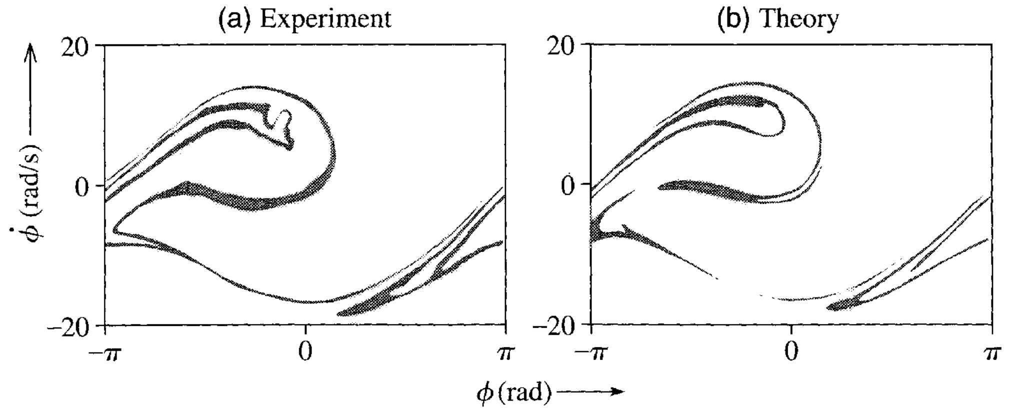

Figure 12.32 Poincaré section for a DDP. **(a)** Experimental results using the Daedalon Chaotic Pendulum. **(b)** Theoretical prediction using the same parameters as in part (a). Courtesy of Professors H.J.T. Smith and James Blackburn and the Daedalon Corporation.

understood. Nevertheless, it is undeniably fascinating that the strange geometrical structure of the fractal appears in our study of the long-term behavior of chaotic systems. This discovery has stimulated much research on both the physics of chaotic systems and the mathematics of fractals.

To observe a strange attractor with a real pendulum would obviously be challenging, but once again the experimentalists have risen to the challenge. Figure 12.32 shows a Poincaré section made with the Daedalon chaotic pendulum. Part (a) shows the experimental results and part (b) the theoretical prediction (that is, a numerical solution of the equation of motion using the experimental values of the parameters). Considering the great subtlety of these graphs, the agreement is outstanding.[21]

12.9 The Logistic Map

As I have repeatedly emphasized, the phenomenon of chaos appears in many different situations. In particular, there are certain systems that can exhibit chaos, but whose equations of motion — called *maps* — are simpler than the equations of any mechanical system. Although these systems are not strictly part of classical mechanics, they are worth mentioning here, for several reasons: Because their equations of motion are simple, several aspects of their motion can be understood using quite elementary methods. Any understanding of chaos that we get from studying these simpler systems can shed light on the corresponding behavior of mechanical systems. In particular, there is an intimate connection between these "maps" and the Poincaré sections of mechanical systems. Finally, a discussion of chaos in this new context highlights the diversity of systems that exhibit the phenomenon.

[21] There are, nevertheless, differences. One possible cause is the difficulty of making a drive motor that is perfectly sinusoidal.

Discrete Time and Maps

In almost all problems of mechanics, one is concerned with the evolution of a system as time advances continuously. However, there are systems for which time is a discrete variable. The history of any event that occurs just once a year, such as the Super Bowl, is an example. The score at Super Bowl games is defined only at a sequence of discrete times starting in 1967 and spaced at one-year intervals:

$$t = 1967, 1968, 1969, \cdots. \tag{12.31}$$

The attendance at your weekly lunch group is defined only for discrete times spaced a week apart. The total rainfall in the annual Indian monsoon is defined only for discrete times spaced a year apart.

Even when a variable is defined as a function of continuous time, we may find that we need its values only at certain discrete times. For example, entymologists studying the population of a particular bug may have no interest in the population's day-by-day evolution; rather they may need to record the bug population just once a year, immediately after the year's new arrivals have hatched. Another example of this situation is the Poincaré section for a mechanical system, as described in Section 12.8. Our ultimate interest is to know the state of the system for all (continuous) times, but we saw that it is sometimes useful to record just the state at discrete, one-cycle intervals. To the extent that we are prepared to settle for this smaller amount of information, the Poincaré section reduces the problem of the pendulum (or whatever) to a discrete-time problem, and anything we can learn about discrete-time systems should shed light on the possible behavior of Poincaré sections.

In the case of the driven damped pendulum, we know that the state of the system, as given by the pair $[\phi(t), \dot{\phi}(t)]$, at any time t determines uniquely the state at any later time. In particular, it determines the state $[\phi(t+1), \dot{\phi}(t+1)]$ one cycle later. This means that there exists a function f (which we don't know, but which certainly exists) that, acting on any chosen pair $[\phi(t), \dot{\phi}(t)]$, gives the corresponding pair $[\phi(t+1), \dot{\phi}(t+1)]$. That is,

$$[\phi(t+1), \dot{\phi}(t+1)] = f\left([\phi(t), \dot{\phi}(t)]\right). \tag{12.32}$$

In the same way, we could imagine a bug species with the property that the population n_{t+1} in year $t+1$ is uniquely determined[22] by the population n_t in the preceding year t. Again this would imply the existence of a function f that carries any n_t onto its corresponding n_{t+1}:

$$n_{t+1} = f(n_t). \tag{12.33}$$

[22] This is, of course, a very simplified model. In the real world, the population n_{t+1} certainly depends on n_t, but also on many other factors, such as the rainfall in year t, the supply of bug food, and the population of birds that like to eat the bug. Nevertheless, we can imagine a temperate island, with a constant supply of bug food and no bug predators, on which n_{t+1} is uniquely determined by n_t alone.

We can call an equation of this form the **growth equation** for the population concerned.

