Homework 9

Due date: April 9th, 2025

1. Since $\{w_1, w_2, \ldots, w_k\}$ is an orthonormal basis for W, we can represent w using $a = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_k \end{bmatrix} \in \mathbb{R}^k$.

$$w = a_1 w_1 + \cdots + a_k w_k = Aa$$

w is the orthogonal projection of v onto $W \iff v - w$ is orthogonal to all vectors in W. Specifically it must be orthogonal to each basis vector, meaning for every $j = 1, 2, \ldots, k$

$$\langle v - w \ , \ w_j \rangle = 0$$

Since the matrix A has the basis vectors as columns, we can say that

$$A^T (v - w)_j = \langle w_j \ , \ v - w \rangle$$

And since all inner products are zero, this is equivalent to

$$A^T (v - w) = 0$$

Using our previous representation of w, this can be rewritten as

$$A^T (v - Aa) = 0$$

$$= A^T v - A^T A a = 0$$

And because the columns of A form an orthonormal set, $A^T A = I_k$, we get

$$A^T v - I_k a = A^T v - a = 0$$

$$a = A^T v$$

So going back to the previous representation of w,

$$w = Aa = A(A^T v) = AA^T v$$

Thus this vector w is the closest approximation of v.

2. First we note that for all $0 \leq k \leq n$, we have that

$$\left\langle \frac{1}{\sqrt{2}}, \cos(kx) \right\rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{1}{\sqrt{2}} \cos(kx)\, dx \ = \frac{1}{\pi} \left[\frac{\sin(kx)}{k} \right] \Big|_{-\pi}^{\pi} = 0$$

where we used the fact that $\sin(k\pi) = 0$ for all integers k.

Furthermore, note that

$$\left\langle \frac{1}{\sqrt{2}}, \sin(kx) \right\rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{1}{\sqrt{2}} \sin(kx)\, dx \; = 0$$

since $\sin(kx)$ is an odd function for all integers k.

Next, note that

$$\left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{1}{2}\, dx \; = \frac{1}{\pi} \frac{2\pi}{2} = 1.$$

Now for all positive integers m and n, we first note that $\cos(mx)\sin(nx)$ is an odd function, so they will have an inner product of 0. Considering $\cos(nx)$ and $\cos(mx)$, we have two cases. If $n = m$, we have that

$$\langle \cos(mx), \cos(mx) \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} \cos^2(mx)\, dx \; = \frac{1}{\pi} \int_{0}^{\pi} \left(1 + \cos(2mx)\right)\, dx \; = \frac{1}{\pi} \left(x + \frac{1}{2m} \sin(2mx) \right) \Bigg|_0^{\pi}$$

$$= \frac{1}{\pi} \pi = 1.$$

If $n \neq m$, then we have

$$\langle \cos(nx), \cos(mx) \rangle = \frac{2}{\pi} \int_{0}^{\pi} \cos(nx)\cos(mx)\, dx \; = \frac{1}{\pi} \int_{0}^{\pi} \left(\cos((n-m)x) + \cos((n+m)x)\right)\, dx$$

$$= \frac{1}{\pi} \left(\frac{1}{n-m} \sin((n-m)x) + \frac{1}{n+m} \sin((n+m)x) \right) \Bigg|_0^{\pi} = 0$$

where we once again used that $\sin(k\pi) = 0$.

We finally consider $\sin(nx)$ and $\sin(mx)$, which once again has two cases. If $n = m$, then

$$\langle \sin(mx), \sin(mx) \rangle = \frac{2}{\pi} \int_{0}^{\pi} \sin^2(mx)\, dx \; = \frac{1}{\pi} \int_{0}^{\pi} \left(1 - \cos(2mx)\right)\, dx \; = \frac{1}{\pi} \left(x - \frac{1}{2m} \sin(2mx) \right) \Bigg|_0^{\pi}$$

$$= \frac{1}{\pi} \pi = 1.$$

If $n \neq m$, then we have

$$\langle \sin(nx), \sin(mx) \rangle = \frac{2}{\pi} \int_{0}^{\pi} \sin(nx)\sin(mx)\, dx \; = \frac{1}{\pi} \int_{0}^{\pi} \left(\cos((n-m)x) - \cos((n+m)x)\right)\, dx$$

$$= \frac{1}{\pi} \left(\frac{1}{n-m} \sin((n-m)x) - \frac{1}{n+m} \sin((n+m)x) \right) \Bigg|_0^{\pi} = 0$$

by the same reasons as above. Therefore, the set is an orthonormal set, which is the desired result.

