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Robust Portfolio Optimization

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Abstract

The objective of robust portfolio optimization is to find a way to allocate capital to some financial assets such that portfolio return is maximized in the worst-case scenario, which is desirable for investors with a low tolerance for risk. This study aims to apply the robust approach to asset allocation based on 30 of the biggest stocks on the Stockholm Stock Exchange. Three models with different constraints on portfolio return and variance are obtained and solved using the Gurobi Optimizer. The result of any one of the models could be proposed as a low-risk portfolio. The choice between the models is a trade-off between higher expected return and lower variance, and it depends on the individual preferences of the investor.

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1 Introduction

One of the main challenges of investment portfolio management is making decisions that balance a high expected payoff and a tolerable risk level. *Portfolio optimization* uses mathematical optimization methods to find the best distribution of wealth among a set of financial assets with respect to some objective reflecting the investor's goals. Modern portfolio theory originates from the 1950s and Harry Markowitz's mean-variance portfolio selection, which allows an investor to either maximize the expected return for a given risk or minimize the risk for a given expected return [9].

Something to be considered when modeling any real-life scenario is parameter uncertainty and how this can affect the solution. In the case of portfolio optimization, measures of future asset behavior such as return and variance are unknown in practice and must be estimated. A drawback of many portfolio optimization models, including Markowitz's mean-variance approach, is that small changes in the parameter estimation can have a significant impact on the solution. One way to deal with this issue is to use optimization methods that directly incorporate the parameter uncertainty into the model. Two such methods are *stochastic optimization* — where the uncertainty is assumed to follow some probabilistic distribution — and *robust optimization* — where the uncertainty is assumed to be deterministic and set-based [15]. This thesis will focus on the latter.

1.1 Problem Formulation

The purpose of this thesis is to present and discuss various robust optimization models addressing the following question.

Suppose that a risk averse asset management company wants to distribute its money into 30 assets on the Swedish stock market with protection against return uncertainties. How should the asset management company distribute its money in order to gain the most return while keeping risk low?

2 Theoretical Background

This section introduces the mathematical portfolio and optimization theory.

2.1 Asset Return

The *annual rate of return* on an investment in a financial asset is given by

$$r = \frac{P_1 - P_0}{P_0}, \tag{1}$$

where P_0 is the current purchase price of the asset and P_1 is the price after a year [1].

2.2 n-asset Portfolio

An n*-asset portfolio* is a collection of investments that distributes some total wealth among n financial assets such as stocks, bonds, and cash [2].

The fraction of the total wealth allocated to each asset i is denoted by x_i, where each $x_i \geq 0$ and $\sum_{i=1}^{n} x_i = 1$ [1].

Each asset has a return r_i. The total return of the portfolio is

$$\sum_{i=1}^{n} x_i r_i = \mathbf{r}^\top \mathbf{x}, \tag{2}$$

where $\mathbf{x} = (x_1, ..., x_n)^\top$ and $\mathbf{r} = (r_1, ..., r_n)^\top$ are vectors representing the asset weights and returns respectively.

2.3 Expected Return, Variance, and Diversification

The future return r_i of an asset i is uncertain. The *expected return* $\mathbb{E}(r_i)$ measures the average of the potential return outcomes, and the *variance* $\mathrm{Var}(r_i)$ measures the spread of the potential return outcomes around the expected return [11]. The *covariance* $\mathrm{Cov}(r_i, r_j)$ between two asset returns r_i and r_j measures how the assets change together. It is positive if they change in the same direction and negative otherwise. Note that $\mathrm{Cov}(r_i, r_i) = \mathrm{Var}(r_i)$.

Consider a portfolio with asset weights $\mathbf{x} = (x_1, ..., x_n)^\top$ and returns $\mathbf{r} = (r_1, ..., r_n)^\top$. The expected portfolio return $\mathbb{E}(\mathbf{r}^\top \mathbf{x})$ is given by

$$\mathbb{E}(\mathbf{r}^\top \mathbf{x}) = \mathbb{E}\left(\sum_{i=1}^{n} x_i r_i\right) = \sum_{i=1}^{n} x_i \mathbb{E}(r_i). \tag{3}$$

The portfolio variance $\mathrm{Var}(\mathbf{r}^\top \mathbf{x})$ is given by

$$\begin{aligned}
\mathrm{Var}(\mathbf{r}^\top \mathbf{x}) &= \mathrm{Var}\left(\sum_{i=1}^{n} x_i r_i\right) \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} x_i x_j \mathrm{Cov}(r_i, r_j) \\
&= \sum_{i=1}^{n} x_i^2 \mathrm{Var}(r_i) + 2\sum_{i \neq j} x_i x_j \mathrm{Cov}(r_i, r_j)
\end{aligned} \tag{4}$$

and is a measure of the risk associated with the portfolio.

One common strategy to reduce risk without sacrificing return is *diversification*. In simple terms, the concept of diversification is analogous to the saying "don't place all your eggs in one basket" [10, 11]. Markowitz quantifies this idea by showing that portfolio variance can be reduced by combining assets with low—ideally negative— covariances. Intuitively, this protects against the possibility of many assets in the portfolio performing poorly at the same time.

2.4 Risk Attitude

Investors can be characterized by three types of risk attitude: *risk averse, risk loving, and risk neutral* [14]. An investor who is risk averse tends to choose less risky investments even though these usually come with a lower reward [1]. When making decisions under high uncertainty, a highly risk averse investor may apply the *maximin principle*, which seeks to maximize the return of the worst possible outcome [8]. A risk loving individual, on the other hand, desires higher rewards even if it involves a riskier gamble. A risk neutral investor values outcomes objectively by their expected monetary value.

