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Some algebraic properties of the Reticular Action Model for moment structures

J. Jack McArdle and **Roderick P. McDonald**

A number of models for the analysis of moment structures, such as LISREL, have recently been shown to be capable of being given a particularly simple and economical representation, in terms of the Reticular Action Model (RAM). In contrast to previous treatments, a formal algebraic treatment is provided which shows that RAM directly incorporates many common structural models, including models describing the structure of means. It is also shown here that RAM treats coefficient matrices with patterned inverses simply and generally.

1. Introduction

There has recently been some interest in relationships among models that have been proposed for the structural analysis of covariance matrices and mean vectors (i.e. moment structures). Seminal developments in statistical factor analysis (e.g. Anderson & Rubin, 1956) led to Bock & Bargmann's (1966) introduction of a group of models for the *analysis of covariance structures*. This basic foundation was further expanded by Jöreskog to include a remarkably flexible second-order common factor model (ACOVS, 1970), and a variant accounting also for first-moments (ACOVSM, 1973*a*). The introduction of a patterned inverse matrix in the model was indicated in the treatment by Keesling & Wiley (see Wiley, 1973) and by Jöreskog (1973*b*) of a model for *linear structural relations* (*LISREL*). More recently, McDonald (1978, 1979) described an mth-order COSAN model in which the inverse of any matrix in the model could be patterned as desired, and which, as a consequence, yields the ACOVS model of any order and models of the LISREL type.

The general chronological sequence outlined above shows a broad tendency toward increasing complexity of the matrix representation of the models accompanying an apparent increase in generality. Most recently, however, McArdle (1978, 1979*a*, 1980) used graphic concepts from *latent variable path analysis* (see Wright, 1934; Wold, 1980) to develop a structural equation system termed the Reticular Action Model (RAM), whose algebraic representation required just three matrices. McArdle (1979*b*, *c*) and Horn & McArdle (1980) have previously shown that many seemingly more complex models for the structural analysis of multivariate data, such as LISREL, can be represented in terms of the more compact RAM algebra.

The object of this paper is to show how the original RAM logic (*a*) may be presented in a formal and compact algebraic notation, (*b*) provides an algebraic device for the specification of a broad class of linear models, (*c*) incorporates many models for structures on means, (*d*) treats models involving patterned inverses generally, and, hence, (*e*) includes all necessary and sufficient parameter matrices for a general structural equation system with these useful features.

2. Basic RAM algebra

Let

$$\mathbf{v} = \mathbf{A}\mathbf{v} + \mathbf{u}, \tag{1}$$

where $\mathbf{A}$ is a $(t \times t)$ matrix of *asymmetric* coefficients, and $\mathbf{v}$ and $\mathbf{u}$ are $(t \times 1)$ vectors of random variables. In most applications, the diagonal of $\mathbf{A}$ consists of zeros, and each component of $\mathbf{v}$ is expressed as a linear combination of the remaining variables, plus a residual $\mathbf{u}$. This resembles and is a generalization of Guttman's (1953) image theory; the components of $\mathbf{Av}$ are the images of the t variables in terms of the other $t-1$ variables while the components of $\mathbf{u}$ are anti-images. If it also happens that the ith row of $\mathbf{A}$ consists of zeros, then the variable $\mathbf{v}_i$ is the same as its own residual $\mathbf{u}_i$. We let

$$\mathbf{S} = \mathscr{E}\{\mathbf{uu}'\}, \tag{2}$$

where $\mathbf{S}$ is a $(t \times t)$ matrix of *symmetric* coefficients, and where $\mathscr{E}$ denotes the expectation operator. We also let the $(t \times t)$ symmetric matrix

$$\mathbf{C} = \mathscr{E}\{\mathbf{vv}'\}. \tag{3}$$

It follows that equation (1) may now be rewritten as

$$\mathbf{u} = (\mathbf{I}-\mathbf{A})\,\mathbf{v} \tag{4}$$

so, by assuming that $\mathbf{A}$ is patterned so $(\mathbf{I}-\mathbf{A})$ is non-singular,

$$\mathbf{v} = (\mathbf{I}-\mathbf{A})^{-1}\,\mathbf{u}. \tag{5}$$

We may also use the well-known identity

$$\mathbf{E} = (\mathbf{I}-\mathbf{A})^{-1} = \sum_{r=0}^{\infty} \mathbf{A}^r = \mathbf{I}+\mathbf{A}+\mathbf{A}^2+\ldots+\mathbf{A}^{\infty}, \tag{6}$$

where $\mathbf{A}^0 = \mathbf{I}$, and which may be verified by $(\mathbf{I}-\mathbf{A})$ multiplication (e.g. Hohn, 1958, pp. 217–224). The geometric series terminates at $r+1$ terms if $\mathbf{A}^r \neq \mathbf{0}$ while $\mathbf{A}^{r+1} = \mathbf{0}$. That is, $\mathbf{A}$ is nilpotent of index r (Stein, 1967, p. 24). In those cases where the ith main diagonal of any $\mathbf{A}^r$ is non-null, the geometric series is infinite and conditions for its 'stability' may be required (e.g. Fox, 1980).

Using these basic definitions it now follows that

$$\mathbf{S} = (\mathbf{I}-\mathbf{A})\,\mathbf{C}(\mathbf{I}-\mathbf{A})' = \mathbf{E}^{-1}\,\mathbf{C}\mathbf{E}^{-1\prime} \tag{7}$$

and

$$\mathbf{C} = (\mathbf{I}-\mathbf{A})^{-1}\,\mathbf{S}(\mathbf{I}-\mathbf{A})^{-1\prime} = \mathbf{ESE}'. \tag{8}$$

Now let $\mathbf{v}$ be partitioned into two subvectors; $\mathbf{g}$ of p components, and $\mathbf{h}$ of q components. That is, $t = p+q$ and

$$\mathbf{v} = \begin{bmatrix} \mathbf{g} \\ \mathbf{h} \end{bmatrix}. \tag{9}$$

The p components of $\mathbf{g}$ may be considered observed, manifest or *given* variables, and the q components of $\mathbf{h}$ may be considered unobserved, latent or *hidden* variables (common factors, latent traits, true scores, etc.). We define

$$\mathbf{F} = [\mathbf{I} \;:\; \mathbf{0}] \tag{10}$$

as a $(p \times t)$ matrix with a $(p \times p)$ identity submatrix and a $(p \times q)$ null submatrix so that

$$\mathbf{g} = \mathbf{Fv}, \tag{11}$$

and so, following (5),

$$\mathbf{g} = \mathbf{F}(\mathbf{I}-\mathbf{A})^{-1}\mathbf{u} = \mathbf{FEu}. \tag{12}$$

Finally, we define the $(p \times p)$ symmetric matrix

$$\mathbf{M} = \mathscr{E}\{\mathbf{gg}'\}, \tag{13}$$

whereby applying (8) to (12) then yields

$$\mathbf{M} = \mathbf{F}(\mathbf{I}-\mathbf{A})^{-1}\mathbf{S}(\mathbf{I}-\mathbf{A})^{-1\prime}\mathbf{F}' = \mathbf{FESE'F'} = \mathbf{FCF'}. \tag{14}$$

Equations (1) to (14) completely define RAM. In most applications we will wish to explain the structure of the relations of the manifest variables, as given by **M**, in terms of the structure of the relations among all variables, as given by **A** and **S** in **C**. In all cases, the matrices **A** and **S** are *patterned*; the elements of these matrices could be prescribed constants, parameters that are free to be individually estimated, or parameters that are constrained to be a function of one or more other parameters (for details see McDonald, 1978; Jöreskog & Sörbom, 1979). In contrast to **A** and **S**, **F** is a fixed known matrix of prescribed unity and zero constants that acts to filter or select the manifest variables out of the full set of manifest and latent variables. If, for any reason, the components of **v** are permuted to some mixed order, the columns of **F** can be correspondingly permuted. Of course, the rows and columns of **C** that are filtered out by **F** commonly contain useful information about the latent variable structure.

