Homework 8

Due date: April 3rd, 2025

1. (a)
   (b)
   (c)

2.

3. *Proof.* First, note that since $x^2 \geq 0$ for all $x \in \mathbb{R}$ and $f \in C[a,b]$, we have that

$$\langle f, f \rangle = \int_a^b f(x)^2 \, dx \geq 0.$$

Furthermore, we have that

$$\langle f, f \rangle = 0 \iff \int_a^b f(x)^2 \, dx = 0 \iff f(x)^2 = 0 \iff f(x) = 0$$

for all $x \in [a,b]$. This is also due to the non-negativity of $f(x)^2$. Next, for any $f, g \in C[a,b]$ we have that

$$\langle f, g \rangle = \int_a^b f(x)g(x) \, dx = \int_a^b g(x)f(x) \, dx = \langle g, f \rangle.$$

Finally, for any $f, g, h \in C[a,b]$ and $c_1, c_2 \in \mathbb{R}$, by the linearity of the Riemann integral, we have that

$$\langle c_1 f(x) + c_2 g(x), h(x) \rangle = \int_a^b (c_1 f(x) + c_2 g(x)) h(x) \, dx = \int_a^b (c_1 f(x)h(x) + c_2 g(x)h(x)) \, dx$$

$$= c_1 \int_a^b f(x)h(x) \, dx + c_2 \int_a^b g(x)h(x) \, dx = c_1 \langle f, h \rangle + c_2 \langle g, h \rangle.$$

Therefore, $\langle \cdot, \cdot \rangle$ is an inner product, which is the desired result. ∎

4. We are given that $\mathbf{v} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 0 \end{pmatrix}$ and $\mathbf{w} = \begin{pmatrix} 1 \\ -1 \\ 2 \\ -2 \end{pmatrix}$.

   (a) $\langle \mathbf{v}, \mathbf{w} \rangle = 1 * 1 + 3 * (-1) + 2 * 2 + 0 * (-2) = 2$

   (b) $||\mathbf{v}|| = \sqrt{1^2 + 3^2 + 2^2} = \sqrt{14}$

   (c) $\mathbf{v} - \mathbf{w} = \begin{pmatrix} 0 \\ 4 \\ 0 \\ 2 \end{pmatrix}$

   $||\mathbf{v} - \mathbf{w}|| = \sqrt{0^2 + 4^2 + 0^2 + 2^2} = 2\sqrt{5}$

(d) $\cos(\theta) = \frac{\langle \mathbf{v},\mathbf{w}\rangle}{||\mathbf{v}||\cdot||\mathbf{w}||} = \frac{2}{\sqrt{14}\cdot\sqrt{1^2+(-1)^2+2^2+(-2)^2}} = \frac{2}{\sqrt{140}}$
$\theta = \cos^{-1}(\frac{2}{\sqrt{140}}) \approx 1.40095$

5. (a)

$$\begin{aligned}\langle f, g\rangle &= \int_0^1 f(x)g(x)dx \\ &= \int_0^1 (x-1)(x+1)dx \\ &= \int_0^1 (x^2-1)dx \\ &= [\frac{x^3}{3} - x]_0^1 \\ &= (\frac{1}{3} - 1) - (\frac{0^3}{3} - 0) = \frac{-2}{3}\end{aligned}$$

(b)

$$||f-g|| = \sqrt{\int_0^1 |(f-g)|dx}$$

$$f(x) - g(x) = (x-1) - (x+1) = x - 1 - x - 1 = -2$$

$$|f(x) - g(x)| = 2$$

$$\int_0^1 2dx = [2x]_0^1 = 2 - 0 = 2$$

(c) The angle between f and g is

$$\cos(\theta) = \frac{\langle f, g\rangle}{||f||\cdot||g||}$$

We already know that $\langle f, g\rangle = \frac{-2}{3}$, so we need to find the norm $||f||$.

$$\begin{aligned}||f|| &= (\int_0^1 (x-1)^2 dx)^{\frac{1}{2}} = (\int_0^1 (x^2 - 2x + 1)dx)^{\frac{1}{2}} \\ &= ([\frac{x^3}{3} - 2\frac{x^2}{2} + x]_0^1)^{\frac{1}{2}} = ((\frac{1}{3} - 2\frac{x^2}{2} + x) - (0))^{\frac{1}{2}} \\ &= (\frac{1}{3} - 1 + 1)^{\frac{1}{2}} = \frac{1}{\sqrt{3}}\end{aligned}$$

.

