In this equation, you will recognize the coefficient b/m as the constant that we renamed as 2β in Chapter 5,

$$\frac{b}{m} = 2\beta,$$

where β was called the **damping constant**. Similarly the coefficient g/L is just ω_o^2,

$$\frac{g}{L} = \omega_o^2,$$

where ω_o is the **natural frequency** of the pendulum. Finally, the coefficient F_o/mL must have the dimensions of $(\text{time})^{-2}$; that is, F_o/mL has the same units as ω_o^2. It is convenient to rewrite this coefficient as $F_o/mL = \gamma\omega_o^2$. That is, we introduce a dimensionless parameter

$$\gamma = \frac{F_o}{mL\omega_o^2} = \frac{F_o}{mg}, \tag{12.10}$$

which I shall call the **drive strength** and is just the ratio of the drive amplitude F_o to the weight mg. This parameter γ is a dimensionless measure of the strength of the driving force. When $\gamma < 1$, the drive force is less than the weight and we would expect it to produce a relatively small motion. (For instance, the drive force is insufficient to hold the pendulum out at $\phi = 90°$.) Conversely, if $\gamma \geq 1$, the drive force exceeds the pendulum's weight, and we should anticipate that it will produce larger scale motions (for instance, motion in which the pendulum is pushed all the way over the top at $\phi = \pi$).

Making all these substitutions, we get our final form of the equation of motion (12.9) for a driven damped pendulum

$$\ddot{\phi} + 2\beta\dot{\phi} + \omega_o^2 \sin\phi = \gamma\omega_o^2 \cos\omega t. \tag{12.11}$$

This is the equation whose solutions we shall be studying for the next several sections.

12.3 Some Expected Features of the DDP

Properties of the Linear Oscillator

To appreciate the extraordinary richness of the chaotic motion of our driven damped pendulum, we must first review what sort of behavior we might *expect*, based on our experiences with linear oscillators. Specifically, if we release the pendulum near the equilibrium position $\phi = 0$ with a small initial velocity and if the drive strength is small, $\gamma \ll 1$, we would expect ϕ to remain small at all times. Thus we should be

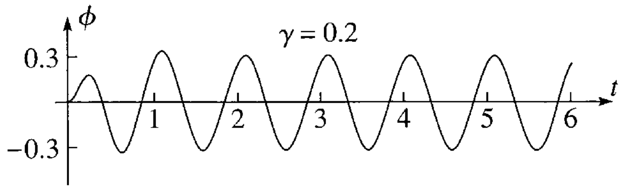

Figure 12.2 The motion of a DDP with a relatively weak drive strength of $\gamma = 0.2$. The drive period was chosen to be $\tau = 1$, so that the horizontal axis gives the time in units of the drive period. You can see clearly that after about two cycles the motion has settled down to a perfectly sinusoidal motion with period equal to the drive period.

able to approximate the term $\sin\phi$ in (12.11) as ϕ, and the equation of motion would become the linear equation

$$\ddot{\phi} + 2\beta\dot{\phi} + \omega_o^2\phi = \gamma\omega_o^2 \cos\omega t \tag{12.12}$$

which has exactly the form of (5.57) for the linear oscillator of Chapter 5. Thus the "expected" behavior of the driven damped pendulum, at least for a weak enough driving force, is just the behavior described in Section 5.5. This behavior can be quickly summarized: The initial behavior of the pendulum depends on the initial conditions, but any differences (or "transients") due to the initial conditions die out rapidly, and the motion approaches a unique "attractor," in which the pendulum oscillates sinusoidally with exactly the frequency of the driving force:

$$\phi(t) = A\cos(\omega t - \delta). \tag{12.13}$$

These predictions are nicely illustrated in Figure 12.2, which shows the actual motion of the driven damped pendulum for a fairly weak drive strength of $\gamma = 0.2$. [Since the exact equation of motion (12.11) cannot be solved analytically, this and all subsequent plots of the motion of the DDP were made from numerical solutions of (12.11).[5]] The drive frequency was chosen to be $\omega = 2\pi$, so that the drive period is $\tau = 2\pi/\omega = 1$. This means that the horizontal axis shows time in units of the drive period.The natural frequency was chosen to be $\omega_o = 1.5\omega$, so that the system is fairly close to resonance, since this is where chaotic motion is usually easiest to find.[6] The most striking feature of this plot is that after about two cycles the motion has settled down to a purely sinusoidal motion with exactly the period of the driving force, $\tau = 1$. The initial conditions chosen for this plot were that $\phi = \dot{\phi} = 0$ at $t = 0$. It is a fact (though not one that our one plot can show) that, whatever initial conditions we were to choose, the motion of a linear oscillator would always approach the same unique attractor as the initial transients die out.

---

[5] All of these plots were made using Mathematica's numerical solver NDSolve. For many plots the default precision of 15 digits was more than sufficient, but where there was any reason for doubt, the precision was increased in integer steps until two successive calculations were indistinguishable.

[6] The other parameters used were as follows: Damping constant $\beta = \omega_o/4$, and initial conditions $\phi = \dot{\phi} = 0$ at $t = 0$.

To summarize, for a linear damped oscillator, with a sinusoidal driving force: **(1)** There is a unique attractor which the motion approaches, irrespective of the chosen initial conditions. **(2)** The motion of this attractor is itself sinusoidal with frequency exactly equal to the drive frequency.