3. Basic RAM specification

We now illustrate the simplicity and flexibility of RAM by the specification of five alternative and well-known structural equation models for four vectors of manifest variables. For simplicity in this section we provide only a few basic details on each model, and we assume all means (i.e. first-moments) are zero.

Let us first consider a *recursive path analysis* model written as

$$\mathbf{w} = \mathbf{Hx}+\mathbf{e}, \quad \mathbf{x} = \mathbf{Jy}+\mathbf{f}, \quad \mathbf{y} = \mathbf{Kz}+\mathbf{m}, \tag{15}$$

where **w**, **x**, **y** and **z** are observed vectors, **H**, **J** and **K** are matrices of regression coefficients, and **e**, **f** and **m** are residuals. This particular model configuration may be recognized as a 'causal chain' (Wold, 1954) or a 'simplex process' (Guttman, 1954). To complete the model we write

$$\mathbf{P} = \mathscr{E}\{\mathbf{ee}'\}, \quad \mathbf{Q} = \mathscr{E}\{\mathbf{ff}'\}, \quad \mathbf{R} = \mathscr{E}\{\mathbf{mm}'\}, \quad \mathbf{T} = \mathscr{E}\{\mathbf{zz}'\}. \tag{16}$$

To express this model in terms of (1) to (14) we write

$$\mathbf{v} = \begin{bmatrix} \mathbf{w} \\ \mathbf{x} \\ \mathbf{y} \\ \mathbf{z} \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} \mathbf{0} & \mathbf{H} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{J} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{K} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}, \quad \mathbf{u} = \begin{bmatrix} \mathbf{e} \\ \mathbf{f} \\ \mathbf{m} \\ \mathbf{z} \end{bmatrix}, \tag{17}$$

which yields, corresponding to (1),

$$\begin{bmatrix} \mathbf{w} \\ \mathbf{x} \\ \mathbf{y} \\ \mathbf{z} \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{H} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{J} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{K} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{w} \\ \mathbf{x} \\ \mathbf{y} \\ \mathbf{z} \end{bmatrix} + \begin{bmatrix} \mathbf{e} \\ \mathbf{f} \\ \mathbf{m} \\ \mathbf{z} \end{bmatrix}, \tag{18}$$

and, corresponding to (2),

$$\mathbf{S} = \mathscr{E}\left\{ \begin{bmatrix} \mathbf{e} \\ \mathbf{f} \\ \mathbf{m} \\ \mathbf{z} \end{bmatrix} [\mathbf{e}' \quad \mathbf{f}' \quad \mathbf{m}' \quad \mathbf{z}'] \right\} = \begin{bmatrix} \mathbf{P} & & & \text{sym} \\ \mathbf{0} & \mathbf{Q} & & \\ \mathbf{0} & \mathbf{0} & \mathbf{R} & \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{T} \end{bmatrix}. \tag{19}$$

By (6), the required inverse may be represented as the geometric series

$$\mathbf{E} = \begin{bmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} \end{bmatrix} + \begin{bmatrix} \mathbf{0} & \mathbf{H} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{J} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{K} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} + \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{HJ} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{JK} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}$$
$$+ \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{HJK} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{H} & \mathbf{HJ} & \mathbf{HJK} \\ \mathbf{0} & \mathbf{I} & \mathbf{J} & \mathbf{JK} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{K} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} \end{bmatrix}, \tag{20}$$

which is nilpotent of index 3 (i.e. $\mathbf{A}^4 = \mathbf{0}$), as may be verified on multiplication by $(\mathbf{I}-\mathbf{A})$. In traditional path analytic terms, $\mathbf{E}$ represents a matrix of *total effects*, which may be further decomposed into *direct effects* (here $\mathbf{A}$), *indirect effects* (here $\mathbf{A}^2+\mathbf{A}^3$), and more generally interpreted in terms of *effects of length r* (i.e. $\mathbf{A}^r, r = 0, 3$; see Alwin & Hauser, 1975; Fox, 1980).

Corresponding to (5), we can rewrite (18) as

$$\begin{bmatrix} \mathbf{w} \\ \mathbf{x} \\ \mathbf{y} \\ \mathbf{z} \end{bmatrix} = \left[\begin{bmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} \end{bmatrix} - \begin{bmatrix} \mathbf{0} & \mathbf{H} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{J} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{K} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \right]^{-1} \begin{bmatrix} \mathbf{e} \\ \mathbf{f} \\ \mathbf{m} \\ \mathbf{z} \end{bmatrix}$$
$$= \begin{bmatrix} \mathbf{I} & \mathbf{H} & \mathbf{HJ} & \mathbf{HJK} \\ \mathbf{0} & \mathbf{I} & \mathbf{J} & \mathbf{JK} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{K} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{e} \\ \mathbf{f} \\ \mathbf{m} \\ \mathbf{z} \end{bmatrix} \tag{21}$$

and, corresponding to (8), the model structure may be expressed as

$$\mathbf{C}=\begin{bmatrix}\mathbf{I} & \mathbf{H} & \mathbf{HJ} & \mathbf{HJK}\\ \mathbf{0} & \mathbf{I} & \mathbf{J} & \mathbf{JK}\\ \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{K}\\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I}\end{bmatrix}\begin{bmatrix}\mathbf{P} & \mathbf{0} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{Q} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{R} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{T}\end{bmatrix}\begin{bmatrix}\mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0}\\ \mathbf{H}' & \mathbf{I} & \mathbf{0} & \mathbf{0}\\ \mathbf{J}'\mathbf{H}' & \mathbf{J}' & \mathbf{I} & \mathbf{0}\\ \mathbf{K}'\mathbf{J}'\mathbf{H}' & \mathbf{K}'\mathbf{J}' & \mathbf{K}' & \mathbf{I}\end{bmatrix} \tag{22}$$

$$=\begin{bmatrix}\mathbf{P}+\mathbf{HQH}'+\mathbf{HJRJ}'\mathbf{H}'+\mathbf{HJKTK}'\mathbf{J}'\mathbf{H}' & & & \text{sym}\\ \mathbf{QH}'+\mathbf{JRJ}'\mathbf{H}'+\mathbf{JKTK}'\mathbf{J}'\mathbf{H}' & \mathbf{Q}+\mathbf{JRJ}'+\mathbf{JKTK}'\mathbf{J}' & & \\ \mathbf{RJ}'\mathbf{H}'+\mathbf{KTK}'\mathbf{J}'\mathbf{H}' & \mathbf{RJ}'+\mathbf{KTK}'\mathbf{J}' & \mathbf{R}+\mathbf{KTK}' & \\ \mathbf{TK}'\mathbf{J}'\mathbf{H}' & \mathbf{TK}'\mathbf{J}' & \mathbf{TK}' & \mathbf{T}\end{bmatrix}$$