Now we need to find $||g||$.

$$||g|| = (\int_0^1 (x+1)^2 dx)^{\frac{1}{2}} = (\int_0^1 (x^2 + 2x + 1)dx)^{\frac{1}{2}} = ([\frac{x^3}{3} + x^2 + x]_0^1)^{\frac{1}{2}}$$

$$= (\frac{1}{3} + 1 + 1)^{\frac{1}{2}} = \sqrt{\frac{7}{3}}$$

We can now use both of these norms to find θ:

$$\cos(\theta) = \frac{\frac{-2}{3}}{\frac{-1}{3} \cdot \sqrt{\frac{7}{3}}} = \frac{\frac{-2}{3}}{\sqrt{\frac{7}{9}}} = \frac{\frac{-2}{3}}{\frac{\sqrt{7}}{3}} = \frac{-2}{\sqrt{7}}$$

Now we can solve for θ:

$$\theta = \cos^{-1}(\frac{-2}{\sqrt{7}}) \approx \cos^{-1}(-0.7559) \approx 139.1^\circ$$

(d) To find a nonzero h that is perpendicular to f, we would have to find a function $h(x)$ that satisfies the following:

$$\int_0^1 (x-1) \cdot h(x) dx = 0$$

We can try a simple polynomial $h(x) = ax + b$, and plug it into the above orthogonality condition:

$$\int_0^1 (x-1)(ax+b) dx = 0$$

$$(x-1)(ax+b) = (ax^2 + bx - ax - b) = ax^2 + (b-a)x - b$$

$$\int_0^1 (ax^2 + (b-a)x - b) dx = a\frac{1}{3} + (b-a)\frac{1}{2} - b = 0$$

$$2a + 3(b-a) - 6b = 0 \to 2a + 3b - 3a - 6b = 0 \to -a - 3b = 0 \to a = -3b$$

Let us pick $b = 1$, then $a = -3$, so $h(x) = -3x + 1$

6. We know that $\mathbf{v_1} = (1, 2, 2), \mathbf{v_2} = (-1, 0, 2)$, and $\mathbf{v_3} = (0, 0, 1)$

(a)

$$\mathbf{v_1} = \mathbf{x_1} \text{ and } ||\mathbf{x_1}||_2 = \sqrt{1^2 + 2^2 + 2^2}$$

$$||\mathbf{w}|| = \frac{\mathbf{x_1}}{||\mathbf{x_1}||_2} = (\frac{1}{3}, \frac{2}{3}, \frac{2}{3})$$

$$\mathbf{x_2} = \mathbf{v_2} - \langle \mathbf{v_2}, \mathbf{w_1} \rangle \mathbf{w_1} = (-1, 0, 2) - \langle (-1, 0, 2), (\frac{1}{3}, \frac{2}{3}, \frac{2}{3}) \rangle$$

$$= (-1, 0, 2) - 1(\frac{1}{3}, \frac{2}{3}, \frac{2}{3})$$

$$= (\frac{-4}{3}, \frac{-2}{3}, \frac{4}{3})$$

$$||\mathbf{x^2}||_2 = \sqrt{(\frac{-4}{3})^2 + (\frac{-2}{3})^2 + (\frac{4}{3})^2} = 2$$

$$\mathbf{w_2} = \frac{\mathbf{x_2}}{||\mathbf{x_2}||_2} = (\frac{-2}{3}, \frac{-1}{3}, \frac{2}{3})$$

$$\mathbf{x_3} = \mathbf{v_3} - \langle \mathbf{v_3}, \mathbf{w_1} \rangle \mathbf{w_1} - \langle \mathbf{v_3}, \mathbf{w_2} \rangle \mathbf{w_2}$$

$$\mathbf{x_3} = (0,0,1) - \langle (0,0,1), (\frac{1}{3}, \frac{2}{3}, \frac{2}{3}) \rangle (\frac{1}{3}, \frac{2}{3}, \frac{2}{3}) - \langle (0,0,1)(\frac{-2}{3}, \frac{-1}{3}, \frac{2}{3}) \rangle (\frac{-2}{3}, \frac{-1}{3}, \frac{2}{3})$$

$$= (0,0,1) - \frac{2}{3}(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}) - \frac{2}{3}(\frac{-2}{3}, \frac{-1}{3}, \frac{2}{3})$$

$$= (\frac{2}{9}, \frac{-2}{9}, \frac{1}{9})$$

$$||\mathbf{x_3}||_2 = \frac{1}{3}$$

$$\mathbf{w_3} = \frac{\mathbf{x_3}}{||\mathbf{x_3}||_2} = (\frac{2}{3}, \frac{-2}{3}, \frac{1}{3})$$

(b) We know $\mathbf{x} = (7,5,1)$

$$\langle (7,5,1), (\frac{1}{3}, \frac{2}{3}, \frac{2}{3}) \rangle = \frac{19}{3}$$

$$\langle (7,5,1), (\frac{-2}{3}, \frac{-1}{3}, \frac{2}{3}) \rangle = \frac{-17}{3}$$

$$\langle (7,5,1), (\frac{2}{3}, \frac{-2}{3}, \frac{1}{3}) \rangle = \frac{5}{3}$$

.

Now, we can write

$$\frac{19}{3}(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}) - \frac{17}{3}(\frac{-2}{3}, \frac{-1}{3}, \frac{2}{3}) + \frac{5}{3}(\frac{2}{3}, \frac{-2}{3}, \frac{1}{3})$$