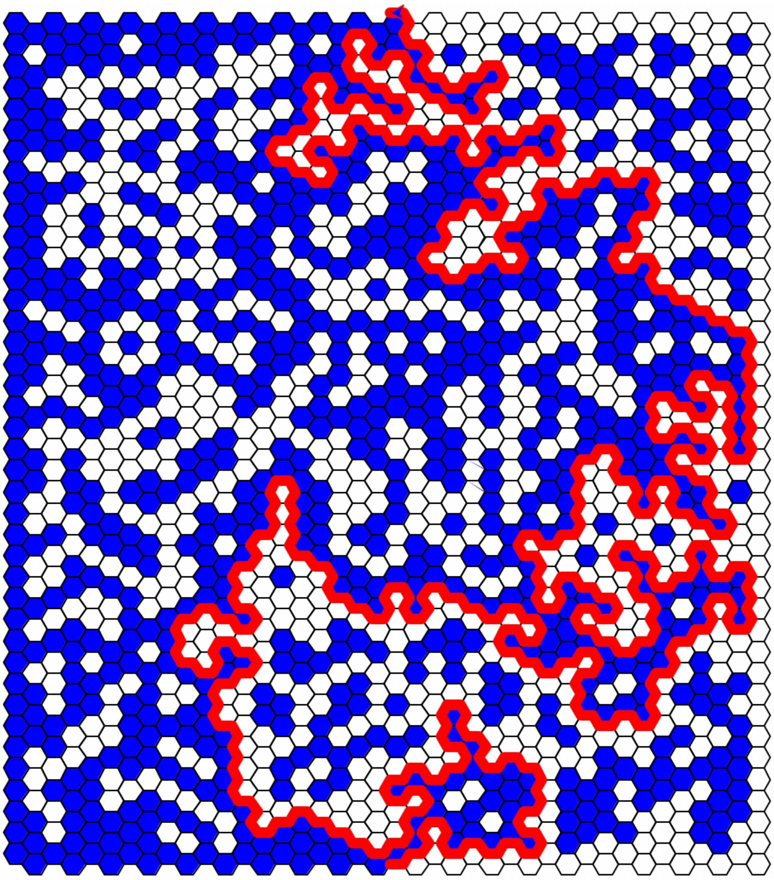
KTH ROYAL INSTITUTE
OF TECHNOLOGY

Degree Project in Mathematics

First cycle, 15 credits

# Random curves and their scaling limits


**JONATAN S. WÄCHTER**

Stockholm, Sweden 2022

# Abstract

We focus on planar Random Walks and some related stochastic processes. The discrete models are introduced and some of their core properties examined. We then turn to the question of continuous analogues, starting with the well-known convergence of the Random Walk to Brownian Motion. For the Harmonic Explorer and the Loop Erased Random Walk, we discuss the idea for convergence to SLE($\kappa$) and carry out parts of the proof in the former case using a martingale observable to pin down the Loewner driving process.

## Acknowledgements

I want to thank my supervisor Fredrik Viklund for an interesting topic and for making some relatively recent developments of the subject accessible to me. It has been a pleasure to discuss ideas with him. I also want to thank my programme director Sara Zahedi for her advice and guidance over the past three years, and for encouraging me to contact Fredrik.

# Contents

# 1 Introduction

In the modelling of physical systems it is common to start with something discrete, where the individual components can be clearly separated and their interdependencies described. Once this has been done, it might be possible to make the discretization finer and finer to obtain something continuous. Some of the models we will consider in this thesis draw inspiration from statistical physics, and specifically the subfield of critical phenomena. Physiscists try to understand the behaviour of such systems near a critical point, i.e. what happens to certain observables of the system, and how they change with respect to the distance to the critical point. They are also interested in symmetries that arise in the model. This could be scale and rotational invariance, conformal invariance as well as different notions of fractality. Since the models are stochastic, this self-similarity is also in a probabilistic sense.

Many interesting symmetries become apparent first in the continuous model and that is a large part of the motivation to come up with continuous analogues of the discrete ones. At the same time, the technical difficulties tend to increase in the continuum and so 'taking the limit' is not always straightforward.

After a brief recap of some Probability Theory and Complex Function Theory, the main part of this thesis will start by introducing the random walk, which can be seen as a lattice model for random movement. This example is interesting because of the relative simplicity, and for the fact that the scaling limit, i.e. the continuous analogue, Brownian Motion, is well understood. Both the discrete and the continuous model are strongly related to harmonic functions, and this theoretical direction is developed. When we start building new models based on the Random Walk, harmonic functions will play a very important role.

One such model we will consider is the Harmonic Explorer, which can be seen as a sequence of random walks determining the evolution of a domain by cutting out a curve from one point in the boundary to another point. We will derive some of its properties and make a comparison with the Loop Erased Random Walk, which has an even clearer connection to Random Walk. While the latter of the two has some relation to polymer formation, these models are examined for mathematical interest only, and have little connection to physical questions.

Both models are non-Markovian, in the sense that the transition probabilities governing the tip point of the curve at time $n$ to a new point at time $n+1$, do depend on all previous times, and not just, as in the case of the Random Walk, the current time. We will see that this makes the analysis a lot harder, but that some results can still be derived by transferring questions to the underlying Random Walk.

In the last chapter we will see the sense in which a Random Walk, properly rescaled in time and space converges in distribution to a Brownian Motion. The latter model is conformally invariant, providing one example where the continuous model has a symmetry that its discrete counterpart is lacking. It will not be immediate to transfer the results of this limit to the two other models mentioned, a big problem being the non-Markovian property. By realizing that the two discrete models can in fact be viewed as Markov Chains on a state space of point-domain-tuples we will get some intuition for the approach taken instead: In the late 90s it was shown that a continuous analogue of this so called Markov Domain Property, together with conformal invariance, determines a one-parameter family of random curves, the Schramm-Loewner-Evolution with parameter $\kappa$, abbreviated $SLE(\kappa)$.

A lot of work consequently went into understanding the nature of the convergence to $SLE(\kappa)$. We will develop some of the theory of SLE and then take first steps towards a proof of convergence for the Harmonic Explorer to an SLE-curve with $\kappa = 4$. The proof idea is the same as for the convergence of Loop Erased Random Walk to $SLE(2)$ and uses a 'martingale observable' to pin down the so called Loewner driving process of the randomly evolving curve.

# 2 Preliminaries

## 2.1 Probability Theory

We assume familiarity with measure-theoretic probability theory; a good reference is [3].

### 2.1.1 Stochastic processes and martingales

In this paper we aim to study a particular class of stochastic processes in the plane. In so doing, we will not restrict ourselves to martingales, but these turn out to be very important in the theoretical development.

**Definition 2.1.** *Given a probability space $(\Omega, \mathcal{A}, P)$ and an index set $\mathcal{J}$ we call collections of random variables $\{X_t, t \in \mathcal{J}\}$ a stochastic process.*

The set $\mathcal{J}$ we will generally take to be $\mathbb{N}$ or $[0, \infty)$, representing time.

**Definition 2.2.** *Let $(\Omega, \mathcal{F}, P)$ be a probability space and $\mathcal{J}$ a totally ordered index set. Let $\mathcal{F}_t$ be a sequence of sigma-algebras on $\Omega$, such that $\mathcal{F}_s \subseteq \mathcal{F}_t$ for all $s \leq t$, where $s, t \in \mathcal{J}$. We call the sequence $\mathcal{F}_t$ a filtration.*

**Definition 2.3.** *A family of random variables $(M_n)_{n\geq 0}$ is called a martingale with respect to the filtration $(\mathcal{F}_n)$ if*

- *$M_n$ is $\mathcal{F}_n$-measurable and in $L^1$.*
- $E[M_{n+1}|\mathcal{F}_n] = M_n$

**Definition 2.4** (Stopping time)**.** *A random variable $T$, taking values in $\mathcal{J} \cup \{\infty\}$ is a stopping time with respect to $(\mathcal{F}_t)_{t\in\mathcal{J}}$ if $\{T \leq t\} \in \mathcal{F}_t$ for all $t \in \mathcal{J}$.*

### 2.1.2 Convergence

We will need some notions of convergence: Almost sure convergence, convergence in probability and $L^p$-convergence. We also assume familiarity with uniform integrability and how this property together with convergence in probability implies convergence in $L^1$. Convergence in distribution is what we will chiefly be interested in. To highlight the last two points we record:

**Definition 2.5** (Uniform integrability (UI))**.** *We say that the sequence of random variables $(X_n)$ is UI provided that $\sup_n E[|X_n|\mathbf{1}(|X_n| \geq K)] \to 0$ as $K \to \infty$.*

**Theorem 2.6.** *UI + CV Prob $\Leftrightarrow$ CV $L^1$*

**Definition 2.7** (Weak convergence). *We say that a sequence of probability measures $(\mu_n)_{n\geq 0}$ converges weakly to $\mu$ if for any continuous and bounded function $f$ it holds*

$$\int f d\mu_n \to \int f d\mu \tag{1}$$

**Remark 2.8.** *If the probability measures $(\mu_n)$ associated to the random variables $(X_n)$ converge weakly to $\mu$ associated to $X$, then we say that $X_n$ converges in distribution or law to $X$, denoted $X_n \xrightarrow{d} X$*

The following theorem gives an equivalence between weak convergence and other more or less intuitive possible starting points for a definition and will be used in later sections:

**Theorem 2.9** (Portmanteau). *Let $\mu_n$ be probability measures on a metric space. The following are equivalent*

- $\mu_n \to \mu$ *weakly*
- $\liminf_n \mu_n(O) \geq \mu(O)$ *for all open sets $O$*
- $\limsup_n \mu_n(C) \leq \mu(C)$ *for all closed sets $C$*
- $\lim_n \mu_n(A) = \mu(A)$ *for all measurable $A$ with $\mu(\partial A) = 0$*

**Theorem 2.10.** *Using the shorthand CV for 'convergence in', the hierarchy among different types of convergence can be written as follows:*

$$CV L^q \implies CV L^p \implies CV L^1 \implies CV \textit{ Probability} \impliedby CV \textit{ a.s.} \tag{2}$$
$$CV \textit{ Probability} \implies CV \textit{ Distribution} \tag{3}$$

*where $q > p$.*

## 2.2 Complex Analysis

We assume familiarity with complex analysis up to the level of the first eight chapters of [12]. Below we highlight some of the most important results for our purposes.

### 2.2.1 Conformal maps

The Riemann mapping theorem can be used to transfer problems from a proper domain of the complex plane to the unit disk. In our case, this proper domain will be the geometric object that our stochastic process operates on. In the very last chapter we will first map the process to the unit disk and then to the upper halfplane. This will allow us to i.) obtain some geometric estimates and ii.) compare the process to the canonical representation of the 'universal object' hinted at in the introduction.

**Theorem 2.11** (Riemann mapping theorem)**.** *Let $D$ be a proper simply connected domain. Then there exists a conformal isomorphism $\varphi : D \to \mathbb{D}$.*

### 2.2.2 Area and distortion

For control over how conformal maps stretch and distort regions of the complex plane, we collect some key theorems that will be used extensively in the last chapter.

Let $U$ be a region of the complex plane.

**Definition 2.12** (Univalent functions)**.** *A function $f : U \to \mathbb{C}$ is called univalent if it is holomorphic and injective.*

The following classes of univalent functions are central to geometric function theory.

**Definition 2.13** (The Class $\mathcal{S}$)**.** *Let $\mathcal{S}$ be the class of univalent functions $f$ from the unit disk $\mathbb{D}$ to the complex plane, normalized to have $f(0) = 0$ and $f'(0) = 1$.*

Functions in the class $\mathcal{S}$ have expansions of the form $f(z) = z + a_2 z^2 + a_3 z^3 + \ldots$. By the Riemann mapping theorem, proper simply connected domains give rise to conformal maps into $\mathbb{D}$. Under proper rescaling and translation of the domain, the inverse of such a map will be an element of $\mathcal{S}$. So the function class just defined codifies equivalence classes of proper simply connected domains. This correspondence makes it possible to study geometry from a function perspective, or vice versa.

Let $\Delta = \{z \in \mathbb{C} : |z| > 1\}$. This is the image of $\mathbb{D}$ under the inverse map, provided that we recognize $\infty$ and associate it with 0. The natural correspondence between these two domains can be extended to a natural correspondence between functions on them:

**Definition 2.14** (The Class $\Sigma$)**.** *Let $\Sigma$ be the class of univalent functions $g$ from $\Delta$ to the complex plane, such that $\lim_{z\to\infty} g(z) = \infty$ and $\lim_{z\to\infty} g'(z) = 1$.*

Then $\mathcal{S}$ is in bijective correspondence with $\Sigma$. For $f \in \mathcal{S}$, $g(z) = f(z^{-1})^{-1}$ defines a $g \in \Sigma$, and for any $g \in \Sigma$, $f(z) = g(z^{-1})^{-1}$ defines an $f \in \mathcal{S}$. This correspondence will be useful. It holds that $g(z) = z + b_0 + b_1 z^{-1} + b_2 z^{-2} + ....$ One can obtain a bound on the coefficients of $g$, and then translate to derive similar bounds on functions in the class $\mathcal{S}$. These often have geometric interpretations, like infinitesimal distortion. A first step and a key theorem is the following:

**Theorem 2.15** (Area Theorem). *For $g \in \Sigma$ with $g(z) = z + b_0 + b_1 z^{-1} + b_2 z^{-2} + ...$ it holds that*

$$\sum_{n=1}^{\infty} n|b_n|^2 \leq 1 \tag{4}$$

*Proof.* Let $g \in \Sigma$ and $C_r = \{g(re^{i\theta}) : \theta \in [0, 2\pi)\}$. $C_r$ can be associated with a simple curve, separating the plane into an interior and exterior. Let $D_r$ be the interior. By Green's theorem it holds that

$$\begin{aligned} Area(D_r) &= \frac{1}{2i} \int_{\partial D_r} \bar{z} dz & (5)\\ &= \frac{1}{2i} \int_{|z|=r} g(\bar{z}) g'(z) dz & (6)\\ &= \frac{1}{2i} \int_0^{2\pi} g(re^{-i\theta}) g(re^{i\theta}) d\theta & (7)\\ &= \pi \left( r^2 - \sum_{n=1}^{\infty} n|b_n|^2 r^{-2n} \right) & (8) \end{aligned}$$

The left hand side is nonnegative, so that $\sum_{n=1}^{\infty} n|b_n|^2 r^{-2n} \leq r^2$ for $r > 1$. Using domainated convergence we take the limit $r \to 1$ to get the result. ∎

As a first consequence of this theorem, using that all terms in the sum are positive, we have $|b_n| \leq \frac{1}{\sqrt{n}}$, so for example $|b_1| \leq 1$.

Many results can be derived by considering transformations with respect to which $\mathcal{S}$ is invariant and using the connection to the class $\Sigma$.

**Theorem 2.16** (Bieberbach's Theorem). *For functions $f \in \mathcal{S}$ with expansion $f(z) = z + a_1 z + a_2 z^2 + ...$ it holds that $|a_2| \leq 2$.*

The condition that $f(0) = 0$ for $f \in \mathcal{S}$, together with the fact that conformal maps are open, means that $f(\mathbb{D})$ necessarily contains a small ball centered at zero. We seek a lower bound on the radius, i.e. an answer to the question, which ball is always contained in the image, regardless of the particular choice of $f$. It is a priori not clear that there is such a ball, but it turns out that:

**Theorem 2.17.** *For $f \in \mathcal{S}$, $B(0, \frac{1}{4}) \subset f(\mathbb{D})$, i.e. the ball of radius $\frac{1}{4}$ centered at zero is always contained in the image.*

The following theorems give us the promised control over distortion and distances to the boundary:

**Theorem 2.18** (Distortion Theorem)**.** *For $f \in \mathcal{S}$, letting $r = |z| < 1$, it holds*

$$\frac{1-r}{(1+r)^3} \leq |f'(z)| \leq \frac{1+r}{(1-r)^3} \tag{9}$$

**Theorem 2.19.** *For $f \in \mathcal{S}$, letting $r = |z| < 1$, it holds*

$$\frac{r}{(1+r)^2} \leq |f(z)| \leq \frac{r}{(1-r)^2} \tag{10}$$

**Corollary 2.20.** *For a univalent function $f : \mathbb{D} \to \mathbb{C}$, let $d_f(z) := \operatorname{dist}(f(z), f(\partial\mathbb{D}))$. We have the following:*

$$\frac{1}{4}(1-|z|^2)|f'(z)| \leq d_f(z) \leq (1-|z|^2)|f'(z)| \tag{11}$$

**Corollary 2.21.** *Given $f : D \to D'$ a conformal isomorphism, with domain (and consequently also codomain) conformally equivalent to $\mathbb{D}$, then*

$$1/4|f'(z)| \operatorname{dist}(z, \partial D) \leq \operatorname{dist}(f(z), \partial D') \leq 4|f'(z)| \operatorname{dist}(z, \partial D) \tag{12}$$

# 3 Random Walk

## 3.1 Alternative definitions, Markov Chains

**Definition 3.1** (Lattice random walk)**.** *Let $L$ be a lattice with generating set $\sigma$ and $(X_n)_{n\geq 0}$ be a sequence of identically distributed and independent $\sigma$-valued random variables. Set $Y_n = \sum_{i=0}^{n} X_i$. We call $(Y_n)$ a lattice random walk.*

**Example 1.** *The simple random walk on $L = \mathbb{Z}$ is obtained by letting $X_n$ take values in $\{1, -1\}$. The simplest distribution is that which assigns each choice equal probability $1/2$. We call the walk symmetric in that case.*

Another way to view random walks, is in the setting of Markov chains.

**Definition 3.2** (Markov chain)**.** *A sequence of random variables $(X_n)_{n\geq 0}$ is called a discrete time Markov Chain if*

$$P(X_{n+1} = x_{n+1}|X_0 = x_0, ..., X_n = x_n) = P(X_{n+1} = x_{n+1}|X_n = x_n) \tag{13}$$

A Markov Chain is therefore characterised by its transition probabilities $p(x, y) = P(X_1 = y|X_0 = x)$ and the set of values $X_n$ may attain, called the state space. In the case of finite state spaces, the transition probabilities can be codified in a finite dimensional linear transformation and represented as a matrix $P$. This is for instance the case with Random Walk considered in a finite subset of a lattice. We will be interested in stopping the walk once it hits the boundary. In the framework of Markov Chains, the boundary points are then absorbing in the sense that $p(x, x) = 1$ for $x \in \partial D$.

## 3.2 Connection to Dirichlet problem

In the following we will see some first hints of the connection between complex analysis and probability theory. The first discipline offers many tools to study harmonic functions. The setting is a region $D \in \mathbb{R}^d$ with a smooth boundary $\partial D$ and a function $f : \partial D \to \mathbb{R}$. The Dirichlet problem asks us to find a function $u : \bar{D} \to \mathbb{R}$ with $u|_{\partial D} = f$ and $\Delta u = 0$ inside $D$. It turns out that such a function exists, and that it is unique. An equivalent formulation of harmonicity is the mean value property $u(x) = \frac{1}{|\partial B_r(x)|} \int_{\partial B_r(x)} u dS$, integrating over the surface of a ball centered at $x$.

