Homework

Due date: February 27, 2025

1. Let $f(x) = 3xe^x - cosx$. By using the forward-difference formula, the three-point midpoint formula, and the five-point midpoint formula with $h = 0.1, 0.05, 0.01$, compute approximations of $f'(1.3)$.

For $x_0 = 1.3$, the function at the values modified by h are as follows.

h	Rule	x	$f(x)$
0.1	$x_0 - h$	1.2	11.5901
	$x_0 + h$	1.4	16.8619
	$x_0 - 2h$	1.1	9.4602
	$x_0 + 2h$	1.5	20.0969
0.05	$x_0 - h$	1.25	12.7735
	$x_0 + h$	1.35	15.4036
	$x_0 - 2h$	1.2	11.5901
	$x_0 + 2h$	1.4	16.8619
0.01	$x_0 - h$	1.29	13.7818
	$x_0 + h$	1.31	14.3074
	$x_0 - 2h$	1.28	13.5244
	$x_0 + 2h$	1.32	14.5758

Forward Difference

$$f'(x_0) = \frac{1}{h}(f(x_0 + h) - f(x_0)) + O(h)$$

For $h = 0.1$

$$f'(1.3) \approx \frac{1}{0.1}(f(1.4) - f(1.3)) = \frac{16.8619 - 14.0427}{0.1} = 28.191$$

For $h = 0.05$

$$f'(1.3) \approx \frac{1}{0.05}(f(1.35) - f(1.3)) = \frac{15.4036 - 14.0427}{0.05} = 27.216$$

For $h = 0.01$

$$f'(1.3) \approx \frac{1}{0.01}(f(1.31) - f(1.3)) = \frac{14.3074 - 14.0427}{0.01} = 26.47$$

3 point midpoint

$$f'(x_0) = \frac{1}{2h}(f(x_0 + h) - f(x_0 - h)) + O(h^2)$$

For $h = 0.1$

$$f'(1.3) \approx \frac{1}{2(0.1)}(16.8619 - 11.5901) = 26.359$$

For $h = 0.05$

$$f'(1.3) \approx \frac{1}{2(0.05)}(15.4036 - 12.7735) = 26.301$$

For $h = 0.01$

$$f'(1.3) \approx \frac{1}{2(0.01)}(14.3074 - 13.7818) = 26.28$$

5 point midpoint

$$f'(x_0) = \frac{1}{12h}(f(x_0 - 2h) - 8f(x_0 - h) + 8f(x_0 + h) - f(x_0 + 2h)) + O(h^4)$$

For $h = 0.1$

$$f'(1.3) \approx \frac{1}{12(0.1)}((9.4602) - 8(11.5901) + 8(16.8619) - 20.0969) = 26.2814$$

For $h = 0.05$

$$f'(1.3) \approx \frac{1}{12(0.05)}((11.5901) - 8(12.7735) + 8(15.4036) - (16.8619)) = 26.2817$$

For $h = 0.01$

$$f'(1.3) \approx \frac{1}{12(0.01)}((13.5244) - 8(13.7818) + 8(14.3074) - (14.5758)) = 26.2783$$

2.

3. We can differentiate $y = x^3$ as follows:

$$f'(x) = \frac{d}{dx}x^3 = 3x^2$$

. Thus the integrand becomes

$$\sqrt{1 + (3x^2)^2} = \sqrt{1 + 9x^4}$$

. So using Simpson's rule, we need to evaluate

$$L = \int_0^1 \sqrt{1 + 9x^4}dx$$

. We can approximate the integral as follows:

$$\int_a^b f(x)dx \approx \frac{h}{3}[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + 4f(x_5) + f(x_6)]$$

where $h = \frac{b-a}{n}$ and $x_i = a + ih$. For $n = 6$, $a = 0$, and $b = 1$,

$$h = \frac{1 - 0}{6} = \frac{1}{6}$$

The nodes are

$$x_0 = 0, \quad x_1 = \frac{1}{6}, \quad x_2 = \frac{2}{6}, \quad x_3 = \frac{3}{6}, \quad x_4 = \frac{4}{6}, \quad x_5 = \frac{5}{6}, \quad x_6 = 1.$$

