

# Set Theory Notes

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*These notes are not necessarily correct, consistent, representative of the course as it stands today, or rigorous. Any result of the above is not the author's fault.*

**These notes are in progress.**

## 0 Notation

We commonly deal with the following concepts in Set Theory which I will abbreviate as follows for brevity:

Term	Notation
$\{0, 1, 2, \dots\}$	$\mathbb{N}$

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# 1 The Fundamentals

## 1.1 Axiom of Extensionality

For two sets  $a$  and  $b$ , we have that  $a = b$  if and only if for all  $x$  we have that:

$$x \in a \iff x \in b.$$

For two classes  $A$  and  $B$ , we have that  $A = B$  if and only if for all  $x$  we have that:

$$x \in a \iff x \in b.$$

## 1.2 Axiom of Pair Sets

For any sets  $x$  and  $y$ , there is a set  $z = \{x, y\}$ . This is the (unordered) pair set of  $x$  and  $y$ .

## 1.3 Axiom of the Powerset

For each set  $x$ , there exists a set which is the collection of the subsets of  $x$ , the powerset  $\mathcal{P}(x)$ .

For some set  $x$ , we have the powerset defined as follows  $\mathcal{P}(x) = \{z : z \subseteq x\}$ .

## 1.4 Axiom of the Empty Set

There exists a set with no members, the empty set  $\emptyset$ .

We have the empty set defined as follows  $\emptyset = \{x : x \neq x\}$ .

## 1.5 Axiom of Subsets

For some set  $x$ , we have that  $\{y \in x : \Phi(y)\}$  is a set for some well-defined property of sets  $\Phi$ .

## 1.6 Axiom of Unions

We have the basic union of two sets  $x_1$  and  $x_2$ :

$$x_1 \cup x_2 = \{y : y \in x_1 \text{ or } y \in x_2\},$$

but for cases where we want to unify the members of the sets in a set  $X$ , we define:

$$\bigcup X = \{y : \exists x \in X, y \in x\}.$$

This axiom states that for a set  $X$ ,  $\bigcup X$  is a set.

## 1.7 Classes

We have that classes are collection of objects, these could also be sets. Classes that are not sets are called proper classes.

## 1.8 The Set $\omega$

We have the set of natural numbers,  $\mathbb{N} = \{0, 1, 2, \dots\}$ , and from this, we define  $\omega$ :

$$\omega = \{0, 1, 2, \dots\},$$

where for some  $n$  in  $\omega$ ,

$$n = \{0, 1, 2, \dots, n-1\},$$

with  $0_\omega$  being the empty set. We can go beyond this definition, defining:

$$\begin{aligned}\omega + 1 &= \{0, 1, 2, \dots, \omega\}, \\ \omega + 2 &= \{0, 1, 2, \dots, \omega, \omega + 1\}, \\ &\dots \\ \omega + n &= \{0, 1, 2, \dots, \omega, \omega + 1, \dots, \omega + n - 1\}.\end{aligned}$$

## 1.9 Russell's Theorem

We have that  $R = \{x : x \notin x\}$  is not a set.

*Proof.* Suppose we have a set  $z$  such that  $z = R$ , is  $z$  in  $R$ ? If we suppose  $z$  is in  $R$ , we have that  $z$  is not in  $z$  by the definition of  $R$  (as  $z = R$ ) but  $z$  is  $R$  so  $z$  is not in  $R$ , a contradiction. Thus, we have that there is no set  $z$  equal to  $R$ , so  $R$  is not a set but a proper class.  $\square$

## 1.10 The Universe of Sets

We define the universe of sets as  $V = \{x : x = x\}$ . We have that  $V$  is a proper class.

*Proof.* If we suppose  $V$  is a set, we apply the axiom of subsets with  $\Phi(x) = x \notin x$  and reach a contradiction via Russell's theorem.  $\square$



## 2 Relations

We will first state the significant properties relations can have. Taking a relation  $R$  on  $X$  with  $x, y, z$  arbitrary in  $X$ :

Name	Property
Reflexive	$xRx$
Irreflexive	$\neg(xRx)$
Symmetric	$xRy \Rightarrow yRx$
Antisymmetric	$[xRy \text{ and } yRx] \Rightarrow [x = y]$
Connected	$[x = y] \text{ or } [xRy] \text{ or } [yRx]$
Transitive	$[xRy \text{ and } yRz] \Rightarrow [xRz]$

For example, equivalence relations must satisfy reflexivity, symmetry, and transitivity.

### 2.1 Partial Orderings

We say that a relation  $\prec$  on a set  $X$  is a (strict) partial ordering if it is irreflexive and transitive.

Similarly, we say that a relation  $\preceq$  on a set  $X$  is a non-strict partial ordering if it is reflexive, antisymmetric, and transitive.

A partial ordering  $(X, \prec)$  is wellfounded if for any non-empty subset  $Y$  of  $X$ ,  $Y$  has a least element under  $\prec$ .

## 2.2 Bounding

For a partially ordered set  $(X, \prec)$ :

- $x_0$  in  $X$  is the minimum of  $X$  if for all  $x$  in  $X$ ,  $x_0 \preceq x$ ,
- $x'$  in  $X$  is minimal in  $X$  if for all  $x$  in  $X$ ,  $\neg(x \prec x')$ ,
- $x_1$  in  $X$  is the maximum of  $X$  if for all  $x$  in  $X$ ,  $x \preceq x_1$ ,
- $x'$  in  $X$  is maximal in  $X$  if for all  $x$  in  $X$ ,  $\neg(x' \prec x)$ .

Taking a non-empty subset  $Y$  of  $X$ , we consider the subordering  $(Y, \prec)$  and for some  $\alpha$  in  $X$  we say:

- $\alpha$  is a lower bound for  $Y$  if for all  $y$  in  $Y$ ,  $\alpha \prec y$ ,
- $\alpha$  is the infimum of  $Y$  if it's a lower bound and for all lower bounds  $\lambda$  of  $Y$ ,  $\alpha \preceq \lambda$ ,
- $\alpha$  is an upper bound for  $Y$  if for all  $y$  in  $Y$ ,  $y \prec \alpha$ ,
- $\alpha$  is the supremum of  $Y$  if it's an upper bound and for all upper bounds  $\tau$  of  $Y$ ,  $\tau \preceq \alpha$ .

## 2.3 Order Preserving Maps

We say that  $f : (X, \prec_1) \rightarrow (Y, \prec_2)$  is an order preserving map if for each  $x_1, x_2$  in  $X$ :

$$x_1 \prec_1 x_2 \implies f(x_1) \prec_2 f(x_2).$$

Two orderings are (order) isomorphic if there is a bijective order preserving map between them.

## 2.4 Representation Theorem for Partially Ordered Sets

For a partially ordered set  $(X, \prec)$ , there is a set  $Y \subseteq \mathcal{P}(X)$  which is such that  $(X, \preceq)$  is order isomorphic to  $(Y, \subseteq)$ .

*Proof.* For some  $x$  in  $X$ , we set  $X^x = \{x' \in X : x' \preceq x\}$ , the set of elements preceding or equal to  $x$ . For  $x, y$  in  $X$ ,  $x \neq y$  implies that  $X^x \neq X^y$  as these sets contain  $x$  and  $y$  (resp.) so  $x \mapsto X^x$  is injective. This map is surjective trivially (mapping from  $X$  to  $\{X^x : x \in X\}$ ). We have that:

$$x \preceq y \iff X^x \subseteq X^y,$$

by our definition. Thus,  $x \mapsto X^x$  is an order isomorphism. □

## 2.5 Total Orderings

A relation  $\prec$  on a set  $X$  is a (strict) total ordering if it is a connected strict partial ordering.

Similarly, we say that a relation  $\preceq$  on a set  $X$  is a non-strict total ordering if it is a connected non-strict partial ordering.

## 2.6 Well-orderings

A relation  $\prec$  on a set  $X$  is a well-ordering if it is a strict total ordering and for any non-empty subset  $Y$  of  $X$ ,  $Y$  has a least element under  $\prec$ . We denote this with  $(X, \prec) \in WO$ .

