Homework 7

Due date: March 13, 2025

1. Note that
$$||A||_\infty = \max\{|3|+|2|+|4|, |2|+|0|+|2|, |4|+|2|+|3|\} = \max\{9,4,9\} = 9.$$
Furthermore, we have that
$$||A||_1 = \max\{|3|+|2|+|4|, |2|+|0|+|2|, |4|+|2|+|3|\} = \max\{9,4,9\} = 9.$$
Finally, we compute the eigenvalues of A in the following way.
$$\det(A-\lambda I) = \det\begin{pmatrix} 3-\lambda & 2 & 4 \\ 2 & -\lambda & 2 \\ 4 & 2 & 3-\lambda \end{pmatrix} = (3-\lambda)(-\lambda(3-\lambda)-4)-2(2(3-\lambda)-8)+4(4+4\lambda)$$
$$= -\lambda^3+6\lambda^2+15\lambda+8 = -(\lambda+1)^2(\lambda-8).$$
Therefore, the eigenvalues are given by the roots of $-(\lambda+1)^2(\lambda-8)$, which are $\lambda=-1$ and $\lambda=8$. Since A is symmetric, we have that
$$||A||_2 = \rho(A) = \max\{|8|, |-1|\} = 8.$$

2. (a) *Proof.* Let $M_1 = \max_{||x||\neq 0} \frac{||Ax||}{||x||}$ and $M_2 = ||A|| = \max_{||x||=1} ||Ax||$. It suffices to show that $M_1 = M_2$. Note that if $||x|| \neq 0$, then by properties of norms, we have
$$\frac{||Ax||}{||x||} = \left|\left|A\frac{x}{||x||}\right|\right|.$$
Furthermore[1], since $\left|\left|\frac{x}{||x||}\right|\right| = \frac{1}{||x||}||x|| = 1$ we must have that $M_1 \leq M_2$ by definition of M_2. Furthermore, if we fix an arbitrary vector, x, such that $||x|| = 1$. Then we must have
$$||Ax|| = \left|\left|A\frac{x}{1}\right|\right| = \left|\left|A\frac{x}{||x||}\right|\right| = \frac{||Ax||}{||x||}$$
by definition of M_1, we must have that $M_2 \leq M_1$, and therefore, $M_1 = M_2$, which was the desired result. ■

   (b) *Proof.* Note that by part (a), we have that for any vector $x \neq 0$,
$$||A|| = \max_{y\neq 0} \frac{||Ay||}{||y||} \geq \frac{||Ax||}{||x||}.$$
Multiplying by $||x||$ gives that $||Ax|| \leq ||A||||x||$ for all $x \neq 0$. Furthermore, note that the inequality also holds when $x = 0$, since $Ax = 0$. Therefore,
$$||Ax|| \leq ||A||||x||$$
for all x. We then have that
$$||AB|| = \max_{||x||=1||} ||ABx|| \leq \max_{||x||=1} ||A||||Bx|| = ||A|| \max_{||x||=1} ||Bx|| = ||A||||B||$$
which is the desired result. ■

[1] It should be noted that $\frac{x}{||x||}$ is intended to mean $\frac{1}{||x||}x$, to agree with multiplying x by a scalar.

(c) *Proof.* By repetedly applying part (b), we have that for all $k \in \mathbb{N}$

$$||A^k|| = ||A \cdot A^{k-1}|| \leq ||A||||A^{k-1}|| = ||A||||A \cdot A^{k-2}|| \leq ||A||^2||A^{k-2}|| \leq \dots ||A||^k.$$

■

(d) *Proof.* Note that by part (b), we have

$$||A||||A^{-1}|| \geq ||AA^{-1}|| = ||I|| = \max_{||x||=1} ||Ix|| = \max_{||x||=1} ||x|| = 1$$

which is the desired result. ■

3. *Proof.* **Step 1.** $\|A\|_1 \leq \max_{1\leq j\leq n} \sum_{i=1}^{n} |a_{ij}|$.

