Homework 3

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1. *Proof.* It is sufficient to show that

$$\lim_{n\to\infty} \frac{|p_{n+1}-p|}{|p_n-p|} = g'(p)$$

since letting $\lambda = g'(p)$ would complete the proof. By the mean value theorem on $[p_n, p]$, or $[p, p_n]$, there exists a $\xi_n \in (p_n, p)$ such that

$$\frac{|g(p_n)-g(p)|}{|p_n-p|} = \frac{|p_{n+1}-p|}{|p_n-p|} = g'(\xi_n).$$

Since $p_n \xrightarrow{n\to\infty} p$, we must have $\xi_n \xrightarrow{n\to\infty} p$. Therefore, by the continuity of g', we must have

$$g'(\xi_n) \xrightarrow{n\to\infty} g'(p).$$

Therefore, taking the limit of both sides of the first equation gives

$$\lim_{n\to\infty} \frac{|p_{n+1}-p|}{|p_n-p|} = \lim_{n\to\infty} g'(\xi_n) = g'(p)$$

which is the desired result. □

2. (a)
   (b)
   (c)

3. (a)
   (b)
   (c)

4. *Proof.* Suppose there exists, two degree n polynomials, p_1 and p_2, such that

$$p_1(x_i) = y_i = p_2(x_i)$$

for all $0 \le i \le n$. It suffices to show that $p_1(x) = p_2(x)$. Therefore, the polynomial,

$$f(x) = p_1(x) - p_2(x)$$

is a polynomial of degree at most n, with $n+1$ distinct roots, $x_0, x_1, \ldots, x_n$. However, by the Fundamental Theorem of Algebra, f must be the 0 polynomial.[1] Therefore,

$$f(x) = 0,$$

which means that $p_1(x) = p_2(x)$, which is the desired result. □

[1] Since otherwise it would be a non-zero degree n polynomial with more than n distinct roots.

5. (a) We can take $(0,7),(2,11),(3,28)$, and $(4,63)$ as $(x_0,y_0),(x_1,y_1),...(x_n,y_n)$ respectively. We find the following:

$$L_0(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}$$

$$L_1(x) = \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}$$

$$L_2(x) = \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}$$

$$L_3(x) = \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}$$

Plugging in the points from above, we get:

$$L_0(x) = \frac{(x-2)(x-3)(x-4)}{(-2)(-3)(-4)} = \frac{-1}{24}(x-2)(x-3)(x-4)$$

$$L_1(x) = \frac{(x-0)(x-3)(x-4)}{(-2)(-1)(-2)} = \frac{1}{4}(x)(x-3)(x-4)$$

$$L_2(x) = \frac{(x-0)(x-2)(x-4)}{(3)(1)(-1)} = \frac{-1}{3}(x)(x-2)(x-4)$$

$$L_3(x) = \frac{(x-0)(x-2)(x-3)}{(4)(2)(1)} = \frac{1}{8}(x)(x-2)(x-3)$$

Using the y-values from above, we get the following interpolation:

$$P(x) = \frac{-7}{24}(x-2)(x-3)(x-4)+\frac{11}{4}(x)(x-3)(x-4)-\frac{28}{3}(x)(x-2)(x-4)+\frac{63}{8}(x)(x-2)(x-3)$$

Which can be simplified to become

$$P(x) = x^3 - 2x + 7$$

(b) To find the approximation of $f(1)$, we just plug in 1 for x in $P(x)$:

$$P(1) = 1^2 - 2(1) + 7 = 1 - 2 + 7 = 6$$

(c)

$$\int_0^4 x^3 - 2x + 7 = 76$$

6.

7. We know that

$$|f(x) - p(x)| = \frac{f^{(n+1)}(\xi)}{(n+1)!} \prod_{i=0}^{n} x - x_i$$

Since $x_0, x_1, ..., x_n$ are evenly spaced,

$$\prod_{i=0}^{n} (x - x_i) \leq \frac{1}{4}((\frac{x_n - x_0}{n})^{n+1}(n!))$$

So,

$$|f(x) - p(x)| = |(\frac{f^{11}(\xi)}{11!})(\frac{1}{4})(\frac{1.6875 - 0}{10})^{11}(10!)$$

$f(x) = \sin(0.16875x)$, so

$$|f(x) - p(x)| = \max(-\frac{((0.16875)^{11}\cos(0.16875)(0))}{11!}, -\frac{((0.16875)^{11}\cos(0.16875)(1.6785))}{11!})$$

.

Since

$$|-((0.16785)^{11}(\cos(0.16785)(x = 0)))| = 3.15996008e^{-9}$$

and

$$|-((0.16785)^{11}(\cos(0.16785)(x = 1.6875)))| = 3.15867894e^{-9}$$

The maximum value of f^{11} is at $x = 0$. Then,

$$|(-\frac{(0.16875)^{11}(\cos(0.16875(1.6875)|))}{11!})\frac{1}{4}(0.16875)^{11}(10!) \leq 2.178e^{-19}$$

.

Then, the error bound of $|f(x) - p(x)| \leq 2.178(10^{-9})$ on $[0, 1.6875]$.