# Chern Classes

Chern classes are important characteristic classes in geometry and topology. They are similar to Stiefel-Whitney classes to some extent, but they are defined for complex vector bundles and integer cohomology. Now, to formally define Chern classes, let $\pi : E \to B$ be a complex vector bundle and $E_0$ be $E$ with the image of the zero section removed. Next, fix a Hermitian metric on $E$, and define a new vector bundle over $E_0$ by letting the fiber be the orthogonal complement under the chosen metric. Call this new bundle $E'$.

We can now finally define Chern classes. Chern classes are elements $c_i(E) \in H^{2i}(B;\mathbb{Z})$, which are defined inductively. If $E$ has rank $n$, then $c_n(E) = e(E)$, where $e(E)$ is understood to be the Euler class of $E$ considered as a real vector bundle. Then, for $i < n$, $c_i(E) = (\pi_0^*)^{-1} c_i(E')$, where $\pi_0 : E_0 \to B$. Finally, $c_i(E) = 0$ if $i > n$.

Some consequences of this definition include that the Chern classes are natural, $c(E_1 \oplus E_2) = c(E_1)$ if $E_2$ is trivial, and $c(E_1 \oplus E_2) = c(E_1) \smile c(E_2)$, where $c(E) = \sum c_j(E)$. Note also that there are two other definitions of Chern classes that are commonly used. The first is similar to the axiomatic definition of Stiefel-Whitney classes, and the second is defined through Chern-Weil theory.

Now we discuss the Chern-Weil theory approach. Fix a connection on $E$, and let $F$ be the curvature of this connection. We can write $\det(\mathrm{id} + A) = 1 + P_1(A) + \ldots + P_n(A)$, where $P_j$ are degree $j$ homogeneous polynomials. Then, define $c_j(E) = [P_j(\frac{i}{2\pi}F)] \in H^{2j}(B;\mathbb{Z})$. The fact that these classes are integral requires a bit of justification, but this definition turns out to be equivalent to the other definitions.