Homework 8

Due date: April 3rd, 2025

1. (a)
   (b)
   (c)
2.
3.
4. We are given that $\mathbf{v} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 0 \end{pmatrix}$ and $\mathbf{w} = \begin{pmatrix} 1 \\ -1 \\ 2 \\ -2 \end{pmatrix}$.

   (a) $\langle \mathbf{v}, \mathbf{w} \rangle = 1 * 1 + 3 * (-1) + 2 * 2 + 0 * (-2) = 2$

   (b) $||\mathbf{v}|| = \sqrt{1^2 + 3^2 + 2^2} = \sqrt{14}$

   (c) $\mathbf{v} - \mathbf{w} = \begin{pmatrix} 0 \\ 4 \\ 0 \\ 2 \end{pmatrix}$

   $||\mathbf{v} - \mathbf{w}|| = \sqrt{0^2 + 4^2 + 0^2 + 2^2} = 2\sqrt{5}$

   (d) $\cos(\theta) = \frac{\langle \mathbf{v}, \mathbf{w} \rangle}{||\mathbf{v}|| \cdot ||\mathbf{w}||} = \frac{2}{\sqrt{14} \cdot \sqrt{1^2 + (-1)^2 + 2^2 + (-2)^2}} = \frac{2}{\sqrt{140}}$

   $\theta = \cos^{-1}(\frac{2}{\sqrt{140}}) \approx 1.40095$

5. (a)
   (b)
   (c)
   (d)
6. We know that $\mathbf{v_1} = (1, 2, 2), \mathbf{v_2} = (-1, 0, 2)$, and $\mathbf{v_3} = (0, 0, 1)$

   (a)

$$\mathbf{v_1} = \mathbf{x_1} \text{ and } ||\mathbf{x_1}||_2 = \sqrt{1^2 + 2^2 + 2^2}$$

$$||\mathbf{w}|| = \frac{\mathbf{x_1}}{||\mathbf{x_1}||_2} = (\frac{1}{3}, \frac{2}{3}, \frac{2}{3})$$

$$\mathbf{x_2} = \mathbf{v_2} - \langle \mathbf{v_2}, \mathbf{w_1} \rangle \mathbf{w_1} = (-1, 0, 2) - \langle (-1, 0, 2), (\frac{1}{3}, \frac{2}{3}, \frac{2}{3}) \rangle$$

$$= (-1, 0, 2) - 1(\frac{1}{3}, \frac{2}{3}, \frac{2}{3})$$

$$= (\frac{-4}{3}, \frac{-2}{3}, \frac{4}{3})$$

$$||\mathbf{x^2}||_2 = \sqrt{(\frac{-4}{3})^2 + (\frac{-2}{3})^2 + (\frac{4}{3})^2} = 2$$

$$\mathbf{w_2} = \frac{\mathbf{x_2}}{||\mathbf{x_2}||_2} = (\frac{-2}{3}, \frac{-1}{3}, \frac{2}{3})$$

$$\mathbf{x_3} = \mathbf{v_3} - \langle \mathbf{v_3}, \mathbf{w_1} \rangle \mathbf{w_1} - \langle \mathbf{v_3}, \mathbf{w_2} \rangle \mathbf{w_2}$$

$$\mathbf{x_3} = (0,0,1) - \langle (0,0,1), (\frac{1}{3}, \frac{2}{3}, \frac{2}{3}) \rangle (\frac{1}{3}, \frac{2}{3}, \frac{2}{3}) - \langle (0,0,1)(\frac{-2}{3}, \frac{-1}{3}, \frac{2}{3}) \rangle (\frac{-2}{3}, \frac{-1}{3}, \frac{2}{3})$$

$$= (0,0,1) - \frac{2}{3}(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}) - \frac{2}{3}(\frac{-2}{3}, \frac{-1}{3}, \frac{2}{3})$$

$$= (\frac{2}{9}, \frac{-2}{9}, \frac{1}{9})$$

$$||\mathbf{x_3}||_2 = \frac{1}{3}$$

$$\mathbf{w_3} = \frac{\mathbf{x_3}}{||\mathbf{x_3}||_2} = (\frac{2}{3}, \frac{-2}{3}, \frac{1}{3})$$

(b) We know $\mathbf{x} = (7,5,1)$

$$\langle (7,5,1), (\frac{1}{3}, \frac{2}{3}, \frac{2}{3}) \rangle = \frac{19}{3}$$

$$\langle (7,5,1), (\frac{-2}{3}, \frac{-1}{3}, \frac{2}{3}) \rangle = \frac{-17}{3}$$

$$\langle (7,5,1), (\frac{2}{3}, \frac{-2}{3}, \frac{1}{3}) \rangle = \frac{5}{3}$$

.

Now, we can write

$$\frac{19}{3}(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}) - \frac{17}{3}(\frac{-2}{3}, \frac{-1}{3}, \frac{2}{3}) + \frac{5}{3}(\frac{2}{3}, \frac{-2}{3}, \frac{1}{3})$$