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A non homogeneous Riemann Solver for shallow water and two phase flows

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Abstract In this work we consider a two steps finite volume scheme, recently developed to solve nonhomogeneous systems. The first step of the scheme depends on a diffusion control parameter which we modulate, using the limiters theory. Results on Shallow water equations and two phase flows are presented.

Keywords Non homogeneous systems · Finite volumes · *SRNHR* scheme

1. Introduction

This paper corresponds to a lecture given at the conference "Numerical Simulations of Multiphase and Complex Flows" that was held from 18 to 22 April 2005 in Porquerolles, France.

Complex fluid flow phenomena such as multiphase flows or flows submitted to external forces (friction, gravity for shallow water flows), are represented by nonhomogeneous systems of PDE. Classical numerical schemes can not be used for the numerical simulation of such problems. As a matter of fact, multiphase system of equations can be non hyperbolic. It is therefore not easy to extend the usual Riemann solvers based on eigenvalues and eigenvectors computations. To overcome the above mentioned difficulties, some valuable works have been carried out nevertheless (see [1, 3, 5–8, 13, 14], for instance). Finite Volume schemes obtained by this methods are often costly, due to exact or approximate calculus of jacobian field decompositions. To propose an alternative, we consider in this work a particular class of non conservative systems. We assume that the solution of the associated Riemann problem is self-similar. Assuming this hypothesis, a new Non Homogeneous Riemann Solver (*SRNH*), using flux values instead of eigenvectors, was developed [4]. The *SRNHR* scheme depends on a local parameter allowing to control numerical diffusion. We show in this contribution, that this parameter can be adapted using a method based on limiters theory. As an illustration

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of the scheme efficiency both in 1D and 2D, we present some results of a dam break over a step, and the classical Ransom Faucet problem.

2. Governing equations and *SRNHR* scheme

Consider a system of balance laws, represented by the following set of equations:

$$\frac{\partial W(x,t)}{\partial t} + \sum_{j=1}^{d} \frac{\partial F_j(W(x,t))}{\partial x_j} = Q(x, W) \tag{1}$$

$$x = (x_1, x_2, \ldots, x_d) \in D \subset \mathbb{R}^d, \quad t > 0,$$

To equation (1), one adds initial condition $W(x, 0) = W_0(x)$ and boundary conditions. In the subsections below, u and v being the x and y velocities of the fluid, we give the structure of W and the fluxes F_j, for each physical problem we will consider in this work.

2.1. $2D$ Shallow Water equations

Let us note g the gravity acceleration, h the water level, and $z = z(x)$ the bottom topography. The 2D Saint Venant system is obtained with:

$$W(x, y) = (h, hu, hv)^T, \quad F_1(W) = \left(hu, hu^2 + \frac{1}{2}gh^2, huv\right)^T,$$

$$F_2(W) = \left(hv, huv, hv^2 + \frac{1}{2}gh^2\right)^T, \quad Q(x, y, W) = \left(0, -gh\frac{dz}{dx}, -gh\frac{dz}{dy}\right)^T$$

2.2. $2D$ two phase flows

Let ρ_k, μ_k, u_k, v_k, be the density, presence fraction, and velocities, respectively for liquid ($k = l$), and for gas ($k = v$). Then the two phase flow system is given by:

$$W = (\mu_l\rho_l,\ \mu_l\rho_l u_l,\ \mu_l\rho_l v_l,\ \mu_v\rho_v,\ \mu_v\rho_v u_v,\ \mu_v\rho_v v_v)^T,$$

$$F_1(W) = \left(\mu_l\rho_l u_l,\ \mu_l\rho_l u_l^2,\ \mu_l\rho_l u_l v_l,\ \mu_v\rho_v u_v,\ \mu_v\rho_v u_v^2,\ \mu_v\rho_v u_v v_v\right)^T,$$

$$F_2(W) = \left(\mu_l\rho_l v_l,\ \mu_l\rho_l u_l v_l,\ \mu_l\rho_l v_l^2,\ \mu_v\rho_v v_v,\ \mu_v\rho_v u_v v_v,\ \mu_v\rho_v v_v^2\right)^T,$$

$$Q_1(x, y, W) = \left(0,\ -\mu_l\frac{\partial P}{\partial x},\ -\mu_l\frac{\partial P}{\partial y},\ 0,\ -\mu_v\frac{\partial P}{\partial x},\ -\mu_v\frac{\partial P}{\partial y}\right)^T$$
$$-\delta\left(0, (P - P_i)\frac{\partial\mu_l}{\partial x}, (P - P_i)\frac{\partial\mu_l}{\partial y}, 0, (P - P_i)\frac{\partial\mu_v}{\partial x}, (P - P_i)\frac{\partial\mu_v}{\partial y}\right)^T,$$

$$Q_2(x, y, W) = (0,\ \mu_l\rho_l g,\ 0,\ 0,\ \mu_v\rho_v g,\ 0)^T, \quad \text{and} \quad Q = Q_1 + Q_2.$$

