decision. Many important applications of chaos theory (in astronomy and statistical mechanics, for instance) concern nondissipative systems, as did the pioneering work of Poincaré. Nevertheless, it is my experience that the most accessible topics for the beginner concern dissipative systems, and, to keep the length of this chapter reasonably finite, I decided to treat only these. Please see this chapter as a sampler of the good things in chaos theory; it certainly makes no claim to tell the whole story.

This chapter is very different from all the other chapters of this book. The theory of chaos is new and not at all elementary. (And parts of the theory have yet to be discovered!) It requires a much deeper understanding of differential equations than I am assuming in this book, and a proper exposition of chaos theory requires a whole book rather than one chapter.[1] Therefore, I shall restrict myself here to simply *describing* the fascinating main properties of chaotic motion, without much attempt to prove that the motion is as I claim. This is actually a reasonably satisfactory situation. Before you try to read any of the more advanced books, it is almost certainly a good thing to have some idea of what chaos involves and some familiarity with the tools used to describe it, and these are what I hope to communicate.

12.1 Linearity and Nonlinearity

For a system to exhibit chaos its equations of motion must be *nonlinear*. We have noted examples of linear and nonlinear equations from time to time in this book, but let us review the two concepts now. A differential equation is linear if it involves the dependent variable or variables and their derivatives only linearly. The equation of motion of a cart (mass m) on a spring (force constant k),

$$m\ddot{x} = -kx, \tag{12.1}$$

is a linear differential equation for the cart's position x. Similarly, the equations of motion for the two carts discussed in Chapter 11 [Equations (11.2) for example] are linear equations for the two carts' positions x_1 and x_2. If we apply a driving force $F(t)$ to the cart of Equation (12.1), the resulting equation,

$$m\ddot{x} = -kx + F(t), \tag{12.2}$$

is still linear [though no longer homogeneous, since the "inhomogeneous" term $F(t)$ does not involve the dependent variable x at all]. By contrast, the equation of motion for a simple pendulum (mass m, length L) is $I\ddot{\phi} = \Gamma$ or

$$mL^2\ddot{\phi} = -mgL\sin\phi, \tag{12.3}$$

which is a *nonlinear* equation for ϕ, since $\sin\phi$ is not linear in ϕ. (If the oscillations are small, then $\sin\phi \approx \phi$, and the equation is well approximated by a linear equation;

[1] Several such books exist, of which my favorite is *Nonlinear Dynamics and Chaos* by Steven H. Strogatz, Addison-Wesley, Reading, MA (1994), but be warned, it takes eight chapters of mathematical preliminaries to get to the chaos.

in general, however, the equation for the simple pendulum is definitely nonlinear.) Another example is the equation of motion of a single planet in the field of the sun,

$$m\ddot{\mathbf{r}} = -GmM\hat{\mathbf{r}}/r^2, \tag{12.4}$$

which is a nonlinear equation for the variables $\mathbf{r} = (x, y, z)$ because the force term is nonlinear in x, y, and z. These two examples show that nonlinear equations are not especially unusual. On the contrary, many perfectly everyday systems have equations of motion that are nonlinear.

So far in this book the main difference between linear and nonlinear differential equations has been that the former have been easily solved analytically, whereas most of the latter have been *impossible* to solve analytically. In fact, our experience in this regard reflects the true state of affairs: Almost all of the linear equations of mechanics *are* analytically solvable, and almost none of the nonlinear ones are.[2] This circumstance is largely to blame for the failure until recently of scientists to recognize that chaos is an important and widespread phenomenon. Because nonlinear equations are so intractable, textbooks always focussed on linear problems. When nonlinear problems *had* to be addressed, they were often solved using approximations that reduced then to linear problems. In this way, the astonishingly rich variety of complications that occur for nonlinear systems went almost completely unrecognized. The first person to notice some of the symptoms of chaos was the French mathematician Poincaré (1854–1912) in his studies of the gravitational three-body problem — the motion of three bodies (such as the sun, earth, and moon) interacting via the gravitational force. The equation of motion for this system is nonlinear, like its two-body counterpart (12.4), and Poincaré observed that it exhibits the phenomenon now called *sensitivity to initial conditions* that is one of the characteristics of chaotic motion, as we shall see.

That Poincaré's observation of chaos went nearly unnoticed by physicists until the 1970s is probably due to several factors. The discoveries of relativity (1905) and then of quantum mechanics (around 1925) diverted most physicists' attention away from classical mechanics. And the difficulty of solving nonlinear equations without the aid of computers certainly discouraged the pursuit of nonlinear problems. In any case it was only in the 1970s that computer solutions of various nonlinear problems[3] drew the attention of significant numbers of scientists (physicists and many others) to the phenomenon that we now call chaos.

Nonlinearity is essential for chaos — if a system's equations of motion are linear, it cannot exhibit chaos. But nonlinearity does not guarantee chaos. For example, the equation (12.3) for a simple pendulum is nonlinear, but even when the amplitude is large (and the linear approximation is definitely not good) the simple pendulum never

[2] One of the rare examples of a solvable nonlinear equation is (12.4) for a planet, whose orbit we found in Chapter 8. But notice that we did this by a cunning change of variables that reduced the nonlinear equation (8.37) for r to the linear equation (8.45) for u.