EXAMPLE 12.1 Exponential Population Growth

The simplest example of a growth equation of the type (12.33) is the case where n_{t+1} is proportional to n_t:

$$n_{t+1} = f(n_t) = rn_t. \tag{12.34}$$

That is, the function $f(n)$ that gives next year's population in terms of this year's is just

$$f(n) = rn \tag{12.35}$$

where the positive constant r could be called the **growth rate** or **growth parameter** of the population. [For example if every bug alive this spring dies before next spring but leaves two surviving offspring, then the population would satisfy (12.34) with $r = 2$.] Solve the equation (12.34) for n_t in terms of n_0 and discuss the long-term behavior of n_t.

The solution to (12.34) is easily seen by inspection. Observe that

$$n_1 = f(n_0) = rn_0$$

and

$$n_2 = f(n_1) = f(f(n_0)) = r^2 n_0$$

from which it is clear that

$$n_t = f(n_{t-1}) = \overbrace{f(f(\cdots f}^{t \text{ terms}}(n_0)\cdots)) = r^t n_0. \tag{12.36}$$

We see that if $r > 1$, the population n_t grows exponentially, approaching infinity as $t \to \infty$. If $r = 1$, the population stays constant, and if $r < 1$ it decreases exponentially to zero.

Before we discuss a more interesting growth equation, I need to introduce some terminology. In mathematics, the words "function" and "map" are used as almost exact synonyms. Thus we can say that Equation (12.33) defines n_{t+1} as a *function* of n_t. Or we can say that (12.33) is a **map** in which f carries n_t onto the corresponding[23] n_{t+1}, a relationship that we can represent thus:

$$n_t \xrightarrow{f} n_{t+1} = f(n_t). \tag{12.37}$$

[23] The origin of this rather strange usage seems to be in cartography. A cartographer's map of the US, for example, establishes a correspondence between each actual point of the US and the corresponding point on a piece of paper, in somewhat the same way the function $y = f(x)$ establishes a correspondence between each value x and the corresponding value $y = f(x)$.

The whole sequence of numbers, $n_0, n_1, n_2, \cdots$ can be written similarly as

$$n_0 \xrightarrow{f} n_1 \xrightarrow{f} n_2 \xrightarrow{f} n_3 \xrightarrow{f} \cdots \tag{12.38}$$

and is naturally described as an **iterated map**, or just **map** for short. For some reason the word "map" (as opposed to "function") is used almost universally to describe relationships like (12.37) between successive values of any discrete-time variable. The map (12.37) is a *one-dimensional map* since it carries the single number n_t onto the single number n_{t+1}. The corresponding relationship (12.32) for the Poincaré section of a DDP defines a *two-dimensional map*, since it carries the pair of numbers $[\phi(t), \dot{\phi}(t)]$ onto the pair $[\phi(t+1), \dot{\phi}(t+1)]$.

The Logistic Map

The exponential map of Example 12.1 with $r > 1$ is a reasonably realistic model for the initial growth of many populations, but no real population can grow exponentially for ever. Something — overcrowding or shortage of food, for example — eventually slows down the growth. There are many ways to modify the map (12.34) to give a more realistic model of population growth. One of the simplest is to replace the function $f(n) = rn$ of (12.34) by

$$f(n) = rn(1 - n/N) \tag{12.39}$$

where N is a large positive constant, whose significance we shall see directly. That is, we replace the exponential map (12.34) with the so-called **logistic map**

$$n_{t+1} = f(n_t) = rn_t(1 - n_t/N). \tag{12.40}$$

As long as the population remains small compared to N, the term n_t/N in (12.40) is unimportant, and our new map produces the same exponential evolution as the exponential map. But if n_t grows toward N, the term n_t/N becomes important, and the parenthesis $(1 - n_t/N)$ begins to diminish and "kill off" some of the excess growth. Thus this "mortality factor" $(1 - n/N)$ in (12.39) produces exactly the expected slowing of the population growth as n becomes large, and overcrowding or starvation become important. In particular, if n_t were to reach the value N, the parenthesis $(1 - n_t/N)$ in (12.40) would vanish, and next year's population n_{t+1} would be zero. If n_t were greater than N, then n_{t+1} would be negative — which is impossible. In other words, a population governed by the logistic map (12.40) can never exceed the value N, the maximum or **carrying capacity** of the model. Notice that, because of the term involving n in the mortality factor, the logistic map (12.39) (unlike the exponential map) is *nonlinear*. It is this nonlinearity that makes possible the chaotic behavior of the logistic map.

Figure 12.33 compares exponential and logistic growth, for a growth parameter $r = 2$ and an initial population $n_0 = 4$. The upper curve (gray dots) shows the unending doubling of the exponential case; the lower curve (black dots) shows the growth predicted by the logistic map (12.40), with the same growth parameter $r = 2$ and with a carrying capacity $N = 1000$. As long as n remains small (much less than 1000), the

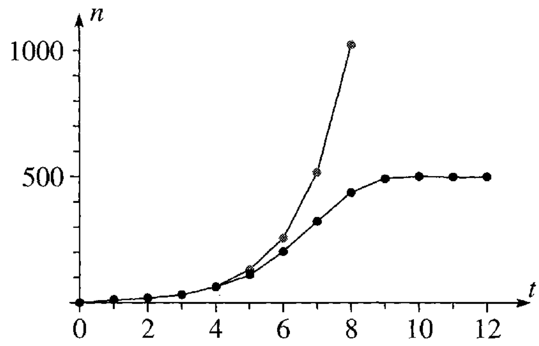

Figure 12.33 Exponential and logistic growth, both with growth parameter $r = 2$. The gray dots show the exponential growth, increasing without limit; the black dots show the logistic growth, which eventually slows down and approaches an equilibrium at $n = 500$. The lines joining the dots are just to guide the eye.

logistic growth is indistinguishable from pure exponential growth, but once n reaches 100 or so, the mortality factor visibly slows the logistic growth, which eventually levels off at around $n = 500$.

