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Exotic Hadrons

Classification of Mass Models and Predictions for Non-Strange Dibaryons

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Abstract

In this report we study theoretical models for calculating mass spectra of exotic hadrons and carry out numerical predictions for selected states. A brief introduction to the Standard Model and other key concepts are presented in order to contextualize and aid the reader. Four mass models are described and classified, applicable to different multiquark systems. Then, predictions for mass spectra of six non-strange dibaryon candidates are performed, using a simple mass formula with parameters fixed from experimentally determined baryon masses. Finally, results are discussed in relation to other existing work on the subject.

Sammanfattning

I den här rapporten studerar vi teoretiska modeller för beräkning av masspektra för exotiska hadroner samt genomför numeriska beräkningar för några särskilt utvalda tillstånd. En kort introduktion till standardmodellen och andra centrala koncept inom partikelfysik ges för att kontextualisera och underlätta för läsaren. Fyra massmodeller beskrivs och klassifieras, vilka är applicerbara på olika multikvarksystem. Sedan görs prediktioner av masspektra för sex potentiella icke-särdibaryoner med hjälp av en enkel massformel, vars parametrar bestäms utifrån anpassning till experimentellt uppmätta värden på baryonmassor. Resultaten diskuteras slutligen i relation till tidigare arbeten inom området.

Contents

1. Introduction

The history of physics is clustered with attempts to break down our complex macroscopic world in the search for the answer to one simple, yet elusive question: *What is it all made of?* Is there one common building block that unites the buzzing bumblebee and the flaring Sun, as well as the Great Barrier Reef and our bleeping phones? When matter is picked apart to infinitesimal size, is there a final unit that we cannot divide any further? The Greek philosophers Democritus and Leucippus were the first to propose such an indivisible object around 500 B.C., which they named *atomos* [1]. At that time, the idea had to compete with the Aristotelian view of everything being made up of four elements – a battle which it lost, whereafter the atomos was quickly suppressed to the sidelines of science. More than two millennia would pass before the *atom* made its return as a valuable concept in physics [2].

Now we know that even the atom is not the smallest building block in our Universe. After the discovery of the electron in 1897, a whole new subatomic world revealed itself, and 20th century physics was quickly flooded with new particles. This led to the introduction of what is called the *The Standard Model of Particle Physics* – as a way of arranging and categorizing the fundamental particles and interactions. Of particular interest for this report are subatomic particles known as *quarks*, which were proposed independently by Gell-Mann [3] and Zweig [4] in 1964, and verified by experiments a few years later.

Quarks are never found isolated in Nature – rather they exist in composite structures which are called *hadrons*. Although observed hadrons normally consist of a quark-antiquark pair or a collection of three quarks, the quark model does not impose any limitation for the number of quarks constituting a hadron. Thus, the existence of so-called *exotic hadrons*, comprised of more than three quarks, have long been theorized: particles having possibly four or five constituent quarks, dubbed tetraquarks and pentaquarks, respectively, have been observed by various particle accelerator experiments in recent years [5]. Other observations might also indicate the existence of hexaquark states [6], being composed of six quarks.

In this report, we begin by presenting some theoretical background in Sec. 2, summarizing the Standard Model and other key concepts in elementary particle physics. Then follows the investigative Sec. 3, which consists of two parts: in Sec. 3.1 we present a selection of established models for calculating the mass spectrum of hadrons and in Secs. 3.2–3.3 we apply one of these models to a particular hexaquark system. Results are presented and discussed in Secs. 3.4–3.5, and finally, Sec. 4 is devoted to conclusions and an overall summary.

2. Background Material

For the following background, most of the theory is found in Griffiths' *Introduction to Elementary Particles* [7] and Amsler's *Nuclear and Particle Physics* [8]. If no other citations are explicitly given, these are the references that have been used.

2.1 The Standard Model of Particle Physics

The theory of fundamental particles and how they interact was originally developed in the 1970s and given the name *the Standard Model*. Even to this day it is one of the most complete theories in all of physics, categorizing all 17 known fundamental particles into three distinct types: spin 1/2 *fermions* that make up all matter, spin 1 *gauge bosons* that mediate interactions and the spin 0 *Higgs boson* that gives particles mass. Commonly illustrated as shown in Fig. 2.1, the Standard Model also accounts for three out of the four fundamental forces of Nature: *electromagnetic interaction, strong interaction* and *weak interaction*. The fourth and final, being *gravity*, is yet to be reconciled with the rest of the theory.

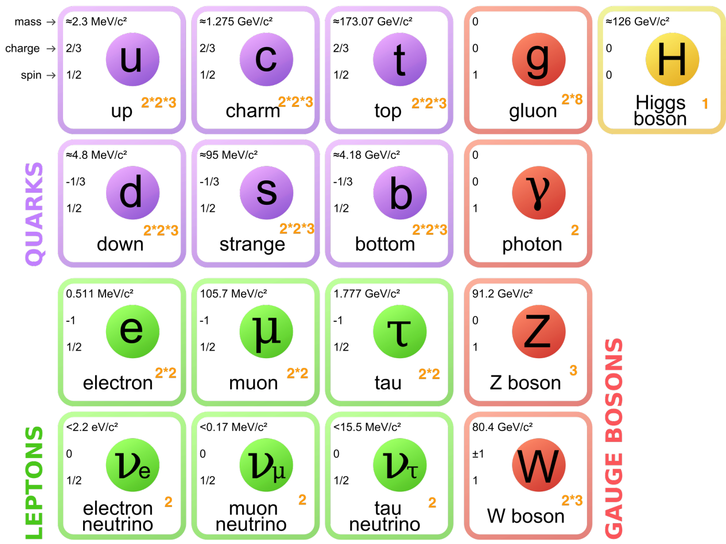


Figure 2.1. The Standard Model of Particle Physics. Image: `pngkey.com`.

2.1.1 Fundamental Particles

Fermions are divided into *quarks* and *leptons*, and while the latter can exist independently, the former are only observed in bound states. There are three different *generations* of quarks, each comprising two *flavors*: (u, d), (c, s) and (t, b). The flavors are *up, down, charm, strange, top* and *bottom*, in natural correspondence with the symbolic letters. Quarks get progressively heavier across the generations, hence u, d, s are often referred to as *light quarks* and c, b, t as *heavy quarks*. In addition to spin, quarks also carry *electric charge* and *color*. In units of the elementary charge (e) each quark carries the fractional value $-1/3$ or $+2/3$, depending on flavor. *Color* or *color charge* is similarly to spin an intrinsic property; the name being a label that has nothing to do with the word as used in everyday language. Quarks can exist in three different color states, or some linear combination thereof, which are accordingly named *red, green* and *blue*.

Leptons also comprise six distinct flavors divided into three generations: (e^-, ν_e), (μ^-, ν_μ) and (τ^-, ν_τ). The *electron, muon*, and *tau* all carry electric charge -1, hence called *charged leptons*, and in analogy to the quarks these get progressively heavier across the generations. For each charged lepton there is also a neutrally charged *neutrino*, ν, subscripted by the appropriate charged lepton. Neutrinos have very small mass and hardly interact with anything, making them difficult to observe.

For every type of particle there exists a corresponding antiparticle, denoted by the same symbol and an additional over-bar, for instance the anti-up quark $\overline{u}$ or the anti-electron neutrino $\overline{\nu_e}$. Antiparticles carry the same mass and spin but opposite charge and color relative to the regular particles.

2.1.2 Interactions

Forces or interactions in the Standard Model arise from *symmetries* and *gauge invariance* and are mediated by so-called gauge bosons. Most familiar is probably the *electromagnetic interaction*, which acts between charged particles and is mediated by *photons*, denoted by γ. Photons carry neither mass, charge nor color, and consequently do not interact with each other. The quantum mechanical theory associated with electromagnetic interactions is called *quantum electrodynamics* (QED) and originated in the 1940s.

