1 Main equations

The main equation to be solved are the continuity, momentum and energy equations

$$
\begin{aligned}
\frac{\partial \left\langle \rho \right\rangle \{u\}}{\partial t} + \frac{\partial \left\langle \rho \right\rangle \{u_i\}\{u_j\}}{\partial x_j} &= -\frac{\partial \left\langle p \right\rangle}{\partial x_j} + \frac{\partial}{\partial x_j}\left[\left(\frac{\left\langle \mu \right\rangle}{Re} + \mu_t\right)\frac{\partial \{u_j\}}{\partial x_j}\right] \\
\frac{\partial \left\langle \rho \right\rangle \{H\}}{\partial t} + \frac{\partial \left\langle \rho \right\rangle \{u_j\}\{H\}}{\partial x_j} &= \frac{\partial}{\partial x_j}\left[\left(\frac{1}{RePr}\frac{\left\langle \lambda \right\rangle}{\left\langle c_p \right\rangle} + \left\langle \rho \right\rangle \alpha_t\right)\frac{\partial \{H\}}{\partial x_j}\right]
\end{aligned}
\tag{1}
$$

2 Turbulent Viscosity models

To model the reynolds stress the following approximation is used:

$$- \left\langle \rho u_i'' u_j'' \right\rangle \approx \mu_t \left(\frac{\partial \left\langle u_i \right\rangle}{\partial x_j} + \frac{\partial \left\langle u_j \right\rangle}{\partial x_i} \right) - \frac{2}{3} \left\langle \rho \right\rangle k \delta_{ij} \tag{2}$$

2.1 Abe Model

This model is presented in [1]. The following transport equations are implemented in the code:

$$\begin{aligned}
\frac{\partial \left\langle \rho \right\rangle k}{\partial t} + \frac{\partial \left\langle \rho \right\rangle \{u_j\} k}{\partial x_j} &= \frac{\partial}{\partial x_j} \left[\left(\frac{\mu}{Re} + \frac{\mu_t}{\sigma_k} \right) \frac{\partial k}{\partial x_j} \right] + P_k + B_k - \left\langle \rho \right\rangle \varepsilon \\
\frac{\partial \left\langle \rho \right\rangle \varepsilon}{\partial t} + \frac{\partial \left\langle \rho \right\rangle \{u_j\} \varepsilon}{\partial x_j} &= \frac{\partial}{\partial x_j} \left[\left(\frac{\mu}{Re} + \frac{\mu_t}{\sigma_\varepsilon} \right) \frac{\partial \varepsilon_t}{\partial x_j} \right] + C_{\varepsilon 1} \frac{\varepsilon}{k} \left(P_k + B_k \right) - C_{\varepsilon 2} f_\varepsilon \left\langle \rho \right\rangle \frac{\varepsilon^2}{k}
\end{aligned} \tag{3}$$

With the following functions:

$$\begin{aligned}
f_\mu &= \left[1 - \exp\left(-\frac{y^*}{14} \right) \right]^2 \left[1 + \frac{5}{Re_t^{3/4}} \exp\left[-\left(\frac{{Re_t}^2}{200} \right) \right] \right] \\
f_\varepsilon &= \left[1 - \exp\left(-\frac{y^*}{3.1} \right) \right]^2 \left[1 - 0.3 \exp\left(-\frac{Re_t}{6.5} \right)^2 \right]
\end{aligned} \tag{4}$$

where:

$$y^* = \frac{y(\left\langle \nu \right\rangle \varepsilon)^{1/4}}{\left\langle \nu \right\rangle}, \quad Re_t = \frac{k^2}{\left\langle \nu \right\rangle \varepsilon} \tag{5}$$

$$\mu_t = \left\langle \rho \right\rangle C_\mu f_\mu \frac{k^2}{\varepsilon} \tag{6}$$

and the following constants:

$$C_\mu = 0.09, \quad \sigma_k = 1.4, \quad , \sigma_\varepsilon = 1.4, \quad C_{\varepsilon 1} = 1.45, \quad C_{\varepsilon 2} = 1.9 \tag{7}$$

