Principal G-Bundles

Principal G-bundles are fiber bundles $\pi : P \to X$ such that G acts freely and transitively as a right action on the fibers. There are many important examples of such bundles. Now, let $\pi : E \to X$ be a rank n real vector bundle. If we construct the fiber over $x \in X$ as the set of bases of $\pi^{-1}(x)$, we get a principal $\mathrm{GL}(n; \mathbb{R})$-bundle. Similarly, if E is oriented, we get a principal $\mathrm{GL}^{+}(n; \mathbb{R})$-bundle by this same process. If we can fix a metric on E, we get a principal $\mathrm{O}(n)$-bundle. Finally, if X is oriented, we can get a principal $\mathrm{SO}(n)$-bundle.

An important property of principal G-bundles is that $H^1(X; G)$ can be identified with the set of principal G-bundles for a given G modulo homeomorphisms that preserve the projection map. If we let $\mathrm{Prin}_G(X)$ denote this moduli space, it follows that $\mathrm{Prin}_G(X) = [X, BG]$, where BG is the classifying space of G.

Another interesting construction involving principal G-bundles is the associated bundle. Associated bundles are useful, especially when considered together with the adjoint representation. If $\pi : P \to X$ is a principal G-bundle and $\phi : G \to \mathrm{Homeo}(F)$ be a left action of G on F. One can define a fiber bundle associated to P by ϕ by setting the total space equal to $P \times F/G$, where G acts as a right action by $(p, f) \cdot g = (p \cdot g, \phi(g^{-1})f)$. The projection map is defined by composing π with the projection onto P. We call this new fiber bundle $P \times_{\phi} F$. In the case in which $\phi = \mathrm{Ad}$, we get the adjoint bundle: $\mathfrak{g}_P = P \times_{\mathrm{Ad}} \mathfrak{g}$, where $\mathfrak{g}$ is the Lie algebra of G. Thus, this is naturally a bundle of Lie algebras.