A short introduction the finite element method

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Chapter 1

The Finite Element Method and Lax-Milgram's Theorem

This section presents the finite element method, including error estimates, together with the basic related theory on existence and stability of linear elliptic partial differential equations. The motivation to introduce finite element methods is the computational simplicity and efficiency for construction of stable higher order discretizations for elliptic and parabolic differential equations, modelling for instance diffusion, including general boundary conditions and domains. Finite element methods require somewhat more work per degree of freedom as compared to finite difference methods on a uniform mesh. On the other hand, construction of higher order finite difference approximations including general boundary conditions and general domains is troublesome.

In one space dimension such an elliptic problem can, for given functions $a, f, r : (0,1) \to \mathbf{R}$, take the form of the following equation for $u : [0,1] \to \mathbf{R}$,

$$\begin{aligned} &(-au')' + ru = f \quad \text{on} \quad (0,1) \\ &u(x) = 0 \qquad\qquad\ \ \text{for } x = 0,\ x = 1, \end{aligned} \tag{1.1}$$

where $a > 0$ and $r \geq 0$. The basic existence and uniqueness result for general elliptic differential equations is based on Lax-Milgram's Theorem, which we will describe in section 1.4. We shall see that its stability properties, based on so called energy estimates, is automatically satisfied for finite element methods in contrast to finite difference methods.

1.1 The Finite Element Method

A derivation of the finite element method can be divided into:

(1) variational formulation in an infinite dimensional space V,

(2) variational formulation in a finite dimensional subspace, $V_h \subset V$,

(3) choice of a basis for V_h, and

(4) solution of the discrete system of equations.

Step 1. *Variational formulation in an infinite dimensional space, V.*
Consider the following Hilbert space,

$$V = \left\{ v : (0,1) \to \mathbf{R} \ : \ \int_0^1 \left(v^2(x) + (v'(x))^2\right) \mathrm{d}x < \infty, \ v(0) = v(1) = 0 \right\},$$

with the corresponding norm $\|v\|_V = \sqrt{\int_0^1 (v^2(x) + (v'(x))^2)\, \mathrm{d}x}$. Multiply equation (1.1) by $v \in V$ and integrate by parts to get

$$\begin{aligned} \int_0^1 fv \ \mathrm{d}x &= \int_0^1 ((-au')' + ru)v \ \mathrm{d}x \\ &= [-au'v]_0^1 + \int_0^1 (au'v' + ruv) \ \mathrm{d}x \\ &= \int_0^1 (au'v' + ruv) \ \mathrm{d}x. \end{aligned} \tag{1.2}$$

Therefore the variational formulation of (1.1) is to find $u \in V$ such that

$$A(u,v) = L(v) \qquad \forall v \in V, \tag{1.3}$$

where

$$\begin{aligned} A(u,v) &= \int_0^1 (au'v' + ruv) \ \mathrm{d}x, \\ L(v) &= \int_0^1 fv \ \mathrm{d}x. \end{aligned}$$

Remark 1.1. The integration by parts in (1.2) shows that a smooth solution of equation (1.1) satisfies the variational formulation (1.3). For a solution of the variational formulation (1.3) to also be a solution of the equation (1.1), we need additional conditions on the regularity of the functions a, r and f so that u'' is continuous. Then the following integration by parts yields, as in (1.2),

$$0 = \int_0^1 (au'v' + ruv - fv) \ \mathrm{d}x = \int_0^1 (-(au')' + ru - f)v \ \mathrm{d}x.$$

Since this holds for all $v \in V$, it implies that

$$-(au')' + ru - f = 0,$$

provided $-(au')' + ru - f$ is continuous.

