Homework

Due date: February 27, 2025

1. Let $f(x) = 3xe^x - cosx$. By using the forward-difference formula, the three-point midpoint formula, and the five-point midpoint formula with $h = 0.1, 0.05, 0.01$, compute approximations of $f'(1.3)$.

For $x_0 = 1.3$, the function at the values modified by h are as follows.

h	Rule	x	$f(x)$
0.1	$x_0 - h$	1.2	11.590063167374895
	$x_0 + h$	1.4	16.86187271784739
	$x_0 - 2h$	1.1	9.460151757597654
	$x_0 + 2h$	1.5	20.096863614853586
0.05	$x_0 - h$	1.25	12.773463728086636
	$x_0 + h$	1.35	15.403566712229708
	$x_0 - 2h$	1.2	11.590063167374895
	$x_0 + 2h$	1.4	16.86187271784739
0.01	$x_0 - h$	1.29	13.781763095706816
	$x_0 + h$	1.31	14.307412656453414
	$x_0 - 2h$	1.28	13.524381336554086
	$x_0 + 2h$	1.32	14.575773202300644

$$f'(x_0) = \frac{1}{h}(f(x_0 + h) - f(x_0)) + O(h)$$

For $h = 0.1$

$$f'(1.3) = \tfrac{1}{0.1}(f(1.4) - f(1.3)) = \tfrac{16.86187271784739 - 14.042758175090468}{0.1} \approx 28.1911$$

For $h = 0.05$

$$f'(1.3) = \tfrac{1}{0.05}(f(1.35) - f(1.3)) = \tfrac{15.403566712229708 - 14.042758175090468}{0.05} \approx 27.2162$$

For $h = 0.01$

$$f'(1.3) = \tfrac{1}{0.01}(f(1.31) - f(1.3)) = \tfrac{14.307412656453414 - 14.042758175090468}{0.01} \approx 26.4654$$

Three-Point Midpoint Approximation

$$f'(x_0) = \frac{1}{2h}(f(x_0 + h) - f(x_0 - h)) + O(h^2)$$

For $h = 0.1$

$$f'(1.3) \approx \frac{1}{2(0.1)}(16.86187271784739 - 11.590063167374895) \approx 26.3590$$

For $h = 0.05$

$$f'(1.3) \approx \frac{1}{2(0.05)}(15.403566712229708 - 12.773463728086636) \approx 26.3010$$

For $h = 0.01$

$$f'(1.3) \approx \frac{1}{2(0.01)}(14.307412656453414 - 13.781763095706816) \approx 26.2825$$

Five-Point Midpoint Approximation

$$f'(x_0) = \frac{1}{12h}\big(f(x_0 - 2h) - 8f(x_0 - h) + 8f(x_0 + h) - f(x_0 + 2h)\big) + O(h^4)$$

For $h = 0.1$

$$f'(1.3) \approx \frac{(9.460151757597654) - 8(11.590063167374895) + 8(16.86187271784739) - (20.096863614853586)}{12(0.1)}$$

$$\approx 26.2815$$

For $h = 0.05$

$$f'(1.3) \approx \frac{(11.590063167374895) - 8(12.773463728086636) + 8(15.403566712229708) - (16.86187271784739)}{12(0.05)}$$

$$\approx 26.2817$$

For $h = 0.01$

$$f'(1.3) \approx \frac{(13.524381336554086) - 8(13.781763095706816) + 8(14.307412656453414) - (14.575773202300644)}{12(0.01)}$$

$$\approx 26.2817$$

Conclusion

All of these methods improve as h decreases, but those with higher orders converge much faster. The Five-Point Midpoint method is the most precise, but it may be computationally intensive.

2. Note that using taylor expansions around x, we get that

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2}f''(x) + O(h^3)$$

and

$$f(x+2h) = f(x) + 2hf'(x) + 2h^2 f''(x) + O(h^3).$$

Therefore, we have

$$4f(x+h) - 3f(x) - f(x+2h) =$$

$$4\left(f(x) + hf'(x) + \frac{h^2}{2}f''(x)\right) - 3f(x) - \big(f(x) + 2hf'(x) + 2h^2 f''(x)\big) + O(h^3)^{1} = 2hf'(x) + O(h^3).$$

Dividing by $2h$ gives that the error term equals $O(h^2)$, which is the desired result.

[1] If $f = O(h^3)$ and $g = O(h^3)$, then $f + g = O(h^3)$.