## 2.7 Ordered Pairs

For  $x, y$  sets, the ordered pair of  $x$  and  $y$  is the set:

$$\langle x, y \rangle = \{\{x\}, \{x, y\}\}.$$

### 2.7.1 Uniqueness of Ordered Pairs

For  $x, y, u, v$  sets, we have that:

$$\langle x, y \rangle = \langle u, v \rangle \iff (x = u) \text{ and } (y = v).$$

*Proof.* Suppose the former, if  $x = y$  then  $\langle x, y \rangle = \{\{x\}, \{x, x\}\} = \{\{x\}\}$ . Thus,  $\langle u, v \rangle = \{\{u\}\}$  as it is equal to  $\langle x, y \rangle$  which has one element, hence  $u = v$ . By the Axiom of Extensionality, we have that  $x = u$  and so  $y = x = u = v$ .

If  $x \neq y$ , then  $\langle x, y \rangle$  and  $\langle u, v \rangle$  both have the same two elements by our assumption (so  $u \neq v$ ). We cannot have  $\{x\} = \{u, v\}$  so  $\{x\} = \{u\}$  which means  $x = u$  by the Axiom of Extensionality. Thus,  $\{u, v\} = \{x, y\} = \{u, y\}$  so  $y = v$ .

Suppose the latter, then the former holds trivially.  $\square$

### 2.7.2 The Ordered $k$ -tuple

We define the  $k$ -tuple inductively. The 2-tuple is already defined. We define the 3-tuple:

$$\langle x_1, x_2, x_3 \rangle = \langle \langle x_1, x_2 \rangle, x_3 \rangle,$$

and for  $k$  in  $\{3, 4, \dots\}$ :

$$\langle x_1, x_2, \dots, x_k \rangle = \langle \langle x_1, x_2, \dots, x_{k-1} \rangle, x_k \rangle.$$

### 2.7.3 The Product of Sets

For  $A, B$  sets, we define:

$$A \times B = \{\langle a, b \rangle : a \in A, b \in B\}.$$

Similarly to  $k$ -tuples, for  $A_1, A_2, \dots, A_k$  sets, we have  $A_1 \times A_2$  defined, so we define:

$$A_1 \times A_2 \times \dots \times A_k = (A_1 \times A_2 \times \dots \times A_{k-1}) \times A_k,$$

defining the  $k$ -product for  $k$  in  $\{2, 3, \dots\}$ . This is not associative.

## 2.8 Binary Relations

A binary relation  $R$  is a class of ordered pairs. We write  $R^{-1} = \{\langle y, x \rangle : \langle x, y \rangle \in R\}$ .

### 2.8.1 Domain and Range

For a relation  $R$ , we define:

$$\begin{aligned}\text{dom}(R) &= \{x : \exists y \text{ where } \langle x, y \rangle \in R\}, \\ \text{ran}(R) &= \{y : \exists x \text{ where } \langle x, y \rangle \in R\}, \\ \text{Field}(R) &= \text{dom}(R) \cup \text{ran}(R).\end{aligned}$$

## 2.9 Functions

A relation  $F$  is a function if for all  $x$  in  $\text{dom}(F)$ , there is a unique  $y$  in  $\text{ran}(F)$  with  $\langle x, y \rangle$  in  $F$ .

If  $F$  is a function, it is injective if and only if for all  $x, x'$ :

$$(\langle x, y \rangle \in F \text{ and } \langle x', y \rangle \in F) \Rightarrow (x = x').$$

### 2.9.1 Ranges and Restrictions

For  $F : X \rightarrow Y$ :

- $F''A = \{y \in Y : \exists x \in A \text{ such that } F(x) = y\}$  the range of  $F$  on  $A$ ,
- $F \upharpoonright A = \{\langle x, y \rangle \in F : x \in A\}$  the restriction of  $F$  to  $A$ .

We can see that  $F''A = \text{ran}(F \upharpoonright A)$ .

### 2.9.2 The Set of Functions

For  $X, Y$  sets, we have that  ${}^XY = \{F : F : X \rightarrow Y\}$ .

### 2.9.3 Indexed Cartesian Products

For a set  $I$  with each  $i$  in  $I$  corresponding to a non-empty set  $A_i$ :

$$\prod_{i \in I} A_i = \{\text{functions } f : \text{dom}(f) = I \text{ and } f(i) \in A_i \text{ for all } i \in I\}.$$

### 3 Transitive Sets

A set  $x$  is transitive if and only if for all  $y$  in  $x$ ,  $y \subseteq x$ . This can be abbreviated to  $\cup x \subseteq x$ .

#### 3.1 The Successor Function

For a set  $x$ ,  $S(x) = x \cup \{x\}$  is the successor of  $x$ .  $S(x) = x$  is equivalent to saying  $x$  is transitive.

#### 3.2 Transitive Closure

For a set  $x$ , to find a superset of  $x$  which is transitive, the transitive closure  $TC$  of  $x$ , we recurse:

$$\begin{aligned}\bigcup^0 x &= x, \\ \bigcup^{n+1} x &= \bigcup \left( \bigcup^n x \right),\end{aligned}$$

which we can write as:

$$TC(x) = \bigcup \left\{ \bigcup^n x : n \in \mathbb{N} \right\}.$$

The transitive closure is always transitive.

##### 3.2.1 Properties of Transitive Closure

For a set  $x$ :

1.  $x \subseteq TC(x)$ ,
2. If  $t$  is transitive and  $x \subseteq t$  then  $TC(x) \subseteq t$ .  $TC(x)$  is the smallest transitive set containing  $x$ ,
3. By the above,  $TC(x) = x$  if and only if  $x$  is transitive.

*Proof.* (1) This is true as  $\bigcup^0 x = x$ .

(2) If  $x \subseteq t$  then clearly  $\bigcup^0 x \subseteq t$ . We assume  $\bigcup^k x \subseteq t$  and use the fact that:

$$\left[ A \subseteq B \text{ with } B \text{ transitive} \right] \Rightarrow \bigcup A \subseteq B,$$

to deduce that  $\bigcup^{k+1} x \subseteq t$ . By induction we have that  $TC(x) \subseteq t$  as required.

(3) By (1),  $x \subseteq TC(x)$ . If  $x$  is transitive, we substitute it for  $t$  in (2) and get that  $TC(x) \subseteq x$  as required.  $\square$

## 4 Number Systems

### 4.1 Von Neumann Numerals

We have the von Neumann numerals defined as:

$$\begin{aligned}0 &= \emptyset, \\1 &= \{\emptyset\} = \{0\}, \\2 &= \{\emptyset, \{\emptyset\}\} = \{1, 2\}, \\&\dots \\n+1 &= \{0, 1, \dots, n\}.\end{aligned}$$

### 4.2 Inductive Sets

A set  $X$  is called inductive if  $\emptyset$  is in  $X$  and for all  $x$  in  $X$ ,  $S(x)$  is in  $X$ .

### 4.3 Axiom of Infinity

There exists an inductive set.

### 4.4 Natural Numbers

We say that  $x$  is a natural number if for all  $X$ :

$$X \text{ is an inductive set} \Rightarrow x \in X.$$

We define  $\omega$  as the class of natural numbers,  $\omega = \cap\{X : X \text{ is an inductive set}\}$ . We have that  $\omega$  is the smallest inductive set.

*Proof.* Let  $z$  be an inductive set (by the Axiom of Infinity it exists). By the Axiom of Subsets, we define a set  $N$ :

$$N = \{x \in z : \forall Y, Y \text{ is inductive} \Rightarrow x \in Y\},$$

the elements of  $z$  in every inductive set. But  $N = \omega$ , so  $\omega$  is a set.

We know that  $\emptyset$  is in every inductive set by definition, so  $\emptyset$  is in  $\omega$  as it is the intersection of all inductive sets. For any  $x$  in  $\omega$ , we know that for any inductive set  $Y$  that  $x$  is in  $Y$  (by the definition of  $\omega$ ) and thus  $S(x)$  is also in  $Y$  (by the definition of an inductive set). Thus,  $S(x)$  is also in  $\omega$  as  $Y$  was chosen arbitrarily. Hence,  $\omega$  is an inductive set and the smallest such set by its definition.  $\square$

## 4.5 Principle of Mathematical Induction

We suppose  $\Phi$  is a well-defined property of sets, then we have that:

$$\left[ \Phi(0) \text{ and } \forall x \in \omega \text{ we have that } \Phi(x) \Rightarrow \Phi(S(x)) \right] \Rightarrow \left[ \forall x \in \omega \text{ we have that } \Phi(x) \right].$$

*Proof.* We assume the antecedent, it suffices to show that the collection of  $x$  in  $\omega$  where  $\Phi(x)$  holds is inductive (as we assume  $\Phi(0)$  holds).