$$=\begin{bmatrix}\mathbf{P}+\mathbf{HT}^{**}\mathbf{H}' & & & \text{sym}\\ \mathbf{T}^{**}\mathbf{H}' & \mathbf{T}^{**} & & \\ \mathbf{T}^{*}\mathbf{J}'\mathbf{H}' & \mathbf{T}^{*}\mathbf{J}' & \mathbf{T}^{*} & \\ \mathbf{TK}'\mathbf{J}'\mathbf{H}' & \mathbf{TK}'\mathbf{J}' & \mathbf{TK}' & \mathbf{T}\end{bmatrix},$$

where $\mathbf{T}^* = \mathbf{R}+\mathbf{KTK}'$ and $\mathbf{T}^{**} = \mathbf{Q}+\mathbf{JT}^*\mathbf{J}'$. In this model only relations among manifest variables are specified so $t = p$, $\mathbf{g} = \mathbf{v}$, $\mathbf{F} = \mathbf{I}$ and $\mathbf{M} = \mathbf{C}$ as given. This final representation (22) is entirely consistent with Wright's (1934) 'multiplication' or 'tracing' rules for the path analysis decomposition of all model covariances in either extended (e.g. $\mathbf{R}+\mathbf{KTK}'$) or compact (e.g. $\mathbf{T}^*$) form (see Alwin & Hauser, 1975; Duncan, 1975).

Let us next consider a *first-order factor analysis model* written as

$$\mathbf{w} = \mathbf{Ha}+\mathbf{e},\quad \mathbf{x} = \mathbf{Ja}+\mathbf{f},\quad \mathbf{y} = \mathbf{Kb}+\mathbf{m},\quad \mathbf{z} = \mathbf{Lb}+\mathbf{n}, \tag{23}$$

where, again, **w**, **x**, **y** and **z** are observed variables, but **a** and **b** are unobserved common factors, **H**, **J**, **K** and **L** are reinterpreted as factor pattern coefficients, and **e**, **f**, **m** and **n** are reinterpreted as unobserved unique factors. We write

$$\begin{aligned}&\mathbf{P} = \mathscr{E}\{\mathbf{ee}'\},\quad \mathbf{Q} = \mathscr{E}\{\mathbf{ff}'\},\quad \mathbf{R} = \mathscr{E}\{\mathbf{mm}'\},\quad \mathbf{T} = \mathscr{E}\{\mathbf{nn}'\},\\ &\mathbf{V} = \mathscr{E}\{\mathbf{aa}'\},\quad \mathbf{U} = \mathscr{E}\{\mathbf{bb}'\},\quad \mathbf{W} = \mathscr{E}\{\mathbf{ab}'\}.\end{aligned} \tag{24}$$

This particular model configuration may be recognized as a 'non-overlapping, oblique' common factor model (e.g. Cattell, 1965).

To express this model in terms of (1) to (14) we write

$$\mathbf{v}=\begin{bmatrix}\mathbf{w}\\ \mathbf{x}\\ \mathbf{y}\\ \mathbf{z}\\ \mathbf{a}\\ \mathbf{b}\end{bmatrix},\quad \mathbf{A}=\begin{bmatrix}\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{H}&\mathbf{0}\\ \mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{J}&\mathbf{0}\\ \mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{K}\\ \mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{L}\\ \mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}\\ \mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}\end{bmatrix},\quad \mathbf{u}=\begin{bmatrix}\mathbf{e}\\ \mathbf{f}\\ \mathbf{m}\\ \mathbf{n}\\ \mathbf{a}\\ \mathbf{b}\end{bmatrix}, \tag{25}$$

which yields, corresponding to (1),

$$\begin{bmatrix} \mathbf{w} \\ \mathbf{x} \\ \mathbf{y} \\ \mathbf{z} \\ \mathbf{a} \\ \mathbf{b} \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{H} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{J} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{K} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{L} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{w} \\ \mathbf{x} \\ \mathbf{y} \\ \mathbf{z} \\ \mathbf{a} \\ \mathbf{b} \end{bmatrix} + \begin{bmatrix} \mathbf{e} \\ \mathbf{f} \\ \mathbf{m} \\ \mathbf{n} \\ \mathbf{a} \\ \mathbf{b} \end{bmatrix}, \tag{26}$$

and, corresponding to (2),

$$\mathbf{S} = \mathscr{E}\left\{ \begin{bmatrix} \mathbf{e} \\ \mathbf{f} \\ \mathbf{m} \\ \mathbf{n} \\ \mathbf{a} \\ \mathbf{b} \end{bmatrix} [\mathbf{e}' \quad \mathbf{f}' \quad \mathbf{m}' \quad \mathbf{n}' \quad \mathbf{a}' \quad \mathbf{b}'] \right\} = \begin{bmatrix} \mathbf{P} & & & & & \text{sym} \\ \mathbf{0} & \mathbf{Q} & & & & \\ \mathbf{0} & \mathbf{0} & \mathbf{R} & & & \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{T} & & \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{V} & \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{W} & \mathbf{U} \end{bmatrix}. \tag{27}$$

The required inverse may be represented by the $r = 1$ geometric series

$$\mathbf{E} = \begin{bmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} \end{bmatrix} + \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{H} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{J} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{K} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{L} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{H} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{J} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{K} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{L} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} \end{bmatrix}, \tag{28}$$

so, corresponding to (8), the model structure may now be completed as

$$\mathbf{C} = \begin{bmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{H} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{J} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{K} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{L} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{P} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{Q} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{R} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{T} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{V} & \mathbf{W}' \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{W} & \mathbf{U} \end{bmatrix} \begin{bmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{H}' & \mathbf{J}' & \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{K}' & \mathbf{L}' & \mathbf{0} & \mathbf{I} \end{bmatrix} \tag{29}$$

$$= \begin{bmatrix} \mathbf{P}+\mathbf{HVH}' & & & & & \text{sym} \\ \mathbf{JVH}' & \mathbf{Q}+\mathbf{JVJ}' & & & & \\ \mathbf{KWH}' & \mathbf{KWJ}' & \mathbf{R}+\mathbf{KUK}' & & & \\ \mathbf{LWH}' & \mathbf{LWJ}' & \mathbf{LUK}' & \mathbf{T}+\mathbf{LUL}' & & \\ \mathbf{VH}' & \mathbf{VJ}' & \mathbf{W}'\mathbf{K}' & \mathbf{W}'\mathbf{L}' & \mathbf{V} & \\ \mathbf{WH}' & \mathbf{WJ}' & \mathbf{UK}' & \mathbf{UL}' & \mathbf{W} & \mathbf{U} \end{bmatrix}.$$