3. We can differentiate $y = x^3$ as follows:

$$f'(x) = \frac{d}{dx}x^3 = 3x^2$$

. Thus the integrand becomes

$$\sqrt{1 + (3x^2)^2} = \sqrt{1 + 9x^4}$$

. So using Simpson's rule, we need to evaluate

$$L = \int_0^1 \sqrt{1 + 9x^4}dx$$

. We can approximate the integral as follows:

$$\int_a^b f(x)dx \approx \frac{h}{3}[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + 4f(x_5) + f(x_6)]$$

where $h = \frac{b-a}{n}$ and $x_i = a + ih$. For $n = 6$, $a = 0$, and $b = 1$,

$$h = \frac{1 - 0}{6} = \frac{1}{6}$$

The nodes are

$$x_0 = 0, \quad x_1 = \frac{1}{6}, \quad x_2 = \frac{2}{6}, \quad x_3 = \frac{3}{6}, \quad x_4 = \frac{4}{6}, \quad x_5 = \frac{5}{6}, \quad x_6 = 1.$$

So we can evaluate $f(x) = \sqrt{1 + 9x^4}$ at these points:

$$f(0) = \sqrt{1 + 9(0)^4} = \sqrt{1} = 1.$$

$$f\left(\frac{1}{6}\right) = \sqrt{1 + 9\left(\frac{1}{6}\right)^4} = \sqrt{1.00694} \approx 1.00347$$

$$f\left(\frac{2}{6}\right) = \sqrt{1 + 9\left(\frac{2}{6}\right)^4} = \sqrt{1.05556} \approx 1.02747$$

$$f\left(\frac{3}{6}\right) = \sqrt{1 + 9\left(\frac{3}{6}\right)^4} = \sqrt{1.5625} = 1.25$$

$$f\left(\frac{4}{6}\right) = \sqrt{1 + 9\left(\frac{4}{6}\right)^4} = \sqrt{1.7778} \approx 1.3333$$

$$f\left(\frac{5}{6}\right) = \sqrt{1 + 9\left(\frac{5}{6}\right)^4} = \sqrt{2.3403} \approx 1.53$$

$$f(1) = \sqrt{1 + 9(1)^4} = \sqrt{10} \approx 3.1623$$

We can then use the Simpson's rule formula

$$L \approx \frac{h}{3}\left[f(0)+4f\left(\frac{1}{6}\right)+2f\left(\frac{2}{6}\right)+4f\left(\frac{3}{6}\right)+2f\left(\frac{4}{6}\right)+4f\left(\frac{5}{6}\right)+f(1)\right]$$

And substitute values to get the following:

$$L \approx \frac{1}{18}\left[1+4(1.00347)+2(1.02747)+4(1.25)+2(1.3333)+4(1.53)+3.1623\right]$$

$$L \approx \frac{1}{18}\left[1+4.0139+2.0549+5+2.6667+6.12+3.1623\right]$$

$$L \approx \frac{1}{18}\times 24.0178$$

$$L \approx 1.3343$$

4. Note that

$$f^{(4)}(x) = (12-48x^2+16x^4)e^{-x^2}.$$

On $[0,1]$, the above function attains a maximum at $x=0$, with a value of 12. Therefore, we have that for some $\xi \in [0,1]$,

$$|E_n(f)| = \left|\frac{-1}{180}h^4 f^{(4)}(\xi)\right| \leq \frac{12}{180}h^4.$$

Solving for h in the above inequality gives

$$h \leq 0.01967\ldots.$$

Noting that $n=\frac{1}{h}$ then gives

$$n \geq 50.813\ldots.$$

Therefore, $n=52$, is the smallest possible n that gives us an error of $\leq 10^{-8}$.

5. (a) *Proof.* By the extreme value theorem. f has a maximum and minimum on $[a,b]$. Therefore, there exists $x_{\min}, x_{\max} \in [a,b]$ such that $f(x_{\min}) = \min\limits_{a\leq x\leq b} f(x)$ and $f(x\max) = \max\limits_{a\leq x\leq b} f(x)$. Furthermore, by definition of the maximum and minimum, we have

$$f(x_{\min}) = \frac{nf(x_{\min})}{n} = \frac{\sum_{i=1}^{n} f(x_{\min})}{n} \leq \frac{\sum_{i=1}^{n} f(x_i)}{n} \leq \frac{\sum_{i=1}^{n} f(x_{\max})}{n} = f(x_{\max}).$$

Therefore, by the intermediate value theorem, there exists an $c \in [x_{\min}, x_{\max}] \subseteq [a,b]$ such that

$$f(c) = \frac{\sum_{i=1}^{n} f(x_i)}{n}$$

which is the desired result. ∎

(b)