2.5 The Optimization Problem

A continuous *minimization problem* in standard form is given by

$$\begin{aligned} \min_{\mathbf{x}} \quad & f(\mathbf{x}) \\ \text{subject to} \quad & g_i(\mathbf{x}) \leq 0, i = 1, ..., m \\ & h_j(\mathbf{x}) = 0, j = 1, ..., p, \end{aligned} \tag{5}$$

where $\mathbf{x} \in \mathbb{R}^n$ is called the *optimization variable* and $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is called the *objective function* [6]. A *maximization problem* is obtained by negating the objective function. The $g_i(\mathbf{x}) \leq 0$ are called *inequality constraints* and have corresponding *inequality constraint functions* $g_i : \mathbb{R}^n \rightarrow \mathbb{R}$. The $h_j(\mathbf{x})$ are called *equality constraints* and have corresponding *equality constraint functions* $h_j : \mathbb{R}^n \rightarrow \mathbb{R}$.

The *domain* D of the optimization problem is the set of points for which the objective and constraint functions are defined. The set of all points in the domain that satisfy the inequality and equality constraints is called the *feasible region*.

If the objective function as well as the inequality and equality constraint functions are all linear we have a *linear programming* problem. It can be written in the standard form

$$\min_{\mathbf{x}} \left\{ \mathbf{c}^\top \mathbf{x} : \mathbf{A}\mathbf{x} \leq \mathbf{b} \right\}, \tag{6}$$

where $\mathbf{c} \in \mathbb{R}^n$, $\mathbf{b} \in \mathbb{R}^m$, and $\mathbf{A} \in \mathbb{R}^m \times \mathbb{R}^n$.

2.5.1 Duality

The *Lagrangian* L associated with the standard optimization problem (5) is given by

$$L(\mathbf{x}, \boldsymbol{\nu}, \boldsymbol{\lambda}) = f(\mathbf{x}) + \sum_{i=1}^{m} \nu_i g_i(\mathbf{x}) + \sum_{j=1}^{p} \lambda_j h_j(\mathbf{x}), \tag{7}$$

where $\boldsymbol{\nu} = (\nu_1, \ldots, \nu_m)^\top$ and $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_p)^\top$ are the *dual variables*.

The *Lagrange dual function* $\theta(\boldsymbol{\nu}, \boldsymbol{\lambda})$ is defined as

$$\theta(\boldsymbol{\nu}, \boldsymbol{\lambda}) = \inf_{\mathbf{x} \in D} L(\mathbf{x}, \boldsymbol{\nu}, \boldsymbol{\lambda}). \tag{8}$$

For each $(\boldsymbol{\nu}, \boldsymbol{\lambda})$ with $\boldsymbol{\nu} \geq \mathbf{0}$ the dual function $\theta(\boldsymbol{\nu}, \boldsymbol{\lambda})$ places a lower bound on the optimal value of (5).

The *dual problem*

$$\max_{\boldsymbol{\nu} \geq \mathbf{0}, \boldsymbol{\lambda}} \theta(\boldsymbol{\nu}, \boldsymbol{\lambda}) \tag{9}$$

associated with the *primal problem* (5) maximizes this lower bound.

Let p^* be the optimal value of the primal problem (5) and d^* the optimal value of the dual problem (9). The difference $p^* - d^*$ is called the *duality gap*. *Weak duality* states that the duality gap is always greater than or equal to 0. If the duality gap is equal to 0, we additionally have *strong duality*.

If the primal problem (5) is convex, that is of the form

$$\begin{aligned} \min_{\mathbf{x}} \quad & f(\mathbf{x}) \\ \text{subject to} \quad & g_i(\mathbf{x}) \leq 0, i = 1, ..., m \\ & \mathbf{A}\mathbf{x} = \mathbf{b}, \end{aligned} \tag{10}$$

where $f, g_1, \ldots, g_m$ are convex, then *Slater's condition* is a sufficient condition for strong duality. Slater's condition holds if there exists some feasible point $\mathbf{x} \in \text{relint}(D)$, where $\text{relint}(D)$ is the relative interior of the domain D, such that the inequality constraints g_i hold with strict inequality.

2.5.2 Robust Optimization

When an optimization problem models a real-life scenario, some further aspects with potentially suboptimal practical implications must be considered. Ben-Tal and Nemirovski [5] summarize these as

1. There is uncertainty in the data gathered to make the model;
2. Even if the optimal solution can be accurately calculated, it might not be possible to implement;

3. The constraints must lie within the feasible region for all meaningful realizations of the data;

4. The optimization problems might be large-scale;

5. Optimal solutions might be rendered highly infeasible by small changes in the data.

To address the issue of data uncertainty, there is a subfield of optimization known as robust optimization. The objective of robust optimization is to find a solution that is optimal in the worst-case scenario.

We say that the data belongs to some *uncertainty set* $\mathcal{U}$ and require that the optimal solution is inside the feasible region for all points in $\mathcal{U}$ [7]. We then specify a *robust counterpart* to the optimization problem at hand.

