The behavior of the DDP in the linear and nearly linear regimes can be quickly summarized: As the initial transients die out, the motion approaches a unique attractor, in which the pendulum oscillates with the same period as the driver. In the linear regime (driving strength $\gamma \ll 1$) this limiting motion is given by a simple cosine function with frequency equal to the drive frequency ω. In the not-quite-linear regime (γ somewhat larger, but definitely not much greater than 1), the limiting motion is still periodic, with the same period, but it picks up some harmonics and is a sum of cosines with frequencies $n\omega$ as in (12.16). As we shall see in the next section, we have only to increase the drive strength a little above $\gamma = 1$, to encounter some dramatically different behavior.

12.4 The DDP: Approach to Chaos

Let us now continue increasing the strength of our DDP's driver. Figure 12.4 shows the motion (ϕ against t) for all the same parameters and initial conditions as in the last two figures, except that I have increased the drive strength to $\gamma = 1.06$, just a little above the rough boundary at $\gamma = 1$ between weak and strong driving. The most striking thing about this plot is the dramatic oscillation of the initial transient motion. In the first three drive cycles, the pendulum swings from $\phi = 0$ to nearly 5π; that is, it makes nearly two and a half counterclockwise rotations. In the next two cycles it swings back nearly to $\phi = \pi$ and eventually settles down to more-or-less sinusoidal oscillations around $\phi \approx 2\pi$. (The position $\phi = 2\pi$ is, of course, the same as $\phi = 0$, but the statement that ϕ eventually centers on 2π is nonetheless meaningful, indicating that the pendulum has made one net counterclockwise rotation since $t = 0$.)

It is impossible to be completely sure, based on a graph such as Figure 12.4, that the eventual motion really is exactly periodic. One way to examine this question more closely is to print out the positions $\phi(t)$ at successive one-cycle intervals, $t = t_o,\ t_o + 1,\ t_o + 2,\ t_o + 3,\ \cdots$. The larger we choose t_o, the more closely these positions should agree with one another (if, the eventual motion is indeed periodic). For example, starting at $t = 34$ the positions $\phi(t)$ (as given by the same numerical solution on which Figure 12.4 was based) are

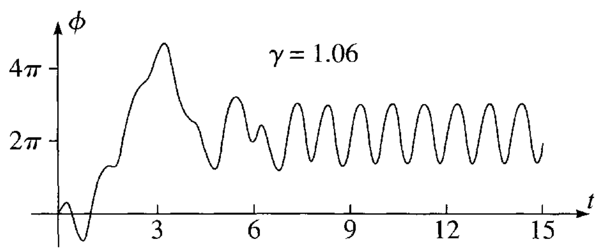

Figure 12.4 The motion of a DDP with drive strength $\gamma = 1.06$. The initial, rather wild, transients die out after about 9 drive cycles, and the motion settles down to an attractor with the same period as the driver.

t	$\phi(t)$
34	6.0366
35	6.0367
36	6.0366
37	6.0366
38	6.0366
39	6.0366

with all subsequent values equal to 6.0366. Evidently, to five significant figures, the motion has settled down to be perfectly periodic after 35 drive cycles.[8] Of course, it is *possible* that the motion does something nonperiodic in between the integer times shown, and certainly no one would accept our data as mathematical proof. Nevertheless, the evidence is overwhelming that for $\gamma = 1.06$ (and with the initial conditions used) $\phi(t)$ does approach an attractor that is periodic with the same period as the driver. In this respect, the motion shown for $\gamma = 1.06$ is not much different from that for $\gamma = 0.9$ as shown in Figure 12.3. However, the dramatic initial swings in Figure 12.4 are harbingers of interesting developments to come.

Period Two

Figure 12.5(a) corresponds exactly to the previous figure except that I have now increased the drive strength to $\gamma = 1.073$. Again the most obvious feature is the wild initial oscillation, which now lasts for nearly 20 drive cycles before the motion settles down to steady oscillations that are at least approximately sinusoidal. However, if you look closely at these oscillations, you will notice that the crests and troughs (especially the troughs) are not all of the same height. Figure 12.5(b) is a many-fold enlargement of the same troughs between $t = 20$ and 30, and you can see clearly that the troughs alternate between two distinct heights. You might wonder if this alternation is itself a transient that will disappear after enough cycles, but this is not in fact so. A plot of the oscillations for $990 \le t \le 1000$ looks exactly the same as that for $20 \le t \le 30$. Another way to show this is to print out the numerical values of $\phi(t)$ at one-cycle intervals. Starting at $t = 30$, this yields

t	$\phi(t)$
30	−6.6438
31	−6.4090
32	−6.6438
33	−6.4090
34	−6.6438
35	−6.4090

[8] Naturally, the motion takes longer to settle down to a constant if we insist on more significant figures. For example, it is not until 46 cycles have passed that $\phi(t)$ starts repeating to six significant figures, after which $\phi(t) = 6.03662$ for $t = 46, 47, 48, \cdots$.