Let  $Y = \{x \in \omega : \Phi(x)\}$ . As we assumed  $\Phi(0)$ , we know that 0 is in  $Y$ . Then, by the second half of our assumption, we can see that  $Y$  is closed under the successor function. Thus,  $Y$  is inductive and as  $\omega$  is the smallest inductive set,  $\omega \subseteq Y$  as required.  $\square$

## 4.6 Representation of Natural Numbers

We have that every natural number is either 0 or  $S(x)$  for some natural number  $x$ .

*Proof.* Let  $Z = \{y \in \omega : y = 0 \text{ or } \exists x \in \omega \text{ such that } S(x) = y\}$ . It suffices to show that  $Z$  is inductive. Clearly, 0 is in  $Z$ . Suppose we have some  $u$  in  $Z$ , then  $u$  is in  $\omega$ . As  $\omega$  is inductive,  $S(u)$  is also in  $\omega$  so  $S(u)$  is in  $Z$ . Thus,  $Z$  is inductive as required.  $\square$

## 4.7 Transitivity of $\omega$

We have that  $\omega$  is transitive.

*Proof.* Let  $X = \{n \in \omega : n \subseteq \omega\}$ . If  $X = \omega$  then by definition  $\omega$  is transitive. It suffices to show that  $X$  is inductive. We know that  $\emptyset$  is in  $X$  as 0 is in  $\omega$ . Taking  $n$  in  $X$ , then clearly  $\{n\} \subseteq \omega$  as  $n$  is in  $\omega$ . Furthermore,  $n \subseteq \omega$  as  $n$  is in  $X$ . Thus,  $n \cup \{n\} \subseteq \omega$  so  $S(n) \in X$  which means  $X$  is inductive as required.  $\square$

## 4.8 Ordering on the Naturals

For  $m, n$  in  $\omega$ , we define:

$$\begin{aligned} m < n &\iff m \in n, \\ m \leq n &\iff m = n \text{ or } m \in n. \end{aligned}$$

By definition,  $n < S(n)$ .



We have that:

1. This ordering is transitive,
2. For all  $n$  in  $\omega$  and for all  $m$  we have that  $m < n$  if and only if  $S(m) < S(n)$ ,
3. For all  $n$  in  $\omega$ ,  $n \not< n$ .

*Proof.* (1) This follows from the transitivity of set inclusion.

(2) We take  $\Phi(k) = [(m < k) \Rightarrow (S(m) < S(k))]$ . We see  $\Phi(0)$  holds. Supposing  $\Phi(k)$  holds for some  $k$ , let  $m < S(k)$  then  $m$  is in  $k \cup \{k\}$ . If  $m$  is in  $k$  then by  $\Phi(k)$  we have that  $S(m) < S(k) < S(S(k))$ . If  $m = k$  then  $S(m) = S(k) < S(S(k))$ . Thus, by induction, we have our result.

Assume  $S(m) < S(n)$ ,  $m$  is in  $S(m) = m \cup \{m\}$  which is in  $S(n) = n \cup \{n\}$ . If  $S(m) = n$ , then  $m$  is in  $n$  so  $m < n$ . If  $S(m)$  is in  $n$  then  $m$  is in  $n$  as  $n$  is transitive.

(3) We know that  $0 \not< 0$  as  $0 \notin 0$ . If  $k \notin k$  then  $S(k) \notin S(k)$  by Part (ii). Thus,  $X = \{k \in \omega : k \notin k\}$  is inductive which makes it equal to  $\omega$  as required.  $\square$

## 4.9 Total Ordering on the Naturals

We have that  $<$  is a (strict) total ordering on the naturals.

## 4.10 Well-ordering Theorem for $\omega$

Let  $X \subseteq \omega$ , then either  $X = \emptyset$  or there is some  $n_0$  in  $X$  such that for any  $m$  in  $X$  either  $n_0 = m$  or  $n_0 < m$ .

*Proof.* Suppose  $X \subseteq \omega$  but has no least element. Let  $Z = \{k \in \omega : \forall n < k, n \notin X\}$ . We want to show  $Z$  is inductive, meaning  $Z = \omega$  and so  $X = \emptyset$ .

Vacuously,  $0$  is in  $Z$ . Suppose we have  $k$  in  $Z$ , we let  $n < S(k) = k \cup \{k\}$  and consider:

- If  $n \in k$  then  $n \notin X$  as  $n < k \in Z$ ,
- If  $n = k$  then  $n \notin X$  because if  $n$  was in  $X$  then it would be the least element of  $X$ , a contradiction.

Thus,  $S(k)$  is in  $Z$  so  $Z$  is inductive.  $\square$

### 4.11 Recursion Theorem on $\omega$

Let  $A$  be any set with  $a$  in  $A$  and  $f : A \rightarrow A$  any function. There exists a unique function  $h : \omega \rightarrow A$  such that for any  $n$  in  $\omega$ :

$$\begin{aligned} h(0) &= a, \\ h(S(n)) &= f(h(n)). \end{aligned}$$

*Proof.* We will find  $h$  as a union of  $k$ -approximations where  $u$  is a  $k$ -approximation if it is a function with  $\text{dom}(u) = k$  and for:

- If  $k > 0$  then  $u(0) = a$ ,
- If  $k > S(n)$  then  $u(S(n)) = f(u(n))$ .

From this, we see that  $\{\langle 0, a \rangle\}$  is a 1-approximation in particular. Furthermore, if  $u$  is a  $k$ -approximation and  $l \leq k$  then  $u \upharpoonright l$  is an  $l$ -approximation, finally if  $u(k-1) = c$  for some  $c$ , then  $u' = u \cup \{\langle k, f(c) \rangle\}$  is a  $(k+1)$ -approximation.

#### Agreement on Domain

If  $u$  is a  $k$ -approximation and  $v$  is a  $k'$ -approximation for some  $k \leq k'$  then  $v \upharpoonright k = u$  (hence  $u \subseteq v$ ).

*Proof.* We appeal to the contrary with  $0 \leq m < k$  being the least natural such that  $u(m) \neq v(m)$ . We know that  $m \neq 0$  as  $u(0) = a = v(0)$ . So,  $m = S(m')$  for some  $m'$ . As  $m$  is chosen minimally,  $u(m') = v(m')$ . We can then see that  $u(m) = f(u(m')) = f(v(m')) = v(m)$ , a contradiction.  $\square$

#### Uniqueness

If  $h$  exists, it is unique.

*Proof.* Suppose  $h$  and  $h'$  are two different functions with domain  $\omega$  satisfying the theorem. We take  $0 \leq m < \omega$  to be the least natural such that  $h(m) \neq h'(m)$  and apply the same reasoning to the above.  $\square$

#### Existence

Let  $u$  be in  $B$  if and only if there exists  $k$  in  $\omega$  such that  $u$  is a  $k$ -approximation. For any  $u, v$  in  $B$  either  $u \subset v$  or vice-versa by our previous results. We take  $h = \bigcup B$ .

We have that  $h$  is a function:

*Proof.* We appeal to the contrary. If  $\langle n, c \rangle$  and  $\langle n, d \rangle$  are in  $h$  with  $c \neq d$ , then we have  $u, v$  in  $B$  with  $u(n) = c$  and  $v(n) = d$  but this is impossible by **Agreement on Domain**.  $\square$

We have that  $\text{dom}(h) = \omega$ :

*Proof.* We appeal to the contrary and suppose  $\emptyset \neq X = \{n \in \omega : n \notin \text{dom}(h)\}$ . By the definition of  $h$  this means that:

$$X = \{n \in \omega : \text{There's no } u\text{-approximation with } n \in \text{dom}(u)\}.$$

We saw that there is a 1-approximation, so 0 is not the least element of  $X$ . We suppose  $n_0 = S(m)$  is the least element of  $X$ . As  $m$  is not in  $X$ , there must be an  $n_0$ -approximation  $n$  with  $n(m) = c$  for some  $c$ . But, we saw that we can extend  $k$ -approximations, so we can generate a  $(n_0 + 1)$ -approximation which is a contradiction. Thus,  $X = \emptyset$ .  $\square$

Thus, we have that  $h$  exists and is a unique function as required.  $\square$

## 5 Well-orderings and Ordinals

### 5.1 The Principle of Transfinite Induction

Let  $\langle X, \prec \rangle$  be a well-ordering. We have that:

$$[\forall x \in X, (\forall y \prec x, \Phi(y)) \Rightarrow \Phi(x)] \Rightarrow \forall x \in X, \Phi(x).$$

*Proof.* We appeal to the contrary and assume the antecedent but suppose that  $\emptyset \neq Z = \{x \in X : \neg \Phi(x)\}$ . As  $\langle Z, \prec \rangle$ , there is  $\prec$ -least element  $z_0$ . But then for all  $x \prec z_0$ ,  $\Phi(x)$  holds. But, by the antecedent, this means  $\Phi(z_0)$  holds, a contradiction.  $\square$

### 5.2 Initial Segments

For a well-ordering  $\langle X, \prec \rangle$ , the  $\prec$ -initial segment of some element  $z$  in  $X$  is the set of predecessors of  $z$ , denoted by  $X_z$ . Note that  $X_z$  does not contain  $z$ .