The *strong interaction* acts between colored objects and is mediated by vector bosons known as *gluons*, denoted by g. Gluons are massless and electrically neutral but do carry color, thus in addition to mediate interactions between quarks they also interact with each other. This fact makes the quantum theory of the strong interaction, *quantum chromodynamics* (QCD), vastly more complicated. Its present form was developed in the 1970s.

The *weak interaction* involves both quarks and leptons and is mediated by three so-called *intermediary vector bosons*, $W^\pm$ and Z^0. In addition to all three being massive and colorless, the former two carry charge ± 1 while the latter is electrically neutral. The weak force is the only interaction that can change the flavor of particles, for instance being the mechanism responsible for radioactive decay. In contrast to the aforementioned interactions, the quantum theory of the weak interaction is not commonly referred to as *quantum flavordynamics*. Instead it has been successfully unified with the electromagnetic interaction at high energies, in what since the 1970s is known as *electroweak theory* (EWT).

2.1.3 Composite Structures

As stated above, fermions make up all matter in the Universe. Paradoxically, quarks and gluons have never been observed in isolated states, but rather in composite structures known as *hadrons*. The mechanism responsible is called *color confinement* and arises from the fact that the mediators in QCD – the gluons – themselves carry color. This imposes the constraint that naturally occurring hadrons can exist only in colorless states, meaning that the color of its constituent quarks sum up to zero. Zero color is achieved either by combining equal amounts of each kind, that is $(red + green + blue)$, or by combining color and anticolor, for instance $(blue + \overline{blue})$. Three quarks in a colorless state form what is called a *baryon*. Being composed of an odd number of spin $1/2$ particles, it also carries half-integer spin and consequently constitutes a fermion, i.e. a matter particle. Examples of baryons include the proton $(p = uud)$ and the neutron $(n = udd)$. A colorless combination of the type quark-antiquark is called a *meson*, conversely carrying integer spin and hence being a boson. Examples of mesons include the pion $(\pi^+ = u\bar{d})$ and the upsilon meson $(\Upsilon = b\bar{b})$. Hadrons are classified in terms of their valence quarks, being the quarks and antiquarks that give rise to the resulting quantum numbers of the composite structure, in what is called the *quark model*. They are organized by a scheme named the *Eightfold Way* [3,4], based on group theoretical $SU(3)$ symmetries, which will be discussed in Sec. 2.2 below.

2.2 Symmetries, Groups and Conservation Laws

2.2.1 Noether's Theorem

The idea of symmetry is of utmost importance in physics. Not only does it often reduce complicated problems to more manageable ones – symmetries also imply something deep about certain quantities in Nature. This is formulated in *Noether's theorem*, which informally stated says that for every continuous symmetry of a physical system, there exists a corresponding conserved quantity. Conversely, every conserved quantity implies some underlying symmetry. Some examples are: homogeneity of space implies the conservation of linear momentum, isotropy of space implies the conservation of angular momentum, and homogeneity of time implies the conservation of energy [9].

2.2.2 Group Theory and Representation

A symmetry of a system is an operation that when performed on said system leaves it invariant. The set of all symmetry operations $\mathcal{S} = \{S_i\}$ of a particular system must satisfy:

- *Closure* – if $S_i, S_j \in \mathcal{S}$, then the composition $S_k = S_j S_i \in \mathcal{S}$.
- *Identity* – there exists $I \in \mathcal{S}$ such that $IS = SI = S$, for all $S \in \mathcal{S}$.
- *Inverse* – for all $S \in \mathcal{S}$ there exists an inverse S^{-1}, such that $SS^{-1} = S^{-1}S = I$.
- *Associativity* – $S_i(S_j S_k) = (S_i S_j)S_k$, for all $S_i, S_j, S_k \in \mathcal{S}$.

These are precisely the defining properties of a *group*, and hence symmetries are conveniently described by the mathematics of group theory [10]. A useful concept when studying groups is a *representation*, defined as a mapping of group elements onto a set of linear operators. The mapping needs to fulfill two conditions, namely: (i) the identity element is mapped onto the identity operator, and (ii) the group multiplication law is mapped onto the natural multiplication in the space on which the linear operators act. For instance, the cyclic group of order 3, $\mathbf{Z}_3 = \{I, S, S^2\}$ with $S^2 = S^{-1}$, can be represented by a set of three 3×3 matrices. A representation is *reducible* if it has an invariant subspace, meaning that elements of a subspace remain in that same subspace after being acted on by the representation, otherwise it is *irreducible*. Furthermore, it is said to be *completely reducible* if it can be decomposed into a direct sum of irreducible representations. In the case of matrices this corresponds to a block-diagonal form [10]. The type of groups that will be of most relevance when studying symmetries are so-called *Lie groups*, and in particular the *special unitary group* $SU(N)$. It can be represented by the set of unitary $N \times N$ matrices with determinant $\mathbf{1}$. These $N^2 - 1$ elements are then called the *generators* of the group. In particle physics, the most prominent are $SU(2)$ and $SU(3)$, the former with regards to spin and the latter to both color and flavor.

The lowest order non-empty representation is $SU(2)$, containing $2^2 - 1 = 3$ generators. These are usually taken to be the *Pauli matrices* σ_i listed in Appendix A.1. It is convenient to use the properties of $SU(2)$ when dealing with spin systems. Consider two fermions that each can be in either one of two spin states, usually labeled $\uparrow$ and $\downarrow$. The product spin state can symbolically be written as $\mathbf{2} \otimes \mathbf{2}$, accounting for the four possible combinations. This is a completely reducible representation, since it can be decomposed into a direct sum

$$\mathbf{2} \otimes \mathbf{2} = \mathbf{3} \oplus \mathbf{1}, \tag{2.1}$$

where the triplet $\mathbf{3}$ contains the three symmetric states $\{\uparrow\uparrow, \downarrow\downarrow, \frac{1}{\sqrt{2}}(\uparrow\downarrow + \downarrow\uparrow)\}$ having total spin one and the singlet $\mathbf{1}$ contains the antisymmetric state $\{\frac{1}{\sqrt{2}}(\uparrow\downarrow - \downarrow\uparrow)\}$ having total spin zero.

Analogously, $SU(3)$ contains $3^2 - 1 = 8$ generators known as the *Gell-Mann matrices* λ_i, see Appendix A.2. As mentioned in Sec. 2.1.1, quarks can exist in three possible color states and the properties of color are thus conveniently described by the properties of $SU(3)$. Additionally, the light quarks u, d, s also form an approximate symmetry of flavor, known as flavor $SU(3)$ and sometimes denoted $SU(3)_f$. As in the case of $SU(2)$ we can label states symbolically with $\mathbf{3}$ representing quarks and $\overline{\mathbf{3}}$ representing antiquarks. In Sec. 2.1.3, it was stated that naturally occurring hadrons are always found to be color neutral. This turns out to be emergent from an underlying law, saying that a hadron needs to be a color singlet under $SU(3)$ symmetry – the decomposition of a product color state must contain an irreducible representation having dimension one. This allows for the combination quark-antiquark to exist since

$$\mathbf{3} \otimes \overline{\mathbf{3}} = \mathbf{8} \oplus \mathbf{1} \tag{2.2}$$

and indeed the meson is observed in Nature. Since these are invariant subspaces, the eight states in the octet are expected to be able to change into each other under transformations, but not change into the singlet. The combination of two quarks is not allowed as $\mathbf{3} \otimes \mathbf{3} = \mathbf{6} \oplus \overline{\mathbf{3}}$ does not decompose into a singlet, however adding a third results in the stable baryon

$$\mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3} = \mathbf{10} \oplus \mathbf{8} \oplus \mathbf{8} \oplus \mathbf{1}. \tag{2.3}$$

Using these types of arguments, Gell-Mann predicted and organized hadrons in a classification scheme known as the *Eightfold Way* [3].