Step 2. *Variational formulation in the finite dimensional subspace, V_h.*
First divide the interval $(0,1)$ into $0 = x_0 < x_2 < ... < x_{N+1} = 1$, i.e. generate the mesh. Then define the space of continuous piecewise linear functions on the mesh with zero boundary conditions

$$\begin{aligned} V_h = \{v \in V \quad : \quad & v(x)\mid_{(x_i,x_{i+1})} = c_i x + d_i, \text{ i.e. } v \text{ is linear on } (x_i, x_{i+1}), i = 0, \cdots, N \\ & \text{and } v \text{ is continuous on } (0,1)\}. \end{aligned}$$

The variational formulation in the finite dimensional subspace is to find $u_h \in V_h$ such that

$$A(u_h, v) = L(v) \qquad \forall v \in V_h. \tag{1.4}$$

The function u_h is a finite element solution of the equation (1.1). Other finite element solutions are obtained from alternative finite dimensional subspaces, e.g. based on piecewise quadratic approximation.

Step 3. *Choose a basis for V_h.*
Let us introduce the basis functions $\phi_i \in V_h$, for $i = 1, ..., N$, defined by

$$\phi_i(x_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases} \tag{1.5}$$

A function $v \in V_h$ has the representation

$$v(x) = \sum_{i=1}^{N} v_i \phi_i(x),$$

where $v_i = v(x_i)$, i.e. each $v \in V_h$ can be written in a unique way as a linear combination of the basis functions ϕ_i.

Step 4. *Solve the discrete problem (1.4).*
Using the basis functions ϕ_i, for $i = 1, ..., N$ from Step 3, we have

$$u_h(x) = \sum_{i=1}^{N} \xi_i \phi_i(x),$$

where $\xi = (\xi_1, ..., \xi_N)^T \in \mathbf{R}^N$, and choosing $v = \phi_j$ in (1.4), we obtain

$$\begin{aligned} L(\phi_j) &= A(u_h, \phi_j) \\ &= A(\sum_i \phi_i \xi_i, \phi_j) = \sum_i \xi_i A(\phi_i, \phi_j), \end{aligned}$$

so that $\xi \in \mathbf{R}^N$ solves the linear system

$$\tilde{A}\xi = \tilde{L}, \tag{1.6}$$

where

$$\begin{aligned} \tilde{A}_{ji} &= A(\phi_i, \phi_j), \\ \tilde{L}_j &= L(\phi_j). \end{aligned}$$

The $N \times N$ matrix $\tilde{A}$ is called the stiffness matrix and the vector $\tilde{L} \in \mathbf{R}^N$ is called the load vector.

Example 1.2. Consider the following two dimensional problem,

$$\begin{aligned} -\mathrm{div}(k\nabla u) + ru &= f \quad \text{in } \Omega \subset \mathbb{R}^2 \\ u &= g_1 \quad \text{on } \Gamma_1 \\ \frac{\partial u}{\partial n} &= g_2 \quad \text{on } \Gamma_2, \end{aligned} \tag{1.7}$$

where $\partial\Omega = \Gamma = \Gamma_1 \cup \Gamma_2$ and $\Gamma_1 \cap \Gamma_2 = \emptyset$ and $\frac{\partial u}{\partial n} := n \cdot \nabla u$ is the derivative in the outward normal direction. The variational formulation has the following form.

(i) Variational formulation in the infinite dimensional space.

Let

$$V_g = \left\{ v(x) : \int_\Omega (v^2(x) + |\nabla v(x)|^2) \; \mathrm{d}x < \infty, v|_{\Gamma_1} = g \right\}.$$

Take a function $v \in V_0$, i.e. $v = 0$ on Γ_1, then by (1.7)

$$\begin{aligned} \int_\Omega fv \; \mathrm{d}x &= -\int_\Omega \mathrm{div}(k\nabla u)v \; \mathrm{d}x + \int_\Omega ruv \; \mathrm{d}x \\ &= \int_\Omega k\nabla u \cdot \nabla v \; \mathrm{d}x - \int_{\Gamma_1} k\frac{\partial u}{\partial n} v \; \mathrm{d}s - \int_{\Gamma_2} k\frac{\partial u}{\partial n} v \; \mathrm{d}s + \int_\Omega ruv \; \mathrm{d}x \\ &= \int_\Omega k\nabla u \cdot \nabla v \; \mathrm{d}x - \int_{\Gamma_2} kg_2 v \; \mathrm{d}s + \int_\Omega ruv \; \mathrm{d}x. \end{aligned}$$

