Homework 6

Due date: March 6, 2025

1. To solve the system of linear equations, we can create the following matrix:

$$\left[\begin{array}{ccc|c} 3 & 2 & -1 & 7 \\ 5 & 3 & 2 & 4 \\ -1 & 1 & -3 & -1 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|c} 5 & 3 & 2 & 4 \\ 3 & 2 & -1 & 7 \\ -1 & 1 & -3 & -1 \end{array}\right] \xrightarrow{R_2 + 3R_3} \left[\begin{array}{ccc|c} 5 & 3 & 2 & 4 \\ 0 & 5 & -10 & 4 \\ -1 & 1 & -3 & -1 \end{array}\right] \xrightarrow{R_3 + \frac{1}{5}R_1}$$

$$\left[\begin{array}{ccc|c} 5 & 3 & 2 & 4 \\ 0 & 5 & -10 & 4 \\ -1 & \frac{8}{5} & \frac{-13}{5} & \frac{-1}{5} \end{array}\right] \xrightarrow{R_3 - \frac{8}{25}R_2} \left[\begin{array}{ccc|c} 5 & 3 & 2 & 4 \\ 0 & 5 & -10 & 4 \\ 0 & 0 & \frac{3}{5} & \frac{-37}{25} \end{array}\right]$$

Which gives us our partially pivoted matrix. We can now solve the system of equations!

$$\frac{3}{5}x_3 = \frac{-37}{25}$$

$$x_3 = \frac{-37}{25} \times \frac{5}{3} = \frac{-37}{15}$$

$$5x_2 - 10 \times \frac{-37}{15} = 4$$

$$75x_2 + 370 = 60$$

$$75x_2 + 370 = -310$$

$$x_2 = \frac{-62}{15}$$

$$5x_1 + 3 \times \frac{-62}{15} + 2 \times \frac{-37}{15} = 4$$

$$75x_1 + 3 \times -62 + 2 \times -37 = 60$$

$$75x_1 - 260 = 60$$

$$75x_1 = 320$$

$$x_1 = \frac{64}{15}$$

2. To solve the system of linear equations using the inverse matrix, we can do the following:

$$\left[\begin{array}{ccc|ccc} 3 & 2 & -1 & 1 & 0 & 0 \\ 5 & 3 & 2 & 0 & 1 & 0 \\ -1 & 1 & -3 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_3} \left[\begin{array}{ccc|ccc} -1 & 1 & -3 & 0 & 0 & 1 \\ 5 & 3 & 2 & 0 & 1 & 0 \\ 3 & 2 & -1 & 1 & 0 & 0 \end{array}\right] \xrightarrow{R_1 \leftarrow (-1)R_1} \left[\begin{array}{ccc|ccc} 1 & -1 & 3 & 0 & 0 & -1 \\ 5 & 3 & 2 & 0 & 1 & 0 \\ 3 & 2 & -1 & 1 & 0 & 0 \end{array}\right]$$

$$\xrightarrow[R_3 \leftarrow R_3 - 3R_1]{R_2 \leftarrow R_2 - \frac{1}{5}R_1} \left[\begin{array}{ccc|ccc} 1 & -1 & 3 & 0 & 0 & -1 \\ 0 & 1 & -13 & 0 & 1 & 5 \\ 0 & 5 & -10 & 1 & 0 & 3 \end{array}\right] \xrightarrow[R_3 \leftarrow \frac{1}{5}R_3]{R_2 \leftarrow \frac{1}{8}R_2} \left[\begin{array}{ccc|ccc} 1 & -1 & 3 & 0 & 0 & -1 \\ 0 & 1 & \frac{-13}{8} & 0 & \frac{1}{8} & \frac{5}{8} \\ 0 & 1 & -2 & \frac{1}{5} & 0 & \frac{3}{5} \end{array}\right] \xrightarrow{R_3 \leftarrow R_3 - R_2}$$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 3 & 0 & 0 & -1 \\ 0 & 1 & \frac{-13}{8} & 0 & \frac{1}{8} & \frac{5}{8} \\ 0 & 1 & \frac{-3}{8} & \frac{1}{5} & \frac{-1}{8} & \frac{-1}{40} \end{array}\right] \xrightarrow{R_3 \leftarrow \frac{-8}{3}R_3} \left[\begin{array}{ccc|ccc} 1 & -1 & 3 & 0 & 0 & -1 \\ 0 & 1 & 0 & \frac{-13}{15} & \frac{2}{3} & \frac{11}{5} \\ 0 & 0 & 1 & \frac{-8}{15} & \frac{1}{3} & \frac{1}{15} \end{array}\right]$$