In the discrete case, one further ingredient is a lattice $L = (V, E)$, where the vertices are a subset of $\mathbb{R}^d$. Let $\hat{D} = V \cap D^o$ be the vertices of the lattice that are in the interior of $D$. Let $\partial\hat{D} = \{v \in V : (v, w) \in E, w \in \hat{D}\}$. Given a function $\hat{f} : \partial\hat{D} \to \mathbb{R}$ we have that a function $\hat{u} : \hat{D} \cup \partial\hat{D} \to \mathbb{R}$ is discrete harmonic if $\hat{u}|_{\partial\hat{D}} = \hat{f}$ and that $\hat{u}$ satisfies the mean value property $\hat{u}(v) = \frac{1}{N_v} \sum_{w:(v,w)\in E} \hat{u}(w)$, where $N_v$ denotes the number of neighbors of $v$, i.e. the number of $w$ such that $(v, w) \in E$. In the case of the lattice $\mathbb{Z}^d$, the vertices are points with integer coordinates and the edges connect all vertices with unit distance from one another.

Given mild conditions on the setup just described, it turns out that there is a unique discrete harmonic function $\hat{u}$.

**Theorem 3.3** (Uniqueness discrete harmonic function)**.** *For a given domain $\hat{D} \cup \partial\hat{D}$ and a bounded function $\hat{f} : \partial\hat{D} \to \mathbb{R}$, there is a unique discrete harmonic $\hat{u}$.*

We will make use of a lemma that has a well known counterpart in the continuous case.

**Lemma 3.4** (Discrete Maximum Principle)**.** *If $\hat{u} : \hat{D} \cup \partial\hat{D} \to \mathbb{R}$ is a discrete harmonic function, then either $\hat{u}$ attains its maximum and minimum on $\partial\hat{D}$ or $\hat{u}$ is constant.*

*Proof.* Suppose otherwise that $v_0$ is a maximum point and that $v_0 \in \hat{D}$. By the mean value property $\hat{u}(v_0) = \frac{1}{N_v} \sum_{w:(v,w)\in E} \hat{u}(w)$. Since all terms in the sum are less than or equal to the left hand side and equality holds after taking the mean, it must be the case that $\hat{u}(v_0) = \hat{u}(w)$ for all neighbors $w$. Since all neighbors are maximum points as well, we may iterate the argument to conclude that all vertices are in fact maximum points, hence $\hat{u}$ is constant. The case of minimum points follows in the same way. ■

*Proof.* Let $\hat{u}_1$ and $\hat{u}_2$ be two solutions of the same discrete Dirichlet problem. By linearity, $\hat{u}_1 - \hat{u}_2$ solves the Dirichlet problem with boundary values zero on the same domain. By the maximum principle, $\hat{u}_1 - \hat{u}_2 = 0$. We conclude uniqueness. ■

There is a probabilistic representation of discrete harmonic functions using random walk on the underlying lattice.

**Theorem 3.5.** *Let $\hat{u}$ be the solution to the discrete Dirichlet problem with boundary values $\hat{f}$. Consider a simple lattice random walk $(X_n)$ and let $T = \min\{n \geq 0 :$*

$X_n \in \partial\hat{D}\}$. *Denote by* $E^v$ *the expectation operator corresponding to the measure* $P^v$ *on random walks starting at the vertex* $v$. *Then it holds that*

$$\hat{u}(v) = E^v[\hat{f}(X_T)] \tag{14}$$

**Remark 3.6.** *It is important that* $T$ *is almost surely finite for the above expressions to make sense. For an irreducible Markov chain this is the case.*

*Proof.* If $v \in \partial\hat{D}$ a walk started at $v$ is stopped immediately, and so $E^v[\hat{f}(X_T)] = \hat{f}(X_0) = \hat{f}(v)$, i.e. the boundary values are correct. For $v \in \hat{D}$ we may appply the Markov property of the random walk and the fact that it is simple, $p(v,w) = \frac{1}{N_v}$, to obtain that

$$E^v[\hat{f}(X_T)] = \sum_{w:(v,w)\in E} p(v,w)E^w[\hat{f}(X_T)] \tag{15}$$

$$= \frac{1}{N_v} \sum_{w:(v,w)\in E} E^w[\hat{f}(X_T)] \tag{16}$$

so the mean value property holds as well. ∎

Consider a random walk $X_n$ started at $x$ and a discrete harmonic function $\hat{u}$. Then if all neighbors are equally likely to be visited next, the mean value property means that $\hat{u}(x) = \hat{u}(X_0) = E[\hat{u}(X_1)]$. If we denote by $P$ the operator sending $\hat{u}(X_0)$ to $\sum_{(x_0,x_1)\in E} p(x_0,x_1)\hat{u}(x_1) = E[\hat{u}(X_1)]$, we can write this as $\mathcal{L} := (I-P)\hat{u}(x) = 0$, where $\mathcal{L}$ is called the Laplacian. Note that $P$ depends on the Markov chain, and that this framework has the classical mean value property as a special case, but is in fact more general.

If the set $\hat{D}\cup\partial\hat{D}$ is finite, any function defined on this set can be viewed as a vector in $\mathbb{R}^{|\hat{D}|}$. Suppose $\hat{f} : \partial D \to \mathbb{R}$. We now seek a function $\hat{u}$ with the same values on $\partial\hat{D}$, being in the kernel of $\mathcal{L}$. Viewing $\hat{u}$ as a vector and $\mathcal{L}$ as a matrix linear transformation, we get, depending on the ordering of the vertices in $\hat{D}\cup\partial\hat{D}$ a right hand side $b$ and a matrix equation $\mathcal{L}_{\hat{D}}\hat{u}_{\hat{D}} = b$ for the vertex values not specified by $\hat{f}$. More specifically for the row corresponding to the vertex $x$:

$$\mathcal{L}\hat{u}(x) = u(x) - \sum_{(x,y)\in E} p(x,y)\hat{u}(y) \tag{17}$$

$$= u(x) - \sum_{(x,y)\in E, y\in\hat{D}} p(x,y)\hat{u}(y) - \sum_{(x,y)\in E, y\in\partial\hat{D}} p(x,y)\hat{f}(y) \tag{18}$$

$$= \mathcal{L}_{\hat{D}}\hat{u}_{\hat{D}}(x) - b(x) = 0 \tag{19}$$

The transformation $\mathcal{L}_{\hat{D}}$ is invertible, and this inverse is given by $G_{\hat{D}}$ which is the matrix with entries $G(x,y) = E^x[V_y]$, i.e. the expected number of visits to $y$ by the chain started at $x$. Here $x, y \in \hat{D}$. To see this, we note that for the row $x$ and column $y$:

$$G(x,y) = E^x[V_y] = \mathbf{1}(x=y) + \sum_{(x,z)\in E} p(x,y)E^z[V_y] \tag{20}$$

$$= \mathbf{1}(x=y) + \sum_{(x,z)\in E} p(x,z)G(z,y) \tag{21}$$

which can be written in matrix form as $G_{\hat{D}} = I + P_{\hat{D}}G_{\hat{D}}$. Thus $G_{\hat{D}} = (I - P_{\hat{D}})^{-1} = \mathcal{L}_{\hat{D}}^{-1}$.

# 4 Harmonic Explorer

## 4.1 Combinatorial setting

We are now ready to introduce the Harmonic Explorer, which is a random curve in the plane generated by a sequence of random walks. The setup consists of a domain $D \subset \mathbb{C}$, the underlying triangular lattice, with vertices $V(TG)$, and the discrete domain $D \cap V(TG) = \hat{D}$. The boundary $\partial \hat{D}$ is defined as in the previous section, and we assume that we may connect the vertices therein by a simple grid path. There is a function $\hat{f} : \partial \hat{D} \to \{0, 1\}$ that partitions the set $\partial \hat{D} = V_0^+ \cup V_0^- =: V_0$ according to the function value, i.e. $V_0^+ = \{v \in \partial \hat{D} : \hat{f}(v) = 1\}$ and similarly for $V_0^-$. We further impose that $\hat{f}$ is such that this partition has the property that one may reach all vertices $V_0^+$ by traversing the grid path induced by $\partial \hat{D}$, without ever touching $V_0^-$. Geometrically, the two sets in the partition lie on two disjoint arcs of this grid path. There are two points $\bar{v}_0$ and $\bar{v}_{end}$ that lie on the center of the edges of the grid path that connect the arcs of $V_0^-$ with $V_0^+$.

Heuristically, the Harmonic Explorer (HE) starts at the center of the triangle with an edge containing $\bar{v}_0$, lying in $\hat{D}$, and sends out a random walker from the vertex $v_1$ opposing the edge of $\bar{v}_0$. This walk is stopped upon hitting $\partial \hat{D}$ and the value of $\hat{f}$ at the final vertex is reported back to HE. If the value is zero, HE takes a right turn into the neighboring triangle, that is the triangle sharing the edge from a vertex of $V_0^+$ and $v_1$. If the value of $\hat{f}$ is instead one, the HE takes a left turn into the triangle sharing an edge from a vertex from $v_1$ to $V_0^-$. In both cases the vertex $v_1$ gets added to form the new boundary $V_1 = V_0 \cup \{v_1\}$ where the added boundary value at $v_1$ is the same as the value of $\hat{f}$ of the stopped random walk. This gives a new $\hat{f}_1 : V_1 \to \{0, 1\}$.

By letting $V_1$ take the role of $V_0$ we can similarly define $\bar{v}_1$, taking the previous role of $\bar{v}_0$ as well as $v_2$ playing the role of $v_1$. Note also that the extended $\hat{f}$ induces a partition $V_1 = V_1^+ \cup V_1^-$, which shares the qualities of the partition of $V_0$. We can therefore repeat the previous step, i.e sending a walker from $v_2$ and recording the value at the boundary, letting HE take the corresponding turn into a neighboring triangle and updating from $V_1$ to $V_2$ etc. This process can be repeated indefinitely.

Since the Harmonic Explorer always takes a turn away from the component of $V_n = V_n^+ \cup V_n^-$ that the random walk was stopped at, it can only ever exit the domain $\hat{D}$ by going through $\bar{v}_{end}$. We denote by $N$ the time step at which this exit takes place. By the boundedness of $\hat{D}$ it is also possible to see that this exit must happen in finite time.

## 4.2 Description using harmonic extensions

There is an alternative combinatorial description of the HE process, specifically for the way the turns are decided and the domain updated, which is perhaps less vivid and intuitive, but serves our later theoretical purposes better. Recall from the previous section the strong correspondence between random walks on a lattice and harmonic functions. Seeing that we have a discrete $\hat{f}$ defined on the boundary vertices and a random walk that records and reports back the value of this function at its first exit site, we might try to give an alternative characterisation of HE using a harmonic function.

To this end, let $h = \hat{u}$ be the discrete harmonic extension of $\hat{f}$, i.e. the solution of the discrete Dirichlet problem corresponding to $\hat{D}$, $\partial\hat{D}$ and $\hat{f}$. There is nothing stochastic here, since everything needed for the setup of this problem is determined from the start. Since HE is a random curve, we will need to introduce the randomness at some other level. To this end, let $X_1 \sim U(0,1)$ be a uniform random variable on the unit interval. In the notation used above, let $p_1 := h(v_1)$, i.e. the value at the vertex $v_1$ of the discrete harmonic extension $h$. The claim is that it is equivalent to let HE send out a random walk from $v_1$ and take a turn depending on the value of $\hat{f}$ at exit site of the walker, and to sample $X_1$ and then, if $X_1 \leq p_1$ or not, take a turn. More precisely the claim is that if $X_1 \leq p_1$, HE should take the same turn as if the stopped walk $Y_\tau \in V_0^+$, and that this alternative description of HE gives the same stochastic process. We would then, after successive turns and updates to the domain, define $h_n$ similarly as $h$ but with respect to the Dirichlet problem with boundary $V_n$ and updated $\hat{f}_n$. Let $p_n = h_{n-1}(v_n)$ and $(X_n)$ a sequence of uniform random variables on the unit interval. To see that this is indeed an equivalent description, the following is needed:

**Theorem 4.1** (HE and random walk)**.** *Let $(Y_m)$ be a random walk on $V(TG) \cap D$ started at $v_n$ and define $\tau = \inf\{k \geq 0 : Y_k \in V_{n-1}\}$. Then the following holds,*

$$P(X_n \leq p_n) = P(Y_\tau \in V_n^+) \tag{22}$$

*Proof.* We compute

$$\begin{aligned} P(Y_\tau \in V_n^+) &= E^{v_n}\left[\mathbf{1}(Y_\tau \in V_n^+)\right] && (23)\\ &= E^{v_n}\left[\hat{f}(Y_\tau)\right] && (24)\\ &= h_{n-1}(v_n) && (25)\\ &= p_n = P(X_n \leq p_n) && (26)\end{aligned}$$

where the last line follows from the definition of $p_n$ and the distribution of $X_n$. ∎

The turn to the left or right determining the one-step evolution of the HE is clearly dependent on the past, since the probability depends on the geometry of the domain, which in turn is decided by all previous turns. The geometric information we need at time $n$ is contained in $\hat{D}$, $V_n^+$ and $V_n^-$. Equivalently, we may collect $\tilde{D}_n = \hat{D} \setminus V_n$ for the interior of the domain, as well as $\gamma(n)$ and $\bar{v}_{end}$ which together imply $V_n^+$ and $V_n^-$. With this information captured in a triple $(\tilde{D}_n, \gamma(n), \bar{v}_{end})$, and considering the space of such triples, we get a possible state space for a Markov Chain. From any such triple we have probability zero of moving to all but two other triples. The 'neighbors' in this state space are the triples implied by a turn to the left and right respectively. The possibility of viewing the Harmonic Explorer as a Markov Chain on this state space is called the Domain Markov Property.

## 4.3 A martingale observable

The description in terms of harmonic $h_n$ will be useful for proving the convergence to SLE(4). It turns out that by indexing over $n \in \mathbb{N}$ we get a martingale for every interior vertex $v$. More precisely, we have the following:

**Theorem 4.2** (HE Martingale Observable)**.** *Let $h_n$ denote the discrete harmonic extension of $\hat{f}_n$. For $v \in V(TG) \cap D$, $h_n(v)$ is a martingale. Furthermore, if $N$ is the terminal time, $h_N(v) \in \{0, 1\}$.*

*Proof.* Let $\mathcal{F}_n = \sigma(X_1, ..., X_n)$. Then we have that $\hat{f}_n$ and $h_n$ are $\mathcal{F}_n$-measurable. $h_n$ is bounded, hence in $L^1$. To check the relation $E[h_{n+1}(v)|\mathcal{F}_n] = h_n(v)$ there are three cases to consider, listed here in the order of increasing complexity: i.) $v \in V_n$ , ii.) $v = v_{n+1}$ and iii.) $v \in V(TG) \cap D \backslash V_{n+1}$.

For the first case we note that $h_{n+1}(v)$ is actually $\mathcal{F}_n$-measurable, since for these $v$ it needs to respect previously assigned values. We get $E[h_{n+1}(v)|\mathcal{F}_n] = h_{n+1}(v) = h_n(v)$. For the second case we can compute the conditional probability $P(h_{n+1}(v_{n+1}) = 1|\mathcal{F}_n) = P(X_{n+1} \leq p_{n+1}) = h_n(v_{n+1})$. The other possible value for $h_{n+1}(v_{n+1})$ is zero, so we can conclude this case.

For the third case, we will make use of some properties of the harmonic extension, which is an operation we will denote by $\mathcal{H}(\cdot)$. So with this notation $h_n = \mathcal{H}(\hat{f}_n)$. Consider a function $\hat{g}$ on vertices $v \in \hat{V} \subset V$. This is the setting for the harmonic extension problem, and $\mathcal{H}(\hat{g})$ is the extension defined on $V$. In our case $\hat{g}$ is a random variable. If we take the conditional expectation of $\hat{g}$ with respect to a filtration generated by a finite partition of the underling space, then $E[\hat{g}|\mathcal{G}] = \sum_{A \in \mathcal{P}} E[\hat{g}|A]\mathbf{1}_A$. Where $E[\hat{g}|A]$ is the expected value of $\hat{g}$ (a set of values, one for each vertex) under the event $A$. But $E[\hat{g}|A]$ is a function defined on $\hat{V}$, so it can be extended via $\mathcal{H}(E[\hat{g}|A])$. The first property we will use relies on the linearity of $\mathcal{H}$, which gives

$$\begin{aligned} \mathcal{H}(E[\hat{g}|\mathcal{G}]) &= \mathcal{H}(\sum_{A \in \mathcal{P}} E[\hat{g}|A]\mathbf{1}_A) && (27) \\ &= \sum_{A \in \mathcal{P}} \mathcal{H}(E[\hat{g}|A])\mathbf{1}_A && (28) \\ &= E[\mathcal{H}(\hat{g})|\mathcal{G}] && (29) \end{aligned}$$

so there is a form of commutativity with $E[\cdot|\mathcal{G}]$.

Secondly, if we look at $h_n$ restricted to $V_{n+1}$ and apply the harmonic extension we get the unrestricted $h_n$. We may write this relation as $\mathcal{H}(h_n|_{V_{n+1}}) = h_n$.

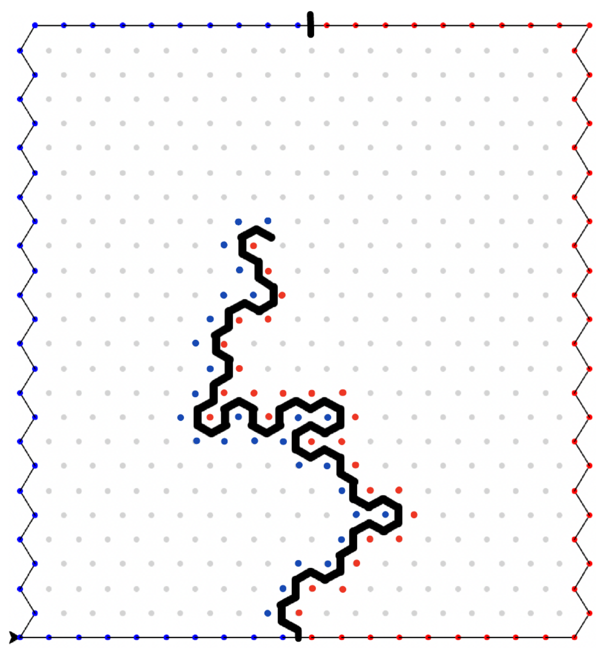
By case i.) and ii.) it holds that $h_n|_{V_{n+1}} = E[h_{n+1}(\cdot)|\mathcal{F}_n]|_{V_{n+1}}$. Now

$$h_n = \mathcal{H}(h_n|_{V_{n+1}}) \tag{30}$$
$$= \mathcal{H}(E[h_{n+1}(\cdot)|\mathcal{F}_n]|_{V_{n+1}}) \tag{31}$$
$$= E[\mathcal{H}(h_{n+1}(\cdot)|_{V_{n+1}})|\mathcal{F}_n] \tag{32}$$
$$= E[h_{n+1}(\cdot)|\mathcal{F}_n] \tag{33}$$

so that the crucial relation holds for all $v \in V(TG) \cap D$.

The domain contains a finite number of vertices and the self avoiding HE curve exits at $\bar{v}_{end}$ in finite time $N$. This creates a cut through the original domain, separating it into two connected components. By the assignment of boundary values that determine $h_N$, we see that one of these components is the interior of $V_N^-$ and the other is the interior of $V_N^+$. By the maximum principle, $h_N$ is constant within these two components, taking the same values in the interior as on the respective boundary. Hence $h_N \in \{0, 1\}$.