For example, consider the problem (6) but suppose that the data $(\mathbf{c}, \mathbf{A}, \mathbf{b})$ lies in some uncertainty set $\mathcal{U}$. When we consider the problem for every $(\mathbf{c}, \mathbf{A}, \mathbf{b}) \in \mathcal{U}$, we get a family of optimization problems

$$\min_{\mathbf{x}\in\mathbb{R}^n} \left\{\mathbf{c}^\top \mathbf{x} : \mathbf{A}\mathbf{x} \le \mathbf{b}\right\}_{(\mathbf{c},\mathbf{A},\mathbf{b})\in\mathcal{U}} . \tag{11}$$

We can tranform the family of problems (11) into a single linear optimization problem

$$\min_{\mathbf{x}\in\mathbb{R}^n} \left\{\mathbf{c}^\top \mathbf{x} : \mathbf{A}\mathbf{x} \le \mathbf{b}, (\mathbf{c}, \mathbf{A}, \mathbf{b}) \in \mathcal{U}\right\} , \tag{12}$$

and we call this the robust counterpart to (6).

We can rewrite (12) so that the uncertainty lies only in the constraints and get

$$\min_{\mathbf{x},t} \left\{t : \mathbf{c}^\top \mathbf{x} - t \le 0, a_i^T \mathbf{x} - b_i \le 0, i = 1, ..., m, (\mathbf{c}, \mathbf{A}, \mathbf{b}) \in \mathcal{U}\right\} . \tag{13}$$

As observed by Ben-Tal et al. in [4], the robust counterpart (13) has infinitely many constraints. This means that (13) is a semi-infinite problem and therefore not computationally tractable in general. Being able to identify the cases where it is possible to solve the robust counterpart of an uncertain problem is one of the major theoretical challenges of robust optimization.

3 Method

This section describes the method of the project in three parts: data collection, parameter estimation, and optimization models.

3.1 Data Collection

The portfolio analysis is based on 30 stocks on the Stockholm Stock Exchange (.ST) and a time frame of 15 years. We selected the 30 largest stocks according to *Avanza*, excluding companies with initial public offerings more recent than 2007-01-01 due to insufficient historical data [3]. Some companies have both A and B stocks and for these companies, we chose only the one with the highest market value since it is unusual to buy both A and B stocks from the same company. The 30 stocks chosen are shown in table A.1.

To acquire market data for the stocks, we used the Yahoo Finance Python API *yfinance*. For each stock, we selected the *adjusted closing price* on the first trading day of every year from 2007 to 2023. The reason for using the adjusted closing price rather than the raw closing price is that it adjusts for corporate actions such as splits and dividend distributions and therefore tends to be a more accurate measure of a stock's value [16].

3.2 Parameter Estimation

Let $P_i^{(t)}$ denote the adjusted closing price of stock i on the first trading day of year t and $r_i^{(t)}$ the annual return of stock i from year $t-1$ to year t.

We used the historical price data and the formula (1) to calculate the annual returns for stocks $i = 1, \dots, 30$ and years $t \in T := \{2007, \dots, 2023\}$.

For every stock i we estimated the lower and upper bounds on the return

$$r_i^- = \min_{t \in T} r_i^{(t)} \text{ and } r_i^+ = \max_{t \in T} r_i^{(t)}.$$

We also estimated the entries

$$\mu_i = \mathbb{E}(r_i)$$

of the mean return vector $\boldsymbol{\mu}$ for $i = 1, \ldots, n$ and the entries

$$\sigma_{ij} = \text{Cov}(r_i, r_j)$$

of the covariance matrix $\boldsymbol{\Sigma}$ for $i, j = 1, \ldots, n$.

In the same manner, we acquired yearly price data for the OMX Stockholm 30 index and estimated the annual returns $r_{\text{OMX}}^{(t)}$, expected return $\mathbb{E}(r_{\text{OMX}})$, and variance Var$(r_{\text{OMX}})$. The OMXS30 index consists of the 30 most actively traded stocks on the Stockholm Stock Exchange, and we used this data as an indicator of general market performance.

3.3 Optimization Models

We used the *gurobipy* library in Python to solve our optimization models. The Gurobi Optimizer can solve linear and quadratic programming problems.

3.3.1 Robust Optimization Model

The problem of maximizing the return of a portfolio of *n* risky assets as defined by (2) is

$$\max_{\mathbf{x}} \left\{ \mathbf{r}^\top \mathbf{x} : \mathbf{x}^\top \mathbf{1} = 1, \mathbf{x} \geq \mathbf{0} \right\}. \tag{14}$$

Clearly, this is a linear programming problem.

Since there is great uncertainty in the returns vector $\mathbf{r}$, this is a problem well-suited for robust optimization, especially if the investor is highly risk averse.

We defined the uncertainty set $\mathcal{U} = [\mathbf{r}^-, \mathbf{r}^+]$, where $\mathbf{r}^- = (r_1^-, \ldots, r_n^-) \in \mathbb{R}^n$ and $\mathbf{r}^+ = (r_1^+, \ldots, r_n^+) \in \mathbb{R}^n$ are lower and upper limits on the returns respectively, and constructed the robust counterpart to (14)

$$\max_{\mathbf{x}} \left\{ \min_{\mathbf{r}} \left\{ \mathbf{r}^\top \mathbf{x} : \mathbf{r} \in [\mathbf{r}^-, \mathbf{r}^+] \right\} : \mathbf{x}^\top \mathbf{1} = 1, \mathbf{x} \geq \mathbf{0} \right\}. \tag{15}$$

This model can be solved by inspection. The inner problem

$$\min_{\mathbf{r}} \left\{ \mathbf{r}^\top \mathbf{x} : \mathbf{r} \in [\mathbf{r}^-, \mathbf{r}^+] \right\} \tag{16}$$

has optimal solution $\mathbf{r}^{(*)} = \mathbf{r}^-$ since for $\mathbf{x} \geq \mathbf{0}$, $\mathbf{r}^\top\mathbf{x}$ is strictly increasing in $\mathbf{r}$.