$$\xrightarrow[R_1 \leftarrow R_1 - 3R_3]{R_1 \leftarrow R_1 + R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{11}{15} & \frac{-1}{3} & \frac{-7}{15} \\ 0 & 1 & 0 & \frac{-13}{15} & \frac{2}{3} & \frac{11}{5} \\ 0 & 0 & 1 & \frac{-8}{15} & \frac{1}{3} & \frac{1}{15} \end{array}\right]$$

Therefore,

$$A^{-1} = \begin{bmatrix} \frac{11}{15} & \frac{-1}{3} & \frac{-7}{15} \\ \frac{-13}{15} & \frac{2}{3} & \frac{11}{15} \\ \frac{-8}{15} & \frac{1}{3} & \frac{1}{15} \end{bmatrix}$$

Let

$$b = \begin{bmatrix} 7 \\ 4 \\ 1 \end{bmatrix}$$

then

$$x = A^{-1}b = \begin{bmatrix} \frac{11}{15} & \frac{-1}{3} & \frac{-7}{15} \\ \frac{-13}{15} & \frac{2}{3} & \frac{11}{15} \\ \frac{-8}{15} & \frac{1}{3} & \frac{1}{15} \end{bmatrix} \begin{bmatrix} 7 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{64}{15} \\ \frac{-62}{15} \\ \frac{-37}{15} \end{bmatrix}$$

. Therefore,

$$x_1 = \frac{64}{15}, x_2 = \frac{-62}{15}, x_3 = \frac{-37}{15}$$

3. 26 for Gaussian. 43 for matrix inversion. There are significantly less operations for Gaussian elimination therefore making it the more efficient method.

4. We first note that $||x||_\infty = \max 1 \leq i \leq n |x_i| \geq 0$ for all $x \in \mathbb{R}^n$. Furthermore, we have

$$||x||_\infty = 0 \iff \max_{1 \leq i \leq n} |x_i| = 0 \iff |x_i| = 0 \text{ for all } i \iff x = 0.$$

Furthermore, for all $a \in \mathbb{R}$, we have

$$||ax||_\infty = \max_{1 \leq i \leq n} |ax_i| = \max_{1 \leq i \leq n} |a||x_i| = |a| \max_{1 \leq i \leq n} |x_i| = |a|||x||_\infty.$$

Finally, for all $x, y \in \mathbb{R}^n$, by the triangle inequality for the absolute value, we have that

$$||x+y||_\infty = \max_{1 \leq i \leq n} |x_i + y_i| \leq \max_{1 \leq i \leq n} (|x_i| + |y_i|) \leq \max_{1 \leq i \leq n} |x_i| + \max_{1 \leq i \leq n} |y_i| = ||x||_\infty + ||y||_\infty.$$

Therefore, $|| \cdot ||_\infty$ is a norm on $\mathbb{R}^n$.

5. (a) *Proof.* By the proposition, there exists $C, C_\infty \in \mathbb{R}_{>0}$ such that

$$||x|| \leq C_\infty ||x||_\infty \leq C_\infty C ||x||'$$

For all $x \in \mathbb{R}^n$. Similarly, there exists $C', C'_\infty \in \mathbb{R}_{>0}$ such that

$$||x||' \leq C'_\infty ||x||_\infty \leq C'_\infty C' ||x||$$

For all $x \in \mathbb{R}^n$. Therefore, letting $D = C_\infty C$ and $D' = C'_\infty C'$ completes the proof. $\square$

(b)

