12.5 Chaos and Sensitivity to Initial Conditions

If we increase the drive strength γ beyond the critical value $\gamma_c = 1.0829$, then our DDP begins to exhibit the behavior that has come to be called "chaos." Figure 12.10 shows the first thirty drive cycles of the DDP with $\gamma = 1.105$. The pendulum is obviously "trying" to oscillate with the period of the driver. Nevertheless, the actual oscillations wander around erratically and never repeat themselves exactly. Of course you might wonder if I have not given the oscillations time to settle down; perhaps at some later time they would converge to a periodic motion. In fact, however, a graph of any time interval is just as erratic, but never an exact repetition of any other interval. Even though the driving force is perfectly periodic and even after the transients have all died out, the long-term motion is definitely nonperiodic. This erratic, nonperiodic long-term behavior is one of the defining characteristics of chaos. The other defining characteristic is the phenomenon called sensitivity to initial conditions.

Sensitivity to Initial Conditions

The issue of sensitivity to initial conditions arises in connection with the following questions: Imagine two identical DDP's, with all parameters the same, but launched at $t = 0$ with slightly different initial conditions. [Perhaps the initial angles $\phi(0)$ differ by a fraction of a degree.] As time goes by, do the motions of the two pendulums remain nearly the same? Do they perhaps get closer to one another? Or do they diverge and become more and more different?

To make these questions more precise, let us denote the positions of the two pendulums by $\phi_1(t)$ and $\phi_2(t)$. These two functions satisfy exactly the same equation of motion, but have slightly different initial conditions. If now $\Delta\phi(t)$ denotes the difference between our two solutions,

$$\Delta\phi(t) = \phi_2(t) - \phi_1(t), \tag{12.21}$$

the issue is the time dependence of $\Delta\phi(t)$. Does $\Delta\phi(t)$ stay more-or-less constant? Does it decrease as time goes by? Or does it increase?

For the linear oscillator discussed in Chapter 5, the answer is that $\Delta\phi(t)$ goes to zero, since we proved that all solutions of the equation of motion approach the

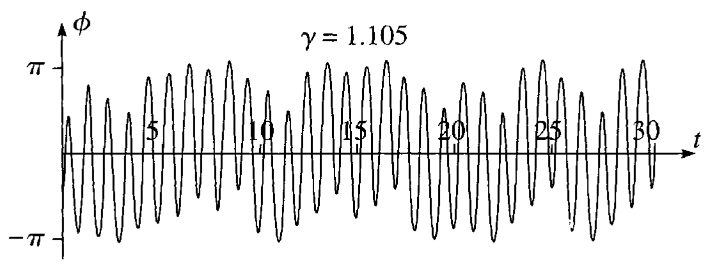


Figure 12.10 Chaos. The first 30 drive cycles of the DDP with $\gamma = 1.105$ are erratic and show no signs of periodicity. In fact the oscillations never do settle down to a regular periodic motion, and this erratic, nonperiodic long-term motion is one of the defining characteristics of chaos.

same attractor as $t \to \infty$. Therefore the *difference* of any two solutions must approach zero. Further, the difference must approach zero *exponentially*. To see this, recall from Equation (5.67) that any solution has the form

$$\phi(t) = A\cos(\omega t - \delta) + C_1 e^{r_1 t} + C_2 e^{r_2 t} \tag{12.22}$$

where the cosine term is the same for all solutions, whereas the two decaying exponential terms have coefficients C_1 and C_2 that depend on the initial conditions. This implies that when we take the difference of two solutions the cosine term drops out and we are left with

$$\Delta\phi(t) = B_1 e^{r_1 t} + B_2 e^{r_2 t}, \tag{12.23}$$

where the constants B_1 and B_2 depend on the two sets of initial conditions. The precise behavior of this difference depends on the relative sizes of the damping constant β and the natural frequency ω_o. In all of the examples so far in this chapter I have chosen $\beta = 0.25\omega_o$, so that $\beta < \omega_o$ (the situation called underdamping). In this case, we saw in Section 5.4 that the coefficients r_1 and r_2 have the form $-\beta \pm i\omega_1$. Some simple algebra then puts (12.23) in the form [compare Equation (5.38)]

$$\Delta\phi(t) = De^{-\beta t}\cos(\omega_1 t - \delta). \tag{12.24}$$

That is, $\Delta\phi(t)$ is the exponential $e^{-\beta t}$ times an oscillatory cosine.

