12.8 Poincaré Sections

For the periodic motion of a DDP, a state-space orbit is a descriptive way of viewing the pendulum's history. For chaotic motion, a state-space orbit conveys a sense of the dramatic nature of chaos, but is too full of information to be of much serious use. One way around this difficulty is a trick which we have used earlier and was suggested by Poincaré: Instead of following the motion as a function of the continuous variable t, we look at the position just once per cycle at times $t = t_o, t_o + 1, t_o + 2, \cdots$. The **Poincaré section** for a DDP is just a plot showing the pendulum's "position" $[\phi(t), \dot{\phi}(t)]$ in state space at one-cycle intervals

$$t = t_o,\ t_o + 1,\ t_o + 2, \cdots, \tag{12.30}$$

with t_o usually chosen so that the initial transients have died out.[20] To illustrate this, consider the state-space orbit shown in Figure 12.28(a) for a DDP with $\gamma = 1.078$ (and the damping constant restored to our usual value $\beta = \omega_o/4$). The two loops of this orbit indicate that (as we already knew) the long-term motion has period two. To emphasize this I have drawn dots to show the position $[\phi(t), \dot{\phi}(t)]$ at one-cycle intervals, $t = 20, 21, 22, \cdots$. Since the motion has period two, these alternate between just two distinct positions and show up as just two dots. In a Poincaré section one dispenses with the orbit and draws just the dots at one-cycle intervals as in Figure 12.28(b). When the motion is periodic, there is no particular advantage to the Poincaré section over the complete state-space orbit of Figure 12.28(a), although the Poincaré section does show the period very clearly. (A Poincaré section with four dots would show period-four motion, and so on.) On the other hand, when the motion is chaotic,

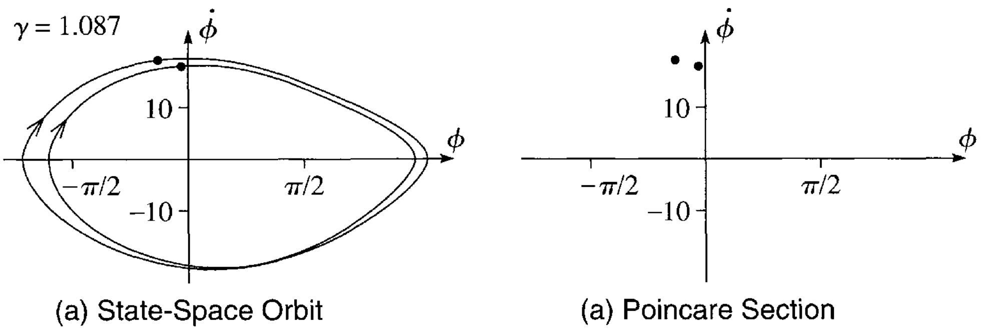

Figure 12.28 **(a)** State-space orbit of a DDP with $\gamma = 1.078$ for $20 \leq t \leq 60$. The dots show the positions at $t = 20, 21, 22, \cdots$, but, since the motion has period two, these alternate between just two fixed points. The right point shows the positions for $t = 20, 22, \cdots$; the left one, those for $t = 21, 23, \cdots$. **(b)** In the corresponding Poincaré section, one omits the orbit and draws only the dots showing the positions at $t = 20, 21, 22, \cdots$. The presence of just two dots is a clear indication of period-two motion.

[20] For a multidimensional system, the Poincaré section involves taking a two-dimesional slice, or *section*, through the multidimensional state space. Hence the word "section."

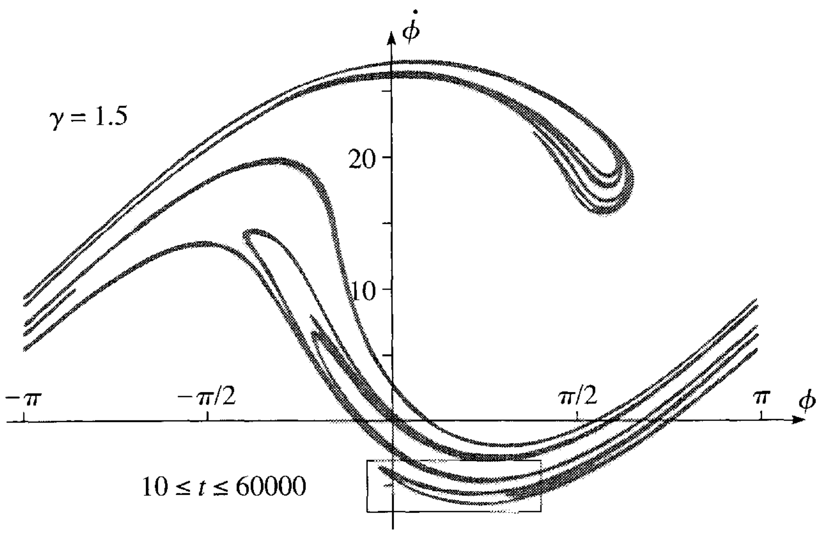

Figure 12.29 Poincaré section for a pendulum with $\gamma = 1.5$ and damping constant $\beta = \omega_0/8$ for times $10 \le t \le 60000$. This figure is made of nearly 60000 points showing the "position" $[\phi(t), \dot{\phi}(t)]$ at one-cycle intervals, $t = 10, 11, \cdots, 60000$. The rectangular box indicates the region that is enlarged in Figure 12.30.