Fix $\mathbf{x}$ with $\|\mathbf{x}\|_1 = 1$. $\quad (\Rightarrow \sum_{j=1}^{n} |x_j| = 1)$

$$\begin{aligned}
\|A\mathbf{x}\|_1 &= \sum_{i=1}^{n} \left| \sum_{j=1}^{n} a_{ij}x_j \right| \\
&\leq \sum_{i=1}^{n} \sum_{j=1}^{n} |a_{ij}x_j| \\
&\leq \sum_{j=1}^{n} |x_j| \sum_{i=1}^{n} |a_{ij}| \\
&\leq \max_{1\leq j\leq n} \sum_{i=1}^{n} |a_{ij}|.
\end{aligned}$$

Taking the maximum over all $\mathbf{x}$ with $\|\mathbf{x}\|_1 = 1$, we get:

$$\|A\|_1 = \max_{\|\mathbf{x}\|_1=1} \|A\mathbf{x}\|_1 \leq \max_{1\leq j\leq n} \sum_{i=1}^{n} |a_{ij}|.$$

Step 2. $\|A\|_1 \geq \max_{1\leq j\leq n} \sum_{i=1}^{n} |a_{ij}|$.

There exists some p such that:

$$\sum_{i=1}^{n} |a_{ip}| = \max_{1\leq j\leq n} \sum_{i=1}^{n} |a_{ij}|.$$

Define $\mathbf{x} = [x_j]$ as:

$$x_j = \begin{cases} 1 & j = p, \\ 0 & \text{else} \end{cases}$$

($\|\mathbf{x}\|_1 = 1.$ and $a_{ij}x_j = |a_{ip}|$.)

$$\begin{aligned}\|A\mathbf{x}\|_1 &= \sum_{i=1}^{n} \left| \sum_{j=1}^{n} a_{ij}x_j \right| \\ &= \sum_{i=1}^{n} |a_{ip}| \\ &= \max_{1\le j\le n} \sum_{i=1}^{n} |a_{ij}|.\end{aligned}$$

Thus,

$$\|A\|_1 = \max_{\|\mathbf{x}\|_1=1} \|A\mathbf{x}\|_1 \ge \max_{1\le j\le n} \sum_{i=1}^{n} |a_{ij}|.$$

Since both inequalities hold, we conclude:

$$\|A\|_1 = \max_{1\le j\le n} \sum_{i=1}^{n} |a_{ij}|.$$

■

4. For the Gauss-Seidel method, we get the following after each iteration:

Iteration 1: $x[0] = 0.5000000, x[1] = 2.833333, x[2] = -1.083333$

Iteration 2: $x[0] = 1.916667, x[1] = 2.944444, x[2] = -1.027778$

Iteration 3: $x[0] = 1.972222, x[1] = 2.981481, x[2] = -1.009259$

Iteration 4: $x[0] = 1.990741, x[1] = 2.993827, x[2] = -1.003086$

Iteration 5: $x[0] = 1.996914, x[2] = 2.997942, x[3] = -1.001029.$

Leaving the final solution as $[1.996914, 2.997924, -1.001029]$

Here is the code that was used:

```
import numpy as np

def gauss_seidel(A, b, x0=None, iterations=5):
    n = len(b)
    x = np.zeros(n) if x0 is None else np.array(x0, dtype=float)

    for iteration in range(1, iterations + 1):
        print(f"Iteration {iteration}:")
        for i in range(n):
            sum1 = sum(A[i][j] * x[j] for j in range(i))
            sum2 = sum(A[i][j] * x[j] for j in range(i + 1, n))
```

```
                x[i] = (b[i] - sum1 - sum2) / A[i][i]
                print(f"x[{i}] = {x[i]:.6f}")
            print()

        return x

    A = np.array([[2, -1, 0], [-1, 3, -1], [0, -1, 2]], dtype=float)
    b = np.array([1, 8, -5], dtype=float)
    x0 = [0, 0, 0]

    solution = gauss_seidel(A, b, x0, iterations=5)
    print("Final solution after 5 iterations:", solution)
```

For the Jacobi method, we got the following:

