Dynamic Analysis of Stadium Structures under Crowd-Induced Vibrations

1 Beskrivning

När man designar stadionkonstruktioner är det viktigt att förstå och kontrollera de dynamiska påfrestningarna som kan uppstå p.g.a. publikinducerade vibrationer. Som exempel kan nämnas att Bruce Springsteens konsert på Ullevi stadion i Göteborg den 8 juni 1985 drog en publik på över 64000 besökare med över 10000 personer på läktardelen. Under denna konsert så bidrog publikens rytmiska rörelse till att det mjuka underlaget under stadion bestående av göteborgsk lera sattes i självsvängning, vilket bidrog till strukturella skador som uppgick till över 36 miljoner kronor.

I det här projektet är du ansvarig för att skapa en enklare matematisk modell av en stadionläktare och analysera dess respons på en populär fotbollsläktarsång. Det har visat sig genom en förstudie att en läktare kan modelleras i en frihetsgrad med massan M, styvheten k och dämpningen c. När en person med massan m står på läktaren så skapar den sammanlagda massan, av personen och läktaren, en förskjutning δ. Antag att personen börjar gunga med frekvensen f och därmed utöva en periodisk kraft som är proportionell mot personens vikt med proportionalitetskonstanten α.

2 Del 1

Gör en friläggning av ovan beskrivna modell, formulera kraftekvationen och ta fram den bestående delen av lösningen samt bestäm förstorningsfaktorn M.

Kraftekvationen:

$$(m+M)\ddot{y} = -ky - c\dot{y} + F_0\sin(\omega t)$$

?

$$(m+M)\ddot{y} = -ky - c\dot{y} + F_0\sin(\omega t)$$

$$\ddot{y} + \frac{c}{(m+M)}\dot{y} + \frac{k}{(m+M)}(y+\delta_0) = g + \alpha\left(\frac{m}{M+m}\right)\cdot g\sin(\omega t)$$
$$\ddot{y} + 2\zeta\omega_n\dot{y} + \omega_n^2 y = g + \frac{\alpha m g}{M+m}\sin(\omega t)$$

Help:

$$y = A\sin(\omega t-\alpha) + K = A\sin(\omega t)\cos(\alpha) - A\cos(\omega t)\sin(\alpha) + K$$
$$\dot{y} = A\omega\cos(\omega t-\alpha) + K = A\omega\cos(\omega t)\cos(\alpha) + A\omega\sin(\omega t)\sin(\alpha)$$
$$\ddot{y} = -A\omega^2\sin(\omega t-\alpha) = -A\omega^2\sin(\omega t)\cos(\alpha) - A\omega^2\cos(\omega t)\sin(\alpha)$$

$$F_0 = \alpha\cdot m\cdot g$$
$$k = \frac{(m+M)g}{\delta_0}$$
$$2\zeta\omega_n = \frac{c}{M+m} \Rightarrow \zeta = \frac{c}{(M+m)2\omega_n}$$
$$\omega_n^2 = \frac{k}{(M+m)} = \frac{g}{\delta_0}$$

Svängningsekvationen:

$$\ddot{y} + \frac{c}{(m+M)}\dot{y} + \frac{gy}{\delta_0} + g = g + \alpha\left(\frac{m}{M+m}\right)g\sin(\omega t)$$

Svängningsekvationen final:

$$\ddot{y} + 2\zeta\omega_n\dot{y} + \frac{gy}{\delta_0} = \frac{\alpha mg\sin(\omega t)}{(m+M)}$$

Separate:

For $\sin(\omega t)$:

$$(\omega_n^2 - \omega^2)A\cos(\alpha) + 2\zeta\omega_n\omega A\sin(\alpha) = \frac{\alpha mg}{M+m}$$

For $\cos(\omega t)$:

$$-(\omega_n^2 - \omega^2)A\sin(\alpha) + 2\zeta\omega_n\omega A\cos(\alpha) = 0$$