no two cycles of the motion are the same, and the state-space orbit can be a real mess, as we saw in Figure 12.27. In this case, the Poincaré section reveals some totally unexpected structures.

To illustrate a Poincaré section for chaotic motion, I chose the pendulum whose chaotic state-space orbit was shown in Figure 12.27. It is clear that, since this motion never repeats itself, the Poincaré section will contain infinitely many points, and these infinitely many points will comprise a subset of the points of the full orbit. It is probably fair to say that no one could ever *guess* what this subset would look like, but with the aid of a high-speed computer we can find out. The result is shown in Figure 12.29. Although it is certainly not obvious exactly what this elegant figure signifies, it certainly *is* obvious that it signifies something. By selecting from Figure 12.27 just those points at one-cycle intervals, we have reduced the dense, and nearly solid, tangle of Figure 12.27 to the elegant curve of Figure 12.29. Actually, while Figure 12.29 *looks* like a relatively simple curve, it is not a curve at all, but rather a **fractal**. A fractal can be defined in various ways, but a characteristic feature of fractals is that when one enlarges the scale and zooms in on a portion of the picture, one uncovers further structures that are in some ways similar to the original picture (somewhat like a photograph of a person holding a photograph of a person holding a photograph . . . and so on). To illustrate this property of our Poincaré section, I have zoomed in on the region indicated by the rectangular box at the bottom of Figure 12.29. Notice that (at the scale of Figure 12.29) this region comprises a prominent "tongue" pointing to the left near the left of the box, with a second tongue inside the first near the right of the box. Figure 12.30 is a fourfold enlargement of this box. This enlargement makes clear that the apparently single tongue on the right of the box of 12.29 is actually four tongues, while that on the left is actually at least five.

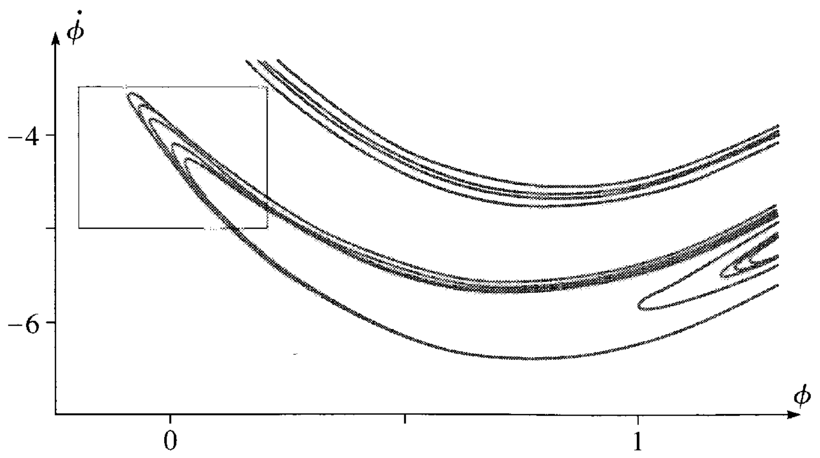

Figure 12.30 Enlargement of the small box at the bottom of the Poincaré section of Figure 12.29. Each of the two tongues in the box of 12.29 is seen to be made up of several tongues. The box on the left is the region that is further enlarged in Figure 12.31.