2.2.3 Conservation of Quantum Numbers

To explain why the mass of the neutron was so similar to the mass of the proton, Heisenberg proposed that the two should be regarded as different states of a single particle, called the *nucleon*, N [11]. In analogy to spin being thought of as intrinsic angular momentum, this introduced the vector quantity $\vec{I}$, called *isospin*, with components I_1, I_2, I_3 in an abstract *isospin space*. The nucleon is said to carry isospin $1/2$ with the component I_3 having eigenvalues $+1/2$ for the proton and $-1/2$ for the neutron, implying that a half rotation in isospin space converts protons to neutrons and vice versa. Heisenberg's claim was that the strong interaction were invariant under such rotations due to an *internal symmetry*. From Noether's theorem it then follows that isospin is conserved under strong interaction. In the language of Sec. 2.2.2: the strong interactions are invariant under an internal $SU(2)$ symmetry group, with the nucleon belonging to the $\mathbf{2}$ representation. When the quark model was later introduced, quarks were assigned isospin $+1/2$ for u, $-1/2$ for d and 0 for all other flavors.

Another quantity that is conserved under strong interactions is *hypercharge* Y, often defined as

$$Y = B + S, \tag{2.4}$$

where $B = \frac{1}{3}(n_q - n_{\overline{q}})$ is called *baryon number* and counts the difference between the number of quarks and antiquarks, whereas $S = (n_{\overline{s}} - n_s)$ is called *strangeness* and serves a similar purpose for s quarks specifically. In general, Eq. (2.4) can also include terms C, B' and T' for *charm, bottomness* and *topness*, respectively, analogous to S. For light quarks, hypercharge is related to isospin by the Gell-Mann–Nishijima formula

$$Q = I_3 + \frac{1}{2}Y, \tag{2.5}$$

where Q is the electric charge in units of e.

The operation of inverting all spatial coordinates is called a *parity transformation*, and the associated *parity operator* is denoted P. Since two consecutive inversions returns the original configuration it is evident that $P^2 = I$, hence the eigenvalues of P are ± 1. In contrast to many other quantum numbers, parity is multiplicative, meaning that the parity of a composite system is the product of the parities of its constituents. The electromagnetic and the strong interaction are invariant under parity, while the weak interaction is not.

There is also an operation called *charge conjugation* which effectively converts particles to antiparticles, and vice versa, by changing the sign of all internal quantum numbers. The charge conjugation operator C has eigenvalues ± 1, with eigenstates being for instance the photon, which is its own antiparticle. Just as for parity, charge conjugation is invariant under the electromagnetic and the strong interaction, but not under weak interactions.

Various quantum numbers are used to classify hadrons in different ways. Typically seen notations are $I(J^P)$ and J^{PC}, where, in addition to I, P and C defined above, J denotes the *total angular momentum.*

2.3 Exotic Hadrons

Apart from the constraint of having zero net color, the quark model imposes no limitation on the number of quarks that can constitute a hadron. The existence of so-called *exotic hadrons* consisting of more than three quarks was proposed as early as 1964 by Gell-Mann [3], although experimental observations were not made until the beginning of the 21st century [12]. The most commonly discussed exotic hadrons are those composed by four, five or six quarks, called *tetraquarks*, *pentaquarks* and *hexaquarks*, respectively.

2.3.1 Tetraquarks

The first claimed observation of an exotic state composed of four constituent quarks was made by the Belle experiment in 2003 [13]. Showing a resonance peak at (3872 ± 0.6) MeV, the discovery was named $X(3872)$ – the X signifying uncertainty about the nature of the state. In contrast to many other claimed observations of exotic hadrons at the time, usually seen only in one experiment or in one decay mode, the $X(3872)$ was shortly after the discovery by Belle confirmed by the BaBar, CDF and D0 experiments [5,14–16]. Once the LHC came online, experiments such as ATLAS, LHCb and CMS were able to gather an enormous amount of data by studying the production rate of $X(3872)$ in $p\bar{p}$ collisions. Results include more precise measurements of its mass, (3871.69 ± 017) MeV, and definite determination of its quantum numbers, $J^{PC} = 1^{++}$ [5]. Despite being the most studied exotic hadron, the true nature of the $X(3872)$ is yet to be discovered. Initially it was hypothesized, based on its mass and quantum numbers, to be the excited charmonium state $\chi_{c1}(2P)$, hitherto unobserved. However, extensive studies of the decay mode $X(3872) \to J/\psi\pi^+\pi^-$, indicate violation of isospin symmetry which is not to be expected from a conventional charmonium meson. Considering the $X(3872)$ as an exotic state would then suggest it being composed of two quarks and two antiquarks ($c\bar{c}u\bar{u}$), produced in B^+ decay, although the way in which its constituent parts are bound together is not definitively determined [5].

2.3.2 Pentaquarks

For hadrons containing five quarks, the name pentaquarks was suggested in 1987 by Gignoux *et al.* [17], as well as Lipkin [18]. Although experimental evidence eluded researchers for a long time, the LHCb experiment in 2015 made a convincing discovery of two pentaquark states, when studying the decay $\Lambda_b^0 \to J/\psi K^- p$. The two states, denoted $P_c(4380)^+$ and $P_c(4450)^+$, respectively, were found to have constituent quark content $uudc\bar{c}$ [19]. Additional pentaquark states were discovered in 2019 when a more extensive analysis by the LHCb-collaboration, showed that the $P_c(4450)$ was actually consisting of two overlapping resonance peaks, named $P_c(4440)^+$ and $P_c(4457)^+$ [20].

The exploration of pentaquark states has been hampered somewhat in the last years, since both the colliders Tevatron and LHC produce Λ_b^0 baryons, but only the LHCb-detector is efficient enough to separate final states of $J/\psi K^- p$ from the more copious background modes [5]. A new method of creating pentaquarks, called *photoproduction*, is therefore being tested today. Photons and J/ψ mesons have identical quantum numbers, so the idea behind photoproduction is that a photon beam allowed to collide with target protons may create pentaquark states. Currently, experiments using this method are carried out at the Jefferson Laboratory CEBAF with the hope that this way of producing pentaquarks will unlock a whole new branch of pentaquark studies [5].

2.3.3 Hexaquarks

The proposition of six quarks combined into a single structure, although not necessarily called hexaquark states, was initially proposed by Dyson and Xuong [23] in 1964, as so-called dibaryon states. They predicted non-strange S-wave dibaryon candidates D_{IJ}, with I and J denoting isospin and spin, respectively. The proposal was that the deuteron, $D_{01}(NN)$, and a virtual state, $D_{10}(NN)$, were contained in the $\overline{\mathbf{10}}$ and $\mathbf{27}$ representations of $SU(6)$. They also predicted four additional states $D_{03}(\Delta\Delta), D_{30}(\Delta\Delta), D_{12}(N\Delta)$ and $D_{21}(N\Delta)$, based on group theoretical symmetry arguments.

Another hexaquark state was proposed by Jaffe in 1976 [21]. It was assumed to have the quark content *uuddss* and given the name *dihyperon*, or the H-particle. The term hyperon is sometimes used for baryons containing one or more strange quarks. The stability of the H-particle and many other dibaryon candidates was later studied further in several works by Silvestre-Brac and Leandri, see Ref. [22] and references therein. For the pure hexaquark sector, simply meaning six quarks and no antiquarks, a large systematic study of dibaryon candidates was carried out by means of a schematic chromomagnetic model (see Sec. 3.1.2). Results pointed to dibaryon candidates most favourable for stability being composed of three different pairs of identical quarks and these states in particular are studied in Ref. [22]. Despite the possibility of such resonances not being excluded, the investigations entail that stable dibaryon states are not favored – there always exists a two-baryon channel with lower energy.