This model includes a mixture of manifest and latent variables so, following (10) and (11), we define

$$\mathbf{g} = \begin{bmatrix} \mathrm{w} \\ \mathrm{x} \\ \mathrm{y} \\ \mathrm{z} \end{bmatrix}, \quad \mathbf{F} = \begin{bmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} \end{bmatrix}, \tag{30}$$

and, corresponding to (12), we now write

$$\begin{bmatrix} \mathrm{w} \\ \mathrm{x} \\ \mathrm{y} \\ \mathrm{z} \end{bmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{H} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{J} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{K} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{L} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{e} \\ \mathbf{f} \\ \mathbf{m} \\ \mathbf{n} \\ \mathbf{a} \\ \mathbf{b} \end{bmatrix}, \tag{31}$$

and finally, corresponding to (12), the application of (30) to (29) yields

$$\mathbf{M} = \begin{bmatrix} \mathbf{P}+\mathbf{HVH}' & & & \mathrm{sym} \\ \mathbf{JVH}' & \mathbf{Q}+\mathbf{JVJ}' & & \\ \mathbf{KWH}' & \mathbf{KWJ}' & \mathbf{R}+\mathbf{KUK}' & \\ \mathbf{LWH}' & \mathbf{LWJ}' & \mathbf{LUK}' & \mathbf{T}+\mathbf{LUL}' \end{bmatrix}, \tag{32}$$

which, by (14), is equivalent to the first $p = 4$ rows and columns of **C** in (29). If the latent variables **a** or **b** are not listed last as in **v** above, the columns of **F** need to be correspondingly permuted. In this model the rows and columns of **C** that are filtered out by **F** contain the traditional factor structure and inter-factor covariance submatrices.

The two previous examples illustrate the basic features of RAM specification. These principles can be used for the specification of many seemingly more complex model configurations. Let us briefly consider a *restricted canonical analysis* model written as

$$\mathbf{w} = \mathbf{Ha}+\mathbf{e}, \quad \mathbf{x} = \mathbf{Ja}+\mathbf{f}, \quad \mathbf{a} = \mathbf{Ky}+\mathbf{Lz}+\mathbf{d}, \tag{33}$$

where, again, **w**, **x**, **y** and **z** represent observed variables, **a** represents an unobserved common factor, **H** and **J** represent factor pattern coefficients, and **e** and **f** are unobserved unique factors, while **K** and **L** are reinterpreted as coefficients for the regression of unobserved **a** upon observed **y** and **z** with residual **d**. We write

$$\begin{aligned} &\mathbf{P} = \mathscr{E}\{\mathbf{ee}'\}, \quad \mathbf{Q} = \mathscr{E}\{\mathbf{ff}'\}, \quad \mathbf{R} = \mathscr{E}\{\mathbf{yy}'\}, \quad \mathbf{T} = \mathscr{E}\{\mathbf{zz}'\}, \\ &\mathbf{V} = \mathscr{E}\{\mathbf{yz}'\}, \quad \mathbf{U} = \mathscr{E}\{\mathbf{ef}'\}, \quad \mathbf{W} = \mathscr{E}\{\mathbf{dd}'\}, \end{aligned} \tag{34}$$

This configuration, with the single latent variable **a** serving as a 'mediator' for multiple manifest variable regressions, has been used to represent a 'multiple indicators & multiple causes' model (e.g. MIMIC, Hauser & Goldberger, 1971) and, usually given that $\mathbf{W} = \mathbf{0}$, a 'structural canonical' model (e.g. Bagozzi *et al.*, 1981).

To express this model in terms of (1) to (14) we can write

$$\mathbf{v} = \begin{bmatrix} \mathbf{w} \\ \mathbf{x} \\ \mathbf{a} \\ \mathbf{z} \\ \mathbf{y} \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{H} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{J} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{K} & \mathbf{L} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}, \quad \mathbf{u} = \begin{bmatrix} \mathbf{e} \\ \mathbf{f} \\ \mathbf{d} \\ \mathbf{z} \\ \mathbf{y} \end{bmatrix}, \quad \mathbf{S} = \begin{bmatrix} \mathbf{P} & & & & \text{sym} \\ \mathbf{U} & \mathbf{Q} & & & \\ \mathbf{0} & \mathbf{0} & \mathbf{W} & & \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{R} & \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{V} & \mathbf{T} \end{bmatrix} \tag{35}$$

and whose required inverse may be represented by the $r = 2$ termed geometric series

$$\mathbf{E} = \begin{bmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} \end{bmatrix} + \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{H} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{J} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{K} & \mathbf{L} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} + \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{HK} & \mathbf{HL} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{JK} & \mathbf{JL} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{I} & \mathbf{0} & \mathbf{H} & \mathbf{HK} & \mathbf{HL} \\ \mathbf{0} & \mathbf{I} & \mathbf{J} & \mathbf{JK} & \mathbf{JL} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{K} & \mathbf{L} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} \end{bmatrix}. \tag{36}$$

It follows directly that the overall model structure **C** may be simply created from **S** and **E** by (8) and, upon defining

$$\mathbf{F} = \begin{bmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} \end{bmatrix}, \tag{37}$$

the manifest structure **M** may be completed as (14).

Let us also briefly consider the *second-order factor analysis model* written as

$$\mathbf{w} = \mathbf{Ha}+\mathbf{e}, \quad \mathbf{x} = \mathbf{Ja}+\mathbf{f}, \quad \mathbf{y} = \mathbf{Kb}+\mathbf{m}, \quad \mathbf{z} = \mathbf{Lb}+\mathbf{n}, \quad \mathbf{a} = \mathbf{Gc}+\mathbf{d}, \quad \mathbf{b} = \mathbf{Dc}+\mathbf{i}, \tag{38}$$

where **w**, **x**, **y** and **z** represent observed variables, **a** and **b** represent unobserved common first-order factors, **H**, **J**, **K** and **L** are factor pattern coefficients, **e**, **f**, **m** and **n** are unobserved first-order unique factors, and, in addition, **c** is an unobserved common second-order factor, and **d** and **i** are unobserved second-order unique factors. We write

$$\begin{gathered} \mathbf{P} = \mathscr{E}\{\mathbf{ee}'\}, \quad \mathbf{Q} = \mathscr{E}\{\mathbf{ff}'\}, \quad \mathbf{R} = \mathscr{E}\{\mathbf{mm}'\}, \quad \mathbf{T} = \mathscr{E}\{\mathbf{nn}'\}, \\ \mathbf{V} = \mathscr{E}\{\mathbf{dd}'\}, \quad \mathbf{U} = \mathscr{E}\{\mathbf{ii}'\}, \quad \mathbf{W} = \mathscr{E}\{\mathbf{cc}'\}. \end{gathered} \tag{39}$$

This particular configuration is a special case of a hierarchical or higher-order common factor model (e.g. Cattell, 1965; Jöreskog, 1973*a*).

To express this model in terms of (1) to (14) we write

$$\mathbf{v}=\begin{bmatrix} w \\ x \\ y \\ z \\ a \\ b \\ c \end{bmatrix},\quad \mathbf{A}=\begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{H} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{J} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{K} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{L} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{G} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{D} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix},\quad \mathbf{u}=\begin{bmatrix} e \\ f \\ m \\ n \\ d \\ i \\ c \end{bmatrix},$$

$$\mathbf{S}=\begin{bmatrix} \mathbf{P} & & & & & & \text{sym} \\ \mathbf{0} & \mathbf{Q} & & & & & \\ \mathbf{0} & \mathbf{0} & \mathbf{R} & & & & \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{T} & & & \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{V} & & \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{U} & \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{W} \end{bmatrix}. \tag{40}$$

The required inverse may be represented by the $r = 2$ termed geometric series

$$\mathbf{E}=\begin{bmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} \end{bmatrix} + \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{H} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{J} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{K} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{L} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{G} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{D} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}$$

$$+\begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{HG} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{JG} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{KD} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{LD} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{H} & \mathbf{0} & \mathbf{HG} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{J} & \mathbf{0} & \mathbf{JG} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{K} & \mathbf{KD} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{L} & \mathbf{LD} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{G} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{D} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} \end{bmatrix}, \tag{41}$$

and the resulting structure may be obtained by an application of (8) to (14) (compare Weeks, 1978*a*, *b*, 1980).

