Homework 4

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1. (a) We are given the points $(0,7),(2,11),(3,28),(4,63)$, so $f(x_0)=7, f(x_1)=11, f(x_2)=28,\ f(x_3)=63$.

The first order divided differences are as follows:

$$f[x_0,x_1]=\frac{7-11}{0-2}=2$$

$$f[x_1,x_2]=\frac{28-11}{3-2}=17$$

$$f[x_2,x_3]=\frac{63-28}{4-3}=35$$

The second order differences are as follows:

$$f[x_0,x_1,x_2]=\frac{17-2}{3-0}=5$$

$$f[x_1,x_2,x_3]=\frac{35-17}{4-2}=4$$

And the third order difference is:

$$f[x_0,x_1,x_2,x_3]=\frac{9-5}{4-0}=1$$

Using the Newton formula,

$$p_n(x)=f[x_0]+f[x_0,x_1](x-x_0)+f[x_0,x_1,x_2](x-x_0)(x-x_1)+...$$

We can write p_1 as

$$p_1(x)=7+2(x-0)=7+2x$$

p_2 as

$$p_2(x)=7+2x+5(x-0)(x-2)=5x^2-8x+7$$

and p_3 as

$$p_3(x)=5x^2-8x+7+1(x-0)(x-2)(x-3)=x^3-2x+7$$

(b)

$$\int_0^4 (x^3-2x+7)dx$$

$$=\int_0^4 x^3dx-\int_0^4 2xdx+\int_0^4 7dx$$

$$=(\frac{x^4}{4}-x^2+7)|_0^4$$

$$=(64-16+28)-0=76$$

2. (a) We know that the formula for the k^{th} divided difference is as follows:

$$f[x_0, x_1, ..., x_k] = \frac{f[x_1, x_2, ..., x_k] - f[x_0, x_1, ..., x_{k-1}]}{x_k - x_0}$$

Given the table, we can calculate the first divided differences as

$$f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{4-1}{1-(-2)} = \frac{3}{1} = 3$$

$$f[x_1, x_2] = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{11-4}{0-(-1)} = \frac{7}{1} = 7$$

$$f[x_2, x_3] = \frac{f(x_3) - f(x_2)}{x_3 - x_2} = \frac{16-11}{1-0} = \frac{5}{1} = 5$$

$$f[x_3, x_4] = \frac{f(x_4) - f(x_3)}{x_4 - x_3} = \frac{13-16}{2-1} = \frac{-3}{1} = -3$$

$$f[x_4, x_5] = \frac{f(x_5) - f(x_4)}{x_5 - x_4} = \frac{-4-13}{3-2} = \frac{-17}{1} = -17$$

The second divided differences are:

$$f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0} = \frac{7-3}{0-(-2)} = \frac{4}{2} = 2$$

$$f[x_1, x_2, x_3] = \frac{f[x_2, x_3] - f[x_1, x_2]}{x_3 - x_1} = \frac{5-7}{1-(-1)} = \frac{-2}{2} = -1$$

$$f[x_2.x_3, x_4] = \frac{f[x_3, x_4] - f[x_2, x_3]}{x_4 - x_2} = \frac{-3-5}{2-0} = \frac{-8}{2} = -4$$

$$f[x_3, x_4, x_5] = \frac{f[x_4, x_5] - f[x_3, x_4]}{x_5 - x_3} = \frac{-17-(-3)}{3-1} = \frac{-14}{2} = -7$$

The third divided differences are

$$f[x_0, x_1, x_2, x_3] = \frac{f[x_1, x_2, x_3] - f[x_0, x_1, x_2]}{x_3 - x_0} = \frac{1-2}{1-(-2)} = \frac{-3}{3} = -1$$

$$f[x_1, x_2, x_3, x_4] = \frac{f[x_2, x_3, x_4] - f[x_1, x_2, x_3]}{x_4 - x_1} = \frac{-4-(-1)}{2-(-1)} = \frac{-3}{3} = -1$$

$$f[x_2, x_3, x_4, x_5] = \frac{f[x_3, x_4, x_5] - f[x_2, x_3, x_4]}{x_5 - x_2} = \frac{-7-(-4)}{3-0} = \frac{-3}{3} = -1$$

The fourth divided differences are:

$$f[x_0, x_1, x_2, x_3, x_4] = \frac{f[x_1, x_2, x_3, x_4] - f[x_0, x_1, x_2, x_3]}{x_4 - x_0} = \frac{-1-(-1)}{2-(-2)} = \frac{0}{4} = 0$$

$$f[x_1, x_2, x_3, x_4, x_5] = \frac{f[x_2, x_3, x_4, x_5] - f[x_1, x_2, x_3, x_4]}{x_5 - x_1} = \frac{-1-(-1)}{3-(-1)} = \frac{0}{4} = 0$$

