

Homework 7

Due date: March 13, 2025

1. Note that

$$\|A\|_\infty = \max\{|3| + |2| + |4|, |2| + |0| + |2|, |4| + |2| + |3|\} = \max\{9, 4, 9\} = 9.$$

Furthermore, we have that

$$\|A\|_1 = \max\{|3| + |2| + |4|, |2| + |0| + |2|, |4| + |2| + |3|\} = \max\{9, 4, 9\} = 9.$$

Finally, we compute the eigenvalues of A in the following way.

$$\begin{aligned} \det(A - \lambda I) &= \det \begin{pmatrix} 3 - \lambda & 2 & 4 \\ 2 & -\lambda & 2 \\ 4 & 2 & 3 - \lambda \end{pmatrix} = (3 - \lambda)(-\lambda(3 - \lambda) - 4) - 2(2(3 - \lambda) - 8) + 4(4 + 4\lambda) \\ &= -\lambda^3 + 6\lambda^2 + 15\lambda + 8 = -(\lambda + 1)^2(\lambda - 8). \end{aligned}$$

Therefore, the eigenvalues are given by the roots of $-(\lambda + 1)^2(\lambda - 8)$, which are $\lambda = -1$ and $\lambda = 8$. Since A is symmetric, we have that

$$\|A\|_2 = \rho(A) = \max\{|8|, |-1|\} = 8.$$

2. (a) *Proof.* Let $M_1 = \max_{\|x\| \neq 0} \frac{\|Ax\|}{\|x\|}$ and $M_2 = \|A\| = \max_{\|x\|=1} \|Ax\|$. It suffices to show that $M_1 = M_2$. Note that if $\|x\| \neq 0$, then by properties of norms, we have

$$\frac{\|Ax\|}{\|x\|} = \left\| A \frac{x}{\|x\|} \right\|.$$

Furthermore¹, since $\left\| \frac{x}{\|x\|} \right\| = \frac{1}{\|x\|} \|x\| = 1$ we must have that $M_1 \leq M_2$ by definition of M_2 . Furthermore, if we fix an arbitrary vector, x , such that $\|x\| = 1$. Then we must have

$$\|Ax\| = \left\| A \frac{x}{1} \right\| = \left\| A \frac{x}{\|x\|} \right\| = \frac{\|Ax\|}{\|x\|}$$

by definition of M_1 , we must have that $M_2 \leq M_1$, and therefore, $M_1 = M_2$, which was the desired result. ■

- (b) *Proof.* Note that by part (a), we have that for any vector $x \neq 0$,

$$\|A\| = \max_{y \neq 0} \frac{\|Ay\|}{\|y\|} \geq \frac{\|Ax\|}{\|x\|}.$$

Multiplying by $\|x\|$ gives that $\|Ax\| \leq \|A\|\|x\|$ for all $x \neq 0$. Furthermore, note that the inequality also holds when $x = 0$, since $Ax = 0$. Therefore,

$$\|Ax\| \leq \|A\|\|x\|$$

for all x . We then have that

$$\|AB\| = \max_{\|x\|=1} \|ABx\| \leq \max_{\|x\|=1} \|A\|\|Bx\| = \|A\| \max_{\|x\|=1} \|Bx\| = \|A\|\|B\|$$

which is the desired result. ■

¹It should be noted that $\frac{x}{\|x\|}$ is intended to mean $\frac{1}{\|x\|}x$, to agree with multiplying x by a scalar.

(c) *Proof.* By repeatedly applying part (b), we have that for all $k \in \mathbb{N}$

$$\|A^k\| = \|A \cdot A^{k-1}\| \leq \|A\| \|A^{k-1}\| = \|A\| \|A \cdot A^{k-2}\| \leq \|A\|^2 \|A^{k-2}\| \leq \dots \|A\|^k.$$

■

(d) *Proof.* Note that by part (b), we have

$$\|A\| \|A^{-1}\| \geq \|AA^{-1}\| = \|I\| = \max_{\|x\|=1} \|Ix\| = \max_{\|x\|=1} \|x\| = 1$$

which is the desired result. ■

3.

4. For the Gauss-Seidel method, we get the following after each iteration:

Iteration 1: $x[0] = 0.5000000, x[1] = 2.833333, x[2] = -1.083333$

Iteration 2: $x[0] = 1.916667, x[1] = 2.944444, x[2] = -1.027778$

Iteration 3: $x[0] = 1.972222, x[1] = 2.981481, x[2] = -1.009259$

Iteration 4: $x[0] = 1.990741, x[1] = 2.993827, x[2] = -1.003086$

Iteration 5: $x[0] = 1.996914, x[2] = 2.997942, x[3] = -1.001029$.