Before we discuss the logistic map in any detail it is convenient to simplify it by changing variables from the population n to the *relative population*,

$$x = n/N, \tag{12.41}$$

the ratio of the actual population n to its maximum possible value N. Dividing both sides of (12.40) by N, we see that x_t obeys the growth equation

$$x_{t+1} = f(x_t) = rx_t(1 - x_t) \tag{12.42}$$

where I have redefined the map f as a function of x to be

$$f(x) = rx(1 - x). \tag{12.43}$$

Since the population n is confined to the range $0 \leq n \leq N$, the relative population $x = n/N$ is restricted to

$$0 \leq x \leq 1. \tag{12.44}$$

Within this range, the function $x(1 - x)$ has a maximum of 1/4 (at $x = 1/2$). Thus, to guarantee that x_{t+1}, as given by (12.42), never exceeds 1, we must limit the growth factor to $0 \leq r \leq 4$. Therefore, we shall be studying the map (12.42) in the ranges $0 \leq x \leq 1$ and $0 \leq r \leq 4$.

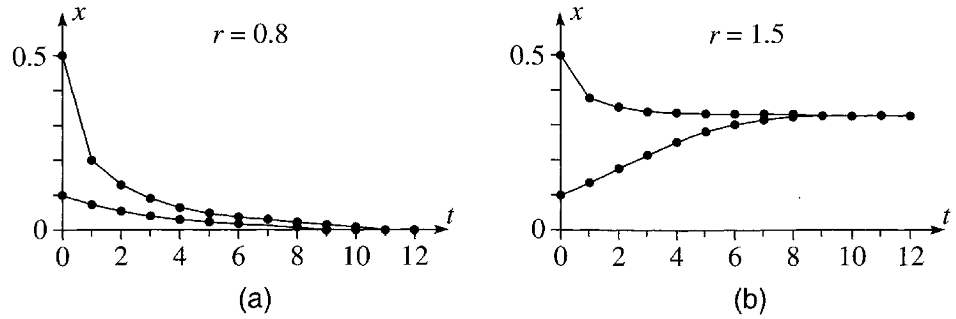

Figure 12.34 The relative population $x_t = n_t/N$ for the logisitic map (12.42), with two different initial conditions for each of two different growth rates. **(a)** With the growth parameter $r = 0.8$, the population rapidly approaches zero whether $x_0 = 0.1$ or $x_0 = 0.5$. **(b)** With $r = 1.5$ and the same two initial conditions, the population approaches the fixed value 0.33.

Before we look at some of the exotic aspects of the logistic map, let us first look at a couple of cases where it behaves just as one might expect. Figure 12.34(a) shows the logistic population for a growth parameter $r = 0.8$ and for two different initial values, $x_0 = 0.1$ and $x_0 = 0.5$. You can see that in either case $x_t \to 0$ as $t \to \infty$. In fact it is easy to see that as long as $r < 1$, the population eventually goes to zero whatever its initial value: From (12.42) we see that $x_t \le r x_{t-1}$ and hence that $x_t \le r^t x_0$; therefore, if $r < 1$, we conclude that $x_t \to 0$ as $t \to \infty$.

Figure 12.34(b) shows the logistic population for a growth parameter $r = 1.5$ and for the same two initial conditions. For $x_0 = 0.1$ we see that the population increases at first. On the other hand, for the larger initial value $x_0 = 0.5$ the mortality factor causes the population to *decrease* at first. In either case, it eventually levels out at $x = 0.33$.

Fixed Points

In both the cases shown in Figure 12.34 we can say that the logistic map has a constant **attractor** towards which the population eventually moves, namely $x = 0$ for any $r < 1$, and $x = 0.33$ for $r = 1.5$. If the population happens to start out equal to such a constant attractor, $x_0 = x^*$ say, then it simply remains fixed there for all time; that is, $x_t = x^*$ for all t. This obviously happens if and only if

$$f(x^*) = x^*. \tag{12.45}$$

Any value x^* which satisfies this equation is called a **fixed point** of the map f. These fixed points are analogous to the equilibrium points of a mechanical system, in that a system which starts at a fixed point remains there for ever.

For a given map, we can solve the equation (12.45) to find the map's fixed points. For example, the fixed points of the logistic map must satisfy

$$r x^*(1 - x^*) = x^*, \tag{12.46}$$

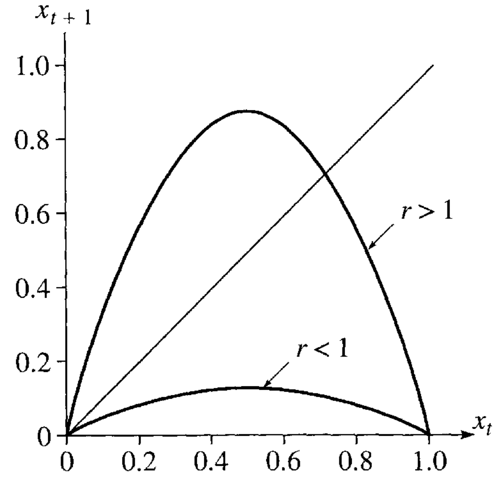

Figure 12.35 Graphs of x against x (the 45° line) and of the logistic function $f(x) = rx(1 - x)$ (the two curves) for two choices of r, one less than and one greater than 1. The fixed points of the logistic map lie at the intersections of the line with the curve. When $r < 1$ there is just one intersection, at $x^* = 0$; when $r > 1$ there are two intersections, one still at $x^* = 0$ and the other at an $x^* > 0$.

which is easily solved to give

$$x^* = 0 \qquad \text{or} \qquad x^* = \frac{r-1}{r}. \tag{12.47}$$

The first solution is the fixed point $x^* = 0$ that we have already noted. The second solution depends on the value of r. For $r < 1$ it is negative and hence irrelevant. For $r = 1$ it coincides with the first solution $x^* = 0$, but for $r > 1$ it is a distinct, second fixed point. For example, for $r = 1.5$ it gives the fixed point we have already noted at $x^* = 1/3$.