Combined:

$$\omega_n^2 K = g \to \frac{g}{\delta_0}K = g \to K = \delta_0$$

2.1 Amplituden

$$Y = \frac{F_0/k}{\sqrt{\left(1-(\omega/w_n)^2\right)^2 + 4\delta_0^2\left(\omega/\omega_n\right)^2}} = \frac{\left(\frac{\alpha mg\delta_0}{(m+M)g}\right)}{\sqrt{\left(1-\frac{\omega^2\delta_0}{g}\right)^2 + \frac{4\zeta^2\omega^2\delta_0}{g}}} =$$

$$\frac{\frac{\alpha m\delta_0}{(m+M)}}{\sqrt{\left(1-\frac{\omega^2\delta_0}{g}\right)^2 + \left(\frac{C}{m+M}\right)^2\frac{\omega^2}{\omega_n^4}}} = \frac{\frac{\alpha m\delta_0}{(m+M)}}{\sqrt{\left(1-\frac{\omega^2\delta_0}{g}\right)^2 + \left(\frac{c}{m+n}\right)^2\frac{\omega^2\delta_0^2}{g^2}}} =$$

$$\frac{\frac{\alpha m\delta_0}{(m+M)}}{\sqrt{1-\frac{2\omega^2\delta_0}{g}+\frac{\omega^4\delta_0^2}{g^2}+\left(\frac{c}{m+M}\right)^2\frac{\omega^2\delta_0^2}{g^2}}} = \frac{\frac{\alpha mg\delta_0}{(m+M)}}{\sqrt{g^2-2\omega^2 g\delta_0+\omega^4\delta_0^2+\left(\frac{c}{m+M}\right)^2\omega^2\delta_0^2}}$$

$$\frac{\frac{\alpha mg\delta_0}{m+M}}{\delta_0\sqrt{\frac{g^2}{\delta_0^2}-\frac{2\omega^2 g}{\delta_0}+\omega^4+\left(\frac{c\omega}{m+M}\right)^2}} =$$

$$= \frac{\frac{\alpha mg}{m+M}}{\sqrt{\left(\frac{g}{\delta_0}-\omega^2\right)^2+\left(\frac{c\omega}{m+M}\right)^2}}$$

2.2 Förstoringsfaktor

$$M = \frac{X}{(X)_{\omega=0}}$$

vilket med $(X)_{\omega=0} = F_0/k$ ger:

$$M = \frac{\frac{\frac{\alpha mg}{m+M}}{\sqrt{\left(\frac{g}{\delta_0}-\omega^2\right)^2+\left(\frac{c\omega}{m+M}\right)^2}}}{\frac{F_0}{k}} =$$

$$= \frac{\frac{\frac{\alpha mg}{m+M}}{\sqrt{\left(\frac{g}{\delta_0}-\omega^2\right)^2+\left(\frac{c\omega}{m+M}\right)^2}}}{\frac{\alpha\cdot m\cdot g}{\frac{(m+M)g}{\delta_0}}} =$$

$$= \frac{\frac{\frac{\alpha m g}{m+M}}{\sqrt{\left(\frac{g}{\delta_0}-\omega^2\right)^2+\left(\frac{c\omega}{m+M}\right)^2}}}{\frac{\alpha \cdot m \cdot g}{\frac{(m+M)g}{\delta_0}}}$$

$$= \frac{\frac{\alpha m g}{m+M}}{\frac{\alpha \cdot m \cdot g}{\frac{(m+M)g}{\delta_0}} \cdot \sqrt{\left(\frac{g}{\delta_0}-\omega^2\right)^2+\left(\frac{c\omega}{m+M}\right)^2}}$$

$$= \frac{\frac{\alpha m g}{m+M}}{\frac{\alpha \cdot m \cdot g}{\frac{(m+M)g}{\delta_0}} \cdot \sqrt{\left(\frac{g}{\delta_0}-\omega^2\right)^2+\left(\frac{c\omega}{m+M}\right)^2}} \cdot \frac{\frac{(m+M)g}{\delta_0}}{\frac{(m+M)g}{\delta_0}}$$

$$= \frac{\alpha m g}{m+M} \cdot \frac{\frac{(m+M)g}{\delta_0}}{\alpha \cdot m \cdot g} \cdot \frac{1}{\sqrt{\left(\frac{g}{\delta_0}-\omega^2\right)^2+\left(\frac{c\omega}{m+M}\right)^2}}$$

$$= \frac{\alpha m g}{\alpha \cdot m \cdot g} \cdot \frac{(m+M)g}{\frac{(m+M)g}{\delta_0}} \cdot \frac{1}{\sqrt{\left(\frac{g}{\delta_0}-\omega^2\right)^2+\left(\frac{c\omega}{m+M}\right)^2}}$$

$$= \frac{\alpha m g}{\alpha \cdot m \cdot g} \cdot \frac{(m+M)g}{\frac{(m+M)g}{\delta_0}} \cdot \frac{1}{\sqrt{\left(\frac{g}{\delta_0}-\omega^2\right)^2+\left(\frac{c\omega}{m+M}\right)^2}}$$

$$= \frac{\delta_0}{\sqrt{\left(\frac{g}{\delta_0}-\omega^2\right)^2+\left(\frac{c\omega}{m+M}\right)^2}}$$

