Homework 8

Due date: April 3rd, 2025

1. (a)
   (b)
   (c)
2.
3.
4. We are given that $\mathbf{v} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 0 \end{pmatrix}$ and $\mathbf{w} = \begin{pmatrix} 1 \\ -1 \\ 2 \\ -2 \end{pmatrix}$.
   (a) $\langle \mathbf{v}, \mathbf{w} \rangle = 1 * 1 + 3 * (-1) + 2 * 2 + 0 * (-2) = 2$
   (b) $||\mathbf{v}|| = \sqrt{1^2 + 3^2 + 2^2} = \sqrt{14}$
   (c) $\mathbf{v} - \mathbf{w} = \begin{pmatrix} 0 \\ 4 \\ 0 \\ 2 \end{pmatrix}$
   $||\mathbf{v} - \mathbf{w}|| = \sqrt{0^2 + 4^2 + 0^2 + 2^2} = 2\sqrt{5}$
   (d) $\cos(\theta) = \frac{\langle \mathbf{v}, \mathbf{w} \rangle}{||\mathbf{v}|| \cdot ||\mathbf{w}||} = \frac{2}{\sqrt{14} \cdot \sqrt{1^2 + (-1)^2 + 2^2 + (-2)^2}} = \frac{2}{\sqrt{140}}$
   $\theta = \cos^{-1}(\frac{2}{\sqrt{140}}) \approx 1.40095$
5. (a)

$$\langle f, g \rangle = \int_0^1 f(x)g(x)dx$$

$$= \int_0^1 (x-1)(x+1)dx$$

$$= \int_0^1 (x^2 - 1)dx$$

$$= [\frac{x^3}{3} - x]_0^1$$

$$= (\frac{1}{3} - 1) - (\frac{0^3}{3} - 0) = \frac{-2}{3}$$

   (b)

$$||f - g|| = \sqrt{\int_0^1 |(f - g)| dx}$$

$$f(x) - g(x) = (x - 1) - (x + 1) = x - 1 - x - 1 = -2$$

$$|f(x) - g(x)| = 2$$

$$\int_0^1 2dx = [2x]_0^1 = 2 - 0 = 2$$

(c) The angle between f and g is

$$\cos(\theta) = \frac{\langle f, g\rangle}{||f|| \cdot ||g||}$$

We already know that $\langle f, g\rangle = \frac{-2}{3}$, so we need to find the norm $||f||$.

$$||f|| = (\int_0^1 (x-1)^2 dx)^{\frac{1}{2}} = (\int_0^1 (x^2 - 2x + 1)dx)^{\frac{1}{2}}$$

$$= ([\frac{x^3}{3} - 2\frac{x^2}{2} + x]_0^1)^{\frac{1}{2}} = ((\frac{1}{3} - 2\frac{x^2}{2} + x) - (0))^{\frac{1}{2}}$$

$$= (\frac{1}{3} - 1 + 1)^{\frac{1}{2}} = \frac{1}{\sqrt{3}}$$

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Now we need to find $||g||$.

$$||g|| = (\int_0^1 (x+1)^2 dx)^{\frac{1}{2}} = (\int_0^1 (x^2 + 2x + 1)dx)^{\frac{1}{2}} = ([\frac{x^3}{3} + x^2 + x]_0^1)^{\frac{1}{2}}$$

$$= (\frac{1}{3} + 1 + 1)^{\frac{1}{2}} = \sqrt{\frac{7}{3}}$$

We can now use both of these norms to find θ:

$$\cos(\theta) = \frac{\frac{-2}{3}}{\frac{-1}{3} \cdot \sqrt{\frac{7}{3}}} = \frac{\frac{-2}{3}}{\sqrt{\frac{7}{9}}} = \frac{\frac{-2}{3}}{\frac{\sqrt{7}}{3}} = \frac{-2}{\sqrt{7}}$$

Now we can solve for θ:

$$\theta = \cos^{-1}(\frac{-2}{\sqrt{7}}) \approx \cos^{-1}(-0.7559) \approx 139.1°$$