There is a problem in trying to display the time dependence of a function like (12.24). The exponential factor decays so fast that one cannot easily show its range of values on a conventional graph. For example, with the values I have been using, $\beta = 0.25\omega_o = 0.75\pi = 2.356$, after just one drive cycle ($t = 1$) the exponential factor is $e^{-\beta t} = e^{-2.356} \approx 0.09$, and $\Delta\phi(t)$ has diminished by an order of magnitude. If we wanted to plot $\Delta\phi(t)$ against t over 10 cycles, say, then $\Delta\phi(t)$ would shrink by about ten orders of magnitude — a range that cannot possibly be shown on a simple linear plot of $\Delta\phi(t)$ against t.

As you probably know, the solution to this problem is to make a logarithmic plot; that is, we plot the *log* of $\Delta\phi(t)$ against t. Actually since $\Delta\phi(t)$ can be negative, we must plot $\ln|\Delta\phi(t)|$ against t. According to (12.24) this should obey

$$\ln|\Delta\phi(t)| = \ln D - \beta t + \ln|\cos(\omega_1 t - \delta)|. \tag{12.25}$$

The first term on the right is a constant, and the second is linear in t with slope $-\beta$. The third is a little complicated: Since $|\cos(\omega_1 t - \delta)|$ oscillates between 1 and 0, its natural log oscillates between 0 and $-\infty$. Thus a graph of $\ln|\Delta\phi(t)|$ against t should bounce up and down (going to $-\infty$ each time the cosine term vanishes), underneath an envelope that decreases linearly with slope $-\beta$. This is clearly visible in Figure 12.11, which shows a plot of $\log|\Delta\phi(t)|$ against t for the relatively weak driving strength $\gamma = 0.1$, for which the linear approximation is certainly good. [I plotted the log to base 10 rather than the natural log, because the former is easier to interpret on a graph. Since $\log(x)$ is just the constant $\ln(10)$ times $\ln(x)$, this changes none of our qualitative predictions.] To plot this graph, I gave the first pendulum the same initial conditions as in Figures 12.8 and 12.10; the second pendulum was released with its

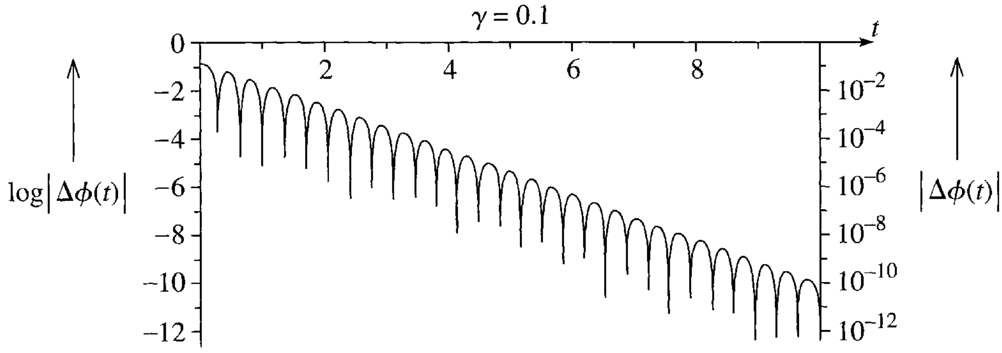


Figure 12.11 Logarithmic plot of $\Delta\phi(t)$, the separation of two identical DDP's, with a weak drive strength $\gamma = 0.1$, that were released with initial positions that differ by 0.1 radians (or about 6°). The vertical axis on the left shows $\log|\Delta\phi(t)|$, while that on the right shows $|\Delta\phi(t)|$ itself. The picture shows clearly that the maxima of $\log|\Delta\phi(t)|$ decrease linearly, and hence that $\Delta\phi(t)$ decays exponentially.

