A striking feature of Figure 12.18 is the long interval of period-1 motion, from just below $\gamma = 1.3$ to just above $\gamma = 1.4$. This period-1 motion is actually a rolling motion, as you can see in Figure 12.19 which shows the motion for $\gamma = 1.4$. In part (a), which shows $\phi(t)$ as a function of t, you can see that the pendulum is rolling clockwise at a rate of one complete revolution per drive cycle (ϕ decreases by 20π in 10 cycles). That the motion really is periodic is even more evident in part (b), which shows $\dot{\phi}(t)$ as a function of t. After about two drive cycles, $\dot{\phi}(t)$ is clearly periodic with period 1.

12.7 State-Space Orbits

In the next two sections I give a brief introduction to the *Poincaré section*, which is an important alternative way to view the motion of chaotic (and nonchaotic) systems. The Poincaré section is a simplification of the so-called *state-space orbit*. This simplification is especially helpful for complicated multidimensional systems but can be introduced in the context of our one-dimensional driven damped pendulum. Thus I shall start in this section by describing state-space orbits for the DDP.

In our discussion of the DDP we have focussed almost exclusively on the position $\phi(t)$ as a function of t. It turns out, however, that it is sometimes an advantage to follow *both* the position $\phi(t)$ *and* the angular velocity $\dot{\phi}(t)$ as time evolves. In principle, if one knows $\phi(t)$ for all t, then one can calculate $\dot{\phi}(t)$ by straightforward differentiation. Thus to follow $\dot{\phi}(t)$ as well as $\phi(t)$ is, in this sense, redundant. Nevertheless, following both variables can provide new insights into the motion, and this is what we shall now discuss.

There is an immediate problem in plotting the two variables $\phi(t)$ and $\dot{\phi}(t)$ as functions of the third variable t, since this requires a three-dimensional plot — something which is hard to make, and not especially illuminating when made. The usual procedure is to draw the pair of values $[\phi(t), \dot{\phi}(t)]$ as a point in a two-dimensional plane where the horizontal axis labels ϕ and the vertical axis $\dot{\phi}$. (For reasons I'll discuss in a moment, this plane with coordinates ϕ and $\dot{\phi}$ is called *state space*.) As time passes, the point $[\phi(t), \dot{\phi}(t)]$ moves in this two-dimensional space and traces out a curve, which is called a **state-space orbit** (or phase-space trajectory). Once you get used to interpreting these state-space orbits, you will find that they give a rather clear picture of the system's motion.

As a first example, let us consider a DDP with $\gamma = 0.6$ (a drive strength for which the linear approximation is still fairly good) and with our favorite initial conditions $\phi(0) = -\pi/2$ and $\dot{\phi}(0) = 0$. Figure 12.20 shows a conventional plot of $\phi(t)$ against t for this case. To interpret this picture one has to know (as you certainly do) that the changing position $\phi(t)$ is shown by the vertical displacement of the graph while the time t advances from left to right. With this understood, you can clearly see the motion starting at $t = 0$ with $\phi(0) = -\pi/2$ and quickly approaching the expected sinusoidal attractor, with $\phi(t)$ of the form $\phi(t) = A\cos(\omega t - \delta)$.

Figure 12.21 shows the state-space orbit for the same DDP with the same initial conditions. Part (a) shows the first twenty cycles, $0 \le t \le 20$. To interpret this picture,

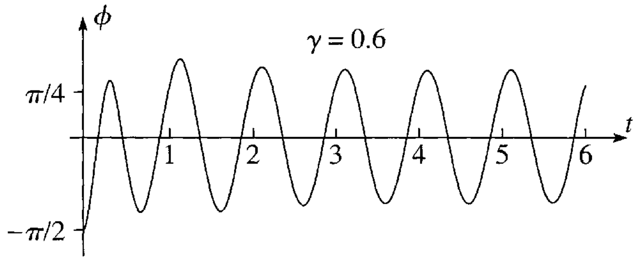


Figure 12.20 Conventional plot of $\phi(t)$ against t for a DDP with drive strength $\gamma = 0.6$. The motion quickly settles down to almost perfectly sinusoidal oscillation.