It is a fortunate circumstance that we can actually solve the equation (12.45) analytically to find the fixed points of the logistic map, but it is also instructive to examine the equation graphically, since graphical considerations give additional insight and can be applied to many different maps, some of which cannot necessarilly be solved analytically. To solve Equation (12.45) graphically, we just plot the two functions x and $f(x)$ against x as in Figure 12.35 and read off the fixed points as those values x^* where the two graphs intersect. When r is small, the curve of $f(x)$ lies below the 45° line of x against x, and the only intersection is at $x = 0$; that is, the only fixed point is $x^* = 0$. When r is large the curve of $f(x)$ bulges above the 45° line and there are two fixed points. The boundary between the two cases is easily found by noting that the slope of $f(x)$ at $x = 0$ is just r. Thus as we increase r, the curve moves across the 45°line (whose slope is 1) when $r = 1$. Thus for $r < 1$ there is just one fixed point at $x^* = 0$; but when $r > 1$ there are two fixed points, one at $x^* = 0$ and the other at an $x^* > 0$. The advantage of this graphical argument is that it works equally well for any similar function $f(x)$ as long as it is a single concave-down arch. [For example, $f(x)$ could be the function $f(x) = r\sin(\pi x)$ of Problem 12.23.]

A Test for Stability

That x^* is a fixed point (that is, an equilibrium value) for the logistic map guarantees that when the population starts out at x^* it will stay there. By itself this is not enough to ensure that the value x^* is an attractor for the map. We need to check in addition that x^* is a **stable fixed point**, that is, that if the population starts out *close* to x^* it will evolve towards x^* not away from it. (This issue has an exact parallel in the study of equilibrium points of a mechanical system: If a system starts out exactly at an equilibrium point then it will — in principle — stay there indefinitely. But only if the equilibrium is stable will the system move back to equilibrium if disturbed a little away.)

There is a simple test for stability, which we can derive as follows: If x_t is close to a fixed point x^*, we naturally write

$$x_t = x^* + \epsilon_t. \tag{12.48}$$

That is, we define ϵ_t as the *distance* of x_t from the fixed point x^*. If ϵ_t is small, this lets us evaluate x_{t+1} as

$$\begin{aligned} x_{t+1} &= f(x_t) = f(x^* + \epsilon_t) \\ &\approx f(x^*) + f'(x^*)\epsilon_t = x^* + \lambda\epsilon_t \end{aligned} \tag{12.49}$$

where in the last expression I have used the fact that x^* is a fixed point [so that $f(x^*) = x^*$] and I have introduced the notation λ for the derivative of $f(x)$ at x^*,

$$\lambda = f'(x^*). \tag{12.50}$$

Now, according to (12.48), $x_{t+1} = x^* + \epsilon_{t+1}$. Comparing this with the last expression of (12.49) we see that

$$\epsilon_{t+1} \approx \lambda\epsilon_t\,. \tag{12.51}$$

Because of this simple relation, the number $\lambda = f'(x^*)$ is called the **multiplier** or **eigenvalue** of the fixed point. It shows that if $|\lambda| < 1$, then once x_t is close to x^*, successive values get closer and closer to x^*. On the other hand, if $|\lambda| > 1$, then when x_t is close to x^*, the succeeding values move *away* from x^*. This is our required test for stability:

Stability of Fixed Points

Let x^* be a fixed point of the map $x_{t+1} = f(x_t)$; that is, $f(x^*) = x^*$. If $|f'(x^*)| < 1$, then x^* is stable and acts as an attractor. If $|f'(x^*)| > 1$, then x^* is unstable and acts as a repeller.

We can immediately apply this test to the two fixed points of the logistic map: Since $f(x) = rx(1 - x)$, its derivative is

$$f'(x) = r(1 - 2x).$$

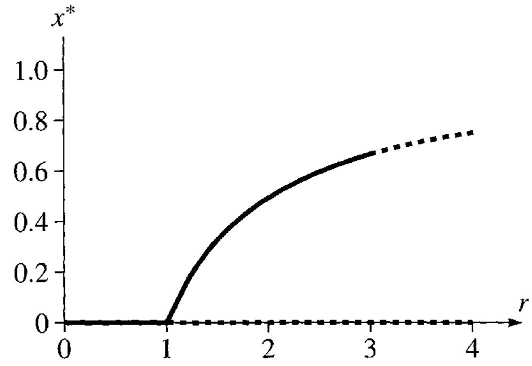

Figure 12.36 The fixed points x^* of the logistic map as functions of the growth parameter r. The solid curves show the stable fixed points and the dashed curves the unstable. Note how the fixed point $x^* = 0$ becomes unstable at precisely the place ($r = 1$) where the nonzero fixed point first appears.

At the fixed point $x^* = 0$, this means the crucial derivative is

$$f'(x^*) = r\,;$$

therefore, the fixed point $x^* = 0$ is stable for $r < 1$ but unstable for $r > 1$. At the fixed point $x^* = (r - 1)/r$, the derivative is

$$f'(x^*) = 2 - r\,,$$

so that this fixed point is stable for $1 < r < 3$, but unstable for $r > 3$. These results are summarised in Figure 12.36, which shows the fixed-point values x^* as functions of the growth parameter r with the stable fixed points shown as solid curves and the unstable as dashed curves.

The arguments just given show exactly when each of the two fixed points becomes unstable and that the second one appears exactly when the first becomes unstable. On the other hand, it would be nice to have an argument that made clearer *why* the fixed points behave the way they do and showed more clearly (what is true) that the same qualitative conclusions apply to any other one-dimensional map with the same general features. Such an argument can be found by examining the graph of Figure 12.35. In that figure, we saw that the fixed points of the map correspond to intersections of the curve of $f(x)$ against x with the 45° line (x against x). When r is small, it was clear that there is only one such intersection at $x^* = 0$. As r increased the curve bulged up more and eventually crossed the 45° line producing a second intersection and hence a second, nonzero fixed point. We saw that this second intersection appears when the slope $f'(0)$ of the curve at $x = 0$ is exactly 1 (that is, when it is tangent to the 45° line). In light of our test for stability, this means the second fixed point has to appear at exactly the moment when the first one at $x^* = 0$ changes from stable to unstable.