Proposition 1. *Let $a, b \in \mathbb{R}_{\geq 0}$. Then*

$$\sqrt{a+b} \leq \sqrt{a} + \sqrt{b}.$$

Proof of Proposition 1. Note that since a and b are non-negative, we have that

$$a + b \leq a + b + 2\sqrt{a}\sqrt{b} = (\sqrt{a} + \sqrt{b})^2.$$

Taking the square root on both sides completes the proof. □

Note that by Proposition 1, we have the following for any $x \in \mathbb{R}^2$:

$$||x||_2 = \sqrt{x_1^2 + x_2^2} \leq \sqrt{x_1^2} + \sqrt{x_2^2} = |x_1| + |x_2| = ||x||_1.$$

Furthermore, by the Cauchy-Schwarz-Bunyakovsky Inequality, we have that

$$||x||_1 = |x_1| + |x_2| = 1 \cdot |x_1| + 1 \cdot |x_2| \leq \sqrt{1+1}\sqrt{|x_1|^2 + |x_2|^2} = \sqrt{2}||x||_2.$$

Therefore, $C_1 = 1$ and $C_2 = \sqrt{2}$ gives us the desired result.

6. (a) For a continuous function f $\in V = C[a, b]$ the L^1-norm is

$$||f||_1 = \int_a^b |f(x)|dx$$

Substituting, we have

$$||f||_1 = \int_0^1 |x|dx$$

Since x is ≥ 0 on $[0, 1]$,

$$\int_0^1 |x|dx = \int_0^1 xdx$$
$$= \frac{1}{2}x^2|_0^1$$
$$= \frac{1}{2}$$

For a continuous function f $\in V = C[a, b]$ the L^2-norm is

$$||f||_2 = \sqrt{\int_a^b f(x)^2 dx}$$

Substituting, we have

$$||f||_2 = \sqrt{\int_0^1 x^2 dx}$$

$$= \sqrt{\frac{1}{3}x^3|_0^1}$$
$$= \frac{1}{\sqrt{3}}$$

For a continuous function f $\in V = C[a,b]$ the L^∞-norm is

$$||f||_\infty = \max_{a \leq x \leq b} |f(x)|$$

Substituting, we have

$$||f||_\infty = \max_{0 \leq x \leq 1} \{|x|\}$$
$$= 1$$

(b) For continuous functions f, g $\in V = C[a,b]$ the distance between f and g with respect to the L^1-norm is

$$||f-g||_1 = \int_a^b |f-g|dx$$

Substituting, we have

$$||f-g||_1 = \int_0^1 |x-(1-x)|dx$$
$$= \int_0^1 |2x-1|dx$$

Since 2x-1 is < 0 on $[0, 0.5)$ and geq 0 on $[0.5, 1]$, we can perform piecewise integration, splitting into $\int_0^{0.5} 1-2xdx + \int_{0.5}^1 2x-1dx$

$$\int_0^{0.5} 1-2xdx = x - x^2|_0^{0.5}$$
$$= \frac{1}{4}$$
$$\int_{0.5}^1 2x-1dx = x^2 - x|_{0.5}^1$$
$$= (1-1) - (\frac{1}{4} - \frac{1}{2})$$
$$= \frac{1}{4}$$

$\frac{1}{4} + \frac{1}{4} = \frac{1}{2}$, thus $||f-g||_1 = \int_0^1 |2x-1|dx = \frac{1}{2}$

For continuous functions f, g $\in V = C[a,b]$ the distance between f and g with respect to the L^2-norm is

$$||f-g||_2 = \sqrt{\int_a^b (f-g)^2 dx}$$

Substituting, we have

$$||f-g||_2 = \sqrt{\int_0^1 (x-(1-x))^2 dx}$$

$$= \sqrt{\int_0^1 (2x-1)^2 dx}$$

$$= \sqrt{\int_0^1 4x^2 - 4x + 1 dx}$$

$$= \sqrt{\frac{4}{3}x^3 - 2x^2 + x|_0^1}$$

$$= \sqrt{\frac{4}{3} - 2 + 1}$$

$$= \sqrt{\frac{1}{3}} = \frac{1}{\sqrt{3}}$$

For continuous functions f, g $\in V = C[a,b]$ the distance between f and g with respect to the L^∞-norm is

$$||f-g||_\infty = \max_{a \le x \le b} |f-g|$$

Substituting, we have

$$||f-g||_\infty = \max_{0 \le x \le 1} |x-(1-x)|$$

$$= \max_{0 \le x \le 1} |2x-1|$$

$$= 1$$