### 5.3 Order Preserving Maps on Well-orderings

For a well-ordering  $\langle X, \prec \rangle$  with  $f : \langle X, \prec \rangle \rightarrow \langle X, \prec \rangle$  an order preserving map, we have that for all  $x$  in  $X$ ,  $x \prec f(x)$ .

*Proof.* We appeal to the contrary, that for some  $x$  in  $X$ , we have  $f(x) \prec x$ . As  $\langle X, \prec \rangle$  is a well-ordering, there's a  $\prec$ -least  $x_0$  in  $X$  with the property that  $f(x_0) \prec x_0$ . But  $f(f(x_0)) \prec f(x_0)$  as  $f$  is order preserving. Thus, a contradiction to the minimality of  $x_0$ .  $\square$

#### 5.3.1 Uniqueness of Order Isomorphisms

For well-orderings  $\langle X, \prec_x \rangle$ ,  $\langle Y, \prec_y \rangle$  with  $f : \langle X, \prec_x \rangle \rightarrow \langle Y, \prec_y \rangle$  an order isomorphism. We have that  $f$  is unique.

*Proof.* Suppose we have two such isomorphisms  $f$  and  $g$ . We have that  $(f^{-1} \circ g)$  is also an order isomorphism. Taking  $x$  arbitrary in  $X$ :

$$\begin{aligned} & x \preceq_x (f^{-1} \circ g)(x) \\ \implies & f(x) \preceq_y f(f^{-1} \circ g)(x) \\ \implies & f(x) \preceq_y g(x). \end{aligned}$$

By applying this argument again with the roles of  $f$  and  $g$  swapped, we can also see that  $g(x) \preceq_y f(x)$ . Thus,  $f(x) = g(x)$ .

In particular, if  $\langle X, \prec_x \rangle = \langle Y, \prec_y \rangle$  then this isomorphism is the identity map.  $\square$

### 5.3.2 Non-existence of Order Isomorphisms to Segments

A well-ordered set is not order isomorphic to any segment of itself.

*Proof.* We appeal to the contrary and suppose there is such an order isomorphism on a well-ordering  $\langle X, \prec \rangle$  to  $\langle X_z, \prec \rangle$  for some  $z$  in  $X$ . But, we have that  $x \preceq f(x)$  for any  $x$  in  $X$  and  $f(z) \prec z$  as  $f(z)$  is in  $X_z$ . Thus, we have that  $z \preceq f(z)$  and  $z \succ f(z)$ , a contradiction.  $\square$

### 5.3.3 Order Isomorphism to Set of Segments

A well-ordered set  $\langle X, \prec \rangle$  is order isomorphic to the set of its initial segments ordered by  $\subset$ .

*Proof.* We let  $Y = \{X_a : a \in X\}$ , we have that  $\varphi$  characterised by  $a \mapsto X_a$  is an injective map as segments do not contain the element which determines it. As  $a \prec b \Leftrightarrow X_a \subset X_b$ , the mapping is order preserving.  $\square$

## 5.4 Ordinal Numbers

We say that  $\langle X, \in \rangle$  is an ordinal if and only if  $X$  is transitive, and where  $\prec = \in$ ,  $\langle X, \prec \rangle$  is a well-ordering. We have that  $\langle \omega, \in \rangle$  is an ordinal.

### 5.4.1 Elements and Segments

For an ordinal  $\langle X, \in \rangle$ , then every element  $z$  in  $X$  is identical to  $X_z$ .

*Proof.* Suppose  $X$  is transitive and  $\in$  well-orders  $X$ . Taking  $z$  in  $X$ :

$$\begin{aligned} w \in X_z &\iff w \in X \text{ and } w \in z \\ &\iff w \in z, & (\text{as } z \subseteq X) \end{aligned}$$

thus,  $X_z = z$  as required.  $\square$

So, for any elements  $a, b$  of an ordinal:

$$a \in b \iff a \subset b \iff X_a \subset X_b.$$

### 5.4.2 Subsets and Segments

For an ordinal  $\langle X, \in \rangle$  with  $Y \subset X$ , if  $\langle Y, \in \rangle$  is also an ordinal, then  $Y$  is an  $\in$ -initial segment of  $X$ .

*Proof.* Taking  $a$  in  $Y$  as supposed, as  $Y$  is an ordinal so  $Y_a = a$ . As  $Y \subset X$ ,  $a$  is in  $X$  so  $X_a = a$ . Thus,  $X_a = Y_a$ . Furthermore, as  $Y \neq X$ , we consider  $c = \inf\{z \in X : z \notin Y\}$  which exists as the set is non-empty and  $\langle X, \in \rangle$  is a well-ordering. Hence,  $Y = X_c$ .  $\square$

### 5.4.3 Segments

For an ordinal  $\langle X, \in \rangle$  any  $\in$ -initial segment of  $\langle X, \in \rangle$  is an ordinal.

*Proof.* We take some  $u$  in  $X$  and  $w$  in  $X_u$ . As  $\in$  well-orders  $X$ , it well-orders any subset of  $X$  so  $\langle X_u, \in \rangle$  is a well-ordering. We have that:

$$t \in w \in u \implies t \in u = X_u,$$

thus  $X_u$  is transitive as required.  $\square$

### 5.4.4 The Intersection of Ordinals

For ordinals  $X, Y$ ,  $X \cap Y$  is also an ordinal.

*Proof.* We take  $\in$  to be the ordering on  $X$ . We know that  $X \cap Y$  is transitive as  $X$  and  $Y$  are transitive. Any subset of  $X$  is a well-ordering under  $\in$ , in particular  $X \cap Y$  is well-ordered by  $\in$ .  $\square$

## 5.5 Classification Theorem for Ordinals

For two ordinals  $X, Y$ , either  $X = Y$  or one is an initial segment of the other (or equivalently a member).

*Proof.* Suppose that  $X \neq Y$ . We know that  $X \cap Y$  is an ordinal also. We have two cases.

If  $X = X \cap Y$  or  $Y = X \cap Y$ , one must be an initial segment of the other as it must be a proper subset under our assumption  $X \neq Y$ .

If  $X \cap Y$  is a proper subset of  $X$  and  $Y$ , it is an initial segment of  $X$  and  $Y$  simultaneously we set  $X \cap Y = X_a = Y_b$  for some  $a$  in  $X$  and  $b$  in  $Y$ . But, we know that as  $X$  and  $Y$  are ordinals,  $a = X_a = Y_b = b$ . However, this means:

$$a = b \in X \cap Y = X_a,$$

but  $a \notin X_a$ , a contradiction.  $\square$

## 5.6 Equality under Isomorphisms

For two ordinals  $X$  and  $Y$ , if  $X$  is order isomorphic to  $Y$  then  $X = Y$ .

*Proof.* Suppose  $X \neq Y$ , then without loss of generality we take  $X$  to be an initial segment of  $Y$ . But, this would mean  $Y$  is order isomorphic to a segment of itself which is a contradiction.  $\square$

## 5.7 Bound on Isomorphisms

A well-ordering is order isomorphic to at most one ordinal.

*Proof.* If a well-ordering is isomorphic to more than one ordinals, then these ordinals are isomorphic to each other and thus, equal.  $\square$

## 5.8 Criterion for Ordinals

If every segment of a well-ordered set  $\langle A, \prec \rangle$  is order isomorphic to some ordinal,  $\langle A, \prec \rangle$  itself is order isomorphic to an ordinal.