3. (a) We seek the continuous Fourier approximation $S_3(x)$ of the function $f(x) = e^x$ on the interval $[-\pi, \pi]$, using the formula:

$$S_n(x) = \langle f(x), 1\rangle \cdot \frac{1}{2} + \sum_{k=1}^{n} \left(\langle f(x), \sin(kx)\rangle \cdot \sin(kx) + \langle f(x), \cos(kx)\rangle \cdot \cos(kx)\right)$$

with the inner product defined as:

$$\langle f(x), g(x)\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)g(x)\,dx$$

$$a_0 = \langle f(x), 1\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} e^x\,dx = \frac{1}{\pi}(e^{\pi} - e^{\pi})$$

$$\Rightarrow \frac{a_0}{2} = \frac{1}{2\pi}(e^{\pi} - e^{-\pi})$$

$$a_1 = \frac{1}{\pi}\int_{-\pi}^{\pi} e^x \cos(x)\,dx = \frac{1}{\pi}\left[\frac{e^x}{2}(\sin(x) + \cos(x))\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left(-\frac{e^{\pi}}{2} + \frac{e^{-\pi}}{2}\right) = -\frac{1}{2\pi}(e^{\pi} - e^{-\pi})$$

$$b_1 = \frac{1}{\pi}\int_{-\pi}^{\pi} e^x \sin(x)\,dx = \frac{1}{\pi}\left[\frac{e^x}{2}(\sin(x) - \cos(x))\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left(\frac{e^{\pi}}{2} - \frac{e^{-\pi}}{2}\right) = \frac{1}{2\pi}(e^{\pi} - e^{-\pi})$$

Using:

$$\int e^x \cos(ax)dx = \frac{e^x(a\sin(ax) + \cos(ax))}{a^2+1}, \qquad \int e^x \sin(ax)dx = \frac{e^x(\sin(ax) - a\cos(ax))}{a^2+1}$$

We get:

$$a_2 = \frac{1}{\pi}\left[\frac{e^x(2\sin(2x) + \cos(2x))}{5}\right]_{-\pi}^{\pi} = \frac{1}{5\pi}(e^{\pi} - e^{-\pi})$$

$$b_2 = \frac{1}{\pi}\left[\frac{e^x(\sin(2x) - 2\cos(2x))}{5}\right]_{-\pi}^{\pi} = -\frac{2}{5\pi}(e^{\pi} - e^{-\pi})$$

$$a_3 = \frac{1}{\pi}\left[\frac{e^x(3\sin(3x) + \cos(3x))}{10}\right]_{-\pi}^{\pi} = -\frac{1}{10\pi}(e^{\pi} - e^{-\pi})$$

$$b_3 = \frac{1}{\pi}\left[\frac{e^x(\sin(3x) - 3\cos(3x))}{10}\right]_{-\pi}^{\pi} = \frac{3}{10\pi}(e^{\pi} - e^{-\pi})$$

Evaluating numerically, we have:

$$S_3(x) \approx 3.6767 - 3.6767\cos(x) + 3.6767\sin(x) + 1.4707\cos(2x) - 2.9414\sin(2x) - 0.7353\cos(3x) + 2.2060\,\mathrm{s}$$

(b) We must calculate $\langle y, \phi_k\rangle$ and $\langle y, \psi_k\rangle$ for $k = 0, 1, 2, 3$ First, calculate $\langle y, \phi_0\rangle$,

$$\langle y, \phi_k\rangle = \frac{1}{10}\sum_{j=0}^{19} y_j \cos(kx_j)$$

$\langle y, \phi_1\rangle$, $\langle y, \phi_2\rangle$, and $\langle y, \phi_3\rangle$.

$$\langle y, \phi_0\rangle = \frac{1}{10}\sum_{j=0}^{19} y_j \cos(0) = \frac{1}{10}\sum_{j=0}^{19} y_j$$

$$\langle y, \phi_1\rangle = \frac{1}{10}\sum_{j=0}^{19} y_j \cos(x_j)$$

$$\langle y, \phi_2\rangle = \frac{1}{10}\sum_{j=0}^{19} y_j \cos(2x_j)$$

$$\langle y, \phi_3\rangle = \frac{1}{10}\sum_{j=0}^{19} y_j \cos(3x_j)$$

Next, calculate $\langle y, \psi_k\rangle$ for $k = 1, 2, 3$

$$\langle y, \psi_k\rangle = \frac{1}{10}\sum_{j=0}^{19} y_j \sin(kx_j)$$

$$\langle y, \psi_1\rangle = \frac{1}{10}\sum_{j=0}^{19} y_j \sin(x_j)$$

$$\langle y, \psi_2\rangle = \frac{1}{10}\sum_{j=0}^{19} y_j \sin(2x_j)$$

$$\langle y, \psi_3\rangle = \frac{1}{10}\sum_{j=0}^{19} y_j \sin(3x_j)$$

$$S_3(x) = \frac{1}{2}\langle y, \phi_0\rangle + \sum_{k=1}^{3}(\langle y, \phi_k\rangle \cos(kx)) + \sum_{k=1}^{3}(\langle y, \psi_k\rangle \sin(kx))$$