(d) To find a nonzero h that is perpendicular to f, we would have to find a function $h(x)$ that satisfies the following:

$$\int_0^1 (x-1) \cdot h(x)dx = 0$$

We can try a simple polynomial $h(x) = ax + b$, and plug it into the above orthogonality condition:

$$\int_0^1 (x-1)(ax+b)dx = 0$$

$$(x-1)(ax+b) = (ax^2 + bx - ax - b) = ax^2 + (b-a)x - b$$

$$\int_0^1 (ax^2 + (b-a)x - b)dx = a\frac{1}{3} + (b-a)\frac{1}{2} - b = 0$$
$$2a + 3(b-a) - 6b = 0 \to 2a + 3b - 3a - 6b = 0 \to -a - 3b = 0 \to a = -3b$$

Let us pick $b = 1$, then $a = -3$, so $h(x) = -3x + 1$

6. We know that $\mathbf{v_1} = (1,2,2), \mathbf{v_2} = (-1,0,2)$, and $\mathbf{v_3} = (0,0,1)$

(a)
$$\mathbf{v_1} = \mathbf{x_1} \text{ and } ||\mathbf{x_1}||_2 = \sqrt{1^2 + 2^2 + 2^2}$$
$$||\mathbf{w}|| = \frac{\mathbf{x_1}}{||\mathbf{x_1}||_2} = (\frac{1}{3}, \frac{2}{3}, \frac{2}{3})$$
$$\mathbf{x_2} = \mathbf{v_2} - \langle \mathbf{v_2}, \mathbf{w_1} \rangle \mathbf{w_1} = (-1,0,2) - \langle (-1,0,2), (\frac{1}{3}, \frac{2}{3}, \frac{2}{3}) \rangle$$
$$= (-1,0,2) - 1(\frac{1}{3}, \frac{2}{3}, \frac{2}{3})$$
$$= (\frac{-4}{3}, \frac{-2}{3}, \frac{4}{3})$$
$$||\mathbf{x^2}||_2 = \sqrt{(\frac{-4}{3})^2 + (\frac{-2}{3})^2 + (\frac{4}{3})^2} = 2$$
$$\mathbf{w_2} = \frac{\mathbf{x_2}}{||\mathbf{x_2}||_2} = (\frac{-2}{3}, \frac{-1}{3}, \frac{2}{3})$$
$$\mathbf{x_3} = \mathbf{v_3} - \langle \mathbf{v_3}, \mathbf{w_1} \rangle \mathbf{w_1} - \langle \mathbf{v_3}, \mathbf{w_2} \rangle \mathbf{w_2}$$
$$\mathbf{x_3} = (0,0,1) - \langle (0,0,1), (\frac{1}{3}, \frac{2}{3}, \frac{2}{3}) \rangle (\frac{1}{3}, \frac{2}{3}, \frac{2}{3}) - \langle (0,0,1)(\frac{-2}{3}, \frac{-1}{3}, \frac{2}{3}) \rangle (\frac{-2}{3}, \frac{-1}{3}, \frac{2}{3})$$
$$= (0,0,1) - \frac{2}{3}(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}) - \frac{2}{3}(\frac{-2}{3}, \frac{-1}{3}, \frac{2}{3})$$
$$= (\frac{2}{9}, \frac{-2}{9}, \frac{1}{9})$$
$$||\mathbf{x_3}||_2 = \frac{1}{3}$$
$$\mathbf{w_3} = \frac{\mathbf{x_3}}{||\mathbf{x_3}||_2} = (\frac{2}{3}, \frac{-2}{3}, \frac{1}{3})$$

(b) We know $\mathbf{x} = (7,5,1)$
$$\langle (7,5,1), (\frac{1}{3}, \frac{2}{3}, \frac{2}{3}) \rangle = \frac{19}{3}$$
$$\langle (7,5,1), (\frac{-2}{3}, \frac{-1}{3}, \frac{2}{3}) \rangle = \frac{-17}{3}$$
$$\langle (7,5,1), (\frac{2}{3}, \frac{-2}{3}, \frac{1}{3}) \rangle = \frac{5}{3}$$

.

Now, we can write
$$\frac{19}{3}(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}) - \frac{17}{3}(\frac{-2}{3}, \frac{-1}{3}, \frac{2}{3}) + \frac{5}{3}(\frac{2}{3}, \frac{-2}{3}, \frac{1}{3})$$