Homework 6

Due date: March 6, 2025

1.

2.

3.

4.

5. (a) *Proof.* By the proposition, there exists $C, C_\infty \in \mathbb{R}_{>0}$ such that

$$||x|| \leq C_\infty ||x||_\infty \leq C_\infty C ||x||'$$

For all $x \in \mathbb{R}^n$. Similarly, there exists $C', C'_\infty \in \mathbb{R}_{>0}$ such that

$$||x||' \leq C'_\infty ||x||_\infty \leq C'_\infty C' ||x||$$

For all $x \in \mathbb{R}^n$. Therefore, letting $D = C_\infty C$ and $D' = C'_\infty C'$ completes the proof. □

(b)

Proposition 1. *Let $a, b \in \mathbb{R}_{\geq 0}$. Then*

$$\sqrt{a+b} \leq \sqrt{a} + \sqrt{b}.$$

Proof of Proposition 1. Note that since a and b are non-negative, we have that

$$a + b \leq a + b + 2\sqrt{a}\sqrt{b} = \left(\sqrt{a} + \sqrt{b}\right)^2.$$

Taking the square root on both sides completes the proof. □

Note that by Proposition 1, we have the following for any $x \in \mathbb{R}^2$:

$$||x||_2 = \sqrt{x_1^2 + x_2^2} \leq \sqrt{x_1^2} + \sqrt{x_2^2} = |x_1| + |x_2| = ||x||_1.$$

Furthermore, by the Cauchy-Schwarz-Bunyakovsky Inequality, we have that

$$||x||_1 = |x_1| + |x_2| = 1 \cdot |x_1| + 1 \cdot |x_2| \leq \sqrt{1+1}\sqrt{|x_1|^2 + |x_2|^2} = \sqrt{2}||x||_2.$$

Therefore, $C_1 = 1$ and $C_2 = \sqrt{2}$ gives us the desired result.

6. (a)

(b)