Combinatorics Notes

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*An important note, these notes are absolutely **NOT** guaranteed to be correct, representative of the course, or rigorous. Any result of this is not the author's fault.*

1 Bipartite Graphs

1.1 Definition of a Bipartite Graph

A graph $G = (V, E)$ is bipartite if V can be partitioned into two vertex sets V_1, V_2 such that each edge connects a vertex from V_1 to a vertex in V_2.

1.2 Characterisation of Bipartite Graphs

A graph is bipartite if and only if it contains no odd cycle.

1.3 The Handshake Lemma for Bipartite Graphs

We have that for $G = (V, E)$ a bipartite graph with bipartition V_1, V_2:

$$\sum_{v \in V_1} \deg(v) = \sum_{v \in V_2} \deg(v).$$

1.4 Hall's Marriage Problem

1.4.1 Definition of a Matching

For $G = (V, E)$ a bipartite graph with bipartition X, Y, a matching from X to Y is a set of edges:

$$M = \{(x, y) : x \in X, y \in Y\},$$

such that $f : X \to Y$ defined by:

$$f(x) := y \qquad \text{where } (x, y) \in M,$$

is injective.

In other words, $|M| = |X|$ and each y in Y appears in at most one edge in M.

1.4.2 Hall's Marriage Theorem

For $G = (V, E)$ a bipartite graph with bipartition X, Y:

$$\begin{gathered} G \text{ has a matching from } X \text{ to } Y \\ \iff \\ \text{For all } S \subseteq X, |N(S)| \geq |S|. \end{gathered}$$

We also have that if:

$$\min_{x \in X}\big[\deg(x)\big] \geq \max_{y \in Y}\big[\deg(y)\big],$$

then G has a matching from X to Y.

2 Trees and Forests

2.1 Definition of a Forest

A graph $F = (V, E)$ is a forest if it has no cycles (is **acyclic**).

2.2 Definition of a Tree

A graph is a tree if it is a forest and is connected.

2.3 Definition of a Leaf

For a vertex v in a tree, v is a leaf if it has degree one.

2.4 Existence of Leaves

For a tree T of order 2 or more, we have that T has a leaf.

2.5 Characterisation of Trees

We have that for a graph $G = (V, E)$, the following is equivalent:

- G is a tree
- G is maximally acyclic (G is acyclic and the addition of any edge forms a cycle)
- G is minimally connected (G is connected and the removal of any edge disconnects it)
- G is connected and $|E| = |V| - 1$
- G is acyclic and $|E| = |V| - 1$
- Any two vertices in G are connected by a unique path.

2.6 Minimum Spanning Trees

In a connected, undirected graph $G = (V, E)$, we have that a spanning tree $T = (V, E')$ of G is a subgraph of G where T is a tree and $E' \subseteq E$.

A spanning tree on G is minimal if there is no other spanning tree on G with a lower weight.

2.6.1 Existence of spanning trees

We have that there is a spanning tree in a graph G if and only if G is connected.

2.6.2 Kruskal's algorithm

For a graph $G = (V, E)$, we have the following steps to the algorithm:

1. Generate a graph $T = (V, \emptyset)$
2. Sort the edges by weight
3. For each edge (u, v) (in increasing order):
   - If u or v are not connected in T, add (u, v) to T
   - Stop if there are $|V| - 1$ edges in T or if we have run out of edges.

When this terminates, if the order of T is $|V| - 1$ then T is a minimum spanning tree. Otherwise, T is an acyclic graph with $n - k$ components.

3 Cliques and Independent Sets

3.1 Definition of a Triangle

We often call K_3 (the complete graph on three vertices) a triangle. A graph G contains a triangle if a subgraph of G is isomorphic to K_3.

3.2 Mantel's Theorem

For $G = (V, E)$ a graph of order n that contains no triangles, we have that:

$$|E| \leq \left\lfloor \frac{n^2}{4} \right\rfloor = \left\lfloor \left(\frac{n}{2}\right)^2 \right\rfloor .$$

We also have that:

$$\left[|E| = \left\lfloor \frac{n^2}{4} \right\rfloor \right] \Rightarrow \left[G \cong K_{k,n-k} \text{ where } k = \left\lfloor \frac{n}{2} \right\rfloor \right],$$

there always exists a graph where the equality above holds.

4 Planar Graphs

The motivator for understanding planar graphs is the problem of drawing graphs in the plane without intersecting edges.

4.1 Definition of an Arc

An arc is a subset of $\mathbb{R}^2$ of the type $\sigma : [0,1] \to \mathbb{R}^2$ where σ is an injective, continuous map and $\sigma(0), \sigma(1)$ are the endpoints of the arc. Injectivity here ensures the arc does not cross itself.

4.2 Definition of a Drawing

For a graph $G = (V, E)$, drawing it is equivalent to assigning:

- A point p in $\mathbb{R}^2$ for each v in V (such that the map from vertices to points is injective)
- An arc σ for each $e = (x, y)$ in E (such that σ intersects exactly two points, the points corresponding to x and y).