*Proof.* Each segment must be order isomorphic to at most one ordinal (thus exactly one). We define a function  $F$  from the segments of  $A$  to the ordinal  $F(A_b)$  such that  $\langle A_b, \prec \rangle \cong \langle F(b), \in \rangle$ . We know that this ordinal is unique as non-identical segments have differing sizes. We let  $Z = \text{ran}(F)$ :

$$Z = \{F(b) : \exists b \in A, \exists \text{ an isomorphism } g_b \text{ from } \langle A_b, \prec \rangle \text{ to } \langle F(b), \in \rangle\},$$

noting that the isomorphism between  $A_b$  and  $F(b)$  is unique. If  $c, b$  are in  $A$  with  $c \prec b$  then  $A_c = (A_b)_c$  implying that  $F(c) \neq F(b)$  so  $F$  is injective and thus bijective between  $A$  and  $Z$ . Continuing with  $c \prec b$ :

$$g_b \upharpoonright A_c : \langle A_c, \prec \rangle \cong \langle (F(b))_{g_b(c)}, \in \rangle,$$

by the uniqueness of order isomorphisms  $(g_b \upharpoonright A_c) = g_c$  and  $F(c) = (F(b))_{g_b(c)}$ . Thus,  $F(c) \in F(b)$ .

We know that  $Z$  is well-ordered by  $\in$  as  $A$  is well-ordered by  $\prec$  and  $F$  is an order isomorphism. For  $u \in F(b) \in Z$ , as  $g_b$  is surjective,  $u = g_b(c)$  for some  $c \prec b$ . Then,  $u = F(b)_u = F(b)_{g_b(c)} = F(c)$ . Hence,  $u$  is in  $Z$ , so  $Z$  is transitive. Thus,  $Z$  is an ordinal.  $\square$

## 5.9 Representation Theorem for Well-orderings

Every well-ordering is order isomorphic to exactly one ordinal.

*Proof.* For a well-ordering  $\langle X, \prec \rangle$ , we know that if it is isomorphic to an ordinal, this is the only such ordinal. We take:

$$Z = \{v \in X : X_v \text{ is not isomorphic to an ordinal}\},$$

and want to show it's empty as this will suffice combined with our criterion. Supposing the contrary, we take  $v_0$  to be the  $\prec$ -least element of  $Z$ . We have that  $\langle X_{v_0}, \prec \rangle$  is a well-ordering with each element preceding  $v_0$ , as such  $(X_{v_0})_w = X_w$  for each  $w$  in  $X_{v_0}$  is order isomorphic to an ordinal. Which means  $X_{v_0}$  must also be order isomorphic to an ordinal via our criterion, a contradiction. Thus,  $Z$  is empty, as required.  $\square$

## 5.10 Order Type of Well-orderings

For a well-ordering  $\langle X, \prec \rangle$ , the order type of  $\langle X, \prec \rangle$  is the unique ordinal isomorphic to  $\langle X, \prec \rangle$ , written as  $\text{ot}(\langle X, \prec \rangle)$ .

## 5.11 Classification Theorem for Well-orderings

For two well-orderings  $\langle A, \prec_A \rangle$  and  $\langle B, \prec_B \rangle$  we have that exactly one of the following holds:

- $\langle A, \prec_A \rangle \cong \langle B, \prec_B \rangle$ ,
- There exists  $b$  in  $B$  such that  $\langle A, \prec_A \rangle \cong \langle B_b, \prec_B \rangle$ ,
- There exists  $a$  in  $A$  such that  $\langle A_a, \prec_A \rangle \cong \langle B, \prec_B \rangle$ .

*Proof.* We take  $\langle X, \in \rangle$  and  $\langle Y, \in \rangle$  to be the unique ordinals isomorphic to  $\langle A, \prec_A \rangle$  and  $\langle B, \prec_B \rangle$  (resp.) via maps:

$$\begin{aligned} f : \langle X, \in \rangle &\rightarrow \langle A, \prec_A \rangle, \\ g : \langle Y, \in \rangle &\rightarrow \langle B, \prec_B \rangle. \end{aligned}$$

We know that either these ordinals are order isomorphic or order isomorphic to an initial segment of the other. If the former is true, then we have that our well-orderings are isomorphic via  $f$  and  $g$  and their inverses. If the latter is true, we know that (without loss of generality)  $\langle X, \in \rangle \cong \langle Y_y, \in \rangle$  for some  $y$  in  $Y$ . Thus:

$$f(\langle X, \in \rangle) \cong g(\langle Y_y, \in \rangle) \implies \langle A, \prec_A \rangle \cong \langle B_{g(y)}, \prec_B \rangle,$$

as required.  $\square$



## 6 Properties of Ordinals

We collate the properties of ordinals covered so far for some ordinals  $\alpha$ ,  $\beta$ , and  $\gamma$ :

- Ordinals are transitive,
- Ordinals are well-ordered by  $\in$ ,
- $\alpha \in \beta \in \gamma$  implies that  $\alpha \in \gamma$ ,
- If  $X \in \alpha$  then  $X$  is an ordinal with  $X = \alpha_X$ ,
- If  $\alpha \cong \beta$  then  $\alpha = \beta$ ,
- Exactly one of the following holds:
  - $\alpha = \beta$ ,
  - $\alpha \in \beta$ ,
  - $\beta \in \alpha$ .

### 6.1 Principle of Transfinite Induction

For  $\Phi$  a well-defined and definite property of ordinals, we have that for all ordinals  $\alpha$ :

$$[\forall \beta < \alpha, \Phi(\beta) \Rightarrow \Phi(\alpha)] \Rightarrow [\Phi(\alpha)].$$

Hence, the class of ordinals is well-ordered.

*Proof.* We consider  $C = \{\alpha \text{ an ordinal} : \neg\Phi(\alpha)\}$  and suppose it's non-empty. We take  $\alpha_0$  in  $C$ , if it is not the least element we have that  $\alpha_0 \cap C$  is non-empty as there is some  $\beta$  in  $C$  with  $\beta < \alpha_0$  which is equivalent to saying that  $\beta \in \alpha_0$ . As  $\alpha_0$  is an ordinal, we have that  $\alpha_0$  is well-ordered by  $\in$ , hence  $C \cap \alpha_0 \subseteq \alpha_0$  has an  $\in$ -least element  $\alpha_1$  which is the least element of  $C$ .

Thus, we have that  $\neg\Phi(\alpha_1)$  holds, but as this is the least element of  $C$ , for all ordinals  $\beta$  less than  $\alpha_1$  we have that  $\Phi(\beta)$  holds. However, by our antecedent, this means  $\Phi(\alpha_1)$  holds, a contradiction. Thus,  $C$  is empty as required.

Note that our argument showed that any non-empty class on the ordinals has a least element. Thus, the class of ordinals is well-ordered by  $\in$ .  $\square$

## 6.2 The Class of Ordinals

The class of ordinals is a proper class.

*Proof.* Suppose the class of ordinals is a set  $z$ . We have that  $\langle z, \in \rangle$  is transitive and well-ordered by  $\in$ . Thus,  $z$  is an ordinal, as such  $z$  is in  $z$ . But, this contradicts the strict ordering of  $\in$ .  $\square$

## 6.3 Sum of Orderings

For  $\langle A, R \rangle$  and  $\langle B, S \rangle$  strict total orderings with  $A \cap B$  empty, we define the sum ordering  $\langle C, T \rangle$  as:

$$C = A \cup B,$$

$$xTy \Leftrightarrow \begin{cases} xRy & \text{for } x, y \in A \\ xSy & \text{for } x, y \in B \\ x \in A \text{ and } y \in B & \text{otherwise.} \end{cases}$$

We can avoid the disjoint constraint by taking the sum of  $\langle A \times \{0\}, R \rangle$  and  $\langle B \times \{1\}, S \rangle$ . We can built this functionality into our operation. We name this operation  $+$ ' and for  $\alpha, \beta$  ordinals:

$$\alpha +' \beta = \text{ot}(\langle \alpha \times \{0\} \cup \beta \times \{1\} \rangle, T),$$

$$\langle \gamma, i \rangle T \langle \delta, j \rangle \Leftrightarrow (i = j \text{ and } \gamma < \delta) \text{ or } (i < j).$$

## 6.4 Product of Orderings

For  $\langle A, R \rangle$  and  $\langle B, S \rangle$  strict total orderings. We define the product of these orderings  $\langle A, R \rangle \times \langle B, S \rangle$  to be the ordering  $\langle C, U \rangle$ :

$$C = A \times B$$

$$\langle x, y \rangle U \langle x', y' \rangle \Leftrightarrow (ySy') \text{ or } (y = y' \text{ and } xRx'),$$

taking the latter and replacing each member by the former. We can again construct an operation for ordinals  $\alpha$  and  $\beta$ :

$$\alpha \cdot' \beta = \text{ot}(\langle A \times B, U \rangle),$$

with  $U$  defined as above.