The variational formulation for the model problem (1.7) is to find $u \in V_{g_1}$ such that

$$A(u, v) = L(v) \qquad \forall v \in V_0, \tag{1.8}$$

where

$$\begin{aligned} A(u, v) &= \int_\Omega (k\nabla u \cdot \nabla v + ruv)\, \mathrm{d}x, \\ L(v) &= \int_\Omega fv\, \mathrm{d}x + \int_{\Gamma_2} kg_2 v \mathrm{d}s. \end{aligned}$$

(ii) Variational formulation in the finite dimensional space.

Assume for simplicity that Ω is a polygonal domain which can be divided into a triangular mesh $T_h = \{K_1, ...K_N\}$ of non overlapping triangles K_i and let $h = \max_i$(length of longest side of K_i). Assume also that the boundary function g_1 is continuous and that its restriction to each edge $K_i \cap \Gamma_1$ is a linear function. Define

$$\begin{aligned} V_0^h &= \{v \in V_0 \;:\; v|_{K_i} \text{ is linear } \forall K_i \in T_h,\; v \text{ is continuous on } \Omega\}, \\ V_{g_1}^h &= \{v \in V_{g_1} \;:\; v|_{K_i} \text{ is linear } \forall K_i \in T_h,\; v \text{ is continuous on } \Omega\}, \end{aligned}$$

and the finite element method is to find $u_h \in V_{g_1}^h$ such that

$$A(u_h, v) = L(v), \qquad \forall v \in V_0^h. \tag{1.9}$$

(iii) Choose a basis for V_0^h.

As in the one dimensional problem, choose the basis $\phi_j \in V_0^h$ such that

$$\phi_j(x_i) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \qquad j = 1, 2, ..., N,$$

where $x_i,\; i = 1, \ldots, N$, are the vertices of the triangulation.

(iv) Solve the discrete system.

Consider first the case g_1=0. Let

$$u_h(x) = \sum_{i=1}^{N} \xi_i \phi_i(x), \quad \text{and} \quad \xi_i = u_h(x_i).$$

Then (1.9) can be written in matrix form,

$$\tilde{A}\xi = \tilde{L}, \quad \text{where} \quad \tilde{A}_{ji} = A(\phi_i, \phi_j) \ \text{ and } \ \tilde{L}_j = L(\phi_j).$$

In the case general case with non zero g_1, we have

$$u_h(x) = \sum_{i=1}^{N} \xi_i \phi_i(x) + \sum_{x_i \in \Gamma_1} g_1(x_i)\phi_i(x)$$

and $\tilde{L}$ is modified to $\tilde{L}_j = L(\phi_j) - \sum_{x_i \in \Gamma_1} g_1(x_i) A(\phi_i, \phi_j)$.

□

1.2 Lax-Milgram's Theorem and an Error Estimate

We shall now study existence and stability of solutions to elliptic partial differential equations together with error estimates for finite element methods. We first state the following theorem, which we will prove later,

Theorem 1.3 (Lax-Milgram). *Let V be a Hilbert space with norm $\|\cdot\|_V$ and scalar product $(\cdot,\cdot)_V$ and assume that A is a bilinear functional and L is a linear functional that satisfy:*

(1) A is symmetric, i.e. $A(v,w) = A(w,v) \quad \forall v, w \in V$;

(2) A is V-elliptic, i.e. $\exists\ \alpha > 0$ such that $A(v,v) \geq \alpha \|v\|_V^2 \quad \forall v \in V$;

(3) A is continuous, i.e. $\exists\ C \in \mathbb{R}$ such that $|A(v,w)| \leq C\|v\|_V \|w\|_V$; and

(4) L is continuous, i.e. $\exists\ \Lambda \in \mathbb{R}$ such that $|L(v)| \leq \Lambda \|v\|_V \quad \forall v \in V$.