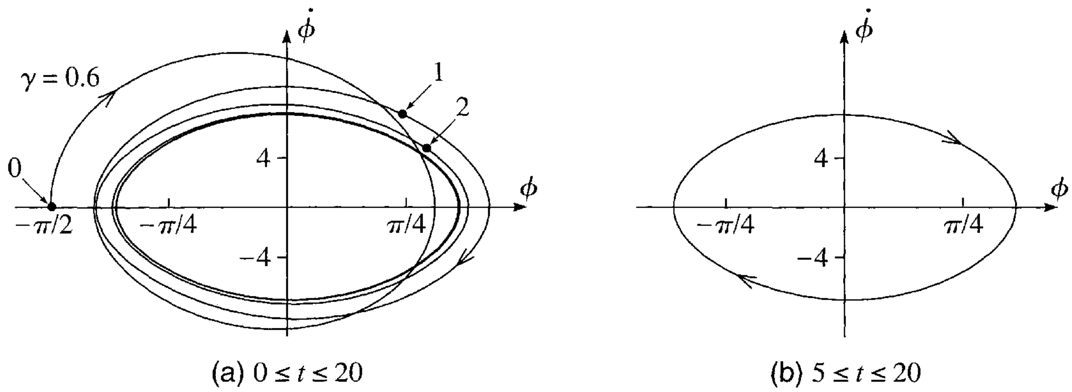


Figure 12.21 State-space orbit for a DDP with drive strength $\gamma = 0.6$. State space is the two-dimensional plane with coordinates ϕ and $\dot{\phi}$; the state-space orbit is just the path traced by the point $[\phi(t), \dot{\phi}(t)]$ as time passes. **(a)** The first 20 cycles, starting from the initial values $\phi(0) = -\pi/2$ and $\dot{\phi}(0) = 0$. The three dots labelled 0, 1, and 2 show the positions of $[\phi(t), \dot{\phi}(t)]$ at $t = 0$, 1, and 2. The orbit spirals inward and rapidly approaches the period-one attractor, which appears as an ellipse in state space. **(b)** The same as (a) but with the first 5 cycles omitted so that only the elliptical attractor is seen. Between the times $5 \le t \le 20$, the point $[\phi(t), \dot{\phi}(t)]$ moves 15 times around the same elliptical path.

one has to know that as t advances the curve is traced, in the direction of the arrows, by the pair $[\phi(t), \dot{\phi}(t)]$. With this understood, you can see clearly that the orbit starts out from $\phi(0) = -\pi/2$ and $\dot{\phi}(0) = 0$. Since the initial acceleration $\ddot{\phi}$ is positive,[19] $\dot{\phi}(t)$ increases from the outset, and $\phi(t)$ begins to increase as soon as $\dot{\phi}$ is nonzero. Thus the point $[\phi(t), \dot{\phi}(t)]$ moves up initially, curving to the right. The oscillation of $\phi(t)$ is evidenced by the back and forth, left-right, motion of the orbit; the oscillation of $\dot{\phi}(t)$ by the up and down vertical motion. Eventually, as the transients die out, the motion approaches its long-term attractor, in which (in the linear approximation) we know that $\phi(t)$ has the form

$$\phi(t) = A\cos(\omega t - \delta). \tag{12.28}$$

[19] As you can easily check, with the given initial conditions, both gravity and the drive force give the pendulum a positive acceleration at first.

