Homework 8
Due date: April 9th, 2025

1.

2. First we note that for all $0 \le k \le n$, we have that

$$\left\langle \frac{1}{\sqrt{2}}, \cos(kx) \right\rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{1}{\sqrt{2}} \cos(kx)\, dx \; = \; \frac{1}{\pi} \left[\frac{\sin(kx)}{k} \right] \Bigg|_{-\pi}^{\pi} = 0$$

where we used the fact that $\sin(k\pi) = 0$ for all integers k.

Furthermore, note that

$$\left\langle \frac{1}{\sqrt{2}}, \sin(kx) \right\rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{1}{\sqrt{2}} \sin(kx)\, dx \; = \; 0$$

since $\sin(kx)$ is an odd function for all integers k.

Next, note that

$$\left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{1}{2}\, dx \; = \; \frac{1}{\pi} \frac{2\pi}{2} = 1.$$

Now for all positive integers m and n, we first note that $\cos(mx)\sin(nx)$ is an odd function, so they will have an inner product of 0. Considering $\cos(nx)$ and $\cos(mx)$, we have two cases. If $n = m$, we have that

$$\langle \cos(mx), \cos(mx) \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} \cos^2(mx)\, dx \; = \; \frac{1}{\pi} \int_0^{\pi} (1 + \cos(2mx))\, dx \; = \; \frac{1}{\pi} \left(x + \frac{1}{2m} \sin(2mx) \right) \Bigg|_0^{\pi}$$

$$= \frac{1}{\pi} \pi = 1.$$

If $n \ne m$, then we have

$$\langle \cos(nx), \cos(mx) \rangle = \frac{2}{\pi} \int_0^{\pi} \cos(nx)\cos(mx)\, dx \; = \; \frac{1}{\pi} \int_0^{\pi} (\cos((n-m)x) + \cos((n+m)x))\, dx$$

$$= \frac{1}{\pi} \left(\frac{1}{n-m} \sin((n-m)x) + \frac{1}{n+m} \sin((n+m)x) \right) \Bigg|_0^{\pi} = 0$$

where we once again used that $\sin(k\pi) = 0$.

We finally consider $\sin(nx)$ and $\sin(mx)$, which once again has two cases. If $n = m$, then

$$\langle \sin(mx), \sin(mx) \rangle = \frac{2}{\pi} \int_0^{\pi} \sin^2(mx)\, dx \; = \; \frac{1}{\pi} \int_0^{\pi} (1 - \cos(2mx))\, dx \; = \; \frac{1}{\pi} \left(x - \frac{1}{2m} \sin(2mx) \right) \Bigg|_0^{\pi}$$

$$= \frac{1}{\pi} \pi = 1.$$

If $n \neq m$, then we have

$$\langle \sin(nx), \sin(mx) \rangle = \frac{2}{\pi} \int_0^\pi \sin(nx)\sin(mx)\,dx = \frac{1}{\pi} \int_0^\pi \left(\cos((n-m)x) - \cos((n+m)x)\right)\,dx$$

$$= \frac{1}{\pi} \left(\frac{1}{n-m}\sin((n-m)x) - \frac{1}{n+m}\sin((n+m)x) \right) \Big|_0^\pi = 0$$

by the same reasons as above. Therefore, the set is an orthonormal set, which is the desired result.

3. (a) We seek the continuous Fourier approximation $S_3(x)$ of the function $f(x) = e^x$ on the interval $[-\pi, \pi]$, using the formula:

$$S_n(x) = \langle f(x), 1 \rangle \cdot \frac{1}{2} + \sum_{k=1}^n \left(\langle f(x), \sin(kx) \rangle \cdot \sin(kx) + \langle f(x), \cos(kx) \rangle \cdot \cos(kx) \right)$$

with the inner product defined as:

$$\langle f(x), g(x) \rangle = \frac{1}{\pi} \int_{-\pi}^\pi f(x)g(x)\,dx$$

$$a_0 = \langle f(x), 1 \rangle = \frac{1}{\pi} \int_{-\pi}^\pi e^x\,dx = \frac{1}{\pi}(e^\pi - e^\pi)$$