The following closure relations and parameters specifications (SI system) are used: $\mu_v + \mu_l = 1$, $P = A_v\rho_v^\gamma$, $\rho_l = K_l P^a$, with $A_v = 10^5$, $\gamma = 1.4$, $a = 4.37 \times 10^{-5}$ and $K_l =$

987, 57. $P - P_i = \rho_v(u_v - u_l)^2$ is the interfacial pressure, $\delta = 0$ gives a non hyperbolic non conservative system, while $\delta \neq 0$ enlarges the domain of hyperbolicity.

2.3. The *SRNHR* scheme

Consider the $1D$ system of balance laws:

$$\begin{cases} \dfrac{\partial W}{\partial t} + \dfrac{\partial F(W)}{\partial x} = Q(x, W) \quad \text{in } X = \mathbb{R}\times]0, T[\\ W(x, 0) = W_0(x), \end{cases} \tag{2}$$

with $Q(x, W) = H(W)\frac{\partial G(x,W)}{\partial x}$.

Suppose that the corresponding Riemann problem: $W_0(x) = W_L$ if $x < 0,$ and $W_0(x) = W_R$ if $x > 0$, admits a selfsimilar solution: $W(x, t) = R_s(\frac{x}{t}, W_L, W_R)$ (see the example of Shallow Water equation in [2]).

In [4], using the above property, a two step Non Homogeneous Approximate Riemann Solver was developed. Let us sketch the main steps of this scheme construction.

Recall that in the framework of finite volume methods, at each time step the approximate solution is a piecwise constant function over the volume $]\, x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}[$. So we can see the transition from time t_n to time t_{n+1} as the resolution of the local Riemann problems defined on the interfaces $x_{i+\frac{1}{2}}$.

Integrating the equation (2-1) a first time over the domain $]x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}[\times]t_n, t_n + 1[$, one can write:

$$W_i^{n+1} = W_i^n - \frac{\Delta t}{\Delta x}\big[F\big(W^n_{i+\frac{1}{2}}\big) - F\big(W^n_{i-\frac{1}{2}}\big)\big] + \Delta t Q_i^n$$

where Q_i^n is an approximation, to define in a judicious way, of $\frac{1}{\Delta x \Delta t}\int_{t_n}^{t_{n+1}}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} Q(x, W)\, dxdt$, and $W^n_{i+\frac{1}{2}}$ is an approximation of $R_s(O, W_i^n, W_{i+1}^n)$, the self similar solution of the local Riemann problem at the interface $x_{i+\frac{1}{2}}$.

The question is how to devise a good approximation of this solution?

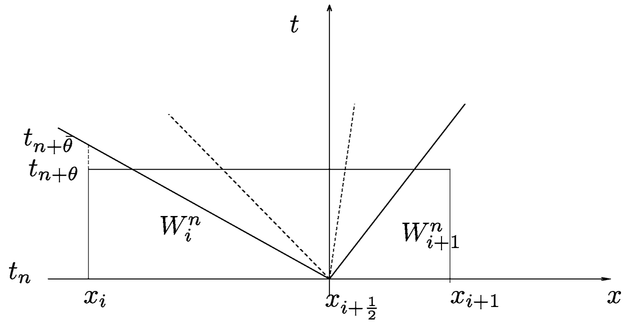
The idea proposed here is to integrate, once more, the equation (2-1) over the domain $\pi_\theta =]x_i, x_{i+1}[\times]t_n, t_n^+[$, where $t_n^+ = t_n + \theta^n_{i+\frac{1}{2}}$.

