Chapter 4
Pricing market securities

The Black–Scholes model we have seen so far has a simple mathematical side but it has an even simpler financial side. The asset we considered was a stock which could be held without additional cost or benefit and was freely tradable at the price quoted. Even leaving aside the issues of transaction costs and illiquidity, not much of the financial market is like that. Even vanilla products – foreign exchange, equities and bonds – don't actually fit the simple asset class we devised. Foreign exchange involves two assets which pay interest, equities pay dividends, and bonds pay coupons.

Just retreading the same mathematics for each of these will be enough to keep us busy. The sophistication we have to peddle now is financial.

4.1 Foreign exchange

In the foreign exchange market, like the stock market, holding the basic asset, currency, is a risky business. The dollar value of, say, one pound sterling varies from moment to moment just as a US stock does. And with this risk comes demand for derivatives: claims based on the future value of one unit of currency in terms of another.

Forwards

Consider, though, a forward transaction: a dollar investor wanting to agree the cost in dollars of one pound at some future date T. As with stocks, the replicating strategy to guarantee the forward claim is static. We buy pounds now and sell dollars against them. But cash in both currencies attracts interest. And just as in the simple Black–Scholes model, our cash holding wasn't cash but a cash bond, so our cash holdings here will be cash bonds as well.

Let's make things concrete. Suppose the constant dollar interest rate is r, the sterling interest rate is u, and C_0 dollars buy a pound now. Consider the following static replicating strategy. At time t we

- own e^{-uT} units of sterling cash bonds, and
- go short $C_0 e^{-uT}$ units of dollar cash bonds.

At time zero the portfolio has nil value, and at time T the sterling holding will be one pound as required and the dollar short holding will be $C_0 e^{(r-u)T}$ – the forward price we require.

Contrast this with the stock forward price $S_0 e^{rT}$. We must be careful in extending our simple model to foreign exchange – *both* instruments now make payments. And that makes a difference.

Black–Scholes currency model

There are three instruments and processes to model – two local currency cash bonds and the exchange rate itself. Following the mathematical simplicity of Black–Scholes, our market will be:

Black–Scholes currency model

We let B_t be the dollar cash bond, D_t its sterling counterpart, and C_t be the dollar worth of one pound. Then our model is

$$\begin{array}{rl} \text{Dollar bond} & B_t = e^{rt}, \\ \text{Sterling bond} & D_t = e^{ut}, \\ \text{Exchange rate} & C_t = C_0 \exp(\sigma W_t + \mu t), \end{array}$$

for some W_t a $\mathbb{P}$-Brownian motion and constants r, u, σ and μ.

The dollar investor

The underlying finance dictates that there are two tradables available to the dollar investor. One is uncomplicated – the dollar bond is straightforwardly a dollar tradable much as the cash bond was in the basic account of Black–Scholes. But the other is not.

We would like to think of the stochastic process C_t, the exchange rate, as a tradable but it isn't. The process C_t represents the dollar value of one pound sterling, but sterling cash isn't a tradable instrument in our market. To hold cash naked would be to set up an arbitrage against the cash bond – to put it another way, the existence of the sterling cash bond D_t sets an interest rate for sterling cash by arbitrage, and that rate is u not zero.

On the other hand, D_t by itself isn't a dollar tradable either – it is the price of a tradable instrument, but it's a *sterling* price.

Fortunately, the product of the two $S_t = C_t D_t$ *is* a dollar tradable. The dollar investor can hold sterling cash bonds, and the dollar value of the holding will be given by the translation of the sterling price D_t into dollars, that is by multiplication by C_t.

Translation, then, yields two processes, B_t and S_t, which mirror the basic Black–Scholes set up.

Three steps to replication (foreign exchange)

(1) Find a measure $\mathbb{Q}$ under which the sterling bond discounted by the dollar bond $Z_t = B_t^{-1} S_t = B_t^{-1} C_t D_t$ is a martingale.

(2) Form the process $E_t = \mathbb{E}_{\mathbb{Q}}(B_T^{-1} X \mid \mathcal{F}_t)$.

(3) Find a previsible process ϕ_t, such that $dE_t = \phi_t \, dZ_t$.

Step one

The dollar discounted worth of the sterling bond is

$$Z_t = C_0 \exp\big(\sigma W_t + (\mu + u - r)t\big).$$

Can we make this into a martingale under some new measure $\mathbb{Q}$? Only if $\tilde{W}_t = W_t + \sigma^{-1}(\mu + u - r + \frac{1}{2}\sigma^2)t$ is a $\mathbb{Q}$-Brownian motion, which is made possible as before by the Cameron–Martin–Girsanov theorem. Then, under $\mathbb{Q}$

$$
\begin{aligned}
Z_t &= C_0 \exp\left(\sigma \tilde{W}_t - \tfrac{1}{2}\sigma^2 t\right), \\
\text{and thus} \qquad C_t &= C_0 \exp\left(\sigma \tilde{W}_t + (r - u - \tfrac{1}{2}\sigma^2)t\right).
\end{aligned}
$$

Step two

Given this $\mathbb{Q}$, define the process E_t to be the conditional expectation process $\mathbb{E}_{\mathbb{Q}}(B_T^{-1} X | \mathcal{F}_t)$, which as noted before is a $\mathbb{Q}$-martingale.

Step three

The martingale representation theorem produces an $\mathcal{F}$-previsible process ϕ_t linking E_t with Z_t, such that

$$E_t = E_0 + \int_0^t \phi_s \, dZ_s.$$

Now where? We need a replicating strategy (ϕ_t, ψ_t) detailing holdings of our two dollar tradables S_t and B_t, so we try

- holding ϕ_t units of sterling cash bond, and
- holding $\psi_t = E_t - \phi_t Z_t$ units of dollar cash bond.

The dollar value of the replicating portfolio at time t is $V_t = \phi_t S_t + \psi_t B_t = B_t E_t$. This portfolio is only self-financing if changes in its value are only due to changes in the assets' prices, that is $dV_t = \phi_t \, dS_t + \psi_t \, dB_t$, or as was shown to be equivalent in section 3.7, if $dE_t = \phi_t \, dZ_t$ – which is precisely what the martingale representation theorem guarantees.

Since $V_T = B_T E_T$, and E_T is the discounted claim $B_T^{-1} X$, we have a self-financing strategy (ϕ_t, ψ_t) which replicates our arbitrary claim X.

Option price formula (foreign exchange)

All claims have arbitrage prices and those prices are given by the portfolio value

$$V_t = B_t \, \mathbb{E}_{\mathbb{Q}}\left(B_T^{-1} X \mid \mathcal{F}_t\right).$$

where $\mathbb{Q}$ is the measure under which the discounted asset Z_t is a martingale.

Example – forward contract

A sterling forward contract. At what price should we agree to trade sterling at a future date T? If we agree to buy a unit of sterling for an amount k of dollars, our payoff at time T is

$$X = C_T - k.$$

Its worth at time t is $V_t = B_t \mathbb{E}_{\mathbb{Q}}(B_T^{-1} X | \mathcal{F}_t)$ which is $e^{-r(T-t)} \mathbb{E}_{\mathbb{Q}}(C_T - k | \mathcal{F}_t)$. So the forward price at time zero for purchasing sterling at time T is $k = \mathbb{E}_{\mathbb{Q}}(C_T)$ or

$$F = \mathbb{E}_{\mathbb{Q}}\Big(C_0 \exp\big(\sigma \tilde{W}_T + (r - u - \tfrac{1}{2}\sigma^2)T\big)\Big) = e^{(r-u)T} C_0.$$

That is, the current price for sterling discounted by a factor depending on the difference between the interest rates of the two currencies. With this strike, the contract's value at time t is

$$V_t = e^{-uT}\big(e^{ut} C_t - e^{rt} C_0\big).$$

The discounted portfolio value is $E_t = B_t^{-1} V_t = e^{-uT} Z_t - e^{-uT} C_0$, thus $dE_t = e^{-uT}\, dZ_t$, and so the required hedge ϕ_t is the constant e^{-uT}, and ψ_t is the constant $-e^{-uT} C_0$.