2.2 MK Model

This model is presented in [8]. The following transport equations are implemented in the code:

$$\begin{aligned}
\frac{\partial \left\langle \rho \right\rangle k}{\partial t} + \frac{\partial \left\langle \rho \right\rangle \{u_j\} k}{\partial x_j} &= \frac{\partial}{\partial x_j} \left[\left(\frac{\mu}{Re} + \frac{\mu_t}{\sigma_k} \right) \frac{\partial k}{\partial x_j} \right] + P_k + B_k - \left\langle \rho \right\rangle \varepsilon \\
\frac{\partial \left\langle \rho \right\rangle \varepsilon}{\partial t} + \frac{\partial \left\langle \rho \right\rangle \{u_j\} \varepsilon}{\partial x_j} &= \frac{\partial}{\partial x_j} \left[\left(\frac{\mu}{Re} + \frac{\mu_t}{\sigma_\varepsilon} \right) \frac{\partial \varepsilon_t}{\partial x_j} \right] + C_{\varepsilon 1} \frac{\varepsilon}{k} \left(P_k + B_k \right) - C_{\varepsilon 2} f_\varepsilon \left\langle \rho \right\rangle \frac{\varepsilon^2}{k}
\end{aligned} \tag{8}$$

With the following functions:

$$\begin{aligned} f_\mu &= \left[1-\exp\left(-\frac{y^+}{70}\right)\right]^2\left[1+\frac{3.45}{\sqrt{Re_t}}\right] \\ f_\varepsilon &= \left[1-\frac{2}{9}\exp\left(-\left(\frac{Re_t}{6}\right)^2\right)\right]\left[1-\exp\left(\frac{y^+}{5}\right)\right]^2 \end{aligned} \tag{9}$$

$$Re_t = \frac{k^2}{\langle\nu\rangle\varepsilon}, \quad y^+ = \frac{\langle\rho_w\rangle u_\tau y}{\langle\mu_w\rangle}, \quad u_\tau = \sqrt{\frac{\tau_w}{\rho_w}}, \quad \tau_w = \left(\langle\mu\rangle\frac{\partial\{u\}}{\partial y}\right)_w \tag{10}$$

$$\mu_t = \langle\rho\rangle C_\mu f_\mu \frac{k^2}{\varepsilon} \tag{11}$$

and the following constants:

$$C_\mu = 0.09, \quad \sigma_k = 1.4, \quad ,\sigma_\varepsilon = 1.3, \quad C_{\varepsilon 1} = 1.4, \quad C_{\varepsilon 2} = 1.8 \tag{12}$$

2.3 V2F Model

This model is presented in [4]. The following transport equations are implemented in the code:

$$\begin{aligned} \frac{\partial\langle\rho\rangle k}{\partial t} + \frac{\partial\langle\rho\rangle\{u_j\}k}{\partial x_j} &= \frac{\partial}{\partial x_j}\left[\left(\frac{\langle\mu\rangle}{Re}+\frac{\mu_t}{\sigma_k}\right)\frac{\partial k}{\partial x_j}\right] + P_k + B_k - \langle\rho\rangle\varepsilon \\ \frac{\partial\langle\rho\rangle\varepsilon}{\partial t} + \frac{\partial\langle\rho\rangle\{u_j\}\varepsilon}{\partial x_j} &= \frac{\partial}{\partial x_j}\left[\left(\frac{\langle\mu\rangle}{Re}+\frac{\mu_t}{\sigma_\varepsilon}\right)\frac{\partial \varepsilon}{\partial x_j}\right] + \frac{C_{\varepsilon 1}(P_k+B_k) - C_{\varepsilon 2}\langle\rho\rangle\varepsilon}{\tau_u} \\ \frac{\partial\langle\rho\rangle\overline{v^2}}{\partial t} + \frac{\partial\langle\rho\rangle\{u_j\}\overline{v^2}}{\partial x_j} &= \frac{\partial}{\partial x_j}\left[\left(\frac{\langle\mu\rangle}{Re}+\frac{\mu_t}{\sigma_{\overline{v^2}}}\right)\frac{\partial \overline{v^2}}{\partial x_j}\right] + \langle\rho\rangle kf - 6\langle\rho\rangle\frac{\varepsilon}{k}\overline{v^2} \\ L^2\nabla^2 f - f &= \frac{1}{T}\left[(C_{f1}-6)\frac{\overline{v^2}}{k} - \frac{2}{3}(C_{f1}-1)\right] - C_{f2}\frac{P_k}{\langle\rho\rangle k} \end{aligned} \tag{13}$$

With the production P_k and Reynolds stress, calculated according to:

$$P_k = \mu_t\left(\frac{\partial\{u_i\}}{\partial x_j} + \frac{\partial\{u_j\}}{\partial x_i} - \frac{2}{3}\delta_{ij}\nabla\cdot\{u\}\right) - \frac{2}{3}\langle\rho\rangle k\delta_{ij} \tag{14}$$

