

## Homework 8

Due date: April 3rd, 2025

1. (a)  
(b)  
(c)
2. We are given the data points:  $(1, 2.31), (2, 2.01), (3, 1.80), (4, 1.66), (5, 1.55), (6, 1.47), (7, 1.41)$   
We seek a quadratic function of the form:  $f(x) = a + bx + cx^2$  The normal equations are as such:

$$\sum y = an + b \sum x + c \sum x^2, \quad (1)$$

$$\sum xy = a \sum x + b \sum x^2 + c \sum x^3, \quad (2)$$

$$\sum x^2y = a \sum x^2 + b \sum x^3 + c \sum x^4. \quad (3)$$

The values of  $x$ ,  $y$ , and the required powers of  $x$  are:

$x$	$y$	$x^2$	$x^3$	$x^4$	$xy$	$x^2y$
1	2.31	1	1	1	2.31	2.31
2	2.01	4	8	16	4.02	8.04
3	1.80	9	27	81	5.40	16.20
4	1.66	16	64	256	6.64	26.56
5	1.55	25	125	625	7.75	38.75
6	1.47	36	216	1296	8.82	52.92
7	1.41	49	343	2401	9.87	69.03
$\Sigma$		140	784	4676	44.81	213.81

The computed sums are:

$$\sum x = 28, \quad \sum x^2 = 140, \quad \sum x^3 = 784, \quad \sum x^4 = 4676, \quad (4)$$

$$\sum y = 12.21, \quad \sum xy = 44.81, \quad \sum x^2y = 213.81. \quad (5)$$

Substituting the computed sums into the normal equations:

$$12.21 = 7a + 28b + 140c, \quad (6)$$

$$44.81 = 28a + 140b + 784c, \quad (7)$$

$$213.81 = 140a + 784b + 4676c. \quad (8)$$

We solve using Gaussian elimination:

$$\begin{bmatrix} 7 & 28 & 140 \\ 28 & 140 & 784 \\ 140 & 784 & 4676 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 12.21 \\ 44.81 \\ 213.81 \end{bmatrix}. \quad (9)$$

Solving this system gives:

$$a \approx 2.5929, \quad (10)$$

$$b \approx -0.3258, \quad (11)$$

$$c \approx 0.0227. \quad (12)$$

Thus, the quadratic least squares approximation is:  $f(x) = 2.5929 - 0.3258x + 0.0227x^2$ .

3. *Proof.* First, note that since  $x^2 \geq 0$  for all  $x \in \mathbb{R}$  and  $f \in C[a, b]$ , we have that

$$\langle f, f \rangle = \int_a^b f(x)^2 dx \geq 0.$$

Furthermore, we have that

$$\langle f, f \rangle = 0 \iff \int_a^b f(x)^2 dx = 0 \iff f(x)^2 = 0 \iff f(x) = 0$$

for all  $x \in [a, b]$ . This is also due to the non-negativity of  $f(x)^2$ . Next, for any  $f, g \in C[a, b]$  we have that

$$\langle f, g \rangle = \int_a^b f(x)g(x) dx = \int_a^b g(x)f(x) dx = \langle g, f \rangle.$$

Finally, for any  $f, g, h \in C[a, b]$  and  $c_1, c_2 \in \mathbb{R}$ , by the linearity of the Riemann integral, we have that

$$\begin{aligned} \langle c_1 f(x) + c_2 g(x), h(x) \rangle &= \int_a^b (c_1 f(x) + c_2 g(x))h(x) dx = \int_a^b (c_1 f(x)h(x) + c_2 g(x)h(x)) dx \\ &= c_1 \int_a^b f(x)h(x) dx + c_2 \int_a^b g(x)h(x) dx = c_1 \langle f, h \rangle + c_2 \langle g, h \rangle. \end{aligned}$$

Therefore,  $\langle \cdot, \cdot \rangle$  is an inner product, which is the desired result. ■

4. We are given that  $\mathbf{v} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 0 \end{pmatrix}$  and  $\mathbf{w} = \begin{pmatrix} 1 \\ -1 \\ 2 \\ -2 \end{pmatrix}$ .

