

Homework 8

Due date: April 9th, 2025

- 1.
- 2.
3. (a) We seek the continuous Fourier approximation $S_3(x)$ of the function $f(x) = e^x$ on the interval $[-\pi, \pi]$, using the formula:

$$S_n(x) = \langle f(x), 1 \rangle \cdot \frac{1}{2} + \sum_{k=1}^n (\langle f(x), \sin(kx) \rangle \cdot \sin(kx) + \langle f(x), \cos(kx) \rangle \cdot \cos(kx))$$

with the inner product defined as:

$$\langle f(x), g(x) \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)g(x) dx$$

$$\begin{aligned} a_0 &= \langle f(x), 1 \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} e^x dx = \frac{1}{\pi} (e^{\pi} - e^{-\pi}) \\ \Rightarrow \frac{a_0}{2} &= \frac{1}{2\pi} (e^{\pi} - e^{-\pi}) \end{aligned}$$

$$\begin{aligned} a_1 &= \frac{1}{\pi} \int_{-\pi}^{\pi} e^x \cos(x) dx = \frac{1}{\pi} \left[\frac{e^x}{2} (\sin(x) + \cos(x)) \right]_{-\pi}^{\pi} \\ &= \frac{1}{\pi} \left(-\frac{e^{\pi}}{2} + \frac{e^{-\pi}}{2} \right) = -\frac{1}{2\pi} (e^{\pi} - e^{-\pi}) \end{aligned}$$

$$\begin{aligned} b_1 &= \frac{1}{\pi} \int_{-\pi}^{\pi} e^x \sin(x) dx = \frac{1}{\pi} \left[\frac{e^x}{2} (\sin(x) - \cos(x)) \right]_{-\pi}^{\pi} \\ &= \frac{1}{\pi} \left(\frac{e^{\pi}}{2} - \frac{e^{-\pi}}{2} \right) = \frac{1}{2\pi} (e^{\pi} - e^{-\pi}) \end{aligned}$$

Using:

$$\int e^x \cos(ax) dx = \frac{e^x (a \sin(ax) + \cos(ax))}{a^2 + 1}, \quad \int e^x \sin(ax) dx = \frac{e^x (\sin(ax) - a \cos(ax))}{a^2 + 1}$$

We get:

$$\begin{aligned} a_2 &= \frac{1}{\pi} \left[\frac{e^x (2 \sin(2x) + \cos(2x))}{5} \right]_{-\pi}^{\pi} = \frac{1}{5\pi} (e^{\pi} - e^{-\pi}) \\ b_2 &= \frac{1}{\pi} \left[\frac{e^x (\sin(2x) - 2 \cos(2x))}{5} \right]_{-\pi}^{\pi} = -\frac{2}{5\pi} (e^{\pi} - e^{-\pi}) \\ a_3 &= \frac{1}{\pi} \left[\frac{e^x (3 \sin(3x) + \cos(3x))}{10} \right]_{-\pi}^{\pi} = -\frac{1}{10\pi} (e^{\pi} - e^{-\pi}) \end{aligned}$$

$$b_3 = \frac{1}{\pi} \left[\frac{e^x(\sin(3x) - 3\cos(3x))}{10} \right]_{-\pi}^{\pi} = \frac{3}{10\pi}(e^{\pi} - e^{-\pi})$$

Evaluating numerically, we have:

$$S_3(x) \approx 3.6767 - 3.6767 \cos(x) + 3.6767 \sin(x) + 1.4707 \cos(2x) - 2.9414 \sin(2x) - 0.7353 \cos(3x) + 2.2060 \sin(3x)$$

(b) We must calculate $\langle y, \phi_k \rangle$ and $\langle y, \psi_k \rangle$ for $k = 0, 1, 2, 3$ First, calculate $\langle y, \phi_0 \rangle$,

$$\langle y, \phi_k \rangle = \frac{1}{10} \sum_{j=0}^{19} y_j \cos(kx_j)$$

$\langle y, \phi_1 \rangle$, $\langle y, \phi_2 \rangle$, and $\langle y, \phi_3 \rangle$.