Setting $W^n_{i+\frac{1}{2}} = \frac{1}{\Delta x}\int_{t_n}^{t_n^+} R_s(\frac{x}{t_n^+}, W_i^n, W_{i+1}^n)\, dx$, one obtains:

$$W^n_{i+\frac{1}{2}} = \frac{1}{2}\big(W_i^n + W_{i+1}^n\big) - \frac{\theta^n_{i+\frac{1}{2}}}{\Delta x}\big[F\big(W_{i+1}^n\big) - F\big(W_i^n\big)\big] + \theta^n_{i+\frac{1}{2}} Q^n_{i+\frac{1}{2}}$$

where $Q^n_{i+\frac{1}{2}} = Q\,(x_i, x_{i+1}, W_i^n, W_{i+1}^n)$ is an approximation of $\frac{1}{\Delta x \theta^n_{i+\frac{1}{2}}}\int_{\pi_\theta} Q(x, W)\, dxdt$.

In [4], the intermediate time step $\theta^n_{i+\frac{1}{2}}$ was expressed as a fraction of the current time step Δt, writing $\theta^n_{i+\frac{1}{2}} = \frac{\alpha^n_{i+\frac{1}{2}}}{2}\Delta t$, where $\alpha^n_{i+\frac{1}{2}}$ is a real positive number. This choice has one drawback. It is the apparition of the metric Δx in the intermediate state $W^n_{i+\frac{1}{2}}$. It is not easy to find a natural analogy of this metric in the 2D case. In the present work, we write $\theta = \alpha^n_{i+\frac{1}{2}}\bar{\theta}$, and $\bar{\theta}$ is defined by the local Rusanov velocity (see figure bellow): $\bar{\theta} = \frac{\Delta x}{2S^n_{i+\frac{1}{2}}}$, where $S^n_{i+\frac{1}{2}} = \max_{p=1,\ldots,m}(\max(|\lambda^n_{p,i}|, |\lambda^n_{p,i+1}|))$, $\lambda^n_{p,i}$ being the pth eigenvalue of the system, corresponding to the state W^n_i.



Remark 2.1. As an example, for Shallow Water systems, the term $Q^n_{i+\frac{1}{2}}$ can be written: $Q^n_{i+\frac{1}{2}} = -g\frac{h^n_i+h^n_{i+1}}{2}\frac{z_{i+1}-z_i}{\Delta x}$. We then eliminate the difficulty of the metric Δx in the intermediate state.

Hence, in the case of 1D systems, *SRNHR* scheme writes under the two steps form [10]:

$$\begin{cases} W^n_{i+\frac{1}{2}} = \frac{1}{2}\left(W^n_i + W^n_{i+1}\right) - \frac{\alpha^n_{i+\frac{1}{2}}}{2S^n_{i+\frac{1}{2}}}\left[f\left(W^n_{i+1}\right) - f\left(W^n_i\right)\right] + \frac{\alpha^n_{i+\frac{1}{2}}}{2}\frac{\Delta x}{S^n_{i+\frac{1}{2}}}\hat{Q}^n_{i+\frac{1}{2}} \\ W^{n+1}_i = W^n_i - r^n\left[f\left(W^n_{i+\frac{1}{2}}\right) - f\left(W^n_{i-\frac{1}{2}}\right)\right] + \Delta t^n Q^n_i, \end{cases} \tag{3}$$

where $\alpha^n_{i+\frac{1}{2}}$ a real positive parameter, and $r^n = \frac{\Delta t^n}{\Delta x}$, Δt^n and Δx being the time step and mesh size.

3. How to fix the parameter $\alpha^n_{i+\frac{1}{2}}$

The analysis of the scheme in the $1D$ homogeneous scalar case, leads to the following results:

Define:

$$S^n_{i+\frac{1}{2}} = \max\left(\left|f'\left(W^n_i\right)\right|, \left|f'\left(W^n_{i+1}\right)\right|\right) \quad \text{and} \quad s^n_{i+\frac{1}{2}} = \min\left(\left|f'\left(W^n_i\right)\right|, \left|f'\left(W^n_{i+1}\right)\right|\right),$$

Proposition 3.1. ([10]) *If one makes the choice* $\alpha^n_{i+\frac{1}{2}} = (\alpha^n_{i+\frac{1}{2}})_1 = \frac{S^n_{i+\frac{1}{2}}}{s^n_{i+\frac{1}{2}}}$, $\forall i$, $\forall n$, *then under some* CFL *condition the SRNHR scheme is a first order, stable and convergent scheme.*

Proposition 3.2. *Suppose now that* $\alpha^n_{i+\frac{1}{2}} = (\alpha^n_{i+\frac{1}{2}})_2 = r^n S^n_{i+\frac{1}{2}}$. *Then the scheme SRNHR becomes the second order Richtmeyer scheme [9].*

Remark 3.3. One can consider $\alpha^n_{i+\frac{1}{2}}$ as a local diffusion control parameter.