initial position 0.1 radians lower, so that the initial difference was $\Delta\phi(0) = 0.1$ rad, or about 6 degrees. The most important feature of the plot is that the successive maxima of $\log|\Delta\phi(t)|$ decrease perfectly linearly, confirming that $\Delta\phi(t)$ decays exponentially, dropping by about 10 orders of magnitude in the first ten drive cycles (as you can easily check from the graph).[13]

So far we have proved that, in the linear regime, the separation $\Delta\phi(t)$ of two identical DDP's, launched with different initial conditions, decreases exponentially. This has an important practical consequence: In practice, we cannot possibly know the initial conditions of any system *exactly*. Therefore, when we try to predict the future behavior of our DDP we must recognize that the initial conditions we use may differ a little from the true initial conditions. This means that our predicted motion for $t > 0$ may differ from the true motion. But because $\Delta\phi(t)$ goes to zero exponentially, we can be sure that our error will never be worse than the initial error and will, in fact, rapidly approach zero. We can say that the linear oscillator is *insensitive to its initial conditions*. To achieve any prescribed accuracy in our predictions, we have only to ascertain the initial condition to this same accuracy.

What happens as we increase the drive strength γ out of the linear regime? Naturally, we can no longer depend on our proofs for the linear oscillator. However, we can reasonably expect that the difference $\Delta\phi(t)$ will continue to decay exponentially for at least some range of drive strengths. The question is "How large is this range?" and the answer is quite surprising: Provided the difference in initial conditions is sufficiently small, the difference $\Delta\phi(t)$ continues to decay exponentially for all values of γ up to the critical value γ_c at which chaos sets in. For example, if $\gamma = 1.07$ we

[13] A second noteworthy feature is that the points where $|\Delta\phi(t)|$ vanishes (and, hence, $\log|\Delta\phi(t)|$ goes to $-\infty$) show up on the logarithmic plot as sharp downward spikes. This is because the plotting program can only sample a finite number of points and naturally misses the points where $\log|\Delta\phi(t)| = -\infty$. Instead it can only detect that there are points where $\log|\Delta\phi(t)|$ has a precipitous minimum, and this is what it shows.

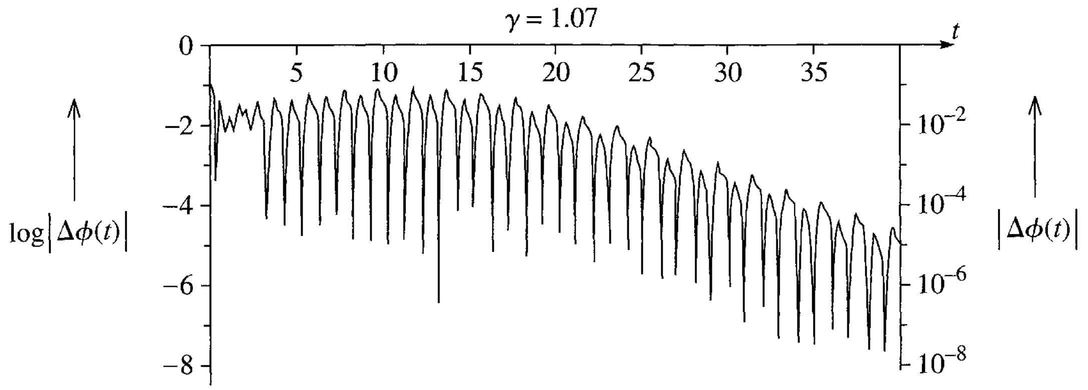


Figure 12.12 Logarithmic plot of $\Delta\phi(t)$, the separation of two identical DDP's, with drive strength $\gamma = 1.07$, that were released with initial positions that differ by 0.1 radians (or about 6°). For the first 15 or so drive cycles, $\Delta\phi(t)$ holds fairly constant in amplitude, but then the maxima of $\log|\Delta\phi(t)|$ decrease linearly, implying that $\Delta\phi(t)$ decays exponentially.