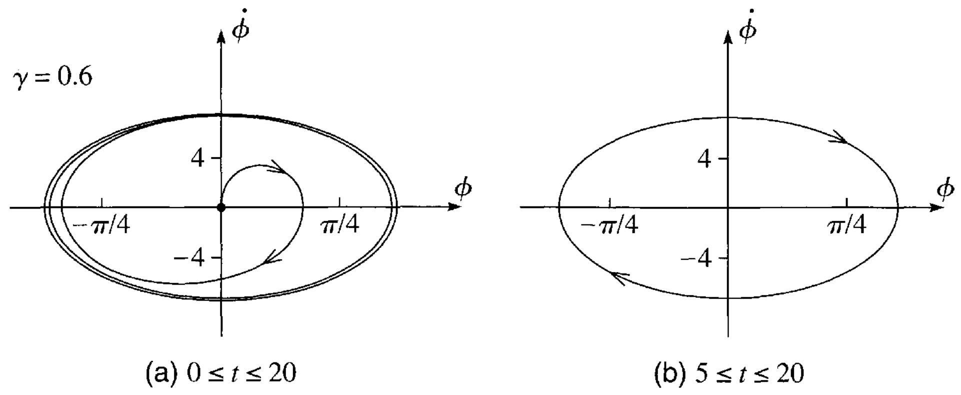


Figure 12.22 State-space orbit for a DDP with drive strength $\gamma = 0.6$ and initial conditions $\phi(0) = \dot\phi(0) = 0$. **(a)** The first 20 cycles, starting from the origin and spiralling out toward the elliptical attractor. **(b)** In the 15 cycles $5 \le t \le 20$, the orbit moves 15 times around the elliptical attractor to give exactly the same picture as in Figure 12.21(b).

This implies that the angular velocity $\dot\phi(t)$ approaches the form

$$\dot\phi(t) = -\omega A \sin(\omega t - \delta). \tag{12.29}$$

The two equations (12.28) and (12.29) are the parametric equations for an ellipse drawn clockwise in the plane of $(\phi, \dot\phi)$, with semimajor and semiminor axes A and ωA. Thus, once the transients have died out, the point $[\phi(t), \dot\phi(t)]$ moves around this ellipse with angular frequency equal to the drive frequency ω; that is, the state-space orbit completes one revolution per drive cycle. In Figure 12.21(a), the state-space orbit spirals in toward this ellipse, merging with it after about three cycles. [This already illustrates one small advantage of the state-space orbit over the conventional plot of $\phi(t)$ against t: In the conventional Figure 12.20 the actual motion has become indistinguishable from the limiting sinusoidal motion after little more than 1 cycle; in the state-space plot of Figure 12.21(a) the actual and limiting orbits can be told apart for some three cycles. Thus the state-space orbit gives a more sensitive picture of the approach to the attractor.] Figure 12.21(b) is the same as part (a), except that I have omitted the first 5 cycles; that is, in part (b), $5 \le t \le 20$ and only the elliptical attractor shows up. Since our main interest is usually in the limiting motion, state-space plots are usually drawn as in part (b), with enough initial cycles omitted so that only the limiting motion is visible.

Figure 12.22 shows the state-space orbit for exactly the same DDP as in Figure 12.21, but with initial conditions $\phi(0) = \dot\phi(0) = 0$. In part (a) you can easily see that the orbit starts out with the stated initial conditions, and spirals outward, completing some 2.5 cycles before merging with the elliptical attractor. Part (b) shows the 15 cycles starting from $t = 5$, by which time the orbit is indistinguishable from its long-term attractor. In particular, Figure 12.22(b) is exactly the same as Figure 12.21(b), because for $\gamma = 0.6$ all initial conditions lead to the same attractor.

