Homework 3

Due date: February 6th, 2025

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1. *Proof.* It is sufficient to show that

$$\lim_{n\to\infty} \frac{|p_{n+1}-p|}{|p_n-p|} = g'(p)$$

since letting $\lambda = g'(p)$ would complete the proof. By the mean value theorem on $[p_n, p]$, or $[p, p_n]$, there exists a $\xi_n \in (p_n, p)$ such that

$$\frac{|g(p_n)-g(p)|}{|p_n-p|} = \frac{|p_{n+1}-p|}{|p_n-p|} = g'(\xi_n).$$

Since $p_n \xrightarrow{n\to\infty} p$, we must have $\xi_n \xrightarrow{n\to\infty} p$. Therefore, by the continuity of g', we must have

$$g'(\xi_n) \xrightarrow{n\to\infty} g'(p).$$

Therefore, taking the limit of both sides of the first equation gives

$$\lim_{n\to\infty} \frac{|p_{n+1}-p|}{|p_n-p|} = \lim_{n\to\infty} g'(\xi_n) = g'(p)$$

which is the desired result. □

2. (a) Note that

$$f'(x) = 5x^4 - 7.2x^2 + 1.28x + 1.536 \implies f'(0.8) = 0$$
$$f''(x) = 20x^3 - 14.4x + 1.28 \implies f''(0.8) = 0$$
$$f'''(x) = 60x^2 - 14.4 \implies f'''(0.8) = -14.4 \neq 0$$

Therefore, 0.8 is a root of multiplicity 3 of f.

(b)

n	x_n
0	2
1	1.68
2	1.4363
3	1.2536
4	1.1190
5	1.0216
6	0.9038
7	0.9038
8	0.8703
9	0.8474
10	0.8318

(c) Using $\mu(x) = \frac{f(x)}{f'(x)}$, we see the following:

n	x_n
0	2
1	0.6286
2	0.78835
3	0.79995
4	0.80000
5	0.80000
6	0.80000
7	0.80000
8	0.80000
9	0.80000
10	0.80000

which clearly converges much faster.

3. (a) Since f is a "suitably" differentiable function, we may assume that $f^{(i)}(x)$ is continuous in some "suitable" region around p, for all $0 \le i \le m$. Since p is a root of multiplicity m, We must have that $f^{(m-1)}(p) = 0$ and $f^{(m)}(p) \neq 0$. Therefore, by repeated application of L'Hopital's Rule $m-1$ times, and by the fact that $f \in C^m$, we have

$$\lim_{x\to p} \mu(x) = \lim_{x\to p} \frac{f(x)}{f'(x)} = \lim_{x\to p} \frac{f^{(m-1)}(x)}{f^{(m)}(x)} = \frac{f^{(m-1)}(p)}{f^{(m)}(p)} = 0.$$

which us the desired result.

(b) Note that since p is a root of f with multiplicity m, there exists a function[1] $g(x)$ such that

$$f(x) = (x-p)^m g(x).$$

where $g(p) \neq 0$. Therefore, we have

$$\mu(x) = \frac{f(x)}{f'(x)} = \frac{(x-p)^m g(x)}{m(x-p)^{p-1}g(x) + (x-p)^m g'(x)} = \frac{(x-p)g(x)}{mg(x) + (x-p)g'(x)}.$$

If we let $h(x) := \frac{g(x)}{mg(x)+(x-p)g'(x)}$, we then have

$$h(p) = \frac{g(p)}{mg(p)} = \frac{1}{m} \neq 0.$$

Thus, since $\mu(x) = (x-p)h(x)$, and $h(p) \neq 0$, p must be a simple root of $\mu(x)$. Therefore, $\mu'(p) \neq 0$, which is the desired result.

(c) For $x \neq p$, by the quotient rule, we have

$$\frac{\mu(x)}{\mu'(x)} = \frac{f(x)}{f'(x)} \frac{f'(x)^2}{f'(x)^2 - f(x)f''(x)} = \frac{f(x)f'(x)}{f'(x)^2 - f(x)f''(x)}$$

which is the desired result.

[1] which is also differentiable in a suitable interval.

4. *Proof.* Suppose there exists, two degree n polynomials, p_1 and p_2, such that

$$p_1(x_i) = y_i = p_2(x_i)$$

for all $0 \le i \le n$. It suffices to show that $p_1(x) = p_2(x)$. Therefore, the polynomial,

$$f(x) = p_1(x) - p_2(x)$$

is a polynomial of degree at most n, with $n+1$ distinct roots, $x_0, x_1, \ldots, x_n$. However, by the Fundamental Theorem of Algebra, f must be the 0 polynomial.[2] Therefore,

$$f(x) = 0,$$

which means that $p_1(x) = p_2(x)$, which is the desired result. $\square$

5. (a) We can take $(0,7), (2,11), (3,28)$, and $(4,63)$ as $(x_0,y_0), (x_1,y_1), \ldots(x_n,y_n)$ respectively. We find the following:

