Selected Exercises

Complex Analysis
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Here are selected problems I solved while self-studying *Complex Analysis* by Stein and Sharkarchi.
1.9. We first deduce the Cauchy-Riemann equations in polar coordinates. Notice that:

$$\begin{aligned}
\frac{\partial u}{\partial r} &= \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} \\
&= \frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta \\
&= \frac{1}{r}\frac{\partial v}{\partial y} r\cos\theta - \frac{1}{r}\frac{\partial v}{\partial x} r\sin\theta \\
&= \frac{1}{r}\frac{\partial v}{\partial \theta} \\
\frac{1}{r}\frac{\partial u}{\partial \theta} &= \frac{1}{r}\left(\frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta}\right) \\
&= \frac{1}{r}\left(-\frac{\partial u}{\partial x} r\sin\theta + \frac{\partial u}{\partial y} r\cos\theta\right) \\
&= -\frac{\partial v}{\partial y}\sin\theta - \frac{\partial v}{\partial x}\cos\theta \\
&= -\frac{\partial v}{\partial r}.
\end{aligned}$$

Write $\log z = \log r + i\theta = u(r,\theta) + iv(r,\theta)$. By Theorem 2.4, it suffices to show u and v satisfy the Cauchy-Riemann equations. Clearly, for the given domain, we have

$$\frac{\partial u}{\partial r} = \frac{1}{r} \quad \frac{\partial u}{\partial \theta} = 0 \quad \frac{\partial v}{\partial r} = 0 \quad \frac{\partial v}{\partial \theta} = 1.$$

Thus, we get that

$$\frac{\partial u}{\partial r} = \frac{1}{r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$$

and

$$\frac{1}{r}\frac{\partial u}{\partial \theta} = 0 = -\frac{\partial v}{\partial r}.$$

Thus, $\log z$ is holomorphic for $r > 0$ and $\theta \in (-\pi, \pi)$.

1.10. Applying the differentiation operators as usual, we compute

$$\begin{aligned}
4\frac{\partial}{\partial z}\frac{\partial}{\partial \bar{z}} &= 2\frac{\partial}{\partial z}\frac{\partial}{\partial x} - \frac{2}{i}\frac{\partial}{\partial z}\frac{\partial}{\partial y} \\
&= \frac{\partial^2}{\partial x^2} + \frac{1}{i}\frac{\partial^2}{\partial y \partial x} - \frac{1}{i}\frac{\partial^2}{\partial x \partial y} + \frac{\partial^2}{\partial y^2} \\
&= \Delta
\end{aligned}$$

$$\begin{aligned}
4\frac{\partial}{\partial \bar{z}}\frac{\partial}{\partial z} &= 2\frac{\partial}{\partial \bar{z}}\frac{\partial}{\partial x} + \frac{2}{i}\frac{\partial}{\partial \bar{z}}\frac{\partial}{\partial y} \\
&= \frac{\partial^2}{\partial x^2} - \frac{1}{i}\frac{\partial^2}{\partial y \partial x} + \frac{1}{i}\frac{\partial^2}{\partial x \partial y} + \frac{\partial^2}{\partial y^2} \\
&= \Delta.
\end{aligned}$$

1.24. Using the reverse parameterization defined in the book, we observe

$$\begin{aligned}
\int_{\gamma^-} f(x)\ dz &= \int_a^b f(z^-(t))(z^-)'(t)\ dt \\
&= -\int_a^b f(z(a+b-t))z'(b+a-t)\ dt \\
&= \int_b^a f(z(u))z'(u)\ du \\
&= -\int_a^b f(z(u))z'(u)\ du
\end{aligned}$$

$$= -\int_{\gamma} f(z)\, dz.$$