4.3 Definition of a Planar Drawings and Graphs

A drawing with a set of arcs A is planar if for each σ_1, σ_2 in A, we have that σ_1, σ_2 either intersect at their endpoints or not at all.

A graph is planar if it admits at least one planar drawing. We have that K_5 and $K_{3,3}$ are not planar.

4.3.1 Non-Planar Subgraphs

For a graph G with G' a subgraph of G that is not planar, we have that G is not planar.

4.4 Definition of a Jordan Curve

An arc in the plane whose endpoints conincide is called a Jordan curve.

4.5 Jordan Curve Theorem

For any Jordan curve C, C divides the plane into exactly two connected regions called the 'interior' and 'exterior'. The curve is the boundary of these regions.

4.6 Definition of a Face

For a planar graph G, a face of a drawing of G is a connected region bound by the drawing. The region going off to infinity is the outer face and the rest are inner faces.

4.7 Euler's Formula

For a connected graph $G = (V, E)$ where F is the set of faces of a given drawing of G, we have that:

$$|V| - |E| + |F| = 2.$$

4.8 Edge Bound on Planar Graphs

For $G = (V, E)$ a planar graph on at least three vertices:

$$|E| \leq 3(|V| - 2).$$

5 Graph Colouring

5.1 k-colouring

A k-colouring of a graph $G = (V, E)$ is an assignment of $[k]$ to V performed by $c : V \to [k]$ such that for u, v adjacent vertices in V, $c(u) \neq c(v)$. A graph is k colourable if a k-colouring exists for it.

5.2 Chromatic Number

The chromatic number of a graph G denoted by $\chi(G)$ is the smallest k such that G is k colourable.

5.3 Bound on Chromatic Number

For a graph G, we have that for some k in $\mathbb{Z}$:

$$\Delta(G) \leq k \qquad \Rightarrow \qquad \chi(G) \leq k + 1.$$

5.4 Definition of a Map

A map is a graph derived from some traditional map where regions correspond to faces, points where at least three regions border each other are vertices and, the border between exactly two regions are edges. We assume these regions are connected and they do not touch solely at a point (or several points)

5.5 Five Colour Theorem

Every map with corresponding graph G can be coloured with five colours, that is $\chi(G) \leq 5$.

5.6 Dual Graphs

Given a planar graph $G = (V, E)$ and a fixed planar drawing of G, the dual graph $G^* = (V^*, E^*)$ relative to this drawing is a planar graph obtained by assigning a vertex to each face and connecting these vertices by and edge if their corresponding faces border.

5.6.1 k-colourability of the dual graph

We have that for a graph G, G is k-colourable if and only if G^* is k-colourable.

6 Order from Disorder

6.1 Definition of the Ramsey Number

For some s in $\mathbb{Z}_{>1}$, we let $r(s)$ be the smallest n in $\mathbb{N}$ such that whenever the edges of K_n are 2-coloured, there exists a monochromatic K_s. We have that this exists for all s as chosen above.

Equivalently, $r(s)$ is the smallest n such that for any graph G on n vertices satisfies either:

$$K_s \subseteq G$$
$$\text{or}$$
$$K_s \subseteq \bar{G}.$$

6.2 Definition of the Off-diagonal Ramsey Number

For some s, t in $\mathbb{Z}_{>1}$, we let $r(s, t)$ be the least n in $\mathbb{N}$ such that whenever the edges of K_n are 2-coloured with colour set $\{A, B\}$, there exists an A-monochromatic K_s or a B-monochromatic K_t. We have that this exists for all s, t as chosen above.

6.2.1 Properties of the Off-diagonal Ramsey Number

We have for all s, t in $\mathbb{Z}_{>1}$:

- $r(s, s) = r(s)$
- $r(s, t) = r(t, s)$
- $r(2, s) = s$.

6.3 Ramsey's Theorem

We have that for all s, t in $\mathbb{Z}_{>2}$:

$$r(s,t) \leq r(s-1,t) + r(s,t-1).$$

6.4 An Upper Bound on Ramsey Numbers

For all s, t in $\mathbb{Z}_{>1}$, we have that:

$$r(s,t) \leq 2^{s+t},$$

an consequence of this is that:

$$r(s) \leq 4^s.$$

6.5 A Lower Bound on Ramsey Numbers

For all s in $\mathbb{Z}_{>1}$, we have that:

$$r(s) \geq 2^{\frac{s}{2}}.$$

6.6 The k-colour Ramsey Number

For some k in $\mathbb{Z}_{>0}$, s in $\mathbb{Z}_{>1}$, we let $r_k(s)$ be the smallest n in $\mathbb{N}$ such that whenever the edges of K_n are k-coloured, there exists a monochromatic K_s. We have that this exists for all k, s as chosen above.

6.7 Infinite Ramsey

For a set A and k in $\mathbb{Z}_{>0}$, we have that:

$$A^{(k)} = \{\{a, b\} : a, b \in A, a \neq b\},$$

the set of subsets of A of size two not containing duplicates.

Let $\mathbb{N}^{(2)}$ be 2-coloured, we have that there exists an infinite set $M \subseteq \mathbb{N}$ such that $M^{(2)}$ in $\mathbb{N}^{(2)}$ is monochromatic.