This remark and the two propositions above, lead us to define the control parameter as follows: $\alpha^n_{i+\frac{1}{2}} = \Phi^n_{i+\frac{1}{2}}(\alpha^n_{i+\frac{1}{2}})_2 + (1 - \Phi^n_{i+\frac{1}{2}})(\alpha^n_{i+\frac{1}{2}})_1$ where $\Phi^n_{i+\frac{1}{2}}$ is a limiter function (for example Superbee or Van-Leer).

For instance, in the case of shallow water equations, $\Phi^n_{i+\frac{1}{2}}$ is a function of local Riemann invariants. Recall that for Saint-Venant equations, Riemann invariants are given by $I_k = u + (-1)^k 2\sqrt{gh}$, for $k = 1, 2$.

4. Stationary states preserving for Shallow Water problems

Consider the Saint-Venant system defined above.

Definition 1. $W(x, t)$ is a static stationary solution of the system if $\frac{\partial W}{\partial t} = 0$ and $u(x, t) = 0$. In this case, one has $h(x, t) + z(x) = \text{Cste}$.

Definition 2. A finite volume scheme is said to verify the exact $\mathcal{C}$-property [13] if it preserves the equilibrium state: $h_i^n + z_i = c\,\forall i \in \mathbb{Z},\ n \in \mathbb{N}$.

The *SRNHR* scheme has the following property:

Proposition 4.1. *If source terms in the scheme, are discretized as follows:*

$$\hat{Q}^n_{i+\frac{1}{2}} = -\frac{1}{2\Delta x} g\big(h_i^n + h_{i+1}^n)(z_{i+1} - z_i\big),$$

and $Q_i^n = -\frac{1}{8\Delta x} g(h_{i-1}^n + 2h_i^n + h_{i+1}^n)(z_{i+1} - z_{i-1})$*, then SRNHR scheme satisfies the exact C-property [13], and then stationnary states are preserved.*

5. Numerical results

5.1. 1D homogeneous dam break

Consider a dam break represented by the system of Section 2.1 (in the 1D case), where $z \equiv 0$, and initial conditions are:

$$h_0(x) = \begin{cases} 6 & \text{if } 0 \le x \le 6 \\ 2 & \text{if } 6 < x \le 12 \end{cases}, \quad \text{and} \quad u_0(x) = 0,\ \forall x.$$

Results are given at $t = 0.4$ on a mesh of 100 points, and are compared to the exact solution [2]. We see (Figure 1) that *SRNHR* scheme with limiters gives more accurate results than Roe scheme [12].

5.2. 1D dam break over a step and contact discontinuity

Consider now a dam break over a step. Source term here is the bottom slope:

$$z(x) = \begin{cases} 0 & \text{if } x \le 6 \\ 1 & \text{if } x > 6 \end{cases}, \quad h_0(x) = \begin{cases} 5 & \text{if } x \le 6 \\ 1 & \text{if } x > 6. \end{cases}, \quad u_0(x) = 0.$$

We compare results obtained with *SRNHR* scheme to those obtained with Vazquez equilibrium scheme [13]. The mesh contains 400 points and results are given at $t = 0.5$. Figure 2 shows the efficiency of *SRNHR* scheme which approximates the jump over the step with no point in this stationary contact discontinuity.