So, the problem (15) reduces to

$$\max_{\mathbf{x}} \left\{ (\mathbf{r}^-)^\top \mathbf{x} : \mathbf{x}^\top \mathbf{1} = 1, \mathbf{x} \geq \mathbf{0} \right\} \tag{17}$$

with optimal solution

$$\mathbf{x}^{(*)} = (x_1^{(*)}, \ldots, x_n^{(*)}) : \begin{cases} x_i^{(*)} = 1, \text{ for } i = \text{argmax}_i(r_i^-) \\ x_i^{(*)} = 0, \text{ otherwise} \end{cases}.$$

That is, the optimal solution allocates the entire wealth to the asset with the highest lower bound on return r^-.

3.3.2 Sum of Returns Constraints

We improved the model (15) by introducing additional constraints. First of all, the model optimizes the portfolio return in the scenario where all assets simultaneously perform at their worst. This approach is overly conservative and does not reflect a realistic worst-case scenario.

To that end, we introduced an additional constraint on the returns

$$\sum_{i=1}^{n} r_i = \mathbf{r}^\top \mathbf{1} \geq R := \min_{t \in T} \sum_{i=1}^{n} r_i^{(t)}, \tag{18}$$

where R is the historical minimum yearly sum of individual asset returns.

The robust optimization model becomes

$$\max_{\mathbf{x}} \left\{ \min_{\mathbf{r}} \left\{ \mathbf{r}^\top \mathbf{x} : \mathbf{r} \in [\mathbf{r}^-, \mathbf{r}^+], \mathbf{r}^\top \mathbf{1} \geq R \right\} : \mathbf{x}^\top \mathbf{1} = 1, \mathbf{x} \geq \mathbf{0} \right\}. \tag{19}$$

Notice that the max-min problem (19) can be separated such that the uncertainty is contained in the convex inner problem

$$\min_{\mathbf{r}} \left\{ \mathbf{r}^\top \mathbf{x} : \mathbf{r} \in [\mathbf{r}^-, \mathbf{r}^+], \mathbf{r}^\top \mathbf{1} \geq R \right\}. \tag{20}$$

This is possible because the uncertainty lies in the objective function.

To solve the problem (19) we can replace the inner problem (20) with its dual

$$\max_{\nu}\left\{\inf_{\mathbf{r}\in[\mathbf{r}^-,\mathbf{r}^+]}\left\{\mathbf{r}^\top\mathbf{x}+\nu\left(R-\mathbf{r}^\top\mathbf{1}\right)\right\}:\nu\geq 0\right\}. \tag{21}$$

The infimum over r evaluates to

$$\begin{aligned}\inf_{\mathbf{r}\in[\mathbf{r}^-,\mathbf{r}^+]}\left\{\mathbf{r}^\top\mathbf{x}+\nu\left(R-\mathbf{r}^\top\mathbf{1}\right)\right\} &= \inf_{\mathbf{r}\in[\mathbf{r}^-,\mathbf{r}^+]}\left\{\mathbf{r}^\top(\mathbf{x}-\nu\mathbf{1})\right\}+\nu R\\ &= \mathbf{r}^{(*)\top}(\mathbf{x}-\nu\mathbf{1})+\nu R,\end{aligned}$$

where $r_i^{(*)}=\begin{cases} r_i^-, \text{ if } x_i-\nu>0\\ r_i^+, \text{ if } x_i-\nu\leq 0\end{cases}.$

The problem is convex and Slater's condition holds. Therefore, strong duality holds, and the optimal value of the dual problem (21) equals the optimal value of the primal problem (20).

Therefore, (19) is equivalent to

$$\begin{aligned}&\max_{\mathbf{x}}\left\{\max_{\nu}\left\{\mathbf{r}^{(*)\top}(\mathbf{x}-\nu\mathbf{1})+\nu R:\nu\geq 0\right\}:\mathbf{x}^\top\mathbf{1}=1,\mathbf{x}\geq\mathbf{0}\right\}\\ &\max_{\mathbf{x},\nu}\left\{\mathbf{r}^{(*)\top}(\mathbf{x}-\nu\mathbf{1})+\nu R:\mathbf{x}^\top\mathbf{1}=1,\mathbf{x}\geq\mathbf{0},\nu\geq 0\right\}.\end{aligned} \tag{22}$$

The reformulation (22) of the robust counterpart (19) shows that the problem is convex and computationally tractable. So, for this specific robust optimization problem, we do not have to deal with the issue of semi-infiniteness as discussed in section 2.5.2.

3.3.3 Portfolio Variance Constraints

For a low-risk portfolio, it may also be desirable for the portfolio variance as defined by (4) to be kept within reasonable limits.

We considered the following limits:

1. The variance of the portfolio should be less than 90% of the minimum asset variance. That is,

$$\mathrm{Var}\left(\mathbf{r}^\top \mathbf{x}\right) = \mathbf{x}^\top \mathbf{\Sigma} \mathbf{x} \leq 0.9 \min_i \{\mathrm{Var}(r_i)\}. \tag{23}$$

This ensures that the optimal portfolio has a lower variance than any single asset.

2. The variance of the portfolio should be less than some constant α times the variance of the OMX Stockholm 30 index. That is,

$$\mathrm{Var}\left(\mathbf{r}^\top \mathbf{x}\right) = \mathbf{x}^\top \mathbf{\Sigma} \mathbf{x} \leq \alpha \mathrm{Var}(r_{\mathrm{OMX}}). \tag{24}$$

The purpose of this constraint is to compare the robust optimal portfolio with a market portfolio that uses the OMXS30 index proportions as weights. Choosing α as small as possible such that the model is feasible gives us a robust portfolio with variance close to the general trend on the stock market.