know that the motion has period 2 (for the initial conditions of Figures 12.8 through 12.11), and the motion is distinctly nonlinear; nonetheless, the difference $\Delta\phi(t)$ still decays exponentially, as is clearly visible in Figure 12.12, which shows the difference $\Delta\phi(t)$ for two solutions with the same initial conditions as in Figure 12.11. In this case, $\Delta\phi(t)$ remains pretty well constant in amplitude for the first 15 or 20 drive cycles, but then the crests of $\log|\Delta\phi(t)|$ drop perfectly linearly, indicating that $\Delta\phi(t)$ decays exponentially as $t \to \infty$. Notice, however, that the exponential decay is considerably slower than in the linear case: Here the amplitude drops by about 4 orders of magnitude in the last 25 cycles; in the linear case of Figure 12.11, it dropped by 10 orders of magnitude in just 10 cycles. Nevertheless, the main point is that $\Delta\phi(t)$ goes to zero exponentially, and, as in the linear regime, we can predict the future behavior of our DDP, confident that any uncertainties in our predictions will be not much larger (and usually much smaller) than our uncertainty in the initial conditions.

If we now increase the drive strength past $\gamma_c = 1.0829$ into the chaotic regime, the picture changes completely. Figure 12.13 shows $\Delta\phi(t)$ for the same DDP as in Figures 12.11 and 12.12, except that the drive strength is now $\gamma = 1.105$, the same value used in our first plot of chaotic motion in Figure 12.10. The most obvious feature of this graph is that $\Delta\phi(t)$ clearly *grows* with time. In fact, you will notice that to highlight the growth of $\Delta\phi(t)$ I chose the initial difference to be just $\Delta\phi(0) = 0.0001$ radians. Starting from this tiny value, $|\Delta\phi|$ has increased in 16 drive cycles by more than 4 orders of magnitude to about $|\Delta\phi| \approx 3.5$.

From $t = 1$ through $t = 16$ (where the graph is leveling off), the maxima in Figure 12.13 grow almost perfectly linearly, implying that $\Delta\phi(t)$ grows exponentially.[14] This exponential growth spells disaster for any attempt at accurate prediction of the DDP's

[14] The eventual leveling of the curve is easily understood. You can see from Figure 12.10 that the angle $\phi(t)$ [actually, $\phi_1(t)$, but the same applies to $\phi_2(t)$] oscillates between about $\pm\pi$. That is, neither $\phi_1(t)$ nor $\phi_2(t)$ ever exceeds magnitude π. Therefore their difference $\Delta\phi(t)$ can never exceed 2π. Thus the curve *has* to level off before $\Delta\phi(t)$ reaches 2π.