$$\Rightarrow \frac{a_0}{2} = \frac{1}{2\pi}(e^\pi - e^{-\pi})$$

$$a_1 = \frac{1}{\pi} \int_{-\pi}^\pi e^x \cos(x)\,dx = \frac{1}{\pi} \left[\frac{e^x}{2}(\sin(x) + \cos(x)) \right]_{-\pi}^\pi$$

$$= \frac{1}{\pi} \left(-\frac{e^\pi}{2} + \frac{e^{-\pi}}{2} \right) = -\frac{1}{2\pi}(e^\pi - e^{-\pi})$$

$$b_1 = \frac{1}{\pi} \int_{-\pi}^\pi e^x \sin(x)\,dx = \frac{1}{\pi} \left[\frac{e^x}{2}(\sin(x) - \cos(x)) \right]_{-\pi}^\pi$$

$$= \frac{1}{\pi} \left(\frac{e^\pi}{2} - \frac{e^{-\pi}}{2} \right) = \frac{1}{2\pi}(e^\pi - e^{-\pi})$$

Using:

$$\int e^x \cos(ax)dx = \frac{e^x(a\sin(ax) + \cos(ax))}{a^2 + 1}, \quad \int e^x \sin(ax)dx = \frac{e^x(\sin(ax) - a\cos(ax))}{a^2 + 1}$$

We get:

$$a_2 = \frac{1}{\pi} \left[\frac{e^x(2\sin(2x) + \cos(2x))}{5} \right]_{-\pi}^\pi = \frac{1}{5\pi}(e^\pi - e^{-\pi})$$

$$b_2 = \frac{1}{\pi} \left[\frac{e^x(\sin(2x) - 2\cos(2x))}{5} \right]_{-\pi}^\pi = -\frac{2}{5\pi}(e^\pi - e^{-\pi})$$

$$a_3 = \frac{1}{\pi} \left[\frac{e^x (3\sin(3x) + \cos(3x))}{10} \right]_{-\pi}^{\pi} = -\frac{1}{10\pi}(e^\pi - e^{-\pi})$$

$$b_3 = \frac{1}{\pi} \left[\frac{e^x (\sin(3x) - 3\cos(3x))}{10} \right]_{-\pi}^{\pi} = \frac{3}{10\pi}(e^\pi - e^{-\pi})$$

Evaluating numerically, we have:

$$S_3(x) \approx 3.6767 - 3.6767\cos(x) + 3.6767\sin(x) + 1.4707\cos(2x) - 2.9414\sin(2x) - 0.7353\cos(3x) + 2.2060\,\text{s}$$

(b) We must calculate $\langle y, \phi_k \rangle$ and $\langle y, \psi_k \rangle$ for $k = 0, 1, 2, 3$ First, calculate $\langle y, \phi_0 \rangle$,

$$\langle y, \phi_k \rangle = \frac{1}{10} \sum_{j=0}^{19} y_j \cos(kx_j)$$

$\langle y, \phi_1 \rangle$, $\langle y, \phi_2 \rangle$, and $\langle y, \phi_3 \rangle$.

$$\langle y, \phi_0 \rangle = \frac{1}{10} \sum_{j=0}^{19} y_j \cos(0) = \frac{1}{10} \sum_{j=0}^{19} y_j$$

