

Chern Classes

Chern classes are important characteristic classes in geometry and topology. They are similar to Stiefel-Whitney classes to some extent, but they are defined for complex vector bundles and integer cohomology. Now, to formally define Chern classes, let $\pi : E \rightarrow B$ be a complex vector bundle and E_0 be E with the image of the zero section removed. Next, fix a Hermitian metric on E , and define a new vector bundle over E_0 by letting the fiber be the orthogonal complement under the chosen metric. Call this new bundle E' .

We can now finally define Chern classes. Chern classes are elements $c_i(E) \in H^{2i}(B; \mathbb{Z})$, which are defined inductively. If E has rank n , then $c_n(E) = e(E)$, where $e(E)$ is understood to be the Euler class of E considered as a real vector bundle. Then, for $i < n$, $c_i(E) = (\pi_0^*)^{-1}c_i(E')$, where $\pi_0 : E_0 \rightarrow B$. Finally, $c_i(E) = 0$ if $i > n$.

Some consequences of this definition include that the Chern classes are natural, $c_i(E_1 \oplus E_2) = c_i(E_1)$ if E_2 is trivial, and $c(E_1 \oplus E_2) = c(E_1) \smile c(E_2)$, where $c(E) = \sum c_j(E)$. Note also that there are two other definitions of Chern classes that are commonly used. The first is similar to the axiomatic definition of Stiefel-Whitney classes, and the second is defined through Chern-Weil theory.

Now we discuss the Chern-Weil theory approach. Fix a connection on E , and let F be the curvature of this connection. We can write $\det(\text{id} + A) = 1 + P_1(A) + \dots + P_n(A)$, where P_j are degree j homogeneous polynomials. Then, define $c_j(E) = [P_j(\frac{i}{2\pi}F)] \in H^{2j}(B; \mathbb{Z})$. The fact that these classes are integral requires a bit of justification, but this definition turns out to be equivalent to the other definitions.