta + \frac{\partial u}{\partial y} r \cos \theta \right) \\&= -\frac{\partial v}{\partial y} \sin \theta - \frac{\partial v}{\partial x} \cos \theta \\&= -\frac{\partial v}{\partial r}.\end{aligned}$$

Write  $\log z = \log r + i\theta = u(r, \theta) + iv(r, \theta)$ . By Theorem 2.4, it suffices to show  $u$  and  $v$  satisfy the Cauchy-Riemann equations. Clearly, for the given domain, we have

$$\frac{\partial u}{\partial r} = \frac{1}{r} \quad \frac{\partial u}{\partial \theta} = 0 \quad \frac{\partial v}{\partial r} = 0 \quad \frac{\partial v}{\partial \theta} = 1.$$

Thus, we get that

$$\frac{\partial u}{\partial r} = \frac{1}{r} = \frac{1}{r} \frac{\partial v}{\partial \theta}$$

and

$$\frac{1}{r} \frac{\partial u}{\partial \theta} = 0 = -\frac{\partial v}{\partial r}.$$

Thus,  $\log z$  is holomorphic for  $r > 0$  and  $\theta \in (-\pi, \pi)$ .

**1.10.** Applying the differentiation operators as usual, we compute

$$\begin{aligned} 4 \frac{\partial}{\partial z} \frac{\partial}{\partial \bar{z}} &= 2 \frac{\partial}{\partial z} \frac{\partial}{\partial x} - \frac{2}{i} \frac{\partial}{\partial z} \frac{\partial}{\partial y} \\ &= \frac{\partial^2}{\partial x^2} + \frac{1}{i} \frac{\partial^2}{\partial y \partial x} - \frac{1}{i} \frac{\partial^2}{\partial x \partial y} + \frac{\partial^2}{\partial y^2} \\ &= \Delta \end{aligned}$$

$$\begin{aligned} 4 \frac{\partial}{\partial \bar{z}} \frac{\partial}{\partial z} &= 2 \frac{\partial}{\partial \bar{z}} \frac{\partial}{\partial x} + \frac{2}{i} \frac{\partial}{\partial \bar{z}} \frac{\partial}{\partial y} \\ &= \frac{\partial^2}{\partial x^2} - \frac{1}{i} \frac{\partial^2}{\partial y \partial x} + \frac{1}{i} \frac{\partial^2}{\partial x \partial y} + \frac{\partial^2}{\partial y^2} \\ &= \Delta. \end{aligned}$$