Homework 4

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1. (a) We are given the points $(0,7),(2,11),(3,28),(4,63)$, so $f(x_0)=7, f(x_1)=11, f(x_2)=28,\ f(x_3)=63$.

The first order divided differences are as follows:

$$f[x_0,x_1]=\frac{7-11}{0-2}=2$$

$$f[x_1,x_2]=\frac{28-11}{3-2}=17$$

$$f[x_2,x_3]=\frac{63-28}{4-3}=35$$

The second order differences are as follows:

$$f[x_0,x_1,x_2]=\frac{17-2}{3-0}=5$$

$$f[x_1,x_2,x_3]=\frac{35-17}{4-2}=4$$

And the third order difference is:

$$f[x_0,x_1,x_2,x_3]=\frac{9-5}{4-0}=1$$

Using the Newton formula,

$$p_n(x)=f[x_0]+f[x_0,x_1](x-x_0)+f[x_0,x_1,x_2](x-x_0)(x-x_1)+...$$

We can write p_1 as

$$p_1(x)=7+2(x-0)=7+2x$$

p_2 as

$$p_2(x)=7+2x+5(x-0)(x-2)=5x^2-8x+7$$

and p_3 as

$$p_3(x)=5x^2-8x+7+1(x-0)(x-2)(x-3)=x^3-2x+7$$

(b)

$$\int_0^4(x^3-2x+7)dx$$

$$=\int_0^4 x^3dx-\int_0^4 2xdx+\int_0^4 7dx$$

$$=(\frac{x^4}{4}-x^2+7)|_0^4$$

$$=(64-16+28)-0=76$$

2. (a) We know that the formula for the k^{th} divided difference is as follows:

$$f[x_0, x_1, ..., x_k] = \frac{f[x_1, x_2, ..., x_k] - f[x_0, x_1, ..., x_{k-1}]}{x_k - x_0}$$

Given the table, we can calculate the first divided differences as

$$f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{4 - 1}{1 - (-2)} = \frac{3}{1} = 3$$

$$f[x_1, x_2] = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{11 - 4}{0 - (-1)} = \frac{7}{1} = 7$$

$$f[x_2, x_3] = \frac{f(x_3) - f(x_2)}{x_3 - x_2} = \frac{16 - 11}{1 - 0} = \frac{5}{1} = 5$$

$$f[x_3, x_4] = \frac{f(x_4) - f(x_3)}{x_4 - x_3} = \frac{13 - 16}{2 - 1} = \frac{-3}{1} = -3$$

$$f[x_4, x_5] = \frac{f(x_5) - f(x_4)}{x_5 - x_4} = \frac{-4 - 13}{3 - 2} = \frac{-17}{1} = -17$$

The second divided differences are:

$$f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0} = \frac{7 - 3}{0 - (-2)} = \frac{4}{2} = 2$$

$$f[x_1, x_2, x_3] = \frac{f[x_2, x_3] - f[x_1, x_2]}{x_3 - x_1} = \frac{5 - 7}{1 - (-1)} = \frac{-2}{2} = -1$$

$$f[x_2.x_3, x_4] = \frac{f[x_3, x_4] - f[x_2, x_3]}{x_4 - x_2} = \frac{-3 - 5}{2 - 0} = \frac{-8}{2} = -4$$

$$f[x_3, x_4, x_5] = \frac{f[x_4, x_5] - f[x_3, x_4]}{x_5 - x_3} = \frac{-17 - (-3)}{3 - 1} = \frac{-14}{2} = -7$$

The third divided differences are

$$f[x_0, x_1, x_2, x_3] = \frac{f[x_1, x_2, x_3] - f[x_0, x_1, x_2]}{x_3 - x_0} = \frac{1 - 2}{1 - (-2)} = \frac{-3}{3} = -1$$

$$f[x_1, x_2, x_3, x_4] = \frac{f[x_2, x_3, x_4] - f[x_1, x_2, x_3]}{x_4 - x_1} = \frac{-4 - (-1)}{2 - (-1)} = \frac{-3}{3} = -1$$

$$f[x_2, x_3, x_4, x_5] = \frac{f[x_3, x_4, x_5] - f[x_2, x_3, x_4]}{x_5 - x_2} = \frac{-7 - (-4)}{3 - 0} = \frac{-3}{3} = -1$$

The fourth divided differences are:

$$f[x_0, x_1, x_2, x_3, x_4] = \frac{f[x_1, x_2, x_3, x_4] - f[x_0, x_1, x_2, x_3]}{x_4 - x_0} = \frac{-1 - (-1)}{2 - (-2)} = \frac{0}{4} = 0$$

$$f[x_1, x_2, x_3, x_4, x_5] = \frac{f[x_2, x_3, x_4, x_5] - f[x_1, x_2, x_3, x_4]}{x_5 - x_1} = \frac{-1 - (-1)}{3 - (-1)} = \frac{0}{4} = 0$$

(b) We will construct the Newton Interpolation formula

$$P(x) = f[x_0] + \sum_{k=1}^{n} f[x_0, x_1, ..., x_k](x - x_1)(x - x_2)...(x - x_{k-1})$$

using the divided differences calculated in (a).

$$P(x) = 1 + 3(x + 2) + 2(x + 2)(x + 1) - 1(x + 2)(x + 1)x$$

$$P(x) = 1 + 3x + 6 + 2x^2 + 6x + 4 - x^3 - 3x^2 - 2x$$
$$P(x) = -x^3 + (2x^2 - 3x^2) + (3x + 6x - 2x) + (1 + 6 + 4)$$
$$P(x) = -x^3 - x^2 + 7x + 11$$

.