This confirms our earlier intuition.

Example – call option

A sterling call. Suppose we have a contract which allows us the option of buying a pound at time T in the future for the price of k dollars. The dollar payoff at time T is

$$X = (C_T - k)^+.$$

The value of the payoff at time t is $V_t = B_t \mathbb{E}_{\mathbb{Q}}(B_T^{-1} X | \mathcal{F}_t)$. Because C_T is log-normally distributed we can evaluate this easily using a probabilistic result:

Log-normal call formula

If Z is a normal $N(0, 1)$ random variable, and F, $\bar{\sigma}$ and k are constants, then

$$\mathbb{E}\Big(\big(F \exp(\bar{\sigma} Z - \tfrac{1}{2}\bar{\sigma}^2) - k\big)^+\Big) = F\Phi\left(\frac{\log\frac{F}{k} + \frac{1}{2}\bar{\sigma}^2}{\bar{\sigma}}\right) - k\Phi\left(\frac{\log\frac{F}{k} - \frac{1}{2}\bar{\sigma}^2}{\bar{\sigma}}\right).$$

As the forward price F is $\mathbb{E}_{\mathbb{Q}}(C_T)$, the value of C_T can be written in the form $F\exp(\bar{\sigma}Z - \frac{1}{2}\bar{\sigma}^2)$, where $\bar{\sigma}^2$ is the variance of $\log C_T$, namely $\sigma^2 T$, and Z is a normal $N(0,1)$ under $\mathbb{Q}$.

The option price at time zero is then $\mathbb{E}\big((F\exp(\bar{\sigma}Z - \frac{1}{2}\bar{\sigma}^2) - k)^+\big)$, which the theorem tells us is

$$V_0 = e^{-rT}\left\{F\Phi\left(\frac{\log\frac{F}{k} + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}\right) - k\Phi\left(\frac{\log\frac{F}{k} - \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}\right)\right\}.$$

The hedge is

$$\phi_t = e^{-uT}\Phi\left(\frac{\log\frac{F_t}{k} + \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}}\right),$$

$$\psi_t = -ke^{-rT}\Phi\left(\frac{\log\frac{F_t}{k} - \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}}\right),$$

where F_t is the forward sterling price at time t, $F_t = e^{(r-u)(T-t)}C_t$.

The sterling investor

A sterling investor sees things differently. Were we operating in pounds we would not be wanting dollar price processes of tradable instruments but sterling ones. The first of these is simply the sterling bond $D_t = e^{ut}$, which will be our basic unit of account. There is also the inverse exchange rate process C_t^{-1} – the worth in pounds of one dollar. This has the value

$$C_t^{-1} = C_0^{-1}\exp(-\sigma W_t - \mu t),$$

but it is not the sterling price of a tradable instrument, any more than C_t was for the dollar investor. Our other actual sterling tradable price process is the sterling value of the *dollar bond* $C_t^{-1}B_t$.

With our two sterling tradable prices, D_t and $C_t^{-1}B_t$, we can follow again our three-step replication programme. The sterling discounted value of the dollar bond is

$$Y_t = D_t^{-1}C_t^{-1}B_t = C_0^{-1}\exp\big(-\sigma W_t - (\mu + u - r)t\big).$$

This discounted price process Y_t will be a martingale under the new measure $\mathbb{Q}^{\pounds}$, if

$$\tilde{W}_t^{\pounds} = W_t + \sigma^{-1}(\mu + u - r - \tfrac{1}{2}\sigma^2)t$$

is $\mathbb{Q}^{£}$-Brownian motion. Then hedging will be possible as before.

Option price formula (sterling investor)
The value to the sterling investor of a sterling payoff X at time T is

$$U_t = D_t \, \mathbb{E}_{\mathbb{Q}^{£}}\big(D_T^{-1} X \mid \mathcal{F}_t\big).$$

where $\mathbb{Q}^{£}$ is the measure under which the sterling discounted asset Y_t is a martingale.

Change of numeraire

A worrying possibility now surfaces – the measures $\mathbb{Q}$ and $\mathbb{Q}^{£}$ are different. Will the dollar and sterling investors disagree about the price of the same security?

Suppose X is a dollar claim which pays off at time T. To the dollar investor, the claim is worth at time t

$$V_t = B_t \, \mathbb{E}_{\mathbb{Q}}\big(B_T^{-1} X \mid \mathcal{F}_t\big) \quad \text{dollars.}$$

To the sterling investor, the claim pays off $C_T^{-1} X$ pounds, rather than X dollars, at time T, and its sterling worth at time t is

$$U_t = D_t \, \mathbb{E}_{\mathbb{Q}^{£}}\big(D_T^{-1}(C_T^{-1} X) \mid \mathcal{F}_t\big) \quad \text{pounds.}$$

Do these two prices agree? That is, is the dollar worth of the sterling valuation, $C_t U_t$, the same as the original dollar valuation V_t?

The $\mathbb{Q}^{£}$-Brownian motion $\tilde{W}_t^{£}$ is equal to $\tilde{W}_t - \sigma t$, so that by the converse of the Cameron–Martin–Girsanov theorem the Radon–Nikodym derivative of $\mathbb{Q}^{£}$ with respect to $\mathbb{Q}$ (up to time T) must be

$$\frac{d\mathbb{Q}^{£}}{d\mathbb{Q}} = \exp\big(\sigma \tilde{W}_T - \tfrac{1}{2}\sigma^2 T\big).$$

The $\mathbb{Q}$-martingale associated with the Radon–Nikodym derivative, formed by conditional expectation is

$$\zeta_t = \mathbb{E}_{\mathbb{Q}}\Big(\frac{d\mathbb{Q}^{£}}{d\mathbb{Q}} \;\Big|\; \mathcal{F}_t\Big) = \exp(\sigma \tilde{W}_t - \tfrac{1}{2}\sigma^2 t).$$

Note that ζ_t is (up to a constant) the dollar discounted worth of the sterling bond. Concretely, $C_0\zeta_t = Z_t = B_t^{-1}C_tD_t$. Recall also (Radon–Nikodym fact (ii) of section 3.4) that for any random variable X which is known by time T,

$$\mathbb{E}_{\mathbb{Q}^{\pounds}}(X|\mathcal{F}_t) = \zeta_t^{-1}\mathbb{E}_{\mathbb{Q}}(\zeta_T X|\mathcal{F}_t).$$

So the dollar worth of the sterling investor's valuation is

$$C_tU_t = C_tD_t\mathbb{E}_{\mathbb{Q}^{\pounds}}\big(D_T^{-1}C_T^{-1}X \mid \mathcal{F}_t\big) = C_tD_t\zeta_t^{-1}\mathbb{E}_{\mathbb{Q}}\big(\zeta_TD_T^{-1}C_T^{-1}X \mid \mathcal{F}_t\big),$$

which is (substituting in the ζ_t expression) equal to

$$C_tU_t = B_t\,\mathbb{E}_{\mathbb{Q}}\big(B_T^{-1}X \mid \mathcal{F}_t\big) = V_t.$$

Thus the payoff of X dollars at time T is worth the same to either investor at any time beforehand. Similar calculations show that the dollar and sterling investors' replicating strategies for X are identical. So they agree not only on the prices but also on the hedging strategy.