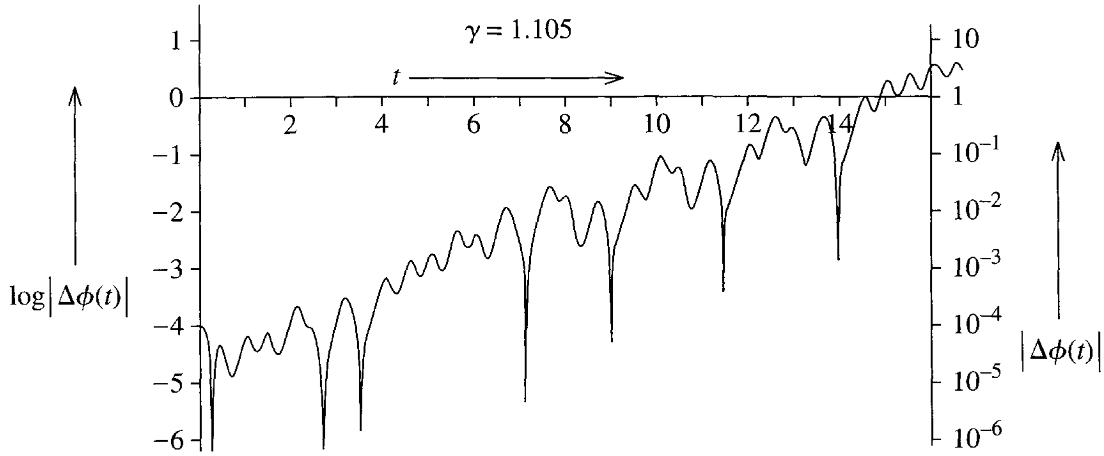


Figure 12.13 The separation $\Delta\phi(t)$ of two identical pendulums, both with drive strength $\gamma = 1.105$ and with an initial separation $\Delta\phi(0) = 10^{-4}$ rad. After a small initial drop, the crests of $\log|\Delta\phi(t)|$ *increase linearly*, showing that $\Delta\phi(t)$ itself *grows exponentially*.

long-term motion. For the present case, an error as small as 10^{-4} radians in our initial conditions will have grown in 16 cycles to an error of about 3.5, or more than π radians. Thus an uncertainty of $\pm 10^{-4}$ radians in the initial conditions grows to an uncertainty of $\pm\pi$, and an uncertainty of $\pm\pi$ in the angle of a pendulum means that we have *no idea at all* where the pendulum is! I chose this example because it is especially dramatic. Nevertheless, in any chaotic motion, $\Delta\phi(t)$ grows exponentially for a while at least. Even if this growth levels out before $\Delta\phi(t)$ reaches π, the exponential growth means that a tiny uncertainty in the initial conditions quickly grows into a large uncertainty in the predicted motion. It is in this sense that we say chaos exhibits **extreme sensitivity to initial conditions**, and this sensitivity is what can make the reliable prediction of chaotic motion a practical impossibility.

The Liapunov Exponent

What we have seen in the preceding three examples can be rephrased to say that the difference $\Delta\phi(t)$ between two identical DDP's released with slightly different initial conditions behaves exponentially:

$$\Delta\phi(t) \sim K e^{\lambda t} \tag{12.26}$$

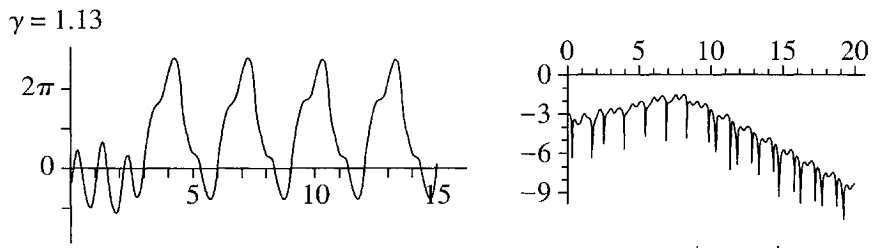
(where the symbol "$\sim$" signifies that $\Delta\phi(t)$ may oscillate underneath an envelope with the advertized behavior, and K is a positive constant). The coefficient λ in the exponent is called the **Liapunov exponent.**[15] If the long-term motion is nonchaotic (settles down to periodic oscillation) the Liapunov exponent is negative; if the long-term motion is chaotic (erratic and nonperiodic) the Liapunov exponent is positive.

[15] Strictly speaking there are several Liapunov exponents, of which the one discussed here is the largest.

Higher Values of γ

So far we have seen that as we increased the drive strength γ, the motion of our DDP became more and more complicated — from the linear regime, with its pure sinusoidal response, to the nearly linear regime, with the addition of harmonics, to the appearance of subharmonics and (for certain initial conditions at least) a period-doubling cascade, and finally to chaos. You might naturally anticipate that, if we were to increase γ still further, the chaos would continue and intensify, but as usual our nonlinear system defies predictions. As γ increases, the DDP actually alternates between intervals of chaos separated by intervals of periodic, nonchaotic motion. I shall illustrate this with just two examples.

We saw that with $\gamma = 1.105$, the DDP exhibits chaotic behavior (Figure 12.10) and exponential divergence of neighboring solutions (Figure 12.13). We have only to increase the drive strength to $\gamma = 1.13$ to enter a narrow "window" of *nonchaotic* period-3 oscillation with exponential *convergence* of neighboring solutions, as shown in Figure 12.14. In part (a), you can see that within three drive cycles the motion settles into regular period-3 oscillations. Part (b) shows the separation $\Delta\phi(t)$ of two pendulums released with an initial difference $\Delta\phi(0)$ of 0.001 radians; in the first eight drive cycles, $\Delta\phi(t)$ actually increases, but from then on it decreases exponentially to zero, dropping by six orders of magnitude in the next twelve cycles.