[3] The first such calculation, of atmospheric convection, was made by the meteorologist Edward Lorenz at MIT in 1963, but this work did not attract widespread attention for another decade. For an exhaustive, but very readable, history of chaos theory see *Chaos, Making a New Science* by James Gleick, Viking-Penguin, New York (1987).

exhibits chaos. On the other hand, if we add in a damping force $-bv = -bL\dot{\phi}$ and a driving force $F(t)$, (12.3) becomes the equation of the *driven, damped pendulum*:

$$mL^2\ddot{\phi} = -mgL\sin\phi - bL^2\dot{\phi} + LF(t) \tag{12.5}$$

and this equation *does* lead to chaos for some values of the parameters. Loosely speaking, the requirement for chaos is that the equations of motion be nonlinear and *somewhat complicated*. Equation (12.3) for the simple pendulum is not sufficiently complicated, but Equation (12.5) for the driven damped pendulum is. Unfortunately, a discussion of precisely what is "sufficiently complicated" to produce chaos would be well beyond the scope of this book.[4]

Another relatively simple example of a nonlinear system that exhibits chaos is the double pendulum of Section 11.4. In the small angle approximation, the equations of motion for the double pendulum are linear equations for the two angles ϕ_1 and ϕ_2 [see (11.41) and (11.42)], but in general they are nonlinear (see Problem 11.15), and they are sufficiently complicated to produce chaos. The driven damped pendulum and the double pendulum are two of the simplest mechanical systems that exhibit chaos. The driven damped pendulum has just one degree of freedom (one coordinate, ϕ, needed to specify its configuration), whereas the double pendulum has two (two coordinates, ϕ_1 and ϕ_2, needed). For this reason the driven damped pendulum is the simpler one to analyze and will be the main focus of our discussion here.

What's Special about Nonlinearity?

In the enormous set of all possible differential equations, the linear equations form a miniscule subset, with many simple properties that are not shared by the general nonlinear equation. Thus it is really the linear equations which are "special." Nevertheless, for the reasons already mentioned, many physicists are much more conversant with the linear case, and they are sometimes tempted to assume that familiar properties of linear equations will carry over to nonlinear equations. This dangerous assumption is frequently wrong. In particular, the main message of this chapter is that chaos, which never appears in linear systems, is a common occurrence in nonlinear systems. Unfortunately, the underlying theory of this particular difference is beyond the scope of this book, and we shall have to be content with seeing some simple examples of chaotic motion, without a detailed understanding of *why* chaos occurs. Here, I would like to mention just one huge difference between linear and nonlinear equations that high-

[4] For the record, the criterion is this: As we shall see in Chapter 13, a set of second-order differential equations (like Newton's second law) for n variables can usually be rewritten as a set of first-order equations for N variables, $\xi_1, \cdots, \xi_N$ where $N > n$, with the general form $\dot{\xi}_i = f_i(\xi_1, \cdots, \xi_N)$ for $i = 1, \cdots, N$. For instance, if we write $\dot{\phi} = \omega$, the one equation (12.3) for the angle ϕ of the simple pendulum becomes two first-order equations, one for ϕ and one for ω, namely $\dot{\phi} = \omega$ and $\dot{\omega} = -(g/L)\sin\phi$. When the right-hand sides of these equations are independent of t (as they are here) the equations are said to be *autonomous*. For a dissipative system to exhibit chaos, its equations of motion, when put in this standard autonomous form, must be nonlinear and have N variables with $N \geq 3$. Nondissipative systems need nonlinearity and $N \geq 4$.

lights the importance of not letting the linear prejudice that many of us share mislead us in our study of nonlinear equations.

Nonlinear Equations Don't Obey the Superposition Principle

We saw in Chapter 5 that linear homogeneous equations satisfy the superposition principle — that any linear combination of solutions gives us another solution. We have used this result several times, particularly in Chapters 5 and 11, but let me refresh your memory with the example of a second-order equation of the form

$$p(t)\ddot{x}(t) + q(t)\dot{x}(t) + r(t)x(t) = 0, \tag{12.6}$$

where $x(t)$ is the unknown and $p(t)$, $q(t)$, and $r(t)$ are known fixed functions. [An example of such an equation is (12.1) for a cart on a spring.] Notice first that, because every term in this equation is linear in $x(t)$ (or its derivatives), we can multiply through by any constant a and see at once that if $x(t)$ is a solution, then so is $ax(t)$. Second, if $x_1(t)$ and $x_2(t)$ are both solutions, then we can add the two corresponding equations, one for $x_1(t)$ and one for $x_2(t)$, and conclude that $x_1(t) + x_2(t)$ is also a solution. Thus any linear combination

$$x(t) = a_1x_1(t) + a_2x_2(t)$$

is also a solution of (12.6) — the result called the superposition principle. On the other hand, it is easy to see that neither of the arguments just given works if the equation is nonlinear. [Make sure you can see this. Suppose, for example, the last term in (12.6) was $r(t)\sqrt{x(t)}$; see Problem 12.3.] Therefore, the superposition principle does not apply to nonlinear equations.

An important consequence of the superposition principle that we have used repeatedly is this : To find all the solutions of (12.6) we have only to find two independent solutions $x_1(t)$ and $x_2(t)$; then *every* solution can be expressed as a linear combination of $x_1(t)$ and $x_2(t)$. More generally, to find all the solutions of an nth order homogeneous linear differential equation, we have only to find n independent solutions and then every solution can be expressed as a linear combination of these n solutions. Since the superposition principle does not apply to nonlinear equations, this dramatic simplification does not apply to nonlinear equations.

There is a corresponding situation for inhomogeneous equations, such as (12.2) and (12.5), again as we saw in Chapter 5. If $x_p(t)$ is any one particular solution of a *linear* nth order inhomogeneous equation, then *every* solution can be written as $x_p(t)$ plus a linear combination of n independent solutions of the corresponding homogeneous equation. For nonlinear equations, there is no corresponding result (Problem 12.4). Thus every solution of any nth order linear equation (homogeneous or inhomogeneous) can be expressed simply in terms of n independent functions, but for nonlinear equations there is no such simple expression.

With these general observations on nonlinear equations, let us take up the one nonlinear equation that we shall discuss in detail, Equation (12.5) for a driven damped