Then there is a unique function $u \in V$ such that $A(u,v) = L(v) \quad \forall v \in V$, and the stability estimate $\|u\|_V \leq \Lambda/\alpha$ holds.

By (3) and (4) we see that the bilinear form A introduces a corresponding norm $\|v\|_A := \sqrt{A(v,v)}$ since we have

$$\alpha^{1/2}\|v\|_V \leq \|v\|_A \leq C^{1/2}\|v\|_V\,.$$

The norm $\|\cdot\|_A$ is called the energy norm. The basic error estimate for the finite element method is the best approximation property

$$\|u - u_h\|_A \leq \min_{v \in V_h} \|u - v\|_A\,, \tag{1.10}$$

which is based on the orthogonality

$$A(u - u_h, v) = 0\,, \quad \text{for all } v \in V_h \subset V$$

as follows: since u and u_h solve the variational problems

$$\begin{aligned} A(u, v) &= L(v)\,, \quad \forall v \in V\,, \\ A(u_h, v) &= L(v)\,, \quad \forall v \in V_h \subset V\,, \end{aligned}$$

we have

$$A(u - u_h, v) = A(u, v) - A(u_h, v) = L(v) - L(v) = 0\,, \quad \text{for } v \in V_h \subset V. \tag{1.11}$$

Therefore we obtain for any $v \in V_h$

$$\begin{aligned} A(u - u_h, u - u_h) &= A(u - u_h, u - v + v - u_h) \\ &= A(u - u_h, u - v) + A(u - u_h, v - u_h) \\ &= A(u - u_h, u - v) \end{aligned}$$

since the last term vanishes by the orthogonality (1.11). Cauchy-Schwartz's inequality applied to the energy norm implies

$$\|u - u_h\|_A^2 = A(u - u_h, u - u_h) = A(u - u_h, u - v) \le \|u - u_h\|_A \|u - v\|_A$$

and by dividing by $\|u - u_h\|_A$ we have proved (1.10).

1.3 Error Estimates Based on the Interpolant

The approximation property of the space V_h can be characterized by

Lemma 1.4. *Suppose V_h is the piecewise linear finite element space (1.4), which discretizes the functions in V, defined on $(0,1)$, with the interpolant $\pi : V \to V_h$ defined by*

$$\pi v(x) = \sum_{i=1}^{N} v(x_i)\phi_i(x), \tag{1.12}$$

where $\{\phi_i\}$ is the basis (1.5) of V_h. Then

$$\begin{aligned} \|(v - \pi v)'\|_{L^2(0,1)} &\le \sqrt{\int_0^1 h^2 v''(x)^2 \,\mathrm{d}x} \le Ch, \\ \|v - \pi v\|_{L^2(0,1)} &\le \sqrt{\int_0^1 h^4 v''(x)^2 \,\mathrm{d}x} \le Ch^2, \end{aligned} \tag{1.13}$$

where $h = \max_i \ (x_{i+1} - x_i)$ and $\|w\|_{L^2(0,1)} := \sqrt{\int_0^1 \big(w(x)\big)^2 \mathrm{d}x}$.