With the turbulent visocisty μ_t calculated as:

$$\mu_t = C_\mu\langle\rho\rangle\overline{v^2}\tau_u \tag{15}$$

Where the turbulent time scale τ_u and length scale L are given by:

$$\tau_u = \max\left(\frac{k}{\varepsilon}, 6\sqrt{\frac{\langle\nu\rangle}{\varepsilon}}\right), \quad L = C_L\max\left[\frac{k^{3/2}}{\varepsilon}, C_\eta\frac{\langle\nu\rangle^{3/4}}{\varepsilon^{1/4}}\right] \tag{16}$$

$$
\begin{aligned}
C_\mu = 0.19, \quad \sigma_k = 1, \quad \sigma_\varepsilon = 1.3 \\
C_{\varepsilon 1} = 1.4\left[1 + 0.045\sqrt{\frac{k}{\overline{v^2}}}\right], \quad C_{\varepsilon 2} = 1.9 \\
C_1 = 1.4, \quad C_2 = 0.3, \quad C_L = 0.3, \quad C_\eta = 70
\end{aligned}
\tag{17}
$$

3 Turbulent Heat Flux Models

$$- \left\langle \rho u_j'' H'' \right\rangle \approx \langle \rho \rangle \alpha_t \frac{\partial \{H\}}{\partial x_j} \tag{18}$$

3.1 Turbulent Prandtl models

The Turbulent Prandtl models, estimate the turbulent diffusivity coefficient α_t in the following way:

$$\alpha_t = \frac{\mu_t}{\langle \rho \rangle Pr_t} \tag{19}$$

3.1.1 Irrenfried Model

This model is presented in [5].

$$Pr_t = \left[\gamma_{IS} + CPe_t \sqrt{2 \left(\frac{1}{Pr_{t,b}} - \gamma_{IS} \right)} - (CPe_t)^2 \left[1 - \exp \left(- \frac{1}{CPe_t} \sqrt{2 \left(\frac{1}{Pr_{t,b}} - \gamma_{IS} \right)} \right) \right] \right]^{-1} \tag{20}$$

with

$$\gamma_{IS} = \frac{1}{Pr_{t,b} + 0.1 Pr^{0.83}}, \quad Pe_t = \frac{\mu_t}{\langle \mu \rangle} Pr, \quad Pr_{t,b} = 1.0, \quad C = 3 \tag{21}$$

3.1.2 Kays

This model is presented in [6].

$$Pr_t = \begin{cases} 1.07, & \mu_t / \langle \mu \rangle < 0.2 \\ \frac{2}{Pe} + 0.85 & 0.2 \leq \mu_t / \langle \mu \rangle \end{cases} \tag{22}$$

with $Pe_t = (\mu_t / \langle \mu \rangle) Pr$

3.1.3 Kays-Crawford Model

This model is presented in [7].

$$Pr_t = \frac{1}{C_1 + C_2 \mu_\gamma - C_3 \mu_\gamma^2 (1 + \exp(-C_4 / \mu_\gamma))} \tag{23}$$

with $\mu_\gamma = \mu_t / \langle \mu \rangle$.

Where the following constants are used:

$$C_1 = 0.5882, \quad C_2 = 0.228, \quad C_3 = 0.0441, \quad C_4 = 5.165 \tag{24}$$

3.1.4 Tang Model

This model is presented in [9].

$$Pr_t = \begin{cases} 1.0, & \mu_t / \langle \mu \rangle < 0.2 \\ 0.85 + \frac{Pr}{A}, & 0.2 \leq \mu_t / \langle \mu \rangle \leq 10 \\ 0.85, & 10 < \mu_t / \langle \mu \rangle \end{cases} \tag{25}$$

with $A = 15$.

3.1.5 Bae Model

This model is presented in [2].

$$Pr_{t,0} = \frac{1 + \frac{\langle \mu \rangle}{\langle \rho \rangle} \left| \left(\frac{\partial \langle \rho \rangle}{\partial y} \right) / \left(\frac{\partial \{u\}}{\partial y} \right) \right|}{1 + \frac{\langle T \rangle}{\langle \rho \rangle} \left| \left(\frac{\partial \langle \rho \rangle}{\partial y} \right) / \left(\frac{\partial \langle T \rangle}{\partial y} \right) \right| + \frac{\langle T \rangle}{\langle c_p \rangle} \left| \left(\frac{\partial \langle c_p \rangle}{\partial y} \right) / \left(\frac{\partial \langle T \rangle}{\partial y} \right) \right|} \tag{26}$$

$$\begin{aligned} f_1 &= 1 - \exp\left(-\frac{y^+}{A^+} \right) \\ f_2 &= 0.5 \left[1 + \tanh\left(\frac{B - y^+}{10} \right) \right] \end{aligned} \tag{27}$$

