Finite Element Methods for Implementing the Black-Scholes Pricing Model

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Overview

The Black-Scholes Model and Options Pricing

The PDE:

$$\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial^2 S} - rV = 0 \tag{1}$$

Parameters: V is the option price, r is the given risk-free interest rate, S is the underlying asset price, σ is the given volatility or risk of the underlying asset, and K is exercise price

European Call Options

$$V(S, T) = \max\left(S - K, 0\right)$$

Intrinsic Value of an European Call Option

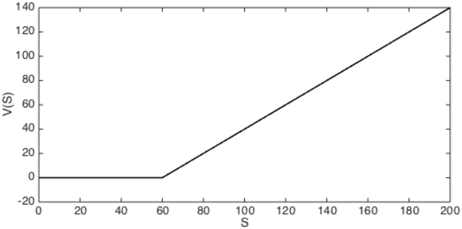

Figure: European Call with Exercise Price = \$60

We have the following boundary conditions for an European call option:

$$V(0, t) = 0,$$
$$\frac{\partial V}{\partial S}(V_\infty, t) = 1, \; \forall t$$

Finite Element Method (FEM)

Goal:

- to discretize space using FEM and to discretize time using a first order approximation
- dividing the domain into smaller pieces and using locally valid approximations to generate an approximate solution

Linear Finite Element

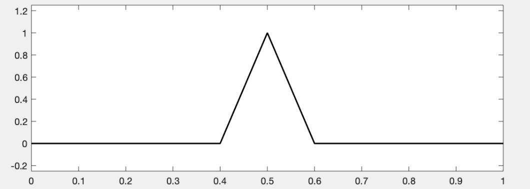

Figure: $\varphi_{0.5}$

Norms

1. **Energy Norm**
   - $\left\| u' - u_h' \right\|$
2. L^2 **Norm**
   - $\left\| u - u_h \right\|$

Boundary Conditions

- Dirichlet conditions:
$$u(0) = u(1) = 0$$
- Neumann conditions:
$$u'(0) = u'(1) = 0$$
- Robin conditions:
$$u(0) = u'(0), u(1) = -u'(1)$$

The Continuous Problem

$$a_1(x)u'' + a_2(x)u' + a_3(x)u = f(x) \tag{2}$$

with boundary conditions:

$$\alpha_1 u(a) + \beta_1 u'(a) = \gamma_1$$
$$\alpha_2 u(b) + \beta_2 u'(b) = \gamma_2$$

where the type of boundary conditions depend on the values of the parameters: $\alpha_1, \alpha_2, \beta_1, and \beta_2$

The Model Problem

Let $\Omega = (0{,}1)$ and $f \in \mathrm{L}^2(\Omega)$. In Ω :

$$-u'' = f \tag{3}$$

$$u(0) = u(1) = 0$$

Multiply by a test function, v, and integrate to get:

$$a(u, v) = F(v) \;\; \forall v \in V$$

where $a(u, v) = \int_\Omega u'v'$ and $F(v) = \int_\Omega fv$, for all $u, v \in V$. This is the *weak or variational formulation*

Lax-Milgram Lemma

Let V be a Hilbert space and let $a\colon VxV \to \mathbb{R}$ be a bilinear form and $F\colon V \to \mathbb{R}$ be a linear form. Assume

1. a is bounded: $\mid a(u,v) \mid \leq M \left\| u \right\|_V \left\| v \right\|_V \; \forall u, v \in V$
2. F is bounded: there exists > 0 such that $\mid F(v) \mid \leq \Lambda \left\| v \right\|_V \; \forall v \in V$
3. a is V-elliptic: there exists $\alpha > 0$ such that $a(v,v) \geq \alpha \left\| v \right\|_V^2 \; \forall v \in V$

Then, there exists a unique $u \in V$ such that

$$a(u,v) = F(v) \text{ where } \forall v \in V$$

This lemma shows that a variational formulation is well-posed.

Other Key Inequalities for FEM

1. **Cauchy-Schwartz Lemma:**
   Let $(V, (\cdot, \cdot))$ be a pre-Hilbert space. Then,
   $$\mid (u, v) \mid \leq \|u\|_V \|v\|_V \ \forall u, v \in V$$
2. **Friedrisch-Poincaré Inequality:**
   There exists $c > 0$, depending only on Ω, such that
   $$\|v\|_{L^2(\Omega)} \leq c \|v'\|_{L^2(\Omega)} \ \forall v \in H_0^1(\Omega)$$
3. **Trace Inequality:**
   There exists $c > 0$, depending only on $\Omega = (a, b)$ such that
   $$\mid v(a) \mid \leq c \|v\|_{H^1(\Omega)}$$
   $$\mid v(b) \mid \leq c \|v\|_{H^1(\Omega)}$$

The Discrete Problem

- Instead of an infinitely dimensional space V, we use V_h, a finite dimensional space.
- The domain, V_h, will be discretized to approximate the solution. This is the spatial discretization.
- In a steady-state case of a problem, there is no time derivative to worry about.

