Homework 10

Due date: April 24th, 2025

1. We have matrix A as

$$\begin{bmatrix} 3 & 1 & 0 \\ 1 & 4 & 2 \\ 0 & 2 & 3 \end{bmatrix}$$

.

We want to find Q and R through the Gram-Schmidt process.

First we can start with a_1 as

$$\begin{bmatrix} 3 \\ 1 \\ 10 \end{bmatrix}$$

. And compute the norm

$$||a_1|| = \sqrt{(3^2 + 1^2 + 0^2)} = \sqrt{10}$$

. We can then normalize a_1 to get the first column of our matrix Q

$$q_1 = \frac{1}{\sqrt{10}} \begin{bmatrix} 3 \\ 1 \\ 10 \end{bmatrix} = \begin{bmatrix} \frac{3}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} \\ 0 \end{bmatrix}$$

.

We can then repeat this process for

$$a_2 = \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix}$$

However, in order to make a_2 orthogonal to q_1, we have to subtract the projection of a_2 onto q_1.

$$\text{proj}_{q1}(a_2) = (a_2 \cdot q_1)q_1.$$

First we can compute

$$a_2 \cdot q_1 = (1 \times \frac{3}{\sqrt{10}} + 4 \times \frac{1}{\sqrt{10}} + 2 \times 0) = \frac{3+4}{\sqrt{10}} = \frac{7}{\sqrt{10}}$$

Now, we can compute the projection

$$\text{proj}_{q_1}(a_2) = \frac{7}{\sqrt{10}} \begin{bmatrix} \frac{3}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{21}{10} \\ \frac{7}{10} \\ 0 \end{bmatrix}$$

And now we can subtract the projection from a_2 to get u_2.

$$u_2 = \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix} - \begin{bmatrix} \frac{21}{10} \\ \frac{7}{10} \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{21}{10} \\ \frac{7}{10} \\ 9 \end{bmatrix}$$

.

We now must normalize u_2 to get

$$||u_2|| = \sqrt{\frac{-11}{10}^2 + \frac{33}{10}^2 + 2^2} = \sqrt{\frac{121}{100} + \frac{1089}{100} + 4} = \sqrt{\frac{1610}{100}} = \frac{\sqrt{1610}}{10}.$$

Then, we can normalize u_2 to get

$$q_2 = \frac{10}{\sqrt{1610}} \begin{bmatrix} \frac{-11}{10} \\ \frac{33}{10} \\ 2 \end{bmatrix} = \begin{bmatrix} \frac{-11}{\sqrt{1610}} \\ \frac{33}{\sqrt{1610}} \\ \frac{21}{\sqrt{1610}} \end{bmatrix}$$

.

We can take the third column of A and repeat.

$$a_3 \cdot q_1 = 0 \times \frac{3}{\sqrt{10}} + 2\frac{1}{\sqrt{10}} + 3 \times 0 = \frac{2}{\sqrt{10}}$$

$$\text{proj}_{q_1}(a_3) = \frac{2}{\sqrt{10}} \begin{bmatrix} \frac{3}{\sqrt{(10)}} \\ \frac{1}{\sqrt{10}} \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{6}{10} \\ \frac{2}{10} \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{3}{5} \\ \frac{1}{5} \\ 0 \end{bmatrix}$$

$$q_2 = \begin{bmatrix} \frac{-11}{\sqrt{1610}} \\ \frac{33}{\sqrt{1610}} \\ \frac{20}{\sqrt{1610}} \end{bmatrix}$$

$$a_3 \cdot q_2 = 0 \times \frac{-11}{\sqrt{1610}} + 2 \times \frac{33}{\sqrt{1610}} + 3 \times \frac{20}{\sqrt{1610}} = \frac{126}{\sqrt{1610}}$$

.

$$\text{proj}_{q_2}(a_3) = \frac{126}{\sqrt{1610}} = \begin{bmatrix} \frac{-11}{\sqrt{1610}} \\ \frac{33}{\sqrt{1610}} \\ \frac{20}{\sqrt{1610}} \end{bmatrix} = \begin{bmatrix} \frac{-1386}{1610} \\ \frac{4158}{1610} \\ \frac{2520}{1610} \end{bmatrix} = \begin{bmatrix} \frac{-693}{805} \\ \frac{2079}{805} \\ \frac{1260}{805} \end{bmatrix}$$

$$u_3 = \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix} - \begin{bmatrix} \frac{3}{5} \\ \frac{1}{5} \\ 0 \end{bmatrix} - \begin{bmatrix} \frac{-693}{805} \\ \frac{2079}{805} \\ \frac{1260}{805} \end{bmatrix}$$

so

$$u_3 = \begin{bmatrix} \frac{210}{805} \\ \frac{-1934}{805} \\ \frac{1155}{805} \end{bmatrix}$$