■



# 5 Loop Erased Random Walk

In this section we will introduce the Loop Erased Random Walk and derive some of its properties. It is perhaps more explicitly related to the random walk than is the Harmonic Explorer, since it can be obtained by a deterministic transformation applied to one single random walk in a domain $A$, stopped at some time $T$, which is typically the exit time from said domain. In the terminology of Markov Chains we let $\partial A$ be absorbing.

## 5.1 Notation

Before proceeding, it will be useful to establish some further notation. The setting is a Markov chain on $A \cup \partial A$, which might be some set of vertices, e.g. in a lattice, connected by a set of edges. Many of the notions introduced in the following passages can be extended to allow for negative or even complex numbers as 'weights', meaning that we replace the probabilistic perspective with a generalized signed or even complex measure perspective. A good reference for this chapter, with some generalizations is given in [8].

A path is a sequence of vertices, denoted $\omega = [\omega_0\omega_1...\omega_n]$. Note that for a sequence of $n+1$ vertices, the length is $n$. We may also write $\omega = e_1 \oplus ... \oplus e_n$ where $e_i = (\omega_{i-1}, \omega_i)$ are edges. Corresponding to the length of $\omega$, there are $n$ such edges in the representation. The Markov chain has transition probabilities $p(x, y) = P(X_{i+1} = y|X_i = x)$. We let $q$ be the measure on edges given by $q(e_i) = p(\omega_{i-1}, \omega_i)$, and note that this can be naturally extended to multiple edges by proclaiming that $q(e_1 \oplus ... \oplus e_n) = q(e_1)...q(e_n)$. This is natural because it gives the probability that the Markov chain takes the path $\omega$, and so we have a probability measure on paths. Since paths can be viewed as curves if the space of vertices has geometric structure, and random curves are the central object of study in this thesis, such a measure will be useful.

We can talk about individual paths, but also about collections or sets of paths. We denote by $\mathcal{K}_n(x, y)$ the set of paths of length $n$ starting in $x$ and ending in $y$. This set might be empty. Let $\mathcal{K}(x, y) = \bigcup_{n \geq 0} \mathcal{K}_n(x, y)$, i.e. all paths from $x$ to $y$ and $\mathcal{K} = \bigcup_x \bigcup_y \mathcal{K}(x, y)$ be all paths in $A \cup \partial A$. A collection that will be of special importance is $\mathcal{K}(x, \partial A) = \bigcup_{y \in \partial A} \mathcal{K}(x, y)$ which are the paths starting at $x$ an ending at a boundary point $y \in \partial A$. For the particular stochastic process we are about to consider, it will be crucial to distinguish between paths in different domains $B \subset A$. We will then employ the same notation as above but add a subscript like $\mathcal{K}_B$ to make it clear that $(A \backslash B) \cup \partial A$ is now viewed as the boundary, and that the paths stay completely in $B$, except possibly for the starting and ending points.

We can measure collections of paths $\mathcal{K}_1$ by declaring that $q(\mathcal{K}_1) = \sum_{\omega \in \mathcal{K}_1} q(\omega)$. A note of caution is warranted: In the context of Markov chains we have an underlying probability space for the stochastic process. If $\mathcal{K}_1$ is defined in such a way that some of its elements are overlapping from the perspective of this probability space, the sum above will not make sense probabilistically. An example would be the set of paths starting at $x_0$, then going to $x_1$, together with those starting at $x_0$ and ending at $\partial A$. If we sum these paths as above, we would count paths starting at $x_0$, then going to $x_1$ and finally ending at $\partial A$ twice.

Keeping this in mind, we define the concatenation operation for paths and for collection of paths:

$$\omega \oplus \omega' = [\omega_0 ... \omega_n] \oplus [\omega'_0 ... \omega'_n] = [\omega_0 ... \omega_n \omega'_0 ... \omega'_n] \tag{34}$$

$$\mathcal{K}_1 \oplus \mathcal{K}_2 = \{\omega \oplus \omega' : \omega \in \mathcal{K}_1, \omega' \in \mathcal{K}_2\} \tag{35}$$

We will want to measure such concatenated paths and sets as well. It is immediate from the definition that $q(\omega \oplus \omega') = q(\omega)q(\omega')$. The following computation verifies the same nice formula for concatenation of sets of paths:

$$q(\mathcal{K}_1 \oplus \mathcal{K}_2) = \sum_{\omega_1 \oplus \omega_2 \in \mathcal{K}_1 \oplus \mathcal{K}_2} q(\omega_1 \oplus \omega_2) = \sum_{\omega_1 \in \mathcal{K}_1} \sum_{\omega_2 \in \mathcal{K}_2} q(\omega_1)q(\omega_2) = q(\mathcal{K}_1)q(\mathcal{K}_2) \tag{36}$$

## 5.2 Loop erased random walk

Given a path $\omega$ from $\omega_0$ to $\omega_n$, we are interested in applying a transformation to obtain a new path that is self-avoiding, i.e. such that $\omega_i \neq \omega_j$ for $i \neq j$. As the following example shows, it is not enough to consider subpaths of $\omega$ for this transformation to be well defined.

**Example 2.** *Consider the path $\omega = [abcdabc]$. Then $\omega' = [abc]$ and $\omega'' = [adc]$ are both self-avoiding subpaths of $\omega$.*

Whenever a path is not self avoiding this induces a nonempty set of loops of the form $l = [\omega_i \omega_{i+1} ... \omega_j]$ with $\omega_i = \omega_j$. The following definition introduces the notion of erasing these loops in chronological order to obtain a self-avoiding path.

**Definition 5.1** (Chronological loop erasure)**.** *Given a path $\omega = [\omega_0 ... \omega_n]$ we obtain the loop erased path $LE(\omega) := \eta = [\eta_0 ... \eta_k]$ by the following procedure.*

- *Set $\eta_0 = \omega_0$.*
- *For $k \geq 0$*
  - *Let $j_k$ be the largest index $j$ such that $\omega_j = \eta_k$.*
  - *If $j_k = n$, set $\eta = [\eta_0 ... \eta_k]$*
  - *else, set $\eta_{k+1} = \omega_{j_k+1}$.*

**Example 3.** *Using the previous example path $\omega = [abcdabc]$ we obtain $\eta_0 = \omega_0 = a$, $j_0 = 4$, $\eta_1 = \omega_5 = b$. Then $j_1 = 5$ and $\eta_2 = \omega_6 = c$. Since $j_2 = 6 = n$ the process terminates and the result is $\eta = [abc]$.*

Let $(X_n)$ be an irreducible Markov chain with transition matrix $P$, state space $\bar{A} = A \cup \partial A$ and suppose furthermore that $A$ is finite and $\partial A$ nonempty.

Under these assumptions the stopping time $T = \min\{n \geq 0 : X_n \in \partial A\}$ is almost surely finite and consequently the associated stopped process $(X_n^T)$ almost surely gives rise to finite paths $\omega$. Therefore $\eta = LE(\omega)$ is well defined.

Suppose $X_0 = x$. The matrix $P$ induces a measure on paths $\omega = [x...y]$, where $y \in \partial A$, which in turn induces a measure on loop erased paths $\eta$ in the following sense:

**Definition 5.2.** *Loop erased random walk from $x$ to the boundary $\partial A$ is the probability measure $\hat{p}$ on loop erased paths $\eta$ given by*

$$\hat{p}(\eta) = \sum_{\omega \in \mathcal{K}(x, \partial A) : LE(\omega) = \eta} q(\omega) \tag{37}$$

Recall the Green's function and the special case $G_B(x,x)$, where $B \subset A$. This is the expected number of visits to $x$ by a walk started at that same point, stopped when exiting $B$. The following theorem establishes a connection between such quantities and the measure just introduced.

**Theorem 5.3** (LERW and the Green's function)**.** *It holds that*

$$\hat{p}(\eta) = q(\eta) \prod_{i=0}^{k-1} G_{A_i}(\eta_i, \eta_i) \tag{38}$$

*Proof.* For loop erased $\eta = LE(\omega)$ we want to recover the set of possible original $\omega$ and measure this set. For $\eta = [\eta_0 ... \eta_k]$ these are given by $\omega = l_0 \oplus [\eta_0\eta_1] \oplus l_1 \oplus ... \oplus l_{k-1} \oplus [\eta_{k-1}\eta_k]$. Note that since $\eta_k \in \partial A$ the stopped process can give rise to no loop $l_k$. Here $l_j \in \mathcal{K}_{A_j}(\eta_j, \eta_j)$, i.e. the loop is in $A \backslash \{\eta_0, ..., \eta_{j-1}\}$.

Using that $q(A \oplus B) = q(A)q(B)$ we obtain

$$\begin{aligned}
\hat{p}(\eta) &= \sum_{\omega \in \mathcal{K}(x, \partial A): LE(\omega) = \eta} q(\omega) && (39)\\
&= q(\mathcal{K}_{A_0}(\eta_0, \eta_0) \oplus [\eta_0\eta_1] \oplus \mathcal{K}_{A_1}(\eta_1, \eta_1) \oplus ... && (40)\\
&\qquad ... \oplus \mathcal{K}_{A_{k-1}}(\eta_{k-1}, \eta_{k-1}) \oplus [\eta_{k-1}\eta_k]) && (41)\\
&= q(\eta) \prod q(\mathcal{K}_{A_j}(\eta_j, \eta_j)) && (42)\\
&= q(\eta) \prod G_{A_j}(\eta_j, \eta_j) && (43)
\end{aligned}$$

To see the last step, we note that

$$\begin{aligned}
G_{A_j}(\eta_j, \eta_j) &= E^{\eta_j}[V_{\eta_j}] && (44)\\
&= E^{\eta_j}[\sum_{n \geq 0} \mathbf{1}(X_n = \eta_j, n < T)] && (45)\\
&= \sum_{n \geq 0} P^{\eta_j}(X_n = \eta_j, n < T) && (46)\\
&= \sum_{\omega \in \mathcal{K}_{A_j}(\eta_j, \eta_j)} q(\omega) && (47)
\end{aligned}$$

using definitions and monotone convergence. The stopping time $T$ referred to in this last step of equations depends on $j$ since removal of states to go from $A$ to $A_j$ amounts to changing the boundary $\partial A$ with our conventions.

■

The product $F(A; x_1, ..., x_k) := \prod_{i=1}^{k} G_{A_i}(x_i, x_i)$ in the expression of the previous theorem looks very different depending on how we choose to order the visited vertices of the loop-erased path. We will see shortly that there is in fact no such dependence.

The symmetric group $S_k$ is generated by transpositions of the form $(a_i a_{i+1})$, $i = 1, ..., k-1$. The idea is to use this fact to reduce the proof of permutation invariance to a simpler case and finish using induction.

**Theorem 5.4** (Permutation invariance Green's function product)**.** *It holds that* $F(A; x_1, ..., x_k) = F(A; x_{\sigma(1)}, ..., x_{\sigma(k)})$ *for* $\sigma \in S_k$*.*

*Proof.* The case $k = 1$ is clear. For $k = 2$ we let $\eta = [xy]$. We need to show that $G_A(x, x)G_{A\setminus\{x\}}(y, y) = G_A(y, y)G_{A\setminus\{y\}}(x, x)$.

It holds that $G_A(x, x) = q(\mathcal{K}_A(x, x))$. Let $L_{x,A}$ be the set of loops rooted at $x$ that are either trivial or visit $y$ before the first return to $x$. Let $\bar{L}_{y,A}$ the set of loops rooted at $y$ that are either trivial or don't visit $y$ after their last visit to $x$. The claim is that we may make the following decompositions:

$$\mathcal{K}_A(x, x) = \mathcal{K}_{A\setminus\{y\}}(x, x) \oplus L_{x,A} \tag{48}$$
$$\mathcal{K}_A(y, y) = \bar{L}_{y,A} \oplus \mathcal{K}_{A\setminus\{x\}}(y, y) \tag{49}$$

To see that this is the case, consider first a loop in $\mathcal{K}_A(x, x)$. It may never visit $y$, and it can then be represented as in the first line above if we let the loop from $L_{x,A}$ be trivial. If it does visit $y$ at some point, there is always first a loop in $\mathcal{K}_{A\setminus\{y\}}(x, x)$, possibly trivial, and then a loop where we ensure that we visit $y$ before returning to $x$. We can let $j_y$ be the smallest index $j$ such that $\omega_j = y$ and then let $j_x$ be the largest index $j < j_y$ such that $\omega_j = x$. This last index gives the point of the loop where we make the 'cut' into the two new loops seen in the decomposition.

For the case of loops in $\mathcal{K}_A(y, y)$ it is similarly the case that it may never visit $x$, i.e. that the loop from $\bar{L}_{y,A}$ is taken to be the trivial one. If it does visit $x$, we can ensure that this loop does not end in a loop rooted at $y$ that does not visit $x$, since this latter loop could then be absorbed into some loop from $\mathcal{K}_{A\setminus\{x\}}(y, y)$. So it is possible to have it never visit $y$ after the last visit to $x$ and before the last visit to $y$. Using indices to identify the 'cut' point, we may this time let $j_x$ be the largest index $j$ such that $\omega_j = x$. Then let $j_y$ be the smallest index $j > j_x$ such that $\omega_j = y$. The latter index is now the point of separation.

The key point is that $q(L_{x,A}) = q(\bar{L}_{y,A})$, which follows if we can establish a bijection between the loops in these two sets. The trivial loop with measure $q(\omega_{tr}) = 1$ is in both sets. For nontrivial loops in $L_{x,A}$, we have that they visit $y$ at some point. If we start the loop at the first visit to $y$ by cyclically permuting the vertices in the sequence forming the loop, we obtain a new loop rooted at $y$. Since this is the first visit to $y$, this loop visits $x$ after the second to last, i.e. terminal visit to $y$. In other words, it never visits $y$ twice after visiting $x$ for the last time, so is an element in $\bar{L}_{y,a}$. A reverse cyclical permutation of an element in $\bar{L}_{y,A}$ gives an element from $L_{x,A}$. Such permutations of the elements in the loop do not change its measure. We therefore have that

$$\begin{aligned} G_A(x,x)G_{A\setminus\{x\}}(y,y) &= q(\mathcal{K}_A(x,x))q(\mathcal{K}_{A\setminus\{x\}}(y,y)) && (50)\\ &= q(\mathcal{K}_{A\setminus\{y\}}(x,x))q(L_{x,A})q(\mathcal{K}_{A\setminus\{x\}}(y,y)) && (51)\\ &= q(\mathcal{K}_{A\setminus\{y\}}(x,x))q(\bar{L}_{y,A})q(\mathcal{K}_{A\setminus\{x\}}(y,y)) && (52)\\ &= G_{A\setminus\{y\}}(x,x)G_A(y,y) && (53)\end{aligned}$$

So it holds for $k=2$ elements $x_i$. For general $k$ and a transposition $\sigma = (a_{i-1}, a_i)$ we see that

$$\begin{aligned} F(A; x_{\sigma(1)}, ..., x_{\sigma(k)}) &= F(A; x_1, ..., x_i, x_{i-1}, ..., x_k) && (54)\\ &= \left(\prod_{j=1}^{i-2} G_{A_j}(x_j, x_j)\right)\left(\prod_{j=i+1}^{k} G_{A_j}(x_j, x_j)\right) \cdot && (55)\\ &\quad \cdot G_{A_{i-2}\setminus\{x_{i-2}\}}(x_i, x_i) G_{A_{i-2}\setminus\{x_{i-2}, x_i\}}(x_{i-1}, x_{i-1}) && (56)\end{aligned}$$

Let $\tilde{A} := A_{i-2}\setminus x_{i-2} = A_{i-1}$. Since $A_i = \tilde{A}\setminus\{x_{i-1}\}$ we can apply the case $k=2$ to the two factors affected by the transposition to get

$$G_{A_{i-2}\setminus\{x_{i-2}\}}(x_i, x_i) G_{A_{i-2}\setminus\{x_{i-2}, x_i\}}(x_{i-1}, x_{i-1}) = G_{A_{i-1}}(x_{i-1}, x_{i-1}) G_{A_i}(x_i, x_i) \tag{57}$$

By using the fact that $S_k$ is generated by transpositions on this form, we conclude. ■

From the above lemma we are led naturally to the following definition

**Definition 5.5.** *For a collection of vertices $V = \{x_1, ..., x_k\}$ let $F_V(A) = F(A; x_1, ..., x_k)$. For the path $\eta$ visiting vertices $\eta_0, ..., \eta_n$ with the last point in $\partial A$, we define $F_\eta(A) = F(A; \eta_0, ..., \eta_{n-1})$.*

Invariance under permutations is a property enjoyed by determinants. In light of the previous results regarding $F_V(A)$ and the strong connection between lattice random walks and linear algebra, the following result unifies these ideas, expressing $F_V(A)$ as a determinant.

**Theorem 5.6** (Determinant of Laplacian)**.** *The quantitiy $F_V(A)$ is related to a determinant of the Laplacian $\mathcal{L}$ or Green's function $G$ by the following formula*

$$F_V(A) = \frac{\det G_A}{\det G_{A\setminus V}} = \frac{\det \mathcal{L}_{A\setminus V}}{\det \mathcal{L}_A}. \tag{58}$$

*with the convention that $G_\emptyset = 1$.*

**Remark 5.7.** *This gives another way to see how the permutation invariance of $F_V(A)$ arises. We will however use the previously derived invariance result to shorten this proof.*

*Proof.* We begin to consider the special case $F_A(A) = \det G_A$. When $A = \{x\}$ we have

$$G_A = E^x[V_x] = E[\sum_{n\geq 0} \mathbf{1}(X_n = x)] = \sum_{n\geq 0} p(x,x)^n = \frac{1}{1-p(x,x)} = \frac{1}{\mathcal{L}_A} \tag{59}$$

For $|A| > 1$, assuming $A = A' \cup \{x\}$ and that $|A'|$ is such that the statement is true, we can order this set so that $x$ is the first element. Using 5.4 and the definition we get $F_A(A) = F_{\{x\}}(A)F_{A'}(A') = G_A(x,x)\det G_{A'}$.

$G_A(x,x)$ is the entry in the row and column corresponding to $x$ and $x$ of the matrix $G_A$. Recall that this in the inverse of $\mathcal{L}_A$. If we let $g$ be the column corresponding to $x$ of this matrix, we have that $\mathcal{L}_A g$ is a vector of zeros, except for a one in the position of $x$. Call this vector $e_x$. So $\mathcal{L}_A g(y) = \mathbf{1}(y = x)$. Using Cramer's rule we can solve the system $\mathcal{L}g = e_x$ to obtain $g(y) = \frac{\det \mathcal{L}_{A,y}}{\det \mathcal{L}_A}$ where $\mathcal{L}_{A,y}$ is the matrix $\mathcal{L}_A$ where the column corresponding to $y$ is replaced by $e_x$. Expansion of the determinant along the column associated to $x$ gives that $\det \mathcal{L}_{A,x} = \det \mathcal{L}_{A\backslash\{x\}} = \det \mathcal{L}_{A'}$. So we have $G_A(x,x) = g(x) = \frac{\det \mathcal{L}_{A'}}{\det \mathcal{L}_A}$ and $F_A(A) = \det G_A$ as desired.

Now note that $F_A(A) = F_V(A)F_{A\backslash V}(A\backslash V)$ and solve for $F_V(A)$ to obtain the statement of the theorem.

