3D Navier-Stokes Equation: A Step Toward Millennium Problem Resolution

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1 Introduction

Establishing the existence and smoothness of solutions to the 3D Navier-Stokes equations for all time remains a Millennium Problem. This document outlines a detailed derivation, aiming to provide a foundation for such a proof by simplifying and solving the equations step-by-step.

2 Helmholtz Decomposition and Pressure Calculation

Initiating the derivation with Helmholtz decomposition to compute pressure independently.

$$\nabla^2 \phi = 0, \quad p = -\nabla \phi \tag{1}$$

- ϕ: Velocity potential, the "driving force" behind pressure.
- $\nabla^2 \phi$: Laplacian measuring the spread across dimensions.

Expanding to 3D, pressure is determined by the gradient of ϕ in x, y, and z directions.

3 Euler Equation (Viscosity Ignored)

Deriving the Euler equation by neglecting viscosity, focusing on inertial effects.

$$\rho \frac{\partial \mathbf{u}}{\partial t} = -\nabla p + \rho \mathbf{f} \tag{2}$$

- $\rho \frac{\partial \mathbf{u}}{\partial t}$: Mass $\times$ acceleration (Newton's second law).
- $-\nabla p$: Force due to pressure gradient.
- $\rho \mathbf{f}$: External forces like gravity.

Simplifying to 1D with constant pressure gradient G:

$$\rho\frac{\partial u}{\partial t} = G + \rho f \tag{3}$$

Integrating with respect to time yields:

$$u(t) = \frac{G}{\rho}t + u_0 \tag{4}$$

4 Stokes Equation (Inertia Ignored)

Focusing on steady-state flow by neglecting time-dependent terms.

$$0 = -\nabla p + \mu\nabla^2\mathbf{u} + \rho\mathbf{f} \tag{5}$$

In 1D:

$$0 = -\frac{\partial p}{\partial x} + \mu\frac{\partial^2 u}{\partial y^2} + \rho f \tag{6}$$

Assuming $G = -\frac{\partial p}{\partial x} + \rho f$:

$$\frac{\partial^2 u}{\partial y^2} = \frac{G}{\mu} \tag{7}$$

Integrating twice and applying boundary conditions $u(0) = 0$, $u(h) = 0$:

$$u(y) = -\frac{G}{2\mu}y(h-y) \tag{8}$$

This is the Poiseuille flow solution.

5 Momentum Equation (External Forces Ignored)

Neglecting external forces and pressure gradients.

$$\rho\frac{\partial\mathbf{u}}{\partial t} = \mu\nabla^2\mathbf{u} \tag{9}$$

Assuming separation of variables $u(y,t) = \phi(y)T(t)$:

$$\frac{1}{T}\frac{dT}{dt} = \nu\frac{1}{\phi}\frac{d^2\phi}{dy^2} = -\nu\lambda \tag{10}$$

Solutions are $T(t) = e^{-\nu\lambda t}$ and $\phi(y) = A\cos(\sqrt{\lambda}y) + B\sin(\sqrt{\lambda}y)$, with $\phi(0) = 0$ implying $A = 0$.

6 Linearized Convection Equation

Linearizing the nonlinear convection term around mean velocity $\mathbf{u}_0$.

$$\rho\frac{\partial \mathbf{u}}{\partial t} + \rho(\mathbf{u}_0 \cdot \nabla\mathbf{u} + \mathbf{u} \cdot \nabla\mathbf{u}_0) = -\nabla p + \mu\nabla^2\mathbf{u} \tag{11}$$

Assuming a wave solution $u(x, y, t) = A\sin(k_y y)e^{i(k_x x - \omega t)}$:

$$-i\omega A + ik_x u_0 A = -\frac{1}{\rho}\frac{\partial p}{\partial x} - \nu k_y^2 A \tag{12}$$

7 Helmholtz Decomposition for Pressure

Reiterating $\nabla^2\phi = 0$, with $\phi(x, y, t) = C\sin(k_y y)e^{i(k_x x - \omega t)}$:

$$p = -\frac{\partial \phi}{\partial x} = ik_x C\sin(k_y y)e^{i(k_x x - \omega t)} \tag{13}$$

8 Final 3D Oscillatory Solution

Combining all steps, the solution is:

$$u(x, y, t) = A\sin(k_y y)\exp[i(k_x x - \omega t)]\exp[-\nu k^2 t] \tag{14}$$

Where $\omega = k_x u_0 - i\nu k^2$, and $k^2 = k_x^2 + k_y^2 + k_z^2$.
Using Euler's formula, the real part is:

$$u_{\text{real}}(x, y, t) = A\sin(k_y y)\cos(k_x x - \omega_r t)e^{-\nu k^2 t} \tag{15}$$

9 Conclusion

This solution demonstrates a smooth, decaying oscillatory behavior, a critical step toward proving the existence and smoothness required by the Navier-Stokes Millennium Problem. Further nonlinear analysis is needed for complete turbulence modeling.