Let us finally consider a *non-recursive path analysis* model,

$$\mathbf{w} = \mathbf{Hx}+\mathbf{Ky}+\mathbf{e}, \quad \mathbf{x} = \mathbf{Jw}+\mathbf{Lz}+\mathbf{f},$$

where, again, **w**, **x**, **y** and **z** represent observed variables, **H**, **J**, **K** and **L** represent regression coefficients, and **e**, **f** and **m** represent unobserved regression residuals. The critical feature of this model is that **w** is regressed upon **x** and, simultaneously, **x** is regressed upon **w**. Due to the mathematically non-recursive configuration of this model it has been used in representations of 'feedback and servo-mechanism' models (e.g. Cattell, 1966), as well as 'crossed and lagged panel' models (e.g. Duncan, 1975). We write

$$\begin{aligned} &\mathbf{P} = \mathscr{E}\{\mathbf{ee}'\}, \quad \mathbf{Q} = \mathscr{E}\{\mathbf{ff}'\}, \quad \mathbf{U} = \mathscr{E}\{\mathbf{ef}'\}, \\ &\mathbf{R} = \mathscr{E}\{\mathbf{yy}'\}, \quad \mathbf{T} = \mathscr{E}\{\mathbf{zz}'\}, \quad \mathbf{V} = \mathscr{E}\{\mathbf{yz}'\}. \end{aligned} \tag{42}$$

To express this model in terms of (1) to (14) we write

$$\mathbf{v} = \begin{bmatrix} \mathbf{w} \\ \mathbf{x} \\ \mathbf{y} \\ \mathbf{z} \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} \mathbf{0} & \mathbf{H} & \mathbf{K} & \mathbf{0} \\ \mathbf{J} & \mathbf{0} & \mathbf{0} & \mathbf{L} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}, \quad \mathbf{u} = \begin{bmatrix} \mathbf{e} \\ \mathbf{f} \\ \mathbf{y} \\ \mathbf{z} \end{bmatrix}, \quad \mathbf{S} = \begin{bmatrix} \mathbf{P} & & & \text{sym} \\ \mathbf{U} & \mathbf{Q} & & \\ \mathbf{0} & \mathbf{0} & \mathbf{R} & \\ \mathbf{0} & \mathbf{0} & \mathbf{V} & \mathbf{T} \end{bmatrix}. \tag{43}$$

Following (6), the required inverse may be expanded as a full r-termed geometric series

$$\begin{aligned} \mathbf{E} = {} & \begin{bmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} \end{bmatrix} + \begin{bmatrix} \mathbf{0} & \mathbf{H} & \mathbf{K} & \mathbf{0} \\ \mathbf{J} & \mathbf{0} & \mathbf{0} & \mathbf{L} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \\ & + \begin{bmatrix} \mathbf{HJ} & \mathbf{0} & \mathbf{0} & \mathbf{HL} \\ \mathbf{0} & \mathbf{JH} & \mathbf{JK} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} + \begin{bmatrix} \mathbf{0} & \mathbf{HJH} & \mathbf{HJK} & \mathbf{0} \\ \mathbf{JHJ} & \mathbf{0} & \mathbf{0} & \mathbf{JHL} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} + \dots \\ & + \begin{bmatrix} (1-\phi)\,\mathbf{Z^*HJ} & (\phi)\,\mathbf{Z^*H} & (\phi)\,\mathbf{Z^*K} & (1-\phi)\,\mathbf{Z^*HL} \\ (\phi)\,\mathbf{Z^*J} & (1-\phi)\,\mathbf{Z^*JH} & (1-\phi)\,\mathbf{Z^*JK} & (\phi)\,\mathbf{Z^*L} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \\ = {} & \begin{bmatrix} \mathbf{Z^{**}} & \mathbf{Z^{**}H} & \mathbf{Z^{**}K} & \mathbf{Z^{**}HL} \\ \mathbf{Z^{**}J} & \mathbf{Z^{**}} & \mathbf{Z^{**}JK} & \mathbf{Z^{**}L} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} \end{bmatrix}, \end{aligned} \tag{44}$$

where for $r>2$, $\phi = \text{mod}\,[r/2]$ (i.e. $\phi = 0$ when r is even and $\phi = 1$ when r is odd), $\mathbf{Z}^* = (\mathbf{HJ})^{r-3}$, and $\mathbf{Z}^{**} = (\mathbf{I}-\mathbf{HJ})^{-1}$, the geometric series representation of **HJ** (see equation 6). The expansion (44), which may be verified by $(\mathbf{I}-\mathbf{A})$ multiplication, yields an infinite sequence, resulting from terms in the main diagonal of $\mathbf{A}^2$, which shows the influence of the **HJ** product term. As before, the resulting structure may be obtained by the application of (8) to (14), yielding a number of well-known results in path analysis directly (e.g. Heise, 1975; Fox, 1980).

These five different models for the same four vectors of manifest variables show how RAM specification is both simple and flexible. In any RAM specification: (*a*) manifest or latent variables are identified by the choice of unities and zeros in the filter matrix **F**, (*b*) asymmetric relations among variables are indicated by the choice of parameters in the asymmetric matrix **A**, and (*c*) symmetric relations among variables are indicated by the choice of parameters in the symmetric matrix **S**. As the previous examples illustrate, alternative variables and parameters are included or deleted with minimal changes in matrix elements and orders, and interpretation depends on the particular configuration of the **F**, **A** and **S** matrices.

4. First-moment RAM specification

In typical applications all manifest variables may be scaled to have zero means so that **M**, **S** and **C** become covariance matrices. But, more generally, **M** can be considered a raw product-moment matrix, without 'correction for means', whose scale can be used to determine the scale of **A**, **S** and **C**. It will now be shown that the simple RAM model (1) to (14) directly takes account of both first and second moments.

