# Homework 8

Due date: April 9th, 2025

1.

2.

3. (a)

   (b)

4. (a) Q is orthogonal.

$$Q^tQ = 1$$

$$\det(Q^tQ) = 1$$

$$\det(Q^t)\det(Q) = 1$$

   Since $\det(Q^t) = \det(Q)$,

$$det(Q)^2 = 1$$

$$\det(Q) \pm 1$$

   (b)

$$(PQ)^tPQ = Q^tP^tPQ = Q^t(I)Q = Q^tQ = I$$

   . So $PQ$ is orthogonal.

5. (a)

   (b)

   (c)

6. We are given the symmetric matrix:

$$A = \begin{bmatrix} 1 & -1 & 2 \\ -1 & 1 & 2 \\ 2 & 2 & 2 \end{bmatrix}$$

   We want to decompose it as:

$$A = QDQ^T$$

   To find the eigenvalues, we solve the characteristic polynomial:

$$\det(A - \lambda I)$$

$$A - \lambda I = \begin{bmatrix} 1 & -1 & 2 \\ -1 & 1 & 2 \\ 2 & 2 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

$$= \begin{bmatrix} 1-\lambda & -1 & 2 \\ -1 & 1-\lambda & 2 \\ 2 & 2 & 2-\lambda \end{bmatrix}$$

We know can find the determinant of the matrix:

$$(1-\lambda)\begin{bmatrix} 1-\lambda & 2 \\ 2 & 2-\lambda \end{bmatrix} - (-1)\begin{bmatrix} -1 & 2 \\ 2 & 2-\lambda \end{bmatrix} + 2\begin{bmatrix} -1 & 1-\lambda \\ 2 & 2 \end{bmatrix} = -\lambda^3+4\lambda^2+4\lambda-16 = -(\lambda-4)(\lambda-2)(\lambda+2)$$

The eigenvalues are:

$$\lambda_1 = 4, \quad \lambda_2 = 2, \quad \lambda_3 = -2$$

Solve $(A - 4I)\vec{v} = 0$, we get:

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \quad \|\vec{v}_1\| = \sqrt{6} \Rightarrow q_1 = \frac{1}{\sqrt{6}}\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

Solve $(A - 2I)\vec{v} = 0$, we get:

$$\vec{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \quad \|\vec{v}_2\| = \sqrt{2} \Rightarrow q_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

Solve $(A + 2I)\vec{v} = 0$, we get:

$$\vec{v}_3 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \quad \|\vec{v}_3\| = \sqrt{3} \Rightarrow q_3 = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

Orthogonal matrix $Q$ (columns are the normalized eigenvectors):

$$Q = \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \frac{2}{\sqrt{6}} & 0 & -\frac{1}{\sqrt{3}} \end{bmatrix}$$

Diagonal matrix $D$ of eigenvalues:

$$D = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$