3.3.4 Final Models

The final three models are listed below. For each model we obtained the optimal portfolio weights, the objective value, the expected portfolio return, and the portfolio variance.

1. Basic Model

$$\max_{\mathbf{x},\nu} \left\{ \mathbf{r}^{(*)\top}(\mathbf{x} - \nu \mathbf{1}) + \nu R : \mathbf{x}^\top \mathbf{1} = 1, \mathbf{x} \geq \mathbf{0}, \nu \geq 0 \right\} \tag{25}$$

2. Minimum Variance Model

$$\max_{\mathbf{x},\nu} \left\{ \mathbf{r}^{(*)\top}(\mathbf{x} - \nu \mathbf{1}) + \nu R : \mathbf{x}^\top \mathbf{1} = 1, \mathbf{x}^\top \mathbf{\Sigma} \mathbf{x} \leq \min_i \{\mathrm{Var}(r_i)\}, \mathbf{x} \geq \mathbf{0}, \nu \geq 0 \right\} \tag{26}$$

3. OMX Model

$$\max_{\mathbf{x},\nu} \left\{ \mathbf{r}^{(*)\top}(\mathbf{x} - \nu \mathbf{1}) + \nu R : \mathbf{x}^\top \mathbf{1} = 1, \mathbf{x}^\top \mathbf{\Sigma} \mathbf{x} \leq \alpha \mathrm{Var}(r_{\mathrm{OMX}}), \mathbf{x} \geq \mathbf{0}, \nu \geq 0 \right\} \tag{27}$$

4 Results

This section presents the results obtained for the models (25), (26), and (27) presented in the previous section.

4.1 Basic Model

For the model (25), the Gurobi Optimizer suggests that the best portfolio consists of a single stock, as shown in Figure 4.1.

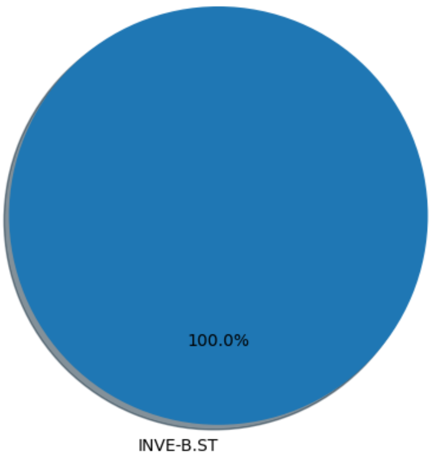

Figure 4.1: Optimal Portfolio Weights $\mathbf{x}^{(*)}$ for Basic Model (%)

Figures 4.2 and 4.3 show the ranges and variances of the annual returns respectively, with the stock chosen by the optimizer marked in red.

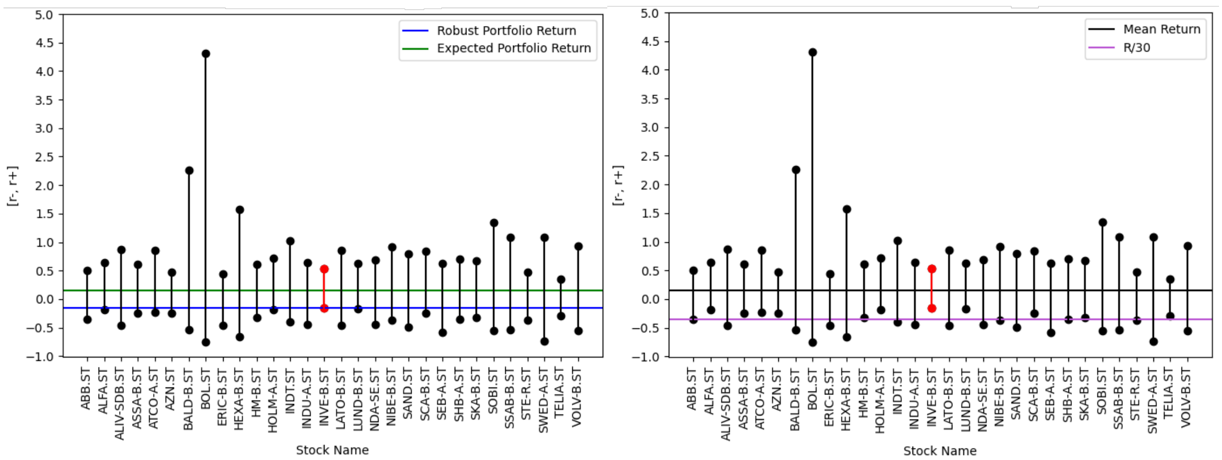

Figure 4.2: Stock Annual Return Ranges

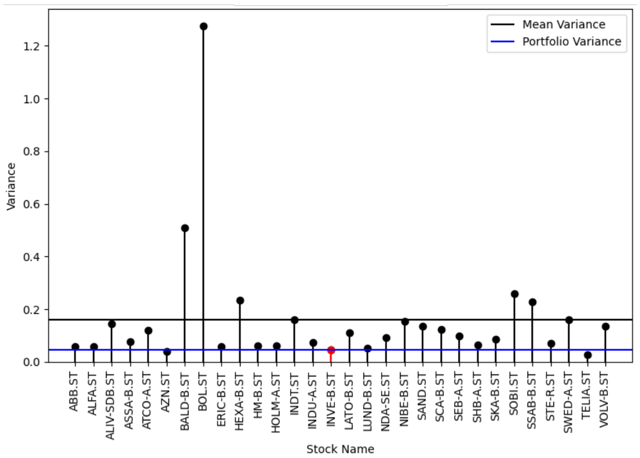

Figure 4.3: Stock Annual Return Variances

The robust objective value return is $\mathbf{r}^{(*)\top}\mathbf{x}^{(*)} = -0.15556$.