The difference of martingale measures only reflected the different numeraires of the two investors rather than any fundamental disagreement over prices. Further details on the effect, or lack of it, of changing numeraires are in section 6.4.

All investors, whatever their currency of account, will agree on the current value of a derivative or other security.

4.2 Equities and dividends

An equity is a stock which makes periodic cash payments to the current holder. Our previous models treated a stock as a pure asset, but they can be modified to handle dividend payments.

It is simplest to begin with a dividend which is paid continuously.

Equity model with continuous dividends
Let the stock price S_t follow a Black–Scholes model, $S_t = S_0 \exp(\sigma W_t + \mu t)$ and B_t be a constant-rate cash bond $B_t = \exp(rt)$. The dividend payment made in the time interval of length dt starting at time t is

$$\delta S_t \, dt,$$

where δ is a constant of proportionality.

Just as with foreign exchange, our problem is that the process S_t is not a tradable asset. If we buy the stock for S_0, by the time we come to sell it at time t, what we bought is worth not just the price of the stock itself, namely S_t, but also the total accumulated dividends, which under the model will depend on all the different values that the stock has taken up until time t. The process S_t is no longer the value of the asset as a whole, because it is not enough.

We need to translate S_t somehow, and to find a new process as we did in foreign exchange, which involves S_t but *is* a tradable. Consider the following simple portfolio strategy. The portfolio starts with one unit of stock, costing S_0, and at every instant when the cash dividend is paid out, that cash is immediately used to buy a little more stock. That is, we are continuously reinvesting the dividends in the stock. The infinitesimal payout is $\delta S_t \, dt$ per unit of stock, which will purchase $\delta \, dt$ more units of stock. At time t, the number of stock units held by the portfolio will be $\exp(\delta t)$, and the worth of the portfolio is

$$\tilde{S}_t = S_0 \exp\big(\sigma W_t + (\mu + \delta)t\big).$$

Note how the structure of the model's assumptions made the translation straightforward. We assumed that the dividend payments were a constant proportion of the stock price. As a consequence it made it natural to construct the tradable by reinvesting in the stock. If we had assumed that the dividend stream was known in advance, independent of the stock price, then we would have reinvested in the cash bond (for an example of this see section 4.3 on bonds). Assumptions are all.

Replicating strategies – equities

Our definition of a portfolio of stock and bond (ϕ_t, ψ_t) can be rewritten as a portfolio of the reinvested stock and bond $(\tilde{\phi}_t, \psi_t)$, where $\tilde{\phi}_t = e^{-\delta t}\phi_t$, with value $V_t = \phi_t S_t + \psi_t B_t = \tilde{\phi}_t \tilde{S}_t + \psi_t B_t$. The advantage of the new framework is that the self-financing equation retains the familiar form

$$dV_t = \tilde{\phi}_t \, d\tilde{S}_t + \psi_t \, dB_t,$$

whereas in the plain stock/bond notation, this equation would need to be modified by the dividend cash stream, becoming $dV_t = \phi_t \, dS_t + \psi_t \, dB_t + \phi_t \delta S_t \, dt$. That is, changes in the portfolio value are due both to trading profits and losses (the dS_t and dB_t terms) and also to dividend payments.

Working now with our reinvested stock, as usual we want to make the discounted asset $\tilde{Z}_t = B_t^{-1}\tilde{S}_t$ into a martingale. Now $\tilde{Z}_t$ has SDE

$$d\tilde{Z}_t = \tilde{Z}_t\big(\sigma \, dW_t + (\mu + \delta + \tfrac{1}{2}\sigma^2 - r) \, dt\big),$$

so that we want a measure $\mathbb{Q}$ under which $\tilde{W}_t = W_t + \sigma^{-1}(\mu + \delta + \frac{1}{2}\sigma^2 - r)t$ is Brownian motion. So under this martingale measure $\mathbb{Q}$, $d\tilde{Z}_t = \sigma \tilde{Z}_t \, d\tilde{W}_t$. To construct a strategy to hedge a claim X maturing at date T, again we follow the simple Black–Scholes model, and use the martingale representation theorem. That is, there exists a previsible process $\tilde{\phi}_t$ such that

$$E_t = \mathbb{E}_{\mathbb{Q}}\big(B_T^{-1}X \mid \mathcal{F}_t\big) = \mathbb{E}_{\mathbb{Q}}\big(B_T^{-1}X\big) + \int_0^t \tilde{\phi}_s \, d\tilde{Z}_s.$$

The trading strategy is to hold $\tilde{\phi}_t$ units of the translated asset $\tilde{S}_t$ and $\psi_t = E_t - \tilde{\phi}_t \tilde{Z}_t$ units of the cash bond. In terms of our original securities, this amounts to holding $\phi_t = e^{\delta t}\tilde{\phi}_t$ units of the stock S_t and the same ψ_t units of the bond B_t.

Thus, under the martingale measure

$$S_t = S_0 \exp\big(\sigma \tilde{W}_t + (r - \delta - \tfrac{1}{2}\sigma^2)t\big),$$

which is log-normally distributed.

Example – forward

An agreement to buy a unit of stock at time T for amount k has payoff

$$X = S_T - k.$$

Its worth at time t is

$$V_t = \mathbb{E}_{\mathbb{Q}}\big(e^{-r(T-t)}(S_T - k) \mid \mathcal{F}_t\big) = e^{-\delta(T-t)}S_t - e^{-r(T-t)}k.$$

The value of k which gives the contract initial nil value is the forward price of S_T,

$$F = e^{(r-\delta)T}S_0.$$

The hedge is then to hold $\phi_t = e^{-\delta(T-t)}$ units of the stock and $\psi_t = -ke^{-rT}$ units of the bond at time t. Note the slightly surprising dynamic strategy for the forward. Instead of simply holding a certain amount of stock until T, we are continually buying more with the dividend income. Why? Again because of our assumption – if the dividend payments are a known proportion of the stochastic S_t, we have no choice but to hide them in the stock itself.

Example – call option

A call struck at k, exercised at time T has payoff $X = (S_T - k)^+$, and value at time zero of $V_0 = \mathbb{E}_{\mathbb{Q}}\big(e^{-rT}(S_T - k)^+\big)$, which equals

$$V_0 = e^{-rT}\left\{F\Phi\left(\frac{\log\frac{F}{k} + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}\right) - k\Phi\left(\frac{\log\frac{F}{k} - \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}\right)\right\},$$

where F is the forward price above $e^{(r-\delta)T}S_0$. The hedge will be to hold $e^{-\delta(T-t)}\Phi(+)$ units of the stock and have a negative holding of $ke^{-rT}\Phi(-)$ units of the bond. (Here $\Phi(+)$ and $\Phi(-)$ refer respectively to the two Φ terms in the above equation.)

Again the Black–Scholes call option formula re-emerges – if the martingale measure $\mathbb{Q}$ makes the process under study, S_t, have a log-normal distribution, then the theorem in section 4.1 comes into play. Knowing the forward F and the term volatility σ is enough to specify the price.

Example – guaranteed equity profits

A contract pays off according to gains of the UK FTSE stock index S_t, with a guaranteed minimum payout and a maximum payout. More precisely, it is a five-year contract which pays out 90% times the ratio of the terminal and initial values of FTSE. Or it pays out 130% if otherwise it would be less, or 180% if otherwise it would be more. How much is this payout worth?