## 6.5 Supremum of Ordinals

For a set of ordinals  $A$ ,  $\sup(A)$  is the least ordinal  $\gamma$  such that for all  $\delta$  in  $A$ ,  $\delta \leq \gamma$ . Furthermore, we have the strict supremum  $\sup^+(A)$  as the least ordinal  $\gamma^+$  such that for all  $\delta$  in  $A$ ,  $\delta < \gamma^+$ .

We can also write  $\sup(A) = \bigcup A$ .

*Proof.* If we suppose there isn't an ordinal which is an upper bound for  $A$ , there's some  $\delta$  in  $A$  such that  $\delta > \gamma$  for each ordinal  $\gamma$ . But,  $\bigcup A$  must be a set and equal to the set of ordinals, a contradiction.

Take  $S = \sup(A)$  and take  $u$  in  $\bigcup A$ . For some  $a$  in  $A$ ,  $u < a < A$ , so  $u < S$  and so  $u$  is in  $S$ ,  $\bigcup A \subseteq S$ . Conversely, we consider  $s$  in  $S$ , then  $s < S = \sup(A)$  so there is some  $a$  in  $A$  with  $s < a \leq S$ . Thus,  $s$  is in  $A$  so  $s$  is in  $\bigcup A$ ,  $S \subseteq \bigcup A$ . Thus  $S = \bigcup A$ .  $\square$

## 6.6 Types of Ordinals

We can consider three types of ordinals:

- The zero ordinal, 0,
- Successor ordinals, ordinals with immediate predecessors,
- Limit ordinals, ordinals that are not of the other types.

## 6.7 Transfinite Recursion Theorem on Ordinals

For  $F : V \rightarrow V$  a function, there exists a unique function  $H$  from the ordinals to  $V$  such that for all  $\alpha$ :

$$H(\alpha) = F(H \upharpoonright \alpha).$$

*Proof.* We define a function  $u$  to be a  $\delta$ -approximation if  $\text{dom}(u) = \delta$  and for all  $\alpha < \delta$ ,  $u(\alpha) = F(u \upharpoonright \alpha)$ .

### Observations

We consider  $u$  to be a  $\delta$ -approximation. For  $\delta > 0$ , we see that  $u(0) = F(u \upharpoonright 0) = F(\emptyset)$  so a 1-approximation is equal to  $\{\langle 0, F(\emptyset) \rangle\}$  with domain  $\{0\} = 1$ . Additionally, for some  $\gamma < \delta$ ,  $u \upharpoonright \gamma$  is a  $\gamma$ -approximation. Furthermore,  $u \cup \{\langle \delta, F(u) \rangle\}$  is a  $(\delta + 1)$ -approximation. We take  $B = \{u : \exists \delta \text{ such that } u \text{ is a } \delta\text{-approximation}\}$ .

### Agreement on Domain

For  $u$  a  $\delta$ -approximation and  $v$  any  $\gamma$ -approximation with  $\delta < \gamma$ ,  $u = v \restriction \delta$ .

*Proof.* We appeal to the contrary and take  $\tau$  be the least value such that  $u(\tau) \neq v(\tau)$ . Thus,  $(u \restriction \tau = v \restriction \tau)$  but then:

$$u(\tau) = F(u \restriction \tau) = F(v \restriction \tau) = v(\tau),$$

which is a contradiction.  $\square$

### Uniqueness

If such  $H$  exists, it is unique.

*Proof.* We appeal to the contrary, taking  $H'$  to be some differing derivation of  $H$ . We consider the least  $\tau$  such that  $H(\tau) \neq H'(\tau)$  and apply the same reasoning to the above.  $\square$

### Limits

For some limit ordinal  $\lambda$ , if for all  $\alpha < \lambda$  we have that  $u_\alpha$  is an  $\alpha$ -approximation,  $\bigcup_{\alpha < \lambda} u_\alpha$  is a  $\lambda$ -approximation.

*Proof.* This union is a union of an increasing sequence of sets (so  $\alpha < \beta < \lambda \Rightarrow u_\alpha \subseteq u_\beta$ ). As each element is a function, the union is also a function with domain  $\lambda$ . Thus, this union is a  $\lambda$ -approximation.  $\square$

### Existence

We define  $H = \bigcup B$  which is a function with  $\text{dom}(H)$  being the set of ordinals.

*Proof.* We know that  $H$  is a function by **Agreement on Domain**. We take  $C = \{\delta : \text{There's no } \delta\text{-approximation}\}$  and suppose  $C$  is non-empty. By the principle of transfinite induction on the ordinals,  $C$  has a least element  $\psi$ . We know that  $\psi > 1$  as we defined a 1-approximation and by **Limits** it cannot be a limit ordinal. If  $\psi = \mu + 1$  then there's a  $\mu$ -approximation  $v$  by the minimality of  $\psi$ . However, we can extend  $v$  to a  $\psi$ -approximation  $u$  by setting  $u(\mu) = F(v)$ . This is a contradiction.  $\square$

Thus, we have that  $H$  exists and is a unique function as required.  $\square$

## 6.8 Alternative Transfinite Recursion on Ordinals

For  $a$  in  $V$  and  $F_0, F_1 : V \rightarrow V$  functions, there's a unique function  $H$  from the set of ordinals to  $V$  such that:

$$\begin{aligned} H(0) &= a, \\ \text{succ}(\alpha) &\implies H(\alpha) = F_0(H(\beta)) \text{ where } \alpha = \beta + 1, \\ \text{lim}(\alpha) &\implies H(\alpha) = F_1(H \upharpoonright \alpha). \end{aligned}$$

*Proof.* We define  $F : V \rightarrow V$  by:

$$F(u) = \begin{cases} a & \text{for } u = \emptyset \\ F_0(u) & \text{if } u \text{ is a function with a successor domain} \\ F_1(u) & \text{if } u \text{ is a function with a limit domain} \\ \emptyset & \text{otherwise,} \end{cases}$$

and apply the previous recursion theorem. □

## 6.9 Ordinal Addition

We define ordinal addition  $A_\alpha$  for some successor ordinal  $\beta + 1$  and limit ordinal  $\lambda$  as:

$$\begin{aligned} A_\alpha(0) &= \alpha + 0 = \alpha, \\ A_\alpha(\beta + 1) &= S(A_\alpha(\beta)) = A_\alpha(\beta) + 1, \\ A_\alpha(\lambda) &= \sup(\{A_\alpha(x) : x < \lambda\}). \end{aligned}$$

## 6.10 Ordinal Multiplication

We define ordinal multiplication  $M_\alpha$  for some successor ordinal  $\beta + 1$  and limit ordinal  $\lambda$  as:

$$\begin{aligned} M_\alpha(0) &= 0, \\ M_\alpha(\beta + 1) &= M_\alpha(\beta) + \alpha, \\ M_\alpha(\lambda) &= \sup(\{M_\alpha(x) : x < \lambda\}). \end{aligned}$$

## 6.11 Ordinal Exponentiation

We define ordinal exponentiation  $A_\alpha$  for some successor ordinal  $\beta + 1$  and limit ordinal  $\lambda$  as:

$$\begin{aligned} E_\alpha(0) &= 1, \\ E_\alpha(\beta + 1) &= E_\alpha(\beta) \cdot \alpha, \\ E_\alpha(\lambda) &= \sup(\{E_\alpha(x) : x < \lambda\}). \end{aligned}$$

## 6.12 Monotonicity of $A_\alpha$

The functions  $A_\alpha$  are strictly increasing and thus injective.