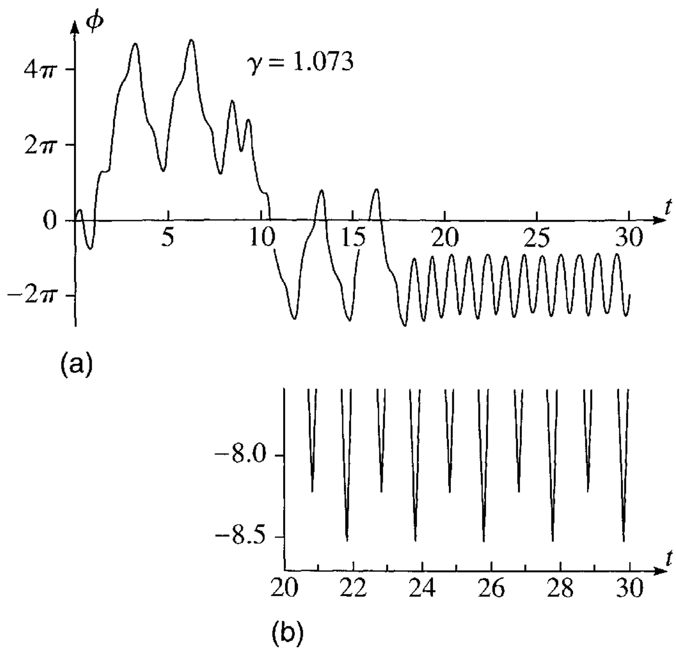

Figure 12.5 **(a)** The first 30 cycles of a DDP with drive strength $\gamma = 1.073$. The wild initial oscillations persist for nearly 20 drive cycles, after which the motion settles down to an attractor that is approximately sinusoidal. However, closer inspection shows that the crests and troughs of this attractor are not all of the same height. **(b)** An enlargement of the attractor for $20 \le t \le 30$ showing just the troughs of part (a). The troughs alternate in height, repeating themselves once every *two* drive cycles.

a pattern that repeats precisely forever. Evidently, by $t = 30$, the motion has settled down so that $\phi(t)$ has the value -6.6438 (to 5 significant figures) for all even values of t and has the distinct value -6.4090 for all odd values of t.

This behavior means that the motion no longer repeats itself with the frequency of the driver. Rather, the motion is periodic with period equal to *twice the drive period*, and we say that the motion has **period two**. (In our units this last statement is literally true; in general it means that the period of the motion is two times the drive period.) It is important to recognise that this development is quite different from the appearance of the harmonics that we noticed in the case of nearly linear motion. A harmonic has frequency $n\omega$, an integer multiple of the drive frequency, and hence period equal to an integer *submultiple* of the drive period. What we have now found has period equal to an integer *multiple* of the drive period, and hence frequency ω/n, which can be described as a **subharmonic** of the drive frequency. Looking at Figure 12.5(a) you can see that the motion is still very nearly sinusoidal with the period of the driver (period 1). Thus the dominant term in $\phi(t)$ is still of the form $A\cos(\omega t - \delta)$; nevertheless, $\phi(t)$ definitely contains a small subharmonic term with period 2.

Period Three

Although the attractor shown in Figure 12.5 has period two, the dominant behavior is still clearly of period one; that is, the new $n = 2$ subharmonic contributes only a small amount to the solution. If we increase the drive strength a little further, we find an

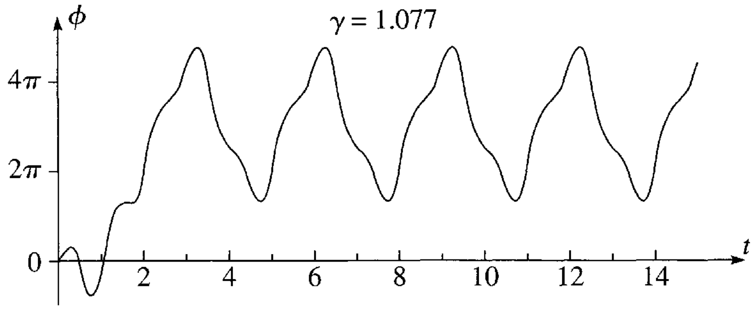

Figure 12.6 The motion of a DDP with drive strength $\gamma = 1.077$. After little more than two drive cycles, the motion has settled down to a periodic attractor which repeats itself every three drive cycles (for example, the troughs come just before $t = 5$, 8, 11, 14, and so on); therefore, the attractor has period three.