Without changing notation we will reinterpret all vectors of variables as matrices containing N columns, consisting of independent observations from a sample of size N. In this way we write the general multivariate linear hypothesis (see Bock, 1975) as

$$\mathbf{y} = \mathbf{B}\mathbf{x} + \mathbf{e}, \tag{45}$$

where $\mathbf{y}$, $(i \times N)$, is a raw score data matrix, $\mathbf{x}$, $(j \times N)$, is a fixed design matrix of known constants of rank $i \leqslant N$, $\mathbf{B}$, $(i \times j)$, is a matrix of parameters to be estimated, and $\mathbf{e}$, $(i \times N)$, is a random matrix. We assume

$$\mathscr{E}\{\mathbf{e}\} = \mathbf{0}, \quad \text{so } \mathscr{E}\{\mathbf{y}\} = (1/N)\,\mathbf{B}\mathbf{x}, \tag{46}$$

and we write

$$\mathbf{M}^* = \mathscr{E}\{(1/N)\,\mathbf{e}\mathbf{e}'\}, \quad \mathbf{W} = (1/N)\,(\mathbf{x}\mathbf{x}'). \tag{47}$$

In terms of (1) we now rewrite this model as

$$\begin{bmatrix} \mathbf{y} \\ \mathbf{x} \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{B} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{y} \\ \mathbf{x} \end{bmatrix} + \begin{bmatrix} \mathbf{e} \\ \mathbf{x} \end{bmatrix} \tag{48}$$

and, corresponding to (47),

$$\mathbf{S} = \begin{bmatrix} \mathbf{M}^* & \\ & \mathbf{W} \end{bmatrix}, \tag{49}$$

which yields both first and second moment information as

$$\mathbf{C} = \begin{bmatrix} \mathbf{M}^* + \mathbf{B}\mathbf{W}\mathbf{B}' & \mathbf{B}\mathbf{W} \\ \mathbf{W}\mathbf{B}' & \mathbf{W} \end{bmatrix} \tag{50}$$

and is fitted to the joint raw product-moment matrix

$$\mathbf{M} = (1/N) \begin{bmatrix} (\mathbf{y}\mathbf{y}') & (\mathbf{x}\mathbf{y}') \\ (\mathbf{y}\mathbf{x}') & (\mathbf{x}\mathbf{x}') \end{bmatrix}. \tag{51}$$

McDonald (1979) has shown, for the equivalent COSAN-based treatment of this type of problem, that if the usual likelihood-based loss function for a covariance matrix is applied to fit model (48) to (50) to a sample joint raw product-moment matrix corresponding to (51), it yields the likelihood estimators both of parameters determining the means and of parameters determining the covariances (also see Jöreskog & Sörbom, 1980, 1981; compare Sörbom, 1978).

This general linear model has been extended for use in a wide variety of applications. For example, Pothoff & Roy (1964) presented a model for growth-curve problems which can be written as

$$\mathbf{y} = \mathbf{QPx} + \mathbf{e}, \tag{52}$$

where $\mathbf{Q}$, $(i \times k)$, of rank $k \leqslant i$ is a within-subjects design matrix, and $\mathbf{P}$, $(k \times j)$, is a parameter matrix of rank $j \leqslant N$. We rewrite (52) as

$$\mathbf{y} = \mathbf{Qx}^* + \mathbf{e}, \quad \mathbf{x}^* = \mathbf{Px}. \tag{53}$$

That is, we interpret the rank factorization $\mathbf{B} = \mathbf{QP}$ as implying a recursive regression of $\mathbf{y}$ on some $\mathbf{x}^*$ that is regressed on $\mathbf{x}$ with zero residuals. In terms of (1) we may now write

$$\begin{bmatrix} \mathbf{y} \\ \mathbf{x}^* \\ \mathbf{x} \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{Q} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{P} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{y} \\ \mathbf{x}^* \\ \mathbf{x} \end{bmatrix} + \begin{bmatrix} \mathbf{e} \\ \mathbf{0} \\ \mathbf{x} \end{bmatrix}, \quad \mathbf{S} = \begin{bmatrix} \mathbf{M}^* & & \\ & \mathbf{0} & \\ & & \mathbf{W} \end{bmatrix}, \tag{54}$$

which is fitted to $\mathbf{M}$ after the appropriate choice of $\mathbf{F}$.

The introduction of latent $\mathbf{x}^*$ with zero residuals illustrates another feature of this class of models. That is, if desirable, we may introduce residuals with zero variances. This kind of specification easily provides for the practically useful simultaneous moment structure models given by Jöreskog (1973*a*) and by Sörbom (1978), as well as the mean structure proposed by Bentler (1976) and Bentler & Weeks (1979, 1980). This device may also be used to provide a matrix specification for some of the complex 'scalar specifications' discussed by McDonald (1980). For example, Horn & McArdle (1980) show how latent $\mathbf{x}^*$ with zero residuals may be patterned with equality constraints to yield a variety of more complex inequality and polynomial constraints on both first and second moment structures. Thus, it is not necessary to make separate algebraic provisions for the first moment structures, or for some of the more common inequality constraints.

5. General RAM specification

Previous published work on RAM (i.e. McArdle, 1979*a*; Horn & McArdle, 1980) has shown how a suitable choice of pattern in $\mathbf{A}$ and $\mathbf{S}$ can yield many commonly used moment structure models, such as LISREL. But now, in order to provide a formal and general treatment of the RAM representation of patterned inverse matrices, we give an account of McDonald's (1978, 1980) simple comprehensive model for the analysis of moment structures (COSAN) in terms of RAM. In the original notation, the COSAN model may be written as

$$\mathbf{M} = \left(\prod_{j=1}^{m} \mathbf{F}_j\right) \mathbf{P} \left(\prod_{j=1}^{m} \mathbf{F}_j\right)', \tag{55}$$

where $\mathbf{F}_j$ is of order $(n_{j-1} \times n_j)$, and $\mathbf{P}$ is of order n_m. In the general case, any of these parameter matrices may have a patterned inverse. The COSAN model yields hierarchical models of arbitrary order, such as the mth-order extension of Jöreskog's (1973*a*) ACOVS model remarked upon by Bentler (1976). Because a patterned matrix does not in general have an inverse of the same pattern, hierarchical models without provision for the inverse of a patterned matrix cannot be regarded as equivalent to models such as LISREL or COSAN (compare Weeks, 1978*a, b*).

Consider, first, the special case of the moment structure (55), in which we write

$$\mathbf{M} = \mathbf{B}_0^{-1}[\mathbf{I} \quad \mathbf{L}_1]\begin{bmatrix} \mathbf{I} & \\ & \mathbf{B}_1 \end{bmatrix}^{-1}\begin{bmatrix} \mathbf{I} & & \\ & \mathbf{I} & \mathbf{L}_2 \end{bmatrix}\mathbf{P}\begin{bmatrix} \mathbf{I} & \\ & \mathbf{I} \\ & \mathbf{L}_2' \end{bmatrix}\begin{bmatrix} \mathbf{I} & \\ & \mathbf{B}_1' \end{bmatrix}^{-1}\begin{bmatrix} \mathbf{I} \\ \mathbf{L}_1' \end{bmatrix}\mathbf{B}_0'^{-1}, \quad (56)$$

corresponding to the model

$$\begin{aligned} \mathbf{x} &= \mathbf{B}_0^{-1}\mathbf{z}_0 + \mathbf{B}_0^{-1}\mathbf{L}_1\mathbf{B}_1^{-1}\mathbf{z}_1 + \mathbf{B}_0^{-1}\mathbf{L}_1\mathbf{B}_1^{-1}\mathbf{L}_2\mathbf{f}_2 \\ &= \mathbf{B}_0^{-1}\{\mathbf{z}_0 + \mathbf{L}_1[\mathbf{B}_1^{-1}(\mathbf{z}_1 + \mathbf{L}_2\mathbf{f}_2)]\}. \end{aligned} \quad (57)$$