*Proof.* We consider  $\beta, \gamma, \delta$  ordinals with:

$$[\beta < \gamma] \implies [A_\alpha(\beta) < A_\alpha(\gamma)], \quad (1)$$

for all  $\gamma \leq \delta$ . The base case is trivial, we consider  $\delta + 1$ . For  $\beta < \delta + 1$ , if  $\beta = \delta$ , then:

$$\begin{aligned} A_\alpha(\delta) &= A_\alpha(\beta) \\ &< A_\alpha(\beta + 1) \\ &= S(A_\alpha(\beta)) \\ &= S(A_\alpha(\delta)). \end{aligned}$$

Otherwise,  $\beta < \delta$  so by our hypothesis:

$$\begin{aligned} A_\alpha(\beta) &< A_\alpha(\delta) \\ &< S(A_\alpha(\delta)) \\ &= A_\alpha(\delta + 1). \end{aligned}$$

Now, we suppose (1) holds for all  $\gamma < \lambda$  for some limit ordinal  $\lambda$ . For  $\beta < \lambda$ , clearly  $\beta < \beta + 1 < \lambda$  as  $\lambda$  has no immediate predecessor. By the hypothesis:

$$\begin{aligned} A_\alpha(\beta) &< A_\alpha(\beta + 1) \\ &\leq \sup(\{A_\alpha(\gamma) : \gamma < \lambda\}) \\ &= A_\alpha(\lambda), \end{aligned}$$

as required. □

Similarly, both  $M_\alpha$  and  $E_\alpha$  are strictly increasing, for ordinals  $\alpha, \beta, \gamma$  with  $\beta > \gamma$ :

- If  $\alpha > 0$  then  $M_\alpha(\beta) < M_\alpha(\gamma)$ ,
- If  $\alpha > 1$  then  $E_\alpha(\beta) < E_\alpha(\gamma)$ .

### 6.13 Remainders

For  $\alpha, \beta$  ordinals with  $0 < \alpha \leq \beta$ :

1. There's a unique ordinal  $\gamma$  such that  $\alpha + \gamma = \beta$ ,
2. There's a unique pair of ordinals  $\zeta, \kappa$  such that  $\alpha \cdot \zeta + \kappa = \beta$  and  $\kappa < \alpha$ .

*Proof.* (i) As  $A_\alpha$  is strictly increasing, we consider  $Z = \{x : \alpha + x \geq \beta\}$  which must be non-empty as  $A_\alpha$  is strictly increasing. We take  $\gamma = \min(Z)$  and see that  $\alpha + \gamma = \beta$  since if  $\alpha + \gamma > \beta$  either:

- $\gamma = \delta + 1$  so  $\alpha + \delta < \beta$  as  $\delta$  is not in  $Z$ . But then,  $(\alpha + \delta) + 1 = \alpha + \gamma \leq \beta$ , a contradiction,
- $\gamma$  is a limit ordinal,  $\alpha + \gamma = \sup(\{\alpha + \delta : \delta < \gamma\})$ . But, if  $\alpha + \gamma > \beta$  then there's some  $\delta < \gamma$  so that  $\delta \geq \beta$ . This contradicts  $\gamma = \min(Z)$ .

(ii) As  $M_\alpha$  is strictly increasing, we again choose the least  $\zeta$  such that  $\alpha \cdot \zeta \leq \beta < \alpha \cdot (\zeta + 1)$ . We apply part (i) to find  $\kappa$  such that  $\alpha \cdot \zeta + \kappa = \beta$ . We suppose  $\zeta'$  and  $\kappa'$  also satisfy (ii), if  $\zeta = \zeta'$  then by the uniqueness of part (i),  $\kappa = \kappa'$ . We suppose  $\zeta < \zeta'$  so  $\zeta + 1 \leq \zeta'$ :

$$\begin{aligned}
 \beta &= \alpha \cdot \zeta + \kappa < \alpha \cdot \zeta + \alpha \\
 &= \alpha \cdot (\zeta + 1) \\
 &\leq \alpha \cdot \zeta' \\
 &\leq \alpha \cdot \zeta' + \kappa' \\
 &= \beta,
 \end{aligned}$$

a contradiction. Hence,  $\zeta = \zeta'$ . □

## 7 Cardinality

### 7.1 Equinumerosity

We say that two sets,  $A$  and  $B$ , are equinumerous if there is a bijection between them, written as  $A \approx B$ .

We have that  $\approx$  is an equivalence relation with equivalence classes as collections of all equinumerous sets of a size.

### 7.2 Finite Sets

A set is finite if it is equinumerous with a natural number. Sets that are not finite are infinite.

### 7.3 Pidgeon-hole Principle

No natural number is equinumerous to a proper subset of itself and thus:

- No finite set is equinumerous to a proper subset of itself,
- Any set equinumerous to a proper subset of itself is infinite,
- Any finite set is equinumerous to a unique natural number,
- $\omega$  is infinite.

*Proof.* We take  $Z = \{n \in \omega : \forall f, (f : n \rightarrow n \text{ and injective}) \Rightarrow (\text{ran}(f) = n)\}$ . Trivially,  $Z$  contains 0. For  $n$  in  $Z$ , we consider  $f : (n+1) \rightarrow (n+1)$  an injective function.

**Case 1** We suppose that  $f \upharpoonright n : n \rightarrow n$  is an injective function and by our inductive hypothesis,  $\text{ran}(f \upharpoonright n) = n$ . Thus,  $\text{ran}(f) = n+1$ .

**Case 2** We suppose that  $f(m) = n$  for some  $m < n$ . As  $f$  is injective, for some  $k < n$ ,  $f(n) = k$ . We define  $g$  identically to  $f$  except  $g(m) = k$  and  $g(n) = n$  so that  $g : (n+1) \rightarrow (n+1)$  and injective so **Case 1** applies to  $g$ . Hence,  $\text{ran}(g) = n+1 = \text{ran}(f)$ .  $\square$



## 7.4 Cantor's Diagonal Argument

The natural numbers are not equinumerous with the real numbers.

*Proof.* We appeal to the contrary and suppose we have some injective map  $f : \omega \rightarrow \mathbb{R}$ :

$$\begin{aligned} f(0) &= 2.72938\dots \\ f(1) &= 3.47000\dots \\ f(2) &= 9.32789\dots \\ &\vdots \end{aligned}$$

We can generate some  $x$  not in  $\text{ran}(f)$  by setting the  $i^{\text{th}}$  decimal place to the  $i^{\text{th}}$  decimal place of  $f(i)$  mapped by:

$$k \mapsto \begin{cases} 1 & k \text{ even} \\ 2 & k \text{ odd.} \end{cases}$$

Thus,  $x$  would differ from every element of  $\text{ran}(f)$ . A contradiction.  $\square$

## 7.5 Cantor's Theorem

No set is equinumerous to its powerset.

*Proof.* We appeal to the contrary and suppose  $f : X \rightarrow \mathcal{P}(X)$  is a bijection for some set  $X$ . We set  $Z = \{u \in X : u \notin f(u)\}$  and see that  $Z \subseteq X$  so  $Z$  is in  $\mathcal{P}(X)$ . As such,  $Z = f(u)$  for some  $u$ , but:

$$\begin{aligned} u \in Z &\implies u \notin f(u), \\ u \notin Z &\implies u \in f(u), \end{aligned}$$

which is a contradiction.  $\square$

## 7.6 Cantor-Schröder-Bernstein Theorem

For sets  $X, Y$ ,  $X \preceq Y$  if there's an injection from  $X$  to  $Y$  and  $X \prec Y$  if  $X \preceq Y$  and  $Y \not\preceq X$ . We have that  $X \preceq Y$  and  $Y \preceq X$  is equivalent to  $X \approx Y$ .

We write  $A \propto B$  if  $A \preceq B$  but  $B \not\preceq A$ .

*Proof.* ( $\Rightarrow$ ) We have  $f : X \rightarrow Y$  and  $g : Y \rightarrow X$  both injective and want to form some  $h : X \rightarrow Y$  bijective. We consider  $C_0 = X \setminus \text{ran}(g)$ , the values suppressing the surjectivity of  $g$ . For  $n$  in  $\mathbb{N}$ , we define:

$$\begin{aligned} D_n &= f''C_n \\ C_{n+1} &= g''D_n = g''(f''C_n), \end{aligned}$$

$$h(v) = \begin{cases} f(v) & \text{if } v \text{ is in } C_n \text{ for some } n \\ g^{-1}(v) & \text{otherwise.} \end{cases}$$

To see that  $h$  is injective, we consider  $u$  and  $v$  in  $X$  and note that as both  $f$  and  $g$  are injective, it's sufficient to show that  $h$  is injective under  $u$  in some  $C_n$  and  $v$  not in any  $C_n$  (without loss of generality). In this case, we take  $u$  to be in  $C_m$  and see that:

$$\begin{aligned} h(u) &= f(u) \in D_m, \\ h(v) &= g^{-1}(v) \notin D_m, \end{aligned}$$

as otherwise  $g(g^{-1}(v)) = v \in C_{m+1}$  which is a contradiction to how we selected  $v$ . So,  $u \neq v$  implies that  $h(u) \neq h(v)$  and as such,  $h$  is injective. To see that  $h$  is surjective, we first note that  $U = \bigcup_{m \in \mathbb{N}} D_m \subseteq \text{ran}(h)$ . We consider  $u$  in  $Y \setminus U$ ,  $g(u)$  is not in  $C_0 = X \setminus \text{ran}(g)$  and not in any  $C_{n+1}$  either as  $u$  is not in any  $D_n$  and as  $g$  is injective, there's no  $v$  in  $D_n$  such that  $g(v) = g(u)$ . As such,  $h(g(u)) = g^{-1}(g(u)) = u$ . So,  $h$  is surjective and as such, bijective.