Theorem 1 (Integral Mean Value Theorem for $g(x) = 1$)**.** *Let f be a continuous function on $[a, b]$. Then there exists a $c \in [a.b]$ such that*

$$f(c)(b - a) = \int_a^b f(x)dx.$$

Lemma 1. *Let f be a Riemann integrable function on $[a, b]$, such that*

$$f(x) \geq 0$$

for all $x \in [a, b]$. Then

$$\int_a^b f(x)dx \geq 0.$$

Proof of Lemma 1. Let $P = \{a = x_0, x_1, \ldots, x_n = b\}$ be an arbitrary partition of $[a, b]$. Since $f(x) \geq 0$ on $[a, b]$, we have that

$$U(f, P) := \sum_{i=1}^{n} (x_i - x_{i-1})M_i \geq 0$$

and

$$L(f, P) := \sum_{i=1}^{n} (x_i - x_{i-1})m_i \geq 0$$

where $M_i := \sup_{x \in [x_{i-1}, x_i]} f(x)$ and $M_i := \inf_{x \in [x_{i-1}, x_i]} f(x)$. Therefore, since P was arbitrary, we must have that

$$\overline{\int_a^b} f(x)dx := \inf\{U(f, P) \mid P \text{ is a partition of } [a, b]\} \geq 0.$$

Since f is Riemann integrable, we finally have that

$$\int_a^b f(x)dx = \overline{\int_a^b} f(x)dx \geq 0$$

which is the desired result. $\square$

Lemma 2. *Let f and g be Riemann integrable functions on $[a, b]$ such that for all $x \in [a, b]$,*

$$f(x) \geq g(x).$$

Then

$$\int_a^b f(x)dx \geq \int_a^b g(x)dx$$

Proof of Lemma 2. Consider $h(x) := f(x) - g(x)$. Since f and g are Riemann integrable on $[a, b]$, h is also Riemann integrable on $[a, b]$. Furthermore, since $f(x) \geq g(x)$ for all $x \in [a, b]$,

$$h(x) \geq 0$$

for all $x \in [a, b]$. Therefore, by Lemma 1 and the additivity of the Riemann integral, we have that

$$\int_a^b f(x)dx - \int_a^b g(x)dx \geq 0.$$

Adding $\int_a^b g(x)dx$ to both sides completes the proof. □

Lemma 3. *Let f and g be Riemann integrable functions on $[a, b]$ such that for all $x \in [a, b]$,*

$$f(x) \leq g(x).$$

Then

$$\int_a^b f(x)dx \leq \int_a^b g(x)dx$$

Proof of Lemma 3. Since $f(x) \leq g(x)$ on $[a, b]$, then $-f(x) \geq -g(x)$ on $[a, b]$. Furthermore, since f and g are both Riemann integrable, then $-f$ and $-g$ are also both Riemann integrable. Therefore, by Lemma 2 and the linearity of the integral, we have

$$-\int_a^b f(x)dx \geq -\int_a^b g(x)dx.$$

Multiplying by -1 completes the proof. □

Lemma 4. *Let f be a continuous function*[2] *on $[a,b]$. Let*[3] $m = \min_{a\le x\le b} f(x)$ *and* $M = \max_{a\le x\le b} f(x)$. *Then*

$$m(b-a) \le \int_a^b f(x)dx \le M(b-a).$$

Proof of Lemma 4. Since $m \le f(x) \le M$ for all $x \in [a,b]$. By Lemma 2, Lemma 3, and properties of the Riemann integral, we have that

$$m(b-a) = \int_a^b m dx \le \int_a^b f(x)dx \le \int_a^b M dx = M(b-a)$$

which was the desired result. □

Proof of Theorem 1. Since f is continuous on $[a,b]$, f attains a maximum and minimum on $[a,b]$, say M and m respectively. Then by Lemma 4, we have that

$$m(b-a) \le \int_a^b f(x)dx \le M(b-a)$$

which implies that

$$m \le \frac{1}{b-a}\int_a^b f(x)dx \le M.$$

Therefore, by the Intermediate Value Theorem, there exists[4] a $c \in [a,b]$ such that

$$f(c) = \frac{1}{b-a}\int_a^b f(x)dx.$$

Multiplying by $(b-a)$ gives the desired result. □

6. (a) We must show that the Simpson Rule S_2 has no error for polynomials $1, x, x^2, x^3$, but has an error for x^4, over a given interval, say, from $x=-i$ to $x=i$. Let us check for exactness for polynomials of increasing degree. We perform the computation directly using $h = \frac{b-a}{2}$:

$$\int_a^b 1dx = b-a$$

and

$$S_2 = \frac{h}{3}(1+4+1) = 2h = 2\frac{b-a}{2} = b-a = \int_a^b 1dx.$$

Next, we have

$$\int_a^b x\,dx = \frac{b^2-a^2}{2} = \frac{(b-a)(b+a)}{2}$$

[2] Continuous on $[a,b]$ also implies Riemann integrable on $[a,b]$.

[3] These values exist by the extreme value theorem.