attractor for which the subharmonic is the dominant term. Figure 12.6 shows the first 15 cycles of the motion of our DDP with the drive strength increased to $\gamma = 1.077$ (and all other parameters the same as before). In this case it is obvious at just a glance that the motion settles down to an attractor that repeats itself every *three* drive cycles and hence has period three. While it would be hard to question that this graph has period three, we can reinforce the conclusion by looking at the values of $\phi(t)$ at one-cycle intervals. Starting from $t = 30$ these are as follows:

t	$\phi(t)$
30	13.81225
31	7.75854
32	6.87265
33	13.81225
34	7.75854
35	6.87265
36	13.81225
37	7.75854
38	6.87265

with exactly the same pattern, repeating once every three drive cycles, continuing indefinitely. Evidently the solution has picked up a period-three term, and this term dominates the solution.

More than One Attractor

For a linear oscillator, with a given set of parameters, we proved in Section 5.5 that there is a unique attractor; that is, whatever the initial values of ϕ and $\dot{\phi}$, the eventual motion will always be the same, once the transients have died out. For a nonlinear oscillator, this is not the case, and the DDP with the drive strength $\gamma = 1.077$ of Figure 12.6 furnishes a clear example. In Figure 12.7, I have shown the motion for a DDP with the same parameters as in Figure 12.6 (including the same drive strength), but with two different sets of initial conditions. The dashed curve is the same solution as

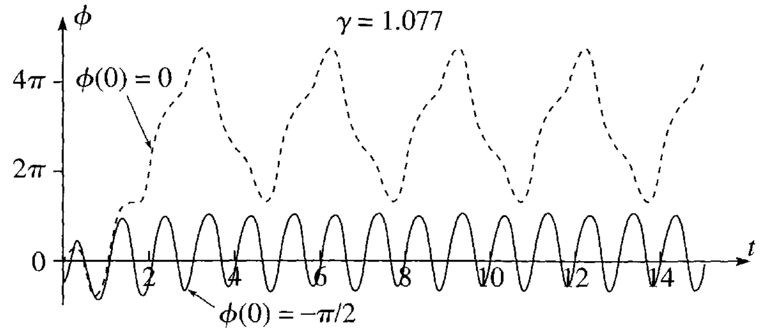

Figure 12.7 Two solutions for the same DDP, with the same drive strengths, but different initial conditions [$\phi(0) = \dot{\phi}(0) = 0$ for the dashed curve, but $\phi(0) = -\pi/2$ and $\dot{\phi}(0) = 0$ for the solid curve]. Even after the transients have died out, the two motions are totally different.

in Figure 12.6, with the same initial conditions as we have used for every graph up to now, $\phi(0) = \dot{\phi}(0) = 0$. The solid curve shows the motion of the exact same DDP, also with $\dot{\phi}(0) = 0$, but with $\phi(0) = -\pi/2$; that is, for the solid curve, the pendulum was released from 90° on the left. As you can clearly see, the two attractors (the curves to which the actual motions converge as the initial transients die out) are totally different. For the dashed curve, the attractor has period three, for the solid curve the eventual period is (as you can see if you look closely) actually two, with alternate troughs (and alternate crests) having slightly different heights. Evidently, for a nonlinear oscillator, different initial conditions can lead to totally different attractors.

A Period-Doubling Cascade

Having recognized that different initial conditions can lead to different attractors, we must anticipate that the evolution of the oscillations as we vary γ may depend on the initial conditions that we choose. In the sequence of Figures 12.2 through 12.6, I used the initial conditions $\phi(0) = 0$ and $\dot{\phi}(0) = 0$ for all five pictures. It turns out that the new initial conditions $\phi(0) = -\pi/2$ and $\dot{\phi}(0) = 0$, introduced in Figure 12.7, lead to a quite different and very interesting evolution. In Figure 12.8, I have shown the motion of the DDP for four successively larger values of γ, all with these new initial conditions. The left-hand pictures show $\phi(t)$ as a function of t for the first ten cycles of the driver. The first graph is for $\gamma = 1.06$, the same value as was used in Figure 12.4, and, as in Figure 12.4, the motion settles down to a steady oscillation with period equal to the drive period; that is, the attractor has period one. To confirm this conclusion, the right-hand picture shows the same motion, but for $28 < t < 40$ (by which time any initial transients have completely disappeared at the scale shown), with the vertical scale magnified to show clearly that successive oscillations are all of equal amplitude.