.

$$||u_3|| = \frac{210^2 + 621^2 + 1155^2}{805^2} = \frac{44100 + 385641 + 1334025}{805^2} = \frac{1768766}{805^2}$$

$$||u_3|| = \frac{\sqrt{1768766}}{805}$$

$$q_3 = \begin{bmatrix} \frac{210}{\sqrt{1768766}} \\ \frac{-621}{\sqrt{1768766}} \\ \frac{1155}{\sqrt{1768766}} \end{bmatrix}$$

.

So

$$Q = \begin{bmatrix} \frac{3}{\sqrt{10}} & \frac{-11}{\sqrt{1610}} & \frac{210}{\sqrt{1768766}} \\ \frac{1}{\sqrt{10}} & \frac{33}{\sqrt{1610}} & \frac{-621}{\sqrt{1768766}} \\ 0 & \frac{20}{\sqrt{1610}} & \frac{1155}{\sqrt{1768766}} \end{bmatrix}$$

$$R = \begin{bmatrix} \sqrt{10} & \frac{7}{\sqrt{10}} & \frac{2}{\sqrt{10}} \\ 0 & \frac{\sqrt{1610}}{10} & \frac{126}{\sqrt{1610}} \\ 0 & 0 & \frac{\sqrt{1768766}}{805} \end{bmatrix}$$

So we can now finally get $A^{(2)}$

$$A^{(2)} = RQ = \begin{bmatrix} 3 & 1 & 0 \\ 1 & 4 & 2 \\ 0 & 2 & 3 \end{bmatrix}$$

.

And since $A^{(2)} = A$, and A is a symmetric matrix, QR-iteration does not alter the matrix. So, $A^3 = A^2 = A$.

2. (a) $A^{(1)} = Q^{(1)}R^{(1)}$. To find $Q^{(1)}$:

$$v_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad ||v_1|| = \sqrt{0^2 + 1^2} = 1, q_1 = \frac{v_1}{||v_1||} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$v_2' = v_2 - (v_2 \cdot q_1)q_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} - (\begin{bmatrix} 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix}) \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} - 0 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$||v_2'|| = \sqrt{1^2 + 0^2} = 1, v_2 = \frac{v_2'}{||v_2'||} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$Q^{(1)} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$R^{(1)} = Q^T A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A^{(2)} = R^{(1)}Q^{(1)} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Since $A^{(2)} = A$, the QR decomposition for $A^{(3)}$ will be the same as for $A^{(2)}$. So

$$A^{(3)} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$A^{(2)}$ and $A^{(3)}$ are both equal to the original matrix A.

(b) In this case, the QR method does not calculate a diagonalization of A because the eigenvalues of A have the same magnitude. The characteristic polynomial is given by $det(A - \lambda I)$, where I is the identity matrix.

$$det(A - \lambda I) = det \begin{pmatrix} -\lambda & 1 \\ 1 & -\lambda \end{pmatrix} = (-\lambda)(-\lambda) - (1)(1) = \lambda^2 - 1$$

so λ is either 1 or -1. Since the magnitudes of the eigenvalues are non-distinct, A does not converge to a diagonal matrix.

(c) See part (b).

3.

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

$$AA^T = \begin{bmatrix} 2 & 1 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 3 & 1 \\ 3 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Eigenvalues:

- $\lambda_1 = 4 - \sqrt{10}$, eigenvector:

$$\vec{u}_1 = \begin{bmatrix} \frac{-5+2\sqrt{10}}{3} \\ \frac{-4+\sqrt{10}}{3} \\ 1 \end{bmatrix}$$

- $\lambda_2 = 4 + \sqrt{10}$, eigenvector:

$$\vec{u}_2 = \begin{bmatrix} \frac{5+2\sqrt{10}}{3} \\ \frac{\sqrt{10}+4}{3} \\ 1 \end{bmatrix}$$