Proof. Take $v \in V$ and consider first (1.13) on an interval (x_i, x_{i+1}). By the mean value theorem, there is for each $x \in (x_i, x_{i+1})$ a $\xi \in (x_i, x_{i+1})$ such that $v'(\xi) = (\pi v)'(x)$. Therefore

$$v'(x) - (\pi v)'(x) = v'(x) - v'(\xi) = \int_\xi^x v''(s)\mathrm{d}s,$$

so that

$$\begin{aligned}
\int_{x_i}^{x_{i+1}} |v'(x) - (\pi v)'(x)|^2 \mathrm{d}x &= \int_{x_i}^{x_{i+1}} (\int_\xi^x v''(s)\mathrm{d}s)^2 \mathrm{d}x \\
&\leq \int_{x_i}^{x_{i+1}} |x - \xi| \int_\xi^x (v''(s))^2 \mathrm{d}s\mathrm{d}x \\
&\leq h^2 \int_{x_i}^{x_{i+1}} (v''(s))^2 \mathrm{d}s, \qquad (1.14)
\end{aligned}$$

which after summation of the intervals proves (1.13).

Next, we have

$$v(x) - \pi v(x) = \int_{x_i}^x (v - \pi v)'(s)\mathrm{d}s,$$

so by (1.14)

$$\begin{aligned}
\int_{x_i}^{x_{i+1}} |v(x) - \pi v(x)|^2 \mathrm{d}x &= \int_{x_i}^{x_{i+1}} (\int_{x_i}^x (v - \pi v)'(s)\mathrm{d}s)^2 \mathrm{d}x \\
&\leq \int_{x_i}^{x_{i+1}} |x - x_i| \int_{x_i}^x ((v - \pi v)')^2(s)\mathrm{d}s\mathrm{d}x \\
&\leq h^4 \int_{x_i}^{x_{i+1}} (v''(s))^2 \mathrm{d}s,
\end{aligned}$$

which after summation of the intervals proves the lemma. □

Our derivation of the a priori error estimate

$$\|u - u_h\|_V \leq Ch,$$

where u and u_h satisfy (1.3) and (1.4), respectivly, uses Lemma 1.4 and a combination of the following four steps:

(1) error representation based on the *ellipticity*

$$\alpha \int_\Omega (v^2(x) + (v'(x))^2)\ \mathrm{d}x \leq A(v, v) = \int_\Omega (a(v')^2 + rv^2)\ \mathrm{d}x,$$

where $\alpha = \inf_{x \in (0,1)}(a(x), r(x)) > 0$,

(2) the *orthogonality*

$$A(u-u_h,v)=0 \quad \forall v\in V_h,$$

obtained by $V_h\subset V$ and subtraction of the two equations

$$\begin{aligned} A(u,v) &= L(v) \quad \forall v\in V \qquad \text{by (1.3)},\\ A(u_h,v) &= L(v) \quad \forall v\in V_h \qquad \text{by (1.4)}, \end{aligned}$$

(3) the *continuity*

$$|A(v,w)|\le C\|v\|_V\|w\|_V \quad \forall v,w\in V,$$

where $C=\sup_{x\in(0,1)}(a(x),r(x))$, and

(4) the *interpolation estimates*

$$\begin{aligned} \|(v-\pi v)'\|_{L^2} &\le Ch,\\ \|v-\pi v\|_{L^2} &\le Ch^2, \end{aligned} \tag{1.15}$$

where $h=\max\ (x_{i+1}-x_i)$.

To start the proof of the error estimate let $e\equiv u-u_h$. Then by Cauchy's inequality

$$\begin{aligned} A(e,e) &= A(e,u-\pi u+\pi u-u_h)\\ &= A(e,u-\pi u)+A(e,\pi u-u_h)\\ &\overset{\text{Step2}}{=} A(e,u-\pi u)\\ &\le \sqrt{A(e,e)}\sqrt{A(u-\pi u,u-\pi u)}, \end{aligned}$$

so that by division of $\sqrt{A(e,e)}$,

$$\begin{aligned} \sqrt{A(e,e)} &\le \sqrt{A(u-\pi u,u-\pi u)}\\ &\overset{\text{Step3}}{=} C\|u-\pi u\|_V\\ &\equiv C\sqrt{\|u-\pi u\|_{L^2}^2+\|(u-\pi u)'\|_{L^2}^2}\\ &\overset{\text{Step4}}{\le} Ch. \end{aligned}$$