Leaving the final solution as $[1.996914, 2.997924, -1.001029]$

Here is the code that was used:

```
import numpy as np

def gauss_seidel(A, b, x0=None, iterations=5):
    n = len(b)
    x = np.zeros(n) if x0 is None else np.array(x0, dtype=float)

    for iteration in range(1, iterations + 1):
        print(f"Iteration {iteration}:")
        for i in range(n):
            sum1 = sum(A[i][j] * x[j] for j in range(i))
            sum2 = sum(A[i][j] * x[j] for j in range(i + 1, n))
            x[i] = (b[i] - sum1 - sum2) / A[i][i]
            print(f"x[{i}] = {x[i]:.6f}")
        print()

    return x

A = np.array([[2, -1, 0], [-1, 3, -1], [0, -1, 2]], dtype=float)
b = np.array([1, 8, -5], dtype=float)
x0 = [0, 0, 0]

solution = gauss_seidel(A, b, x0, iterations=5)
print("Final solution after 5 iterations:", solution)
```

For the Jacobi method, we got the following:

Iteration 1: $x[0] = 0.500000, x[1] = 2.666667, x[2] = -2.500000$

Iteration 2: $x[0] = 1.833333, x[1] = 2.000000, x[2] = -1.166667$

Iteration 3: $x[0] = 1.500000, x[1] = 2.888889, x[2] = -1.500000$

Iteration 4: $x[0] = 1.944444, x[1] = 2.666667, x[2] = -1.055556$

Iteration 5: $x[0] = 1.833333, x[1] = 2.962963, x[2] = -1.166667$

Leaving final solution $[1.833333, 2.962963, -1.166667]$

```
import numpy as np

def jacobi_method(A, b, x0=None, iterations=5):
    n = len(b)
    x = np.zeros(n) if x0 is None else np.array(x0, dtype=float)
    x_new = np.copy(x)

    for iteration in range(1, iterations + 1):
        print(f"Iteration {iteration}:")
        for i in range(n):
            sum1 = sum(A[i][j] * x[j] for j in range(n) if j != i)
            x_new[i] = (b[i] - sum1) / A[i][i]
            print(f"x[{i}] = {x_new[i]:.6f}")
        x[:] = x_new
        print()

    return x

A = np.array([[2, -1, 0], [-1, 3, -1], [0, -1, 2]], dtype=float)
b = np.array([1, 8, -5], dtype=float)
x0 = [0, 0, 0]

solution = jacobi_method(A, b, x0, iterations=5)
print("Final solution after 5 iterations:", solution)
```

5.

6. *Proof.* First, assume that A is strictly diagonally dominant. Namely, suppose that

$$|a_{ii}| > \sum_{j \neq i} |a_{ij}|$$

for all $1 \leq i \leq n$. Then, we must have that

$$\|D^{-1}(L + U)\|_{\infty} \leq \|D^{-1}\|_{\infty} \|L + U\|_{\infty} = \|D^{-1}\|_{\infty} \|D - A\|_{\infty}$$

where we used the fact that $A = D - (L + U)$. Furthermore, since $D = \text{diag}(a_{ii})^2$, we must have that

$$D^{-1} = \text{diag}\left(\frac{1}{a_{ii}}\right).$$

Therefore, since we have that

$$L + U = \begin{cases} a_{ij} & \text{if } i \neq j \\ 0 & \text{else} \end{cases},$$

we must have

$$[D^{-1}(L + U)]_{ij} = \begin{cases} \frac{a_{ij}}{a_{ii}} & \text{if } i \neq j \\ 0 & \text{else} \end{cases}.$$

Therefore, we have that

$$\|D^{-1}(L + U)\|_{\infty} = \max_{1 \leq i \leq n} \sum_{j=1}^n |[D^{-1}(L + U)]_{ij}| = \max_{1 \leq i \leq n} \sum_{j \neq i} \left| \frac{a_{ij}}{a_{ii}} \right| = \max_{1 \leq i \leq n} \frac{1}{|a_{ii}|} \sum_{j \neq i} |a_{ij}|.$$

Since A is strictly diagonally dominant, we have that

$$\|D^{-1}(L + U)\|_{\infty} = \max_{1 \leq i \leq n} \frac{1}{|a_{ii}|} \sum_{j \neq i} |a_{ij}| < \max_{1 \leq i \leq n} \frac{|a_{ii}|}{|a_{ii}|} = 1.$$

Therefore, $\|D^{-1}(L + U)\|_{\infty} < 1$.

Suppose that $\|D^{-1}(L + U)\|_{\infty} < 1$. As discussed before, we have that

$$\|D^{-1}(L + U)\|_{\infty} = \max_{1 \leq i \leq n} \frac{1}{|a_{ii}|} \sum_{j \neq i} |a_{ij}|.$$

Therefore, for all $1 \leq i \leq n$, we have that

$$\frac{1}{|a_{ii}|} \sum_{j \neq i} |a_{ij}| \leq \max_{1 \leq i \leq n} \frac{1}{|a_{ii}|} \sum_{j \neq i} |a_{ij}| < 1.$$

Thus, multiplying by a_{ii} gives that for all $1 \leq i \leq n$,

$$\sum_{j \neq i} |a_{ij}| < |a_{ii}|.$$

Therefore, A is strictly diagonally dominant, which completes the proof. ■

²I am going to make this assumption, since the theorem is not true without it. For example, if $A = I$, then clearly A is strictly diagonally dominant. However, if we let $D = \frac{1}{2}I$, then we must have $L + U = D - A = -\frac{1}{2}I$. This may be achieved with $L = U = -\frac{1}{4}I$. However, this means $\|D^{-1}(L + U)\|_{\infty} = \|2I(-\frac{1}{2}I)\|_{\infty} = \|-I\|_{\infty} = 1$ which is obviously not less than 1. I specify this because we only assumed that D was diagonal, L is lower triangular, and U is upper triangular. However, what I have written is a stronger condition.