However, the search for hexaquarks sparked again in 2014, when the WASA-at-COSY collaboration revealed observations of a narrow resonance-like structure around 2380 MeV, from studying the double-pionic fusion channels $pn \to d\pi^0\pi^0$ and $pn \to d\pi^+\pi^-$. The resonance was named $d^*(2380)$ and data suggested it having quantum numbers $I(J^P) = 0(3^+)$ [6]. A serious candidate for the $d^*(2380)$ has later been thought to be the D_{03} state, proposed fifty years earlier, having the proper quantum numbers and a predicted mass of about 2350 MeV. More recently, Huang *et al.* [24] investigate the six dibaryon candidates proposed by Dyson and Xuong, by use of dynamical calculations as well as a simple mass formula previoulsy applied to baryons and pentaquarks. In particular they investigate the possibility of the $D_{21}(N\Delta)$ as a bound state dibaryon candidate and conclude that models that obtain the experimental $d^*(2380)$ as D_{03} are not compatible with also obtaining D_{21} as a bound state.

3. Investigation

The investigative aspect of this report is mainly divided into two parts. Firstly, in Sec. 3.1 we present an overview of some existing models for calculating the mass spectrum of exotic hadrons, used in modern day particle physics. Then, in Secs. 3.2–3.4, we use an extended version of the Gürsey–Radicati mass formula to calculate the mass spectrum of six non-strange dibaryon candidates. The approach is inspired by previous works on pentaquarks [25, 26] and in particular the study of dibaryon candidates by Huang *et al.* [24]. The results are thereafter discussed and compared to other studies and experiments in Sec. 3.5.

3.1 Models

3.1.1 Two-Body Schrödinger Equation

This model treats a composite structure as a two-body system in a non-relativistic regime and the mass spectrum is obtained by numerically solving the non-relativistic Schrödinger equation for said system. The initial derivation is performed for a quark-antiquark pair and then extrapolated to a tetraquark system, modeled by a diquark-antidiquark pair, following Refs. [27, 28].

Consider a quark-antiquark system with Hamiltonian $\mathcal{H}$, having two potential terms: an unperturbed potential due to one-gluon exchange (OGE) and a perturbation accounting for spin-spin interaction. The unperturbed contribution is usually modeled by the so-called Cornell potential

$$V_{\mathrm{C}}(r) = \kappa\frac{\alpha_s}{r} + br, \tag{3.1}$$

where r is the radial coordinate and the parameters κ, α_s and b are known as the *color factor*, the *fine structure constant of QCD* and the *string tension*, respectively. The first term is of Coulomb type and arises from OGEs between constituent quarks, while the linear term is associated with quark confinement. The perturbation can be taken as

$$V_{\mathrm{S}}(r) = -\frac{2}{3(2\mu)^2}\nabla^2 V_C(r)\boldsymbol{S_1}\cdot\boldsymbol{S_2} = -\frac{2\pi\kappa\alpha_s}{3\mu^2}\delta^3(r)\boldsymbol{S_1}\cdot\boldsymbol{S_2}, \tag{3.2}$$

where $\mu = m_1m_2/(m_1+m_2)$ is the reduced mass of the quark pair, $\mathbf{S}_i$ are spin operators and $\delta^3(r)$ is the Dirac delta function [29]. The delta function needs to be regularized in order to perform numerical calculations, and following Ref. [30] it can be replaced by a smeared Gaussian with a parameter β, resulting in

$$V_{\mathrm{S}}(r) = -\frac{2\pi\kappa\alpha_s}{3\mu^2}\left(\frac{\beta}{\sqrt{\pi}}\right)^3 e^{-\beta^2r^2}\boldsymbol{S_1}\cdot\boldsymbol{S_2}. \tag{3.3}$$

Now the time independent radial Schrödinger equation, $\mathcal{H}\psi(r) = E\psi(r)$, may be formulated in the center-of-mass frame as

$$\left\{ -\frac{1}{2\mu}\left[\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} - \frac{L(L+1)}{r^2}\right] + V_C(r) + V_S(r) \right\}\psi(r) = E\psi(r), \tag{3.4}$$

where $\psi(r)$ is the radial wave function, L is the orbital quantum number and E is the energy eigenvalue. The substitution $\psi(r) \equiv \varphi(r)/r$ transforms Eq. (3.4) into

$$\left[-\frac{d^2}{dr^2} + V_{\text{eff}}(r)\right]\varphi(r) = 2\mu E\varphi(r), \tag{3.5}$$

where the effective potential V_{eff} is given by

$$V_{\text{eff}}(r) = 2\mu\big[V_C(r) + V_S(r)\big] + \frac{L(L+1)}{r^2}. \tag{3.6}$$

Equation (3.5) must be solved numerically for the energy eigenvalue E and the reduced wave function $\varphi(r)$, after which the mass of the system can be determined as $M = m_1 + m_2 + E$.

For a tetraquark system, modeled as a diquark-antidiquark pair, the color structure can be decomposed as

$$\mathbf{3} \otimes \mathbf{3} \otimes \overline{\mathbf{3}} \otimes \overline{\mathbf{3}} = \mathbf{27} \oplus \mathbf{10} \oplus \overline{\mathbf{10}} \oplus \mathbf{8} \oplus \mathbf{1} \tag{3.7}$$

and since the singlet is contained in the decomposition, this state is permitted. It can be shown that the color factor κ of this system is half that of the quark-antiquark system, hence the impact of extrapolating the model is the introduction of a factor 1/2 in the Coulomb part of the potential (3.1). An approach that is often adopted is to consider this a global factor, effectively halving also the string tension b [27]. Hence, the procedure for obtaining the tetraquark mass spectrum is the same as for mesons, except introducing a factor 1/2 in the potential and using the appropriate diquark-antidiquark masses.

3.1.2 Chromomagnetic Interaction Model

To study the mass spectrum of multiquark systems, the chromomagnetic interaction (CMI) model has been widely used by several authors [31–33]. The model describes the mass of a ground state hadron as the sum of the effective constituent quark masses and CMI terms. Despite being a fairly simple approach, the CMI model has been shown to successfully describe the basic features of the hadron mass spectrum, since the mass splittings among hadrons reflect the symmetries of their inner structure. For a hadron in the ground state, the mass can be described by the effective Hamiltonian

$$H = \sum_i m_i + H_{\text{CMI}}, \tag{3.8}$$

where m_i is the effective mass of the ith constituent quark, and the Hamiltonian for the chromomagnetic interaction reads

$$H_{\text{CMI}} = -\sum_{i<j} C_{ij}\lambda_i \cdot \lambda_j \sigma_i \cdot \sigma_j. \tag{3.9}$$

Here, λ_i are the $SU(3)$ Gell-Mann flavor matrices and σ_i the $SU(2)$ Pauli spin matrices, and C_{ij} the coupling constant between two constituent quarks. It has however been found that this does not sufficiently take into account the attractive interactions between constituent quarks, generally resulting in an overestimation for the hadron masses, see Ref. [33] and references therein. Thus, the first term in Eq. (3.8) is replaced according to

$$\sum_i m_i \quad \rightarrow \quad M_{\text{ref}} - \langle H_{\text{CMI}} \rangle, \tag{3.10}$$

so that the mass of the ground state hadrons is instead given by

$$M = M_{\text{ref}} - \langle H_{\text{CMI}} \rangle_{\text{ref}} + \langle H_{\text{CMI}} \rangle, \tag{3.11}$$

where M_{ref} denotes the physical mass of the reference system, $\langle H_{\text{CMI}} \rangle_{\text{ref}}$ the CMI eigenvalue of the reference system and $\langle H_{\text{CMI}} \rangle$ the CMI eigenvalue for the considered multiquark system.