(a)  $\langle \mathbf{v}, \mathbf{w} \rangle = 1 * 1 + 3 * (-1) + 2 * 2 + 0 * (-2) = 2$

(b)  $\|\mathbf{v}\| = \sqrt{1^2 + 3^2 + 2^2} = \sqrt{14}$

(c)  $\mathbf{v} - \mathbf{w} = \begin{pmatrix} 0 \\ 4 \\ 0 \\ 2 \end{pmatrix}$

$$\|\mathbf{v} - \mathbf{w}\| = \sqrt{0^2 + 4^2 + 0^2 + 2^2} = 2\sqrt{5}$$

$$(d) \cos(\theta) = \frac{\langle \mathbf{v}, \mathbf{w} \rangle}{\|\mathbf{v}\| \cdot \|\mathbf{w}\|} = \frac{2}{\sqrt{14} \cdot \sqrt{1^2 + (-1)^2 + 2^2 + (-2)^2}} = \frac{2}{\sqrt{140}}$$

$$\theta = \cos^{-1}\left(\frac{2}{\sqrt{140}}\right) \approx 1.40095$$

5. (a)

$$\begin{aligned} \langle f, g \rangle &= \int_0^1 f(x)g(x)dx \\ &= \int_0^1 (x-1)(x+1)dx \\ &= \int_0^1 (x^2 - 1)dx \\ &= \left[\frac{x^3}{3} - x\right]_0^1 \\ &= \left(\frac{1}{3} - 1\right) - \left(\frac{0^3}{3} - 0\right) = \frac{-2}{3} \end{aligned}$$

(b)

$$\begin{aligned} \|f - g\| &= \sqrt{\int_0^1 |(f - g)|dx} \\ f(x) - g(x) &= (x-1) - (x+1) = x-1-x-1 = -2 \\ |f(x) - g(x)| &= 2 \\ \int_0^1 2dx &= [2x]_0^1 = 2 - 0 = 2 \end{aligned}$$

(c) The angle between  $f$  and  $g$  is

$$\cos(\theta) = \frac{\langle f, g \rangle}{\|f\| \cdot \|g\|}$$

We already know that  $\langle f, g \rangle = \frac{-2}{3}$ , so we need to find the norm  $\|f\|$ .

$$\begin{aligned} \|f\| &= \left(\int_0^1 (x-1)^2 dx\right)^{\frac{1}{2}} = \left(\int_0^1 (x^2 - 2x + 1) dx\right)^{\frac{1}{2}} \\ &= \left(\left[\frac{x^3}{3} - 2\frac{x^2}{2} + x\right]_0^1\right)^{\frac{1}{2}} = \left(\left(\frac{1}{3} - 2\frac{x^2}{2} + x\right) - (0)\right)^{\frac{1}{2}} \\ &= \left(\frac{1}{3} - 1 + 1\right)^{\frac{1}{2}} = \frac{1}{\sqrt{3}} \end{aligned}$$

.

Now we need to find  $\|g\|$ .

$$\|g\| = \left(\int_0^1 (x+1)^2 dx\right)^{\frac{1}{2}} = \left(\int_0^1 (x^2 + 2x + 1) dx\right)^{\frac{1}{2}} = \left(\left[\frac{x^3}{3} + x^2 + x\right]_0^1\right)^{\frac{1}{2}}$$

$$= \left(\frac{1}{3} + 1 + 1\right)^{\frac{1}{2}} = \sqrt{\frac{7}{3}}$$

We can now use both of these norms to find  $\theta$ :

$$\cos(\theta) = \frac{\frac{-2}{3}}{\frac{-1}{3} \cdot \sqrt{\frac{7}{3}}} = \frac{\frac{-2}{3}}{\sqrt{\frac{7}{9}}} = \frac{\frac{-2}{3}}{\frac{\sqrt{7}}{3}} = \frac{-2}{\sqrt{7}}$$

Now we can solve for  $\theta$ :

$$\theta = \cos^{-1}\left(\frac{-2}{\sqrt{7}}\right) \approx \cos^{-1}(-0.7559) \approx 139.1^\circ$$