$$\langle y, \phi_0 \rangle = \frac{1}{10} \sum_{j=0}^{19} y_j \cos(0) = \frac{1}{10} \sum_{j=0}^{19} y_j$$

$$\langle y, \phi_1 \rangle = \frac{1}{10} \sum_{j=0}^{19} y_j \cos(x_j)$$

$$\langle y, \phi_2 \rangle = \frac{1}{10} \sum_{j=0}^{19} y_j \cos(2x_j)$$

$$\langle y, \phi_3 \rangle = \frac{1}{10} \sum_{j=0}^{19} y_j \cos(3x_j)$$

Next, calculate $\langle y, \psi_k \rangle$ for $k = 1, 2, 3$

$$\langle y, \psi_k \rangle = \frac{1}{10} \sum_{j=0}^{19} y_j \sin(kx_j)$$

$$\langle y, \psi_1 \rangle = \frac{1}{10} \sum_{j=0}^{19} y_j \sin(x_j)$$

$$\langle y, \psi_2 \rangle = \frac{1}{10} \sum_{j=0}^{19} y_j \sin(2x_j)$$

$$\langle y, \psi_3 \rangle = \frac{1}{10} \sum_{j=0}^{19} y_j \sin(3x_j)$$

$$S_3(x) = \frac{1}{2} \langle y, \phi_0 \rangle + \sum_{k=1}^3 (\langle y, \phi_k \rangle \cos(kx)) + \sum_{k=1}^3 (\langle y, \psi_k \rangle \sin(kx))$$

$$S_3(x) = \frac{1}{2} \langle y, \phi_0 \rangle + \langle y, \phi_1 \rangle \cos(x) + \langle y, \phi_2 \rangle \cos(2x) + \langle y, \phi_3 \rangle \cos(3x) + \langle y, \psi_1 \rangle \sin(x) + \langle y, \psi_2 \rangle \sin(2x) + \langle y, \psi_3 \rangle \sin(3x)$$

$$S_3(x) \approx 3.676 + 2.963 \cos(x) + 0.946 \cos(2x) + 0.164 \cos(3x) - 1.263 \sin(x) - 0.740 \sin(2x) - 0.214 \sin(3x)$$

4. (a) Q is orthogonal.

$$Q^t Q = I$$

$$\det(Q^t Q) = 1$$

$$\det(Q^t) \det(Q) = 1$$

Since $\det(Q^t) = \det(Q)$,

$$\det(Q)^2 = 1$$

$$\det(Q) = \pm 1$$

- (b)

$$(PQ)^t PQ = Q^t P^t PQ = Q^t (I) Q = Q^t Q = I$$

. So PQ is orthogonal.

5. (a)

- (b)

- (c)

6. We are given the symmetric matrix:

$$A = \begin{bmatrix} 1 & -1 & 2 \\ -1 & 1 & 2 \\ 2 & 2 & 2 \end{bmatrix}$$

We want to decompose it as:

$$A = Q D Q^T$$

To find the eigenvalues, we solve the characteristic polynomial:

$$\det(A - \lambda I)$$

$$\begin{aligned} A - \lambda I &= \begin{bmatrix} 1 & -1 & 2 \\ -1 & 1 & 2 \\ 2 & 2 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} \\ &= \begin{bmatrix} 1-\lambda & -1 & 2 \\ -1 & 1-\lambda & 2 \\ 2 & 2 & 2-\lambda \end{bmatrix} \end{aligned}$$

We know can find the determinant of the matrix:

$$(1-\lambda) \begin{vmatrix} 1-\lambda & 2 \\ 2 & 2-\lambda \end{vmatrix} - (-1) \begin{vmatrix} -1 & 2 \\ 2 & 2-\lambda \end{vmatrix} + 2 \begin{vmatrix} -1 & 1-\lambda \\ 2 & 2 \end{vmatrix} = -\lambda^3 + 4\lambda^2 + 4\lambda - 16 = -(\lambda-4)(\lambda-2)(\lambda+2)$$

The eigenvalues are:

$$\lambda_1 = 4, \quad \lambda_2 = 2, \quad \lambda_3 = -2$$

Solve $(A - 4I)\vec{v} = 0$, we get:

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \quad \|\vec{v}_1\| = \sqrt{6} \Rightarrow q_1 = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

Solve $(A - 2I)\vec{v} = 0$, we get:

$$\vec{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \quad \|\vec{v}_2\| = \sqrt{2} \Rightarrow q_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

Solve $(A + 2I)\vec{v} = 0$, we get:

$$\vec{v}_3 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \quad \|\vec{v}_3\| = \sqrt{3} \Rightarrow q_3 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

Orthogonal matrix Q (columns are the normalized eigenvectors):

$$Q = \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \frac{2}{\sqrt{6}} & 0 & -\frac{1}{\sqrt{3}} \end{bmatrix}$$

Diagonal matrix D of eigenvalues:

$$D = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$