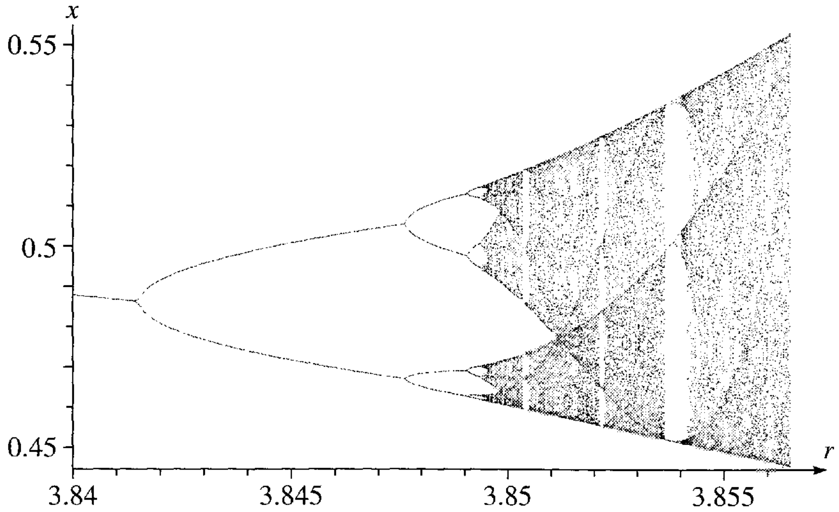

Figure 12.44 A many-fold enlargement of the small rectangle in the logistic bifurcation diagram of Figure 12.41. This tiny section of the original diagram is a perfect, upside-down copy of the whole original. Note that this section is just one of three strands in the original; thus, although this diagram starts out *looking* just like period 1 doubling to period 2, it is actually period 3 doubling to period 6 and so on.

Principal Definitions and Equations of Chapter 12

The Driven Damped Pendulum

A damped pendulum that is driven by a sinusoidal force $F(t) = F_o \cos(\omega t)$ satisfies the nonlinear equation

$$\ddot{\phi} + 2\beta\dot{\phi} + \omega_o^2 \sin\phi = \gamma\omega_o^2 \cos\omega t \qquad \text{[Eq. (12.11)]}$$

where $\gamma = F_o/mg$ is called the **drive strength** and is the ratio of the drive amplitude to the weight.

Period Doubling

For small drive strengths, ($\gamma \lesssim 1$) the long-term response, or **attractor**, of the pendulum has the same period as the drive force. But if γ is increased past $\gamma_1 = 1.0663$, for certain initial conditions and drive frequencies, the attractor undergoes a **period-doubling cascade**, in which the period repeatedly doubles, approaching infinity as $\gamma \to \gamma_c = 1.0829$. [Section 12.4]

Chaos

If the drive strength is increased beyond γ_c, at least for certain choices of drive frequency and initial conditions, the long-term motion becomes nonperiodic, and we say that **chaos** has set in. As γ is increased still further, the long-term motion varies, sometimes chaotic, sometimes periodic. [Sections 12.5 & 12.6]

Sensitivity to Initial Conditions

Chaotic motion is **extremely sensitive to initial conditions**. If two identical chaotic pendulums with identical drive forces are launched with slightly different initial conditions, their separation increases exponentially with time, however small the initial difference. [Section 12.5]

Bifurcation Diagrams

A **bifurcation diagram** is a plot of the system's position at discrete times, t_o, $t_o + 1$, $t_o + 2, \cdots$ (more generally t_o, $t_o + \tau$, $t_o + 2\tau, \cdots$) as a function of the drive strength (more generally the appropriate control parameter). [Figures 12.17 & 12.18]

State-Space Orbits and Poincaré Sections

The **state space** for a system with n degrees of freedom is the $2n$-dimensional space comprising the n generalized coordinates and the n generalized velocities. For the DDP, the points in state space have the form $(\phi, \dot{\phi})$. A **state-space orbit** is just the path traced in state space by a system as t evolves. A **Poincaré section** is a state-space orbit restricted to discrete times t_o, $t_o + 1$, $t_o + 2, \cdots$ (and, when $n > 2$, to a subspace of fewer dimensions). [Sections 12.7 & 12.8]

The Logistic Map

The **logistic map** is a function (or "map") that gives a number x_t at regular discrete intervals (for example, the relative population of a certain bug once each year) as

$$x_{t+1} = rx_t(1 - x_t). \qquad \text{[Eq. (12.42)]}$$

Although this is not a mechanical system, it exhibits many of the features (period doubling, chaos, sensitivity to initial conditions) of nonlinear mechanical systems. [Section 12.9]

Problems for Chapter 12

Stars indicate the approximate level of difficulty, from easiest (⋆) to most difficult (⋆⋆⋆).
Warning: Even when the motion is nonchaotic, it can be very sensitive to tiny errors. In several of the computer problems you may need to increase your working precision to get satisfactory results.

SECTION 12.1 Linearity and Nonlinearity

12.1 ⋆ Consider the nonlinear first-order equation $\dot{x} = 2\sqrt{x - 1}$. **(a)** By separating variables, find a solution $x_1(t)$. **(b)** Your solution should contain one constant of integration k, so you might reasonably expect it to be the general solution. Show, however, that there is another solution, $x_2(t) = 1$, that is