■

## 5.3 Conditional transition probabilities

We have so far looked at the probability of loop erased paths $\eta$, as measured by $\hat{p}$. In this section, we will be interested in probabilities like

$$\hat{p}(x_n|\eta) = P(\eta_n = x_n|\eta = [\eta_0, ..., \eta_{n-1}]). \tag{60}$$

The random walk can be viewed as a Markov chain, and so all quantities like the one above take a very simple form for that process; we only care about where we are at time $n-1$ to determine the probability of ending up at some $x_n$ one step later. It is clear that with $\eta$ being self-avoiding, no value $\eta_i$ $i = 0, ..., n-1$ can be disregarded when computing the transition probability to $x_n$. Towards the end we will see how we can partially overcome these difficulties and realize the LERW as a Markov Chain on marked domains, similar to the case of the Harmonic Explorer.

A path from $x \in A$ to the boundary point $z \in \partial A$ can be decomposed as $\mathcal{K}_A(x, z) = \mathcal{K}_A(x, x) \oplus \mathcal{K}_{A\setminus\{x\}}(x, z)$, or more generally $\mathcal{K}_A(x, \partial A) = \mathcal{K}_A(x, x) \oplus \mathcal{K}_{A\setminus\{x\}}(x, \partial A)$. The LERW is a measure on paths from interior points to the boundary. To get conditional transition probabilties for the LERW, we will use this decomposition principle.

Suppose the LERW at time $n-1$ is given by $\eta = [\eta_0, ..., \eta_{n-1}]$. If $\eta_n = x_n$, this means that the underlying random walk went from $\eta_{n-1}$ to $x_n$, and finally from $x_n$ to the boundary without ever returning to $\eta$. The set of all possible walks, i.e. not restricting to $\eta_n = x_n$, means going from $\eta_{n-1}$ to the boundary without ever visiting $\eta$. So the conditional probability $\hat{p}(x_n|\eta) = P(\eta_n = x_n|\eta = [\eta_0, ..., \eta_{n-1}])$ is the ratio between the probability of these sets of walks

$$\frac{q([\eta_{n-1}, x_n])q(\mathcal{K}_{A_n}(x_n, \partial A))}{q(\mathcal{K}_{A_n}(\eta_{n-1}, \partial A))} \tag{61}$$

Now note that $q(\mathcal{K}_{A_n}(x_n, \partial A)) = P^{x_n}(X_{T_{A_n}} \in \partial A) = E^{x_n}[\mathbf{1}(X_{T_{A_n}} \in \partial A)]$, where $T_{A_n} = \min\{n \geq 0 : X_n \notin A_n\}$. Using the result from the section on random walk and discrete harmonic extensions, we see that this is precisely what we have at hand: The transition probabilities are governed in part by a discrete harmonic function. Since the domain, and therefore also the stopping time $T_{A_n}$ change over time, we have a sequence of harmonic functions that determine the transition probabilities. This is analogous to the Harmonic Explorer.

This also gives

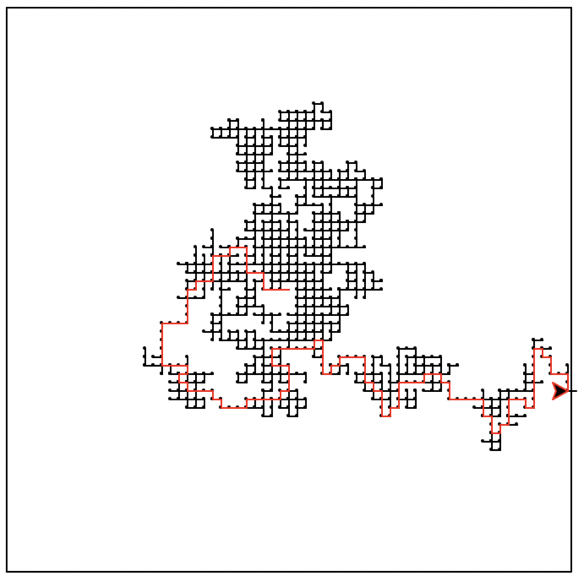
$$\hat{p}(\eta) = \hat{p}(\eta_1|\eta_0)\hat{p}(\eta_2|[\eta_{0,\eta_1}])...\hat{p}(\eta_n|[\eta_0, ..., \eta_{n-1}]) \tag{62}$$

$$= \prod_{i=0}^{n-1} \frac{q([\eta_i, \eta_{i+1}])q(\mathcal{K}_{A_{i+1}}(\eta_{i+1}, \partial A))}{q(\mathcal{K}_{A_{i+1}}(\eta_i, \partial A))}. \tag{63}$$

As in the case of the Harmonic Explorer, what we end up with is a Markov Chain, but not one the state space of vertices. Instead we have transitions between $(A_{n-1}, x_{n-1})$ and $(A_n, x_n)$ where $A_n = A \backslash \{\eta_0, ..., \eta_{n-1}\}$. So the state space is the set of such pairs $\tilde{A}_n$. The transition probabilities are given by

$$P(\tilde{A}_n = (A_n, x_n) | \tilde{A}_{n-1} = (A_{n-1}, x_{n-1})) = \frac{q([x_{n-1}, x_n]) q(\mathcal{K}_{A_n}(x_n, \partial A))}{q(\mathcal{K}_{A_n}(x_{n-1}, \partial A))} \tag{64}$$

and if $x_n \in \partial A$, we have reached an absorbing state.



# 6 Scaling limits

## 6.1 The basic idea

The discrete processes examined so far rely crucially on a lattice. The goal with a scaling limit is to remove this explicit dependence on the lattice and arrive at a continuous time and space object. This might be achieved by scaling both the magnitude of increments and their frequency, as we will see in the case of Random Walk below. Another possibility is to consider larger and larger domains $D$, but with the same lattice spacing and time steps and then map this domain to another domain $D'$ in such a way that the grid lines become very fine in $D'$. Since $D$ is growing and $D'$ is not, the result is a finer and finer spacing of lattice points. In the case of the Harmonic Explorer, we will use this more implicit approach. An open question is how the scaling of time is handled in this setting. In some heuristic sense, this will be taken care of by reparameterizing time to match the geometry of the mapped process. The reference domain $D'$ is convenient because chordal Schramm-Loewner-Evolution lives in this domain. The map from $D$ to $D'$ will be taken to be conformal, since SLE turns out to be invariant in law under such transformations.

## 6.2 Brownian Motion

In the following we will introduce Brownian motion and discuss a symmetry property it enjoys if considered in $\mathbb{C} = \mathbb{R}^2$. We will then see in what sense the random walk converges to Brownian Motion, using a technique that will again be used to show convergence of Harmonic Explorer to $SLE(4)$.

### 6.2.1 Construction and properties

The following properties define a Brownian motion.

**Definition 6.1** (Brownian Motion). *Let $(B(t))_{t\geq 0}$ be a real-valued stochastic process. It is a Brownian motion started at $x \in \mathbb{R}$ if the following conditions hold:*

- *$B(0) = x$,*
- *Independence of increments: For times $0 \leq t_1 \leq t_2 \leq \cdots \leq t_n$, the random variables $B(t_n) - B(t_{n-1}), B(t_{n-1}) - B(t_{n-2}), \ldots, B(t_2) - B(t_1)$ are independent.*
- *For $0 \leq t_0 < t_1$, the increment $B(t_1) - B(t_0)$ is normally distributed, with expectation zero and variance $t_1 - t_0$.*
- *Almost sure continuity of sample paths: $t \mapsto B(t, \omega)$ is continuous, except possibly for $\omega$ in a set of measure zero.*

Henceforth, we will interchangeably use $B(t)$ and $B_t$ for Brownian motion at time $t$, mainly because different notation brings more or less clarity in different contexts.

**Theorem 6.2.** *Brownian motion exists.*

We will not prove this here. A reference for Wiener's construction of Brownian motion is [10].

**Definition 6.3** (Complex Brownian Motion)**.** *Let $B_1$ and $B_2$ be independent one-dimensional Brownian Motions. The process $B = B_1 + iB_2$ is called a complex Brownian Motion.*

Complex Brownian Motion lives in the complex plane, so we will use 'complex' and 'planar' interchangeably.

**Theorem 6.4** (Conformal invariance of planar Brownian motion)**.** *Let $f : D \to D'$ be a conformal isomorphism between domains $D, D' \in \mathbb{C}$. Consider the processes*

- $((B_t)_{t\leq T}, T)$ *where $B_t$ is a planar Brownian Motion started at $z \in D$ and* $T = \inf\{t \geq 0 : B_t \notin D\}$
- $((B'_t)_{t\leq T'}, T')$ *where $B'_t$ is a planar Brownian Motion started at $\phi(z) \in D'$ and* $T' = \inf\{t \geq 0 : B'_t \notin D'\}$

*Now define $\tilde{T} = \int_0^T |\phi'(B_t)|^2 \, dt$ and for $t < \tilde{T}$*

$$\tau(t) = \inf \left\{ s \geq 0 : \int_0^s |\phi'(B_r)|^2 \, dr = t \right\}. \tag{65}$$

*Under time change and mapping under $\phi$ in the sense*

$$\tilde{B}_t = \phi(B_{\tau(t)}) \tag{66}$$

*the processes $((B'_t)_{t\leq T'}, T')$ and $((\tilde{B}_t)_{t\leq \tilde{T}}, \tilde{T})$ have the same distribution.*

For a proof using the same notation, see [2].

#### 6.2.2 Random Walk convergence

We now make precise the way in which a properly rescaled and linearly interpolated random walk in one dimension converges to the one-dimensional Brownian Motion. Depending on the generality of the random walk, we could perform some shortcuts. The fact that the convergence holds for a large class of random walks, i.e. not only the simple random walk, is interesting in and of itself and is an instance of the 'universality' of Brownian motion. Since universality is often the objective when passing from the discrete formulations of a model to the continuum, we choose not to make such shortcuts. We follow [10] closely.

**Theorem 6.5** (Skorokhod embedding). *Let $(B_t)$ be a standard Brownian motion and $X$ a real valued random variable with mean zero and finite variance. Then there exists a stopping time $T$ with respect to the natural filtration of the Brownian motion, such that $B_T$ has the same distribution as $X$ and $E[T] = E[X^2]$.*

We will shortly begin building towards a proof of this fact. But first, we record a generalization of the above that allows to embed martingales, something that will be needed in the very last chapter.

**Theorem 6.6** (Skorokhod embedding for martingales). *Let $(M_n)_{n\leq N}$ be a martingale w.r.t $(\mathcal{F}_n)$ and $\|M_{n+1} - M_n\|_{\sup} = O(\delta)$ with $M_0 = 0$. Then there is a sequence of stopping times $(T_n)$ such that $B_{T_n}$ and $M_n$ have the same law. It is furthermore possible to insist on the following:*

$$E[T_{n+1} - T_n|B[0,T_n]] = E[(B_{T_{n+1}} - B_{T_n})^2|B[0,T_n]] \tag{67}$$

$$T_{n+1} \leq \inf\{t \geq T_n : |B_t - B_{T_n}| \geq O(\delta)\} \tag{68}$$

*with the same constant for $O(\delta)$ in the last equation as in the bound for increments.*

Chapter 3.3 in [9] gives some discussion and useful references. We now move on to prove the special case.

**Definition 6.7.** *Let $X_n$ be a martingale. We call $X_n$ binary splitting if whenever $A(x_0, ..., x_n) := \{X_0 = x_0, ..., X_n = x_n\}$ is an event with positive probability, the support of $X_{n+1}$ conditioned on $A(x_0, ..., x_n)$ contains at most two values.*

**Example 4.** *The simple random walk is a binary splitting martingale. $S_n$ is supported on more than two values but conditional on $\{S_0 = 0, ..., S_{n-1} = x\}$ (and assuming this encodes a possible sequence) it is only supported on $\{x-1, x+1\}$.*

The above example is a somewhat crude one, in that we really only use the value of $S_{n-1}$ to determine the support of $S_n$. In the following theorem, we present a binary splitting martingale that uses progressively more information, as codified by a filtration, to 'pinpoint' and approximate a random variable $X$.

**Theorem 6.8.** *Let $X$ be a random variable with $E[X^2] < \infty$. Then there exists a binary splitting martingale $X_n$ such that $X_n \to X$ almost surely and in $L^2$.*

*Proof.* Define $X_0 = E[X]$, and let $\mathcal{F}_0$ be the trivial $\sigma$-field $\{\Omega, \emptyset\}$. We make the following recursive definitions

$$\xi_n = \mathbf{1}(X \geq X_n) - \mathbf{1}(X < X_n) \tag{69}$$

$$X_n = E[X|\mathcal{F}_n] \tag{70}$$

$$\mathcal{F}_n = \sigma(\xi_0, ..., \xi_{n-1}) \tag{71}$$

Since $\xi_k$ are binary random variables we can see how $\mathcal{F}_n$ is generated by a partition $\mathcal{P}_n$ of $\Omega$ with size $2^n$. Furthermore, elements in $\mathcal{P}_n$ are a union of two elements from $\mathcal{P}_{n+1}$, so the martingale $(X_n)$ is binary splitting.

Furthermore, $X_n$ is $L^2$-bounded, since

$$\infty > E[X^2] = E[(X - X_n)^2] + E[X_n^2] \geq E[X_n^2] \tag{72}$$

where the equality follows from $E[XX_n] = E\left[E[XX_n|\mathcal{F}_n]\right] = E[X_n^2]$.

If we let $\mathcal{F}_\infty = \bigcup_{n\geq 0} \mathcal{F}_n$ we have

$$X_n \to X_\infty := E[X|\mathcal{F}_\infty] \tag{73}$$

almost surely and in $L^2$. The description of this limit is a consequence of Lévy's upward theorem. We can remind ourselves that for $A \in \mathcal{F}_\infty$, $A \in \mathcal{F}_m$ for some $m$. By considering $n \geq m$, we have $E[X_\infty \mathbf{1}_A] = E[\lim_{n\to\infty} X_n \mathbf{1}_A] = \lim_{n\to\infty} E[X_n \mathbf{1}_A] = E[X\mathbf{1}_A]$.

To conclude the proof, it suffices to show that $X = X_\infty$ almost surely. It will be easier once we view this expression from the perspective of an auxiliary construction. The claim is that almost surely

$$\lim_{n\to\infty} \xi_n(X - X_{n+1}) = |X - X_\infty| \tag{74}$$

To see this we partition $\Omega$ into three sets depending on the order relation between $X(\omega)$ and $X_\infty(\omega)$.

Case I: $X_\infty(\omega) = X(\omega)$
The right hand side is zero and with $\xi_n$ uniformly bounded and the term in parentheses $\to 0$, we can almost surely conclude equality for these $\omega$.

Case II: $X_\infty(\omega) < X(\omega)$
This means that $X_n(\omega) < X(\omega)$ for all $n > N$, the latter a large integer. For these $n$, $\xi_n = 1$ and so the left hand side evaluates to $X(\omega) - X_\infty(\omega)$ which is precisely $|X(\omega) - X_\infty(\omega)|$ by the case description.

Case III: $X_\infty(\omega) > X(\omega)$
Follows in the same way as Case II by flipping the appropriate inequalities and some signs.

Using that $X_n$ is $L^2$ bounded we get that $\xi_n(X - X_{n+1})$ is $L^2$-bounded as well, hence uniformly integrable. Together with the almost sure convergence (which in particular implies convergence in probability) we get $L^1$ convergence and as a consequence

$$0 = \lim_{n\to\infty} E[\xi_n(X - X_{n+1})] = E[\lim_{n\to\infty} \xi_n(X - X_{n+1})] \tag{75}$$

so that almost surely $|X - X_\infty|$ is zero, or equivalently, $X = X_\infty$.

■

**Remark 6.9.** *Intuitively, $X_n$ yields a progressively better approximation of $X$ by using information in $\mathcal{F}_n$. In the limit, all information is used, i.e $X_\infty = E[X|\mathcal{F}_\infty]$. The last part of the proof, showing that $X = X_\infty$ almost surely, says that using all the information in $\mathcal{F}_\infty$ is sufficient to pinpoint $X$ exactly in 'essentially all cases'.*

**Theorem 6.10.** *Given a random variable $X$ with $E[X] = 0$ and $E[X^2] < \infty$ there is a stopping time $T$ with respect to the natural filtration such that $B(T) \stackrel{d}{=} X$.*

We will use the previous result 6.8 about binary splitting martingales $X_n$ approximating $X$. Such $X_n$ requires only finite variance. To establish a connection to stopped Brownian motion we will however need to further impose $E[X] = 0$.

The simplest example of embedding a random variable in a stopped Brownian motion is given by letting $a < b$ and considering the time $T_{a,b} = \inf\{t \geq 0 : B_t \in \{a, b\}\}$. It is almost surely a finite time and the stopped process is clearly bounded so we can compute

$$0 = E[B_{T_{a,b}}] = P(B_{T_{a,b}} = a)a + (1 - P(B_{T_{a,b}} = a))b \tag{76}$$

$$P(B_{T_{a,b}} = a) = b/(b - a) \tag{77}$$

So this gives an embedding of a random variable supported on two values provided that this random variable has the 'right' probabilities for $a$ and $b$. It turns out that the sequence $X_n$ conditioned on $A(x_0, ..., x_{n-1})$ is not only supported on two values, but that this relationship between the magnitude of the values and their respective probabilities mirror the $a, b$-example just introduced. This allows us to prove the Skorokhod embedding theorem:

*Proof.* Construct an increasing sequence of stopping times $(T_n)$ such that $B_{T_n}$ is distributed as $X_n$. Since $T_n \to T$ for some stopping time $T$ and by monotone convergence

$$E[T] = \lim_{n\to\infty} E[T_n] = \lim_{n\to\infty} E[X_n^2] = E[X^2] \tag{78}$$

$B_{T_n}$ converges to $X$ in distribution and by the almost sure continuity of Brownian Motion, we have that $B_{T_n}$ converges almost surely to $B_T$, which is therefore distributed as $X$. ■

We are now ready to formulate the setting and sense in which the random walk converges to Brownian motion.

Let $X$ be a random variable with $E[X^2] < \infty$. We can assume that $E[X] = 0$ and $E[X^2] = 1$ by considering the normalization $\frac{X-E[X]}{\sqrt{E[X^2]}}$. Consider the random walk $S_n = \sum_{i=1} X_i$ generated by $(X_i)$, where $X_i \stackrel{d}{=} X$. We interpolate to get

$$S(t) = S_{\lfloor t \rfloor} + (t - \lfloor t \rfloor)(S_{\lfloor t \rfloor + 1} - S_{\lfloor t \rfloor}) \tag{79}$$

and rescale

$$S_n^*(t) = \frac{S(nt)}{\sqrt{n}} \tag{80}$$

to obtain a sequence $(S_n^*)_{n \geq 1}$. The claim is that:

**Theorem 6.11** (Donsker's invariance principle)**.** *On the metric space* $(C([0,1]), \|\cdot\|_{\sup})$ *the sequence* $(S_n^*)$ *converges in distribution to a standard Brownian motion, i.e.*

$$S_n^* \xrightarrow{d} B \tag{81}$$

From the Portmanteau theorem we have that convergence in distribution means $\limsup P(S_n^* \in K) \leq P(B \in K)$, where we take the viewpoint that Brownian motion and the interpolated and scaled random walk define random functions. The set $K$ is a closed set in the space $C([0,1])$ of continuous real-valued functions on the unit interval.

