Homework 7

Due date: March 13, 2025

1. Note that

$$||A||_\infty = \max\{|3|+|2|+|4|, |2|+|0|+|2|, |4|+|2|+|3|\} = \max\{9,4,9\} = 9.$$

Furthermore, we have that

$$||A||_1 = \max\{|3|+|2|+|4|, |2|+|0|+|2|, |4|+|2|+|3|\} = \max\{9,4,9\} = 9.$$

Finally, we compute the eigenvalues of A in the following way.

$$\det(A-\lambda I) = \det\begin{pmatrix} 3-\lambda & 2 & 4 \\ 2 & -\lambda & 2 \\ 4 & 2 & 3-\lambda \end{pmatrix} = (3-\lambda)(-\lambda(3-\lambda)-4) - 2(2(3-\lambda)-8) + 4(4+4\lambda)$$

$$= -\lambda^3 + 6\lambda^2 + 15\lambda + 8 = -(\lambda+1)^2(\lambda-8).$$

Therefore, the eigenvalues are given by the roots of $-(\lambda+1)^2(\lambda-8)$, which are $\lambda = -1$ and $\lambda = 8$. Since A is symmetric, we have that

$$||A||_2 = \rho(A) = \max\{|8|, |-1|\} = 8.$$

2. (a) *Proof.* Let $M_1 = \max_{||x||\neq 0} \frac{||Ax||}{||x||}$ and $M_2 = ||A|| = \max_{||x||=1} ||Ax||$. It suffices to show that $M_1 = M_2$. Note that if $||x|| \neq 0$, then by properties of norms, we have

$$\frac{||Ax||}{||x||} = \left|\left|A\frac{x}{||x||}\right|\right|.$$

Furthermore[1], since $\left|\left|\frac{x}{||x||}\right|\right| = \frac{1}{||x||}||x|| = 1$ we must have that $M_1 \leq M_2$ by definition of M_2. Furthermore, if we fix an arbitrary vector, x, such that $||x|| = 1$. Then we must have

$$||Ax|| = \left|\left|A\frac{x}{1}\right|\right| = \left|\left|A\frac{x}{||x||}\right|\right| = \frac{||Ax||}{||x||}$$

by definition of M_1, we must have that $M_2 \leq M_1$, and therefore, $M_1 = M_2$, which was the desired result. ∎

(b) *Proof.* Note that by part (a), we have that for any vector $x \neq 0$,

$$||A|| = \max_{y\neq 0} \frac{||Ay||}{||y||} \geq \frac{||Ax||}{||x||}.$$

Multiplying by $||x||$ gives that $||Ax|| \leq ||A||||x||$ for all $x \neq 0$. Furthermore, note that the inequality also holds when $x = 0$, since $Ax = 0$. Therefore,

$$||Ax|| \leq ||A||||x||$$

for all x. We then have that

$$||AB|| = \max_{||x||=1||} ||ABx|| \leq \max_{||x||=1} ||A||||Bx|| = ||A|| \max_{||x||=1} ||Bx|| = ||A||||B||$$

which is the desired result. ∎

[1] It should be noted that $\frac{x}{||x||}$ is intended to mean $\frac{1}{||x||}x$, to agree with multiplying x by a scalar.

(c) *Proof.* By repetedly applying part (b), we have that for all $k \in \mathbb{N}$

$$||A^k|| = ||A \cdot A^{k-1}|| \leq ||A||||A^{k-1}|| = ||A||||A \cdot A^{k-2}|| \leq ||A||^2||A^{k-2}|| \leq \dots ||A||^k.$$

∎

(d) *Proof.* Note that by part (b), we have

$$||A||||A^{-1}|| \geq ||AA^{-1}|| = ||I|| = \max_{||x||=1} ||Ix|| = \max_{||x||=1} ||x|| = 1$$

which is the desired result. ∎

3.

4.

5.

6. *Proof.* First, assume that A is strictly diagonally dominant. Namely, suppose that

$$|a_{ii}| > \sum_{j \neq i} |a_{ij}|$$

for all $1 \leq i \leq n$. Then, we must have that

$$||D^{-1}(L+U)||_\infty \leq ||D^{-1}||_\infty ||L+U||_\infty = ||D^{-1}||_\infty ||D-A||_\infty$$

where we used the fact that $A = D - (L+U)$. Furthermore, since $D = \text{diag}(a_{ii})$[2], we must have that

$$D^{-1} = \text{diag}\left(\frac{1}{a_{ii}}\right).$$

Therefore, since we have that

$$L+U = \begin{cases} a_{ij} & \text{if } i \neq j \\ 0 & \text{else} \end{cases},$$

we must have

$$[D^{-1}(L+U)]_{ij} = \begin{cases} \frac{a_{ij}}{a_{ii}} & \text{if } i \neq j \\ 0 & \text{else} \end{cases}.$$

Therefore, we have that

$$||D^{-1}(L+U)||_\infty = \max_{1 \leq i \leq n} \sum_{j=1}^n \left|[D^{-1}(L+U)]_{ij}\right| = \max_{1 \leq i \leq n} \sum_{j \neq i} \left|\frac{a_{ij}}{a_{ii}}\right| = \max_{1 \leq i \leq n} \frac{1}{|a_{ii}|} \sum_{j \neq i} |a_{ij}|.$$

[2] I am going to make this assumption, since the theorem is not true without it. For example, if $A = I$, then clearly A is strictly diagonally dominant. However, if we let $D = \frac{1}{2}I$, then we must have $L+U = D-A = -\frac{1}{2}I$. This may be achieved with $L = U = -\frac{1}{4}I$. However, this means $||D^{-1}(L+U)||_\infty = ||2I\left(-\frac{1}{2}I\right)||_\infty = ||-I||_\infty = 1$ which is obviously not less than 1. I specify this because we only assumed that D was diagonal, L is lower triangular, and U is upper triangular. However, what I have written is a stronger condition.

Since A is strictly diagonally dominant, we have that

$$||D^{-1}(L+U)||_\infty = \max_{1 \le i \le n} \frac{1}{|a_{ii}|} \sum_{j \ne i} |a_{ij}| < \max_{1 \le i \le n} \frac{|a_{ii}|}{|a_{ii}|} = 1.$$

Therefore, $||D^{-1}(L+U)||_\infty < 1$.

Suppose that $||D^{-1}(L+U)||_\infty < 1$. As discussed before, we have that

$$||D^{-1}(L+U)||_\infty = \max_{1 \le i \le n} \frac{1}{|a_{ii}|} \sum_{j \ne i} |a_{ij}|.$$

Therefore, for all $1 \le i \le n$, we have that

$$\frac{1}{|a_{ii}|} \sum_{j \ne i} |a_{ij}| \le \max_{1 \le i \le n} \frac{1}{|a_{ii}|} \sum_{j \ne i} |a_{ij}| < 1.$$

Thus, multiplying by a_{ii} gives that for all $1 \le i \le n$,

$$\sum_{j \ne i} |a_{ij}| < ||a_{ii}||.$$

Therefore, A is strictly diagonally dominant, which completes the proof. ■