Homework 9

Due date: April 9th, 2025

1.

2. (a)

   (b)

   (c)

3.

4. Note that
$$A^t = VS^tU^t.$$
Since U and V are orthogonal, it suffices to show that the entries of S are the singular values of A^t. Since $[S^t]_{ij} = S_{ji} = 0$ whenever $i \neq j$, we can see that the entries of S^t are 0 everywhere except the main diagonal, where $[S^t]_{ii} = S_{ii}$ for all i. Therefore, it suffices to show that the singular values[1] of A^t are equal to the singular values of A. The singular values of A^t are given by the square root of the eigenvalues of $(A^t)^t A^t = (A^t A)^t$. However, we note that for any square matrix B, we have that the eigenvalues of B are equal to the eigenvalues of B^t. Therefore, the eigenvalues of $(A^t A)^t$ are equal to the eigenvalues of $A^t A$, which are precisely the square of the singular values of A. Therefore, the singular values of A and A^t are equal, and thus, $A^t = VS^tU^t$ is a singular value decomposition of A^t, which is the desired result.

5.

6. (a) Note that for all $y_1, y_2 \in [c, d]$ and $t \in [a, b]$, we have that
$$|f(t, y_1) - f(t, y_2)| = |ty_1 - ty_2| = |t||y_1 - y_2| \leq b|y_1 - y_2|.$$
Therefore, f satisfies the Lipschitz condition.

   (b) Separating variables, we see that
$$\int \frac{1}{y}\,dy = \int t\,dt \implies \ln(y) = \frac{t^2}{2} + c \implies y = e^{\frac{t^2}{2}+c} = Ce^{\frac{t^2}{2}}$$
for some constant C. Evaluating our initial condition gives
$$3 = y(0) = e^{\frac{0}{2}}C = C.$$
Therefore, $y(t) = 3e^{\frac{t^2}{2}}$.

   (c) As before, the general solution to $y_\varepsilon' = ty_\varepsilon$ is given by $y_\varepsilon(t) = Ce^{\frac{t^2}{2}}$ for some constant C. Evaluating at the initial condition gives
$$3 + \varepsilon = C.$$

[1]Note that the values are already decreasing by assumption that USV^t is a singular value decomposition.

Therefore, $y_\varepsilon(t) = (3+\varepsilon)e^{\frac{t^2}{2}}$. Note that

$$\lim_{t\to\infty} |y(t) - y_\varepsilon(t)| = \lim_{t\to\infty} \left|3e^{\frac{t^2}{2}} - (3+\varepsilon)e^{\frac{t^2}{2}}\right| = \lim_{t\to\infty} |\varepsilon|\, e^{\frac{t^2}{2}} = \infty.$$

Despite being a slight perturbation of the original solution, the error can still grow quite large out for large values of t.

TESTING