(a) $\phi(t)$ vs t (b) $\log|\Delta\phi(t)|$ vs t

Figure 12.14 Motion of a DDP with $\gamma = 1.13$. **(a)** The graph of $\phi(t)$ quickly settles down to oscillations of period 3 (same initial conditions as in Figures 12.8 and 12.10). **(b)** Logarithmic plot of the distance between two identical pendulums, the first with the same initial conditions as in part (a), the second with its initial angle 0.001 radians lower. After an initial modest increase, $\Delta\phi(t)$ goes to 0 exponentially.

Figure 12.15 shows the corresponding two graphs for a drive strength of $\gamma = 1.503$, where the motion has returned to being chaotic.[16] In part (a) we see a new kind of chaotic motion. The driving force is now strong enough to keep the pendulum rolling right over the top, and in the first 18 drive cycles the pendulum makes 13 complete clockwise rotations [$\phi(t)$ decreases by 26π]. The motion here can be seen as a steady rotation at about one revolution per drive cycle, with an erratic

[16] As we shall see, between the values $\gamma = 1.13$ and 1.503, shown in Figures 12.14 and 12.15, the DDP has passed through several intervals of chaotic and nonchaotic motion, but I omit the details for now.

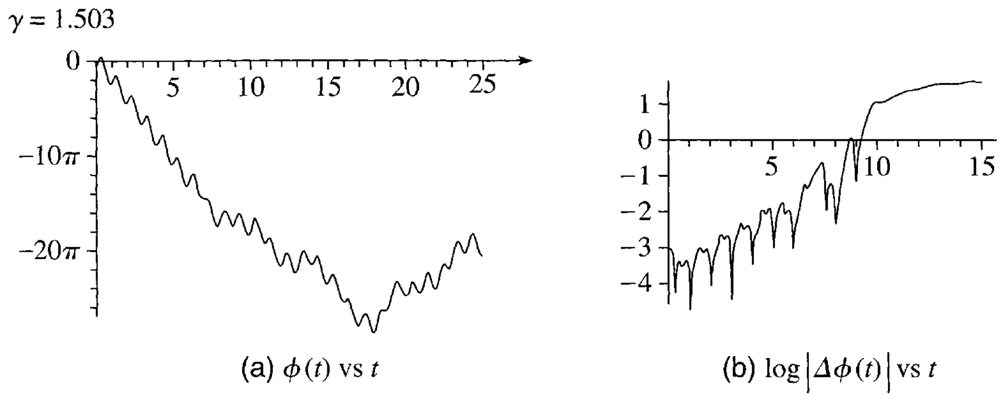


Figure 12.15 Motion of a DDP with $\gamma = 1.503$. (Same initial conditions as in Figure 12.14.) **(a)** The graph of $\phi(t)$ against t oscillates erratically. In the first 18 cycles, it plunges to about -26π; that is, it makes about 13 complete clockwise rotations. It then starts to climb back but never actually repeats itself. **(b)** Logarithmic plot of the distance between two identical pendulums with an initial separation of 0.001 radians. For the first nine or ten cycles, $\Delta\phi(t)$ grows exponentially and then levels out.

oscillation superposed.[17] At $t = 18$, the motion reverses to a more-or-less steady counterclockwise rotation with erratic oscillations superposed, and, as the picture suggests, the motion never settles down to be periodic.