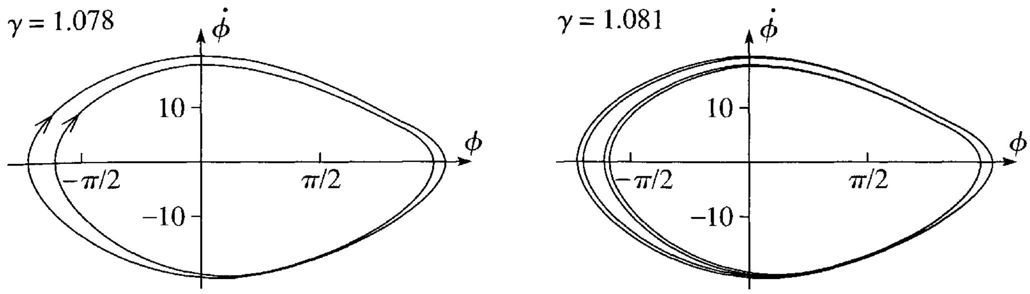
State Space

I shall give a detailed discussion of state space in Chapter 13, but here is a brief explanation of the terminology. For our pendulum, **state space** (also called *phase space*) is the two-dimensional plane defined by the two variables ϕ, the angular position, and $\dot{\phi}$, the angular velocity. This is to be contrasted with the one-dimensional **configuration space** defined by the one variable ϕ that gives the position, or *configuration* of the system. More generally, the configuration space of an n-dimensional mechanical system is the n-dimensional space of its n position coordinates $q_1, \cdots, q_n$, whereas state space is the $2n$-dimensional space of the coordinates $q_1, \cdots, q_n$ *and* velocities $\dot{q}_1, \cdots, \dot{q}_n$. I shall discuss several properties and uses of state space in Chapter 13. Here I mention just one important feature: The "**state**" (or "state of motion" in full) of a mechanical system is often used to mean a specification of the motion (at any chosen time t_0) that is complete enough to determine uniquely the motion at all later times. That is, the state of a system defines the *initial conditions* needed to specify a unique solution of the equation of motion. For our pendulum, specification of the position ϕ at time t_0 is not sufficient to determine a unique solution, but specification of ϕ and $\dot{\phi}$ *is*. That is, the two variables ϕ and $\dot{\phi}$ define the state of the pendulum, and the space of all pairs $(\phi, \dot{\phi})$ is naturally called *state space*.

A **state-space orbit** is simply the path traced in state space by the pair $[\phi(t), \dot{\phi}(t)]$ as time evolves. Natural as this name is, you must recognise that a state-space orbit is very different from the orbit of, say, a planet in ordinary space with coordinates $\mathbf{r} = (x, y, z)$. For example, a planet can have many different orbits passing through a single point $\mathbf{r}$ at a given time t_0. On the other hand, from what was just said about initial conditions, it follows that for any "point" $(\phi, \dot{\phi})$ in state space, our pendulum has exactly one state-space orbit passing through $(\phi, \dot{\phi})$ at any given t_0. Another curious feature of state-space orbits concerns their direction of flow: Since the vertical axis represents the velocity $\dot{\phi}$, the motion at any point above the horizontal axis ($\dot{\phi} > 0$) is always to the right (increasing ϕ), as seen in Figure 12.22. Similarly, the motion at any point below the horizontal axis has to be to the left. If an orbit crosses the horizontal axis then, since $\dot{\phi} = 0$, the orbit must be moving exactly vertically (ϕ not changing). All of these properties are illustrated in Figure 12.22. They imply that any closed state-space orbit, such as the elliptical attractor of Figure 12.22(b), is always traced in a clockwise direction.

More State-Space Orbits

As we increase the drive strength γ, we know that the motion of our DDP undergoes various dramatic changes, some of which show up very nicely in plots of the state-space orbits. For instance, Figure 12.23 shows the state-space orbits for $\gamma = 1.078$ and $\gamma = 1.081$, both in the middle of the period-doubling cascade first shown in Figure 12.8. Both plots show forty cycles starting from $t = 20$, by which time all initial transients have completely died out. That is, both plots show the limiting, long-term motion and are to be compared with Figure 12.22(b). As in that picture, these new orbits move around the origin in more-or-less elliptical loops, but in both cases the



(a) Period 2 (b) Period 4

Figure 12.23 State-space orbits showing the periodic attractors for **(a)** $\gamma = 1.078$ with period 2 and **(b)** $\gamma = 1.081$ with period 4. Both plots show the forty cycles from $t = 20$ to 60. In part (a), the orbit traces just two distinct loops twenty times each; in part (b) it traces four loops ten times each. Compare Figure 12.8 (middle two lines).

orbit makes more than one loop before closing on itself. In part (a) there are two distinct loops, each of which lasts for one drive cycle, so that the motion repeats itself once every two cycles — that is, it has period two. In part (b) there are four distinct loops, indicating very clearly that the period has doubled again to period four. It is important to understand that it makes no difference how many cycles we plot in these two figures [as long as we start after the transients have died out, and plot at least two cycles in part (a) and four in part (b)]. I could have plotted from $t = 20$ to 100 or from 20 to 1000, and part (a) would still have shown the same two loops and part (b) the same four loops.