The strategy is to show that $S_n^*$ is very likely close (in the sup-norm $d := \|\cdot\|_{\sup}$) to a Brownian motion, such that $P(S_n^* \in K)$ can be translated to a similar statement about a Brownian motion with an added error term going to zero. We are looking for a way to connect the realizations of these processes. For this it will be important that we use a Brownian motion $B$ to construct a random walk (by the aid of the Skorokhod embedding) and then take this random walk to construct $S_n^*$. We then compare realizations of $S_n^*$ to realizations of $B$. The processes we compare will then be on the same probability space, and since one gives rise to the other, the realizations themselves will be comparable.

**Lemma 6.12.** *Let* $(B(t))_{t \geq 0}$ *be a linear Brownian motion,* $X$ *a random variable of mean zero and unit variance. Then there is a sequence of stopping times* $0 = T_0 \leq T_1 \leq T_2 \leq \ldots$ *with respect to* $B$ *such that*

- $B(T_n) \stackrel{d}{=} S_n$ *where* $S_n = \sum_{i=1}^n X_i$ *and* $X_i \stackrel{d}{=} X$.

- *For any given $\epsilon > 0$, we have*

$$\lim_{n\to\infty} P(\sup_{0\le t\le 1} |\frac{B(nt)}{\sqrt{n}} - S_n^*(t)| > \epsilon) = 0 \tag{82}$$

**Remark 6.13.** *If we look at realizations, $\frac{B(nt)}{\sqrt{n}}$ is not the same as $B(t)$, but both are standard linear Brownian motions, so they have the same distribution.*

*Proof.* By the Skorokhod embedding we can find the sequence $0 = T_0 \le T_1 \le T_2 \le ...$ inductively by letting $T_1$ be the stopping time such that $B(T_1) \stackrel{d}{=} X$. By the strong Markov property of Brownian motion $B(t) \stackrel{d}{=} B(T_1+t)-B(T_1)$ so that we can construct a stopping time $T_2'$ such that $B(T_1+T_2')-B(T_1) \stackrel{d}{=} X$. Then $T_2 := T_1+T_2'$ is a stopping time and $B(T_2) \stackrel{d}{=} B(T_1)+X \stackrel{d}{=} S_2$. Continuing in this fashion we get $B(T_n) \stackrel{d}{=} S_n$ and since $E[T_i'] = E[T_1] = E[X^2] = 1$ we have $E[T_n] = n$. This gives the first item.

For the second item we need to work creatively with set inclusions to get something we can calculate the probability of. To this end, let $A_n = \{\exists t \in [0,1) : |W_n(t)-S_n^*(t)| > \epsilon\}$ where we introduce $W_n(t) = \frac{B(nt)}{\sqrt{n}}$ for notational convenience.

The unit interval can be partitioned into smaller intervals of the form $I_{k,n} = [(k-1)/n, k/n)$ and for given $t$ and $n$ it is clear that there is a unique $k$ for which $t \in I_{k,n}$. Furthermore we can recall that $S_n^*$ is obtained by scaling time by $n$, so it is precisely on intervals of the form $I_{n,k}$ that $S_n^*$ is linear. For $t \in I_{n,k}$ to record a large distance $|W_n(t) - S_n^*(t)|$ means that the distance of $S_n^*$ from the interpolation endpoints is at least as large, i.e. that

$$\begin{aligned}
A_n &= \{\exists t \in [0,1) : |W_n(t) - S_n^*(t)| > \epsilon\} && (83)\\
&\subset \{\exists t \in [0,1) : |W_n(t) - S_{k-1}/\sqrt{n}| > \epsilon\} && (84)\\
&\quad \cup \{\exists t \in [0,1) : |W_n(t) - S_k/\sqrt{n}| > \epsilon\} && (85)\\
&= \{\exists t \in [0,1) : |W_n(t) - W_n(T_{k-1}/n)| > \epsilon\} && (86)\\
&\quad \cup \{\exists t \in [0,1) : |W_n(t) - W_n(T_k/n)| > \epsilon\} && (87)\\
&=: A_n^* && (88)
\end{aligned}$$

where we used that $S_k = B(T_k) = \sqrt{n}W_n(T_k/n)$.

Brownian motion is almost surely continuous, and as it turns out, also uniformly continuous on compact intervals, with a deterministic modulus of continuity, see the first chapter of [10]. We can therefore drive $W(t) - W(s)$ close by making $t - s$ close. For $\delta \in (0,1)$ this suggests the following intelligent inclusion:

$$A_n^* \subset \{\exists s,t \in [0,2) : |s-t| < \delta, |W_n(s) - W_n(t)| > \epsilon\} \tag{89}$$

$$\cup \{\exists t \in [0,1) : \max(|T_{k-1}/n - t|, |T_k/n - t|) \geq \delta\} \tag{90}$$

$$=: A_\delta \cup B_n(\delta) \tag{91}$$

The aforementioned uniform continuity of Brownian paths gives $\lim_{\delta\to 0} P(A_\delta) = 0$. It is enough to show that after fixing $\delta$, $\lim_{n\to\infty} P(B_n(\delta)) = 0$. Then we would have $P(A_n) \leq P(A_\delta) + P(B_n(\delta))$ and can first pick $\delta$ small enough to make the first term less than $\epsilon/2$ and then an $N$ such that for $n > N$ the same holds for the latter term for this $\delta$.

By the law of large numbers $T_n/n = \frac{1}{n}\sum_{i=1}^n T_i - T_{i-1} \to E[T_i - T_{i-1}] = 1$ almost surely. For $\omega \in \Omega$ we have the corresponding realization $(T_n(\omega))_{n\geq 0}$, which is a real valued sequence. For real valued sequences $(a_n)_{n\geq 0}$, with $a_n/n \to 1$ we have that

$$\sup_{0\leq k\leq n} \frac{a_k - k}{n} = \sup_{0\leq k\leq n} (\frac{a_k}{k} - 1)\frac{k}{n} \to 0 \quad \text{as } n \to \infty \tag{92}$$

since for 'large' $k$, $a_k/k$ is close to 1 and $k/n \leq 1$, while for the finitely many 'small' $k$, the first factor is always bounded and $k/n$ is close to zero by taking $n$ large enough. As a consequence, $\lim_{n\to\infty} \sup_{0\leq k\leq n} \frac{|T_k - k|}{n} < \delta$ almost surely, which we can weaken to conclude

$$\lim_{n\to\infty} P(\sup_{0\leq k\leq n} \frac{|T_k - k|}{n} \geq \delta) = 0 \tag{93}$$

By the partition into intervals $I_{k,n}$ we have

$$B_n(\delta) = \{\exists t \in [0,1) : \max(|T_{k-1}/n - t|, |T_k/n - t|) \geq \delta\} \tag{94}$$

$$\subset \{\sup_{1\leq k\leq n} \max(\frac{T_k - (k-1)}{n}, \frac{k - T_{k-1}}{n}) \geq \delta\} \tag{95}$$

$$\subset \{\sup_{1\leq k\leq n} \frac{T_k - k}{n} \geq \delta/2\} \cup \{\sup_{1\leq k\leq n} \frac{(k-1) - T_{k-1}}{n} \geq \delta/2\} \tag{96}$$

if we take $n > 2/\delta$. So by the union bound $P(B_n(\delta))$ is dominated by two terms going to zero as $n \to \infty$ and we can conclude the proof of the second item. ■

We are ready to tackle Donsker:

*Proof.* Pick a sequence of stopping times as in the preceding lemma to embed the Random Walk in Brownian motion. Recall also that the scaling property gives that with $W_n(t) = B(nt)/\sqrt{n}$, the distribution of $W_n$ is that of a standard Brownian Motion.

We consider a closed set $K \in C([0,1])$. We define $K[\epsilon] := \{f \in C([0,1]) : \|f - g\|_{\sup} \leq \epsilon \text{ for } g \in K\}$, i.e. all $f$ that are $\epsilon$-close to $K$. Then for $\epsilon > 0$

$$\begin{aligned} \{S_n^* \in K\} &= \{S_n^* \in K\} \cap (\{d(S_n^*, W_n) \leq \epsilon\} \cup \{d(S_n^*, W_n) > \epsilon\}) && (97) \\ &\subset \{W_n \in K[\epsilon]\} \cup \{d(S_n^*, W_n) > \epsilon\} && (98) \end{aligned}$$

A union bound gives

$$\begin{aligned} P(S_n^* \in K) &\leq P(W_n \in K[\epsilon]) + P(d(S_n^*, W_n) > \epsilon) && (99) \\ &\to P(W_n \in K[\epsilon]) \quad \text{as } n \to \infty && (100) \\ &= P(B \in K[\epsilon]) && (101) \\ &\to P(B \in \bigcap_{\epsilon > 0} K[\epsilon]) \quad \text{as } \epsilon \to 0 \text{ monotonically} && (102) \\ &= P(B \in K) \quad \text{since } K \text{ closed} && (103) \end{aligned}$$

This means that

$$\limsup_{n \to \infty} P(S_n^* \in K) \leq P(B \in K) \tag{104}$$

and we can conclude by appealing to Portmanteau.

■

## 6.3 Schramm-Loewner-Evolution (SLE)

When we move on to consider models like LERW or HE, the geometry of the domain and the curve up to a point are poised to play a central role. A hint of this is provided by the Domain Markov Property which allows us to view both of these processes as Markov Chains on the state space of 'marked domains'. These marked domains are no longer just points on a lattice, but encode the geometry of both the curve up to time $n$ and the ambient domain. What we are after is a framework to handle randomly evolving geometry. The Schramm-Loewner-Evolution, based mainly on work by Charles Loewner and later by Oded Schramm provides such a framework. Good references for an introduction to this theory are [2] and [5].

### 6.3.1 Loewner's Theory

The setting for Loewner's chordal theory is the upper halfplane $\mathbb{H} \subset \mathbb{C}$. We are interested in sets $K \subset \mathbb{H}$ called hulls:

**Definition 6.14** ($\mathbb{H}$-hulls)**.** *A set $K \subset \bar{\mathbb{H}}$ is called a hull if $K$ is compact and $\mathbb{H}\backslash K$ is simply connected.*

To accomodate evolving geometry, we will allow for hulls indexed by time, that is, families of the form $(K_t)_{t\geq 0}$ where $K_t$ is a hull. For our purposes $K_s \subseteq K_t$ whenever $s \leq t$ and this 'growth' is continuous in the sense that for every $t, \epsilon$ there is a $\delta = \delta(\epsilon, t)$ such that $K_{t+\delta} \subseteq K_t^\epsilon = \mathbb{H} \cap \bigcup_{z\in K_t} \overline{B(z,\epsilon)}$, where the last expression is referred to as the $\epsilon$-thickening of $K_t$. In words, a small enough time step leads to a small change in the hull.

**Theorem 6.15** (Existence and uniqueness of $g_K$)**.** *For any hull $K$, there exists a conformal map $g_K : \mathbb{H}\backslash K \to \mathbb{H}$ with the property that*

$$\lim_{z\to\infty} (g_K(z) - z) = 0. \tag{105}$$

*Furthermore, this map is unique.*

The limit says that when $|z|$ is large, $g_K$ behaves almost like the identity function, a property called hydrodynamic normalization. The following expansion holds

$$g_K(z) = z + a_1 z^{-1} + a_2 z^{-2} + ... \tag{106}$$

and the coefficient $a_K := a_1$ is called the half plane capacity associated with $K$. It obeys the following rules:

**Lemma 6.16** (Properties of $a_K$)**.** *Scaling $a_{\lambda K} = \lambda^2 a_K$, summation $a_{K\cup L} = a_K + a_{g_K(L)}$ and translation invariance: $a_{K+x} = a_K$.*

The halfplane capacity grows continuously with respect to growth of $K$.

**Lemma 6.17.** *Given $K \subset K^\epsilon \subset B(z_0, R)$, there are constants $C = C(R), \alpha > 0$ s.t.*

$$a_{K^\epsilon} - a_K < C(R)\epsilon^\alpha \tag{107}$$

Given a continuously growing family of hulls $(K_t)$ with $\mathbb{H} \cap (K_t \backslash K_s) \neq \emptyset$ given $s < t$, we see by the properties of $a_K$ above that $t \mapsto a_{K_t}$ is increasing and continuous. Consequently there exists an increasing and continuous inverse of this map, say $\varphi^{-1}$. Given $(K_t)$ we may now define $\tilde{K}_t = K_{\varphi^{-1}(2t)}$. So $a_{\tilde{K}_t} = 2t$. What have we done here? In some sense, we have chosen a parameterization of time that connects it to the geometry of $(K_t)$, normalizing for its growth rate. The following theorem from [7], although not part of the standard introductory sequence, sheds some light on this intuition.

**Theorem 6.18.** *It holds that*

$$\frac{1}{66}\,\mathrm{hsiz}(K) \leq a_K \leq \frac{7}{2\pi}\,\mathrm{hsiz}(K) \tag{108}$$

*where* $\mathrm{hsiz}(K) = \mathcal{L}(\bigcup_{z=x+iy\in K} B(x+iy, y))$, *i.e. the Lebesgue measure of balls centered at a point in $K$, tangent to the real line.*

Therefore $a_K$ is comparable to a nice geometric quantity that is some approximation of the size of $K$. If $K_t$ grows really fast, $\varphi^{-1}$ will tend to grow more slowly, and so the parameterization using half plane capacity has a normalizing effect on growth rate. The word 'tend' in the previous sentence is crucial however. The convenience of parameterization in half plane capacity has more to do with the way it works nicely with Loewner's equation (see next section). As the following example shows, the growth rate heuristics can only be taken so far:

**Example 5.** *For any $R > 0$ and $\epsilon > 0$ there is a hull $K$ such that* $\operatorname{diam} K > R$ *while $a_K < \epsilon$. By using comparability to* $\mathrm{hsiz}(K)$ *we can consider a hull extending far in the real direction, so that* $\operatorname{diam} K > R$ *and then drive $a_K$ below any small positive value by insisting on $K$ containing no point with large imaginary part. We can argue that* $\mathrm{hsiz}(K)$ *is bounded by the Lebesgue measure of a long and thin cylinder, which for fixed 'length' can be driven to zero by decreasing 'thickness'.*

A natural set of questions given what we have seen so far concern the behaviour of the family $(g_{K_t})$ for continuously growing hulls $(K_t)$. What properties can be derived for the maps $g_{K_t}$ and for fixed $z$, how does $g_{K_t}(z)$ 'travel'. The next section provides some fascinating answers.

### 6.3.2 Loewner's equation

A special class of growing hulls are those generated by a simple growing curve $\gamma$, in the sense $K_t = \gamma([0,t])$. Under the further assumption that $(K_t)$ is parameterized as $a_{K_t} = 2t$, we arrive in the setting of Loewner's equation. For families of hulls satisfying the above two conditions, the associated maps $g_t = g_{K_t}$ have the expansion $g_t = z + \frac{2t}{z} + a_2 z^{-2} + ....$ They also satisfy the following differential equation:

**Theorem 6.19** (Loewner's equation)**.** *For any $T > 0$, let $\gamma : [0,T] \to \mathbb{H}$ with $\gamma(0) \in \mathbb{R}$ be a simple curve, parameterized by capacity, then*

- $t \mapsto \lim_{z \to \gamma(t)} g_t(z) =: W(t)$ *is well defined and continuous*
- $(g_t)_{t \in [0,T]}$ *satisfies*

$$\partial_t g_t(z) = \frac{2}{g_t(z) - W(t)} \tag{109}$$

  *for $z \in \mathbb{H}$, and $g_0(z) = z$.*

The proof is quite long and a good reference is [5].

By fixing $z \in \mathbb{H}$ and defining $z_t = g_t(z)$ we get an ODE $\dot{z}_t = \frac{2}{z_t - W(t)}$ with initial value $z_0 = z$. This differential equation has a solution in the time interval $[0,T] \cap [0, \tau(z))$ where $\tau(z) = \inf\{t \geq 0 : z \in K_t\}$. We call $\tau(z)$ the lifetime of the solution $z_t$. Another way to look at $\tau(z)$ is as the point in time when $g_t$ maps $z$ to $\mathbb{R}$, hitting the boundary of $\mathbb{H}$. That perspective will be useful in the case of the Harmonic Explorer, where some estimates rely on ensuring a sufficient distance to the boundary.

We will need the following bound on the diameter of $K_t$, see [9] Lemma 2.1:

**Lemma 6.20** (Diameter bounds on $K_t$)**.** *Let $(K_t)_{t \geq 0}$ be generated by $W$ in the sense of Loewner's equation. Then there is a $C > 0$ such that*

$$C^{-1} k(t) \leq \operatorname{diam} K_t \leq C k(t) \quad \forall t \geq 0 \tag{110}$$

*where $k(t) := \sqrt{t} + \max\{|W(s) - W(0)| : s \in [0,t]\}$*

### 6.3.3 Schramm-Loewner-Evolution

If we want to define a stochastic process taking values $(K_t)$, then we need some entry point for the stochasticity. Since $W(t)$ influences the evolution of $K_t$ through Loewner's equation in the case of simple curves with a special parameterization of time, one idea is to let $W(t)$ be stochastic, in the hope of obtaining something meaningful and well-defined for $K_t$. This turns out to be possible. A natural next step is to try different stochastic processes for $W$.

Looking at the setup from the other perspective, we might instead ask for certain qualitative features of $(K_t)$ and check whether there is any $W$ that is able to generate such a process. Oded Schramm considered randomly growing curves $K_t = \gamma([0,t])$ satisfying the Domain Markov Property (in a continuous formulation) and with conformally invariant laws, similar in spirit to the conformal invariance of Brownian Motion that we saw before. He was able to classify all stochastic processes $W$ that generate such families $(K_t)$.

**Theorem 6.21** (Schramm's principle)**.** *Schramm-Loewner-Evolutions are the only random curves with conformally invariant laws, satisfying the Domain Markov Property.*

We will now try to motivate the meaning of the Domain Markov Property in the continuous setting. Let $(P_{D,a,b})$ be a family of probability measures indexed by the marked domains $(D,a,b)$, where $D$ is simply connected and $a,b \in \partial D$. Suppose we have a random curve $\gamma$ and let $\mathcal{F}_t = \sigma(\gamma[0,t])$. Consider some space of curves and a set $C$ in this space. It should now hold that

$$P_{D,a,b}(\gamma[t,\infty] \in C|\mathcal{F}_t) = P_{D\setminus\gamma[0,t],\gamma(t),b}(\gamma \in C) \tag{111}$$

We can think of this as the 'important' information in $\mathcal{F}_t$ being geometric. The family of measures captures precisely this important geometric information.

We can see how this theorem has far-reaching consequences for the basic idea described in the beginning of this section, particularly in the way it connects with some of the discrete models we considered in earlier chapters. The lattice models satisfy the Domain Markov property and so it makes intuitive sense that this would be preserved as the grid gets finer. If one furthermore suspects conformal invariance in the limit, the theorem tells us that the target limiting object must be an SLE curve.

# 7 Harmonic Explorer converges to $SLE(4)$

In the following we use the tools introduced in the previous chapter, as well as some properties of the Harmonic Explorer to establish convergence of the driving process to that of $SLE(4)$. The proof was given in [11] using some results obtained in [9]. Towards the end we briefly comment on how this can be generalized, particularly with a view towards convergence of $LERW$ to $SLE(2)$.

## 7.1 A special property of SLE(4)

We will now derive a special property satisfied only by the $SLE$ with $\kappa = 4$. It is the continuous analogue of 4.2. While it is in no way immediate that the discrete process converges to $SLE(4)$ just because of this analogy, it is a useful heuristic, and as we will see, the convergence does in fact hold. It was presumably this special property of $SLE(4)$ that informed the design of the Harmonic Explorer process.