Iteration 1: $x[0] = 0.500000, x[1] = 2.666667, x[2] = -2.5000000$

Iteration 2: $x[0] = 1.833333, x[1] = 2.000000, x[2] = -1.166667$

Iteration 3: $x[0] = 1.500000, x[1] = 2.888889, x[2] = -1.500000$

Iteration 4: $x[0] = 1.944444, x[1] = 2.666667, x[2] = -1.055556$

Iteration 5: $x[0] = 1.833333, x[1] = 2.962963, x[2] = -1.166667$

Leaving final solution $[1.833333, 2.962963, -1.166667]$

```
import numpy as np

    def jacobi_method(A, b, x0=None, iterations=5):
    n = len(b)
    x = np.zeros(n) if x0 is None else np.array(x0, dtype=float)
    x_new = np.copy(x)

    for iteration in range(1, iterations + 1):
        print(f"Iteration {iteration}:")
        for i in range(n):
            sum1 = sum(A[i][j] * x[j] for j in range(n) if j != i)
            x_new[i] = (b[i] - sum1) / A[i][i]
            print(f"x[{i}] = {x_new[i]:.6f}")
        x[:] = x_new
        print()

    return x

A = np.array([[2, -1, 0], [-1, 3, -1], [0, -1, 2]], dtype=float)
b = np.array([1, 8, -5], dtype=float)
x0 = [0, 0, 0]

solution = jacobi_method(A, b, x0, iterations=5)
```

```
print("Final solution after 5 iterations:", solution)
```

5.

6. *Proof.* First, assume that A is strictly diagonally dominant. Namely, suppose that

$$|a_{ii}| > \sum_{j \neq i} |a_{ij}|$$

for all $1 \leq i \leq n$. Then, we must have that

$$||D^{-1}(L+U)||_\infty \leq ||D^{-1}||_\infty ||L+U||_\infty = ||D^{-1}||_\infty ||D-A||_\infty$$

where we used the fact that $A = D - (L+U)$. Furthermore, since $D = \operatorname{diag}(a_{ii})$[2], we must have that

$$D^{-1} = \operatorname{diag}\left(\frac{1}{a_{ii}}\right).$$

Therefore, since we have that

$$L+U = \begin{cases} a_{ij} & \text{if } i \neq j \\ 0 & \text{else} \end{cases},$$

we must have

$$\left[D^{-1}(L+U)\right]_{ij} = \begin{cases} \frac{a_{ij}}{a_{ii}} & \text{if } i \neq j \\ 0 & \text{else} \end{cases}.$$

Therefore, we have that

$$||D^{-1}(L+U)||_\infty = \max_{1 \leq i \leq n} \sum_{j=1}^{n} \left| \left[D^{-1}(L+U)\right]_{ij} \right| = \max_{1 \leq i \leq n} \sum_{j \neq i} \left| \frac{a_{ij}}{a_{ii}} \right| = \max_{1 \leq i \leq n} \frac{1}{|a_{ii}|} \sum_{j \neq i} |a_{ij}|.$$

Since A is strictly diagonally dominant, we have that

$$||D^{-1}(L+U)||_\infty = \max_{1 \leq i \leq n} \frac{1}{|a_{ii}|} \sum_{j \neq i} |a_{ij}| < \max_{1 \leq i \leq n} \frac{|a_{ii}|}{|a_{ii}|} = 1.$$

Therefore, $||D^{-1}(L+U)||_\infty < 1$.

Suppose that $||D^{-1}(L+U)||_\infty < 1$. As discussed before, we have that

$$||D^{-1}(L+U)||_\infty = \max_{1 \leq i \leq n} \frac{1}{|a_{ii}|} \sum_{j \neq i} |a_{ij}|.$$

[2] I am going to make this assumption, since the theorem is not true without it. For example, if $A = I$, then clearly A is strictly diagonally dominant. However, if we let $D = \frac{1}{2}I$, then we must have $L+U = D-A = -\frac{1}{2}I$. This may be achieved with $L = U = -\frac{1}{4}I$. However, this means $||D^{-1}(L+U)||_\infty = ||2I\left(-\frac{1}{2}I\right)||_\infty = ||-I||_\infty = 1$ which is obviously not less than 1. I specify this because we only assumed that D was diagonal, L is lower triangular, and U is upper triangular. However, what I have written is a stronger condition.

Therefore, for all $1 \leq i \leq n$, we have that

$$\frac{1}{|a_{ii}|}\sum_{j\neq i}|a_{ij}| \leq \max_{1\leq i\leq n}\frac{1}{|a_{ii}|}\sum_{j\neq i}|a_{ij}| < 1.$$

Thus, multiplying by a_{ii} gives that for all $1 \leq i \leq n$,

$$\sum_{j\neq i}|a_{ij}| < |a_{ii}|.$$

Therefore, A is strictly diagonally dominant, which completes the proof. $\blacksquare$