The First Period Doubling

Figure 12.36 tells the whole story of the fixed points of the logistic map. The solid curves show the stable fixed points, which are the constant attractors. We see that for $r < 1$, the logistic population approaches 0 as $t \to \infty$; for $1 < r < 3$, it approaches the other fixed point $x^* = (r - 1)/r$ as $t \to \infty$. But what happens — and this proves the most interesting question — when $r > 3$? In the computer age this is easily answered. Figure 12.37 shows the first 30 cycles of the logistic population with a growth parameter of $r = 3.2$. The striking feature of this graph is that it no longer settles down to a single constant value. Instead, it bounces back and forth between the two fixed values shown as x_a and x_b, repeating itself once every two cycles. In the language developed for the driven damped pendulum, we can say that the period has doubled to period two, and this period-two limiting motion is called a **two-cycle**.

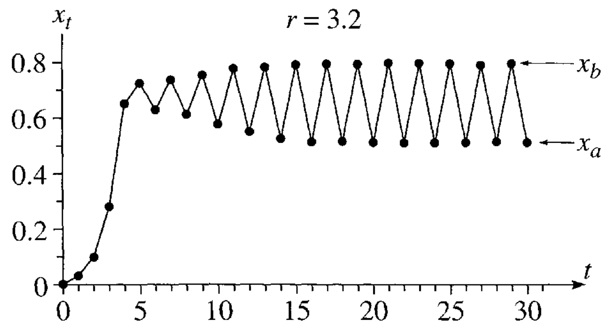

Figure 12.37 A logistic population with growth parameter $r = 3.2$. The population never settles down to a constant value; rather, it oscillates between two values, repeating itself once every *two* cycles. In other words, it has doubled its period to period two.

We can understand the doubling of the period of the logistic map with the graphical methods already developed, although the argument is a bit more complicated. The essential observations are these: First, neither of the two limiting values x_a and x_b is a fixed point of the map $f(x)$. Instead,

$$f(x_a) = x_b \qquad \text{and} \qquad f(x_b) = x_a. \tag{12.52}$$

Let us, however, consider the **double map** (or **second iterate map**)

$$g(x) = f(f(x)), \tag{12.53}$$

which carries the population x_t onto the population two years hence,

$$x_{t+2} = g(x_t). \tag{12.54}$$

It is clear from Figure 12.37 or Equations (12.52) that both x_a and x_b are fixed points of the double map $g(x)$,[24]

$$g(x_a) = x_a \qquad \text{and} \qquad g(x_b) = x_b. \tag{12.55}$$

[24] These points are also called second-order fixed points.

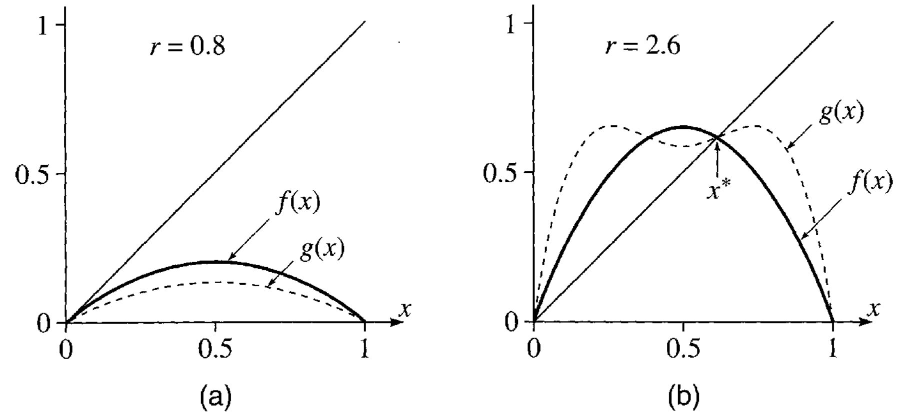

Figure 12.38 The logistic map $f(x)$ and its second iterate $g(x) = f(f(x))$. **(a)** For a growth parameter $r = 0.8$, each function is a single arch which intersects the 45° line just once, at the origin. **(b)** When $r = 2.6$, $f(x)$ is higher than before, but still just a single arch; $g(x)$ has developed two maxima with a valley in between. Both functions have acquired a second intersection with the 45° line, at the same point marked x^*.

Thus to study the two-cycles of the map $f(x)$ we have only to examine the fixed points of the double map $g(x) = f(f(x))$, and for this we can use our understanding of fixed points. Before we do this, it is worth noticing that any fixed point of $f(x)$ is automatically also a fixed point of $g(x)$. (If $x_{t+1} = x_t$ for all t, then certainly $x_{t+2} = x_t$.) Therefore, the two-cycles of $f(x)$ correspond to those fixed points of $g(x)$ that are not also fixed points of $f(x)$.

Since $f(x)$ is a quadratic function of x, it follows that $g(x) = f(f(x))$ is a quartic function, whose explicit form can be written down and studied. However, we can gain a better understanding by considering its graph. When r is small, we know that $f(x)$ is a single low arch (as in Figure 12.35) and you can easily convince yourself that $g(x)$ is an even lower arch, as sketched in Figure 12.38(a), which shows both functions for a growth parameter $r = 0.8$. As r increases, both arches rise, and by the time $r = 2.6$ the function $g(x)$ has developed two maxima as seen in Figure 12.38(b). (You can explore the reason for this development in Problem 12.26.) Also, both curves now intersect the 45° line twice, once at the origin and once at the fixed point $x^* = (r - 1)/r$. That both curves intersect the 45° line at the same points shows two things: First, as we already knew, every fixed point of $f(x)$ is also a fixed point of $g(x)$, and, second, (for the growth parameters shown in this figure) every fixed point of $g(x)$ is also a fixed point of $f(x)$; that is, there are no two-cycles yet.