For the second pair of graphs, the drive strength was increased to $\gamma = 1.078$. At first glance, the motion looks very similar to that for $\gamma = 1.06$, but on closer inspection you can see that the maxima and minima are not all of the same height. This is very visible

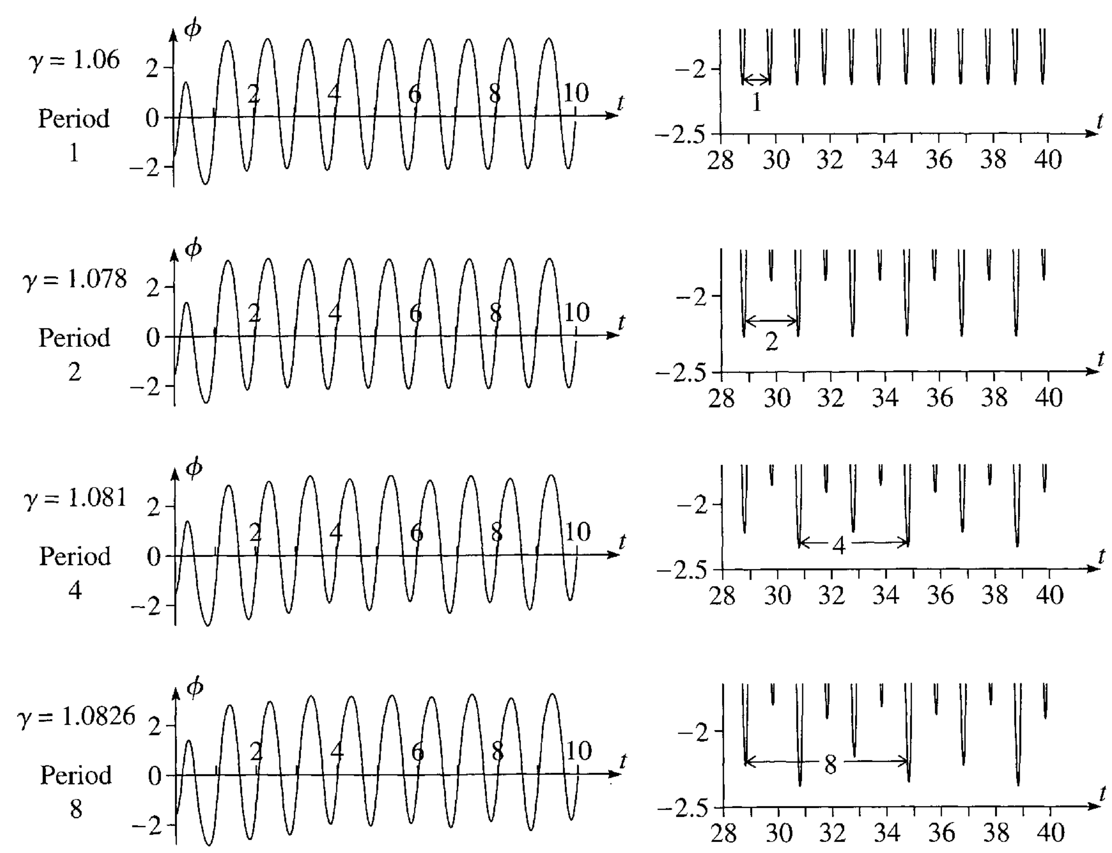

Figure 12.8 A period-doubling cascade. The left-hand pictures show the first ten drive cycles of a DDP with successively larger drive strengths, as indicated on the left. All other parameters, including the initial conditions $\phi(0) = -\pi/2$ and $\dot{\phi}(0) = 0$, are the same in all pictures. In each picture on the right I have enlarged the bottom of the corresponding motion on the left, to show more clearly the differences in extent of successive oscillations; these enlargements show 12 drive cycles, starting from $t = 28$, by which time the motion has settled down to a perfectly periodic attractor (at least at the scale shown). Each double-headed arrow shows one complete cycle of the corresponding motion; the periods of the four attractors are clearly seen to be 1, 2, 4, and 8, as indicated.

in the enlargement on the right where you can see easily that the minima alternate between two distinct, fixed heights, so that the attractor now has period two.