Therefore, by Step 1

$$\alpha\|e\|_V^2\le A(e,e)\le Ch^2,$$

which implies the a priori estimate

$$\|e\|_V\le Ch,$$

where the constant C depends on u. □

1.4 Lax-Milgram's Theorem

Theorem 1.5. *Suppose A is symmetric, i.e. $A(u,v) = A(v,u) \;\; \forall u, v \in V$, then*

$$\text{(Variational problem)} \Longleftrightarrow \text{(Minimization problem)}$$

with

$$\begin{array}{ll} \text{(Variational problem (Var))} & \text{Find } u \in V \text{ such that } A(u,v) = L(v) \quad \forall v \in V, \\ \text{(Minimization problem (Min))} & \text{Find } u \in V \text{ such that } F(u) \leq F(v) \quad \forall v \in V, \end{array}$$

where

$$F(w) \;\; \equiv \;\; \frac{1}{2} A(w,w) - L(w) \qquad \forall w \in V.$$

Proof. Take $\epsilon \in \mathbb{R}$. Then

$$\begin{aligned} (\Rightarrow) \quad F(u \;\; + \;\; \epsilon w) &= \frac{1}{2} A(u + \epsilon w, u + \epsilon w) - L(u + \epsilon w) \\ &= \left(\frac{1}{2} A(u,u) - L(u) \right) + \epsilon A(u,w) - \epsilon L(w) + \frac{1}{2} \epsilon^2 A(w,w) \\ &\geq \left(\frac{1}{2} A(u,u) - L(u) \right) \quad \left(\text{since } \frac{1}{2} \epsilon^2 A(w,w) \geq 0 \text{ and } A(u,w) = L(w) \right) \\ &= F(u). \end{aligned}$$

$(\Leftarrow)$ Let $g(\epsilon) = F(u + \epsilon w)$, where $g : \mathbf{R} \to \mathbf{R}$. Then

$$0 = g'(0) = 0 \cdot A(w,w) + A(u,w) - L(w) = A(u,w) - L(w).$$

Therefore

$$A(u,w) = L(w) \;\; \forall w \in V.$$

□

Theorem 1.6 (Lax-Milgram)**.** *Let V be a Hilbert space with norm $\| \cdot \|_V$ and scalar product $(\cdot, \cdot)_V$ and assume that A is a bilinear functional and L is a linear functional that satisfy:*

(1) A is symmetric, i.e. $A(v,w) = A(w,v) \quad \forall v, w \in V$;

(2) A is V-elliptic, i.e. $\exists \; \alpha > 0$ such that $A(v,v) \geq \alpha \|v\|_V^2 \quad \forall v \in V$;

(3) A is continuous, i.e. $\exists \; C \in \mathbb{R}$ such that $|A(v,w)| \leq C \|v\|_V \|w\|_V$; and

(4) L is continuous, i.e. $\exists \; \Lambda \in \mathbb{R}$ such that $|L(v)| \leq \Lambda \|v\|_V \quad \forall v \in V$.

Then there is a unique function $u \in V$ such that $A(u, v) = L(v) \quad \forall v \in V$, and the stability estimate $\|u\|_V \leq \Lambda/\alpha$ holds.

Proof. The goal is to construct $u \in V$ solving the minimization problem $F(u) \leq F(v)$ for all $v \in V$, which by the previous theorem is equivalent to the variational problem. The energy norm, $\|v\|^2 \equiv A(v, v)$, is equivalent to the norm of V, since by Condition 2 and 3,

$$\alpha\|v\|_V^2 \leq A(v, v) = \|v\|^2 \leq C\|v\|_V^2.$$

Let

$$\beta = \inf_{v \in V} F(v). \tag{1.16}$$

Then $\beta \in \mathbf{R}$, since

$$F(v) = \frac{1}{2}\|v\|^2 - L(v) \geq \frac{1}{2}\|v\|^2 - \Lambda\|v\| \geq -\frac{\Lambda^2}{2}.$$