Our data are

$$\begin{aligned} \text{FTSE drift} \quad & \mu = 7\% \\ \text{FTSE volatility} \quad & \sigma = 15\% \\ \text{FTSE dividend yield} \quad & \delta = 4\% \\ \text{UK interest rate} \quad & r = 6.5\% \end{aligned}$$

As FTSE is composed of 100 different stocks, their separate dividend payments will approximate a continuously paying stream. The claim X is

$$X = \min\{\max\{1.3, 0.9S_T\}, 1.8\},$$

where T is 5 years and the initial FTSE value S_0 is 1. This claim can be rewritten as

$$X = 1.3 + 0.9\{(S_T - 1.444)^+ - (S_T - 2)^+\}.$$

That is, X is actually the difference of two FTSE calls (plus some cash). The forward price for S_T is

$$F = e^{(r-\delta)T} S_0 = 1.133.$$

Using the above call price formula for dividend-paying stocks, we can value these calls (per unit) at 0.0422 and 0.0067 respectively. The worth of X at time zero is then

$$V_0 = 1.3e^{-rT} + 0.9(0.0422 - 0.0067) = 0.9712.$$

Were we to have forgotten that the constituent stocks of FTSE pay dividends, but the dividends are not reflected in the index, we would incorrectly have valued the contract at 1.0183 – about 5% too high.

Periodic dividends

In practice, an individual stock pays dividends at regular intervals rather than continuously, but this presents no real problems for our basic model. Let us assume that the times of dividend payments $T_1, T_2, \ldots$ are known in advance, and at each time T_i, the current holder of the equity receives a payment of a fraction δ of the current stock price. The stock price must also instantaneously decrease by the same amount – or else there would be an arbitrage opportunity. At any time $T = T_i$, then, we can assume the dividend payout exactly equals the instantaneous decrease in the stock price.

Equity model with periodic dividends

At deterministic times $T_1, T_2, \ldots$, the equity pays a dividend of a fraction δ of the stock price which was current just before the dividend is paid. The stock price process itself is modelled as

$$S_t = S_0(1-\delta)^{n[t]} \exp(\sigma W_t + \mu t),$$

where $n[t] = \max\{i : T_i \leqslant t\}$ is the number of dividend payments made by time t. There is also a cash bond $B_t = \exp(rt)$.

We face two problems. The first is the familiar one that S_t is not by itself the price of tradable asset. Translation, however, should provide a cure. The second is more serious. Away from the times T_i, S_t has the usual SDE of $dS_t = S_t\big(\sigma\, dW_t + (\mu + \frac{1}{2}\sigma^2)\, dt\big)$, but at those times it has discontinuous jumps. Thus S_t is discontinuous – it doesn't fit our definition of a stochastic process. Fortunately, translation cures this as well.

Consider the following trading strategy. Starting with one unit of stock, every time the stock pays a dividend we reinvest the dividend by buying more stock. At time t, we will have $(1-\delta)^{-n[t]}$ units of the stock, and the value of our portfolio will be $\tilde{S}_t$, where

$$\tilde{S}_t = (1-\delta)^{-n[t]} S_t = S_0 \exp(\sigma W_t + \mu t).$$

As before, $\tilde{S}_t$ is tradable but our arbitrage justified assumption that the dividend payments match the stock price jumps feeds through into making $\tilde{S}_t$ continuous as well. We are back in familiar territory.

Replicating strategy

Our trading strategy will then be $(\tilde{\phi}_t, \psi_t)$, where $\tilde{\phi}_t$ is the number of units of $\tilde{S}_t$ we hold at time t, and ψ_t is the amount of the cash bond B_t. Such a strategy is equivalent to holding $\phi_t = (1-\delta)^{-n[t]}\tilde{\phi}_t$ units of the actual stock S_t.

The discounted value of the $(\tilde{\phi}_t, \psi_t)$ portfolio is $E_t = \tilde{\phi}_t \tilde{Z}_t + \psi_t$, where $\tilde{Z}_t$ is the discounted value of the reinvested stock price $\tilde{Z}_t = B_t^{-1}\tilde{S}_t$. The portfolio will be self-financing if $dE_t = \tilde{\phi}_t \, d\tilde{Z}_t$.

As before, we want to find a $\mathbb{Q}$ which makes $\tilde{Z}_t$ into a martingale. As $d\tilde{Z}_t = \tilde{Z}_t\big(\sigma\, dW_t + (\mu + \frac{1}{2}\sigma^2 - r)\, dt\big)$, this will have no drift if $\tilde{W}_t = W_t + \sigma^{-1}(\mu + \frac{1}{2}\sigma^2 - r)t$ is $\mathbb{Q}$-Brownian motion. Then $\tilde{Z}_t$ is also a $\mathbb{Q}$-martingale.

We can form the process $E_t = \mathbb{E}_{\mathbb{Q}}(B_T^{-1}X|\mathcal{F}_t)$, where X is the option on the stock which we wish to hedge.

Finally, the martingale representation theorem produces a hedging process $\tilde{\phi}_t$ and the corresponding ψ_t can be set to be $E_t - \tilde{\phi}_t\tilde{Z}_t$. So hedging is still possible in this case, and the value at time zero of the claim X is $\mathbb{E}_{\mathbb{Q}}(B_T^{-1}X)$.

The stock price, under $\mathbb{Q}$, is

$$S_t = S_0(1-\delta)^{n[t]}e^{\sigma\tilde{W}_t + (r - \frac{1}{2}\sigma^2)t}.$$

Since this is log-normal, with the forward price for S_T equal to $F = S_0(1-\delta)^{n[T]}e^{rT}$, the Black–Scholes price for a call option struck at k is equal to

$$V_0 = e^{-rT}\left\{F\Phi\left(\frac{\log\frac{F}{k} + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}\right) - k\Phi\left(\frac{\log\frac{F}{k} - \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}\right)\right\}.$$

4.3 Bonds

A pure discount bond is a security which pays off one unit at some future maturity time T. Were interest rates completely constant at rate r it would have present value at time t of $e^{-r(T-t)}$. We might, however, want to consider the effect of interest rates being stochastic – much as they are in real markets. And with varying interest rates, uncertainty about their future values would cause a discount bond price to move randomly as well.

A full model of discount bonds, or for that matter coupon bonds, will have to wait for chapter five and term structure models. The interplay of interest rates of different maturities and the arbitrage minefield that models have to tiptoe through is not something we want to worry about in a simple Black–Scholes account. As a consequence we will try to take a schizophrenic attitude to interest rates. Bond prices will vary stochastically, but the short-term interest rate will be deterministic. In the real markets there is clearly a link, but then it can be argued that there are links between stock or foreign exchange prices and the cash bonds as well. Over short time horizons most practitioners ignore these links in all three markets.

Discount bonds

The Black–Scholes model for discount bonds is:

Discount bond model

We assume a cash bond $B_t = \exp(rt)$ for some positive constant r, and a discount bond S_t whose price follows

$$S_t = S_0 \exp(\sigma W_t + \mu t),$$

for all times t less than T, some time horizon T long before the maturity time τ of the bond.

In formulation, this model is indistinguishable from the simple Black–Scholes model for stocks. Thus the forward price for purchasing the bond at time $T < \tau$ is

$$F = \mathbb{E}_{\mathbb{Q}}(S_T),$$

where $\mathbb{Q}$ is the measure under which $e^{-rt}S_t$ is a martingale. Since $\sigma^2 T$ is the variance, under $\mathbb{Q}$, of $\log S_T$ (σ is the term volatility), then the price of a call on S_T struck at k is

$$e^{-rT}\left\{F\Phi\left(\frac{\log\frac{F}{k}+\frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}\right) - k\Phi\left(\frac{\log\frac{F}{k}-\frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}\right)\right\}.$$

We have to be careful, though, with our assumption that T is much before the maturity τ. Not only does the distinction between the deterministic cash bond and the stochastic discount bond get harder to maintain as T approaches τ, but for similar reasons it gets harder to justify a simple drift μ and a constant positive σ. The bond promises one unit at time τ, thus its price at time τ must be $S_\tau = 1$. In a good model, the drift and volatility will conspire to ensure this *pull to par* – and indeed this will happen in chapter five. Here if we let $T = \tau$, we would have no such guarantee.