3.1.3 QCD Sum Rules

Another popular way of calculating the mass spectrum of hadrons stems from QCD, mentioned in Sec. 2.1.2. The model is called *QCD sum rules* and was first suggested in 1978 by Shifman, Vainshtein and Zakharov [34], therefore sometimes also referred to as SVZ sum rules [35]. QCD being a field theory, model-dependent constituent quarks are replaced by interpolating *quark currents*, which are operators constructed from quark and gluon fields. From these can be obtained *correlation functions* of the form

$$\Pi_{\mu\nu}(q) = i \int d^4x e^{iq\cdot x} \langle 0 | \mathcal{T}\{j_\mu(x) j_\nu(0)\} | 0 \rangle \tag{3.12}$$

and such are used to derive necessary characteristics of hadronic constellations. Here, q represents the complex four-momentum of the virtual photon and $j_\mu(x)$ the colorless quark current. Eq. (3.12) also contains the so-called time-ordered product, or T-product, defined by

$$\mathcal{T}\{j_\mu(x) j_\nu(0)\} \equiv \begin{cases} j_\mu(x) j_\nu(0) & \text{if } \tau_x > \tau_0 \\ -j_\nu(0) j_\mu(x) & \text{if } \tau_x < \tau_0 \end{cases}, \tag{3.13}$$

which decides the order of operators and is variable in four dimensions.

One method for obtaining a mass formula for hadrons from the correlation function is carried out in Ref. [36]. There, the authors isolate and obtain the ground state contribution to the correlation function as

$$\Pi_{\mu\nu}(q) = \frac{\lambda_Z^2}{M_Z^2 - q^2} + \ldots, \tag{3.14}$$

where M_Z is the sought-after mass and λ_Z is the pole residue. Then, using what is called the QCD spectral density, $\rho_{\text{QCD}}(q)$, and the *Borel transform* with respect to $Q^2 \equiv -q^2$ (see Appendix A.4), the sum rule becomes

$$\lambda_Z^2 \exp\Big(-\frac{M_Z^2}{\mathcal{T}^2}\Big) = \int_{\Delta^2}^{s_0} ds \rho_{\text{QCD}}(s) \exp\Big(\frac{-s}{\mathcal{T}^2}\Big), \tag{3.15}$$

where, the upper bound s_0 denotes the continuum threshold for the spectral density and the lower bound Δ^2 is a quantity that varies for different hadrons. As an example, for hexaquarks $\Delta^2 = 9m_c^2$, with m_c denoting the interpolating current mass [36]. Introducing the new variable $t = \frac{1}{T^2}$, Eq. (3.15) may be differentiated with respect to t, which after rearrangement of terms gives the (squared) mass

$$M_Z^2 = \frac{-\frac{d}{dt}\int_{\Delta^2}^{s_0} ds\rho_{\text{QCD}}(s)e^{-ts}}{\int_{\Delta^2}^{s_0} ds\rho_{\text{QCD}}(s)e^{-ts}}. \tag{3.16}$$

A variety of other works and studies involving the QCD-sum rules, applied to different multiquark systems, can be found in Refs. [37–41].

3.1.4 The Gürsey–Radicati Mass Formula

Already in 1964, Gürsey and Radicati proposed a simple mass formula for baryons, based on the breaking of $SU(6) \supset SU(2) \times SU(3)_f$ spin-flavor symmetry [42]. As pointed out in Ref. [43], the Gürsey–Radicati (GR) formula describes quite well the way in which symmetry breaking effects the mass spectrum of baryons, despite its simplicity. In the original work, the mass formula is given by (up to differences in notation)

$$M = M_0 + AJ(J+1) + BY + C\Big[I(I+1) - \frac{1}{4}Y^2\Big], \tag{3.17}$$

where J, Y and I denote spin, hypercharge and isospin, respectively, and M_0, A, B and C are model parameters. These parameters are obtained from fits to baryon data. An extension to Eq. (3.17) is presented and studied in Ref. [25], and further in Ref. [26], where applications to pentaquark systems are considered. The extended version reads

$$M = M_0 + AJ(J+1) + BY + C\Big[I(I+1) - \frac{1}{4}Y^2\Big] + D\mathcal{C}_2 + EN_c, \tag{3.18}$$

where $\mathcal{C}_2$ is the eigenvalue of the $SU(3)_f$ Casimir operator (see Appendix A.3), N_c is the number of constituent charm quarks in the hadron, and D and E are two additional model parameters. The parameters are fixed using the baryon spectrum and pentaquark masses are then calculated under the assumption that these parameter values are universally valid beyond baryons. Furthermore, the extended GR formula (3.18) is used in the context of hexaquarks by Ref. [24], when studying the possible dibaryon candidates D_{IJ} suggested by Dyson and Xuong. For these dibaryon states, a simple formula for the eigenvalue of the Casimir operator is presented in Ref. [23] as

$$\mathcal{C}_2 = 12 + 2I(I+1). \tag{3.19}$$

3.2 Method

Now that the relevant background information and models have been presented, we move on to the second part of the report. As mentioned in the beginning of this chapter, we use the extended Gürsey–Radicati mass formula (3.18) to calculate the mass spectrum of dibaryons previously predicted by Dyson and Xuong [23], following the approach of Huang *et al.* [24].

Baryon	J	Y	I	$\mathcal{C}_2$	N_c	$M^{\text{Exp.}}$ [MeV]	$\sigma^{\text{Exp.}}$ [MeV]
N	$\frac{1}{2}$	1	$\frac{1}{2}$	3	0	939.565413	$\pm 10^{-6}$
Λ^0	$\frac{1}{2}$	0	0	3	0	1115.683	± 0.006
Σ^0	$\frac{1}{2}$	0	1	3	0	1192.642	± 0.024
Ξ^0	$\frac{1}{2}$	-1	$\frac{1}{2}$	3	0	1314.86	± 0.2
Δ^0	$\frac{3}{2}$	1	$\frac{3}{2}$	6	0	1232	± 2
Σ^{*0}	$\frac{3}{2}$	0	1	6	0	1383.7	± 1.0
Ξ^{*0}	$\frac{3}{2}$	-1	$\frac{3}{2}$	6	0	1531.80	± 0.32
Ω^-	$\frac{3}{2}$	-2	0	6	0	1672.45	± 0.29
Λ_c^+	$\frac{1}{2}$	$\frac{2}{3}$	0	$\frac{4}{3}$	1	2286.46	± 0.14
Ξ_c^0	$\frac{1}{2}$	$-\frac{1}{3}$	$\frac{1}{2}$	$\frac{4}{3}$	1	2470.85	${}^{+0.28}_{-0.31}$
Σ_c^0	$\frac{1}{2}$	$\frac{2}{3}$	1	$\frac{10}{3}$	1	2453.75	± 0.14
$\Xi_c^{0\prime}$	$\frac{1}{2}$	$-\frac{1}{3}$	$\frac{1}{2}$	$\frac{10}{3}$	1	2577.9	± 2.9
Ω_c^0	$\frac{1}{2}$	$-\frac{4}{3}$	0	$\frac{10}{3}$	1	2695.2	± 1.7
Ω_c^{*0}	$\frac{3}{2}$	$-\frac{4}{3}$	0	$\frac{10}{3}$	1	2765.9	± 2.0
Σ_c^{*0}	$\frac{3}{2}$	$\frac{2}{3}$	1	$\frac{10}{3}$	1	2518.5	± 0.2
Ξ_c^{*0}	$\frac{3}{2}$	$-\frac{1}{3}$	$\frac{1}{2}$	$\frac{10}{3}$	1	2649.9	± 0.5

TABLE I. Baryon quantum numbers and experimental masses, retrieved from PDG [44].