As we increase the growth parameter r still further, the two crests of the double map $g(x)$ continue to rise, while the valley between them gets lower. (Again see Problem 12.26 for the reason.) Figure 12.39 shows the curves of $f(x)$ and $g(x)$ for growth parameters $r = 2.8, 3.0$, and 3.4. With $r = 2.8$ [part (a)], the double map $g(x)$ still has just the same two fixed points as $f(x)$, at $x = 0$ and at the point indicated as x^*. By the time $r = 3.4$ [part (c)], the double map has developed two additional fixed points, shown as x_a and x_b; that is, the logistic map now has a two-cycle. The threshold value at which the two-cycle appears is clearly the value for which the curve $g(x)$ is *tangent* to the 45° line — namely $r = 3$ for the logistic map, as in part (b) of the figure. If

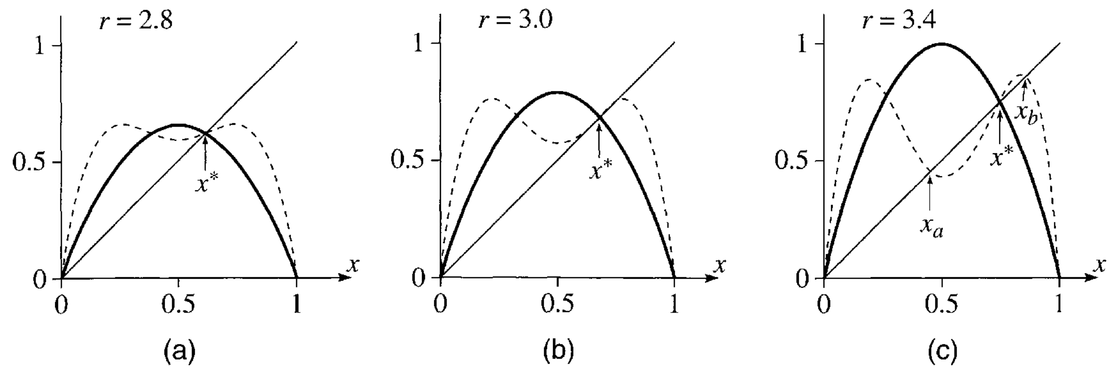

Figure 12.39 The logistic map $f(x)$ (solid curves) and its second iterate $g(x) = f(f(x))$ (dashed) for $r = 2.8$, 3.0, and 3.4. **(a)** With $r = 2.8$, the map $g(x)$ has the same two fixed points as $f(x)$, namely $x = 0$ and $x = x^*$ as shown. **(b)** When r reaches 3.0, the curve of $g(x)$ is tangent to the 45° line, and for any larger value of r, as in **(c)**, the map $g(x)$ has two extra fixed points labelled x_a and x_b.

$r < 3$, the curve $g(x)$ crosses the 45° line just once, at x^*; if $r > 3$ it crosses three times, once at x^* and two more times, at x_a and x_b (one above and one below x^*).

We already know that $r = 3$ is the threshold at which the fixed point x^* becomes unstable. Thus, Figure 12.39 shows that the two-cycle appears at the moment when the "one-cycle" (that is, the fixed point) becomes unstable. Happily, we are now in a position to see *why* this has to be. We have already noted that the two-cycle appears precisely when the curve $g(x)$ is tangent to the 45° line at the point x^*, that is, when

$$g'(x^*) = 1. \tag{12.56}$$

To see what this implies, let us evaluate the derivative $g'(x)$ of the double map $g(x)$ at either of the two-cycle fixed points, x_a say,

$$\begin{aligned} g'(x_a) &= \frac{d}{dx} g(x)\bigg|_{x_a} = \frac{d}{dx} f(f(x))\bigg|_{x_a} = f'(f(x)) \cdot f'(x)\bigg|_{x_a} \\ &= f'(x_b) f'(x_a). \end{aligned} \tag{12.57}$$

Here in the last expression of the first line I have used the chain rule, and in the second line I have used the fact that $f(x_a) = x_b$. Let us apply this result to the birth of the two-cycle in Figure 12.39(b). At the moment of birth, $x^* = x_a = x_b$, and we can combine (12.56) and (12.57) to give

$$[f'(x^*)]^2 = 1.$$

This means that $|f'(x^*)| = 1$, and, by our test for stability, we see that the moment when the two-cycle is born is precisely the moment when the fixed point x^* becomes unstable.

We can use these same techniques to explore what happens as we increase r still further. For example, one can show (Problem 12.28) that the two-cycle that we have just seen appearing at $r = 3$ becomes unstable at $r = 1 + \sqrt{6} = 3.449$ and is succeeded by a stable four-cycle. However, to keep this long chapter from growing totally out of

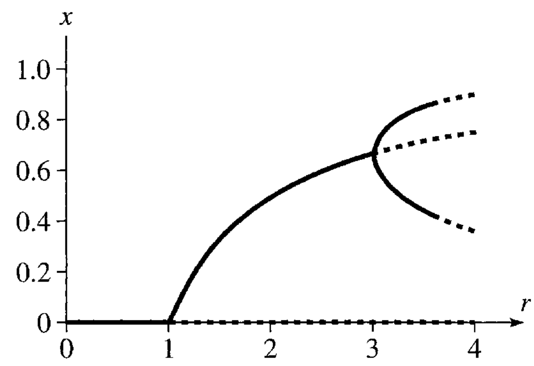

Figure 12.40 The fixed-points and two-cycles of the logistic map as functions of the growth parameter r. The solid curves are stable and the dashed curves unstable.

bounds, I shall just sketch briefly some highlights that can be found by exploring the logistic map numerically.