3 Del 2

Antag följande parametervärden: $M = 1000$ kg; en person ($m = 75$ kg, $\alpha = 0.3$, $\delta = 5$ cm, $c = 2500$Ns/m). Välj en typisk frekvens för en fotbollssång. Skapa en funktion i MATLAB och plotta strukturens dynamiska respons. Hur förhåller sig detta med den förväntade statiska responsen? Beskriv effekten av sångens tempo.

```
function dynamic_response()

    M = 1000; % kg
    m = 75; % kg
    alpha = 0.3;
    delta_0 = 0.05; % m (convert cm to m)
    c = 2500; % Ns/m
    g = 9.81; % m/s^2
    f = 2; % Hz

    % Calculate omega and omega_n
    omega = 2 * pi * f;
    omega_n = sqrt(g / delta_0);

    % Calculate zeta
    zeta=c/((M+m)*2*omega_n);


    % Calculate the amplitude A and phase angle alpha
    A = ((alpha * m * g)/(m+M)) / sqrt((g/delta_0 - omega^2)^2 + (c*omega/(m+M))^2);

    % Calculate the static response K
    K = delta_0;

    % Time range for plotting
    t = linspace(0, 10, 1000); % 0 to 10 seconds with 1000 points
```

```
    % Calculate the dynamic response
    y = K + A * sin(omega * t);

    % Plot the dynamic response
    subplot(2, 1, 1);
    plot(t, y);
    xlabel('Time (s)');
    ylabel('Displacement (m)');
    title('Dynamic Response of the Structure');
    hold on
    plot(t, K * ones(size(t)), 'r--');
    legend('Dynamic Response', 'Static Response');
    hold off

end
```

4 Del 3

Med ovan beskrivna påtvingade kraft (en person som gungar) beslöts det av säkerhetsskäl att amplituden inte fick överskrida 1.5 mm. Hur ska styvheten i strukturen ändras för att uppnå detta krav? Plotta den dynamiska responsen av den justerade strukturen.

We have the amplitude given by the formula derived in step 2.1.

$$Y = \frac{\frac{\alpha m g}{m+M}}{\sqrt{\left(\frac{g}{\delta_0} - \omega^2\right)^2 + \left(\frac{c\omega}{m+M}\right)^2}}$$

Substituting k into the equation gives us the

$$Y = \frac{\frac{\alpha m g}{m+M}}{\sqrt{\left(\frac{k}{\alpha m} - \omega^2\right)^2 + \left(\frac{c\omega}{m+M}\right)^2}}$$

This formula is then defined as an equation in the Matlab program. The equation is solved by defining k as a variable, using the solve() function to solve the equation system and inserting the given values to obtain the value for k. We set the amplitude at 1.5 mm to ensure that the forced vibration does not exceed 1.5 mm.

4.1 Find k

```
% Define symbolic variables
syms k

% Define set variables
g = 9.82;
m = 75;
M = 1000;
A = 1.5*10^(-3);
alpha = 0.3;
f = 2;
c = 2500;
omega = 2*pi*f;

% Define the equation
eqn = A == ((alpha * m * g) / (m + M)) / sqrt(((k / (alpha * m)) - omega^2)^2 + ((c * omega) / (m + M))

% Solve the equation for k
k_val = double(solve(eqn, k));
```

Result: $0.5410 * 10^3$ or $6.5651 * 10^3$

When we run this program we obtain two values for k. This is logical since we will take the squareroot of a k term when caudrating the parenthesis containing k in our calculation of A. We are interested in the bigger value, since the relationship between the stiffness and the amplitude have an inverse relationship. This relationship can be seen more clearly in the following formula, which was the starting point of our derivations of the amplitude formula (in step 2.1) used for our calculations of k (and soon ζ).

$$Y = \frac{F_0/k}{\sqrt{\left(1-(\omega/w_n)^2\right)^2 + 4\delta_0^2\,(\omega/\omega_n)^2}}$$