(b) We will construct the Newton Interpolation formula

$$P(x) = f[x_0] + \sum_{k=1}^{n} f[x_0, x_1, ..., x_k](x - x_1)(x - x_2)...(x - x_{k-1})$$

using the divided differences calculated in (a).

$$P(x) = 1 + 3(x + 2) + 2(x + 2)(x + 1) - 1(x + 2)(x + 1)x$$

$$P(x) = 1 + 3x + 6 + 2x^2 + 6x + 4 - x^3 - 3x^2 - 2x$$
$$P(x) = -x^3 + (2x^2 - 3x^2) + (3x + 6x - 2x) + (1 + 6 + 4)$$
$$P(x) = -x^3 - x^2 + 7x + 11$$

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3. **Cubic Spline Form**

Each spline segment $S_j(x)$ is a cubic polynomial of the form:

$$S_j(x) = a_j + b_j(x - x_j) + c_j(x - x_j)^2 + d_j(x - x_j)^3, \quad x_j \le x \le x_{j+1}.$$

Natural Cubic Spline Conditions

- The spline must pass through each given point:

$$S_j(x_j) = f(x_j), \quad S_j(x_{j+1}) = f(x_{j+1}).$$

- The first derivatives must be continuous at each interior point:

$$S'_j(x_{j+1}) = S'_{j+1}(x_{j+1}).$$

- The second derivatives must also be continuous:

$$S''_j(x_{j+1}) = S''_{j+1}(x_{j+1}).$$

- The second derivatives at the endpoints are zero:

$$S''_0(x_0) = 0, \quad S''_{n-1}(x_n) = 0.$$

For simplicity, we set $h_j = x_{j+1} - x_j$. Since the given points are equally spaced, then h_j is constant ($h_1 = h_2 = h_3 = h_4 = 1$).

System of Equations for c_j

Using the second derivative continuity condition and the natural spline assumption, we get a tridiagonal system:

$$2(h_{j-1}+h_j)c_j + h_j c_{j+1} + h_{j-1}c_{j-1} = \frac{3}{h_j}(a_{j+1}-a_j) - \frac{3}{h_{j-1}}(a_j - a_{j-1}).$$

Since $h_j = h_{j-1} = 1$, this simplifies to:

$$3(a_{j+1}-a_j) - 3(a_j - a_{j-1}).$$

For the interior points where the second derivative is continuous ($j = 2, 3, 4$): For $j = 2$

$$c_1 + 4c_2 + c_3 = 3(2 - 2(1) + 3) = 9.$$

For $j = 3$

$$c_2 + 4c_3 + c_4 = 3(3 - 2(2) + 1) = 3(0) = 0.$$

For $j = 4$

$$c_3 + 4c_4 + c_5 = 3(2 - 2(3) + 2) = 3(-2) = -6.$$

Since this is a natural cubic spline, $c_1 = 0, c_5 = 0$.
This simplifies the system of equations to:

$$\begin{aligned} 4c_2 + c_3 &= 9, \\ c_2 + 4c_3 + c_4 &= 0, \\ c_3 + 4c_4 &= -6. \end{aligned}$$

Now solving:

For c_4:

$$c_4 = -\frac{6 - c_3}{4}.$$

Substituting into the second equation:

$$c_2 + 4c_3 + \frac{-6 - c_3}{4} = 0.$$

Multiplying by 4:

$$4c_2 + 16c_3 - 6 - c_3 = 0.$$

$$4c_2 + 15c_3 = 6.$$

For c_3:

$$c_2 = \frac{6 - 15c_3}{4}.$$

Substituting into the first equation:

$$4\left(\frac{6 - 15c_3}{4}\right) + c_3 = 9.$$

$$6 - 15c_3 + c_3 = 9.$$

$$-14c_3 = 3.$$

$$c_3 = -\frac{3}{14}.$$

Solve for c_4:

$$c_4 = \frac{-6 - (-3/14)}{4} = \frac{-6 + 3/14}{4} = \frac{-84 + 3}{56} = \frac{-81}{56}.$$

Solve for c_2:

$$c_2 = \frac{6 - 15(-3/14)}{4} = \frac{6 + 45/14}{4} = \frac{84 + 45}{56} = \frac{129}{56}.$$