Recall the exit probabilities $h_n(v) = P^v(Y_n \in V_n^+)$ from the discussion of Harmonic Explorer. We now consider $SLE$ in the upper halfplane. Replacing Random Walk with Brownian Motion, we seek its exit probability on the subset of the boundary $\partial(\mathbb{H}\backslash\tilde{\gamma})$ given by the right side of the $SLE$ curve $\tilde{\gamma}$ and $(0, \infty)$.

Conformal invariance can be used to show that $\arg(z)/\pi$ is the probability that a Brownian Motion started at $z \in \mathbb{H}$ exits the upper half plane through the segment $(-\infty, 0) \subset \mathbb{R}$. For a curve at time $t$, the mapping out function in the sense of Loewner's theory, together with conformal invariance gives $1 - \arg X(t,z)/\pi$ as the desired expression for the exit described in the previous paragraph, given that the Brownian motion starts at $z$. Here $X(t,z) = g_t(z) - \sqrt{\kappa}B(t)$ To understand how this probability is expected to evolve for a fixed $z$, we only need to understand $X(t,z)$. By Loewner's equation we have

$$dX = d(g_t(z) - \sqrt{\kappa}B_t) = \frac{2}{X}dt - \sqrt{\kappa}dB_t \tag{112}$$

and then with Itô's formula

$$d\log X = \frac{4-\kappa}{2X^2}dt - \frac{\sqrt{\kappa}}{X}dB_t \tag{113}$$

When $\kappa = 4$, $d\log X = -\frac{2}{X}dB_t$ and so $d\arg X = -\operatorname{Im}(\frac{2}{X})dB_t$. We get that

$$E[\arg X(t_m, z) - \arg X(t_n, z)|\mathcal{F}_{t_n}] = E[\int_{t_n}^{t_m} dB_s|\mathcal{F}_{t_n}] = 0 \tag{114}$$

i.e. $\arg X(t,x)$ is a martingale and $SLE$ produces a qualitatively similar observable as that observed for Harmonic Explorer precisely when $\kappa = 4$.

## 7.2 The driving process of HE

The Harmonic Explorer gives rise to a path in $D$. By mapping this path conformally with $\phi : D \to \mathbb{H}$ we obtain $\gamma^\phi$ which is a process in $\mathbb{H}$.

As we have seen in the sections on Loewner's equation and $SLE$, any such curve $\gamma^\phi$ will correspond to a Loewner transform or driving function $W(t)$. Because the curve is random, this driving function is stochastic. The goal is to show that $W(t)$ converges to an appropriately scaled Brownian motion, namely the driving process of $\tilde{\gamma}$ which is $\tilde{W}(t) = 2B(t)$. More precisely:

**Theorem 7.1.** *For any $T > 0$, $W$ restricted to $[0, T]$ converges in law to the corresponding restriction of $\tilde{W}$.*

We now make some definitions to be used throughout the chapter. Let $t_n := \text{cap}_\infty(\phi \circ \gamma[0, n])$ be the halfplane capacity associated to the curve $\gamma^\phi$ at time $n$. Let $\tilde{D}_n := D\backslash\gamma[0, n]$ and $\phi_n : D_n \to \mathbb{H}$ be the conformal map normalized so that $\phi_n \circ \phi^{-1}(z) - z \to 0$. This normalization corresponds to using the canonical mapping out function $g_{t_n}$, since $\phi_n = g_{t_n} \circ \phi$.

With this, we are ready for a lemma regarding approximation of discrete $h_n$ by a continuous harmonic function $\tilde{h}_n$

**Lemma 7.2.** *Let $\tilde{h}$ be the function given by $\tilde{h}(z) = 1 - \frac{\arg(z)}{\pi}$. For any $\epsilon > 0$, there is an $r = r(\epsilon)$ such that for vertices $v \in V(TG)$ and times $j < N$ such that $rad_v(\tilde{D}_j) > r$ it holds*

$$|h_j(v) - \tilde{h}(\phi_j(v) - W(t_j))| < \epsilon \tag{115}$$

We recall that $h_j(v)$ is the probability that a random walk stopped upon exiting the domain $\tilde{D}_j$, hits the boundary at $V_j^+$. As mentioned in the previous section $\arg(z)/\pi$ features in the context of Brownian motion in the halfplane, stopped when hitting the real line, and denotes the probability of hitting the segment $(-\infty, 0)$. Here we make a translation by $-W(t_j)$ and take the probability of the complement, to compute $\tilde{h}(\phi_j(v) - W(t_j)) = P^{\phi_j(v)}(B_{T_\mathbb{H}} \in \phi_j(V_j^+)) = P^v(B_{T_{\tilde{D}_j}} \in V_j^+)$, where $B_t$ is a Brownian motion and $T_\mathbb{H}, T_{\tilde{D}_j}$ are exit times from the upper half plane and $\tilde{D}_j$ respectively.

We will i.) establish that $z \mapsto \tilde{h}(\phi_j(z) - W(t_j))$ is indeed harmonic and that its boundary values are correct. We would further need to show that ii.) The approximation for vertices not in the boundary is sufficiently accurate.

The second step is quite involved. It relies on coupling Brownian Motion and Random Walk on the triangular lattice $V(TG)$ to obtain guarantees regarding the distance of their exit sites. Depending on what coupling is used, the processes will be more or less close. A complication arises in part because even though the processes might be very close for a long period of time, a time well past the exit times, the erratic nature of these processes precludes any immediate conclusions about the distances of exit sites.

In the appendix, the second item is explored in more detail. The bad news is that this is carried out for a different lattice, but a partial compensation is perhaps provided by the more quantitative convergence rate demonstrated there. The same ideas should be applicable here.

*Proof.* Regarding the first part: To see that $\tilde{h}_j$ given by $\tilde{h}_j(z) = \tilde{h}(\phi_j(z) - W(t_j))$ is harmonic we use that $W(t_j) = \phi_j(\gamma[0, j])$. The function $\tilde{h}_j$ can be realized as a composition of holomorphic $\phi_j$, a holomorphic translation and a harmonic $\tilde{h}$, so by a well known result from complex analysis, it is itself a harmonic function. That $\tilde{h}$, based on $z \mapsto \arg(z)$ is harmonic can be seen by considering $\log(z) = \log|z| + i\arg(z)$ and recognizing that the real part of a holomorphic function (and therefore also the imaginary part) is harmonic. The boundary values are clear from the definition of $\phi_j$.

Some inspiration for item ii.) can be found in the appendix.

■

Let $p_n := \phi_n^{-1}(i + W(t_n))$. The two following events will be of great importance: Let $\mathcal{A}_1(n) := \{n < N\}$, meaning that $n \in \mathbb{N}$ is less than the terminal time of the HE path. Furthermore $\mathcal{A}_2(n) := \{\mathrm{rad}_{p_n}(\tilde{D}_n) > R\}$. Given $n < N$ we define $m$ as the smallest integer greater than $n$ such that $\max\{t_m - t_n, (W(t_m) - W(t_n))^2\} \geq \delta^2$. This $m$ defines a timescale that we will examine more closely. More precisely, we will prove the following theorem, showing that locally, the Loewner transform of $\gamma$ behaves like a Brownian motion scaled by $\sqrt{4}$.

**Theorem 7.3.** *For every $\delta \in (0, 1)$ there is an $R = R(\delta)$ such that on $\mathcal{A}_1 \cap \mathcal{A}_2$ the following estimates hold:*

$$E[W(t_m)|\gamma[0, n]] = W(t_n) + O(\delta^3) \tag{116}$$

$$E[(W(t_m) - W(t_n))^2|\gamma[0, n]] = 4E[t_m - t_n|\gamma[0, n]] + O(\delta^3) \tag{117}$$

For better readability, we will separate the proof into a sequence of steps, beginning with the following lemma:

**Lemma 7.4.** *For $n < N$ and $m, n$ as defined above, there is some $\delta_0 > 0$ such that $\mathrm{rad}_{p_n}(\tilde{D}_m) \geq \mathrm{rad}_{p_n}(\tilde{D}_n)/2 - 1$ if $\delta < \delta_0$.*

*Proof.* By considering the fact that $|\gamma(j + 1) - \gamma(j)| = 1$ it is sufficient to show that $\mathrm{rad}_{p_n}(\tilde{D}_{m-1}) \geq \mathrm{rad}_{p_n}(\tilde{D}_n)/2$. We consider $z \in \{w : |w - p_n| = \mathrm{rad}_{p_n}(\tilde{D}_n)/2\} =: C_{p_n}$, i.e. those complex numbers on the circle centered at $p_n$ with radius similar to the right hand side in the statement of the lemma. Since $z$ inside a compact set in the interior of $D$ the result from 2.21 gives that $\phi_n(z)$ is at a positive distance from $\partial\mathbb{H} = \mathbb{R}$, i.e. that $\mathrm{Im}(\phi_n(z)) > c_1 > 0$. By Loewner's equation it holds that $d/dt\,\mathrm{Im}(g_{t_j})^2 \geq -4$ and so $\mathrm{Im}(\phi_{j+1}(z))^2 - \mathrm{Im}(\phi_j(z))^2 = \mathrm{Im}(g_{t_{j+1}}(\phi(z)))^2 - \mathrm{Im}(g_{t_j}(\phi(z)))^2 \geq -4$. For fixed $z$ we also have the ODE for $z_t$ and that the terminal time $\tau(z)$ is the time $t$ at which $\phi_t(z) \in \mathbb{R}$.

We get $\tau(z) - t_n \geq \mathrm{Im}(\phi_n(z))^2/4$. If we take $\delta_0 = \inf\{\frac{Im(\phi_n(z))}{2} : z \in C_{p_n}\} > 0$ and use the definition of $m$ giving $t_{m-1} - t_n \leq \delta^2$, we have

$$t_{m-1} - t_n < \delta^2 \overset{!}{<} \delta_0^2 \leq \mathrm{Im}(\phi_n(z))^2/4 \leq \tau(z) - t_n \tag{118}$$

$$t_{m-1} < \tau(z) \tag{119}$$

such that $\gamma[0, m-1] \cap C_{p_n} = \emptyset$. Since the growing curve $\gamma$ is the only influence on the decrease of the inradius of $p_n$ between $\tilde{D}_n$ and $\tilde{D}_{m-1}$ we must have $\mathrm{rad}_{p_n}(\tilde{D}_{m-1}) \geq \mathrm{rad}_{p_n}(\tilde{D}_n)/2$, proving the claim.

■

The significance of the previous lemma is that we have now ensured that it is possible to control the inner radius of $p_n$ also in $\tilde{D}_m \subset \tilde{D}_n$, which was not clear a priori, since it could have been the case (and this is still possible if $\delta > \delta_0$) that the evolution of the curve between $t_n$ and $t_m$ destroys such a control. The next lemma, which is somewhat similar in spirit, is instead about controlling $\mathrm{rad}_{p_n}(\tilde{D}_n)$ by means of $\mathrm{rad}_{p_0}(D)$. Once this is accomplished, we are in a setting where we need to worry only about $\rho = \mathrm{rad}_{p_0}(D)$ to ensure that the approximation is good enough for all the time steps that will be needed to get a macroscopic idea of $W$.

**Lemma 7.5.** *Define for $\beta > 0$ the time $\tilde{T}_\beta = \sup\{t \in [0, T] : |W(t)| \leq \beta\}$ and the associated index set $I = \{n \in \mathbb{N} : t_n \leq \tilde{T}_\beta\}$. For all $n \in I$ we have the following control:*

$$\mathrm{rad}_{p_0}(D) \leq O(1)\,\mathrm{rad}_{p_n}(\tilde{D}_n) \tag{120}$$

*Consequently, there is $R'' = R''(R', \beta)$ such that $\mathrm{rad}_{p_0}(D) > R''$ ensures $\mathrm{rad}_{p_n}(\tilde{D}_n) > R'$.*

The central idea here is that as long as $W$ is not too erratic, we have a handle on the geometry of $\tilde{D}_n$.

*Proof.* By 2.21 we have

$$\mathrm{rad}_{p_n}(\tilde{D}_n) \geq |\phi_n^{-1}(\phi_n(p_n))|\frac{1}{4} \tag{121}$$

$$\mathrm{rad}_{p_0}(D) \leq |\phi^{-1}(\phi(p_0))| \tag{122}$$

The first bound is stochastic, since $\phi_n$ is. To overcome this, we first note that for $n \in I$ we have $|\phi_n(p_n)| = |i + W(t_n)| \leq 1 + \beta$, so $\phi_n(p_n)$ lies in a compact subset of $\mathbb{H}$ (take for instance $K = \{z \in \mathbb{H} : \mathrm{Im}(z) \geq 1, |z| \leq 1 + \beta\}$). By invertibility, $\phi_n^{-1}$ is nonzero, and by continuity, it attains its infimum on the compact set. We have that $\sigma(\phi_n)$ is generated by a finite partition of $\Omega$ (since the underlying process is discrete). So we may take the minimum over this partition to replace the first inequality by a deterministic lower bound. The second inequality is deterministic as is. Consequently

$$\operatorname{rad}_{p_n}(\tilde{D}_n) \geq c_1 > 0 \tag{123}$$
$$\operatorname{rad}_{p_0}(D) \leq c_2 \tag{124}$$
$$\operatorname{rad}_{p_n}(\tilde{D}_n)\frac{c_2}{c_1} \geq \operatorname{rad}_{p_0}(D) \tag{125}$$

and the first part is complete. Since $T < \infty$ it holds that $|I| < \infty$, and so the consequence stated at the end of the theorem follows as well.

■

**Lemma 7.6.** *For $n$ and $m$ defined as above, for all $t \in [t_n, t_m]$ the following relations hold*

$$|W(t) - W(t_n)| = O(\delta) \tag{126}$$
$$t_m - t_n = O(\delta^2) \tag{127}$$

*Proof.* We can first notice that the first relation holds in the range $t \in [t_n, t_{m-1}]$, since $|W(t_n) - W(t_{m-1})| < \delta$ by the definition of $m$. The same reasoning gives $t_{m-1} - t_n < \delta^2$ and so the whole proof will be concerned with extending these inequalities (up to multiplicative constants) to the slightly larger time $t_m$. Since we have control in $[t_n, t_j]$ for $j = n, n+1, ..., m-1$ we will be interested in control over intervals $[t_j, t_{j+1}]$, hoping to reach the desired extension.

We pick without loss of generality $R > \delta^{-2}$ large. By the Beurling projection theorem (see [4]) we get that the harmonic measure from $p_n$ to $\gamma[j, j+1]$ in the domain $\tilde{D}_j$ is $O(\delta)$. Conformal invariance of harmonic measure tells us that the same is true for the triplet $\phi_j(p_n), \phi_j \circ \gamma[j, j+1], \mathbb{H}$. The claim is that $\operatorname{diam} \phi_j \circ \gamma[j, j+1] = O(\delta)$. To see why this gives the lemma, we apply 6.20 to get for $\tilde{K}_r := \phi_j \circ \gamma[j, r]$ considered for $r \leq j+1$. We have in particular $\sqrt{\tilde{t}_r} + \sup\{|\tilde{W}(s) - \tilde{W}(0)| : s \in [0, \tilde{t}_r]\} \leq C \operatorname{diam} \phi_j \circ \gamma[j, j+1] = O(\delta)$, where the use of a tilde indicates that we are using times or driving processes related to $\tilde{K}_r$. But in terms of our curve $\phi \circ \gamma$ this gives

$$\sqrt{t_{j+1} - t_j} + |W(t_{j+1}) - W(t_j)| = O(\delta) \tag{128}$$

so separating into two relations, one for each term and then squaring the first gives the statement.

To see that $\operatorname{diam} \phi_j \circ \gamma[j, j+1] = O(\delta)$, we remind ourselves that $\phi_j(p_n) = g_{t_j} \circ \phi(p_n)$. The proof of 7.4 gives a positive lower bound on $\operatorname{Im} \phi_{m-1}(p_n)$. Since $\operatorname{Im} g_t(z)$ is decreasing in $t$, we get that $\operatorname{Im} g_t \circ \phi(p_n)$ is bounded from below by a positive constant for $t \leq t_{m-1}$. Loewner's equation and this lower bound $c$ give

$$|\partial_t g_t \circ \phi(p_n)| = \left| \frac{2}{g_t \circ \phi(p_n) - W(t)} \right| \tag{129}$$

$$= \left| \frac{2}{\mathrm{Re}(g_t \circ \phi(p_n)) - W(t) + i\,\mathrm{Im}(g_t \circ \phi(p_n))} \right| \tag{130}$$

$$\leq \frac{2}{0 + c} = O(1) \tag{131}$$

which we integrate over $[t_n, t_j]$ to get $|\phi_j(p_n) - \phi_n(p_n)| = O(\delta^2)$.

The claim now is that the distance between $\phi_j(p_n)$ and $\phi_j \circ \gamma[j, j+1]$ is $O(1)$. To see this, note that $W(t_j) = \phi_j(\bar{v}_j) \in \phi_j \circ \gamma[j, j+1]$ and consider the following string of inequalities

$$\mathrm{dist}(\phi_j(p_n), \phi_j \circ \gamma[j, j+1]) \leq |\phi_j(\bar{v}_j) - \phi_j(p_n)| \tag{132}$$

$$\leq |\phi_j(\bar{v}_j) - \phi_n(\bar{v}_n)| + |\phi_n(\bar{v}_n) - \phi_n(p_n)| + |\phi_n(p_n) - \phi_j(p_n)| \tag{133}$$

$$\leq |W(t_j) - W(t_n)| + |W(t_n) - (i + W(t_n))| + |\phi_n(p_n) - \phi_j(p_n)| \tag{134}$$

$$= O(\delta) + 1 + O(\delta^2) \tag{135}$$

$$= O(1) \tag{136}$$

The harmonic measure from $\phi_j(p_n)$ to $\phi_j \circ \gamma[j, j+1]$ in $\mathbb{H}$ is estimated to be $O(\delta)$ and since $\mathrm{dist}(\phi_j(p_n), \phi_j \circ \gamma[j, j+1]) = O(1)$ we get $\mathrm{diam}\, \phi_j \circ \gamma[j, j+1] = O(\delta)$. To see this, simply map the quantities under comparison to the unit disk $\mathbb{D}$ by composing the mapping out function for $\mathbb{H} \backslash \phi_j \circ \gamma[j, j+1]$ with a conformal map from $\mathbb{H}$ to $\mathbb{D}$. Then the curve segment is mapped to the boundary and the point $\phi_j(p_n)$ is mapped to the interior. By appealing to the distortion theorems for conformal maps the distances between the points and the diameter of the curve segment are of the same order as their respective preimages. By conformal invariance of harmonic measure and the particularly simple expression for harmonic measure in $\mathbb{D}$ we get the diameter estimate and can conclude. ■

We are now ready for the proof of 7.3.

