## Homework 7

Due date: March 13, 2025

1. Note that

$$||A||_{\infty} = \max\{|3| + |2| + |4|, |2| + |0| + |2|, |4| + |2| + |3|\} = \max\{9, 4, 9\} = 9.$$

Furthermore, we have that

$$||A||_1 = \max\{|3| + |2| + |4|, |2| + |0| + |2|, |4| + |2| + |3|\} = \max\{9, 4, 9\} = 9.$$

Finally, we compute the eigenvalues of A in the following way.

$$\det(A - \lambda I) = \det\begin{pmatrix} 3 - \lambda & 2 & 4 \\ 2 & -\lambda & 2 \\ 4 & 2 & 3 - \lambda \end{pmatrix} = (3 - \lambda)(-\lambda(3 - \lambda) - 4) - 2(2(3 - \lambda) - 8) + 4(4 + 4\lambda)$$
$$= -\lambda^3 + 6\lambda^2 + 15\lambda + 8 = -(\lambda + 1)^2(\lambda - 8).$$

Therefore, the eigenvalues are given by the roots of  $-(\lambda + 1)^2(\lambda - 8)$ , which are  $\lambda = -1$  and  $\lambda = 8$ . Since A is symmetric, we have that

$$||A||_2 = \rho(A) = \max\{|8|, |-1|\} = 8.$$

2. (a) Proof. Let  $M_1 = \max_{||x|| \neq 0} \frac{||Ax||}{||x||}$  and  $M_2 = ||A|| = \max_{||x|| = 1} ||Ax||$ . It suffices to show that  $M_1 = M_2$ . Note that if  $||x|| \neq 0$ , then by properties of norms, we have

$$\frac{||Ax||}{||x||} = \left| \left| A \frac{x}{||x||} \right| \right|.$$

Furthermore<sup>1</sup>, since  $\left|\left|\frac{x}{||x||}\right|\right| = \frac{1}{||x||}||x|| = 1$  we must have that  $M_1 \leq M_2$  by definition of  $M_2$ . Furthermore, if we fix an arbitrary vector, x, such that ||x|| = 1. Then we must have

$$||Ax|| = \left| \left| A\frac{x}{1} \right| \right| = \left| \left| A\frac{x}{||x||} \right| \right| = \frac{||Ax||}{||x||}$$

by definition of  $M_1$ , we must have that  $M_2 \leq M_1$ , and therefore,  $M_1 = M_2$ , which was the desired result.

(b) *Proof.* Note that by part (a), we have that for any vector  $x \neq 0$ ,

$$||A|| = \max_{y \neq 0} \frac{||Ay||}{||y||} \ge \frac{||Ax||}{||x||}.$$

Multiplying by ||x|| gives that  $||Ax|| \le ||A|| ||x||$  for all  $x \ne 0$ . Furthermore, note that the inequality also holds when x = 0, since Ax = 0. Therefore,

$$||Ax|| \le ||A||||x||$$

for all x. We then have that

$$||AB|| = \max_{||x||=1||} ||ABx|| \leq \max_{||x||=1} ||A||||Bx|| = ||A||\max_{||x||=1} ||Bx|| = ||A||||B||$$

which is the desired result.

<sup>&</sup>lt;sup>1</sup>It should be noted that  $\frac{x}{||x||}$  is intended to mean  $\frac{1}{||x||}x$ , to agree with multiplying x by a scalar.

(c) *Proof.* By repetedly applying part (b), we have that for all  $k \in \mathbb{N}$ 

$$||A^k|| = ||A \cdot A^{k-1}|| \le ||A|| ||A^{k-1}|| = ||A|| ||A \cdot A^{k-2}|| \le ||A||^2 ||A^{k-2}|| \le \dots ||A||^k.$$

(d) *Proof.* Note that by part (b), we have

$$||A||||A^{-1}|| \geq ||AA^{-1}|| = ||I|| = \max_{||x||=1} ||Ix|| = \max_{||x||=1} ||x|| = 1$$

which is the desired result.

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4.

5.

6. Proof. First, assume that A is strictly diagonally dominant. Namely, suppose that

$$|a_{ii}| > \sum_{j \neq i} |a_{ij}|$$

for all  $1 \leq i \leq n$ . Then, we must have that

$$||D^{-1}(L+U)||_{\infty} \le ||D^{-1}||_{\infty}||L+U||_{\infty} = ||D^{-1}||_{\infty}||D-A||_{\infty}$$

where we used the fact that A = D - (L + U). Furthermore, since  $D = \text{diag}(a_{ii})^2$ , we must have that

$$D^{-1} = \operatorname{diag}\left(\frac{1}{a_{ii}}\right).$$

Therefore, since we have that

$$L + U = \begin{cases} a_{ij} & \text{if } i \neq j \\ 0 & \text{else} \end{cases},$$

we must have

$$[D^{-1}(L+U)]_{ij} = \begin{cases} \frac{a_{ij}}{a_{ii}} & \text{if } i \neq j \\ 0 & \text{else} \end{cases}.$$

Therefore, we have that

$$||D^{-1}(L+U)||_{\infty} = \max_{1 \le i \le n} \sum_{j=1}^{n} \left| [D^{-1}(L+U)]_{ij} \right| = \max_{1 \le i \le n} \sum_{j \ne i} \left| \frac{a_{ij}}{a_{ii}} \right| = \max_{1 \le i \le n} \frac{1}{|a_{ii}|} \sum_{j \ne i} |a_{ij}|.$$

 $<sup>\</sup>overline{\phantom{a}}^2$ I am going to make this assumption, since the theorem is not true without it. For example, if A=I, then clearly A is strictly diagonally dominant. However, if we let  $D=\frac{1}{2}I$ , then we must have  $L+U=D-A=-\frac{1}{2}I$ . This may be achieved with  $L=U=-\frac{1}{4}I$ . However, this means  $||D^{-1}(L+U)||_{\infty}=||2I\left(-\frac{1}{2}I\right)||_{\infty}=||-I||_{\infty}=1$  which is obviously not less than 1. I specify this because we only assumed that D was diagonal, L is lower triangular, and U is upper triangular. However, what I have written is a stronger condition.

Since A is strictly diagonally dominant, we have that

$$||D^{-1}(L+U)||_{\infty} = \max_{1 \le i \le n} \frac{1}{|a_{ii}|} \sum_{i \ne i} |a_{ij}| < \max_{1 \le i \le n} \frac{|a_{ii}|}{|a_{ii}|} = 1.$$

Therefore,  $||D^{-1}(L+U)||_{\infty} < 1$ .

Suppose that  $||D^{-1}(L+U)||_{\infty} < 1$ . As discussed before, we have that

$$||D^{-1}(L+U)||_{\infty} = \max_{1 \le i \le n} \frac{1}{|a_{ii}|} \sum_{j \ne i} |a_{ij}|.$$

Therefore, for all  $1 \leq i \leq n$ , we have that

$$\frac{1}{|a_{ii}|} \sum_{j \neq i} |a_{ij}| \le \max_{1 \le i \le n} \frac{1}{|a_{ii}|} \sum_{j \neq i} |a_{ij}| < 1.$$

Thus, multiplying by  $a_{ii}$  gives that for all  $1 \leq i \leq n$ ,

$$\sum_{j \neq i} |a_{ij}| < |a_{ii}|.$$

Therefore, A is strictly diagonally dominant, which completes the proof.