The expected return is $\boldsymbol{\mu}^{\top}\mathbf{x}^{(*)} = 0.15054$.

The portfolio variance is $\mathbf{x}^{(*)\top}\boldsymbol{\Sigma}\mathbf{x}^{(*)} = 0.04513$.

4.2 Minimum Variance Model

Figure 4.4 shows the portfolio weights obtained for the model (26) rounded to one decimal place. We disregard values less than 10^{-3} as these have no significant impact on the objective function.

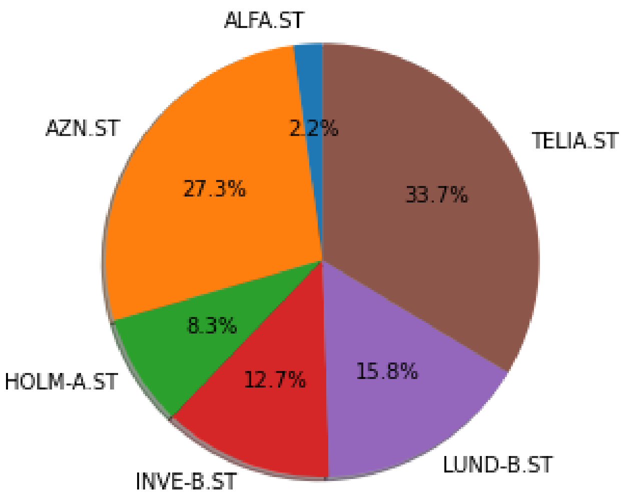

Figure 4.4: Optimal Portfolio Weights $\mathbf{x}^{(*)}$ for Minimum Variance Model (%)

Figures 4.5 and 4.6 show the ranges and variances of the annual returns respectively, with the stocks chosen by the optimizer marked in red.

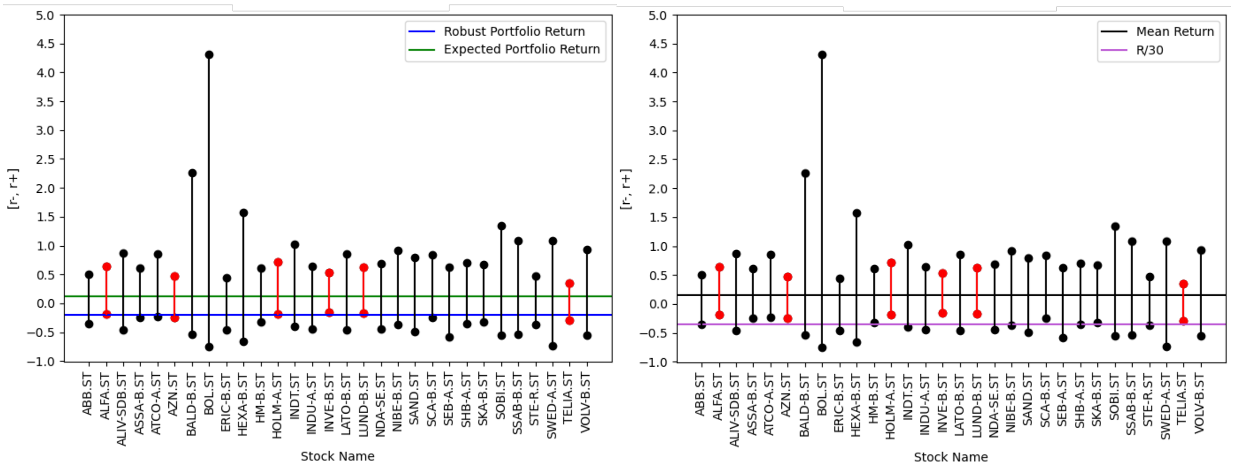

Figure 4.5: Stock Annual Return Ranges

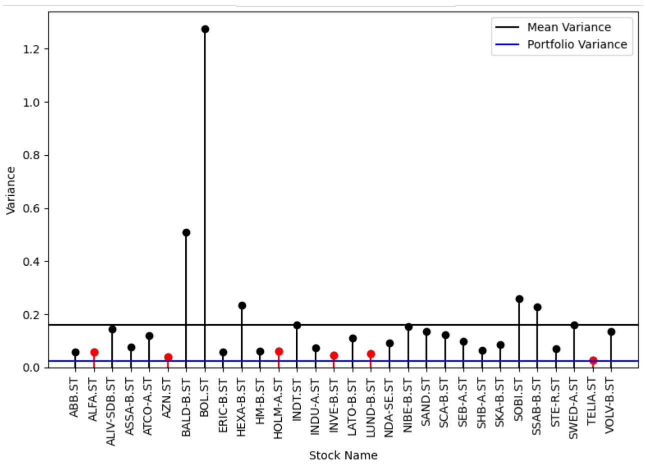

Figure 4.6: Stock Annual Return Variances

The robust objective value return is $\mathbf{r}^{(*)\top}\mathbf{x}^{(*)} = -0.19691$.

The expected return is $\boldsymbol{\mu}^{\top}\mathbf{x}^{(*)} = 0.12654$.

The portfolio variance is $\mathbf{x}^{\top(*)}\boldsymbol{\Sigma}\mathbf{x}^{(*)} = 0.02505$.