The six model parameters are fitted to experimental data on hadron masses, retrieved from Particle Data Group (PDG) [44]. In Ref. [24], the data set used consists of eight non-charmed baryons and six non-strange dibaryons, where of the latter only two have experimentally verified masses. To distinguish our study from theirs, we additionally include eight charmed baryons and furthermore we only use the two verified dibaryon candidates NN and $d^*(2380)$, for D_{01} and D_{03}, respectively. The baryons along with corresponding quantum numbers and masses are found in Tab. I and likewise the dibaryon candidates are found in Tab. II. These data are divided into three sets, for each of which separate fits and calculations will be carried out. The division is as follows:

Set I: Eight non-charmed baryons.
Set II: Eight non-charmed and eight charmed baryons.
Set III: Eight non-charmed, eight charmed baryons and two dibaryon candidates.

The parameters for each data set are then determined by minimizing the χ^2-function

$$\chi^2 = \sum_i \left(\frac{M_i^{\text{Exp.}} - M_i^{\text{GR}}}{\sigma_i^{\text{Exp.}}} \right)^2, \tag{3.20}$$

where $M_i^{\text{Exp.}}$ are the tabulated experimental masses, $\sigma_i^{\text{Exp.}}$ the corresponding experimental uncertainties, and M_i^{GR} are input masses for data sets **I**, **II** and **III**, respectively.

Dibaryon	J	Y	I	$\mathcal{C}_2$	N_c	M^{Exp} [MeV]	$\sigma^{\text{Exp.}}$ [MeV]
$D_{01}(NN)$	1	2	0	12	0	1876.612928	$\pm 10^{-6}$
$D_{10}(NN)$	0	2	1	16	0	–	–
$D_{03}(\Delta\Delta)$	3	2	0	12	0	2380	± 10
$D_{30}(\Delta\Delta)$	0	2	3	36	0	–	–
$D_{12}(N\Delta)$	2	2	1	16	0	–	–
$D_{21}(N\Delta)$	1	2	2	24	0	–	–

TABLE II. Dibaryon quantum numbers as used in Ref. [24]. Masses for NN and $d^*(2380)$ are adopted from Ref. [44] and Ref. [6], respectively.

The minimization is performed using a numerical method called *basin-hopping*, which is an iterative technique for globally optimizing a scalar function of one or several variables. It is a two-phase method consisting of a stochastic global stepping algorithm and local minimization at each step [45]. For the minimization, several numerical methods can be used, but in this work the *BFGS*-algorithm is chosen. Both of these algorithms exist as built-in functions in the SciPy library and additional code is written by one of the authors (`cbeiming@kth.se`).

3.3 Numerical Analysis

After having divided the data into three sets, we fit the parameters of the Gürsey–Radicati formula in two ways for each set. Firstly, we use the experimental uncertainty of the hadron masses – these fits will be called *unprimed* and denoted **I**, **II** and **III**. Secondly, we do what is called a 1 % error fit, i.e. setting each uncertainty σ_i to one percent of the corresponding experimentally determined mass. This can sometimes yield a better fit, though the precision of the results cannot be better than just 1 %. These fits will be called *primed* and denoted **I′**, **II′** and **III′**.

Resulting parameter values for each case are found in Tab. III, along with the value of the minimized χ^2-function (3.20). We then use the GR formula (3.18) with these parameters, along with quantum numbers from Tab. II, to predict the mass spectrum of the dibaryon candidates. Results are presented in Tab. IV and Fig. 3.1. The process is repeated for the baryons in Tab. I, hoping to reproduce the known spectrum, in order to examine the validity of the model. Results are presented in Tab. V.

3.4 Results

The predictions for the dibaryon mass spectrum are presented in Tab. IV and in Fig. 3.1. For fits **III** and **III′**, the experimental masses of D_{01} and D_{03} are used as inputs, hence they are not included as predictions.

Fit	χ^2	M_0 [MeV]	A [MeV]	B [MeV]	C [MeV]	D [MeV]	E [MeV]
I	81590	293.3	−297.1	−195.2	38.04	348.4	–
I′	20.09	280.7	−283.8	−178.6	21.44	351.4	–
II	116400	982.4	17.26	−195.1	38.08	40.12	1378
II′	25.50	966.2	22.23	−175.3	22.53	45.73	1365
III	9362000	1018	169.9	−185.9	15.84	−9.463	1231
III′	51.54	1044	47.06	−187.8	33.49	11.22	1329

TABLE III. Values of the χ^2-function and corresponding parameters obtained for the six different fits. The values are presented with four significant figures.

Dibaryon	**I** [MeV]	**I′** [MeV]	**II** [MeV]	**II′** [MeV]	**III** [MeV]	**III′** [MeV]
$NN(D_{01})$	3745	3832	2052	2153	–	–
$NN(D_{10})$	5809	5848	2254	2336	1530	1926
$\Delta\Delta(D_{03})$	773.7	993.5	2225	2375	–	–
$\Delta\Delta(D_{30})$	13160	13090	3438	3476	1499	2486
$N\Delta(D_{12})$	4026	4145	2358	2469	2549	2209
$N\Delta(D_{21})$	8155	8177	2762	2836	1857	2244

TABLE IV. Predicted mass spectrum of the dibaryon candidates for the six different fits. The values are presented with four significant figures.

3.5 Discussion

Parameters and Fits

We start by examining Tab. III in Sec. 3.3, where the minimized χ^2-values and the parameters for the six different fits are presented. One striking feature is the difference between the values of χ^2 for the primed and the unprimed fits. The unprimed fits **I**, **II** and **III** have $\chi^2 \sim 10^5 - 10^7$, which is very large and indicative of a poor fit, while the primed **I′**, **II′** and **III′** have $\chi^2 \sim 10^1$, suggesting that much better parameter values have been found. This is thought to arise mainly due to N and NN having significantly smaller experimental uncertainties ($\sim 10^{-6}$ MeV) than all other included baryons, making their terms dominate the χ^2-function and hence giving a larger discrepancy for the other data points. Behavior like this is precisely the motivation behind also performing a 1 % error fit in the first place. Concerning the parameter values we note that results for **I** and **I′**, and to lesser extent also **III**, deviate significantly from the rest. For instance the value of M_0 is much lower: 293.3 MeV and 280.7 MeV respectively, compared to all other having $M_0 > 950$ MeV. This is unexpected, since M_0 in a sense should represent the total mass of the constituents. Moreover, the parameters A and D – coefficients for the spin and Casimir eigenvalue terms, respectively – are radically different for the first two fits. There, A is negative: −293.3 MeV for **I** and −283.8 MeV for **I′**, while for the rest A is positive and of smaller magnitude: $\sim$ 20 MeV for **II**, **II′** and 47.02 MeV for **III′**. Fit **III** stands somewhat alone with $A = 169.9$ MeV. The parameter D also differs widely, 348.4 MeV for **I** and 351.4 MeV for **I′**, compared to **II**, **II′** with $(40 - 46)$ MeV

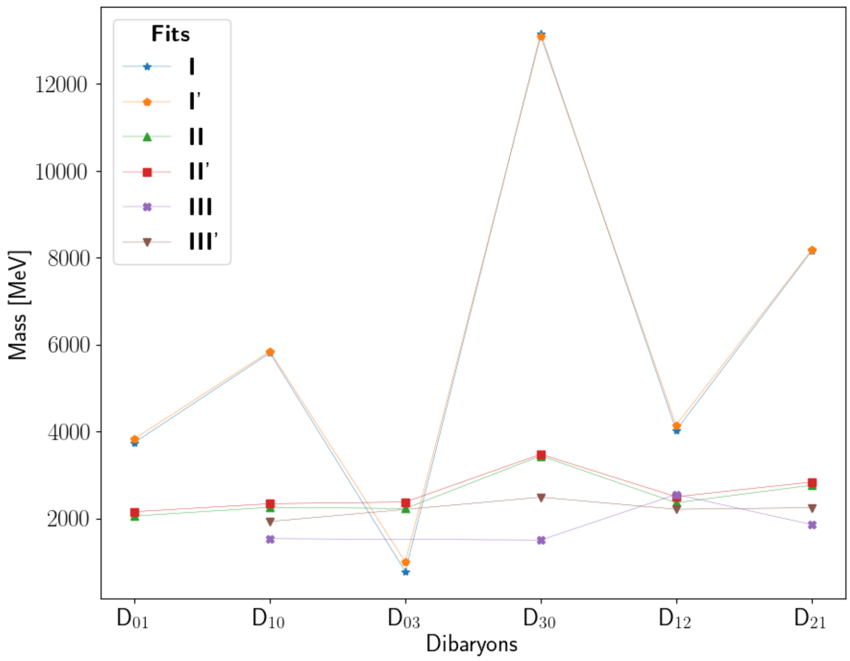


Figure 3.1. Predicted mass spectrum of the dibaryon candidates.

and **III**′ with 11.22 MeV. Again, **III** deviates and has $D = -9.463$ MeV. For reference we note that the three fits showing more collective behavior agree better with Ref. [24] than the other three fits, although their method for obtaining the parameter values is not mentioned.