We want to find a solution to the minimization problem $\min_{v \in V} F(v)$. It is therefore natural to study a minimizing sequence v_i, such that

$$F(v_i) \to \beta = \inf_{v \in V} F(v). \tag{1.17}$$

The next step is to conclude that the v_i infact converge to a limit:

$$\begin{aligned}
\left\|\frac{v_i - v_j}{2}\right\|^2 &= \frac{1}{2}\|v_i\|^2 + \frac{1}{2}\|v_j\|^2 - \left\|\frac{v_i + v_j}{2}\right\|^2 \quad (\text{ by the parallelogram law }) \\
&= \frac{1}{2}\|v_i\|^2 - L(v_i) + \frac{1}{2}\|v_j\|^2 - L(v_j) \\
&\quad - \left(\left\|\frac{v_i + v_j}{2}\right\|^2 - 2L(\frac{v_i + v_j}{2})\right) \\
&= F(v_i) + F(v_j) - 2F\left(\frac{v_i + v_j}{2}\right) \\
&\leq F(v_i) + F(v_j) - 2\beta \quad (\text{ by } (1.16) \text{ }) \\
&\to 0, \qquad (\text{ by } (1.17) \text{ }).
\end{aligned}$$

Hence $\{v_i\}$ is a Cauchy sequence in V and since V is a Hilbert space (in particular V is a complete space) we have $v_i \to u \in V$.

Finally $F(u) = \beta$, since

$$\begin{aligned}
|F(v_i) - F(u)| &= |\frac{1}{2}(\|v_i\|^2 - \|u\|^2) - L(v_i - u)| \\
&= |\frac{1}{2}A(v_i - u, v_i + u) - L(v_i - u)| \\
&\leq (\frac{C}{2}\|v_i + u\|_V + \Lambda)\|v_i - u\|_V \\
&\to 0.
\end{aligned}$$

Therefore there exists a unique (why?) function $u \in V$ such that $F(u) \leq F(v) \quad \forall v \in V$. To verify the stability estimate, take $v = u$ in (Var) and use the ellipticity (1) and continuity (3) to obtain

$$\alpha\|u\|_V^2 \leq A(u, u) = L(u) \leq \Lambda\|u\|_V$$

so that

$$\|u\|_V \leq \frac{\Lambda}{\alpha}.$$

The uniqueness of u can also be verified from the stability estimate. If u_1, u_2 are two solutions of the variational problem we have $A(u_1 - u_2, v) = 0$ for all $v \in V$. Therefore the stability estimate implies $\|u_1 - u_2\|_V = 0$, i.e. $u_1 = u_2$ and consequently the solution is unique. □

Example 1.7. Determine conditions for the functions k, r and $f : \Omega \to \mathbb{R}$ such that the assumptions in the Lax-Milgram theorem are satisfied for the following elliptic partial differential equation in $\Omega \subset \mathbf{R}^2$

$$\begin{aligned}
-\mathrm{div}(k\nabla u) + ru &= f \quad \text{in } \Omega \\
u &= 0 \quad \text{on } \partial\Omega.
\end{aligned}$$

Solution. This problem satisfies (Var) with

$$V = \{v \ : \ \int_\Omega (v^2(x) + |\nabla v(x)|^2)\ \mathrm{d}x < \infty, \text{ and } v|_{\partial\Omega} = 0\},$$

$$\begin{aligned}
A(u, v) &= \int_\Omega (k\nabla u \cdot \nabla v + ruv)\ \mathrm{d}x, \\
L(v) &= \int_\Omega fv\ \mathrm{d}x, \\
\|v\|_V^2 &= \int_\Omega (v^2(x) + |\nabla v|^2)\ \mathrm{d}x.
\end{aligned}$$

Consequently V is a Hilbert space and A is symmetric and continuous provided k and r are uniformly bounded.

