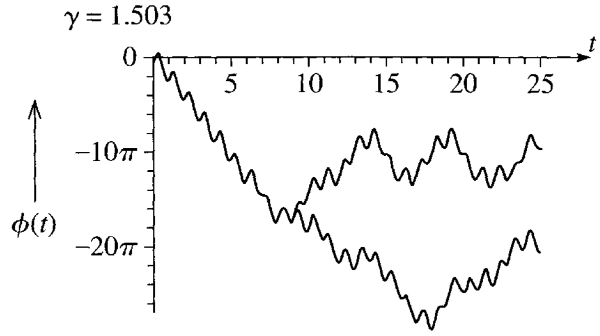

Figure 12.16 Linear plot of the positions of the same two identical DDP's whose separation $\Delta\phi(t)$ was shown in Figure 12.15(b) [$\Delta\phi(0) = 0.001$ rad]. For the first eight and a half drive cycles, the two curves are indistinguishable; after this the difference is dramatically apparent.

12.15(a). **(3)** The erratic motion of chaos always goes along with the sensitivity to initial conditions associated with the exponential divergence of neighboring solutions of the equation of motion.

12.6 Bifurcation Diagrams

So far, each of our pictures of the motion of the driven damped pendulum has shown the motion for one particular value of the drive strength γ. To observe the evolution of the motion as γ changes, we have had to draw several different plots, one for each value of γ. One would like to construct a single plot that somehow displayed the whole story, with its changing periods and its alternating periodicity and chaos as γ varies. This is the purpose of the bifurcation diagram.

A **bifurcation diagram** is a cunningly constructed plot of $\phi(t)$ against γ as in Figure 12.17. Perhaps the best way to explain what this plot shows is just to describe in detail how it was made. Having decided on a range of values of γ to display (from $\gamma = 1.06$ to 1.087 in Figure 12.17) one must first choose a large number of values of γ, evenly spaced across the chosen range. For Figure 12.17, I chose 271 values of γ, spaced at intervals of 0.0001,

$$\gamma = 1.0600,\ 1.0601,\ 1.0602,\ \cdots,\ 1.0869,\ 1.0870.$$

For each chosen value of γ, the next step is to solve numerically the equation of motion (12.11) from $t = 0$ to a time t_{max} picked so that all transients have long since died out. To make Figure 12.17, I chose the same initial conditions as in the last few pictures, namely $\phi(0) = -\pi/2$ and $\dot{\phi}(0) = 0$.[18]

[18] Some authors like to superpose the plots for several different initial conditions. This gives a more complete picture of the many possible motions, but makes the plot harder to interpret. For simplicity, I chose to use just one set of initial conditions.

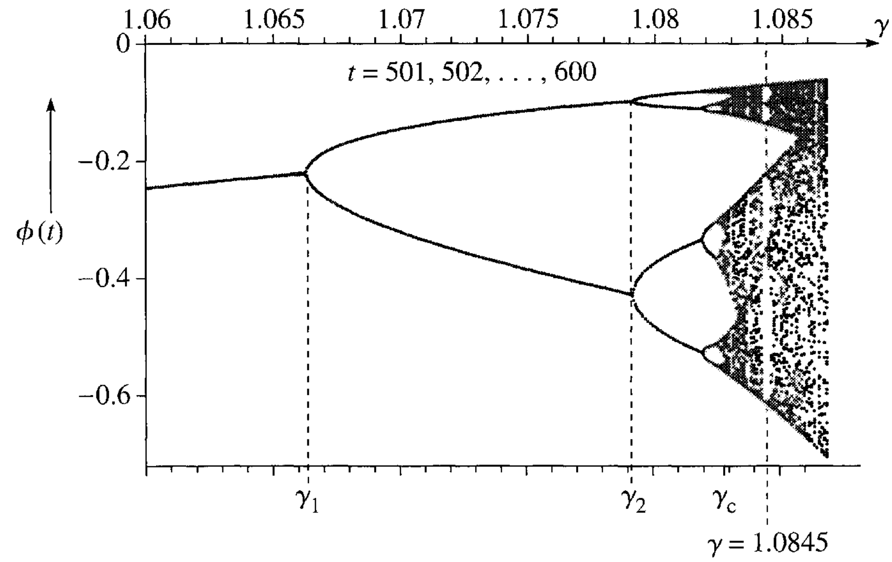

Figure 12.17 Bifurcation diagram for the driven damped pendulum for drive strengths $1.060 \le \gamma \le 1.087$. The period-doubling cascade is clearly visible: At $\gamma_1 = 1.0663$ the period changes from 1 to 2, and at $\gamma_2 = 1.0793$ from 2 to 4. The next bifurcation, from period 4 to 8, is easily seen at $\gamma_3 = 1.0821$, and that from 8 to 16 is just discernable at $\gamma_4 = 1.0827$. To the right of the critical value, $\gamma_c = 1.0829$, the motion is mostly chaotic, although at $\gamma = 1.0845$ you can just make out a brief interval of period-6 motion.