4.3 OMX Model

Starting with $\alpha = 1$ and increasing in steps of 0.25, the smallest α for which the Gurobi Optimizer is able to solve the model (27) is $\alpha = 5$.

Figure 4.7 shows the portfolio weights obtained with $\alpha = 0.5$. The percentages are rounded to one decimal place and values less than 10^{-3} are disregarded.

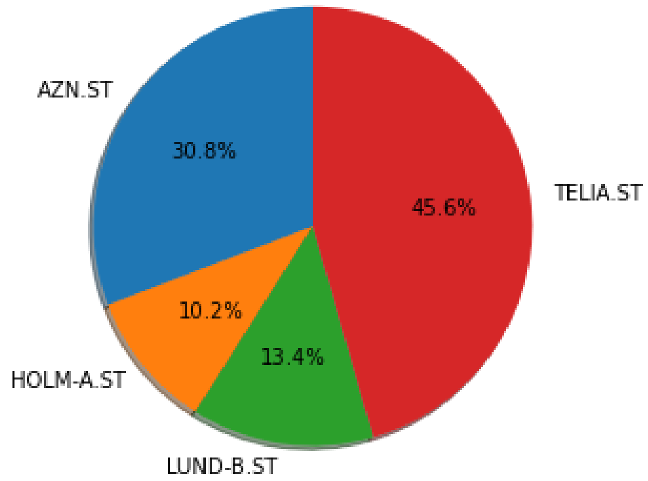

Figure 4.7: Optimal Portfolio Weights $\mathbf{x}^{(*)}$ for OMX Model (%)

Figures 4.8 and 4.9 show the ranges and variances of the yearly returns respectively, with the stocks chosen by the optimizer marked in red.

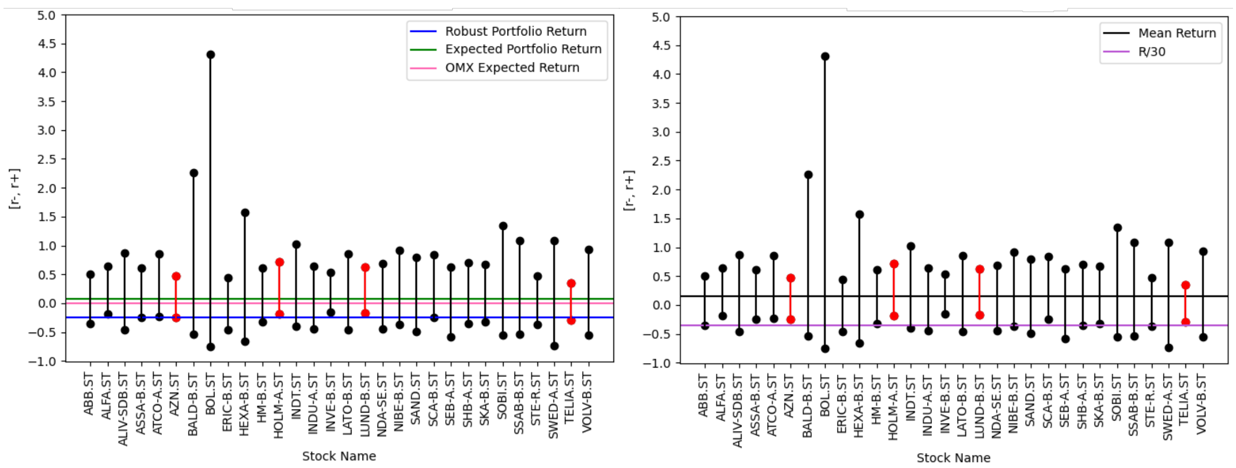

Figure 4.8: Stock Annual Return Ranges

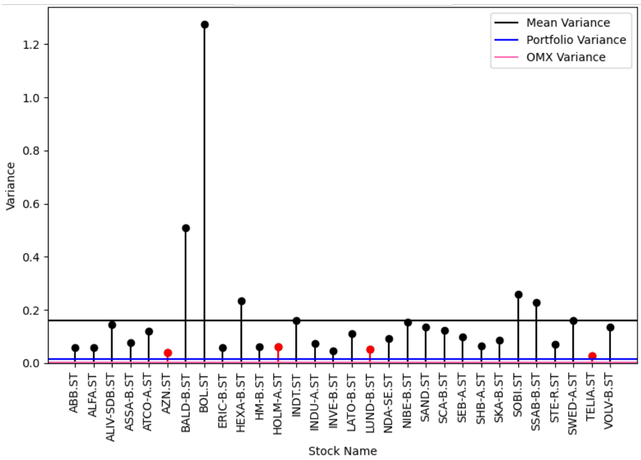

Figure 4.9: Stock Annual Return Variances

The robust objective value return is $\mathbf{r}^{(*)\top}\mathbf{x}^{(*)} = -0.25258$.

The expected return is $\boldsymbol{\mu}^\top \mathbf{x}^{(*)} = 0.07669$.

The portfolio variance is $\mathbf{x}^{(*)\top}\boldsymbol{\Sigma}\mathbf{x}^{(*)} = 0.01461$.

The expected return of the OMX portfolio is $\mathbb{E}(r_{\text{OMX}}) = 0.00602$.

The variance of the OMX portfolio is $\text{Var}(r_{\text{OMX}}) = 0.00292$.

5 Discussion

This section discusses the results presented in the previous section in terms of how reasonable they are and how the different models compare.

