Algebra 2 Notes

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These notes are not necessarily correct, consistent, representative of the course as it stands today, or rigorous. Any result of the above is not the author's fault.

These notes are in progress.

Contents

1 The Fundamentals

1.1 Rings (1.1)

A ring is a set with two binary operations, addition and multiplication, such that they are both commutative, associative, and addition is distributive over multiplication, so for a, b, and c in some ring:

$$(a+b)c = ac + bc.$$

We also have that rings must contain 'zero' and 'one' elements, the additive and multiplicative identities, and every element of the ring has an additive inverse.

1.2 Properties of Rings (1.3)

For a ring R with a, b, and c in R:

- If $a + b = b$ then $a = 0$, 0 is unique,
- If $a \cdot x = x$ for all x in R, then $a = 1$, 1 is unique,
- If $a + b = 0 = a + c$ then $b = c$, $-a$ is unique,
- We have $0 \cdot a = 0$,
- We have $-1 \cdot a = -a$,
- We have $0 = 1$ if and only if $R = \{0\}$.

1.3 Units (1.6-7)

For a ring R, with r in R, if there exists some s such that $rs = 1$ then r is a unit and $s = r^{-1}$ is the multiplicative inverse of r. We write $R^\times$ to be the set of all units in R, which is an abelian group under multiplication.

1.4 Fields (1.9)

A non-zero ring R is a field if $R \setminus \{0\} = R^\times$.

1.5 Subrings (1.14-15)

For a ring R, $S \subseteq R$ is a subring of R if it is a ring and contains zero and one. This is equivalent to saying S is closed under addition, multiplication, and additive inverses, and contains 1.

1.6 The Gaussian Integers (1.17, 1.19)

We define the Gaussian integers as:

$$\mathbb{Z}[i] = \{a + bi : a, b \in \mathbb{Z}\},$$

which is the smallest subring of $\mathbb{C}$ containing i. Generally, for α in $\mathbb{C}$, $\mathbb{Z}[\alpha]$ is the smallest subring containing α and for a ring R with a subring S, for some β in R, we have $S[\beta]$ is the smallest subring of R containing S and β.

1.7 Product Rings (1.20)

For R and S rings, we have that $R \times S$ is a ring under component-wise addition and multiplication.

1.8 Distributivity of Taking Units (1.22)

For rings R and S, $(R \times S)^\times = R^\times \times S^\times$.

Proof. We consider:

$$\begin{aligned}(r, s) \in (R \times S)^\times &\iff (r, s)(p, q) = (1, 1) \text{ for some } (p, q) \in R \times S \\ &\iff rp = 1 \text{ and } sq = 1 \text{ for some } p \in R \text{ and } q \in S \\ &\iff r \in R^\times \text{ and } s \in S^\times,\end{aligned}$$

as required. □

1.9 Polynomials (1.23)

For a ring R and a symbol x, we have that the following is a ring:

$$R[x] = \{a_0 + a_1x + \cdots + a_nx^n : n \in \mathbb{Z}_{\geq 0}, (a_i)_{i \in [n]} \in R^n\}.$$

1.10 Ring Homomorphisms

For R and S rings, a map φ from R to S is a ring homomorphism if it preserves addition and multiplication. This implies that 0 and 1 are fixed points of φ and taking additive inverses is preserved by φ.

1.11 Ring Isomorphisms

A ring isomorphism is a bijective ring homomorphism.