To understand our next move, recall that a good way to check for periodicity (or non-periodicity) is to examine the values

$$\phi(t_o),\ \phi(t_o + 1),\ \phi(t_o + 2),\ \cdots$$

of $\phi(t)$ for a large number of times at one-cycle intervals. If the motion is periodic with period n, these will repeat themselves after n cycles, otherwise not. Therefore, our next step is to use our solutions for $\phi(t)$ to find the values of $\phi(t)$ over a range of times at integer intervals from some chosen $t_{\min}$ to $t_{\max}$ (with $t_{\min}$ large enough that all transients have died out). For Figure 12.17, I found $\phi(t)$ for 100 times,

$$t = 501,\ 502,\ \cdots,\ 600.$$

(Since this had to be done for 271 different values of γ, there were in all 271×100 or nearly 30,000 calculations to do, and the whole process took several hours.) Finally, for each value of γ, these hundred values of $\phi(t)$ were drawn as dots on the plot of ϕ against γ. To see what this accomplishes, consider first a value of γ such as $\gamma = 1.065$ where we know that the motion has period 1. With the period equal to 1, the hundred successive values of $\phi(t)$ are all the same, and the 100 dots all land at the same place in the plot of ϕ against γ. Thus what we see at any γ for which the period is 1 is a *single* dot. From $\gamma = 1.06$ till the threshold value $\gamma_1 = 1.0663$ where the period doubles, our plot is therefore a single curve.

At the threshold $\gamma_1 = 1.0663$, the period changes to 2, and the positions

$$\phi(501),\ \phi(502),\ \phi(503),\ \cdots,\ \phi(600)$$

now alternate between *two different* values. Therefore, these 100 points actually create exactly two distinct dots on the plot, and the single curve *bifurcates* at γ_1 into *two* curves. At $\gamma_2 = 1.0793$ the period doubles again, to period 4, and each of the two curves bifurcates, giving four curves in all. The next doubling, to period 8, is easily seen (though I have not actually indicated it on the picture), and if you look closely you can just pick out some of the bifurcations to period 16. After this, the graph becomes a nearly solid confusion of points, and it is impossible to tell (from the graph, at least) the exact value γ_c where chaos begins, though it is clearly somewhere just below $\gamma = 1.083$. Beyond this point, for the remainder of Figure 12.17, the motion is mostly chaotic, though you can see a small window at $\gamma = 1.0845$, indicated by a vertical dashed line. (The window is especially noticeable in the upper section of the plot, where the dots are otherwise denser.) If you hold a ruler to this vertical line, you will see that at this particular value of γ there are just six distinct points. That is, at $\gamma = 1.0845$, the motion has returned briefly to being periodic, this time with period 6.

A Larger View

Figure 12.17 shows a rather small range of drive strengths ($1.06 \leq \gamma \leq 1.087$) in great detail. Before we examine a larger ranges of drive strengths, we must cope with one small complication. As γ increases, we have seen that the pendulum can start a "rolling" motion in which it makes many complete revolutions. In some cases, it can continue to "roll" indefinitely, so that $\phi(t)$ eventually approaches $\pm\infty$. Even if this rolling motion is perfectly periodic, the successive values

$$\phi(t_o),\ \phi(t_o + 1),\ \phi(t_o + 2),\ \cdots$$

never repeat themselves, since they increase by a multiple of 2π in each cycle. This renders a bifurcation plot, drawn exactly as in Figure 12.17, useless. The most obvious way to get around this difficulty is to redefine ϕ so that it always lies in the range

$$-\pi < \phi \leq \pi.$$

Each time ϕ increases past π, we subtract 2π, and each time it decreases past $-\pi$, we add 2π. With this modification, we can now draw a bifurcation diagram as before. However, keeping ϕ between $\pm\pi$ in this way has the disadvantage that it introduces a meaningless discontinuous jump into $\phi(t)$, each time it passes $\pm\pi$.

A second, and sometimes simpler, way around the problem of the 2π-ambiguity in ϕ is to plot the values of the *angular velocity*

$$\dot{\phi}(t_o),\ \dot{\phi}(t_o + 1),\ \dot{\phi}(t_o + 2),\ \cdots \tag{12.27}$$

instead of the angular position $\phi(t_o)$, $\cdots$. The angular velocity $\dot{\phi}$ is immune to the 2π-ambiguity of ϕ (since $\dot{\phi}$ is unaffected by the addition of any multiple of 2π to ϕ). Thus, if the motion is periodic with period n, then the values (12.27) will repeat themselves after n cycles, and otherwise not. Therefore, a bifurcation diagram drawn using the values of $\dot{\phi}$ instead of ϕ will work just like Figure 12.17, even if the pendulum undergoes a rolling motion.