4.2 Plot results for new k

```
function dynamic_response_k()

    M = 1000; % kg
    m = 75; % kg
    alpha = 0.3;
    delta_0 = 0.05; % m (convert cm to m)
    c = 2500; % Ns/m
    g = 9.81; % m/s^2
    f = 2; % Hz

    % Calculate omega and omega_n
    omega = 2 * pi * f;
    omega_n = sqrt(g / delta_0);

    % Calculate the amplitude A
    A = ((alpha * m * g)/(m+M)) / sqrt((g/delta_0 - omega^2)^2 + (c*omega/(m+M))^2);

    % Calculate the static response K
    K = delta_0;

    % Time range for plotting
    t = linspace(0, 10, 1000); % 0 to 10 seconds with 1000 points

    % Calculate the dynamic response
    y = -K + A * sin(omega * t);

    %----------Plot for the value of k= 6.5651*1.0e+03----------
    k= 6.5651*10^3; %N/m
    delta_new= (((m+M)*g)/k)*10^(-3);

    %New A => with the k= 6.5651*1.0e+03
    A_new = ((alpha * m * g) / (m + M)) / sqrt(((k / (alpha * m)) - omega^2)^2 + ((c * omega) / (m + M)

    % Calculate new dynamic response
    y_new = -delta_new + A_new * sin(omega * t);
    %----------------------------------------------------------

    % Plot the dynamic response
    subplot(2, 1, 1);
    plot(t, y);
    hold on
    plot(t, y_new, 'g');
    xlabel('Time (s)');
    ylabel('Displacement (m)');
    title('Dynamic Response of the Structure');
    hold on
```

```
    plot(t, -delta_new * ones(size(t)), 'r--');
    hold on
    plot(t, -delta_0 * ones(size(t)), 'b--');
    legend('Dynamic Response for old k', 'Dynamic Response for new k');
    hold off

    % Delta range for plotting
    delta_set_old_k = linspace(0.005, 1.5, 1000);
    delta_set_new_k=linspace(0.0016, 1.5, 1000);

    % New omega_n
    omega_new_old_k = sqrt(g ./delta_set_old_k);
    omega_new_k = sqrt(g ./delta_set_new_k);

    % New Amplitude
    A_delta_old_k = ((alpha * m * g) ./ (m + M)) ./ sqrt((g ./ delta_set_old_k - omega^2).^2 + (c * ome
    A_delta_new_k = ((alpha * m * g) ./ (m + M)) ./ sqrt((g ./ delta_set_new_k - omega^2).^2 + (c * ome

    % Calculate A_delta/delta_set
    A_delta_ratio = A_delta_old_k ./ delta_set_old_k;
    A_delta_ratio_newk = A_delta_new_k ./ delta_set_new_k;

    % Plot A_delta/delta_set vs. omega/omega_new
    subplot(2, 1, 2);
    plot(omega ./ omega_new_old_k, A_delta_ratio);
    hold on
    %---------------------------------------
    plot(omega ./ omega_new_k, A_delta_ratio_newk,'r--');
    %---------------------------------------
    xlabel('\omega / \omega_{new}');
    ylabel('A_{\delta} / \delta_{set}');
    title('Amplitude Ratio vs. Frequency Ratio');
    hold off

end
```

5 Del 4

Hur kan dämpningsfaktorn ζ justeras för att uppnå samma krav som i 3)?

From our derivations in step 2.1, the following equation was obtained before reaching the final formula, containing ζ.

$$Y = \frac{\frac{\alpha m g \delta_0}{(m+M)g}}{\sqrt{(1 - \left(\frac{\omega^2 \delta_0}{g}\right))^2 + \left(\frac{4\zeta^2 \omega^2 \delta_0}{g}\right)}}$$

We use this equation similarly to in step 4, i.e. we define the equation in Matlab and use solve() to solve for ζ, setting the amplitude at 1.5 mm to ensure that the forced vibration does not exceed 1.5 mm.

5.1 Find zeta

```
% Define symbolic variables
syms zeta

% Define set variables
delta_0=0.05; %m
g = 9.82;
m = 75;
```

```
M = 1000;
alpha = 0.3;
f = 2;
omega = 2*pi*f;

% Define the amplitude formula
amp_formula = ((alpha * m * g * delta_0) / ((m + M) * g)) / sqrt((1 - ((omega^2 * delta_0) / g))^2 + ((

% Set amplitude to A = 0.0015
A = 0.0015;
eqn = A == amp_formula;

% Solve the equation for zeta
zeta_sym = solve(eqn, zeta);

% Convert the symbolic solution to a numeric value
zeta_val = double(zeta_sym)
```

Result: 0.3734 or -0.3734.