So we can evaluate $f(x) = \sqrt{1+9x^4}$ at these points:

$$f(0) = \sqrt{1+9(0)^4} = \sqrt{1} = 1.$$

$$f\left(\frac{1}{6}\right) = \sqrt{1+9\left(\frac{1}{6}\right)^4} = \sqrt{1.00694} \approx 1.00347$$

$$f\left(\frac{2}{6}\right) = \sqrt{1+9\left(\frac{2}{6}\right)^4} = \sqrt{1.05556} \approx 1.02747$$

$$f\left(\frac{3}{6}\right) = \sqrt{1+9\left(\frac{3}{6}\right)^4} = \sqrt{1.5625} = 1.25$$

$$f\left(\frac{4}{6}\right) = \sqrt{1+9\left(\frac{4}{6}\right)^4} = \sqrt{1.7778} \approx 1.3333$$

$$f\left(\frac{5}{6}\right) = \sqrt{1+9\left(\frac{5}{6}\right)^4} = \sqrt{2.3403} \approx 1.53$$

$$f(1) = \sqrt{1+9(1)^4} = \sqrt{10} \approx 3.1623$$

We can then use the Simpson's rule formula

$$L \approx \frac{h}{3}\left[f(0) + 4f\left(\frac{1}{6}\right) + 2f\left(\frac{2}{6}\right) + 4f\left(\frac{3}{6}\right) + 2f\left(\frac{4}{6}\right) + 4f\left(\frac{5}{6}\right) + f(1)\right]$$

And substitute values to get the following:

$$L \approx \frac{1}{18}\left[1 + 4(1.00347) + 2(1.02747) + 4(1.25) + 2(1.3333) + 4(1.53) + 3.1623\right]$$

$$L \approx \frac{1}{18}\left[1 + 4.0139 + 2.0549 + 5 + 2.6667 + 6.12 + 3.1623\right]$$

$$L \approx \frac{1}{18} \times 24.0178$$

$$L \approx 1.3343$$

4.

5. (a) *Proof.* By the extreme value theorem. f has a maximum and minimum on $[a, b]$. Therefore, there exists $x_{\min}, x_{\max} \in [a, b]$ such that $f(x_{\min}) = \min_{a \leq x \leq b} f(x)$ and $f(x\max) = \max_{a \leq x \leq b} f(x)$. Furthermore, by definition of the maximum and minimum, we have

$$f(x_{\min}) = \frac{nf(x_{\min})}{n} = \frac{\sum_{i=1}^{n} f(x_{\min})}{n} \leq \frac{\sum_{i=1}^{n} f(x_i)}{n} \leq \frac{\sum_{i=1}^{n} f(x_{\max})}{n} = f(x_{\max}).$$

Therefore, by the intermediate value theorem, there exists an $c \in [x_{\min}, x_{\max}] \subseteq [a, b]$ such that

$$f(c) = \frac{\sum_{i=1}^{n} f(x_i)}{n}$$

which is the desired result. □

(b)

Theorem 1 (Integral Mean Value Theorem for $g(x) = 1$)**.** *Let f be a continuous function on $[a, b]$. Then there exists a $c \in [a.b]$ such that*

$$f(c)(b - a) = \int_a^b f(x)dx.$$

Lemma 1. *Let f be a Riemann integrable function on $[a, b]$, such that*

$$f(x) \geq 0$$

for all $x \in [a, b]$. Then

$$\int_a^b f(x)dx \geq 0.$$

Proof of Lemma 1. Let $P = \{a = x_0, x_1, \ldots, x_n = b\}$ be an arbitrary partition of $[a, b]$. Since $f(x) \geq 0$ on $[a, b]$, we have that

$$U(f, P) := \sum_{i=1}^{n} (x_i - x_{i-1})M_i \geq 0$$

and

$$L(f, P) := \sum_{i=1}^{n} (x_i - x_{i-1})m_i \geq 0$$

where $M_i := \sup_{x \in [x_{i-1}, x_i]} f(x)$ and $M_i := \inf_{x \in [x_{i-1}, x_i]} f(x)$. Therefore, since P was arbitrary, we must have that

$$\overline{\int_a^b} f(x)dx := \inf\{U(f, P) \mid P \text{ is a partition of } [a, b]\} \geq 0.$$