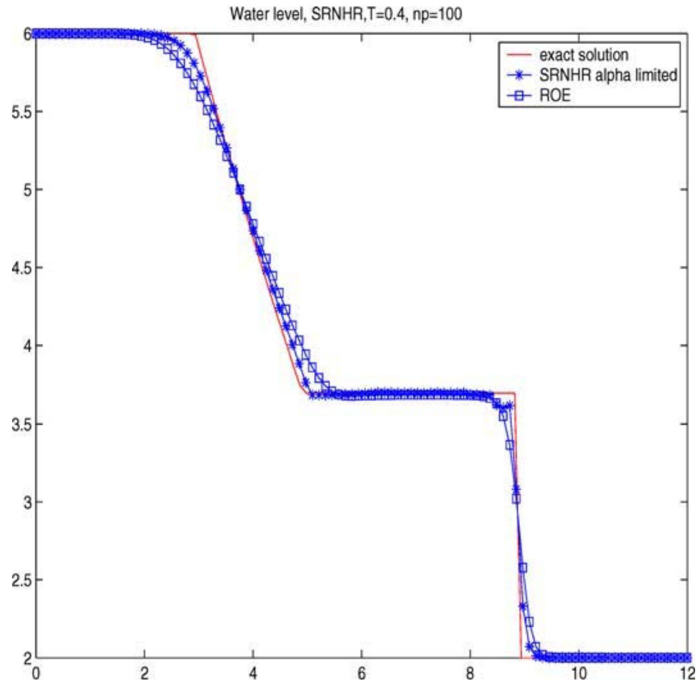


Fig. 1 1D dam break, water level, *SRNHR* and Roe schemes

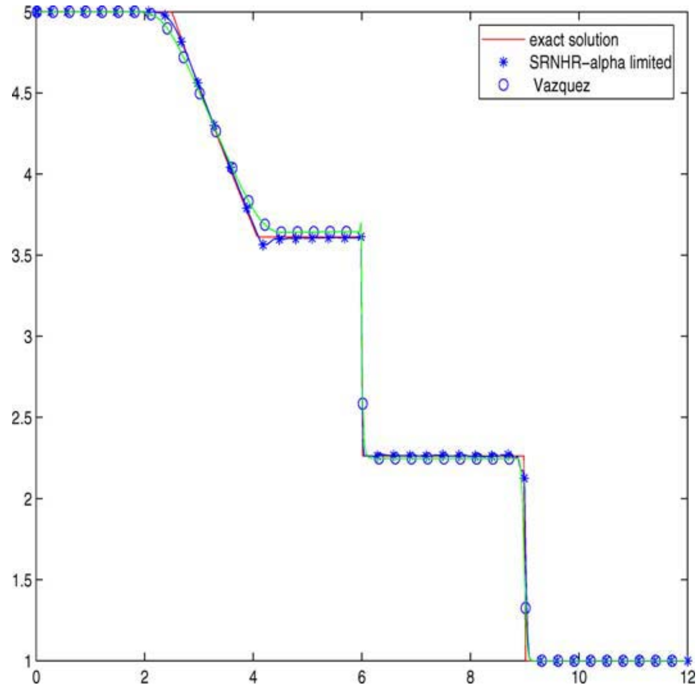


Fig. 2 1D dam break over a step, water level, *SRNHR* and Vazquez schemes.

5.3. $2D$ dam break over a step

In the case of $2D$ systems, one computes the intermediate state in the *SRNHR* scheme first step, using the projection of the PDE system along the normal to the interfaces [1].

Consider 2D Shallow Water equations with the following initial conditions:

$$u_0(x,y) = v_0(x,y) = 0, \quad \forall x \in [0;12], \quad \forall y \in [0;1],$$

$$\text{and } h_0(x,y) = \begin{cases} 6 & \text{if} \quad x \le 6, \quad \forall y \in [0;1] \\ 2 & \text{if} \quad x > 6, \quad \forall y \in [0;1] \end{cases}$$

Results are obtained on an unstructured mesh with 100 points along the x-axis, and 10 points along the y-axis. Figures 3 and 4 show that the shock wave, the rarefaction, and the stationary contact discontinuity are computed accurately, and that the $1D$ behavior is perfectly recovered.

5.4. $1D$ two phase flow

5.4.1. 1D Ransom Faucet experiment specifications

The test case consists in a vertical water jet, contained within a cylindrical channel, and accelerated under the gravity force. The initial gas fraction is $\mu_0 = 0.2$. The exact solution at time $t = 0.6$ is calculated for a well posed problem that is deduced from the initial system,