*Proof.* Assume that $R > 100 \max\{1, r(\delta^3)\}$ and that $\mathcal{A}_2 = \{\mathrm{rad}_{p_n}(\tilde{D}_n) > R\}$. This ensures on the one hand that $\mathrm{rad}_{p_n}(\tilde{D}_m) > r(\delta^3)$ and by 7.4 also that $\mathrm{rad}_{p_n}(\tilde{D}_m) > r(\delta^3)$, but more importantly also that there are multiple vertices inside $\{w : |w - p_n| \leq \mathrm{rad}_{p_n}(\tilde{D}_n)/6\}$, all of which have the following property: For such $w = w_0$ it holds that $\mathrm{rad}_{w_0}(\tilde{D}_k) > r(\delta^3)$ for both $k = n, m$ and so we may apply the estimates from 7.2 to obtain

$$E[\tilde{h}(\phi_m(w_0) - W(t_m))|\gamma[0,n]] \in E[h_m(w_0)|\gamma[0,n]] + (-\delta^3, \delta^3) \tag{137}$$
$$= h_n(w_0) + (-\delta^3, \delta^3) \tag{138}$$
$$\in \tilde{h}(\phi_n(w_0) - W(t_n)) + (-2\delta^3, 2\delta^3) \tag{139}$$

so that $E[\tilde{h}(\phi_m(w_0) - W(t_m))|\gamma[0,n]] = \tilde{h}(\phi_n(w_0) - W(t_n)) + O(\delta^3)$.

Let $z_t := g_t \circ \phi_{w_0}$ so that $\phi_{t_n}(w_0) = z_{t_n}$. It is possible to obtain $\phi_{t_m}(w_0) = z_{t_m}$ by following the ODE trajectory (or 'flow') implied by Loewner's equation. Using 7.6 we have the approximation

$$\frac{2}{z_t - W(t)} = \frac{2}{z_{t_n} - W(t_n)} + O(\delta) \tag{140}$$

for $t \in [t_n, t_m]$. Integrating over this timescale then gives an approximation to the point $z_{t_m}$ in the form of

$$z_{t_m} - z_{t_n} = \phi_m(w_0) - \phi_n(w_0) \tag{141}$$
$$= \left(\frac{2}{\phi_n(w_0) - W(t_n)} + O(\delta)\right)(t_m - t_n) \tag{142}$$
$$= \frac{2}{\phi_n(w_0) - W(t_n)}(t_m - t_n) + O(\delta^3) \tag{143}$$

using 7.6 $t_m - t_n = O(\delta^2)$ in the last step.

From 7.2 we have an associated function $F(z, W) := \tilde{h}(z - W)$. We will estimate $F(z_{t_m}, W(t_m))$ up to an accuracy $O(\delta^3)$ by using a Taylor expansion. From the proof of 7.6 we have $|z_{t_m} - z_{t_n}| = O(\delta^2)$ and from the statement of that lemma also $|W(t_m) - W(t_n)| = O(\delta)$. It therefore suffices to consider only terms up to first order for $z$ and second order for $W$.

$$\tilde{h}(\phi_m(w_0) - W(t_m)) - \tilde{h}(\phi_n(w_0) - W(t_n)) = \tag{144}$$
$$= \partial_z F(z_{t_n}, W(t_n))(z_{t_m} - z_{t_n}) + \tag{145}$$
$$+ \partial_W F(z_{t_n}, W(t_n))(W(t_m) - W(t_n)) \tag{146}$$
$$+ \frac{1}{2}\partial_W^2 F(z_{t_n}, W(t_n))(W(t_m) - W(t_n))^2 + O(\delta^3) \tag{147}$$

Calculating the derivatives $\partial_z F$, $\partial_W F$ and $\partial_W^2 F$ and inserting the estimate for $z_{t_m} - z_{t_n}$ obtained above and then taking the conditional expectation with respect to $\gamma[0, n]$ one gets

$$
\begin{aligned}
O(\delta^3) = 2\,\mathrm{Im}((\phi_n(w_0) - W(t_n))^{-2})E[t_m - t_n|\gamma[0,n]] \qquad (148)\\
- \mathrm{Im}((\phi_n(w_0) - W(t_n))^{-1})E[W(t_m) - W(t_n)|\gamma[0,n]] \qquad (149)\\
- 1/2\,\mathrm{Im}((\phi_n(w_0) - W(t_n))^{-2})E[(W(t_m) - W(t_n))^2|\gamma[0,n]] \qquad (150)
\end{aligned}
$$

The idea now is to use two different vertices $w_0 = w_1, w_2$ to get two equations of this form. The restrictions imposed on these vertices are i.) They should satisfy $|w_i - p_n| < \mathrm{rad}_{p_n}(\tilde{D}_n)/6$, since this is the condition on $w_0$ that guaranteed the approximation quality by means of 7.2, and ii.) The two vertices must be different, since then they together yield two linearly independent equations, ensuring a unique solution for the quantities of interest $E[W(t_m)-W(t_n)|\gamma[0,n]]$ and $E[(W(t_m)-W(t_n))^2|\gamma[0,n]]$ in terms of $E[t_m - t_n|\gamma[0,n]]$ up to $O(\delta^3)$ terms.

Since a prominent feature of the above equation is terms of the form $\mathrm{Im}((\phi_n(w_0) - W(t_n))^{-2})$ it seems reasonable to pick $w_1, w_2$ in terms of the distance $|\phi_n(w_i) - W(t_n)|$. Since $\phi_n(p_n) = i + W(t_n)$, assuming $R > \delta^{-2}$ and using the Koebe distortion theorem, we get for the vertex $w_1$ closest to $p_n$

$$
\begin{aligned}
|\phi_n(w_1) - i - W(t_n)| = |\phi_n(w_1) - \phi_n(p_n)| = O(\delta^2) \qquad (151)\\
\phi_n(w_1) - W(t_n) = i(1 + O(\delta^2)) + O(\delta^2) \qquad (152)
\end{aligned}
$$

and it is then possible to consider the series expansion of $\frac{1}{i+z}$ and $\frac{1}{(i+z)^2}$, discarding quadratic terms (since $z$ is $O(\delta^2)$) and then retaining only the imaginary part.

We obtain the vertex $w_2$ by picking the vertex closest to $\phi_n^{-1}(i + W(t_n) + 1/100)$ and get similarly that $|\phi_n(w_2) - i - W(t_n) - 1/100| = O(\delta^2)$ and that $|w_2 - p_n| < \mathrm{rad}_{p_n}(\tilde{D}_n)/6$. For this case we will therefore want to get the series expansions for $\frac{1}{i+1/100+z}$ and $\frac{1}{(i+1/100+z)^2}$, again only up to the first order.

To ease the burden of notation and let the structure of the equations emerge more clearly, we let $E_t := E[t_m - t_n|\gamma[0,n]]$, $E_W := E[W(t_m) - W(t_n)|\gamma[0,n]]$ and $E_{W^2} := E[(W(t_m) - W(t_n))^2|\gamma[0,n]]$. In these terms 148 takes for $a_i = a_i(w_i)$ and $b_i = b_i(w_i)$ for $i = 1, 2$ the following form

$$
a_i E_W + b_i[4E_t - E_{W^2}] = O(\delta^3) \qquad (153)
$$

where the discussion of expansions above give (with some lengthy calculations) that $a_1 a_2 - b_1 b_2 \neq 0$. Therefore, Gaussian elimination determines

$$
\begin{aligned}
E_W = O(\delta^3) \qquad (154)\\
E_{W^2} = 4E_t + O(\delta^3) \qquad (155)
\end{aligned}
$$

which is what we wanted to prove.

■

**Theorem 7.7.** *For fixed $T \geq 1$, taking the limit $\rho \to \infty$, the restriction of $t \mapsto W(t)$ to $[0,T]$ converges in distribution to the corresponding restriction of a standard Brownian motion scaled by $\sqrt{4}$.*

*Proof.* Let $m_0 = 0$, $m_1 = m$, where $m$ is as defined above in the case $n = 0$ from 7.3. Then define $m_{k+1}$ inductively as $m$ given that $n = m_k$, i.e. $m_{k+1} = \min\{j > m_k : \max\{t_j - t_{m_k}, (W(t_j) - W(t_{m_k}))^2\} \geq \delta^2\}$. Let $(\mathcal{F}_n)$ be the filtration obtained by letting $\mathcal{F}_n = \sigma(\gamma[0, m_n])$, containing all information about the Harmonic Explorer curve up to time step $m_n$.

Consider now the finite subset of this sequence given by those $m_n \in I$, where $I = I(\tilde{T}_\beta) = \{m \in \mathbb{N} : t_m \leq \tilde{T}_\beta\}$ in the notation of 7.5. We will begin by proving the statement for the restriction to $[0, \tilde{T}_\beta]$ and then extend to $[0,T]$. The set $I$ contains a maximal element, which we denote by $m_{n^*}$.

The relations 7.6 and boundedness of $W$ for $t \leq \tilde{T}_\beta$ has geometric implications. Together with 6.18 it implies a worst case lower bound for $t_{m_{k+1}} - t_{m_k}$ in terms of the increment in $W$ and so we get that the set $I$ has size of order $|I| = O(\delta^{-2}T)$.

The claim is that for sufficiently large $\rho$, the key estimate 7.3 holds

$$E[W(t_{m_{n+1}})|\mathcal{F}_n] = W(t_{m_n}) + O(\delta^3) \tag{156}$$

$$E[(W(t_{m_{n+1}}) - W(t_{m_n}))^2|\mathcal{F}_n] = 4E[t_{m_{n+1}} - t_{m_n}|\mathcal{F}_n] + O(\delta^3) \tag{157}$$

for $m_n \in I$.

To see this, it suffices to verify the conditions $\mathcal{A}_1(m_n) \cap \mathcal{A}_2(m_n)$. Recall that $\mathcal{A}_1(m_n) = \{m_n < N\}$, where we recall that $N$ is the random terminal time of the Harmonic Explorer process. Since $I$ is finite, we have that $t_{m_n} < \infty$, by the relations 7.6. On the other hand, by the setup of the map $\phi$, at the terminal time $N$, the diameter of the curve $\gamma^\phi$ is infinite. The diameter bounds from 6.20 yield that either $t_N = \infty$ or $W(t)$ is unbounded for $t \leq t_N$. So from the definition of $I$ as a set of $m_n$ where we have en explicit bound for both $t_{m_n}$ and $W$ up to that point, we must have $t_{m_n} < t_N$ and $m_n < N$. The condition $\mathcal{A}_2(m_n) = \{\mathrm{rad}_{p_{m_n}}(\tilde{D}_{m_n}) > R\}$ for some large $R$ (specifically the choice made in 7.3) can be ensured for $m_n \in I$, by 7.5.

We are interested in convergence to Brownian Motion. As we have seen previously, a common tool is the Skorokhod embedding and we will now construct a new random variable $M_n$ to embed. Heuristically, the aim here is to create a 'true' martingale that is comparable to the 'local' martingale $W(t_{m_n})$. To this end, let

$$M_n = \sum_{j=0}^{n-1} W(t_{m_{j+1}}) - W(t_{m_j}) - E[W(t_{m_{j+1}}) - W(t_{m_j})|\mathcal{F}_n] \tag{158}$$

It is then immediate that $E[M_{n+1} - M_n|\mathcal{F}_n] = 0$ and with $E[|M_n|] \leq nO(\delta^3) + E[|W(t_{m_n})|] < \infty$ we have that $(M_n)$ is indeed a $(\mathcal{F}_n)$-martingale. It holds that

$$\|M_{n+1} - M_n\|_\infty \leq \|W(t_{m_{n+1}}) - W(t_{m_n})\|_\infty + \|E[W(t_{m_{n+1}}) - W(t_{m_n})|\mathcal{F}_n]\|_\infty \tag{159}$$
$$= O(\delta) \tag{160}$$

by using the relations 7.6.

Using the Skorokhod embedding we have $M_n \stackrel{d}{=} B_{\tau_n}$ for a sequence of stopping times $\tau_0 \leq \tau_1 \leq ... \leq \tau_{n^*+1}$.

The claim is that

$$\sup\{|B_t - B_{\tau_n}| : t \in [\tau_n, \tau_{n+1}]\} = O(\delta) \tag{161}$$
$$\sup\{|W(t) - W(t_{m_n})| : t \in [t_{m_n}, t_{m_{n+1}}]\} = O(\delta) \tag{162}$$

where the first relation uses the special property of the embedding mentioned in 6.6 and the second is a consequence of 7.6. We also have that

$$\sup\{|W(t_{m_n}) - M_n| : m_n \in I, n^* + 1\} = O(\delta T) \tag{163}$$

which can be seen by

$$W(t_{m_n}) - M_n = \sum_{j=0}^{n-1} E[W(t_{m_{j+1}}) - W(t_{m_j})|\mathcal{F}_j] \tag{164}$$

and using Jensen's inequality together with 7.6, this time for the driving function differences, to get $O(|I|\delta^3) = O(\delta T)$.

Note that the relations we have just derived give us a way to compare the driving process $W$ to Brownian motion $B$ in the sup-norm provided that we can connect $[t_{m_n}, t_{m_{n+1}}]$ with $[\tau_n, \tau_{n+1}]$. In particular the ratio $t_{m_n}/\tau_n$ will determine the scaling factor of the Brownian motion.

We claim that $4t_{m_n}$ is close to $\tau_n$ for $m_n \in I$ and for $m_{n^*+1}$. To prove this, we introduce

$$Y_n = \sum_{j=0}^{n-1} (M_{j+1} - M_j)^2 \tag{165}$$
$$Z_n = Y_n - 4t_{m_n} \tag{166}$$

Since $Y_n$ is defined in terms of $M_n, M_{n-1}, ..., M_0$ and $M_n$ is defined in terms of $W(t_{m_n}), W(t_{m_{n-1}}), ...W(t_{m_0})$ all the quantitative statements that follow, relying on 7.3

actually hold also for $n^*+1$. From the definition of $Z_n$ it is perhaps not surprising that we will begin by comparing $4t_{m_n}$ and $Y_n$. For $Y_n$ it holds

$$
\begin{aligned}
Y_{n+1} - Y_n &= (M_{n+1} - M_n)^2 && (167)\\
&= (W(t_{m_{n+1}}) - W(t_{m_n}) - E[W(t_{m_{n+1}}) - W(t_{m_n})|\mathcal{F}_n])^2 && (168)\\
&= (W(t_{m_{n+1}}) - W(t_{m_n}))^2 && (169)\\
&\quad - 2(W(t_{m_{n+1}}) - W(t_{m_n}))E[W(t_{m_{n+1}}) - W(t_{m_n})|\mathcal{F}_n] && (170)\\
&\quad + E[W(t_{m_{n+1}}) - W(t_{m_n})|\mathcal{F}_n]^2 && (171)\\
&= (W(t_{m_{n+1}}) - W(t_{m_n}))^2 + O(\delta)O(\delta^3) + O(\delta^3)^2 && (172)\\
&= (W(t_{m_{n+1}}) - W(t_{m_n}))^2 + O(\delta^4) && (173)
\end{aligned}
$$

where we used 7.3 and 7.6. Using 7.3 again we get

$$
\begin{aligned}
E[Z_{n+1} - Z_n|\mathcal{F}_n] &= E[Y_{n+1} - Y_n - 4t_{m_{n+1}} + 4t_{m_n}|\mathcal{F}_n] && (174)\\
&= E[(W(t_{m_{n+1}}) - W(t_{m_n}))^2|\mathcal{F}_n] + O(\delta^3) && (175)\\
&\quad - 4E[t_{m_{n+1}} - t_{m_n}|\mathcal{F}_n] && (176)\\
&= O(\delta^3) && (177)
\end{aligned}
$$

But by the bounds on the increments given by 7.6 we also have $E[(Z_{n+1}-Z_n)^2|\mathcal{F}_n] = O(\delta^4)$. Define $Z'_n = Z_n - \sum_{j=1}^{n} E[Z_j - Z_{j-1}|\mathcal{F}_{j-1}] = \sum_{j=1}^{n}(Z_j - E[Z_j|\mathcal{F}_{j-1}])$ by recognizing $Z_0 = 0$. $Z'_n$ is a martingale. Furthermore, by expanding the square, considering the possible terms and applying expectations $E[\cdot] = E[E[\cdot|\mathcal{F}_k]]$ for a good choice of $k$ we have the orthogonality of increments, which gives

$$
\begin{aligned}
E[(Z'_{n^*+1})^2] &= E[(\sum_{j=1}^{n^*+1} Z'_j - Z'_{j-1})^2] && (178)\\
&= \sum_{j=1}^{n^*+1} E[(Z'_j - Z'_{j-1})^2] && (179)\\
&= O(|I|\delta^4) && (180)
\end{aligned}
$$

We can without loss of generality take $\delta$ small enough to ensure that the trailing sum in the definition of $Z'_n$ (which is of order $O(|I|\delta^3)$) is smaller than $\delta^{1/2}/2$. Then, with Doob's maximal inequality for $L^2$ martingales, we obtain

$$
\begin{aligned}
O(T\delta) &= O(|I|\delta^3) && (181)\\
&= 4\frac{E[|Z'_{n^*+1}|^2]}{\delta} && (182)\\
&\geq P(\max_{m_n\in I, n^*+1} |Z'_n| > \delta^{1/2}/2) && (183)\\
&\geq P(\max_{m_n\in I, n^*+1} |Z_n| > \delta^{1/2}) && (184)
\end{aligned}
$$

But the definition of $Z_n$ gives

$$
P(\max_{m_n\in I, n^*+1} |Y_n - 4t_{m_n}| > \delta^{1/2}) = O(T\delta). \quad (185)
$$

So we have established that $Y_n$ is indeed close to $4t_{m_n}$. The next step is to show that $Y_n$ is close to $\tau_n$ with high probability, for all $n$ such that $m_n \in I$. From the Skorokhod embedding theorem for martingales 6.6, using the bound for $\tau_{n+1} - \tau_n$ we have, taking the conditional expectation with respect to the Brownian motion used for the embedding

$$
E[(\tau_{n+1} - \tau_n)^2 | B[0, \tau_n]] = O(\delta^4) \quad (186)
$$

and by the previous bounds on $Y_{n+1} - Y_n$ as well as relations for the squared difference of driving terms, we immediately get

$$
E[((\tau_{n+1} - Y_{n+1}) - (\tau_n - Y_n))^2 | B[0, \tau_n]] = O(\delta^4) \quad (187)
$$

Then, using the relation for the conditional variance of increments $B_{\tau_{n+1}} - B_{\tau_n}$ in terms of a conditional expectation for a difference $\tau_{n+1} - \tau_n$ we obtain

$$
E[((\tau_{n+1} - Y_{n+1}) - (\tau_n - Y_n)) | B[0, \tau_n]] = 0 \quad (188)
$$

Applying Doob's $L^2$ inequality for martingales once more we have

$$
P(\max_{m_n\in I, n^*+1} |\tau_n - Y_n| > \delta^2) = O(|I|\delta^4)/\delta = O(T\delta) \quad (189)
$$

which we can combine with a previous estimate in the following way

$$
\begin{aligned}
P(\max_{m_n\in I, n^*+1} |\tau_n - 4t_{m_n}| > 2\delta^{1/2}) &\leq P(\max_{m_n\in I, n^*+1} |\tau_n - Y_n| + |Y_n - 4t_{m_n}| > 2\delta^{1/2}) && (190)\\
&\leq P(\max_{m_n\in I, n^*+1} |\tau_n - Y_n| > \delta^{1/2}) + && (191)\\
&\quad + P(\max_{m_n\in I, n^*+1} |Y_n - 4t_{m_n}| > \delta^{1/2}) && (192)\\
&= O(T\delta) && (193)
\end{aligned}
$$

so that $\tau_n$ and $4t_{m_n}$ are indeed close with high probability.