Bifurcation Diagrams

In Figure 12.36 we saw the complete history of the fixed points of the logistic map itself. If we add onto that picture the fixed points x_a and x_b of the double map $g(x) = f(f(x))$ (that is, the two-cycles of the logistic map) we get the graphs shown in Figure 12.40. These curves are reminiscent of the beginnings of the bifurcation diagram, Figure 12.17, for the driven damped pendulum. In fact, we can redraw Figure 12.40 using the same procedure as was used for Figure 12.17: First one picks a large number of equally spaced values of the growth parameter in the range of interest. (I chose the range $2.8 \le r \le 4$, since this is where the excitement lies, and chose 1200 equally spaced values in this range.) Then for each value of r one calculates the populations $x_0, x_1, x_2, \cdots, x_{t\,\max}$, where $t_{\max}$ is some very large time. (I chose $t_{\max} = 1000$.) Next one chooses a time $t_{\min}$ large enough to let all transients die out. (I chose $t_{\min} = 900$.) Finally, in a plot of x against r, one shows the values of x_t for $t_{\min} \le t \le t_{\max}$ as dots above the corresponding value of r. The resulting bifurcation diagram for the logistic map is shown in Figure 12.41.

The similarity of the bifurcation diagram of Figure 12.41 for the logistic map and Figure 12.17 for the driven pendulum is striking indeed. The interpretation of the two pictures is also similar. On the left of Figure 12.41 we see the period-one attractor for $r \le 3$, followed by the first period doubling at $r = 3$. This is followed by the second doubling at $r = 3.449$, and a whole cascade of doublings that end in chaos near $r = 3.570$. With the help of this diagram we can predict the long-term behavior of the logistic population for any particular choice of r (though there is clearly much fine detail that cannot be distinguished at the scale of Figure 12.41). For example at $r = 3.5$, it is clear that the population should have period four, a claim that is borne out in Figure 12.42(a), which shows the twenty cycles from $t = 100$ to 120 for this value of r. Similarly, around $r = 3.84$, sandwiched between wide intervals of chaos, you can see a narrow window that appears to have period three, an observation borne

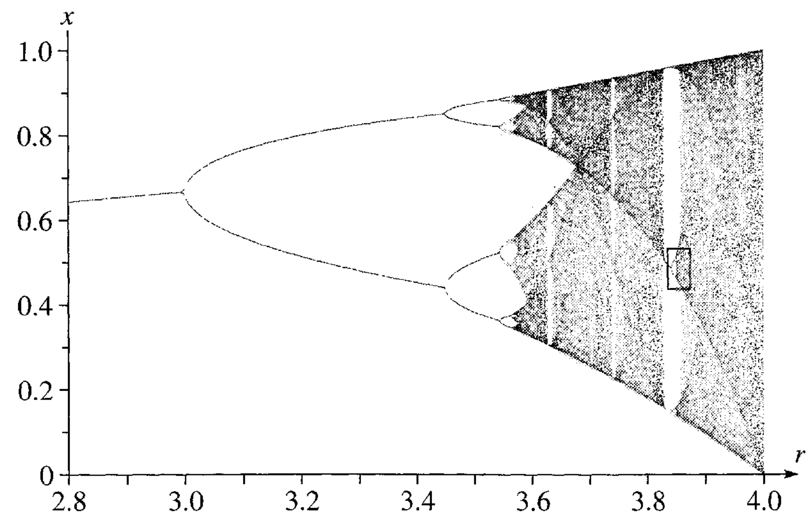

Figure 12.41 Bifurcation diagram for the logistic map. A period-doubling cascade is clearly visible starting at $r_1 = 3$, with a second doubling at $r_2 = 3.449$, and ending in chaos at $r_c = 3.570$. Several windows of periodicity stand out clearly amongst the chaos, especially the period-three window near $r = 3.84$. The tiny rectangle near $r = 3.84$ is the region that is enlarged in Figure 12.44.

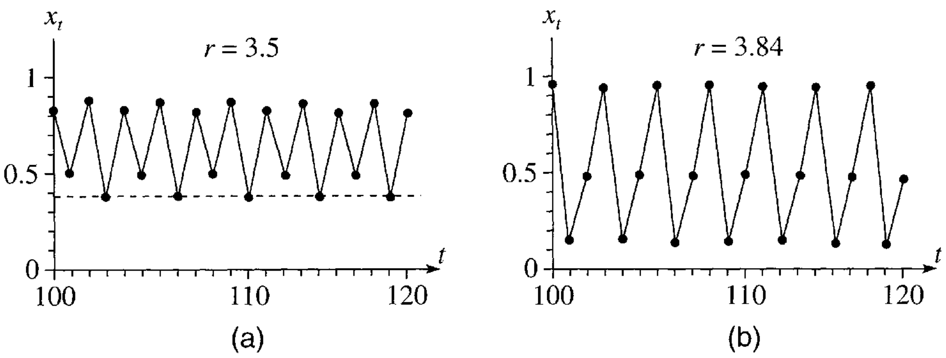

Figure 12.42 Long-term evolution of logistic populations with growth parameters $r = 3.5$ and 3.84. **(a)** Period four. With $r = 3.5$, the twenty cycles $100 \leq t \leq 120$ take on just four distinct values at intervals of four cycles. (The dashed horizontal line is just to highlight the constancy of every fourth dot.) **(b)** Period three. With $r = 3.84$, the population repeats every three cycles.

out in Figure 12.42(b). As an example of chaos, Figure 12.43 shows the evolution of a population with $r = 3.7$; in the eighty cycles shown, there is no evidence of any repetition.

From Figure 12.41 (and careful enlargements), one can read off the threshold values of r at which the period doublings occur. If we denote by r_n the threshold at which the cycle of period 2^n appears, these are found to be

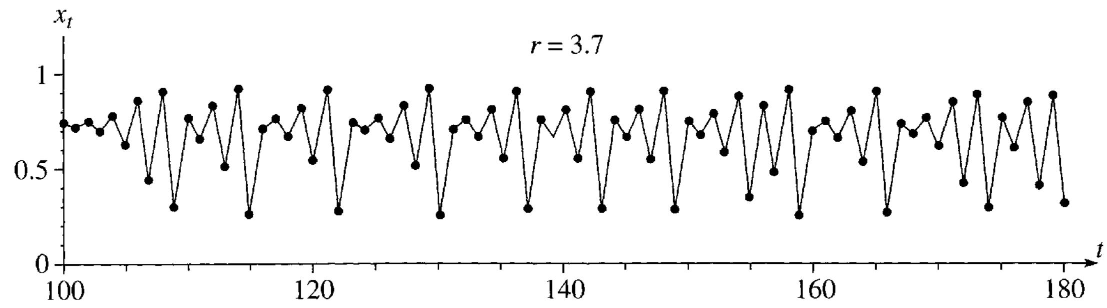

Figure 12.43 Chaos. With $r = 3.7$, the eighty cycles, $100 \leq t \leq 180$, of the logistic map show no tendency to repeat themselves.