Dibaryons

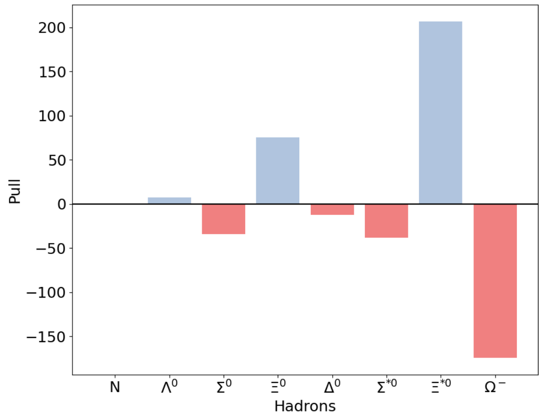
We observe in Fig. 3.1 that the results of **I** and **I**′, using only non-charmed baryons, deviate significantly from the remaining four fits. Since the parameter values obtained for these fits were found to be far off, this is not entirely unexpected. Although most fits follow a general trend for the masses, with D_{30} and D_{21} being the two heaviest, both **I** and **I**′ show a much more extreme spread with practically all predicted values being judged as unrealistic. The remaining fits, however, show a more reasonable behavior, regarding both the primed and the unprimed fits. One explanation for this could be that the heavier charmed baryons contributing with larger weight to the χ^2-function, resulting in more consistent parameter values. This might indicate that including charmed baryons is beneficial, provided that the result from data set **II** reproduces the known dibaryon candidates well. In particular, looking at fit **II**′ we note that the predicted mass of the D_{03} is about 2375 MeV, which is in fact rather close to the experimental value of the $d^*(2380)$, and within our 1 % error margin for the primed fits. However, the prediction for the D_{01}, around 2150 MeV, is too large compared to the experimental value of NN at 1876 MeV. None of the fits were able to describe the deuteron to within even 150 MeV precision.

Baryon	**I** [MeV]	**I′** [MeV]	**II** [MeV]	**II′** [MeV]	**III** [MeV]	**III′** [MeV]
N	939.6	954.1	939.6	956.0	939.7	942.4
Λ^0	1115	1122	1116	1120	1117	1113
Σ^0	1192	1165	1192	1165	1149	1180
Ξ^0	1330	1311	1330	1307	1311	1318
Δ^0	1208	1221	1226	1228	1468	1218
Σ^{*0}	1346	1368	1364	1369	1630	1355
Ξ^{*0}	1598	1578	1616	1578	1840	1593
Ω^-	1622	1661	1640	1652	1955	1630
Λ_c^+	–	–	2293	2290	2239	2295
Σ_c^0	–	–	2449	2426	2252	2384
Ξ_c^0	–	–	2520	2484	2438	2511
$\Xi_c^{0\prime}$	–	–	2600	2575	2419	2533
Ω_c^0	–	–	2751	2724	2586	2682
Ω_c^{*0}	–	–	2802	2791	3096	2823
Σ_c^{*0}	–	–	2501	2492	2761	2525
Ξ_c^{*0}	–	–	2652	2642	2928	2674

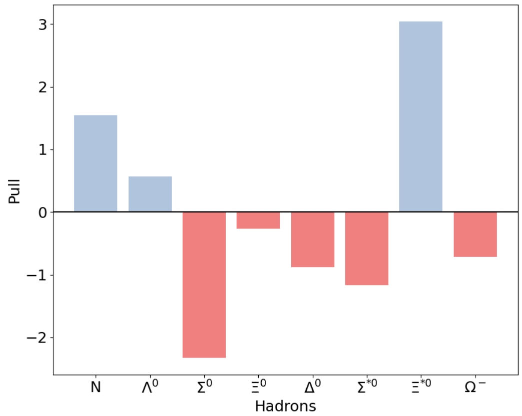
TABLE V. Calculated mass spectrum of baryons for the six different fits. The values are presented with four significant figures.

Model Validity

One way to investigate the validity of the model is to look at how well it reproduces already known results. Since there are only two experimentally determined masses for dibaryon candidates to compare with, we instead look at the well-known baryon spectrum. Tab. V shows the calculated baryon masses for all six fits along with the experimental masses and Figs. 3.2–3.4 show the *pulls*, i.e. the deviation from the experimental values divided by the uncertainty. On a general level, we make one immediate observation. The pull of the unprimed fits are of order $10^2 - 10^3$ for data sets **I**, **II**, and **III**, while the pull for all of the primed fits are of order unity. This agrees with the previously mentioned discrepancies for χ^2, giving an overall better fit, at the expense of losing precision for the baryons that have the smallest uncertainties. Moreover, for all but **III**, the $\Xi^{*0}(1532)$ stands out as the result being farthest from the experimental value, for instance having a pull of $(200 - 250)$ MeV for **I** and **II**. This behaviour of the Ξ^{*0} is also observed in Ref. [26], during a similar analysis concerning pentaquarks. Another noticeable feature, perhaps not unexpected, is that the data sets including charmed baryons and dibaryon candidates, in general yield larger pulls for the non-charmed baryons, e.g. the Σ^0 in **III**. One explanation for this is that the charmed baryons are heavier, due to the charm quark being more massive than the up, down, and strange quarks, and hence shift the spectra towards larger mass. The same logic applies to the dibaryon candidates, naturally being more massive than conventional baryons. One could therefore argue that the inclusion of charmed baryons contributes to making the parameter values more universal.

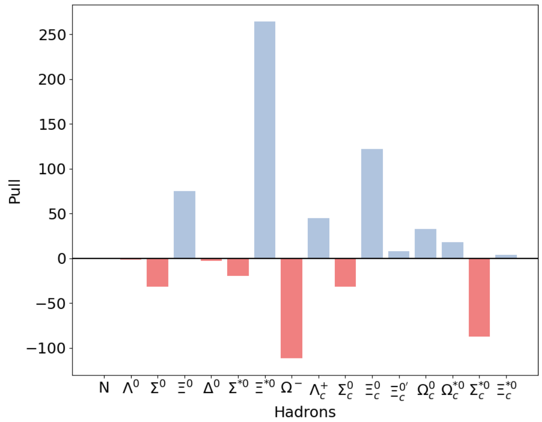


(a) Pulls of Fit **I**.

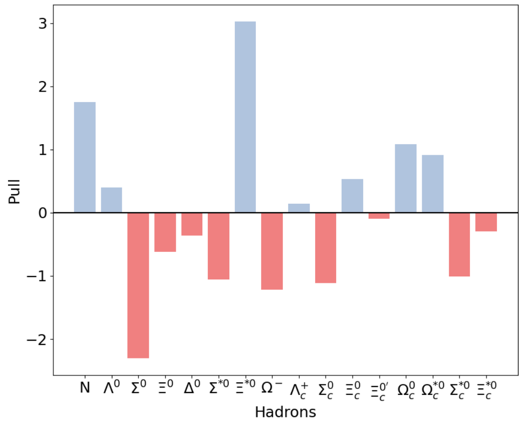


(b) Pulls of Fit **I**′.