3.

4. (a)

(b)

5. A natural cubic spline has second derivatives equal to 0 at the endpoints. The second derivative of a cubic polynomial is a linear function. Let $f(x) = ax^3+bx^2+cx+d$ be a general cubic polynomial, where $a \neq 0$. The first derivative is $f'(x) = 3ax^2+2bx+c$. The second derivative is $f''(x) = 6ax + 2b$. For $f(x)$ to be its own natural cubic spline, $f''(x_0) = 0$ and $f''(x_1) = 0$. This means $6ax_0 + 2b = 0$ and $6ax_1 + 2b = 0$. Subtracting the two equations, we get $6a(x_1 - x_0) = 0$. Since $x_1 \neq x_0$, it must be that $a = 0$. If $a = 0$, then $2b = 0$, so $b = 0$. This contradicts the assumption that $f(x)$ is a cubic polynomial ($a \neq 0$). Therefore, $f(x)$ cannot be its own natural cubic spline unless $f''(x)$ is identically zero, which means $f(x)$ is at most a linear function.

6. (a) *Proof.* Note that by the triangle inequality, we have

$$|m_{kk}x_k| = \left|\sum_{j\neq k} m_{kj}x_j\right| \leq \sum_{j\neq k} |m_{kj}||x_j|.$$

By definition, k was chosen such that $|x_k| = \max_{1\leq j\leq n} |x_j|$. Therefore, since $|m_{kj}|$ is non-negative for all $j \neq k$, we must have

$$|m_{kk}||x_k| = |m_{kk}x_k| \leq \sum_{j\neq k} |m_{kj}||x_j| \leq \sum_{j\neq k} |m_{kj}||x_k| = |x_k| \sum_{j\neq k} |m_{kj}|.$$

Dividing by $|x_k|$ gives

$$|m_{kk}| \leq \sum_{j\neq k} |m_{kj}|$$

which is the desired result. □

(b) We may first assume that $h_i > 0$ for all i, since we may simply choose the nodes x_i in increasing order[1]. Note that the matrix given by

$$A_{ij} = \begin{cases} 2(h_i + h_{i+1}) & \text{if } i = j \\ h_i & \text{if } j = i-1 \text{ or } j = i+1 \\ 0 & \text{otherwise} \end{cases}$$

satisfies the following for each i:

$$|A_{ii}| = |2(h_i + h_{i+1})| = 2(h_i + h_{i+1}) > 2h_i = \sum_{j \neq i} A_{ij}.$$

Therefore, A is strictly diagonally dominant, and thus invertible.

Furthermore, note that since $a_j = f(x_j)$, the constant coefficients, a_j are uniquely determined[2] by f. Since A is invertible, there is a unique solution to the system

$$A\mathbf{x} = \mathbf{b}$$

where

$$\mathbf{x} = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_{n-1} \end{pmatrix} \qquad \text{and} \qquad \mathbf{b} = \begin{pmatrix} \frac{3}{h_1}(a_2 - a_1) - \frac{3}{h_0}(a_1 - a_0) \\ \frac{3}{h_2}(a_3 - a_2) - \frac{3}{h_1}(a_2 - a_1) \\ \vdots \\ \frac{3}{h_{n-1}}(a_n - a_{n-1}) - \frac{3}{h_{n-2}}(a_{n-1} - a_{n-2}) \end{pmatrix}.$$

Therefore, since a_j is uniquely determined by f, so is[3] c_j, and by extension

$$b_j := \frac{1}{h_j}(a_{j+1} - a_j) - \frac{h_j}{3}(c_{j+1} + 2c)$$

and

$$d_j := \frac{1}{3h_j}(c_{j+1} - c_j)$$

are also uniquely determined by f. Therefore, the family $\{S_j\}_{0 \leq j \leq n-1}$ is the unique cubic spline on f, which is the desired result.

[1] Also note that we assume each node is distinct, therefore, $h_i \neq 0$ for all i

[2] That is to say, for the family $\{S_j\}_{0 \leq j \leq n-1}$ to be a cubic spline on f, the conditions derived in class must be satisfied. Showing that the coefficients which satisfy these conditions are unique, then shows that the cubic spline is unique. This will be assumed from now on.

[3] We also note that $c_0 = c_n = 0$, so they are also unique.