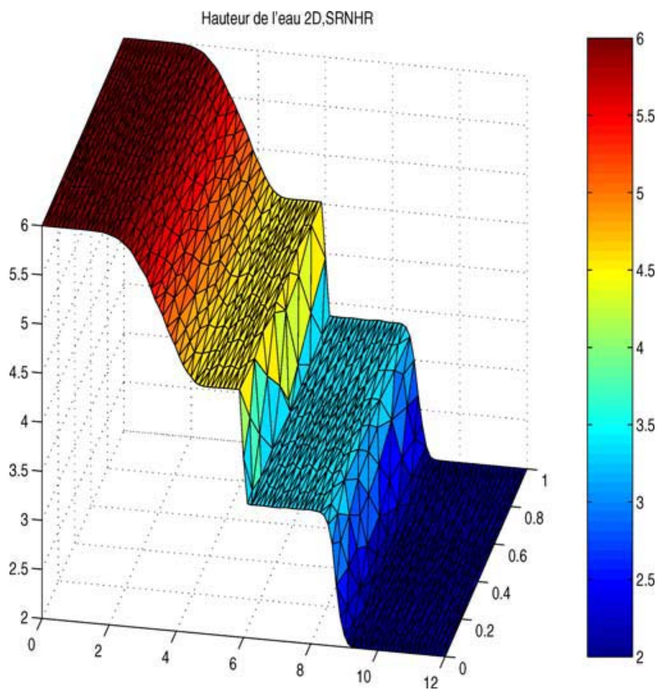


Fig. 3 $2D$ dam break over a step, water level

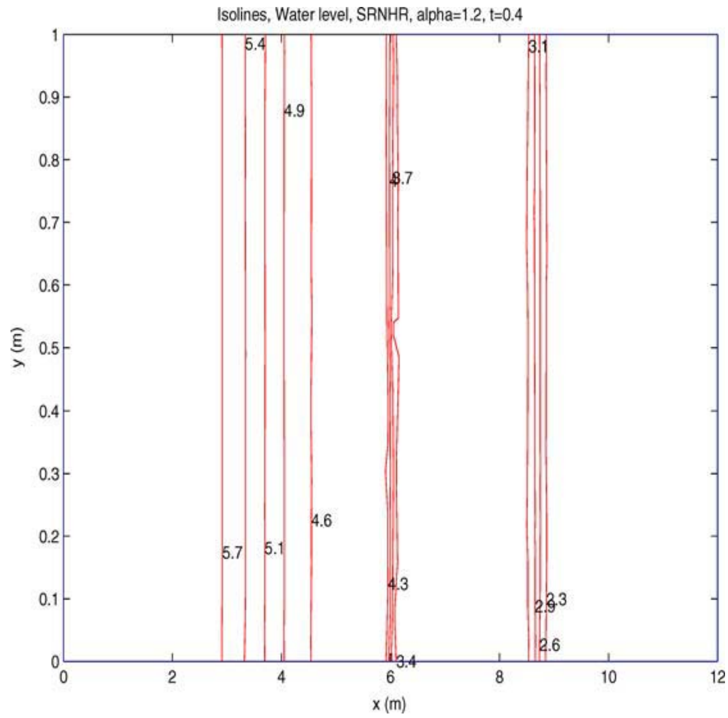
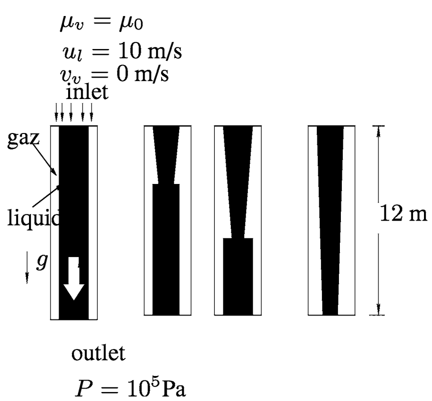


Fig. 4 $2D$ dam break over a step, isolines

when one supposes constant phases densities [11].

Initial conditions:
$\forall x \in [x_0, x_l]$, $\mu_v(t=0)=\mu_0$, $u_l(t=0)=10$, $u_v(t=0)=0$, $p(t=0)=10^5$, $\rho_v(t=0)=1$, $\rho_l(t=0)=988,0638$.

Boundary conditions:
*inlet$(x_0=0)$: $\mu_v(0,t)=\mu_0$, $u_l(0,t)=10$, $u_v(0,t)=0$.
*outlet$(x_l=12)$: $p\,(12,t)=10^5$.



5.4.2. SRNHR algorithm

Consider the system of Section 2.2 in the 1D case. To solve the system, we use the splitting strategy presented in ([4]). The gravity source term Q_2 is treated by an explicit Euler time integration, in a first ODE step, to get W^* from W^n, then the system $W_t + F(W)_x = Q_1$ is solved by *SRNHR* scheme to get W^{n+1} from W^*.