Since f is Riemann integrable, we finally have that

$$\int_a^b f(x)dx = \overline{\int_a^b} f(x)dx \geq 0$$

which is the desired result. $\square$

Lemma 2. *Let f and g be Riemann integrable functions on $[a, b]$ such that for all $x \in [a, b]$,*

$$f(x) \geq g(x).$$

Then

$$\int_a^b f(x)dx \geq \int_a^b g(x)dx$$

Proof of Lemma 2. Consider $h(x) := f(x) - g(x)$. Since f and g are Riemann integrable on $[a, b]$, h is also Riemann integrable on $[a, b]$. Furthermore, since $f(x) \geq g(x)$ for all $x \in [a, b]$,

$$h(x) \geq 0$$

for all $x \in [a, b]$. Therefore, by Lemma 1 and the additivity of the Riemann integral, we have that

$$\int_a^b f(x)dx - \int_a^b g(x)dx \geq 0.$$

Adding $\int_a^b g(x)dx$ to both sides completes the proof. $\square$

Lemma 3. *Let f and g be Riemann integrable functions on $[a, b]$ such that for all $x \in [a, b]$,*

$$f(x) \leq g(x).$$

Then

$$\int_a^b f(x)dx \leq \int_a^b g(x)dx$$

Proof of Lemma 3. Since $f(x) \leq g(x)$ on $[a, b]$, then $-f(x) \geq -g(x)$ on $[a, b]$. Furthermore, since f and g are both Riemann integrable, then $-f$ and $-g$ are also both Riemann integrable. Therefore, by Lemma 2 and the linearity of the integral, we have

$$-\int_a^b f(x)dx \geq -\int_a^b g(x)dx.$$

Multiplying by -1 completes the proof. $\square$

Lemma 4. *Let f be a continuous function*[1] *on $[a,b]$. Let*[2] $m = \min_{a \leq x \leq b} f(x)$ *and* $M = \max_{a \leq x \leq b} f(x)$. *Then*

$$m(b-a) \leq \int_a^b f(x)dx \leq M(b-a).$$

Proof of Lemma 4. Since $m \leq f(x) \leq M$ for all $x \in [a,b]$. By Lemma 2, Lemma 3, and properties of the Riemann integral, we have that

$$m(b-a) = \int_a^b m dx \leq \int_a^b f(x)dx \leq \int_a^b M dx = M(b-a)$$

which was the desired result. □

Proof of Theorem 1. Since f is continuous on $[a,b]$, f attains a maximum and minimum on $[a,b]$, say M and m respectively. Then by Lemma 4, we have that

$$m(b-a) \leq \int_a^b f(x)dx \leq M(b-a)$$

which implies that

$$m \leq \frac{1}{b-a} \int_a^b f(x)dx \leq M.$$

Therefore, by the Intermediate Value Theorem, there exists[3] a $c \in [a,b]$ such that

$$f(c) = \frac{1}{b-a} \int_a^b f(x)dx.$$

Multiplying by $(b-a)$ gives the desired result. □

6. (a) We must show that the Simpson Rule S_2 has no error for polynomials $1, x, x^2, x^3$ over a given interval. Consider the third-degree polynomial $f(x) = ax^3 + bx^2 + cx + d$.

$$S_2 = \int_a^b f(x)dx \approx \frac{h}{3}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$$

where $h = \frac{b-a}{2}$. Plugging in $f(x)$ yields

$$\int_a^b (ax^3+bx^2+cx+d)dx \approx \frac{b-a}{6}\left[(a \cdot a^3 + b \cdot a^2 + c \cdot a + d) + 4\left(a\left(\frac{a+b}{2}\right)^3 + b\left(\frac{a+b}{2}\right)^2 + c\left(\right.\right.\right.$$

. $= \frac{b-a}{6}\left[2a(a^3+b^3) + 2b(a^2+b^2) + 2c(a+b) + 6d\right] = \frac{a}{4}(b^4 - a^4) + \frac{b}{3}(b^3 - a^3) + \frac{c}{2}(b^2 - a^2) + d(b-a)$. The exact integral of f(x) is: $\int_a^b (ax^3 + bx^2 + cx + d)dx = \left[\frac{ax^4}{4} + \frac{bx^3}{3} + \frac{cx^2}{2} + dx\right]_a^b = \frac{a}{4}(b^4 - a^4) + \frac{b}{3}(b^3 - a^3) + \frac{c}{2}(b^2 - a^2) + d(b-a)$ S_2 is equal to $\int_a^b f(x)dx$, so it has a degree of precision 3.