- $\lambda_3 = 0$, eigenvector:

$$\vec{u}_3 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

Singular values W:

$$\sigma_1 = \sqrt{4 - \sqrt{10}}, \quad \sigma_2 = \sqrt{4 + \sqrt{10}}$$

Let:

$$U = [\vec{u}_1 \ \vec{u}_2 \ \vec{u}_3], \quad \Sigma = \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \\ 0 & 0 \end{bmatrix}$$

$$U = \begin{bmatrix} \frac{3-\sqrt{10}}{6} & \frac{3+\sqrt{10}}{6} & \frac{1}{\sqrt{6}} \\ \frac{\sqrt{10}-2}{2\sqrt{6}} & \frac{-\sqrt{10}-2}{2\sqrt{6}} & \frac{-2}{\sqrt{6}} \\ \frac{2\sqrt{2}}{\sqrt{15-\sqrt{10}}} & \frac{2\sqrt{2}}{\sqrt{15+\sqrt{10}}} & \frac{1}{\sqrt{6}} \end{bmatrix}$$

And for V:

$$\vec{v}_i = \frac{1}{\sigma_i} A^T \vec{u}_i$$

Using $v_i = \frac{1}{\sigma_i} A^T u_i$, we get:

$$V = \begin{bmatrix} \frac{14-5\sqrt{10}}{2\sqrt{170-58\sqrt{10}}} & \frac{14+5\sqrt{10}}{2\sqrt{170+58\sqrt{10}}} \\ \frac{3\sqrt{10}-10}{2\sqrt{170-58\sqrt{10}}} & \frac{3\sqrt{10}+10}{2\sqrt{170+58\sqrt{10}}} \end{bmatrix}$$

Where each column is normalized.

So

$$A = U\Sigma V^T$$

4. Note that

$$A^t = VS^tU^t.$$

Since U and V are orthogonal, it suffices to show that the entries of S are the singular values of A^t. Since $[S^t]_{ij} = S_{ji} = 0$ whenever $i \neq j$, we can see that the entries of S^t are 0 everywhere except the main diagonal, where $[S^t]_{ii} = S_{ii}$ for all i. Therefore, it suffices to show that the singular values[1] of A^t are equal to the singular values of A. The singular values of A^t are given by the square root of the eigenvalues of $(A^t)^t A^t = (A^t A)^t$. However, we note that for any square matrix B, we have that the eigenvalues of B are equal to the eigenvalues of B^t. Therefore, the eigenvalues of $(A^t A)^t$ are equal to the eigenvalues of $A^t A$, which are precisely the square of the singular values of A. Therefore, the singular values of A and A^t are equal, and thus, $A^t = VS^tU^t$ is a singular value decomposition of A^t, which is the desired result.

5.

$$y' = \frac{t}{y^2}, y(0) = 1$$

$$y^2 y' = t$$

$$y^2 dy = t dt$$

[1]Note that the values are already decreasing by assumption that USV^t is a singular value decomposition.

$$\int y^2 dy = \int t dt$$

$$y^3 = \frac{3}{2}t^2 + C$$

Since $y(0) = 1, 1^3 = \frac{3}{2} + 0 + C$, so $C = 1$, so $y^3 = \frac{3}{2}t^3 + 1$, then $y = (\frac{3}{2}t^3 + 1)^{\frac{1}{3}}$

6. (a) Note that for all $y_1, y_2 \in [c, d]$ and $t \in [a, b]$, we have that

$$|f(t, y_1) - f(t, y_2)| = |ty_1 - ty_2| = |t||y_1 - y_2| \leq b|y_1 - y_2|.$$

Therefore, f satisfies the Lipschitz condition.

(b) Separating variables, we see that

$$\int \frac{1}{y}\, dy = \int t\, dt \implies \ln(y) = \frac{t^2}{2} + c \implies y = e^{\frac{t^2}{2}+c} = Ce^{\frac{t^2}{2}}$$

for some constant C. Evaluating our initial condition gives

$$3 = y(0) = e^{\frac{0}{2}}C = C.$$

Therefore, $y(t) = 3e^{\frac{t^2}{2}}$.

(c) As before, the general solution to $y'_\varepsilon = ty_\varepsilon$ is given by $y_\varepsilon(t) = Ce^{\frac{t^2}{2}}$ for some constant C. Evaluating at the initial condition gives

$$3 + \varepsilon = C.$$

Therefore, $y_\varepsilon(t) = (3 + \varepsilon)e^{\frac{t^2}{2}}$. Note that

$$\lim_{t\to\infty} |y(t) - y_\varepsilon(t)| = \lim_{t\to\infty} \left|3e^{\frac{t^2}{2}} - (3 + \varepsilon)e^{\frac{t^2}{2}}\right| = \lim_{t\to\infty} |\varepsilon|\, e^{\frac{t^2}{2}} = \infty.$$

Despite being a slight perturbation of the original solution, the error can still grow quite large out for large values of t.