

## Homework 3

Due date: February 6th, 2025

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1. *Proof.* It is sufficient to show that

$$\lim_{n \rightarrow \infty} \frac{|p_{n+1} - p|}{|p_n - p|} = g'(p)$$

since letting  $\lambda = g'(p)$  would complete the proof. By the mean value theorem on  $[p_n, p]$ , or  $[p, p_n]$ , there exists a  $\xi_n \in (p_n, p)$  such that

$$\frac{|g(p_n) - g(p)|}{|p_n - p|} = \frac{|p_{n+1} - p|}{|p_n - p|} = g'(\xi_n).$$

Since  $p_n \xrightarrow{n \rightarrow \infty} p$ , we must have  $\xi_n \xrightarrow{n \rightarrow \infty} p$ . Therefore, by the continuity of  $g'$ , we must have

$$g'(\xi_n) \xrightarrow{n \rightarrow \infty} g'(p).$$

Therefore, taking the limit of both sides of the first equation gives

$$\lim_{n \rightarrow \infty} \frac{|p_{n+1} - p|}{|p_n - p|} = \lim_{n \rightarrow \infty} g'(\xi_n) = g'(p)$$

which is the desired result. □

2. (a) Note that

$$f'(x) = 5x^4 - 7.2x^2 + 1.28x + 1.536 \implies f'(0.8) = 0$$

$$f''(x) = 20x^3 - 14.4x + 1.28 \implies f''(0.8) = 0$$

$$f'''(x) = 60x^2 - 14.4 \implies f'''(0.8) = -14.4 \neq 0$$

Therefore, 0.8 is a root of multiplicity 3 of  $f$ .

(b)

$n$	$x_n$
0	2
1	1.68
2	1.4363
3	1.2536
4	1.1190
5	1.0216
6	0.9038
7	0.9038
8	0.8703
9	0.8474
10	0.8318

- (c) Using  $\mu(x) = \frac{f(x)}{f'(x)}$ , we see the following

$n$	$x_n$
0	2
1	0.6286
2	0.78835
3	0.79995
4	0.80000
5	0.80000
6	0.80000
7	0.80000
8	0.80000
9	0.80000
10	0.80000

which clearly converges much faster.

3. (a)
- (b)
- (c)

4. *Proof.* Suppose there exists, two degree  $n$  polynomials,  $p_1$  and  $p_2$ , such that

$$p_1(x_i) = y_i = p_2(x_i)$$

for all  $0 \leq i \leq n$ . It suffices to show that  $p_1(x) = p_2(x)$ . Therefore, the polynomial,

$$f(x) = p_1(x) - p_2(x)$$

is a polynomial of degree at most  $n$ , with  $n + 1$  distinct roots,  $x_0, x_1, \dots, x_n$ . However, by the Fundamental Theorem of Algebra,  $f$  must be the 0 polynomial.<sup>1</sup> Therefore,

$$f(x) = 0,$$

which means that  $p_1(x) = p_2(x)$ , which is the desired result.  $\square$

5. (a) We can take  $(0, 7), (2, 11), (3, 28)$ , and  $(4, 63)$  as  $(x_0, y_0), (x_1, y_1), \dots, (x_n, y_n)$  respectively. We find the following:

$$L_0(x) = \frac{(x - x_1)(x - x_2)(x - x_3)}{(x_0 - x_1)(x_0 - x_2)(x_0 - x_3)}$$

$$L_1(x) = \frac{(x - x_0)(x - x_2)(x - x_3)}{(x_1 - x_0)(x_1 - x_2)(x_1 - x_3)}$$

$$L_2(x) = \frac{(x - x_0)(x - x_1)(x - x_3)}{(x_2 - x_0)(x_2 - x_1)(x_2 - x_3)}$$

$$L_3(x) = \frac{(x - x_0)(x - x_1)(x - x_2)}{(x_3 - x_0)(x_3 - x_1)(x_3 - x_2)}$$

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<sup>1</sup>Since otherwise it would be a non-zero degree  $n$  polynomial with more than  $n$  distinct roots.

Plugging in the points from above, we get:

$$L_0(x) = \frac{(x-2)(x-3)(x-4)}{(-2)(-3)(-4)} = \frac{-1}{24}(x-2)(x-3)(x-4)$$

$$L_1(x) = \frac{(x-0)(x-3)(x-4)}{(-2)(-1)(-2)} = \frac{1}{4}(x)(x-3)(x-4)$$

$$L_2(x) = \frac{(x-0)(x-2)(x-4)}{(3)(1)(-1)} = \frac{-1}{3}(x)(x-2)(x-4)$$

$$L_3(x) = \frac{(x-0)(x-2)(x-3)}{(4)(2)(1)} = \frac{1}{8}(x)(x-2)(x-3)$$

Using the y-values from above, we get the following interpolation:

$$P(x) = \frac{-7}{24}(x-2)(x-3)(x-4) + \frac{11}{4}(x)(x-3)(x-4) - \frac{28}{3}(x)(x-2)(x-4) + \frac{63}{8}(x)(x-2)(x-3)$$

Which can be simplified to become

$$P(x) = x^3 - 2x + 7$$

(b) To find the approximation of  $f(1)$ , we just plug in 1 for  $x$  in  $P(x)$ :

$$P(1) = 1^2 - 2(1) + 7 = 1 - 2 + 7 = 6$$

(c)

$$\int_0^4 x^3 - 2x + 7 = 76$$

6.

7. We know that

$$|f(x) - p(x)| = \frac{f^{(n+1)}(\xi)}{(n+1)!} \prod_{i=0}^n x - x_i$$

Since  $x_0, x_1, \dots, x_n$  are evenly spaced,

$$\prod_{i=0}^n (x - x_i) \leq \frac{1}{4} \left( \left( \frac{x_n - x_0}{n} \right)^{n+1} (n!) \right)$$

So,

$$|f(x) - p(x)| = \left| \left( \frac{f^{(11)}(\xi)}{11!} \right) \left( \frac{1}{4} \right) \left( \frac{1.6875 - 0}{10} \right)^{11} (10!) \right|$$

$f(x) = \sin(0.16875x)$ , so

$$|f(x) - p(x)| = \max \left( -\frac{((0.16875)^{11} \cos(0.16875)(0))}{11!}, -\frac{((0.16875)^{11} \cos(0.16875)(1.6785))}{11!} \right)$$

.

Since

$$| - ((0.16785)^{11}(\cos(0.16785)(x = 0))) | = 3.15996008e^{-9}$$

and

$$| - ((0.16785)^{11}(\cos(0.16785)(x = 1.6875))) | = 3.15867894e^{-9}$$

The maximum value of  $f^{11}$  is at  $x = 0$ . Then,

$$|(-\frac{(0.16875)^{11}(\cos(0.16875(1.6875)))}{11!})\frac{1}{4}(0.16875)^{11}(10!) \leq 2.178e^{-19}$$

.

Then, the error bound of  $|f(x) - p(x)| \leq 2.178(10^{-9})$  on  $[0, 1.6875]$ .