Nearly Linear Oscillations of the DDP

Let us now imagine increasing the drive strength so that the amplitude of oscillation increases to a value where the approximation

$$\sin\phi \approx \phi$$

is no longer satisfactory. As long as the amplitude is not too large, we would expect to get a satisfactory approximation by including just one more term in the Taylor series for $\sin\phi$ and writing

$$\sin\phi \approx \phi - \tfrac{1}{6}\phi^3.$$

If we use this approximation in the exact equation of motion (12.11), we get the approximate equation

$$\ddot{\phi} + 2\beta\dot{\phi} + \omega_o^2\left(\phi - \tfrac{1}{6}\phi^3\right) = \gamma\omega_o^2\cos\omega t. \tag{12.14}$$

To the extent that the new nonlinear term involving ϕ^3 is small, we can anticipate that the solution of this equation will still be reasonably approximated (once the transients have died out) by an expression of the same form as before,

$$\phi(t) \approx A\cos(\omega t - \delta).$$

When this is put into (12.14), the small term involving ϕ^3 contributes a term proportional to $\cos^3(\omega t - \delta)$. Since

$$\cos^3 x = \tfrac{1}{4}(\cos 3x + 3\cos x) \tag{12.15}$$

(see Problem 12.5) there is now a small term on the left side of (12.14) proportional to $\cos 3(\omega t - \delta)$. Since the right side contains no terms with this time dependence, it follows that at least one of the terms on the left (ϕ, $\dot{\phi}$, or $\ddot{\phi}$, and in fact all three) must. That is, a more exact expression for $\phi(t)$ must have the form

$$\phi(t) = A\cos(\omega t - \delta) + B\cos 3(\omega t - \delta), \tag{12.16}$$

with B much smaller than A. Therefore, we must anticipate that, as we increase the driving force and the amplitude increases, the solution will pick up a small term that oscillates with frequency 3ω.

We can repeat this argument: If we substitute the improved solution (12.16) back into (12.14), then the term in ϕ^3 will give even smaller terms of the form $\cos n(\omega t - \delta)$, with n an integer greater than 3. Therefore we must expect smaller corrections to (12.16) with frequencies $n\omega$, with n equal to various integers. Any term oscillating

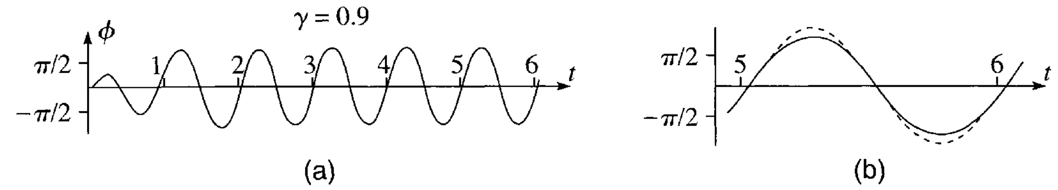

Figure 12.3 **(a)** The motion of a DDP with drive strength $\gamma = 0.9$ (and all other parameters the same as in Figure 12.2). After two or three drive cycles, the motion settles down to a regular oscillation, which has period equal to the drive period and looks at least approximately sinusoidal. **(b)** The solid curve is an enlargement of a single cycle of part (a), from $t = 5$ to 6. The dashed curve, which is a pure cosine with the same frequency, phase, and slope where they cross the axis, shows clearly that the actual motion is no longer perfectly sinusoidal; it is appreciably flatter at the extremes.

with frequency equal to an integer multiple of ω is called a **harmonic** of the drive frequency. Thus our conclusion is that, as the drive strength is increased and the nonlinearity becomes more important, the pendulum's motion will pick up various harmonics of the drive frequency ω, the most important being the $n = 3$ harmonic already included in (12.16).

The nth harmonic, with frequency $n\omega$, is periodic with period $\tau_n = 2\pi/n\omega = \tau/n$, where $\tau = 2\pi/\omega$ is the drive period. Thus in one drive period, the nth harmonic repeats itself n times. In particular, in one drive period, every harmonic will have cycled back to its original value, and a motion that is made up of various harmonics will still be periodic with the same period as the driving force.

The main difference between the motion implied by (12.16) (possibly with other harmonics included) and the motion (12.13) of the linear oscillator is that, with its extra term (or terms), (12.16) is no longer given by a single cosine function. We should be able to see this in a graph of $\phi(t)$ against t, which must deviate slightly from a pure sinusoidal shape. However, in the regime we are considering, the coefficient B in (12.16) and the coefficients of any higher harmonics are all much smaller than A, and the difference between the actual motion and a pure cosine is quite hard to see. Figure 12.3(a) shows the motion of a damped driven pendulum with drive strength $\gamma = 0.9$ (just below our rough boundary at $\gamma = 1$ between weak and strong drive strengths). Just as in Figure 12.2, the motion has quickly settled down to steady oscillation with exactly the period of the driving force. At first glance the curve (after about $t = 2$) appears to be a pure cosine, but on closer examination you may be able to convince yourself that it is a little too flat at the crests and troughs. Figure 12.3(b) is an enlargement of one cycle of the motion (solid curve), with a superposed pure cosine with the same period and phase (dashed curve). This comparison shows clearly that the actual motion is no longer a single pure cosine.[7]

---

[7] The flattened shape at the extremes is nicely consistent with (12.16): Provided B and A have opposite signs, the second term in (12.16) reduces $\phi(t)$ at the crests and troughs and increases it near where it crosses the axis. The behavior is also easy to understand physically: At the extremes, the restoring torque of gravity ($mgL \sin\phi$) is weaker than the linear approximation ($mgL\phi$), and the actual motion is less sharply curved.