With $\gamma = 1.081$, as in the third pair, the graph on the left again looks pretty much like its two predecessors, and it is hard to be sure just what is going on. One of the reasons is that we can't be sure that ten drive cycles (the number shown on the left) are long enough for all transients to have disappeared, but in the right-hand enlargement it is quite clear that the minima are alternating among four different values. That is, the period has doubled again to period four.

In the last pair of pictures, with $\gamma = 1.0826$, it is even harder to be sure what is happening in the left-hand picture, but the enlargement on the right makes clear that the motion eventually repeats once every eight drive cycles. That is, the attractor has period eight. The **period-doubling cascade** seen in these four pairs of pictures

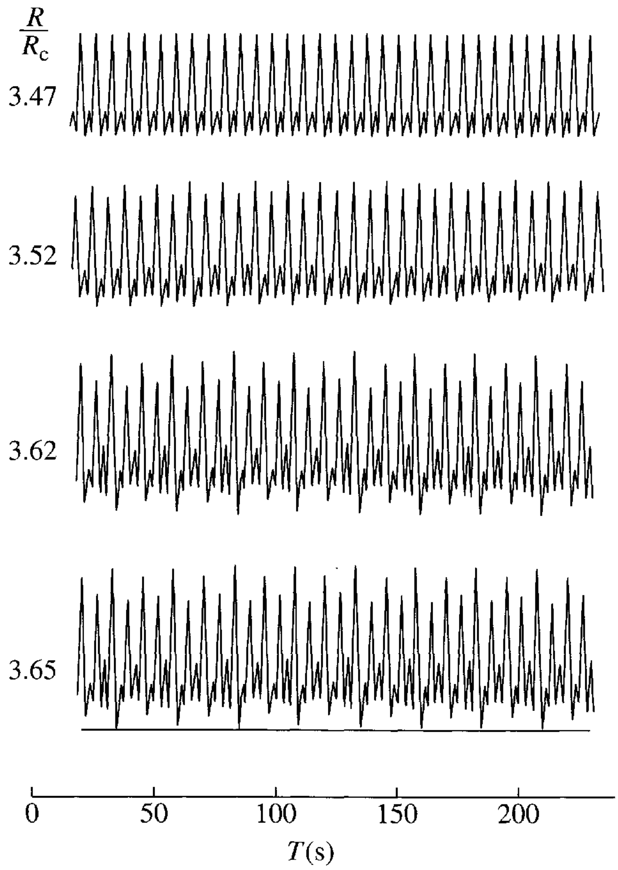

Figure 12.9 A period-doubling cascade in convection of mercury in a small convection cell. The plots show the temperature at one fixed point in the cell as a function of time, for four successively larger temperature gradients as given by the parameter R/R_c.

continues. If we increase the drive strength further, we find motion with period 16, then 32, and so on to infinity.

The period-doubling cascade of Figure 12.8 is a very striking phenomenon, but the quantitative differences between the four successive unenlarged graphs are quite small. You might guess that to build a driven damped pendulum sufficiently precise to observe these subtle differences would be very hard, and this is indeed the case. Nevertheless, such pendulums have been constructed and have been used to observe all of the effects described in this chapter, with amazing agreement between theory and experiment.[9] Perhaps even more remarkable is that the phenomenon of period doubling is found in many completely different nonlinear systems — electrical circuits, chemical reactions, balls bouncing on oscillating surfaces, and many more. In each of these systems, there is a "control parameter" that can be varied (the driving strength of a DDP, a voltage in an electrical circuit, a flow rate in a chemical reaction). The

[9] For a description of three of the commercially available "chaotic pendulums" see J. A. Blackburn and G. L. Baker, "A Comparison of Commercial Chaotic Pendulums," *American Journal of Physics*, Vol. 66, p. 821, (1998). The Daedalon pendulum described there was used to get the data shown in Figure 12.32.

behavior of the system is monitored as this parameter is varied, and it is found that the behavior exhibits period-doubling cascades. Figure 12.9 shows a cascade observed by Libchaber et al.[10] in the convection of mercury in a small box whose bottom is maintained at a slightly higher temperature than its top. This temperature difference is the control parameter and is measured by a number R, called the Rayleigh number. When R is very small, heat is conducted up with no convection. Then at a critical temperature difference, R_c, steady convection sets in, and, as R is increased still further, the convection becomes oscillatory. These oscillations can be observed by measuring the temperature at any fixed point in the cell, and Figure 12.9 shows four plots of the observed temperature (at one fixed point) against time, for four successively larger values of the control parameter R. The period doublings from 1 to 2, from 2 to 4, and from 4 to 8 are beautifully clear.

