Homework 3

Due date: February 6th, 2025

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1.

2. (a)

   (b)

   (c)

3. (a) Since f is a "suitably" differentiable function, we may assume that $f^{(i)}(x)$ is continuous in some "suitable" region around p, for all $0 \leq i \leq m$. Since p is a root of multiplicity m, We must have that $f^{(m-1)}(p) = 0$ and $f^{(m)}(p) \neq 0$. Therefore, by repeated application of L'Hopital's Rule $m-1$ times, and by the fact that $f \in C^m$, we have

$$\lim_{x \to p} \mu(x) = \lim_{x \to p} \frac{f(x)}{f'(x)} = \lim_{x \to p} \frac{f^{(m-1)}(x)}{f^{(m)}(x)} = \frac{f^{(m-1)}(p)}{f^{(m)}(p)} = 0.$$

   which us the desired result.

   (b) Note that since p is a root of f with multiplicity m, there exists a function[1] $g(x)$ such that

$$f(x) = (x-p)^m g(x).$$

   where $g(p) \neq 0$. Therefore, we have

$$\mu(x) = \frac{f(x)}{f'(x)} = \frac{(x-p)^m g(x)}{m(x-p)^{p-1} g(x) + (x-p)^m g'(x)} = \frac{(x-p)g(x)}{mg(x) + (x-p)g'(x)}.$$

   If we let $h(x) := \frac{g(x)}{mg(x)+(x-p)g'(x)}$, we then have

$$h(p) = \frac{g(p)}{mg(p)} = \frac{1}{m} \neq 0.$$

   Thus, since $\mu(x) = (x-p)h(x)$, and $h(p) \neq 0$, p must be a simple root of $\mu(x)$. Therefore, $\mu'(p) \neq 0$, which is the desired result.

   (c) For $x \neq p$, by the quotient rule, we have

$$\frac{\mu(x)}{\mu'(x)} = \frac{f(x)}{f'(x)} \frac{f'(x)^2}{f'(x)^2 - f(x)f''(x)} = \frac{f(x)f'(x)}{f'(x)^2 - f(x)f''(x)}$$

   which is the desired result.

4. *Proof.* Suppose there exists, two degree n polynomials, p_1 and p_2, such that

$$p_1(x_i) = y_i = p_2(x_i)$$

[1] which is also differentiable in a suitable interval.

for all $0 \leq i \leq n$. It suffices to show that $p_1(x) = p_2(x)$. Therefore, the polynomial,

$$f(x) = p_1(x) - p_2(x)$$

is a polynomial of degree at most n, with $n+1$ distinct roots, $x_0, x_1, \ldots, x_n$. However, by the Fundamental Theorem of Algebra, f must be the 0 polynomial.[2] Therefore,

$$f(x) = 0,$$

which means that $p_1(x) = p_2(x)$, which is the desired result. $\square$

5. (a) We can take $(0,7), (2,11), (3,28)$, and $(4,63)$ as $(x_0, y_0), (x_1, y_1), ...(x_n, y_n)$ respectively. We find the following:

$$L_0(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}$$

$$L_1(x) = \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}$$

$$L_2(x) = \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}$$

$$L_3(x) = \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}$$

Plugging in the points from above, we get:

$$L_0(x) = \frac{(x-2)(x-3)(x-4)}{(-2)(-3)(-4)} = \frac{-1}{24}(x-2)(x-3)(x-4)$$

$$L_1(x) = \frac{(x-0)(x-3)(x-4)}{(-2)(-1)(-2)} = \frac{1}{4}(x)(x-3)(x-4)$$

$$L_2(x) = \frac{(x-0)(x-2)(x-4)}{(3)(1)(-1)} = \frac{-1}{3}(x)(x-2)(x-4)$$

$$L_3(x) = \frac{(x-0)(x-2)(x-3)}{(4)(2)(1)} = \frac{1}{8}(x)(x-2)(x-3)$$

Using the y-values from above, we get the following interpolation:

$$P(x) = \frac{-7}{24}(x-2)(x-3)(x-4) + \frac{11}{4}(x)(x-3)(x-4) - \frac{28}{3}(x)(x-2)(x-4) + \frac{63}{8}(x)(x-2)(x-3)$$

Which can be simplified to become

$$P(x) = x^3 - 2x + 7$$

(b) To find the approximation of $f(1)$, we just plug in 1 for x in $P(x)$:

$$P(1) = 1^2 - 2(1) + 7 = 1 - 2 + 7 = 6$$

[2] Since otherwise it would be a non-zero degree n polynomial with more than n distinct roots.

(c)

$$\int_0^4 x^3 - 2x + 7 = 76$$

6.

7.