Note that for all the two phase flow computations, the parameter $\alpha^n_{j+\frac{1}{2}}$ has been kept constant. Moreover, we tested alternatively the classical model ($\delta = 0$), and the model with interfacial pressure ($\delta = 1$). What we are interested in here, is to determine the limit of mesh refinement the scheme can support, before the non hyperbolicity of the physical problem leads to computations blow up. Results are displayed on the Figure 5 and show that one increases this limit from 150 to 500 mesh points, once the interfacial pressure term is added to the system.

5.5. $2D$ two phase flow simulations

We consider the 2D two phase flow model of Section 2.2. Here $\mu_0 = 0.6$, and we aim to perform a numerical simulation of a 2D Ransom Faucet defined in the same way as in Section 5.4.1. Let us precise that no real physical significance is attached to this test case. It just permits to check the robustness of *SRNHR* scheme in the 2D case. As a matter of fact, we could manage to get correct results on a 48×10 so called UK flag mesh (Figures 7 and 8). Nevertheless, for the stiff case $\mu_0 = 0.2$ (imaginary part of the system eigenvalues are not

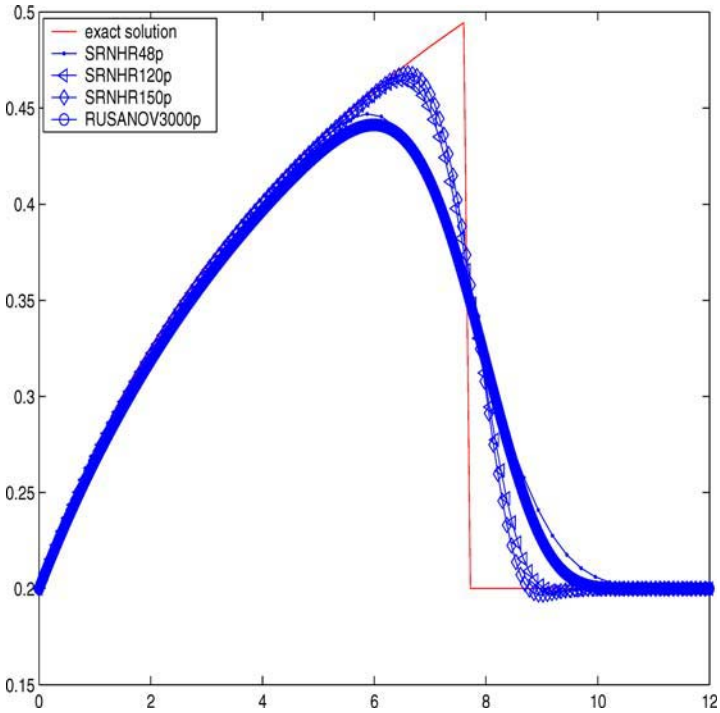


Fig. 5 Ransom Faucet, void fraction

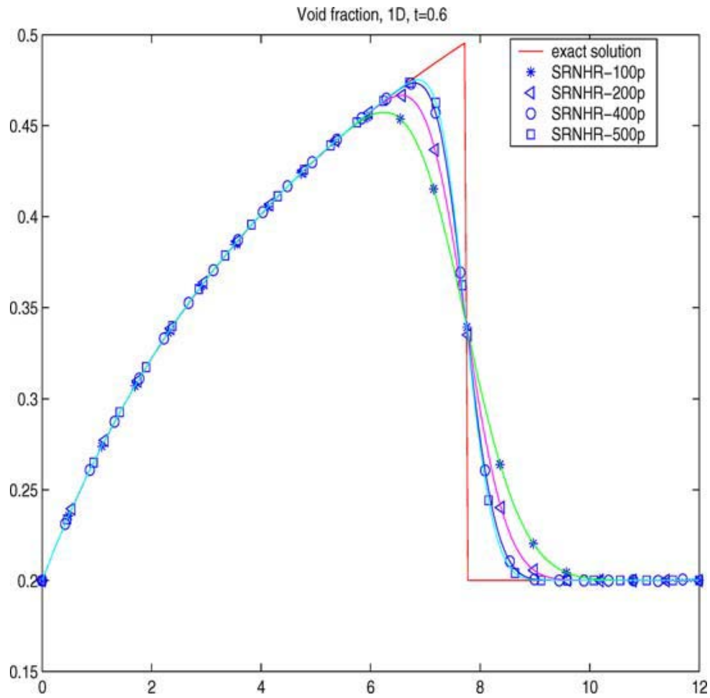


Fig. 6 Ransom Faucet with interfacial pressure, void fraction

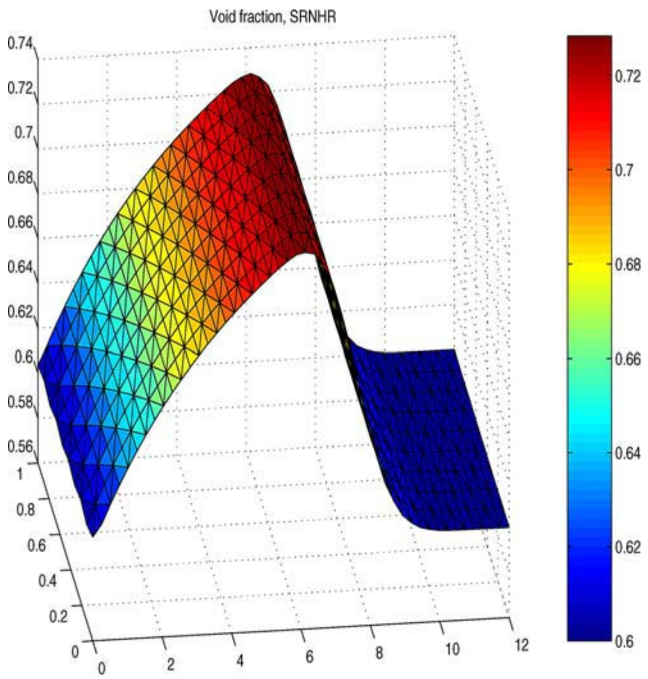


Fig. 7 $2D$ Ransom Faucet, $t = 0.6$, UK 48×10 mesh, void fraction

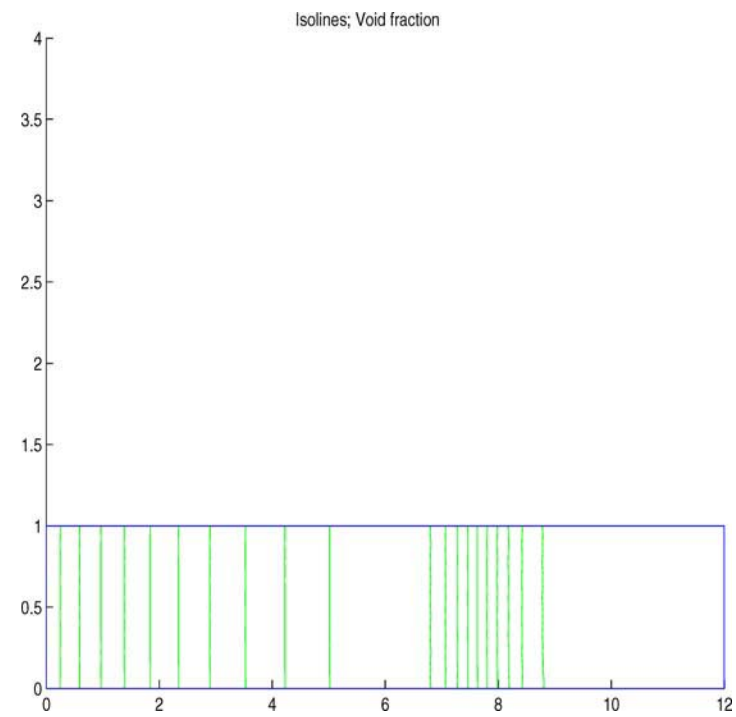


Fig. 8 $2D$ Ransom Faucet, $t = 0.6$, UK 48×10 mesh, isolines

negligible compared to real part in this case), as well as when non structured meshes are used, the computations blow up before reaching the fixed time limit $t = 0.6$.

6. Conclusions

In this work a first approach of the difficulties introduced by non homogeneous systems has been presented. A two step finite volume scheme using physical flux evaluations unstead of Jacobian decompositions has been presented. The diffusion of the mentioned scheme is controled. Two classical examples of non homogeneous systems have been considered numerically. The first one, the shallow water system is hyperbolic, but has a stiff source term, and the second, the two phase flow is non hyperbolic. In both cases the two step scheme gives good results.

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