Bonds with coupons

Most market bonds do not just pay off one unit at maturity, but also pay off a series of smaller amounts c at various pre-determined times $T_1, T_2, \ldots, T_n$ before maturity. Such coupon payments may resemble dividend payments, but unlike the equity model, the amount of the coupon is known in advance. Here the schizophrenia extends to the treatment of coupons before and after the expiry date, T, of the option. The simplest model is to view coupons that occur before time T as coming under the regime of the deterministic cash bond, and coupons occurring after time T (including the redemption payment at maturity) as following a stochastic price process.

Coupon bond model

There is a simple cash bond $B_t = \exp(rt)$, and a coupon bond which pays off an amount c at times T_1, T_2, ..., up to a horizon τ. Denoting $I(t) = \min\{i : t < T_i\}$ to be the sequence number of the next coupon payment after time t, and j to be $I(T) - 1$, the total number of payments before time T, then the price of the bond at time t is then

$$S_t = \sum_{i=I(t)}^{j} ce^{-r(T_i - t)} + A\exp\left(\sigma W_t + \mu t\right), \qquad t < T.$$

Specifically, we model the first sort of coupon (payable at, for example, $T_i < T$) to be worth

$$ce^{-r(T_i - t)} \qquad \text{at time } t \ (t < T_i),$$

and for the sum of all the post-T payments to evolve as an exponential

Brownian motion

$$A\exp(\sigma W_t + \mu t), \qquad \text{for } t < T,$$

for constants A, σ, and μ.

Again S_t is discontinuous at the coupon payment times, and again we can use a translation rather like the one used for equity dividends (section 4.2). But because the coupon payments are known in advance, this time we manufacture a continuous tradable asset by holding one unit of the coupon bond and investing all the coupon payments, as they occur, in the *cash bond.* The value of this asset is $\tilde{S}_t$, where

$$\tilde{S}_t = \sum_{i=1}^{j} ce^{-r(T_i - t)} + A\exp(\sigma W_t + \mu t).$$

This is now a tradable asset with a continuous stochastic process.

Replicating strategy

We describe a portfolio as $(\tilde{\phi}_t, \psi_t)$, where $\tilde{\phi}_t$ is the amount of the asset $\tilde{S}_t$ held at time t, and ψ_t is the direct holding of the cash bond $B_t = e^{rt}$. We let V_t be the value of the portfolio, $V_t = \tilde{\phi}_t\tilde{S}_t + \psi_t B_t$, and E_t be its discounted value $E_t = \tilde{\phi}_t\tilde{Z}_t + \psi_t$, where $\tilde{Z}_t$ is the discounted value of the asset $\tilde{S}_t$. The portfolio is self-financing if $dE_t = \tilde{\phi}_t\, d\tilde{Z}_t$.

As usual, we want to make $\tilde{Z}_t$ into a martingale by changing measure. In fact $\tilde{Z}_t$ is just a constant cash sum of $\sum_{i=1}^{j} ce^{-rT_i}$ plus an exponential Brownian motion $A\exp(\sigma W_t + (\mu - r)t)$. This will be a $\mathbb{Q}$-martingale if $\tilde{W}_t = W_t + \sigma^{-1}(\mu + \frac{1}{2}\sigma^2 - r)t$ is $\mathbb{Q}$-Brownian motion.

For an option X payable at time T, the process $E_t = \mathbb{E}_{\mathbb{Q}}(B_T^{-1}X|\mathcal{F}_t)$ can be represented as $dE_t = \tilde{\phi}_t\, d\tilde{Z}_t$ for some previsible process $\tilde{\phi}_t$. We can set ψ_t to be $E_t - \tilde{\phi}_t\tilde{Z}_t$, so that $(\tilde{\phi}_t, \psi_t)$ is a hedging strategy for X. The value of X at time zero must now be $\mathbb{E}_{\mathbb{Q}}(B_T^{-1}X)$.

Under $\mathbb{Q}$, the price of the bond at time T is just

$$S_T = A\exp\left(\sigma\tilde{W}_T + (r - \tfrac{1}{2}\sigma^2)T\right).$$

This is log-normally distributed, so we can follow the call formula from section 4.1 to see that the forward price for S_T is $F = Ae^{rT}$ and the value of a call on S_T struck at k is

$$e^{-rT}\left\{F\Phi\left(\frac{\log\frac{F}{k} + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}\right) - k\Phi\left(\frac{\log\frac{F}{k} - \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}\right)\right\}.$$

4.4 Market price of risk

Now is the time to tie some loose ends together. The same pattern has been repeating through all the examples so far – the stochastic processes we have been using as models in this chapter have been tied to tradable quantities only indirectly. The foreign exchange process had to be converted from a non-tradable cash process to a tradable discount bond process. For equities, the model process had to have dividends recombined to make it tradable. And for bonds, the coupons had to be reinvested in the numeraire process. Underlying all this was a tradable/non-tradable distinction – we couldn't use the martingale representation theorem to replicate claims until we had something tradable to replicate with. But the distinction has so far been a common sense one – can we do any better?

To some extent, yes. Some of the tradable/non-tradable distinction is going to have to be founded on goodwill. After all whether something can be traded or not in a free market is not a mathematical decision. But if we decide on a particular process S_t representing something truly tradable and select an appropriate discounting process B_t, then we can explore the market they create.

Martingales are tradables

Suppose that there is some measure $\mathbb{Q}$ under which the discounted tradable, $Z_t = B_t^{-1}S_t$, is a $\mathbb{Q}$-martingale, what can we say about another process V_t adapted to the same filtration $\mathcal{F}_t$ such that $E_t = B_t^{-1}V_t$ is also a $\mathbb{Q}$-martingale?

Firstly, the martingale representation theorem gives us that, as long as Z_t has non-zero volatility, we can find an $\mathcal{F}$-previsible process ϕ_t such that

$$dE_t = \phi_t \, dZ_t.$$

Taking our cue from all the examples so far, we could create a portfolio (ϕ_t, ψ_t) where at time t we are

- long ϕ_t of the tradable S_t,
- long $\psi_t = E_t - \phi_t Z_t$ of the tradable B_t.

Then as before we can show that (ϕ_t, ψ_t) is a self-financing strategy, that is changes in the value of the (ϕ_t, ψ_t) portfolio are explainable in terms of changes in value of the tradable constituents alone. And the value of this portfolio at time t is always exactly V_t.

In other words we can make V_t out of S_t and B_t. So it seems reasonable enough to ennoble V_t with the title tradable as well. Being a $\mathbb{Q}$-martingale after discounting is enough to ensure that it can be made costlessly from tradables – so it might as well be tradable itself. Of course all the derivatives that we have been constructing out of claims have this property – $\mathbb{E}_{\mathbb{Q}}(B_T^{-1}X|\mathcal{F}_t)$ is always a $\mathbb{Q}$-martingale.

Non-martingales are non-tradables

What about the other way round? Suppose $B_t^{-1}V_t$ was not a $\mathbb{Q}$-martingale. Then from our definition of a martingale, there must be a positive probability at some times T and s that $\mathbb{E}_{\mathbb{Q}}(B_T^{-1}V_T|\mathcal{F}_s) \neq B_s^{-1}V_s$. What would happen if V_t were tradable and the market stumbled into this possible filtration?

Suppose we define another process U_t by simply setting U_t to be the cost of replicating the claim V_T, that is $U_t = B_t\,\mathbb{E}_{\mathbb{Q}}(B_T^{-1}V_T|\mathcal{F}_t)$. Then the terminal value of U_T will be equal to V_T but at time s, U_s and V_s will be (possibly) different. As $B_t^{-1}U_t$ is a $\mathbb{Q}$-martingale we can view U_t as tradable by dint of being able to construct it from S_t and B_t.