Chaos

If we increase the drive strength γ a little further, we enter a region of chaos. Figure 12.24 shows the state-space orbit for drive strength $\gamma = 1.105$, whose chaotic character was shown in Figures 12.10 and 12.13. Part (a) shows seven cycles from $t = 14$ to $t = 21$, and you can see clearly that in seven cycles the orbit fails to repeat or to close on itself. Thus if the motion is periodic, its period must be greater than 7. To decide whether it *is* periodic, we need to plot more cycles. In part (b), I have plotted from $t = 14$ to 200, and the plot has become an almost solid swath of black but has still not repeated itself. [The evidence for this last claim is that in a plot out to $t = 400$ (not shown) the curve moves into several of the remaining gaps of part (b); thus it has certainly not begun to repeat by $t = 200$.] Therefore, Figure 12.24 adds strong support to our conclusion that the motion never repeats itself and is in fact chaotic.

The black swath of Figure 12.24(b) is very striking, but is too full of information to be of much use. We need a way to extract from this picture a smaller amount of information that might actually tell us more. The technique for doing this is the so-called Poincaré section, but before we take this up, I want to give two more examples of state-space orbits.

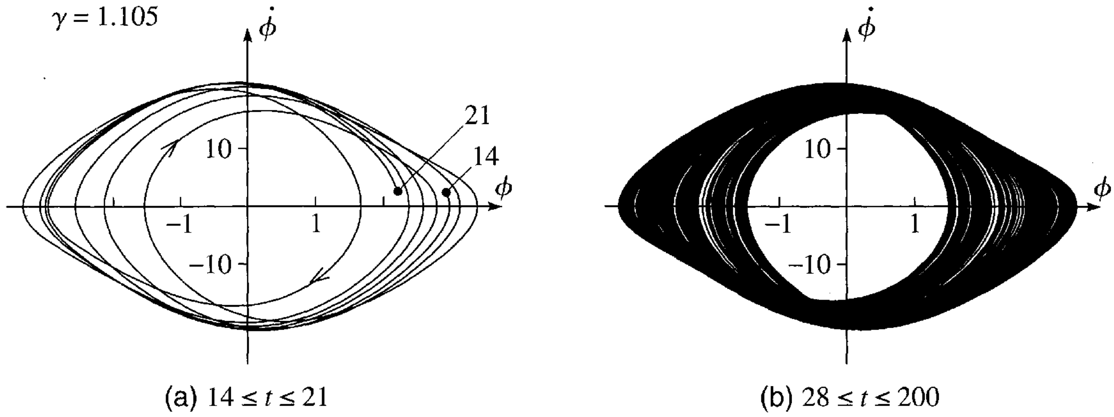


Figure 12.24 State-space orbits for a DDP with $\gamma = 1.105$ showing the chaotic attractor. **(a)** In the seven cycles from $t = 14$ to 21 the orbit does not close on itself. **(b)** The same is true in the 186 cycles from $t = 14$ to 200, and by now it is pretty clear that the motion is never going to repeat itself and is in fact chaotic.

State-Space Orbits for Rolling Motion

We have already seen that for $\gamma = 1.4$ our DDP executes a "rolling motion," making a complete clockwise rotation once each drive cycle (Figure 12.19). The state-space orbit for this motion is shown in Figure 12.25. In this plot you can see clearly how, after a couple of cycles, the pendulum settles down to a periodic motion in which ϕ decreases by 2π and the pendulum makes a complete clockwise rotation once per cycle.

The plot of Figure 12.25 is a very satisfactory way of showing the state-space orbit over a small number of cycles. Sometimes, however, (if the motion is chaotic, for instance) one would like to show the orbit over a very long time interval — several hundred cycles, perhaps — and in this case ϕ may range over many hundreds of complete revolutions. To show this, in the format of Figure 12.25, we would be forced to compress the scale on the ϕ axis to the point where the motion would be completely indecipherable. The usual way around this difficulty is to redefine ϕ so that it always