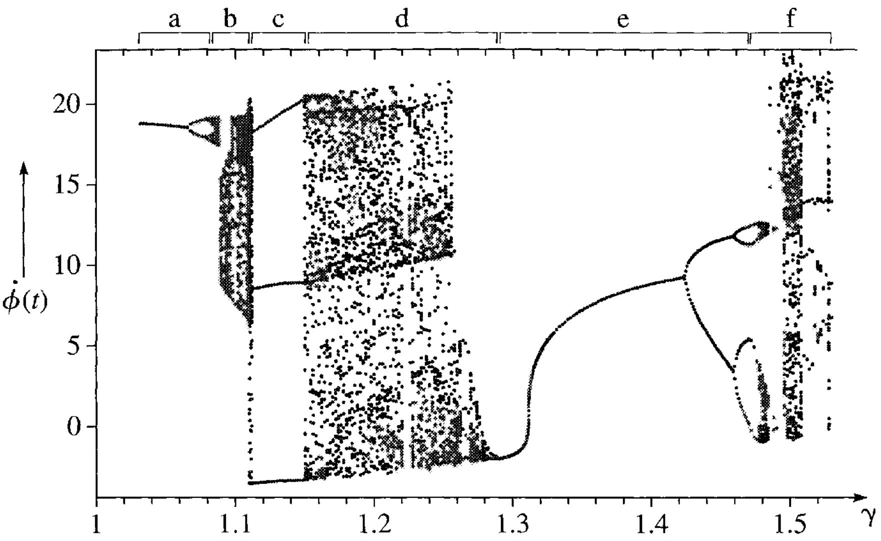

Figure 12.18 Bifurcation diagram showing values of $\dot{\phi}$ for the DDP with drive strengths $1.03 \leq \gamma \leq 1.53$. The intervals labelled $a, b, \cdots, f$ across the top are as follows: **(a)** This interval is the same as was shown in much greater detail in Figure 12.17. It starts with period 1, followed by a period-doubling cascade leading to chaos. **(b)** Mostly chaos. **(c)** Period 3. **(d)** Mostly chaos. **(e)** Period 1, followed by another period-doubling cascade. **(f)** Mostly chaos.

Figure 12.18 is a bifurcation diagram drawn using values of $\dot{\phi}$ over a range from just above $\gamma = 1.0$ to just above $\gamma = 1.5$. The first part of this picture, labelled (a) at the top, is the interval that was shown in much greater detail in Figure 12.17, with a period-doubling cascade that starts from period 1 and ends in chaos. Section (b) is mostly chaos, although we already know that it contains some narrow windows of periodicity (most of which are completely hidden at the scale used here). Section (c) is very clearly period 3 and includes the value $\gamma = 1.13$ that was shown in Figure 12.14. Section (d) is mostly chaos, while (e) starts with a long stretch of period 1, followed by another period-doubling cascade. Finally, section (f) is mostly chaos, although you can just pick out some windows of periodicity.

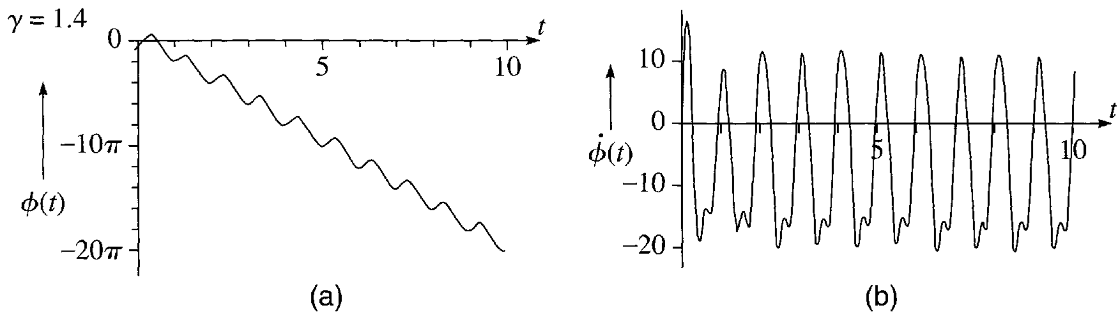

Figure 12.19 Motion of the DDP with drive strength $\gamma = 1.4$. **(a)** The plot of $\phi(t)$ against t shows a periodic rolling motion in which ϕ decreases by 2π in each drive cycle. **(b)** The plot of angular velocity $\dot{\phi}(t)$ against t shows even more clearly that after about two drive cycles the motion becomes periodic, with $\dot{\phi}(t)$ returning to the same value once each cycle.