We obtain two ζ since we're taking the square root of the term containg ζ, which notably are symmetrical. We are looking at the biggest value here as well since the relationship between ζ and Y is inverse.

```
function dynamic_response_zeta()

    M = 1000; % kg
    m = 75; % kg
    alpha = 0.3;
    delta_0 = 0.05; % m (convert cm to m)
    c = 2500; % Ns/m
    g = 9.81; % m/s^2
    f = 2; % Hz

    % Calculate omega and omega_n
    omega = 2 * pi * f;
    omega_n = sqrt(g / delta_0);

    % Calculate the amplitude A
    A = ((alpha * m * g)/(m+M)) / sqrt((g/delta_0 - omega^2)^2 + (c*omega/(m+M))^2);

    % Calculate the static response K
    K = delta_0;

    % Time range for plotting
    t = linspace(0, 10, 1000); % 0 to 10 seconds with 1000 points

    % Calculate the dynamic response
    y = -K + A * sin(omega * t);

    %----------Plot for the value of k= 6.5651*1.0e+03----------
    zeta = 0.3734;

    %New A => with the zeta = 0.3734
    A_new = ((alpha * m * g * delta_0) / ((m + M) * g)) / sqrt((1 - ((omega^2 * delta_0) / g))^2 + ((4

    % Calculate new dynamic response
    y_new = -delta_0 + A_new * sin(omega * t);
    %-----------------------------------------------------------

    % Plot the dynamic response
    subplot(2, 1, 1);
```

```
    plot(t, y);
    hold on
    plot(t, y_new, 'g--');
    xlabel('Time (s)');
    ylabel('Displacement (m)');
    title('Dynamic Response of the Structure');
    hold on
    plot(t, -delta_0 * ones(size(t)), 'r--');
    hold on
    plot(t, -delta_0 * ones(size(t)), 'b--');
    legend('Dynamic Response for old zeta', 'Dynamic Response for new zeta');
    hold off

    % Delta range for plotting
    delta_set = linspace(0.005, 1.5, 1000);

    % New omega_n
    omega_new = sqrt(g ./delta_set);

    % New Amplitude
    A_delta = ((alpha * m * g) ./ (m + M)) ./ sqrt((g ./ delta_set - omega^2).^2 + (c * omega ./ (m + M

    % Calculate A_delta/delta_set
    A_delta_ratio = A_delta ./ delta_set;

    % Plot A_delta/delta_set vs. omega/omega_new
    subplot(2, 1, 2);
    plot(omega ./ omega_new, A_delta_ratio);
    xlabel('\omega / \omega_{new}');
    ylabel('A_{\delta} / \delta_{set}');
    title('Amplitude Ratio vs. Frequency Ratio');

end
```

6 Del 5

Skapa en konturplot av den maximala förskjutningen $(\delta + X)$ som en funktion av hoppfrekvensen (f) och publikstorleken (m). Diskutera denna plot utifrån sammanhanget stor fotbollsmatch.

```
function contour_plot()

    M = 1000; % kg
    alpha = 0.3;
    %delta_0 = 0.05; % m (convert cm to m)
    c = 2500; % Ns/m
    g = 9.81; % m/s^2
    k = 210915; %N/m => Got it from the original by using k=((m+M)*g)/delta_0

    f = linspace(1, 5, 1000); % Hz, range from 1 to 5
    m = linspace(0, 750, 1000); % kg, range from 75 to 750
    delta=((m+M).*g)./k;


    % Calculate omega and omega_n
    [Frekvens, masa] = meshgrid(f, m);

    omega = 2 * pi * Frekvens;

    Amplitude = ((alpha * masa * g) ./ (masa + M)) ./ sqrt((g ./ delta - omega.^2).^2 + ((c * omega) ./
    Displacment = delta + Amplitude;
```

```
    surf(Frekvens, masa, Displacment);
    contourf(Frekvens, masa, Displacment);
    colorbar;

end
```

7 Utmaning 1

Skriv en kort litteraturöversikt där ni diskuterar olika sätt på hur mänsklig excitation kan modelleras numeriskt.

8 Utmaning 2

Ändra i din MATLAB kod för att ta hänsyn till hoppbelastningar.