[1] Continuous on $[a,b]$ also implies Riemann integrable on $[a,b]$.

[2] These values exist by the extreme value theorem.

[3] Technically $c \in [x_{\min}, x_{\max}]$ where $f(x_{\min}) = m$ and $f(x_{\max}) = M$. However, we will skip over that detail as $[x_{\min}, x_{\max}] \subseteq [a,b]$.

(b) We must show that the error when integrating any polynomial of degree three or less using Simpson's rule is zero, while the error for a polynomial of degree four or higher is non-zero. $|E_S| = -\frac{(b-a)^5}{180}h^4 f^4$ Let f(x) = x^3. The fourth derivative of $f(x)$ is $f^4(x) = 0$. Substituting this into the error formula, we get $E = 0$, which means Simpson's rule will give the exact integral for any cubic polynomial. Let f(x) = x^4. The fourth derivative of $f(x)$ is $f^4(x) = 24$. Substituting this into the error formula, we get $E \neq 0$, indicating that Simpson's rule will not give the exact integral for a quartic polynomial. Thus, S_n has degree of precision three.

(c) To find the degree of precision of the approximation formula $\int_{-1}^{1} f(x)dx \approx f(\frac{\sqrt{-3}}{3}) + f(\frac{\sqrt{3}}{3})$, we must test the approximation formula for polynomials $f(x) = 1, x, x^2, x^3, ...$ of increasing degree until the formula is no longer exact. $\int_{-1}^{1} 1dx = 2$ $f(-\frac{\sqrt{3}}{3}) + f(\frac{\sqrt{3}}{3}) = 1+1 = 2$. The formula is exact for $f(x) = 1$. $\int_{-1}^{1} xdx = 0$ $f(-\frac{\sqrt{3}}{3})+f(\frac{\sqrt{3}}{3}) = -\frac{\sqrt{3}}{3} + \frac{\sqrt{3}}{3} = 0$. The formula is exact for $f(x) = x$. $\int_{-1}^{1} x^2dx = \frac{x^3}{3}\Big|_{-1}^{1} = \frac{1}{3} - (-\frac{1}{3}) = \frac{2}{3}$ $f(-\frac{\sqrt{3}}{3}) + f(\frac{\sqrt{3}}{3}) = (-\frac{\sqrt{3}}{3})^2 + (\frac{\sqrt{3}}{3})^2 = \frac{3}{9} + \frac{3}{9} = \frac{2}{3}$. The formula is exact for $f(x) = x^2$. $\int_{-1}^{1} x^3dx = \frac{x^4}{4}\Big|_{-1}^{1} = \frac{1}{4} - \frac{1}{4} = 0$ $f(-\frac{\sqrt{3}}{3}) + f(\frac{\sqrt{3}}{3}) = (-\frac{\sqrt{3}}{3})^3 + (\frac{\sqrt{3}}{3})^3 = -\frac{3\sqrt{3}}{27} + \frac{3\sqrt{3}}{27} = 0$. The formula is exact for $f(x) = x^3$. $\int_{-1}^{1} x^4dx = \frac{x^5}{5}\Big|_{-1}^{1} = \frac{1}{5} - (-\frac{1}{5}) = \frac{2}{5}$ $f(-\frac{\sqrt{3}}{3}) + f(\frac{\sqrt{3}}{3}) = (-\frac{\sqrt{3}}{3})^4 + (\frac{\sqrt{3}}{3})^4 = \frac{9}{81} + \frac{9}{81} = \frac{18}{81} = \frac{2}{9}$ The formula is not exact for $f(x) = x^4$ since $\frac{2}{5} \neq \frac{2}{9}$. Thus, the degree of precision for the approximation formula is three.