Using the following equation the turbulent Prandtl is used:

$$Pr_t = \sigma_t - f_1 f_2 (\sigma - Pr_{t,0}) \tag{28}$$

Where the following constants are used:

$$A^+ = 70, \quad B = 20, \quad \sigma_t = 0.9 \tag{29}$$

3.2 Turbulent diffusivity models

3.2.1 DWX Model

This model is presented by [3]. The following transport equations are implemented in the code:

$$
\begin{aligned}
\frac{\partial \langle\rho\rangle \overline{t^2}}{\partial t} + \frac{\partial \langle\rho\rangle \{u_j\}\overline{t^2}}{\partial x_j} &= \frac{\partial}{\partial x_j}\left[\left(\frac{1}{RePr}\frac{\langle\lambda\rangle}{\langle c_p\rangle} + \frac{\langle\rho\rangle \alpha_t}{\sigma_{\overline{t^2}}}\right)\frac{\partial \overline{t^2}}{\partial x_j}\right] + 2P_{\varepsilon_t} - 2\langle\rho\rangle \varepsilon_t \\
\frac{\partial \langle\rho\rangle \varepsilon_t}{\partial x_j} + \frac{\partial \langle\rho\rangle \{u_j\}\varepsilon_t}{\partial x_j} &= \frac{\partial}{\partial x_j}\left[\left(\frac{1}{RePr}\frac{\langle\lambda\rangle}{\langle c_p\rangle} + \frac{\langle\rho\rangle \alpha_t}{\sigma_{\varepsilon_t}}\right)\frac{\partial \varepsilon_t}{\partial x_j}\right] + C_{P1}\sqrt{\frac{\varepsilon\varepsilon_t}{k\overline{t^2}}}P_{\varepsilon_t} \\
&\quad - C_{D1}f_{D1}\langle\rho\rangle\frac{\varepsilon_t^2}{\overline{t^2}} - C_{D2}f_{D2}\langle\rho\rangle\frac{\varepsilon\varepsilon_t}{k}
\end{aligned}
\tag{30}
$$

With the following functions:

$$
\begin{aligned}
&f_{D1} = 1 - \exp\left(-\frac{y^*}{1.7}\right)^2, \quad f_{D2} = \left(\frac{1}{C_{d2}}\right)(C_{\varepsilon 2}f_\varepsilon - 1)\left[1 - \exp\left(-\frac{y^*}{5.8}\right)^2\right], \\
&f_\varepsilon = 1 - 0.3\exp\left(-\frac{Re_t}{6.5}\right)^2, \quad f_\lambda = \left[1 - \exp\left(-\frac{y^*}{16}\right)\right]^2\left[1 + \frac{3}{Re_t^{3/4}}\right].
\end{aligned}
\tag{31}
$$

Where the production is given by:

$$
P_{\varepsilon_t} = -\langle\rho\rangle\langle u_j' T'\rangle\frac{\partial\langle T\rangle}{\partial x_j} = \alpha_t\langle\rho\rangle\left(\frac{\partial\langle T\rangle}{\partial x_j}\right)^2,
\tag{32}
$$

and the turbulent diffusivity calculated using:

$$
\alpha_t = C_\lambda f_\lambda \frac{k^2}{\varepsilon}(2R)^{0.5},
\tag{33}
$$

where

$$
Re_t = \frac{k^2}{\nu\varepsilon}, \quad y^* = \frac{u_\varepsilon \rho_w y}{\mu_w}, \quad u_\varepsilon = (\nu\varepsilon)^{1/4}, \quad R = \frac{\tau_t}{\tau_u}, \quad \tau_u = \frac{k}{\varepsilon}, \quad \tau_t = \frac{\overline{t^2}}{2\varepsilon_t}.
\tag{34}
$$

The following constants are used:

$$
C_\lambda = 0.1, \quad C_{D1} = 1.5, \quad C_{D1} = 0.9, \quad , C_{P1} = 2.34, \quad \sigma_{\overline{t^2}} = 1.0, \quad \sigma_\varepsilon = 1.0
\tag{35}
$$

The following table shows the units of the important variables:

Heat transfer model	Viscous model
$[k_t] = K^2$	$[k] = m^2/s^2$
$[\varepsilon_t] = K^2/s$	$[\varepsilon] = m^2/s^3$
$[\lambda] = kg\ m/(s^3 K)$	$[\mu] = kg/(m\ s)$
$[\alpha] = m^2/s$	$[\nu] = m^2/s$
$[c_p] = m^2/(s^2 K)$	$-$

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