$$L_0(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}$$

$$L_1(x) = \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}$$

$$L_2(x) = \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}$$

$$L_3(x) = \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}$$

Plugging in the points from above, we get:

$$L_0(x) = \frac{(x-2)(x-3)(x-4)}{(-2)(-3)(-4)} = \frac{-1}{24}(x-2)(x-3)(x-4)$$

$$L_1(x) = \frac{(x-0)(x-3)(x-4)}{(-2)(-1)(-2)} = \frac{1}{4}(x)(x-3)(x-4)$$

$$L_2(x) = \frac{(x-0)(x-2)(x-4)}{(3)(1)(-1)} = \frac{-1}{3}(x)(x-2)(x-4)$$

$$L_3(x) = \frac{(x-0)(x-2)(x-3)}{(4)(2)(1)} = \frac{1}{8}(x)(x-2)(x-3)$$

Using the y-values from above, we get the following interpolation:

$$P(x) = \frac{-7}{24}(x-2)(x-3)(x-4) + \frac{11}{4}(x)(x-3)(x-4) - \frac{28}{3}(x)(x-2)(x-4) + \frac{63}{8}(x)(x-2)(x-3)$$

Which can be simplified to become

$$P(x) = x^3 - 2x + 7$$

[2] Since otherwise it would be a non-zero degree n polynomial with more than n distinct roots.

(b) To find the approximation of $f(1)$, we just plug in 1 for x in $P(x)$:

$$P(1) = 1^2 - 2(1) + 7 = 1 - 2 + 7 = 6$$

(c)

$$\int_0^4 x^3 - 2x + 7 = 76$$

6. Let $g \in C^n[a, b]$, and consider $n+1$ distinct points $x_0, x_1, \ldots, x_n \in [a, b]$ such that

$$x_0 < x_1 < \cdots < x_n.$$

Assume further that

$$g(x_0) = g(x_1) = \cdots = g(x_n) = 0.$$

We aim to prove that there exists a point $\xi \in [x_0, x_n]$ such that $g^{(n)}(\xi) = 0$.

Define the polynomial

$$h(x) = \prod_{i=0}^{n} (x - x_i)$$

which is of degree $n+1$ and satisfies $h(x_i = 0)$ for all $i = 0, 1, ..., n$

Since $g(x_0) = g(x_1) = 0$, by Rolle's Theorem there exists a point $c_1 \in (x_0, x_1)$ such that

$$g'(c_1) = 0.$$

Similarly, $g(x_1) = g(x_2) = 0$ implies there exists a point $c_2 \in (x_1, x_2)$ such that

$$g'(c_2) = 0.$$

Continuing this, we find that $g'(x)$ has zeros in $(x_0, x_1), (x_1, x_2), \ldots, (x_{n-1}, x_n)$.

For each interval where $g'(x)$ has a zero, apply Rolle's Theorem again to $g'(x)$. For example, in (x_0, x_1), there exists a point $d_1 \in (c_1, c_2)$ such that

$$g''(d_1) = 0.$$

Similarly, zeros of $g''(x)$ can be found in $(c_2, c_3), (c_3, c_4), \ldots, (c_{n-1}, c_n)$.

By iteratively applying Rolle's Theorem, we find that:

- $g'(x)$ has zeros in $(x_0, x_1), \ldots, (x_{n-1}, x_n)$,
- $g''(x)$ has zeros in the intervals formed by zeros of $g'(x)$,
- This process continues until the n-th derivative.

After n applications of Rolle's Theorem, there exists a point $\xi \in [x_0, x_n]$ such that

$$g^{(n)}(\xi) = 0.$$

7. We know that

$$|f(x) - p(x)| = \frac{f^{(n+1)}(\xi)}{(n+1)!} \prod_{i=0}^{n} x - x_i$$

Since $x_0, x_1, ..., x_n$ are evenly spaced,

$$\prod_{i=0}^{n}(x - x_i) \leq \frac{1}{4}((\frac{x_n - x_0}{n})^{n+1}(n!))$$

So,

$$|f(x) - p(x)| = |(\frac{f^{11}(\xi)}{11!})(\frac{1}{4})(\frac{1.6875 - 0}{10})^{11}(10!)$$

$f(x) = \sin(0.16875x)$, so

$$|f(x) - p(x)| = \max(-\frac{((0.16875)^{11}\cos(0.16875)(0))}{11!}, -\frac{((0.16875)^{11}\cos(0.16875)(1.6785))}{11!})$$

.

Since

$$| - ((0.16785)^{11}(\cos(0.16785)(x = 0)))| = 3.15996008e^{-9}$$

and

$$| - ((0.16785)^{11}(\cos(0.16785)(x = 1.6875)))| = 3.15867894e^{-9}$$

The maximum value of f^{11} is at $x = 0$. Then,

$$|(-\frac{(0.16875)^{11}(\cos(0.16875(1.6875)|))}{11!})\frac{1}{4}(0.16875)^{11}(10!) \leq 2.178e^{-19}$$

.

Then, the error bound of $|f(x) - p(x)| \leq 2.178(10^{-9})$ on $[0, 1.6875]$.