5.1 Basic Model

Figure 4.2 shows that the single stock INVE-B.ST that the optimizer chooses for the basic model (25) is the one with the highest lower bound on return. That is,

$$r^-_{\text{INVE-B.ST}} = \max(r^-_1, \ldots, r^-_n).$$

This means that the optimal solution is the same as the optimal solution for (15). In other words, the optimal value for (15) is still attainable even with the addition of the constraint (18). This is reasonable because we can see in figure 4.2 that it is still possible to choose $\mathbf{r}$ such that $\max(r_1, \ldots, r_n) = \max(r^-_1, \ldots, r^-_n)$ and $\mathbf{r}^\top \mathbf{1} \geq R$.

That the objective value is negative is expected since the robust model maximizes performance in the worst-case scenario which, as can be seen in figure 4.2, means negative return for all the stocks. The expected return, on the other hand, is positive. From figure 4.3 we observe that the portfolio variance is the same as the variance of INVE-B.ST.

5.2 Minimum Variance Model

For the minimum variance model, the optimizer chooses 6 stocks. In figure 4.5 we see that all the stocks chosen have a high lower bound on return r^-. This is expected for the same reason the optimizer chooses the stock with the highest r^- for the basic model.

Again, the objective value is negative while the expected return is positive. Figure 4.6 shows that, in addition to all having a high r^-, the chosen stocks also have low variance as to not violate the portfolio variance constraint (23).

5.3 OMX Model

The OMX model imposes a significantly stronger constraint (24) on the portfolio variance, and it makes sense that this leads to an infeasible model for small values of α.

Figures 4.8 and 4.9 show that the selected stocks with $\alpha = 5$ all have a relatively high r^- and low variance. We also notice that the optimizer allocates almost half of the total wealth to the stock TELIA.ST. This is reasonable because, as shown in figure 4.9, TELIA.ST has the lowest variance of all the stocks. That is,

$$\text{Var}(r_{\text{TELIA.ST}}) = \min(\text{Var}(r_1), \ldots, \text{Var}(r_n)).$$

The purpose of the OMX model is to compare the robust portfolio to a market portfolio that uses the OMXS30 index proportions as weights. The market portfolio has low variance because it is highly diversified, but it also has low return. The optimal robust portfolio obtained with variance 5 times higher than the OMX portfolio variance has expected return $\frac{0.07669}{0.00602} = 12.7$ times higher than the OMX expected return. In addition to the higher expected return, this robust portfolio has the advantage of performing optimally in the worst-case scenario, which may be desirable for a risk averse investor.

5.4 Comparison of Models

One virtue of the first model is that it is the most simple. This model has the highest objective value, which is expected since it has the fewest constraints. However, this comes at the price of high variance. The actual return of a high-variance portfolio is likely to fluctuate more around the expected return, which may not be desirable for a risk averse investor. Furthermore, a portfolio of a single asset is highly risky because even if this stock has performed well historically, there is no guarantee that it will continue to do so, and there is no backup if it does not.

The variance constraints used in the minimum variance model and the OMX models lead to diversification because limiting the portfolio variance to be less than any single asset variance forces the optimizer to choose more than one stock. The drawback is that the lower the portfolio variance is, the lower both the expected return and the robust objective value return are.

6 Conclusions

We propose that the risk averse asset management company presented in section 1.1 should distribute its assets according to one of the three models presented in this thesis. As discussed in the previous section, the final choice between the models is a trade-off between higher expected return and lower variance, and it depends on the individual preferences of the company.

A suggestion for future work is to improve the models by introducing more advanced constraints. One possibility is to further incorporate the correlation between assets when minimizing over $\mathbf{r}$ to get a less conservative and more realistic worst-case scenario. The constraint (18) aims to address this, but it is still an oversimplification of the real stocks' behavior. The cost of more advanced constraints, however, is that the model becomes more difficult to solve.

The excessive conservatism of robustness is one of the drawbacks of this approach. Roos et. al. present several suggestions for how to reduce this conservatism in [13]. One suggestion is to use a distributional robustness approach instead, as described by Namkoong et. al. in [12]. Future studies could apply this method to portfolio optimization to get a robust portfolio with a higher expected return than the ones presented in this thesis.

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A Appendix

Company Name	Symbol
ABB Ltd	ABB.ST
Alfa Laval Corporate AB	ALFA.ST
Autoliv, Inc.	ALIV-SDB.ST
ASSA ABLOY AB	ASSA-B.ST
Atlas Copco AB	ATCO-A.ST
AstraZeneca PLC	AZN.ST
Fastighets AB Balder	BALD-B.ST
Boliden AB	BOL.ST
Telefonaktiebolaget LM Ericsson	ERIC-B.ST
Hexagon AB	HEXA-B.ST
H&M Hennes & Mauritz AB	HM-B.ST
Holmen AB	HOLM-A.ST
Indutrade AB	INDT.ST
AB Industrivärden	INDU-A.ST
Investor AB	INVE-B.ST
Investment AB Latour	LATO-B.ST
L E Lundbergföretagen AB	LUND-B.ST
Nordea Bank Abp	NDA-SE.ST
NIBE Industrier AB	NIBE-B.ST
Sandvik AB	SAND.ST
Svenska Cellulosa Aktiebolaget SCA	SCA-B.ST
Skandinaviska Enskilda Banken AB	SEB-A.ST
Svenska Handelsbanken AB	SHB-A.ST
Skanska AB	SKA-B.ST
Swedish Orphan Biovitrum AB	SOBI.ST
SSAB AB	SSAB-B.ST
Stora Enso Oyj	STE-R.st
Swedbank AB	SWED-A.ST
Telia Company AB	TELIA.ST
AB Volvo	VOLV-B.ST

Table A.1