Figure 3.2. Pulls for data set **I**.



(a) Pulls of Fit **II**.



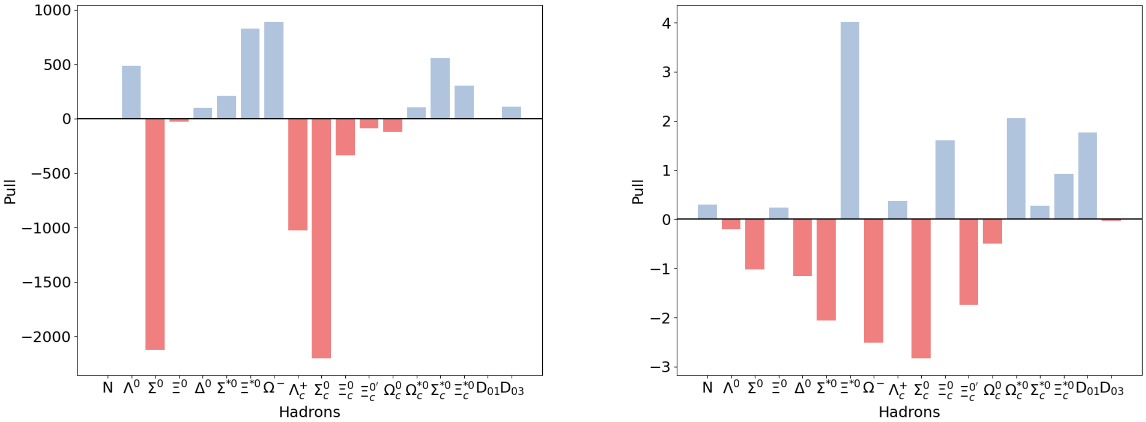
(b) Pulls of Fit **II**′.

Figure 3.3. Pulls for data set **II**.

Although it should be emphasized that the approach of using the GR formula with 1 % error is far from the most precise method conceivable, it is still evident that the basic features of the baryon spectrum is well described by this calculation. The primed fits give pulls in the range ± 5 MeV for all baryons as well as D_{01} and D_{03}, which is of the order one per mille.

Comparison with Other Work

Since the approach of this study is inspired by the work of Huang *et al.* in Ref. [24], it is reasonable to compare our results to theirs. When fixing the parameters of the GR formula, they use experimental masses of the same non-charmed baryons we present in the upper half of Tab. I, as well as the dibaryons listed in Tab. II. In addition to D_{01} and D_{03}, with masses taken to be 1876 MeV and 2380 MeV, respectively, they have also included



(a) Pulls of Fit **III**.

(b) Pulls of Fit **III′**.

Figure 3.4. Pulls for data set **III**.

masses of D_{10}, D_{12} and D_{21}, albeit followed by a question mark (presumably to signal that these are not experimentally verified). Since there is no clear reference to where the values originate, we have chosen to leave them out of our analysis. In any case, the parameter values obtained in their work are (in terms of our notation): $M_0 = 1026.2$ MeV, $A = 56.883$ MeV, $B = -194.70$ MeV, $C = 33.218$ MeV and $D = 9.4085$ MeV. We shall also note that they obtain a separate $M_0 = 2091.9$ MeV for dibaryons, whereas we have simply used $2M_0$ instead. These parameters are in decent agreement with our obtained values for fits **II**–**III′**, in terms of sign and order of magnitude.

Regarding the mass spectra, our results are generally larger than those obtained in Ref. [24]. They obtain 1883 MeV for D_{10}, 2394 MeV for D_{30}, 2168 MeV for D_{12} and 2182 MeV for D_{21}. This is not too surprising since, as we have mentioned, inclusion of heavier charmed baryons in the data sets naturally should shift the mass spectrum somewhat. However, our results do agree to the extent that D_{IJ} is heavier than D_{JI} in all cases, meaning that forming bound states of D_{01}, D_{03} and D_{12} would be preferred. They present a threshold for the D_{21} being a bound state at about 2171 MeV, which is lower than the value they obtain from the GR approach and also lower than what we obtain, except for **III**. This would indicate that no such state is possible. Further, their more extensive dynamical calculations yield concordant results, after which it is concluded that a bound state D_{21} cannot be obtained within these models, while at the same time accounting for the experimentally observed $d^*(2380)$.

4. Summary and Conclusions

The study of exotic hadrons is a relatively new field within physics – especially in comparison to the *atomos* that have been discussed for over two millennia. In this report, we have first reviewed a few of the different models that are used to calculate the mass spectra of multiquark systems. The models are based on different ideas and concept in physics, for instance perturbation theory for the two-body Schrödinger equation in Sec. 3.1.1, chromomagnetic interactions in Sec. 3.1.2, QCD sum rules in Sec. 3.1.3, and internal symmetries for the Gürsey–Radicati mass formula in Sec. 3.1.4. Inspired by other works on the same topic, we have then used the extended Gürsey–Radicati mass formula to predict the mass spectrum for some non-strange dibaryon candidates in Secs. 3.3–3.4. Numerical fits of parameters have been carried out based on three different data sets, using both experimental and 1 % uncertainties for each set, resulting in a total of six fits.

When using only non-charmed baryon masses as input, very large fluctuations in the dibaryon mass spectra have been observed. The other fits have given a more collected prediction, with some results very close to the experimentally observed mass of the hexaquark candidate $d^*(2380)$. In general, the fits that use experimental uncertainties have resulted in very large residual values, $\chi^2 \sim 10^5 - 10^7$, while the 1 % error fits have given $\chi^2 \sim 10^1$. Judging by the calculated the mass spectrum for the dibaryon candidates and comparing with the work of Huang *et al.* [24], results indicate that the existence of these dibaryon states is not compatible with the dibaryon picture for the $d^*(2380)$ within this model.

Research in particle physics is sure to continue in the search for the smallest building blocks of Nature. With exotic hadrons possibly marking new entries in the Standard Model, we are hopefully getting closer to someday answering the elusive question:
What is it all made of?

Acknowledgements

We wish to thank our supervisor Professor Tommy Ohlsson for valuable input during the writing of this report!

A. Appendix

A.1 Pauli Matrices of $SU(2)$

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{A.1}$$

A.2 Gell-Mann Matrices of $SU(3)$

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{A.2}$$

$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \qquad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \tag{A.3}$$

$$\lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad \lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \qquad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}. \tag{A.4}$$

A.3 Casimir Operators

Definition. A *Casimir operator* $\mathcal{C}$ of a group G is an operator that commutes with all other group elements,

$$[\mathcal{C}, g] = 0, \qquad \forall g \in G, \tag{A.5}$$

while not being equal to the identity operator.

For $SU(3)$, two Casimir operators can be constructed according to

$$\mathcal{C}_1(\lambda_i) = \sum_{i=1}^{8} \lambda_i^2, \qquad \mathcal{C}_2(\lambda_i) = \sum_{ijk} d_{ijk}\lambda_i\lambda_j\lambda_k, \tag{A.6}$$

where λ_i are the generators and d_{ijk} are called *structure constants.* These can be found listed, for instance in Ref. [46].

A.4 The Borel Transform

Definition. Let $A(z)$ denote a formal power series

$$A(z) = \sum_{k=0}^{\infty} a_k z^k, \tag{A.7}$$

for a complex number z. Then the *Borel transform* of $A(z)$ is defined to be the equivalent exponential series

$$\mathcal{B}[A(z)](t) = \sum_{k=0}^{\infty} \frac{a_k}{k!} t^k. \tag{A.8}$$

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