pendulum. I shall first describe some features of its motion that we would expect (or at least that are not wholly unexpected), and I shall then take up the surprising features associated with the pendulum's chaotic motion.

12.2 The Driven Damped Pendulum DDP

The equation of motion for the **driven damped pendulum** (or **DDP**) was given in (12.5). Since this equation is going to occupy us for several sections to come, I would like to make quite sure that you are clear where it came from and to tidy it up. The pendulum is sketched in Figure 12.1. The equation of motion is just $I\ddot{\phi} = \Gamma$, where I is the moment of inertia and Γ is the net torque about the pivot. In this case $I = mL^2$, and the torque arises from the three forces shown in Figure 12.1. The resistive force has magnitude bv and hence exerts a torque $-Lbv = -bL^2\dot{\phi}$. The torque of the weight is $-mgL\sin\phi$, and that of the driving force is $LF(t)$. Thus the equation of motion $I\ddot{\phi} = \Gamma$ is

$$mL^2\ddot{\phi} = -bL^2\dot{\phi} - mgL\sin\phi + LF(t) \tag{12.7}$$

exactly as in (12.5).

Throughout this chapter I shall assume that the driving force $F(t)$ is sinusoidal, specifically that

$$F(t) = F_0\cos(\omega t) \tag{12.8}$$

where F_0 is the *drive amplitude* (the amplitude of the driving force) and ω the **drive frequency**. As I argued in Chapter 5, several real and interesting driving forces approximate this sinusoidal form quite closely, and it has proved possible to reproduce such sinusoidal forces with remarkable precision for experiments on chaos. Substituting into (12.7) and reorganizing a little, we find that

$$\ddot{\phi} + \frac{b}{m}\dot{\phi} + \frac{g}{L}\sin\phi = \frac{F_0}{mL}\cos\omega t. \tag{12.9}$$

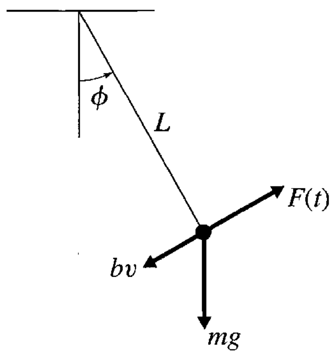

Figure 12.1 The three important forces on the driven damped pendulum are the resistive force with magnitude bv, the weight mg, and the driving force $F(t)$. (There is also a reaction force from the pivot at the top, but this contributes nothing to the torque.)

In this equation, you will recognize the coefficient b/m as the constant that we renamed as 2β in Chapter 5,

$$\frac{b}{m} = 2\beta,$$

where β was called the **damping constant**. Similarly the coefficient g/L is just ω_o^2,

$$\frac{g}{L} = \omega_o^2,$$

where ω_o is the **natural frequency** of the pendulum. Finally, the coefficient F_o/mL must have the dimensions of $(\text{time})^{-2}$; that is, F_o/mL has the same units as ω_o^2. It is convenient to rewrite this coefficient as $F_o/mL = \gamma\omega_o^2$. That is, we introduce a dimensionless parameter

$$\gamma = \frac{F_o}{mL\omega_o^2} = \frac{F_o}{mg}, \tag{12.10}$$

which I shall call the **drive strength** and is just the ratio of the drive amplitude F_o to the weight mg. This parameter γ is a dimensionless measure of the strength of the driving force. When $\gamma < 1$, the drive force is less than the weight and we would expect it to produce a relatively small motion. (For instance, the drive force is insufficient to hold the pendulum out at $\phi = 90°$.) Conversely, if $\gamma \geq 1$, the drive force exceeds the pendulum's weight, and we should anticipate that it will produce larger scale motions (for instance, motion in which the pendulum is pushed all the way over the top at $\phi = \pi$).

Making all these substitutions, we get our final form of the equation of motion (12.9) for a driven damped pendulum

$$\ddot{\phi} + 2\beta\dot{\phi} + \omega_o^2 \sin\phi = \gamma\omega_o^2 \cos\omega t. \tag{12.11}$$

This is the equation whose solutions we shall be studying for the next several sections.

12.3 Some Expected Features of the DDP

Properties of the Linear Oscillator

To appreciate the extraordinary richness of the chaotic motion of our driven damped pendulum, we must first review what sort of behavior we might *expect*, based on our experiences with linear oscillators. Specifically, if we release the pendulum near the equilibrium position $\phi = 0$ with a small initial velocity and if the drive strength is small, $\gamma \ll 1$, we would expect ϕ to remain small at all times. Thus we should be