$$S_3(x) = \frac{1}{2}\langle y, \phi_0\rangle + \langle y, \phi_1\rangle \cos(x) + \langle y, \phi_2\rangle \cos(2x) + \langle y, \phi_3\rangle \cos(3x) + \langle y, \psi_1\rangle \sin(x) + \langle y, \psi_2\rangle \sin(2x) + \langle y, \psi_3\rangle$$

$$S_3(x) \approx 3.676 + 2.963\cos(x) + 0.946\cos(2x) + 0.164\cos(3x) - 1.263\sin(x) - 0.740\sin(2x) - 0.214\sin(3x)$$

4. (a) Q is orthogonal.

$$Q^tQ = 1$$
$$\det(Q^tQ) = 1$$
$$\det(Q^t)\det(Q) = 1$$

Since $\det(Q^t) = \det(Q)$,

$$\det(Q)^2 = 1$$
$$\det(Q) \pm 1$$

(b)

$$(PQ)^t PQ = Q^t P^t PQ = Q^t(I)Q = Q^tQ = I$$

. So PQ is orthogonal.

5. (a) Note that since $\det(P^{-1}) = \det(P)^{-1}$, we have that

$$\det(B) = \det(P^{-1}AP) = \det(P^{-1})\det(A)\det(P) = \det(A)\det(P)\det(P)^{-1} = \det(A).$$

(b) It suffices to show that A and B have the same characteristic polynomial. Indeed, we have that

$$\det(B-\lambda I) = \det(P^{-1}AP-\lambda P^{-1}P) = \det(P^{-1}AP-P^{-1}(\lambda P)) = \det(P^{-1}(A-\lambda I)P) = \det(A-\lambda I).$$

(c) Since A is invertible, then $\det(B) = \det(A) \neq 0$. Therefore, B is also invertible. Furthermore, we have that

$$B^{-1} = (P^{-1}AP)^{-1} = P^{-1}A^{-1}(P^{-1})^{-1} = P^{-1}A^{-1}P.$$

Therefore, A^{-1} and B^{-1} are similar, which is the desired result.

6. We are given the symmetric matrix:

$$A = \begin{bmatrix} 1 & -1 & 2 \\ -1 & 1 & 2 \\ 2 & 2 & 2 \end{bmatrix}$$

We want to decompose it as:

$$A = QDQ^T$$

To find the eigenvalues, we solve the characteristic polynomial:

$$\det(A - \lambda I)$$

$$A - \lambda I = \begin{bmatrix} 1 & -1 & 2 \\ -1 & 1 & 2 \\ 2 & 2 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

$$= \begin{bmatrix} 1-\lambda & -1 & 2 \\ -1 & 1-\lambda & 2 \\ 2 & 2 & 2-\lambda \end{bmatrix}$$

We know can find the determinant of the matrix:

$$(1-\lambda)\begin{bmatrix} 1-\lambda & 2 \\ 2 & 2-\lambda \end{bmatrix} - (-1)\begin{bmatrix} -1 & 2 \\ 2 & 2-\lambda \end{bmatrix} + 2\begin{bmatrix} -1 & 1-\lambda \\ 2 & 2 \end{bmatrix} = -\lambda^3+4\lambda^2+4\lambda-16 = -(\lambda-4)(\lambda-2)(\lambda+2)$$

The eigenvalues are:

$$\lambda_1 = 4, \quad \lambda_2 = 2, \quad \lambda_3 = -2$$

Solve $(A-4I)\vec{v} = 0$, we get:

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \quad \|\vec{v}_1\| = \sqrt{6} \Rightarrow q_1 = \frac{1}{\sqrt{6}}\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

Solve $(A-2I)\vec{v} = 0$, we get:

$$\vec{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \quad \|\vec{v}_2\| = \sqrt{2} \Rightarrow q_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

Solve $(A+2I)\vec{v} = 0$, we get:

$$\vec{v}_3 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \quad \|\vec{v}_3\| = \sqrt{3} \Rightarrow q_3 = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

Orthogonal matrix Q (columns are the normalized eigenvectors):

$$Q = \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \frac{2}{\sqrt{6}} & 0 & -\frac{1}{\sqrt{3}} \end{bmatrix}$$

Diagonal matrix D of eigenvalues:

$$D = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$