The ellipticity follows by

$$\begin{aligned} A(v,v) &= \int_\Omega (k|\nabla v|^2 + rv^2)\ \mathrm{d}x \\ &\geq \alpha \int_\Omega (v^2(x) + |\nabla v|^2)\ \mathrm{d}x \\ &= \alpha \|v\|^2_{H^1}, \end{aligned}$$

provided $\alpha = \inf_{x\in\Omega}(k(x), r(x)) > 0$.

The continuity of A is a consequence of

$$\begin{aligned} A(v,w) &\leq \max(\|k\|_{L^\infty}, \|r\|_{L^\infty}) \int_\Omega (|\nabla v||\nabla w| + |v||w|)\mathrm{d}x \\ &\leq \max(\|k\|_{L^\infty}, \|r\|_{L^\infty}) \|v\|_{H^1}\|w\|_{H^1}, \end{aligned}$$

provided $\max(\|k\|_{L^\infty}, \|r\|_{L^\infty}) = C < \infty$.

Finally, the functional L is continuous, since

$$|L(v)| \leq \|f\|_{L^2}\|v\|_{L^2} \leq \|f\|_{L^2}\|v\|_V,$$

which means that we may take $\Lambda = \|f\|_{L^2}$ provided we assume that $f \in L^2(\Omega)$. Therefore the problem satisfies the Lax-Milgram theorem. □

Example 1.8. Verify that the assumption of the Lax-Milgram theorem are satisfied for the following problem,

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

Solution. This problem satisfies (Var) with

$$\begin{aligned} V = H^1_0 &= \{v \in H^1 \ :\ v|_{\partial\Omega} = 0\}, \\ H^1 &= \{v \ :\ \int_\Omega (v^2(x) + |\nabla v(x)|^2)\ \mathrm{d}x < \infty\}, \end{aligned}$$

$$\begin{aligned} A(u,v) &= \int_\Omega \nabla u \cdot \nabla v\ \mathrm{d}x, \\ L(v) &= \int_\Omega fv\ \mathrm{d}x. \end{aligned}$$

To verify the V-ellipticity, we use the *Poincaré inequality*, i.e. there is a constant C such that

$$v \in H_0^1 \quad \Rightarrow \quad \int_\Omega v^2 \, \mathrm{d}x \le C \int_\Omega |\nabla u|^2 \, \mathrm{d}x. \tag{1.18}$$

In one dimension and $\Omega = (0,1)$, the inequality (1.18) takes the form

$$\int_0^1 v^2(x) \, \mathrm{d}x \le \int_0^1 (v'(x))^2 \, \mathrm{d}x, \tag{1.19}$$

provided $v(0) = 0$. Since

$$v(x) = v(0) + \int_0^x v'(s) \, \mathrm{d}s = \int_0^x v'(s) \, \mathrm{d}s,$$

and by Cauchy's inequality

$$\begin{aligned} v^2(x) = \left(\int_0^x v'(s) \, \mathrm{d}s \right)^2 &\le x \int_0^x v'(s)^2 \, \mathrm{d}s \\ &\le \int_0^1 v'(s)^2 \, \mathrm{d}s \qquad \text{since } x \in (0,1). \end{aligned}$$

The V-ellipticity of A follows by (1.19) and

$$\begin{aligned} A(v,v) = \int_0^1 v'(x)^2 \, \mathrm{d}x &= \frac{1}{2} \int_0^1 \left((v'(x))^2 \, \mathrm{d}x + \frac{1}{2} (v'(x))^2 \right) \, \mathrm{d}x \\ &\ge \frac{1}{2} \int_0^1 (v'(x)^2 + v(x)^2) \, \mathrm{d}x \\ &= \frac{1}{2} \|v\|_{H_0^1}^2 \qquad \forall v \in H_0^1. \end{aligned}$$

The other conditions can be proved similarly as in the previous example. Therefore this problem satisfies the Lax-Milgram theorem. □