This model may be considered as a recursive version of the familiar model for linear structural relations since it may be written

$$\mathbf{B}_0\mathbf{x} = \mathbf{z}_0 + \mathbf{L}_1\mathbf{f}_1, \quad \mathbf{B}_1\mathbf{f}_1 = \mathbf{z}_1 + \mathbf{L}_2\mathbf{f}_2, \quad (58)$$

where $\mathbf{x} = \mathbf{g}$ as in (11), and

$$\mathbf{P} = \begin{bmatrix} \mathscr{E}\{\mathbf{z}_0\mathbf{z}_0'\} & & \\ & \mathscr{E}\{\mathbf{z}_1\mathbf{z}_1'\} & \\ & & \mathscr{E}\{\mathbf{f}_2\mathbf{f}_2'\} \end{bmatrix}. \quad (59)$$

Although not immediately obvious, it is possible to express this mixed sequence model in terms of (1) by writing

$$\mathbf{v} = \begin{bmatrix} \mathbf{x} \\ \mathbf{f}_1 \\ \mathbf{f}_2 \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} (\mathbf{I}-\mathbf{B}_0) & \mathbf{L}_1 & \mathbf{0} \\ \mathbf{0} & (\mathbf{I}-\mathbf{B}_1) & \mathbf{L}_2 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}, \quad \mathbf{u} = \begin{bmatrix} \mathbf{z}_0 \\ \mathbf{z}_1 \\ \mathbf{f}_2 \end{bmatrix}, \quad (60)$$

with $\mathbf{S} = \mathbf{P}$ as given by (59). This result follows from the fact that

$$\mathbf{E} = (\mathbf{I}-\mathbf{A})^{-1} = \begin{bmatrix} \mathbf{B}_0^{-1} & \mathbf{B}_0^{-1}\mathbf{L}_1\mathbf{B}_1^{-1} & \mathbf{B}_0^{-1}\mathbf{L}_1\mathbf{B}_1^{-1}\mathbf{L}_2 \\ \mathbf{0} & \mathbf{B}_1^{-1} & \mathbf{B}_1^{-1}\mathbf{L}_2 \\ \mathbf{0} & \mathbf{0} & \mathbf{I} \end{bmatrix}, \quad (61)$$

as may be verified by multiplication by $(\mathbf{I}-\mathbf{A})$ from (60). Thus, the model (56) or (57) can be rewritten, in the form of (13), as

$$\mathbf{x} = [\mathbf{I} : \mathbf{0} : \mathbf{0}]\begin{bmatrix} \mathbf{B}_0 & -\mathbf{L}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{B}_1 & -\mathbf{L}_2 \\ \mathbf{0} & \mathbf{0} & \mathbf{I} \end{bmatrix}^{-1}\begin{bmatrix} \mathbf{z}_0 \\ \mathbf{z}_1 \\ \mathbf{f}_2 \end{bmatrix}, \quad (62)$$

and the corresponding moment structure given by substitution into (14) is algebraically equivalent to (56).

The mth-order counterpart of (62) follows in the same way. That is, the model

$$\mathbf{x} = [\mathbf{I}\ \mathbf{0}\ \mathbf{0}\ \dots\ \mathbf{0}]\begin{bmatrix} \mathbf{B}_0 & -\mathbf{L}_1 & & & & \\ & \mathbf{B}_1 & -\mathbf{L}_2 & & & \\ & & \mathbf{B}_2 & -\mathbf{L}_3 & & \\ & & & \ddots & \ddots & \\ & & & & \mathbf{B}_{m-1} & -\mathbf{L}_m \\ & & & & & \mathbf{I} \end{bmatrix}^{-1}\begin{bmatrix} \mathbf{z}_0 \\ \mathbf{z}_1 \\ \mathbf{z}_2 \\ \vdots \\ \mathbf{z}_{m-1} \\ \mathbf{f}_m \end{bmatrix} \quad (63)$$

is the same as the model

$$\mathbf{x} = \mathbf{B}_0{}^{-1}\mathbf{z}_0 + \mathbf{B}_0{}^{-1}\mathbf{L}_1\mathbf{B}_1{}^{-1}\mathbf{z}_1 + \mathbf{B}_0{}^{-1}\mathbf{L}_1\mathbf{B}_1{}^{-1}\mathbf{L}_2\mathbf{B}_2{}^{-1}\mathbf{z}_2 + \dots$$
$$+ \mathbf{B}_0{}^{-1}\mathbf{L}_1\mathbf{B}_1{}^{-1}\mathbf{L}_2\mathbf{B}_2{}^{-1}\dots\mathbf{B}_{m-1}{}^{-1}\mathbf{L}_m\mathbf{f}_m. \quad (64)$$

This is shown, straightforwardly but a little tediously, by writing the mth-order counterpart of the inverse (61) in the obvious way, and verifying it by multiplication. Then, by setting any matrix $\mathbf{L}_j$ or $\mathbf{B}_j$ in (63) equal to the identity matrix, we can obtain any mixed sequence of matrices $\mathbf{L}_j$ and inverted matrices $\mathbf{B}_j{}^{-1}$. That is, the RAM model (63) may be used to express *any* COSAN model given by (55). In terms of (63), patterned matrices in (55) appear on the super-diagonal of $\mathbf{A}$, matrices in (55) with patterned inverses appear on the diagonal of $\mathbf{A}$, and it is not necessary to have an inverted submatrix within $\mathbf{A}$ (as in Bentler & Weeks, 1979).

This mixed-sequence representation forms the basis for the interrelations among many structural equation systems. For example, Jöreskog's widely used LISREL model (e.g. Jöreskog, 1973*b*; Jöreskog & Sörbom, 1979, 1981; Wiley, 1973) may be written as

$$\mathbf{y} = \mathbf{\Lambda}_y\boldsymbol{\eta} + \boldsymbol{\varepsilon}, \quad \mathbf{x} = \mathbf{\Lambda}_x\boldsymbol{\xi} + \boldsymbol{\delta}, \quad \mathbf{B}\boldsymbol{\eta} = \mathbf{\Gamma}\boldsymbol{\xi} + \boldsymbol{\zeta}, \quad (65)$$

with expectations

$$\mathscr{E}\{\boldsymbol{\xi}\boldsymbol{\xi}'\} = \mathbf{\Phi}, \quad \mathscr{E}\{\boldsymbol{\zeta}\boldsymbol{\zeta}'\} = \mathbf{\Psi}, \quad \mathscr{E}\{\boldsymbol{\varepsilon}\boldsymbol{\varepsilon}'\} = \mathbf{\Theta}_\varepsilon, \quad \mathscr{E}\{\boldsymbol{\delta}\boldsymbol{\delta}'\} = \mathbf{\Theta}_\delta. \quad (66)$$

As McArdle (1978, 1979*b*; Horn & McArdle, 1980) originally showed, (65) may be represented as

$$\begin{bmatrix} \mathbf{y} \\ \mathbf{x} \end{bmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} \end{bmatrix}\begin{bmatrix} \mathbf{I} & \mathbf{0} & -\mathbf{\Lambda}_y & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} & -\mathbf{\Lambda}_x \\ \mathbf{0} & \mathbf{0} & \mathbf{B} & -\mathbf{\Gamma} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} \end{bmatrix}^{-1}\begin{bmatrix} \boldsymbol{\varepsilon} \\ \boldsymbol{\delta} \\ \boldsymbol{\zeta} \\ \boldsymbol{\xi} \end{bmatrix}, \quad (67)$$

by defining

$$\mathbf{v} = \begin{bmatrix} \mathbf{y} \\ \mathbf{x} \\ \boldsymbol{\eta} \\ \boldsymbol{\xi} \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{\Lambda}_y & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{\Lambda}_x \\ \mathbf{0} & \mathbf{0} & \mathbf{I}-\mathbf{B} & \mathbf{\Gamma} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}, \quad \mathbf{u} = \begin{bmatrix} \boldsymbol{\varepsilon} \\ \boldsymbol{\delta} \\ \boldsymbol{\zeta} \\ \boldsymbol{\xi} \end{bmatrix}, \quad \mathbf{S} = \begin{bmatrix} \mathbf{\Theta}_\varepsilon & & & \\ & \mathbf{\Theta}_\delta & & \\ & & \mathbf{\Psi} & \\ & & & \mathbf{\Phi} \end{bmatrix}. \quad (68)$$