So we have two tradables, U_t and V_t, such that they are identical at time T but different at some earlier time s (with positive probability). We then have an arbitrage engine. If, say, U_s were greater than V_s, we could buy unlimited amounts of V and sell unlimited amounts of U collecting the cash up front. The $V - U$ portfolio can be sold for nothing at time T, leaving just the (invested) cash as a guaranteed profit. And if U_s were less than V_s we would run the engine in reverse.

Thus if V_t were genuinely tradable, the market formed by S_t, B_t and V_t would contain arbitrage opportunities – something we might want to dismiss by *fiat*. To avoid arbitrage engines, then, if $B_t^{-1}V_t$ were not a $\mathbb{Q}$-martingale, it had better not be tradable.

We have something akin to a definition then. Within an established (complete) market of tradable securities, there is a straightforward way of checking whether another process is a tradable security or not. It is tradable if its discounted price is a martingale under the martingale measure $\mathbb{Q}$, and is not tradable if it isn't.

Tradable securities

Given a numeraire B_t and a tradable asset S_t, a process V_t represents a tradable asset if and only if its discounted value $B_t^{-1}V_t$ is actually a $\mathbb{Q}$-martingale, where $\mathbb{Q}$ is the measure under which the discounted asset, $B_t^{-1}S_t$, is a martingale.

One way round, the process is just part of the 'linear span' of S_t and B_t; the other way round, there is only room for two 'independent' tradables in a market defined by one-dimensional Brownian motion – any more and there can be arbitrage.

Exercise 4.1 If S_t is a tradable Black–Scholes stock price under the martingale measure $\mathbb{Q}$, $S_t = \exp\big(\sigma\tilde{W}_t + (r - \frac{1}{2}\sigma^2)t\big)$, with cash bond $B_t = \exp(rt)$, show that
(i) $X_t = S_t^2$ is non-tradable,
(ii) $X_t = S_t^{-\alpha}$, where $\alpha = 2r/\sigma^2$, is tradable.

Tradables and the market price of risk

The market price of risk is best introduced through a slight modification of the simple Black–Scholes model. That model had stock price $S_t = S_0\exp(\sigma W_t + \mu t)$, and SDE

$$dS_t = S_t\big(\sigma\, dW_t + (\mu + \tfrac{1}{2}\sigma^2)\, dt\big).$$

We will find it convenient, however, to define price processes by means of their SDEs, typically

$$dS_t = S_t\big(\sigma\, dW_t + \mu\, dt\big),$$

which has solution $S_t = S_0\exp(\sigma W_t + (\mu - \frac{1}{2}\sigma^2)t)$. The only difference between these two approaches is the subtraction of $\frac{1}{2}\sigma^2$ from the drift, which can be thought of as just a change of notation. Both forms can be equally used to define such geometric Brownian motions, but the SDE formulation allows a greater general class of models to be more easily considered.

Suppose then that we have a couple of tradable risky securities S_t^1 and S_t^2, both in the same market – that is both are functions of the same Brownian motion W_t, and both are defined via their SDEs,

$$dS_t^i = S_t^i(\sigma_i\, dW_t + \mu_i\, dt), \qquad i = 1, 2.$$

Following the discussion on tradables, we want the discounted prices of S_t^1 and S_t^2 to be martingales under the *same* measure $\mathbb{Q}$. So assuming a simple numeraire $B_t = \exp(rt)$, we have that

$$\tilde{W}_t = W_t + \left(\frac{\mu_i - r}{\sigma_i}\right) t$$

must be a $\mathbb{Q}$-Brownian motion for i equal to 1 and 2. But this can only happen if the two changes of drift are the same. That is if

$$\frac{\mu_1 - r}{\sigma_1} = \frac{\mu_2 - r}{\sigma_2}.$$

In one of those coincidences that cause confusion, economists attach a meaning to this quantity – if we interpret μ as the growth rate of the tradable, r as the growth rate of the riskless bond and σ as a measure of the risk of the asset, then

$$\gamma = \frac{\mu - r}{\sigma}$$

is the rate of extra return (above the risk-free rate) per unit of risk. As such it is often called the *market price of risk.*

Using this language then gives us a simple and compelling categorisation of tradables in terms of their SDEs – *all tradables in a market should have the same market price of risk.*

The general market price of risk

We can, in fact, generalise to more sophisticated one-factor models. Rigour will have to wait until section 6.1, but for now we can observe that a general stochastic price process S_t will have SDE

$$dS_t = S_t(\sigma_t \, dW_t + \mu_t \, dt),$$

where σ_t and μ_t are previsible processes.

Then defining

$$\gamma_t = \frac{\mu_t - r}{\sigma_t}$$

gives a time and state dependent market price of risk. Despite this variation, the same as above will hold. All tradable securities must instantaneously have the same market price of risk.

The risk-neutral measure

It is worth reflecting on what we have done – we have provided justification that to be tradable in a market defined by a stock S_t and a numeraire B_t is to share, after discounting by B_t, a martingale measure with S_t. This translates naturally in SDE terms to sharing a market price of risk – the market price of risk is actually the drift change of the underlying Brownian motion given by Cameron–Martin–Girsanov. So we have a natural means for sorting through SDEs for tradables.

We also have a natural explanation for the market terminology of $\mathbb{Q}$ as the *risk-neutral measure*. If we write the SDEs in terms of the $\mathbb{Q}$-Brownian motion $\tilde{W}_t$:

$$dS_t = S_t(\sigma_t \, d\tilde{W}_t + \tilde{\mu}_t \, dt),$$

then S_t is tradable if and only if its market price of risk is zero. All tradables then have the same growth rate under $\mathbb{Q}$ as the cash bond, independent of their riskiness σ_t – the measure $\mathbb{Q}$ is neutral with respect to risk.

But we should not overstretch the economic analogy – within our one-factor market all tradables are instantaneously perfectly correlated. They share a market price of risk not for profound economic reasons or because investors behave with certain risk preferences but for the reason that to do otherwise would produce a non-martingale process with a consequent opportunity for arbitrage. The market price of risk is only a convenient algebraic form for the change of measure from $\mathbb{P}$ to $\mathbb{Q}$, not a new argument for using it.

Non-tradable quantities

But convenient it is. Let's return to our underlying theme – dealing with non-tradable processes. With foreign exchange, equities and bonds we had a model for a process that had a fixed relationship to a tradable but was itself non-tradable. Concretely, we might have a non-tradable X_t which is modelled with the stochastic differential

$$dX_t = \sigma_t \, dW_t + \mu_t \, dt,$$

where σ_t and μ_t are previsible processes and W_t is $\mathbb{P}$-Brownian motion. Here σ_t and μ_t might be constants or constant multiples of X_t, but they needn't be.

We have X_t non-tradable but a deterministic function of X_t, $Y_t = f(X_t)$, is tradable. Then by Itô's formula, Y has differential increment

$$dY_t = \sigma_t f'(X_t) \, dW_t + \left(\mu_t f'(X_t) + \tfrac{1}{2}\sigma_t^2 f''(X_t)\right) dt.$$

As Y_t is tradable, we can write down the market price of risk for Y_t immediately. Assuming the discount rate is constant at r,

$$\gamma_t = \frac{\mu_t f'(X_t) + \frac{1}{2}\sigma_t^2 f''(X_t) - rf(X_t)}{\sigma_t f'(X_t)}.$$

Since this market price of risk is simply the change of measure from $\mathbb{P}$ to $\mathbb{Q}$, we can write down X's behaviour under $\mathbb{Q}$ as

$$dX_t = \sigma_t \, d\tilde{W}_t + \frac{rf(X_t) - \frac{1}{2}\sigma_t^2 f''(X_t)}{f'(X_t)} \, dt.$$

Thus if we have claims on X_t, they can be priced via the normal expectation route, using this risk-neutral SDE for X_t.