$$\langle y, \phi_1 \rangle = \frac{1}{10} \sum_{j=0}^{19} y_j \cos(x_j)$$

$$\langle y, \phi_2 \rangle = \frac{1}{10} \sum_{j=0}^{19} y_j \cos(2x_j)$$

$$\langle y, \phi_3 \rangle = \frac{1}{10} \sum_{j=0}^{19} y_j \cos(3x_j)$$

Next, calculate $\langle y, \psi_k \rangle$ for $k = 1, 2, 3$

$$\langle y, \psi_k \rangle = \frac{1}{10} \sum_{j=0}^{19} y_j \sin(kx_j)$$

$$\langle y, \psi_1 \rangle = \frac{1}{10} \sum_{j=0}^{19} y_j \sin(x_j)$$

$$\langle y, \psi_2 \rangle = \frac{1}{10} \sum_{j=0}^{19} y_j \sin(2x_j)$$

$$\langle y, \psi_3 \rangle = \frac{1}{10} \sum_{j=0}^{19} y_j \sin(3x_j)$$

$$S_3(x) = \frac{1}{2}\langle y, \phi_0\rangle + \sum_{k=1}^{3}(\langle y, \phi_k\rangle \cos(kx)) + \sum_{k=1}^{3}(\langle y, \psi_k\rangle \sin(kx))$$

$$S_3(x) = \frac{1}{2}\langle y, \phi_0\rangle + \langle y, \phi_1\rangle \cos(x) + \langle y, \phi_2\rangle \cos(2x) + \langle y, \phi_3\rangle \cos(3x) + \langle y, \psi_1\rangle \sin(x) + \langle y, \psi_2\rangle \sin(2x) + \langle y, \psi_3\rangle$$

$$S_3(x) \approx 3.676 + 2.963\cos(x) + 0.946\cos(2x) + 0.164\cos(3x) - 1.263\sin(x) - 0.740\sin(2x) - 0.214\sin(3x)$$

4. (a) Q is orthogonal.
$$Q^t Q = 1$$
$$\det(Q^t Q) = 1$$
$$\det(Q^t)\det(Q) = 1$$

Since $\det(Q^t) = \det(Q)$,
$$\det(Q)^2 = 1$$
$$\det(Q) \pm 1$$

(b)
$$(PQ)^t PQ = Q^t P^t PQ = Q^t(I)Q = Q^t Q = I$$

. So PQ is orthogonal.

5. (a)

(b)

(c)

6. We are given the symmetric matrix:

$$A = \begin{bmatrix} 1 & -1 & 2 \\ -1 & 1 & 2 \\ 2 & 2 & 2 \end{bmatrix}$$

We want to decompose it as:

$$A = QDQ^T$$

To find the eigenvalues, we solve the characteristic polynomial:

$$\det(A - \lambda I)$$

$$A - \lambda I = \begin{bmatrix} 1 & -1 & 2 \\ -1 & 1 & 2 \\ 2 & 2 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

$$= \begin{bmatrix} 1-\lambda & -1 & 2 \\ -1 & 1-\lambda & 2 \\ 2 & 2 & 2-\lambda \end{bmatrix}$$

We know can find the determinant of the matrix:

$$(1-\lambda)\begin{bmatrix} 1-\lambda & 2 \\ 2 & 2-\lambda \end{bmatrix} - (-1)\begin{bmatrix} -1 & 2 \\ 2 & 2-\lambda \end{bmatrix} + 2\begin{bmatrix} -1 & 1-\lambda \\ 2 & 2 \end{bmatrix} = -\lambda^3 + 4\lambda^2 + 4\lambda - 16 = -(\lambda-4)(\lambda-2)(\lambda+2)$$

The eigenvalues are:

$$\lambda_1 = 4, \quad \lambda_2 = 2, \quad \lambda_3 = -2$$

Solve $(A - 4I)\vec{v} = 0$, we get:

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \quad \|\vec{v}_1\| = \sqrt{6} \Rightarrow q_1 = \frac{1}{\sqrt{6}}\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

Solve $(A - 2I)\vec{v} = 0$, we get:

$$\vec{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \quad \|\vec{v}_2\| = \sqrt{2} \Rightarrow q_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

Solve $(A + 2I)\vec{v} = 0$, we get:

$$\vec{v}_3 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \quad \|\vec{v}_3\| = \sqrt{3} \Rightarrow q_3 = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

Orthogonal matrix Q (columns are the normalized eigenvectors):

$$Q = \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \frac{2}{\sqrt{6}} & 0 & -\frac{1}{\sqrt{3}} \end{bmatrix}$$

Diagonal matrix D of eigenvalues:

$$D = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$