As previous work on RAM has demonstrated (McArdle, 1978; Horn & McArdle, 1980), and these results verify, the inherent flexibility of the inverse of the patterned matrix permits a large number of ways to illustrate these interrelations among the general models with an inverse of a patterned matrix. An apparent paradox follows; namely, any model that includes the fundamental RAM properties, such as COSAN and LISREL above, should be considered as general as any other model which includes these properties. In fact, as the result (63) demonstrates, it is also possible to find a very large number of ways to express any of these models, including RAM itself, directly within the three matrix RAM scheme. Thus, although the required RAM matrices are small in number, each of these three matrices can be chosen to be indefinitely large and sparse (see McArdle & Horn, 1983). But out of this large and seemingly paradoxical variety of models RAM stands out for one particularly fundamental property—RAM requires only the *necessary* and *sufficient* matrices for general linear structural equation model specification.

6. Final remarks

It should be clear from the above that many popular moment structure models can be represented easily and economically in terms of the simple RAM algebra (1) to (14). We now present a few of the other conceptual and practical features that can be realized by using RAM logic in tandem with more popular model developments.

The algebraic development of RAM represents both a natural simplification and a natural extension of the basic specification features of general models such as LISREL and COSAN. One can therefore use RAM to build upon any of the other generally useful features of these systems. For example, the LISREL model is often presented as a psychometrically based 'measurement' model combined with an econometrically based 'structural' model (Jöreskog & Sörbom, 1979, 1980, 1981). In RAM these decidedly useful concepts are not eliminated—RAM algebra simply indicates that this distinction represents only one of several possible ways to distinguish components of a model (see equations 65–68). In this context the RAM reformulation may be most useful when dealing with, for example: (*a*) arbitrarily patterned measurement models (e.g. Jöreskog, 1973*b*), (*b*) multi-level and hierarchical measurement models (e.g. Cattell, 1978; Weeks, 1978*a*, *b*; compare Weeks, 1980), (*c*) highly extended structural models (e.g. Jöreskog, 1977), (*d*) manifest and latent mean structures (e.g. Horn & McArdle, 1980; Jöreskog & Sörbom, 1981), and (*e*) complex non-linear scalar constraints (e.g. Horn & McArdle, 1980; Rindskopf, 1983). In such models the interpretative distinction between 'measurement' and 'structural' components is typically less useful than the distinction between 'asymmetric' and 'symmetric' patterning. Similar arguments apply to the hierarchical distinctions made in COSAN.

The simplicity and flexibility of RAM specification is a result of a minimum and fundamental set of modelling distinctions expressed in broad structural terms. But, not surprisingly and in a non-trivial sense, the simplicity and flexibility of RAM also carries over into other complex modelling problems, such as in model identification, estimation and comparison (e.g. Browne, 1982; McDonald & Krane, 1979; McDonald, 1982; Jöreskog & Sörbom, 1979). The particularly simple form of RAM moment structure (14), for example, permits the use of existing matrix calculus theorems for the direct calculation of all first and second derivatives required for any non-linear optimization based on any well-defined loss function (e.g. weighted least squares, maximum-likelihood, etc.; McDonald, 1978, pp. 66–69). This result is verified by the particularly simple and flexible specification of RAM matrices **F**, **A** and **S** within the available programs for both COSAN (55) and LISREL (65) and (66) (see Horn &

McArdle, 1980, p. 536). Practical concerns about optimal computational storage and efficiency, however, dictate the programming of a RAM–COSAN 'interface' which reorganizes and partitions the large and sparse matrices **A** and **S** into smaller matrices **B** and **L** following (63) here. A program interface which places minimum user requirements on input and output, and uses available sparse matrix programming (e.g. George & Liu, 1979), has recently been developed by McArdle & Horn (1983), and specific program comparisons are forthcoming.

A modification of this RAM approach has been presented by Bentler & Weeks (1979, 1980). Although McArdle's (1978, 1979*a*) basic RAM equation was recognized by Bentler & Weeks (1979, equations 5 and 6; Weeks, 1980), it is relevant (as a reviewer has pointed out) that these authors were not aware of the broad generality of RAM (cf. Bentler & Weeks, p. 181: 'It appears to be more general than (6) . . .'). It follows that the Bentler & Weeks (1979, 1980) accounts do not follow RAM logic in dealing with the inverses of patterned matrices. They propose a model which can be recognized as a variant of RAM in which a further patterned matrix, further submatrix partitioning and a separate structural equation for the mean vector are all required. In effect, Bentler & Weeks propose separate models for first and second moments and also treat the LISREL structural distinction between 'independent' and 'dependent' model components as essential. These particular distinctions, although traditional in form and useful for certain matrix computations (e.g. Jöreskog & Sörbom, 1979), may be most simply recognized as one particular configuration of the **A** and **S** matrices in RAM. That is, in terms of the original logic of the simpler RAM formulation, these revised features are unnecessary.

Of further conceptual interest is the fact that the basic RAM algebra (1) to (14), unlike most other structural forms, has an exact isomorphic relationship with the mathematical basis of graph theory (e.g. Harary *et al.*, 1965). In virtually all graphic representations a distinction is made between 'nodes' and 'edges'. This parallels the primary distinction between variables and parameters. In some forms of graph theory a further distinction is made between 'directed arrows' and 'undirected spans'. This parallels the secondary distinction between asymmetric and symmetric parameters. Finally, in contemporary path analytic forms of graph theory (e.g. Jöreskog & Sörbom, 1979, 1981) it is common to distinguish manifest variables, symbolized by rectangles, from latent variables, symbolized by ellipses. This simple but general graphic treatment guided the initial development of the simple algebraic treatment presented here (McArdle, 1978, 1979) and will, we hope, lead to further useful graphic–algebraic isomorphisms (e.g. Gilli, 1981).

These conceptual and practical properties lead to a broad interpretation of RAM using the input/output logic typically offered by a 'general systems' philosophy (e.g. Rapoport, 1972). The general nature of this system logic, albeit somewhat abstract, provides a broad and flexible base for future RAM possibilities, and vice versa (McArdle, 1980). In the recent psychonomic literature, only Cattell (1965, 1978) has promoted a general systems foundation which includes manifest and latent variables in a free-form network, or 'reticule', of asymmetric and symmetric action. To emphasize the importance of Cattell's theoretical contribution, we have chosen the mnemonic *Reticular Action Model* (*RAM*) to represent our algebra.

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Requests for reprints should be addressed to J. Jack McArdle, Department of Psychology, University of Virginia, Charlottesville, VA 22901, USA.
Roderick P. McDonald is at Macqarie University, North Ryde, New South Wales, Australia.