Not only are period-doubling cascades observed in numerous different systems. In a sense that I shall describe directly, the cascades occur *in the same way*, a circumstance referred to as "universality."

The Feigenbaum Number and Universality

Returning to the period-doublings of the DDP, you can see from the values of the drive strength γ shown in Figure 12.8 that the doublings occur faster and faster as we increase γ. To make this idea quantitative, we need to examine the **threshold values** of γ at which the period actually doubles. For example, looking at the numbers in Figure 12.8, it seems clear that somewhere between $\gamma = 1.06$ and 1.078, there must be a value γ_1 where the period changes from 1 to 2. Finding where this threshold (or "**bifurcation point**") actually occurs is surprisingly hard, but it turns out that (to 5 significant figures) $\gamma_1 = 1.0663$. Similarly, at $\gamma_2 = 1.0793$, the period changes from 2 to 4. If we let γ_n denote the threshold at which the period changes from 2^{n-1} to 2^n, then the first few thresholds γ_n are as shown in Table 12.1. In the last column of the table, I have shown the distances $\gamma_n - \gamma_{n-1}$ between successive thresholds, which, as you can see, shrink geometrically,[11] each interval being about one fifth of its predecessor.

In the late 1970's, the physicist Mitchell Feigenbaum (born 1944) showed not only that many different nonlinear systems undergo similar period-doubling cascades but that the cascades all show the same geometric acceleration; specifically, the intervals between the thresholds for the control parameter (the drive strength, in our case) satisfy

$$(\gamma_{n+1} - \gamma_n) \approx \frac{1}{\delta}(\gamma_n - \gamma_{n-1}) \tag{12.17}$$

[10] Reproduced with permission from A. Libchaber, C. Laroche, and S. Fauve, *Journal de Physique-Lettres*, vol. 43, p. 211 (1982).

[11] A sequence of numbers, $a_1, a_2, \cdots$, is geometric if $a_{n+1} = ka_n$ for some fixed number k. If $k < 1$, the geometric sequence goes to zero as $n \to \infty$.

Table 12.1 The first four thresholds γ_n at which the period of the DDP [with the initial conditions $\phi(0) = -\pi/2$ and $\dot{\phi}(0) = 0$] doubles from 1 to 2, 2 to 4, 4 to 8, and 8 to 16. The last column shows the widths of the intervals between successive thresholds.

n	period	γ_n	interval
1	$1 \to 2$	1.0663	
			0.0130
2	$2 \to 4$	1.0793	
			0.0028
3	$4 \to 8$	1.0821	
			0.0006
4	$8 \to 16$	1.0827	

where the constant δ has the same value

$$\delta = 4.6692016 \tag{12.18}$$

for all such systems and is called the **Feigenbaum number**.[12] It is the widespread occurrence of period doubling and the fact that δ has the same value for so many different systems that has led to the phenomenon of period doubling being characterized as **universal**. I have written the Feigenbaum relation (12.17) with an "approximately equal" sign, because, strictly speaking, the relation holds only in the limit that $n \to \infty$. For many systems, however, the relation is a very good approximation for *all* values of n. (See Problems 12.11 and 12.29.)

The Feigenbaum relation (12.17) implies that the intervals between successive thresholds approach zero rapidly, and hence that the thresholds themselves approach a finite limit γ_c,

$$\gamma_n \to \gamma_c \qquad (\text{as } n \to \infty). \tag{12.19}$$

Therefore, the sequence of thresholds γ_n satisfies

$$\gamma_1 < \gamma_2 < \cdots < \gamma_n < \cdots < \gamma_c$$

with infinitely many thresholds squeezed in the rapidly narrowing gap between γ_n and γ_c. For our DDP, the limit γ_c is found to be

$$\gamma_c = 1.0829. \tag{12.20}$$

We shall see that beyond the critical value γ_c, chaos sets in, so the period-doubling cascade is called a **route to chaos**. However, I should emphasize that there are systems that exhibit chaos without first going through a period-doubling cascade; that is, the period-doubling cascade is just one of several possible routes to chaos.

[12] Actually there are two Feigenbaum numbers, and this one is often called Feigenbaum's delta.