Examples

(i) If X_t is the logarithm of a tradable asset, then f is the exponential function $f(x) = e^x$. In the simple case where $\sigma_t = \sigma$ and $\mu_t = \mu$ are constants (the basic Black–Scholes model), then the market price of risk for tradables is

$$\gamma_t = \frac{\mu + \frac{1}{2}\sigma^2 - r}{\sigma},$$

and the corresponding risk-neutral SDE for X_t is

$$dX_t = \sigma \, d\tilde{W}_t + (r - \tfrac{1}{2}\sigma^2) \, dt.$$

Time-dependent transforms

More generally, suppose interest rates follow the process r_t, X_t is non-tradable with stochastic differential

$$dX_t = \sigma_t \, dW_t + \mu_t \, dt,$$

and Y is a tradable security which is a deterministic function of X and time, that is $Y_t = f(X_t, t)$. Then under the martingale measure $\mathbb{Q}$, X has differential

$$dX_t = \sigma_t \, d\tilde{W}_t + \frac{r_t f(X_t, t) - \frac{1}{2}\sigma_t^2 f''(X_t, t) - \partial_t f(X_t, t)}{f'(X_t, t)} \, dt,$$

where f' and f'' are derivatives of f with respect to x, and $\partial_t f$ is the derivative of f with respect to t.

(ii) The price process S_t pays dividends at rate δS_t. Let X_t be the process S_t and assume that it follows the Black–Scholes model

$$dX_t = X_t(\sigma\, dW_t + \mu\, dt).$$

The asset $Y_t = \exp(\delta t)X_t$ made from instantaneously reinvesting the dividends back into the stock holding is a tradable asset. The function f is thus chosen to be $f(x,t) = xe^{\delta t}$. The market price of risk for tradables is then

$$\gamma_t = \frac{\mu X_t e^{\delta t} + \delta X_t e^{\delta t} - r X_t e^{\delta t}}{\sigma X_t e^{\delta t}} = \frac{\mu + \delta - r}{\sigma},$$

and thus the risk-neutral SDE for X_t becomes

$$dX_t = X_t\big(\sigma\, d\tilde{W}_t + (r-\delta)\, dt\big).$$

(iii) Foreign exchange, the 'wrong way round'. Let C_t be the dollar/mark exchange rate (worth in deutschmarks of one dollar), then the rate C_t *paid in dollars* is non-tradable. (That is, if C_t is equal to DM 1.45, the process worth \$1.45 is not tradable.) However the process $1/C_t$ is tradable, or more strictly e^{ut}/C_t is a dollar tradable asset if German interest rates are constant at rate u. If $X_t = C_t$ has SDE

$$dX_t = X_t(\sigma_t\, dW_t + \mu_t\, dt),$$

then the time-dependent transform of $f(x,t) = e^{ut}/x$ tells us that its risk-neutral SDE is

$$dX_t = X_t\big(\sigma_t\, d\tilde{W}_t + (\sigma_t^2 + u - r)\, dt\big).$$

4.5 Quantos

British Petroleum, a UK company, has a sterling denominated stock price. But instead of thinking of that stock price just in pounds, we could also consider it as a pure number which could be denominated in any currency.

Contracts like this which pay off in the 'wrong' currency are *quantos*. For instance, if the current stock price were £5.20, we could have a derivative that paid this price in dollars, that is \$5.20. This is not the same as the worth of the BP stock in dollars – that would depend on the exchange rate. What we have done is a purely formal change of units, whilst leaving the actual number unaltered.

Quantos are best described with examples. Here are three:

- a forward contract, namely receiving the BP stock price at time T as if it were in dollars in exchange for paying a pre-agreed dollar amount;
- a digital contract which pays one dollar at time T if the then BP stock price is larger than some pre-agreed strike;
- an option to receive the BP stock price less a strike price, in dollars.

In each case, a simple derivative is given the added twist of paying off in a currency other than in which the underlying security is denominated. And our intuition should warn us that this act of switching currency is not a foreign exchange quibble but something more fundamental. The British Petroleum stock price in dollars is a meaningful concept, but it is not a traded security. The payoffs we describe involve a non-tradable quantity.

Suppose we have a simple two-factor model. We have not actually met multi-factor models yet, but they are no more problematic than single-factor ones if we keep our head. Rigour can be found in the multiple stock models section (6.3). Our two random processes will be the stock price and the exchange rate, which will be driven by two independent Brownian motions $W_1(t)$ and $W_2(t)$.

For the construction, it is helpful to recall exercise 3.2: for ρ lying between -1 and 1, then $\rho W_1(t) + \sqrt{1-\rho^2}W_2(t)$ is also a Brownian motion, and it has correlation ρ with the original Brownian motion $W_1(t)$. This is a useful way to manufacture two Brownian motions which are correlated out of a pair which are independent.

We suppose there exist the following constants: drifts μ and ν, positive volatilities σ_1 and σ_2, and a correlation ρ lying between -1 and 1.

Given these constants, the quanto model is:

Quanto model

The sterling stock price S_t and the value of one pound in dollars C_t follow the processes

$$S_t = S_0 \exp\bigl(\sigma_1 W_1(t) + \mu t\bigr),$$
$$C_t = C_0 \exp\bigl(\rho\sigma_2 W_1(t) + \bar{\rho}\sigma_2 W_2(t) + \nu t\bigr),$$

where $\bar{\rho}$ is the orthogonal complement of ρ, namely $\bar{\rho} = \sqrt{1-\rho^2}$.

In addition there is a dollar cash bond $B_t = \exp(rt)$ and a sterling cash bond $D_t = \exp(ut)$, for some positive constant interest rates r and u.

Before we tease out the tradable instruments in dollars, note the covariance of S_t and C_t. If we write our model in vector form, the vector random variable $(\log S_t, \log C_t)$ is jointly-normally distributed with mean vector $(\log S_0 + \mu t, \log C_0 + \nu t)$ and covariance matrix

$$\begin{pmatrix} \sigma_1 & 0 \\ \rho\sigma_2 & \bar{\rho}\sigma_2 \end{pmatrix} \begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix} \begin{pmatrix} \sigma_1 & 0 \\ \rho\sigma_2 & \bar{\rho}\sigma_2 \end{pmatrix}^{\mathrm{T}} = \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix} t.$$

That is, we have ensured a constant volatility for S_t of σ_1, a constant volatility for C_t of σ_2 and a correlation between them of ρ.

Tradables

What are the dollar tradables? Following the intuition of the foreign exchange section (4.1), there are three: the dollar worth of the sterling bond, C_tD_t; the dollar worth of the stock, C_tS_t; and a dollar numeraire, the dollar cash bond B_t.

Writing down the first two of these tradables after discounting by the third, the numeraire, we have $Y_t = B_t^{-1}C_tD_t$ and $Z_t = B_t^{-1}C_tS_t$ respectively. Their SDEs are

$$dY_t = Y_t\bigl(\rho\sigma_2\, dW_1(t) + \bar{\rho}\sigma_2\, dW_2(t) + (\nu + \tfrac{1}{2}\sigma_2^2 + u - r)\, dt\bigr),$$
$$dZ_t = Z_t\bigl((\sigma_1 + \rho\sigma_2)\, dW_1(t) + \bar{\rho}\sigma_2\, dW_2(t)$$
$$+ (\mu + \nu + \tfrac{1}{2}\sigma_1^2 + \rho\sigma_1\sigma_2 + \tfrac{1}{2}\sigma_2^2 - r)\, dt\bigr).$$

(This can be checked using the n-factor Itô's formula of section 6.3.)