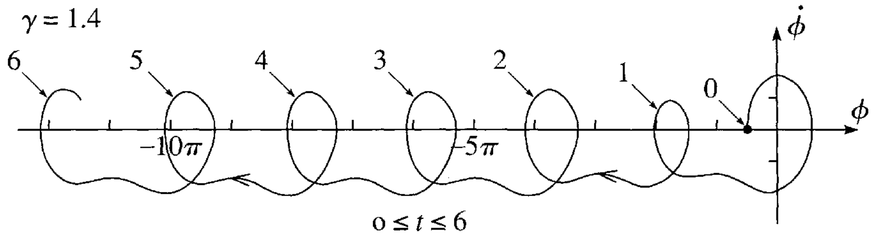


Figure 12.25 First six cycles of the state-space orbit for a DDP with $\gamma = 1.4$, showing the periodic rolling motion, in which ϕ decreases by 2π in each cycle. The numbers $0, 1, \cdots, 6$ indicate the state-space "position" $(\phi, \dot\phi)$ at the times $t = 0, 1, \cdots, 6$.

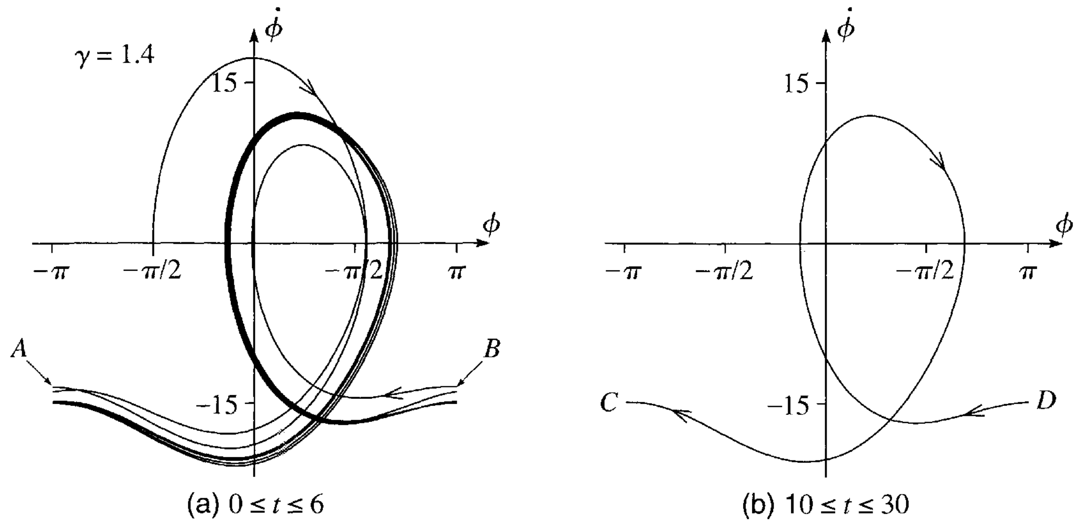


Figure 12.26 **(a)** The exact same orbit as in Figure 12.25 but with ϕ redefined so that it remains between $-\pi$ and π. Each time ϕ decreases to $-\pi$ (at A for example), the orbit disappears and reappears at $+\pi$ (at B for example). **(b)** By the time $t = 10$ the orbit has settled down to perfectly periodic motion, with each successive cycle lying exactly (on this scale) on top of its predecessor.

lies between $-\pi$ and π: Each time ϕ decreases past $-\pi$ we add on 2π and each time it increases past π we subtract 2π. (This is acceptable since any two values of ϕ that differ by a multiple of 2π represent the same position of the pendulum.) With ϕ redefined in this way, the state-space orbit of Figure 12.25 looks as shown in Figure 12.26(a). This new plot is not an obvious improvement on Figure 12.25 (though we shall see that it does have some advantages), but you should study it carefully to understand the relationship of the two kinds of picture. You can think of the new picture as being obtained from Figure 12.25 by cutting apart the intervals $-3\pi < \phi < -\pi$, and $-5\pi < \phi < -3\pi$, and so on, and pasting them all back on top of the interval $-\pi < \phi < \pi$. In the resulting picture, ϕ makes a discontinuous jump each time it arrives at $\phi = \pm\pi$. For example, at about $t = 0.7$, ϕ decreases to $-\pi$ at the point A and jumps to the point B.