As you can check, the separation between successive thresholds shrinks geometrically, just like the corresponding intervals for the period doubling of the driven pendulum. In fact, (see Problem 12.29) the numbers (12.58) give a remarkable fit to the Feigenbaum relation (12.17) that we first met in connection with the DDP, with the same Feigenbaum constant (12.18). Another striking parallel with the driven pendulum is this (Problem 12.30): For those r for which the evolution is non-chaotic, if two populations start out sufficiently close to one another, their difference will converge exponentially to zero as $t \to \infty$. For those r for which the evolution is chaotic, the same difference *diverges* exponentially as $t \to \infty$. That is, the chaotic evolution of the logistic map shows the same sensitive dependence on initial conditions that we found for the driven pendulum.

Perhaps the most striking feature of the logistic bifurcation diagram is that, when one zooms in on certain parts of the diagram, a perfect self similarity emerges. The small rectangle near $r = 3.84$ in Figure 12.41 has been enlarged many fold in Figure 12.44. Apart from the facts that this new picture is upside down and its scale is vastly different, it is a perfect copy of the whole original diagram of which it is a part. This is a striking example of the *self similarity* which appears in many places in the study of chaos and which we met in connection with the Poincaré section of the DDP shown in Figures 12.29, 12.30, and 12.31.

There are many other features of the logistic map and more parallels with the DDP, all worth exploring and some treated in the problems at the end of this chapter. Here, however, I shall leave the logistic map and close this chapter with the hope that you feel at home with some of the main features of chaos and the tools used to explore them. I hope too that your appetite has been whetted to explore this fascinating subject further.[25]

[25] For a comprehensive history, with very little mathematics, see *Chaos, Making a New Science* by James Gleick, Viking-Penguin, New York (1987). For a quite mathematical, but highly readable, account of chaos in many different fields see *Nonlinear Dynamics and Chaos* by Steven H. Strogatz, Addison-Wesley, Reading, MA (1994). Two books which focus mostly on chaos in physical systems are *Chaotic Dynamics: An Introduction* by G.L. Baker and J.P.Gollub, Cambridge Universtiy Press, Cambridge (1996) and *Chaos and Nonlinear Dynamics* by Robert C. Hilborn, Oxford University Press, New York (2000).

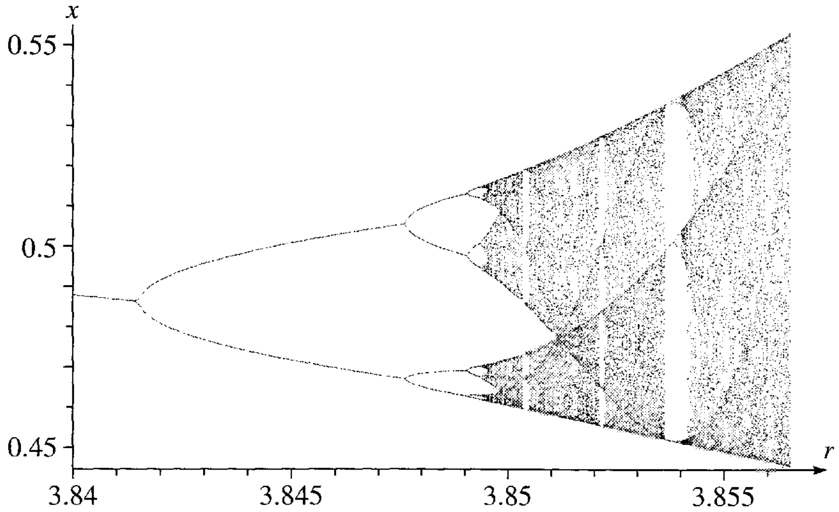

Figure 12.44 A many-fold enlargement of the small rectangle in the logistic bifurcation diagram of Figure 12.41. This tiny section of the original diagram is a perfect, upside-down copy of the whole original. Note that this section is just one of three strands in the original; thus, although this diagram starts out *looking* just like period 1 doubling to period 2, it is actually period 3 doubling to period 6 and so on.

Principal Definitions and Equations of Chapter 12

The Driven Damped Pendulum

A damped pendulum that is driven by a sinusoidal force $F(t) = F_0 \cos(\omega t)$ satisfies the nonlinear equation

$$\ddot{\phi} + 2\beta\dot{\phi} + \omega_0^2 \sin\phi = \gamma\omega_0^2 \cos\omega t \qquad \text{[Eq. (12.11)]}$$

where $\gamma = F_0/mg$ is called the **drive strength** and is the ratio of the drive amplitude to the weight.

Period Doubling

For small drive strengths, ($\gamma \lesssim 1$) the long-term response, or **attractor**, of the pendulum has the same period as the drive force. But if γ is increased past $\gamma_1 = 1.0663$, for certain initial conditions and drive frequencies, the attractor undergoes a **period-doubling cascade**, in which the period repeatedly doubles, approaching infinity as $\gamma \to \gamma_c = 1.0829$. [Section 12.4]

Chaos

If the drive strength is increased beyond γ_c, at least for certain choices of drive frequency and initial conditions, the long-term motion becomes nonperiodic, and we say that **chaos** has set in. As γ is increased still further, the long-term motion varies, sometimes chaotic, sometimes periodic. [Sections 12.5 & 12.6]