

Homework 3

Due date: February 6th, 2025

Martin Coyne, Flora Dedvukaj, Jiahao Gao, Anton Karabushin, Zhihan Lin, Joshua Morales

- 1.
2. (a)
- (b)
- (c)
3. (a) Since f is a “suitably” differentiable function, we may assume that $f^{(i)}(x)$ is continuous in some “suitable” region around p , for all $0 \leq i \leq m$. Since p is a root of multiplicity m , We must have that $f^{(m-1)}(p) = 0$ and $f^{(m)}(p) \neq 0$. Therefore, by repeated application of L'Hopital's Rule $m - 1$ times, and by the fact that $f \in C^m$, we have

$$\lim_{x \rightarrow p} \mu(x) = \lim_{x \rightarrow p} \frac{f(x)}{f'(x)} = \lim_{x \rightarrow p} \frac{f^{(m-1)}(x)}{f^{(m)}(x)} = \frac{f^{(m-1)}(p)}{f^{(m)}(p-1)}.$$

which us the desired result.

- (b) Note that since p is a root of f with multiplicity m , there exists a function¹ $g(x)$ such that

$$f(x) = (x - p)^m g(x).$$

where $g(p) \neq 0$. Therefore, we have

$$\mu(x) = \frac{f(x)}{f'(x)} = \frac{(x - p)^m g(x)}{m(x - p)^{m-1} g(x) + (x - p)^m g'(x)} = \frac{(x - p)g(x)}{mg(x) + (x - p)g'(x)}.$$

If we let $h(x) := \frac{g(x)}{mg(x) + (x - p)g'(x)}$, we then have

$$h(p) = \frac{g(p)}{mg(p)} = \frac{1}{m} \neq 0.$$

Thus, since $\mu(x) = (x - p)h(x)$, and $h(p) \neq 0$, p must be a simple root of $\mu(x)$. Therefore, $\mu'(p) \neq 0$, which is the desired result.

- (c) For $x \neq p$, by the quotient rule, we have

$$\frac{\mu(x)}{\mu'(x)} = \frac{f(x)}{f'(x)} \frac{f'(x)^2}{f'(x)^2 - f(x)f''(x)} = \frac{f(x)f'(x)}{f'(x)^2 - f(x)f''(x)}$$

which is the desired result.

4. *Proof.* Suppose there exists, two degree n polynomials, p_1 and p_2 , such that

$$p_1(x_i) = y_i = p_2(x_i)$$

¹which is also differentiable in a suitable interval.

for all $0 \leq i \leq n$. It suffices to show that $p_1(x) = p_2(x)$. Therefore, the polynomial,

$$f(x) = p_1(x) - p_2(x)$$

is a polynomial of degree at most n , with $n + 1$ distinct roots, x_0, x_1, \dots, x_n . However, by the Fundamental Theorem of Algebra, f must be the 0 polynomial.² Therefore,

$$f(x) = 0,$$

which means that $p_1(x) = p_2(x)$, which is the desired result. \square

5. (a) We can take $(0, 7)$, $(2, 11)$, $(3, 28)$, and $(4, 63)$ as (x_0, y_0) , (x_1, y_1) , \dots , (x_n, y_n) respectively. We find the following:

$$L_0(x) = \frac{(x - x_1)(x - x_2)(x - x_3)}{(x_0 - x_1)(x_0 - x_2)(x_0 - x_3)}$$

$$L_1(x) = \frac{(x - x_0)(x - x_2)(x - x_3)}{(x_1 - x_0)(x_1 - x_2)(x_1 - x_3)}$$

$$L_2(x) = \frac{(x - x_0)(x - x_1)(x - x_3)}{(x_2 - x_0)(x_2 - x_1)(x_2 - x_3)}$$

$$L_3(x) = \frac{(x - x_0)(x - x_1)(x - x_2)}{(x_3 - x_0)(x_3 - x_1)(x_3 - x_2)}$$

Plugging in the points from above, we get:

$$L_0(x) = \frac{(x - 2)(x - 3)(x - 4)}{(-2)(-3)(-4)} = \frac{-1}{24}(x - 2)(x - 3)(x - 4)$$

$$L_1(x) = \frac{(x - 0)(x - 3)(x - 4)}{(-2)(-1)(-2)} = \frac{1}{4}(x)(x - 3)(x - 4)$$

$$L_2(x) = \frac{(x - 0)(x - 2)(x - 4)}{(3)(1)(-1)} = \frac{-1}{3}(x)(x - 2)(x - 4)$$

$$L_3(x) = \frac{(x - 0)(x - 2)(x - 3)}{(4)(2)(1)} = \frac{1}{8}(x)(x - 2)(x - 3)$$

Using the y-values from above, we get the following interpolation:

$$P(x) = \frac{-7}{24}(x - 2)(x - 3)(x - 4) + \frac{11}{4}(x)(x - 3)(x - 4) - \frac{28}{3}(x)(x - 2)(x - 4) + \frac{63}{8}(x)(x - 2)(x - 3)$$

Which can be simplified to become

$$P(x) = x^3 - 2x + 7$$

- (b) To find the approximation of $f(1)$, we just plug in 1 for x in $P(x)$:

$$P(1) = 1^3 - 2(1) + 7 = 1 - 2 + 7 = 6$$

²Since otherwise it would be a non-zero degree n polynomial with more than n distinct roots.

(c)

$$\int_0^4 x^3 - 2x + 7 = 76$$

6.

7.