As in the market price of risk section, we know we want to find a change of measure to make these martingales, or equivalently a market price of risk that represents this change of drift. As there are two sources of risk, $W_1(t)$ and $W_2(t)$, there will be two separate prices of risk. Respectively, $\gamma_1(t)$ will be the price of $W_1(t)$-risk and $\gamma_2(t)$ will be the price of $W_2(t)$-risk. In other words the market price of risk will be a vector $(\gamma_1(t), \gamma_2(t))$. We want to choose these γ so that the drift terms in dY_t and dZ_t vanish simultaneously. Not surprisingly this means solving a pair of simultaneous equations, or equivalently performing the matrix inversion

$$\begin{pmatrix}\gamma_1(t)\\ \gamma_2(t)\end{pmatrix} = \begin{pmatrix}\rho\sigma_2 & \bar{\rho}\sigma_2\\ \sigma_1+\rho\sigma_2 & \bar{\rho}\sigma_2\end{pmatrix}^{-1}\begin{pmatrix}\nu+\frac{1}{2}\sigma_2^2+u-r\\ \mu+\nu+\frac{1}{2}\sigma_1^2+\rho\sigma_1\sigma_2+\frac{1}{2}\sigma_2^2-r\end{pmatrix}.$$

This is a particular case of the more general result that the multi-dimensional market price of risk is

$$\gamma_t = \Sigma^{-1}(\mu - r\mathbf{1}),$$

where Σ is the assets' volatility matrix, μ is their drift vector, and $\mathbf{1}$ is the constant vector $(1, \ldots, 1)$. More details are in section 6.3.

Here then we have a market price of risk $\gamma_t = (\gamma_1(t), \gamma_2(t))$, given by

$$\gamma_1 = \frac{\mu+\frac{1}{2}\sigma_1^2+\rho\sigma_1\sigma_2-u}{\sigma_1}, \quad \text{and} \quad \gamma_2 = \frac{\nu+\frac{1}{2}\sigma_2^2+u-r-\rho\sigma_2\gamma_1}{\bar{\rho}\sigma_2}.$$

Thus under $\mathbb{Q}$ we can write the original processes S_t and C_t as

$$\begin{aligned} S_t &= S_0\exp\left(\sigma_1\tilde{W}_1(t) + (u-\rho\sigma_1\sigma_2-\tfrac{1}{2}\sigma_1^2)t\right),\\ C_t &= C_0\exp\left(\rho\sigma_2\tilde{W}_1(t) + \bar{\rho}\sigma_2\tilde{W}_2(t) + (r-u-\tfrac{1}{2}\sigma_2^2)t\right). \end{aligned}$$

Exercise 4.2 Verify that the measure $\mathbb{Q}$ which has Brownian motions $\tilde{W}_i(t) = W_i(t) + \int_0^t \gamma_i(s)\,ds$ $(i = 1, 2)$ really is the martingale measure for Y_t and Z_t.

Reassuringly the exchange rate process is as it was in section 4.1, given that $\rho\tilde{W}_1(t) + \bar{\rho}\tilde{W}_2(t)$ is another $\mathbb{Q}$-Brownian motion (as was proved in exercise 3.2).

But the stock price S_t is different from what we expected. The drift has an extra term: $-\rho\sigma_1\sigma_2$. For every value of ρ (except one, namely $(u-r)/\sigma_1\sigma_2$) this stops the dollar-discounted stock price being a $\mathbb{Q}$-martingale and thus prevents the price in dollars from being tradable. And that's precisely what our intuition warned us. There isn't a portfolio which is always worth a dollar amount numerically equal to the BP stock price.

Pricing

Since we have a measure $\mathbb{Q}$, under which the dollar tradables are martingales, we can price up our quanto options.

Forward

To price the forward contract, it helps to re-express the stock price at date T as

$$S_T = \exp(-\rho\sigma_1\sigma_2 T)F\exp(\sigma_1\sqrt{T}Z - \tfrac{1}{2}\sigma_1^2 T),$$

where F is the local currency forward price of S_T, $F = S_0 e^{uT}$, and Z is a normal $N(0,1)$ random variable under $\mathbb{Q}$.

Then the value of the forward at time zero in dollars is

$$V_0 = e^{-rT}\mathbb{E}_{\mathbb{Q}}(S_T - k) = e^{-rT}\big(\exp(-\rho\sigma_1\sigma_2 T)F - k\big).$$

For this to be on market, that is to have a value of zero, we must set k to be $F\exp(-\rho\sigma_1\sigma_2 T)$. This is not the same as the simple forward price F for sterling purchase. As σ_1 and σ_2 are both positive, it is clear that this quanto forward price is greater than the simple forward price if and only if the correlation between the stock and the exchange rate is negative.

This actually makes some sense. Suppose we assumed that the quanto forward price was actually the same as the simple forward price F, then we could construct the following portfolio at time zero: by going

- long $C_0\exp\big((r-u)T\big)$ units of the quanto forward struck at F,
- short one unit of the simple sterling forward also struck at F.

If our assumption about the quanto forward price also being F were correct then this portfolio would be costless at time zero. At time T, this static replicating strategy would yield (in dollars)

$$C_0\exp\big((r-u)T\big)(S_T - F) - C_T(S_T - F) = (C_0\exp\big((r-u)T\big) - C_T)(S_T - F).$$

Noting that $C_0 \exp\big((r-u)T\big)$ is the forward FX rate for C_T, consider the effect of negative correlation. If the stock price ends up above its forward and the FX rate is below its forward, then the value of this portfolio is positive. And if the stock price ends up below F and the FX rate is above its forward, then the value is also positive.

Negative correlation makes these win-win situations more likely – perfect negative correlation makes them inevitable. If the quanto forward price really were F under these circumstances it wouldn't be hard to construct an arbitrage. For negative ρ the quanto forward must be greater than F.

Digital

Our digital contract, $I(S_T > k)$ in dollars, has price $V_0 = e^{-rT}\mathbb{Q}(S_T > k)$, or if we write $F_Q = F\exp(-\rho\sigma_1\sigma_2 T)$ the quanto forward price, then

$$V_0 = e^{-rT}\Phi\left(\frac{\log\frac{F_Q}{k} - \frac{1}{2}\sigma_1^2 T}{\sigma_1\sqrt{T}}\right).$$

Again the surprise of the $\exp(-\rho\sigma_1\sigma_2 T)$ term. And in a 'cleaner' option. Surely the event of S_T being greater than k is independent of whether the option is denominated in sterling or dollars. Indeed it is, but again replicating strategies, not expectation under $\mathbb{P}$, price options. And replication involves the exchange rate, which is correlated with the stock price.

Call option

Finally, we can compute the option price of $e^{-rT}\mathbb{E}_{\mathbb{Q}}\big((S_T-k)^+\big)$ as

$$V_0 = e^{-rT}\left(F_Q\Phi\left(\frac{\log\frac{F_Q}{k} + \frac{1}{2}\sigma_1^2 T}{\sigma_1\sqrt{T}}\right) - k\Phi\left(\frac{\log\frac{F_Q}{k} - \frac{1}{2}\sigma_1^2 T}{\sigma_1\sqrt{T}}\right)\right).$$

Perhaps not surprisingly for a log-normal model, this is just the original Black–Scholes formula with the quanto forward.

Exercise 4.3 Suppose everything remains the same, except that the stock S_t is the price in yen of NTT, a Japanese stock, C_t is the dollar/yen exchange rate (the worth in yen of one dollar), and ρ is their correlation. What is the one difference, between the sterling and yen cases, in the expression for the quanto forward price?