An advantage of Figure 12.26(a) over Figure 12.25 is that the new picture gives a more incisive test of the periodicity of the orbit. In the new picture, you can see that the orbit is approaching a periodic attractor, but it is also clear that in the interval $0 \le t \le 6$ the orbit has definitely not *reached* the periodic attractor. (In fact you can just about see that there are 6 distinct loops.) On the other hand, by the time $t = 10$ the successive cycles are indistinguishable on the scale of these pictures. The twenty cycles shown in Figure 12.26(b) all disappear on the left at the same point C, reappear at D, and follow the exact same path back to C twenty times over.

A disadvantage of either plot in Figure 12.26 is the spurious discontinuity each time ϕ jumps from $-\pi$ to π, as at points A and B, for example. We can get rid of these discontinuities (at least in our minds) if we imagine the plot cut out and rolled into a cylinder with the vertical lines $\phi = \pm\pi$ glued together. In this way point A becomes the same as point B, and the state-space orbit moves continuously around the vertical cylinder.

More Chaos

As a final example of a state-space orbit, I have shown the orbit for a DDP with $\gamma = 1.5$ in Figure 12.27. We already know that the motion is chaotic for this value of γ, though for this picture I chose a smaller damping constant, $\beta = \omega_0/8$ instead of the value $\beta = \omega_0/4$ that I used for all previous pictures in this chapter. As it turns out, this smaller damping makes the chaotic motion more wild and produces a Poincaré section that is even more interesting and elegant (as I describe in the next section). With these parameters, the pendulum undergoes an erratic rolling motion, making many complete revolutions, first in one direction and then in the other. Thus we are forced to make a plot with ϕ confined between $-\pi$ and π as in Figure 12.26 — but with dramatically different results. The motion does not repeat itself in the 190 cycles shown, with $10 \le t \le 200$. (The evidence for this claim is that in a plot for $10 \le t \le 250$ — not shown — the orbit moves into some of the unvisited regions of Figure 12.27. If it had begun to repeat itself before $t = 200$, it could not visit new ground after $t = 200$.)

The dense tangle of threads running through Figure 12.27 lend strong support to the claim that for these parameters the motion is chaotic. Unfortunately, one could not claim that the picture sheds much light on the nature of chaotic motion. It is just too densely packed with information to convey any useful message. In the next section I describe the Poincaré section, which is a technique for culling out of pictures like Figure 12.27 enough information to allow an interesting pattern to emerge.

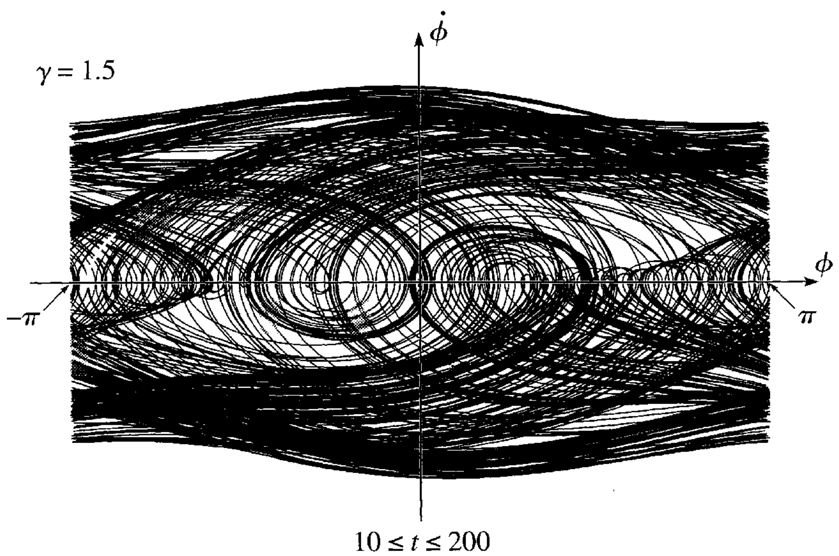


Figure 12.27 The chaotic state-space orbit for a DDP with $\gamma = 1.5$ and $\beta = \omega_0/8$. In the 190 cycles shown, the motion does not repeat itself, and, in fact, it never does.