- (d) To find a nonzero  $h$  that is perpendicular to  $f$ , we would have to find a function  $h(x)$  that satisfies the following:

$$\int_0^1 (x-1) \cdot h(x) dx = 0$$

We can try a simple polynomial  $h(x) = ax + b$ , and plug it into the above orthogonality condition:

$$\int_0^1 (x-1)(ax+b) dx = 0$$

$$(x-1)(ax+b) = (ax^2 + bx - ax - b) = ax^2 + (b-a)x - b$$

$$\int_0^1 (ax^2 + (b-a)x - b) dx = a\frac{1}{3} + (b-a)\frac{1}{2} - b = 0$$

$$2a + 3(b-a) - 6b = 0 \rightarrow 2a + 3b - 3a - 6b = 0 \rightarrow -a - 3b = 0 \rightarrow a = -3b$$

Let us pick  $b = 1$ , then  $a = -3$ , so  $h(x) = -3x + 1$

6. We know that  $\mathbf{v}_1 = (1, 2, 2)$ ,  $\mathbf{v}_2 = (-1, 0, 2)$ , and  $\mathbf{v}_3 = (0, 0, 1)$

(a)

$$\mathbf{v}_1 = \mathbf{x}_1 \text{ and } \|\mathbf{x}_1\|_2 = \sqrt{1^2 + 2^2 + 2^2}$$

$$\|\mathbf{w}\| = \frac{\mathbf{x}_1}{\|\mathbf{x}_1\|_2} = \left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right)$$

$$\mathbf{x}_2 = \mathbf{v}_2 - \langle \mathbf{v}_2, \mathbf{w}_1 \rangle \mathbf{w}_1 = (-1, 0, 2) - \langle (-1, 0, 2), \left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right) \rangle$$

$$= (-1, 0, 2) - 1\left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right)$$

$$= \left(\frac{-4}{3}, \frac{-2}{3}, \frac{4}{3}\right)$$

$$\|\mathbf{x}^2\|_2 = \sqrt{\left(\frac{-4}{3}\right)^2 + \left(\frac{-2}{3}\right)^2 + \left(\frac{4}{3}\right)^2} = 2$$

$$\mathbf{w}_2 = \frac{\mathbf{x}_2}{\|\mathbf{x}_2\|_2} = \left(\frac{-2}{3}, \frac{-1}{3}, \frac{2}{3}\right)$$

$$\begin{aligned}
\mathbf{x}_3 &= \mathbf{v}_3 - \langle \mathbf{v}_3, \mathbf{w}_1 \rangle \mathbf{w}_1 - \langle \mathbf{v}_3, \mathbf{w}_2 \rangle \mathbf{w}_2 \\
\mathbf{x}_3 &= (0, 0, 1) - \langle (0, 0, 1), (\frac{1}{3}, \frac{2}{3}, \frac{2}{3}) \rangle (\frac{1}{3}, \frac{2}{3}, \frac{2}{3}) - \langle (0, 0, 1), (\frac{-2}{3}, \frac{-1}{3}, \frac{2}{3}) \rangle (\frac{-2}{3}, \frac{-1}{3}, \frac{2}{3}) \\
&= (0, 0, 1) - \frac{2}{3} (\frac{1}{3}, \frac{2}{3}, \frac{2}{3}) - \frac{2}{3} (\frac{-2}{3}, \frac{-1}{3}, \frac{2}{3}) \\
&= (\frac{2}{9}, \frac{-2}{9}, \frac{1}{9}) \\
\|\mathbf{x}_3\|_2 &= \frac{1}{3} \\
\mathbf{w}_3 &= \frac{\mathbf{x}_3}{\|\mathbf{x}_3\|_2} = (\frac{2}{3}, \frac{-2}{3}, \frac{1}{3})
\end{aligned}$$

(b) We know  $\mathbf{x} = (7, 5, 1)$

$$\begin{aligned}
\langle (7, 5, 1), (\frac{1}{3}, \frac{2}{3}, \frac{2}{3}) \rangle &= \frac{19}{3} \\
\langle (7, 5, 1), (\frac{-2}{3}, \frac{-1}{3}, \frac{2}{3}) \rangle &= \frac{-17}{3} \\
\langle (7, 5, 1), (\frac{2}{3}, \frac{-2}{3}, \frac{1}{3}) \rangle &= \frac{5}{3}
\end{aligned}$$

.

Now, we can write

$$\frac{19}{3} (\frac{1}{3}, \frac{2}{3}, \frac{2}{3}) - \frac{17}{3} (\frac{-2}{3}, \frac{-1}{3}, \frac{2}{3}) + \frac{5}{3} (\frac{2}{3}, \frac{-2}{3}, \frac{1}{3})$$