The logarithmic plot of Figure 12.15(b) shows the divergence of two pendulums with the same drive strength $\gamma = 1.503$, but an initial separation of 0.001 radians. The separation of the two pendulums increases exponentially for the first 9 or 10 cycles and levels out by about $t = 15$. A dramatic feature of this divergence is that it is big enough to be seen in the conventional linear plot of Figure 12.16, which shows the actual positions $\phi_1(t)$ and $\phi_2(t)$ of the two pendulums. At first sight it is perhaps surprising that for the first 8.5 cycles the two curves are completely indistinguishable, but that the difference is then so abundantly visible. However, you can understand this striking behavior by reference to Figure 12.15(b), where you can see that until $t \approx 8.5$ the separation $\Delta\phi(t)$, although growing rapidly, is nevertheless always less than about 1/3 radian — too small to be seen on the scale of Figure 12.16. By the time $t \approx 9.5$, $\Delta\phi(t)$ has reached about 3 — which is easily visible on the linear plot — and is still climbing rapidly. Thus from $t \approx 9.5$ the two curves are completely distinct.

The main morals to be drawn from these last two examples are these: **(1)** Once the drive strength γ of our DDP is past the critical value $\gamma_c = 1.0829$, there are intervals where the motion is chaotic and others where it is not. These intervals are often quite narrow, so that the chaotic motion comes and goes with startling rapidity. **(2)** The chaos can take on several different forms, such as the erratic "rolling" motion of Figure

[17] If you look closely you can see that for the first 7 cycles, the motion is very close to being a steady rotation of -2π per cycle, with a *regular* period-1 oscillation superposed. This type of motion is actually periodic, since a change of -2π brings the pendulum back to the same place. For some values of the drive strength, the long-term motion settles down exactly this way, a phenomenon called phase locking. (See Problem 12.17.)

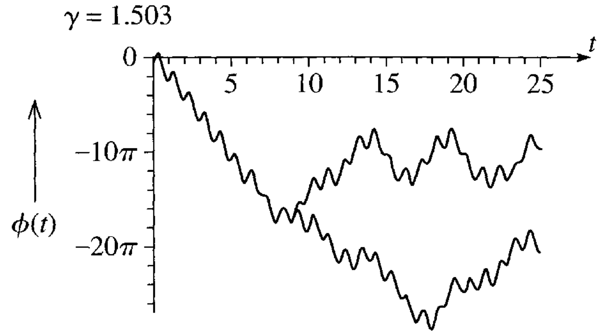


Figure 12.16 Linear plot of the positions of the same two identical DDP's whose separation $\Delta\phi(t)$ was shown in Figure 12.15(b) [$\Delta\phi(0) = 0.001$ rad]. For the first eight and a half drive cycles, the two curves are indistinguishable; after this the difference is dramatically apparent.

12.15(a). **(3)** The erratic motion of chaos always goes along with the sensitivity to initial conditions associated with the exponential divergence of neighboring solutions of the equation of motion.

12.6 Bifurcation Diagrams

So far, each of our pictures of the motion of the driven damped pendulum has shown the motion for one particular value of the drive strength γ. To observe the evolution of the motion as γ changes, we have had to draw several different plots, one for each value of γ. One would like to construct a single plot that somehow displayed the whole story, with its changing periods and its alternating periodicity and chaos as γ varies. This is the purpose of the bifurcation diagram.

A **bifurcation diagram** is a cunningly constructed plot of $\phi(t)$ against γ as in Figure 12.17. Perhaps the best way to explain what this plot shows is just to describe in detail how it was made. Having decided on a range of values of γ to display (from $\gamma = 1.06$ to 1.087 in Figure 12.17) one must first choose a large number of values of γ, evenly spaced across the chosen range. For Figure 12.17, I chose 271 values of γ, spaced at intervals of 0.0001,

$$\gamma = 1.0600,\ 1.0601,\ 1.0602,\ \cdots,\ 1.0869,\ 1.0870.$$

For each chosen value of γ, the next step is to solve numerically the equation of motion (12.11) from $t = 0$ to a time $t_{\max}$ picked so that all transients have long since died out. To make Figure 12.17, I chose the same initial conditions as in the last few pictures, namely $\phi(0) = -\pi/2$ and $\dot{\phi}(0) = 0$.[18]

[18] Some authors like to superpose the plots for several different initial conditions. This gives a more complete picture of the many possible motions, but makes the plot harder to interpret. For simplicity, I chose to use just one set of initial conditions.