With all the needed estimates in place, we are ready to put everything together. The event $\mathcal{A}$ where $\max_{m_n \in I, n^*+1} |\tau_n - 4t_{m_n}| \leq 2\delta^{1/2}$ and also

$$\sup\{|B_t - B_{\tau_n}| : t \in [\tau_n, \tau_{n+1}]\} = O(\delta) \tag{194}$$
$$\sup\{|W(t) - W(t_{m_n})| : t \in [t_{m_n}, t_{m_{n+1}}]\} = O(\delta) \tag{195}$$
$$\sup\{|W(t_{m_n}) - M_n| : m_n \in I, n^* + 1\} = O(\delta T) \tag{196}$$

is an event of probability $1 - O(T\delta)$.

We want to estimate the quantity $\sup\{|B(4t) - W(t)| : t \in [0, \tilde{T}_\beta]\}$ on this event. By the definition of $I$, it clearly holds that the interval $[0, \tilde{T}_\beta]$ is covered in the following way:

$$[0, \tilde{T}_\beta] \subset \bigcup_{m_n \in I} [t_{m_n}, t_{m_{n+1}}]. \tag{197}$$

We may therefore estimate

$$\max_{m_n \in I} \sup\{|B(4t) - W(t)| : t \in [t_{m_n}, t_{m_{n+1}}]\}. \tag{198}$$

Using the fact that $M_n = B_{\tau_n}$, $[t_{m_n}, t_{m_{n+1}}] \subset 1/4[\tau_n - 2\delta^{1/2}, \tau_{n+1} + 2\delta^{1/2}]$ and the triangle inequality this boils down to estimating terms of the form

$$\sup\{|W(t) - W(t_{m_n})| : t \in [t_{m_n}, t_{m_{n+1}}]\} \tag{199}$$
$$\sup\{|W(t_{m_n}) - M_n| : m_n \in I, n^* + 1\} \tag{200}$$
$$\sup\{|B(4t) - B_{\tau_n}| : t \in 1/4[\tau_n - 2\delta^{1/2}, \tau_{n+1} + 2\delta^{1/2}]\} \tag{201}$$

The first and second of these are respectively $O(\delta)$ and $O(\delta T)$ on $\mathcal{A}$, and so the same holds for the maximum of such terms. Recognizing that we scale time of $B$ by 4 and the interval by 1/4, we see that for the second term, we have control over $1/4[\tau_n, \tau_{n+1}]$, where again the order is $O(\delta)$. The perturbed interval endpoints $\tau_n \pm 2\delta^{1/2}$ introduces the need for corrective terms of the form

$$\sup\{|B(4t) - B_{\tau_n}| : t \in 1/4[\tau_n - 2\delta^{1/2}, \tau_n]\} \tag{202}$$

and similarly for $\tau_{n+1}$. Brownian motion is almost surely continuous and $[0, \tilde{T}_\beta]$ is a compact set, hence $B$ is almost surely uniformly continuous on this interval. It turns out, see for instance [10] theorem 1.12, that the random modulus of continuity can be replaced by a deterministic function. Therefore, when taking the maximum over these corrective terms, we can drive its value to zero by letting $\delta \to 0$.

So for fixed $T$ we can indeed find $\rho$ large enough to get

$$P(\sup\{|B(4t) - W(t)| : t \in [0, \tilde{T}_\beta]\} > \epsilon) < \epsilon \tag{203}$$

and the convergence in distribution follows in the same way as at the end of the proof of Donsker's invariance principle 6.11, i.e. we have

$$W|_{[0,\tilde{T}_\beta]} \xrightarrow{d} 2B|_{[0,\tilde{T}_\beta]} \tag{204}$$

as $\rho \to \infty$. To extend this to $[0, T]$ we will show that as $\rho \to \infty$, it actually holds true that $T = \tilde{T}_\beta$. To see this, note first that $T \neq \tilde{T}_\beta$ means that $W$ exceeded the bounds $\{-\beta, \beta\}$ before time $T$. So with the notation $A_\beta := \{f \in C([0, \tilde{T}_\beta]) : \sup_{t\in[0,\tilde{T}_\beta]} |f(t)| > \beta\}$ we can rephrase this as

$$P(T \neq \tilde{T}_\beta) = P(W|_{[0,\tilde{T}_\beta]} \in A_\beta) - P(2B|_{[0,\tilde{T}_\beta]} \in A_\beta) \tag{205}$$
$$+ P(2B|_{[0,\tilde{T}_\beta]} \in A_\beta). \tag{206}$$

But for fixed $T$, Brownian motion is unlikely to exit $[-\beta, \beta]$ if $\beta$ is large. Therefore we can make the last term arbitrarily small. Then for this $\beta$, by the convergence in distribution just shown, we can pick $\rho$ large enough to make the difference represented by the first two terms small. Therefore

$$W|_{[0,T]} \xrightarrow{d} 2B|_{[0,T]} \tag{207}$$

and the proof is complete.

■

## 7.3 Closeness in a topology

We have seen that $W(t)$ can be related to $\tilde{W}(t)$ on arbitrarily large restrictions to $[0, T]$ via convergence in law. The next step is to look at the actual realizations of the curves and not only their driving processes. For a small change in the parameter in Loewner's equation, we need a way to control the change in the resulting curve. Another question is what this does to the nature of convergence and in what topology the convergence holds. The remaining parts of [11] are dedicated to this question and provides a sequence of successive improvements in both quality and topology of convergence.

## 7.4 Martingale observables as a general proof strategy

This last chapter is based on the paper [11], where the convergence of HE to SLE(4) is established. That paper makes many references to [9], where the convergence of LERW to SLE(2) is proved. It should therefore come as no surprise that the proofs share many similarities. The basic idea to derive convergence of the driving processes is to identify a so called martingale observable, which is a collection of martingales, one for each vertex $v$ of the domain. For the Harmonic Explorer we have seen that a natural choice is $h_n(v)$, in fact, it's probably the natural choice, considering that Schramm and Sheffield, inspired by this general proof idea, constructed the discrete process to have precisely such an observable. In the case of Loop Erased Random Walk, one employs instead the expected number of visits of the underlying Random Walk to the vertex $v$. In both of these proofs, the observables are discrete harmonic functions (for fixed time, considering all the vertices $v$). This makes it possible to approximate them by continuous harmonic functions. A lot of work goes into establishing that this is possible and with sufficient accuracy, at sufficiently many future times. Since the evolution of the curve changes the boundary and size of the domain, cutting out a slit from it, one needs to ensure that this doesn't destroy the approximation. But once this has been brought under control, a continuous quantity offers many opportunities for further approximations. In particular, by using Loewner's underlying theory, a Taylor expansion and then taking the conditional expectation, it is possible to arrive at a clean equation for the conditional expectations of $t_{m_{n+1}} - t_{m_n}$, $W(t_{m_{n+1}}) - W(t_{m_n})$ and $(W(t_{m_{n+1}}) - W(t_{m_n}))^2$. By making this kind approximation at two different vertices we get a linear system, which we can solve to get the 'local statement', i.e. that the driving process behaves essentially like we would expect the corresponding scaled Brownian motion to behave, at least on a small timescale. In both cases, there is a somewhat mysterious choice of 'local level' as determined by the sequence $(m_n)$. It is apparently important that the growth in time and the growth in driving process difference is neither too big nor too small. Once this local statement is obtained, the proof of convergence of the restriction relies mainly on comparison with a martingale embedded in the Brownian Motion. Beyond that and as previously mentioned, a large part of the further questions adressed in both papers concern the convergence (in a suitable topology) of not only the driving processes but of the curves themselves.

# 8 Appendix

The proof of approximation of discrete harmonic functions with a continuous counterpart in 7.2 gives no clarity in regards to the rate of convergence. If one wants to establish this rate, it might be possible to use ideas similar to those in [1], where this is established for the case $LERW$ to $SLE(2)$. We record the proof from that paper, with minor adaptations, to see what this could look like. It is important to note however that we use a coupling of planar Brownian Motion and Random Walk on the lattice $\mathbb{Z}^2$ and not on $V(TG)$, the triangular lattice, which we would need if this is to be adapted to the Harmonic Explorer (HE). It is nevertheless an interesting statement in its own right and provides some hints as to what a more quantitative HE to $SLE(4)$ proof could look like.

**Theorem 8.1.** *Let $S_t$ be a random walk in $\mathbb{Z}^2$ and $B_t$ a planar Brownian motion, started at 0 in a domain $D$, with exit times $\tau_D$ and $T_D$. Let $V \subset \partial D$, then for $\epsilon > 0$*

$$P^0(S_{\tau_D} \in V) - P^0(B_{T_D}/\sqrt{2} \in V) = O(R^{-1/4+\epsilon}) \tag{208}$$

*as* $\mathrm{rad}_0(D) \to \infty$.

Note that $V$ is viewed as a set of vertices together with connecting edges. Otherwise the probability for the Brownian motion to exit though $V$ is zero.

We will use a variant of a strong approximation result by Komlos-Major-Tusnady as given in [6].

**Lemma 8.2.** *Let $\sigma_R = \inf\{t \geq 0 : \min\{\sup_{0\leq s\leq t} |S_s|, \sup_{0\leq s\leq t} |B_s|\} \geq R^8\}$. Then there is a $c > 0$ and a coupling of planar Brownian Motion with Random Walk in $\mathbb{Z}^2$ such that*

$$P(\sup_{0\leq t\leq \sigma_R} |S_t - \frac{B_t}{\sqrt{2}}| > c\log R) \to 0 \quad \textit{as } R \to \infty. \tag{209}$$

where $S_t$ is obtained by linear interpolation. Letting $R = \mathrm{rad}_0(D)$ and using the Beurling projection theorem covered in section 2 [1], the approximation valid up to $\sigma_R$ is likely to cover the exit times $T_D = \inf\{t \geq 0 : B_t \in D^C\}$ and $\tau_D = \inf\{t \geq 0 : S_t \in D^C\}$ in the sense that

$$P(\sigma_R < T) = O((\frac{R}{R^8})^{1/2}) \quad \text{for } T = T_D, \tau_D \tag{210}$$

When we use this coupling result in the upcoming proof, we will need to introduce a few more stopping times than what might at first seem reasonable. This is essentially because the strong Markov property holds for $S_t$ and $B_t$ separately, but not for the coupling $X = (S_t, B_t)$. So in some sense, the lemma gives a 'strong approximation' but loses a very desirable property in the process. The added technicalities are also related to the discussion just before the partial proof of 7.2, where it is mentioned that coupling of the processes and closeness of exit sites are related but not entirely similar.

*Proof.* Consider the coupling $(S_t, B_t)$ and let $c_0$ be the $c$ given by 8.2. Introduce the stopping time

$$\eta = \inf\{t \geq 0 : \min\{\mathrm{dist}(S_t, \partial D), \mathrm{dist}(B_t/\sqrt{2}, \partial D)\} \leq 2c_0 \log R\}. \tag{211}$$

which is the first time that one of the processes comes close to the boundary. Define also the event

$$\mathcal{E}_1 = \left\{ \left| S_\eta - B_\eta/\sqrt{2} \right| \leq c_0 \log R \right\} \cap \left\{ \sup_{0 \leq t \leq \sigma_R} |S_t - B_t/\sqrt{2}| \leq c_0 \log R \right\} \tag{212}$$

The above event describes a desirable state from an approximation point of view, and the probability of its complement can be driven low:

$$\begin{aligned} P(\mathcal{E}_1^C) &= P(\mathcal{E}_1^C, \sigma_R > T_D) + P(\mathcal{E}_1^C, \sigma_R \leq T_D) && (213) \\ &\leq P(\mathcal{E}_1^C, \sigma_R > \eta) + O(R^{-\frac{1}{2}(8-1)}) && (214) \\ &= P(\sup_{0 \leq t \leq \sigma_R} |S_t - B_t/\sqrt{2}| > c_0 \log R) + O(R^{-7/2}) && (215) \\ &\to 0 \quad \text{as } R \to \infty && (216) \end{aligned}$$

We further define the stopping times

$$\begin{aligned} \nu_B &= \inf\{t \geq 0 : \mathrm{dist}(B_t/\sqrt{2}, \partial D) \leq 3c_0 \log R\} && (217) \\ \nu_S &= \inf\{n \geq 0 : \mathrm{dist}(S_n, \partial D) \leq 3c_0 \log R\} && (218) \end{aligned}$$

It holds in general that $\eta < \min\{T_D, \tau_D\}$. On the event $\mathcal{E}_1$ we have that $S_t$ and $B_t/\sqrt{2}$ are close at time $\eta$. In particular $\max\{\nu_B, \nu_S\} \leq \eta$ since otherwise one of the processes still hasn't reached a distance of $3c_0 \log R$ from the boundary, while the other is already at $2c_0 \log R$, contradicting the first inequality defining $\mathcal{E}_1$.

We now define two further events $\mathcal{E}_2^B$ and $\mathcal{E}_2^S$, one for each of the two processes $B_t/\sqrt{2}$ and $S_t$. Both events are a subset of $\mathcal{E}_1$, ensuring that we have good control over the distance between the processes. The difference between the events will be given only in terms of one process. This makes it possible to use the Strong Markov Property and a Beurling estimate (see section 2 [1]) to bound the probability of escape from a certain ball before hitting the boundary. By considering the intersection of these events, we will be able to control the difference between the boundary hitting points.

More precisely we define

$$\mathcal{E}_2^B = \mathcal{E}_1 \cap \{T_{\tilde{B}_B} < T_D\} \tag{219}$$

where $\tilde{B}_B = B(B_{\nu_B}/\sqrt{2}, 3c_0 R^\alpha \log R)$ is a ball centered at $B_{\nu_B}/\sqrt{2}$, with a radius that is slightly larger than $\operatorname{dist}(B_{\nu_B}/\sqrt{2}, \partial D)$ for $\alpha \in (0,1)$. We define $\mathcal{E}_2^S$ similarly but in terms of $S_t$. Computing the probability, we get

$$\begin{aligned}
P(\mathcal{E}_2^B \cap \mathcal{E}_2^S) &\geq 1 - P((\mathcal{E}_2^B)^C) - P((\mathcal{E}_2^S)^C) && (220)\\
&\geq 1 - 2P(\mathcal{E}_1^C) - P(T_{\tilde{B}_B} < T_D) - P(T_{\tilde{B}_S} < T_D) && (221)\\
&= 1 - O(R^{-7/2}) - O((\frac{\log R}{R^\alpha \log R})^{1/2}) && (222)\\
&= 1 - O(R^{-\alpha/2}) && (223)
\end{aligned}$$

On $\mathcal{E}_2^B$ we have the control over the difference between the processes afforded by $\mathcal{E}_1$ and that $B_t/\sqrt{2}$ exits through the boundary inside the ball $\tilde{B}_B$. Let $Q_B$ be the component of $\tilde{B}_B \cap D$ containing the point $B_{\nu_B}/\sqrt{2}$ (depending on boundary regularity $\tilde{B}_B \cap D$ may have many components). We have that $B_\eta/\sqrt{2}$ is inside $Q_B$ and by the definition of $\eta$, $B(B_\eta/\sqrt{2}, 3/2c_0 \log R) \subset Q_B$. Since by $\mathcal{E}_1$, $|B_\eta/\sqrt{2} - S_\eta| \leq c_0 \log R$, also $S_\eta \in \tilde{B}_B \cap D$. In effect $B_\eta/\sqrt{2}$, $S_\eta$ and $B_{T_D}$ are in $Q_B$. For $\mathcal{E}_2^S$ we similarly have that $B_\eta/\sqrt{2}, S_\eta, S_{\tau_D} \in Q_S$. On the event $\mathcal{E}_2^B \cap \mathcal{E}_2^S$ it makes sense to look at $Q_B \cap Q_S$ and we have just seen that this set is nonempty, since $B_\eta/\sqrt{2}, S_\eta \in Q_B \cap Q_S$. This gives control over $\operatorname{diam} Q_B \cup Q_S$ allowing us to draw a curve $\beta$ in $D$ beginning and ending in $\partial D$ with the property that it separates $B_{T_D}/\sqrt{2}$ and $S_{\tau_D}$ from zero in $D$ and that $\operatorname{diam} \beta = O(R^\alpha \log R)$. If it were not the case that $Q_B \cap Q_S$ was nonempty, the quantity $\operatorname{diam} Q_B \cup Q_S$ could potentially be quite large, making a bound on $\beta$ impossible.

Let $F \subset \partial D$ be the part of the boundary separated from zero by $\beta$ and note that this set contains the exit points of the two processes. Using a bound on the diameter of the image of $\beta$ as in [1] lemma 2.2, we obtain

$$\begin{aligned}
P^0(B_{T_D} \in F) &\leq P^0(T_\beta < T_D) && (224)\\
&= O((\frac{\operatorname{diam} \beta}{R})^{1/2}) && (225)\\
&= O((\frac{R^\alpha \log R}{R})^{1/2}) && (226)\\
&= O(R^{(\alpha-1)/2}(\log R)^{1/2}) && (227)
\end{aligned}$$

We are now in a position to make the comparison.

$$
\begin{aligned}
P(S_{\tau_D} \in V) &= P(S_{\tau_D} \in V, \mathcal{E}_2^B \cap \mathcal{E}_2^S) + P(S_{\tau_D} \in V, (\mathcal{E}_2^B \cap \mathcal{E}_2^S)^C) && (228)\\
&\leq P(B_{T_D}/\sqrt{2} \in V \cup F) + O(R^{-\alpha/2}) && (229)\\
&\leq P(B_{T_D}/\sqrt{2} \in V) + P(B_{T_D}/\sqrt{2} \in F) + O(R^{-\alpha/2}) && (230)\\
&= P(B_{T_D}/\sqrt{2} \in V) + O(R^{(\alpha-1)/2}(\log R)^{1/2}) + O(R^{-\alpha/2}) && (231)
\end{aligned}
$$

pick $\alpha = 1/2$ to get $(\alpha - 1)/2 = -\alpha/2$. Then

$$
P(S_{\tau_D} \in V) - P(B_{T_D}/\sqrt{2} \in V) = O(R^{-1/4+\epsilon}) \tag{232}
$$

for any $\epsilon > 0$.

■

# References

[1] Christian Beneš, Fredrik Johansson Viklund, and Michael J Kozdron. On the rate of convergence of loop-erased random walk to sle 2. *Communications in Mathematical Physics*, 318:307–354, 2013.

[2] Nathaniel Berestycki and James Norris. Lectures on schramm–loewner evolution. *Lecture notes, available on the webpages of the authors*, 2014.

[3] Rick Durrett. *Probability: theory and examples*, volume 49. Cambridge university press, 2019.

[4] John B Garnett and Donald E Marshall. *Harmonic measure*, volume 2. Cambridge University Press, 2005.

[5] Antti Kemppainen. *Schramm–loewner evolution*. Springer, 2017.

[6] Michael Kozdron and Gregory Lawler. Estimates of random walk exit probabilities and application to loop-erased random walk. 2005.

[7] Steven Lalley, Gregory Lawler, and Hariharan Narayanan. Geometric interpretation of half-plane capacity. 2009.

[8] Gregory F Lawler. Topics in loop measures and the loop-erased walk. 2018.

[9] Gregory F Lawler, Oded Schramm, and Wendelin Werner. Conformal invariance of planar loop-erased random walks and uniform spanning trees. In *Selected Works of Oded Schramm*, pages 931–987. Springer, 2011.

[10] Peter Mörters and Yuval Peres. *Brownian motion*, volume 30. Cambridge University Press, 2010.

[11] Oded Schramm and Scott Sheffield. Harmonic explorer and its convergence to sle4. 2005.

[12] Elias M Stein and Rami Shakarchi. *Complex analysis*, volume 2. Princeton University Press, 2010.