Solving The Model Problem

1. Assemble matrices
2. Solve for RHS (independent of t?)
3. Invert to solve for $\vec{u}$

Solving the Model Problem

$$\int_\Omega u'v' = \int_\Omega fv, \; \forall v_h \in V_h$$

which gives rise to a linear system of equations where $u_h = \sum_{j=1}^{n} u_j \varphi_j$. We get $A\vec{u} = \vec{b}$ where $A = (\int_\Omega \varphi_j' \varphi_i')_{i,j=1}^{n}$, $\vec{b} = (\int_\Omega f \varphi_j)_{j=1}^{n}$

$$A = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \ddots & \vdots \\ 0 & -1 & 2 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & -1 \\ 0 & \cdots & 0 & -1 & 2 \end{pmatrix}$$

1-D Evolution Problems

- Evolution problems require the reintroduction of time discretization since we are no longer working in the steady-state case.
- For our time discretization, we will use a first-order approximation.

The Heat equation:

$$u_t = \kappa u_{xx} + f \text{ in } \Omega x[0, T]$$

with conditions:

$$u(x, 0) = u_o(x),\ x \in \Omega$$

$$u(0, t) = u(1, t) = 0,\ t \in [0, T]$$

Multiplying by a test function $v_h \in V_h \subseteq H_0^1$, we obtain:

$$(\dot{u}_h, v_h) + \kappa(u_h', v_h') = (f, v_h)\ \forall v_h \in V_h$$

Time Discretization and Linear Interpolation for the Heat Equation

We discretize time

$$0 = t_0 < t_1 < \dots < t_M = T$$

and consider for $\theta \in [0, 1]$

$$\left(\frac{u_h^{m+1} - u_h^m}{\Delta t}, v_h\right) + \kappa a(u_h^{m+\theta}, v_h) = (f^{m+\theta}, v_h)$$

and

$$u_h^0 = \pi u_0, \text{ the grid points interpolation}$$

Rearrange to obtain

$$u_h^{m+\theta} = (1-\theta)u_h^m + \theta u_h^{m+1}$$

$$f^{m+\theta} = (1-\theta)f(\cdot, t_m) + \theta f(\cdot, t_{m+1})$$

Setting θ

We want to obtain the max of $\left\|u_h^{m+1}\right\|$:

If $\frac{1}{2} \leq \theta \leq 1$, ie. where $2\theta - 1 \geq 0$:

- $\left\|u_h^{m+1}\right\| \leq C\{\left\|u_h^m\right\| + \Delta t \left\|f^{m-1}\right\|$, $0 \leq m \leq M-1$
- This scheme is unconditionally stable.

If $0 \leq \theta < \frac{1}{2}$:

- CFL condition : $\Delta t \leq \dfrac{h^2}{6(1-2\theta)}(1-\varepsilon)$ for some $\varepsilon \in (0,1)$
- This scheme is conditionally stable.

Setting θ

- Explicit Scheme, $\theta = 0$
  - $Mu_h^{m+1} = (M - \kappa \Delta t S) u_h^m + \Delta t f^m$
  - M is very easy to invert
  - you need $\Delta t \leq ch^2$
- Implicit Scheme, $\theta = 1$
  - $(M + \kappa S) u_h^{m+1} = M u_h^m + \Delta t f^m$
  - unconditionally stable
  - $M + \kappa S$ is hard to invert
- Crank-Nicholson Scheme, $\theta = \frac{1}{2}$
  - unconditionally stable
  - more accurate
  - $M + \kappa \frac{S}{2}$ is hard to invert

MATLAB Implementation

inputs:

- $x =$ the points on the mesh in omega, includes left endpoint but not right
- $n =$ number of interior points on the mesh
- $h =$ space between each point on the mesh
- $T =$ time of maturity of the option and at right endpoint; due to nature of pricing model T is actually the initial condition as time goes in reverse
- $R =$ value of the option at time T
- $\theta =$ parameter of the theta-method, allows us to do a linear interpolation between u at time n and u at time n+1, affects stability of approximation

Implementing Black-Scholes

The PDE:

$$\frac{\partial u}{\partial t} + rx\frac{\partial u}{\partial x} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 u}{\partial^2 x} - ru = 0$$

where we define $b(x) = rx$ and $a(x) = \frac{1}{2}\sigma^2 x^2$.
The *variational formulation*:

$$-\int_0^L \dot{u}v - \int_0^L (b - \frac{da}{dx})\frac{\partial u}{\partial x}v + \int_0^L a\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + r\int_0^L uv = a(L)v(L)$$

where we define $A = \int_0^L (b - \frac{da}{dx})\frac{\partial u}{\partial x}v$ and $S = \int_0^L a\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}$
So we have:

$$-M\dot{\vec{u}}(t) - A\vec{u} + S\vec{u} + rM\vec{u} = \vec{b}(t)$$

but it turns out that the RHS is independent of t.

Takeaways and Future Improvements

- an error estimate and rate of convergence functions in MATLAB for Black-Scholes
- To extend the model to European put option and American call and put options
- To use a higher order of polynomial to approximate the model
- To control the size of the condition number
  - condition number $=$ the ratio between largest and smallest eigenvalues
  - $\frac{1}{h^2} \sim$ condition number
- To extend the model to 2-dimensional space, with a financial interpretation that there are two underlying assets that are somehow related for an option

Thank you!