This yields the final values:

$$c_1 = 0, \quad c_2 = \frac{129}{56}, \quad c_3 = -\frac{3}{14}, \quad c_4 = -\frac{81}{56}, \quad c_5 = 0.$$

4. (a)

   (b)

5. A natural cubic spline has second derivatives equal to 0 at the endpoints. The second derivative of a cubic polynomial is a linear function. Let $f(x) = ax^3 + bx^2 + cx + d$ be a general cubic polynomial, where $a \neq 0$. The first derivative is $f'(x) = 3ax^2 + 2bx + c$. The second derivative is $f''(x) = 6ax + 2b$. For $f(x)$ to be its own natural cubic spline, $f''(x_0) = 0$ and $f''(x_1) = 0$. This means $6ax_0 + 2b = 0$ and $6ax_1 + 2b = 0$. Subtracting the two equations, we get $6a(x_1 - x_0) = 0$. Since $x_1 \neq x_0$, it must be that $a = 0$. If $a = 0$, then $2b = 0$, so $b = 0$. This contradicts the assumption that $f(x)$ is a cubic polynomial ($a \neq 0$). Therefore, $f(x)$ cannot be its own natural cubic spline unless $f''(x)$ is identically zero, which means $f(x)$ is at most a linear function.

6. (a) *Proof.* Note that by the triangle inequality, we have

$$|m_{kk}x_k| = \left|\sum_{j\neq k} m_{kj}x_j\right| \leq \sum_{j\neq k} |m_{kj}||x_j|.$$

By definition, k was chosen such that $|x_k| = \max_{1\leq j\leq n} |x_j|$. Therefore, since $|m_{kj}|$ is non-negative for all $j \neq k$, we must have

$$|m_{kk}||x_k| = |m_{kk}x_k| \leq \sum_{j\neq k} |m_{kj}||x_j| \leq \sum_{j\neq k} |m_{kj}||x_k| = |x_k| \sum_{j\neq k} |m_{kj}|.$$

Dividing by $|x_k|$ gives

$$|m_{kk}| \leq \sum_{j\neq k} |m_{kj}|$$

which is the desired result. $\square$

(b) We may first assume that $h_i > 0$ for all i, since we may simply choose the nodes x_i in increasing order[1]. Note that the matrix given by

$$A_{ij} = \begin{cases} 2(h_i + h_{i+1}) & \text{if } i = j \\ h_i & \text{if } j = i-1 \text{ or } j = i+1 \\ 0 & \text{otherwise} \end{cases}$$

satisfies the following for each $i \neq 1$ and $i \neq n-1$:

$$|A_{ii}| = |2(h_i + h_{i+1})| = 2(h_i + h_{i+1}) > 2h_i = \sum_{j\neq i} A_{ij}.$$

If $i = 1$ or $i = n-1$, then we write that $2h_i > h_i = \sum_{j\neq i} A_{ij}$. Therefore, A is strictly diagonally dominant, and thus invertible.

Furthermore, note that since $a_j = f(x_j)$, the constant coefficients, a_j are uniquely determined[2] by f. Since A is invertible, there is a unique solution to the system

$$A\mathbf{x} = \mathbf{b}$$

where

$$\mathbf{x} = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_{n-1} \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} \frac{3}{h_1}(a_2 - a_1) - \frac{3}{h_0}(a_1 - a_0) \\ \frac{3}{h_2}(a_3 - a_2) - \frac{3}{h_1}(a_2 - a_1) \\ \vdots \\ \frac{3}{h_{n-1}}(a_n - a_{n-1}) - \frac{3}{h_{n-2}}(a_{n-1} - a_{n-2}) \end{pmatrix}.$$

[1] Also note that we assume each node is distinct, therefore, $h_i \neq 0$ for all i

[2] That is to say, for the family $\{S_j\}_{0\leq j\leq n-1}$ to be a cubic spline on f, the conditions derived in class must be satisfied. Showing that the coefficients which satisfy these conditions are unique, then shows that the cubic spline is unique. This will be assumed from now on.

Therefore, since a_j is uniquely determined by f, so is[3] c_j, and by extension

$$b_j := \frac{1}{h_j}(a_{j+1} - a_j) - \frac{h_j}{3}(c_{j+1} + 2c)$$

and

$$d_j := \frac{1}{3h_j}(c_{j+1} - c_j)$$

are also uniquely determined by f. Therefore, the family $\{S_j\}_{0\leq j\leq n-1}$ is the unique cubic spline on f, which is the desired result.

[3]We also note that $c_0 = c_n = 0$, so they are also unique.