[4] Technically $c \in [x_{\min}, x_{\max}]$ where $f(x_{\min}) = m$ and $f(x_{\max}) = M$. However, we will skip over that detail as $[x_{\min}, x_{\max}] \subseteq [a,b]$.

and

$$S_2 = \frac{h}{3}\left(a + 4\frac{a+b}{2} + b\right) = \frac{h}{3}\left(3a + 3b\right) = \frac{(b-a)(b+a)}{2} = \int_a^b x\,dx\,.$$

Next, we have

$$\int_a^b x^2\,dx = \frac{b^3 - a^3}{3} = \frac{(b-a)(b^2+ba+a^2)}{3}$$

and

$$S_2 = \frac{h}{3}\left(a^2 + 4\frac{(a+b)^2}{4} + b^2\right) = \frac{h}{3}(2a^2+2b^2+2ab) = \frac{(b-a)(b^2+ba+a^2)}{3} = \int_a^b x^2\,dx\,.$$

Finally, we have

$$\int_a^b x^3\,dx = \frac{b^4 - a^4}{4} = \frac{(b+a)(b-a)(b^2+a^2)}{4}$$

and

$$S_2 = \frac{h}{3}\left(a^3 + \frac{(a+b)^3}{2} + b^3\right) = \frac{h}{3}\left(\frac{3}{2}a^3 + \frac{3}{2}b^3 + \frac{3}{2}a^2b + \frac{3}{2}ab^2\right)$$

$$= \frac{1}{4}(b-a)(b+a)(a^2+b^2) = \int_a^b x^3\,dx\,.$$

Note however, that if we let $a = 0$ and $b = 1$, and we consider $f(x) = x^4$, we get

$$\int_0^1 x^4\,dx = \frac{1}{5}$$

but

$$S_2 = \frac{1}{6}\left(0 + 4\frac{1}{16} + 1\right) = \frac{5}{24} \neq \frac{1}{5}.$$

Therefore, S_2 has degree of precision 3.

(b) We must show that the error when integrating any polynomial of degree three or less using Simpson's rule is zero, while the error for a polynomial of degree four or higher is non-zero. $|E_n(f)| = -\frac{(b-a)}{180}h^4 f^{(4)}(\xi)$ for some $\xi \in [a, b]$. However, note that for any polynomial of degree less than or equal to 3, we have $f^{(4)}(x) = 0$. Therefore, $E_n(f) = 0$. However, $E_n(f) \neq 0$ for $f(x) = x^4$ on $[1, 2]$. Therefore, S_n has degree of precision 3.

(c) To find the degree of precision of the approximation formula $\int_{-1}^1 f(x)dx \approx f(\frac{\sqrt{-3}}{3}) + f(\frac{\sqrt{3}}{3})$, we must test the approximation formula for polynomials $f(x) = 1, x, x^2, x^3, ...$ of increasing degree until the formula is no longer exact. $\int_{-1}^1 1dx = 2$ $f(-\frac{\sqrt{3}}{3}) + f(\frac{\sqrt{3}}{3}) = 1+1 = 2$. The formula is exact for $f(x) = 1$. $\int_{-1}^1 xdx = 0$ $f(-\frac{\sqrt{3}}{3})+f(\frac{\sqrt{3}}{3}) = -\frac{\sqrt{3}}{3} + \frac{\sqrt{3}}{3} = 0$. The formula is exact for $f(x) = x$. $\int_{-1}^1 x^2dx = \frac{x^3}{3}\Big|_{-1}^{1} = \frac{1}{3} - (-\frac{1}{3}) = \frac{2}{3}$

$f(-\frac{\sqrt{3}}{3})+f(\frac{\sqrt{3}}{3}) = (-\frac{\sqrt{3}}{3})^2+(\frac{\sqrt{3}}{3})^2 = \frac{3}{9}+\frac{3}{9} = \frac{2}{3}$. The formula is exact for $f(x) = x^2$. $\int_{-1}^{1} x^3 dx = \frac{x^4}{4}\Big|_{-1}^{1} = \frac{1}{4} - \frac{1}{4} = 0$ $f(-\frac{\sqrt{3}}{3}) + f(\frac{\sqrt{3}}{3}) = (-\frac{\sqrt{3}}{3})^3 + (\frac{\sqrt{3}}{3})^3 = -\frac{3\sqrt{3}}{27} + \frac{3\sqrt{3}}{27} = 0$. The formula is exact for $f(x) = x^3$. $\int_{-1}^{1} x^4 dx = \frac{x^5}{5}\Big|_{-1}^{1} = \frac{1}{5} - (-\frac{1}{5}) = \frac{2}{5}$ $f(-\frac{\sqrt{3}}{3}) + f(\frac{\sqrt{3}}{3}) = (-\frac{\sqrt{3}}{3})^4 + (\frac{\sqrt{3}}{3})^4 = \frac{9}{81} + \frac{9}{81} = \frac{18}{81} = \frac{2}{9}$ The formula is not exact for $f(x) = x^4$ since $\frac{2}{5} \neq \frac{2}{9}$. Thus, the degree of precision for the approximation formula is three.