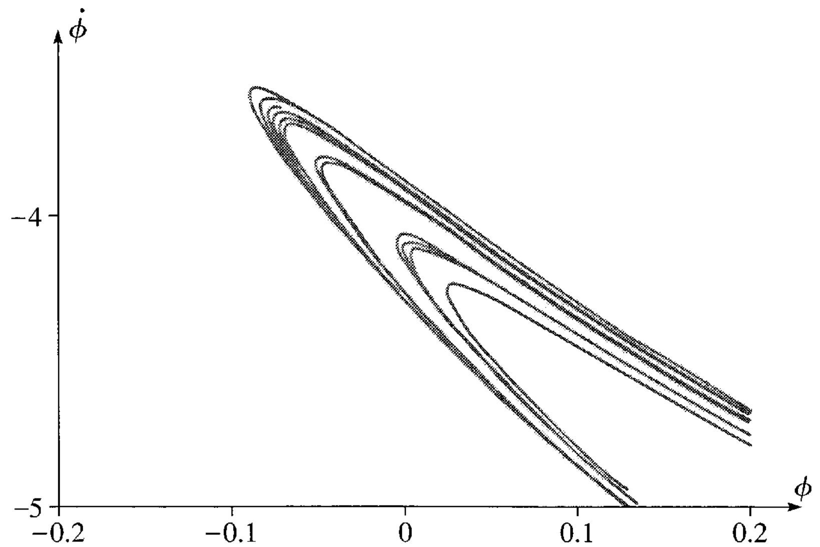

Figure 12.31 A further enlargement of the box at the left of the enlargement of Figure 12.30. Each of the five tongues in the box of 12.30 (except perhaps the innermost one) is seen to be made up of several tongues.

This process of zooming in on successively smaller regions of the Poincaré section can, at least in principle, be continued indefinitely. Figure 12.31 is a further fourfold enlargement of the region shown by the gray rectangle on the left of Figure 12.30. In this enlargement, we see that each of the five tongues on the left of Figure 12.30 (except perhaps the fifth one) actually consists of several separate tongues. This so-called **self similarity** of the figure is one of the characteristic features of a fractal.

When the Poincaré section of the motion of a chaotic system is a fractal, the long-term motion is said to be a **strange attractor**. It would unfortunately be well beyond the scope of this book to explain what it signifies that the Poincaré section of a chaotic attractor is fractal, and indeed there is still much about this phenomenon that is not

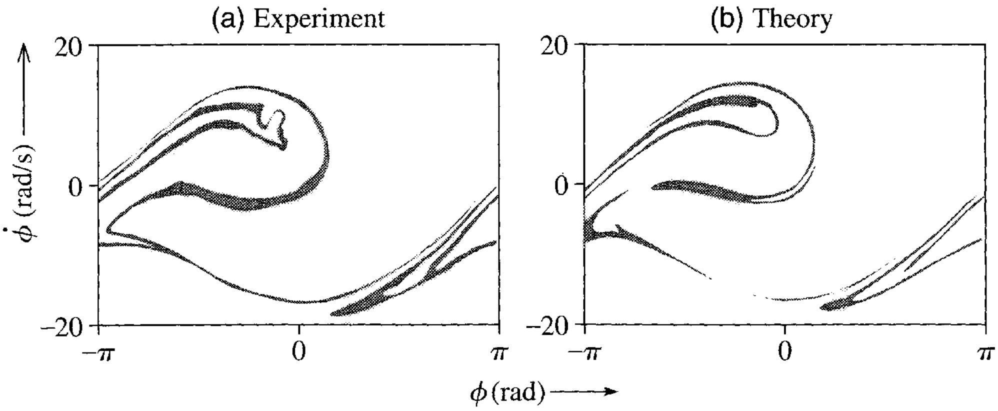

Figure 12.32 Poincaré section for a DDP. **(a)** Experimental results using the Daedalon Chaotic Pendulum. **(b)** Theoretical prediction using the same parameters as in part (a). Courtesy of Professors H.J.T. Smith and James Blackburn and the Daedalon Corporation.

understood. Nevertheless, it is undeniably fascinating that the strange geometrical structure of the fractal appears in our study of the long-term behavior of chaotic systems. This discovery has stimulated much research on both the physics of chaotic systems and the mathematics of fractals.

To observe a strange attractor with a real pendulum would obviously be challenging, but once again the experimentalists have risen to the challenge. Figure 12.32 shows a Poincaré section made with the Daedalon chaotic pendulum. Part (a) shows the experimental results and part (b) the theoretical prediction (that is, a numerical solution of the equation of motion using the experimental values of the parameters). Considering the great subtlety of these graphs, the agreement is outstanding.[21]

12.9 The Logistic Map

As I have repeatedly emphasized, the phenomenon of chaos appears in many different situations. In particular, there are certain systems that can exhibit chaos, but whose equations of motion — called *maps* — are simpler than the equations of any mechanical system. Although these systems are not strictly part of classical mechanics, they are worth mentioning here, for several reasons: Because their equations of motion are simple, several aspects of their motion can be understood using quite elementary methods. Any understanding of chaos that we get from studying these simpler systems can shed light on the corresponding behavior of mechanical systems. In particular, there is an intimate connection between these "maps" and the Poincaré sections of mechanical systems. Finally, a discussion of chaos in this new context highlights the diversity of systems that exhibit the phenomenon.

[21] There are, nevertheless, differences. One possible cause is the difficulty of making a drive motor that is perfectly sinusoidal.