( $\Leftarrow$ ) This direction follows from the definitions. □

## 7.7 Characteristic Function

For a set  $X$ , we define the the characteristic function of any  $Y \subseteq X$  to be  $\chi_Y : X \rightarrow 2$  defined by:

$$\chi_Y(a) = \begin{cases} 1 & \text{if } a \text{ is in } Y \\ 0 & \text{if } a \text{ is in } X \setminus Y. \end{cases}$$

## 7.8 Countability

A set  $X$  is countably infinite if  $X \approx \omega$  and countable if  $X \preceq \omega$ .

The union of countably infinite sets is also countably infinite. Subsets of countable sets are countable.

## 7.9 Countably Infinite Subsets

For an infinite set  $X$  with  $\langle X, R \rangle$  a well-ordering,  $X$  has a countably infinite subset.

## 7.10 Well-ordering Principle

For a set  $X$ , there is a well-ordering  $\langle X, R \rangle$ . This implies that the union of countably infinite sets is countably infinite.

## 7.11 Cardinality

For a set  $X$ , the cardinality of  $X$ ,  $|X|$  is the least ordinal  $\alpha$  such that  $X \approx \alpha$ . We have that for  $X, Y$  sets:

$$\begin{aligned} X \approx Y &\iff |X| = |Y|, \\ X \preceq Y &\iff |X| \leq |Y|, \\ X \prec Y &\iff |X| < |Y|. \end{aligned}$$

## 7.12 Cardinal Numbers

An ordinal  $\alpha$  is a cardinal if  $\alpha = |\alpha|$ .

## 7.13 Cardinality Capture

For  $\alpha, \gamma$  ordinals,  $|\alpha| \leq \gamma < \alpha$  then  $|\alpha| = |\gamma|$ .

*Proof.* By definition, there's a bijection  $f$  from  $\alpha$  to  $|\alpha|$ , so  $||\alpha|| = |\alpha|$ . We know that  $\gamma \subseteq \alpha$ , so we know that  $f \upharpoonright \gamma$  is an injection from  $\gamma$  to  $|\alpha|$  so  $\gamma \preceq |\alpha|$ . But,  $|\alpha| \preceq \gamma$  by our assumption, so  $|\alpha| \approx \gamma$  which implies that  $|\gamma| \approx ||\alpha|| = |\alpha|$ .  $\square$

## 7.14 Cardinal Addition and Multiplication

For cardinals  $\kappa$  and  $\lambda$ , and sets  $K$  and  $L$  with cardinality  $\kappa$  and  $\lambda$  respectively, we define:

$$\begin{aligned} \kappa \oplus \lambda &= |K \cup L|, & (\text{for } K, L \text{ disjoint}) \\ \kappa \otimes \lambda &= |K \times L|. \end{aligned}$$

We note that these operations are commutative and associative. Furthermore, for  $n$  and  $m$  in  $\omega$ :

$$\begin{aligned} n + m &= n \oplus m < \omega, \\ n \cdot m &= n \otimes m < \omega. \end{aligned}$$

## 7.15 Hessenberg's Theorem

For an infinite cardinal  $\kappa$ , there is a bijection from  $\kappa \times \kappa$  to  $\kappa$ . Thus,  $\kappa \otimes \kappa = \kappa$ .

*Proof.* We already know that  $\omega \times \omega \approx \omega$  so we proceed by induction on  $\lambda \geq \omega$ . We assume  $\lambda < \kappa$  so  $\lambda \times \lambda \approx \lambda$ . We consider Gödel's ordering:

$$\begin{aligned} [\langle \alpha, \beta \rangle \triangleleft \langle \lambda, \delta \rangle] &\iff [(\max(\{\alpha, \beta\}) < \max(\{\gamma, \delta\})) \\ &\quad \text{or } [(\max(\{\alpha, \beta\}) = \max(\{\gamma, \delta\})) \\ &\quad \text{and } (\alpha < \gamma \text{ or } (\alpha = \gamma \text{ and } \beta < \delta))]]]. \end{aligned}$$

We know that  $\langle \kappa \times \kappa, \triangleleft \rangle$  is a well-ordering and each  $\langle \alpha, \beta \rangle$  in  $\kappa \times \kappa$  has no more than  $|(\max(\{\alpha, \beta\}) + 1) \times (\max(\{\alpha + \beta\}) + 1)| < \kappa$   $\triangleleft$ -predecessors as shown here:

*Proof.* We can see that:

$$(\kappa \times \kappa)_{\langle \alpha, \beta \rangle} \subset \gamma \times \gamma,$$

where  $\gamma = \max(\{\alpha, \beta\}) + 1$ . But, we know that  $|\gamma \times \gamma| = |\gamma| \otimes |\gamma|$  and as  $\gamma < \kappa$ ,  $|\gamma|$  is an infinite cardinal less than  $\kappa$ ,  $|\gamma| \otimes |\gamma| = \gamma$  by the inductive hypothesis.  $\square$

All initial segments  $\langle (\kappa \times \kappa)_{\langle \alpha, \beta \rangle}, \triangleleft \rangle$  have order type less than  $\kappa$  so  $\text{ot}(\kappa \times \kappa, \triangleleft) \leq \kappa$ . But  $\kappa \times \kappa \geq \kappa$  so  $\text{ot}(\kappa \times \kappa, \triangleleft) = \kappa$ .  $\square$

## 7.16 Confluence of Addition and Multiplication

For infinite cardinals  $\kappa, \lambda$ ,  $\kappa \oplus \lambda = \kappa \otimes \lambda = \max(\{\kappa, \lambda\})$ .

*Proof.* Without loss of generality, we assume  $\lambda \leq \kappa$  so  $\max\{\kappa, \lambda\} = \kappa$ . For  $X$  and  $Y$  disjoint with cardinality  $\kappa$  and  $\lambda$  (resp.):

$$X \preceq X \cup Y \preceq (X \times \{0\}) \cup (X \times \{1\}) = X \times 2 \preceq X \times X.$$

So, in terms of cardinals we have:

$$\kappa \leq \kappa \oplus \lambda \leq \kappa \oplus \kappa = \kappa \otimes 2 \leq \kappa \otimes \kappa.$$

But, by Hessenberg's Theorem,  $\kappa = \kappa \otimes \kappa$  which induces equality on all the above statements. As such,  $\kappa = \kappa \oplus \lambda$  and similarly:

$$\kappa \leq \kappa \otimes \lambda \leq \kappa \otimes \kappa = \kappa,$$

we have that  $\kappa = \kappa \otimes \lambda$ , as required.  $\square$

## 7.17 Cardinality of a Countable Union of Infinite Cardinals

For a set  $A$ ,  ${}^{<\omega}A = \bigcup_{n \in \omega} {}^n A$ . For an infinite cardinal  $\kappa$ ,  $|{}^{<\omega}\kappa| = \kappa$ .

## 7.18 Cardinal Exponentiation

For cardinals  $\kappa$  and  $\lambda$ ,  $\kappa^\lambda = |{}^L K|$  where  $K$  and  $L$  are sets of cardinality  $\kappa$  and  $\lambda$  respectively. For cardinals  $\kappa$ ,  $\lambda$ , and  $\mu$ , we have that:

$$\begin{aligned}\kappa^{\lambda \oplus \mu} &= \kappa^\lambda \otimes \kappa^\mu, \\ (\kappa^\lambda)^\mu &= \kappa^{\lambda \otimes \mu}.\end{aligned}$$

## 7.19 Equinumerosity with Characteristic Functions

For cardinals  $\kappa$  and  $\lambda$  with  $\lambda \geq \omega$  and  $2 \leq \kappa \leq \lambda$ , then  ${}^\lambda \lambda \approx {}^\lambda \kappa \approx {}^\lambda 2 \approx \mathcal{P}(\lambda)$ .

## 7.20 Class of Cardinals

The class of cardinals is a proper class.

*Proof.* We suppose the class of cardinals is a set, as it's the union of ordinals, it's an ordinal  $\tau$ . By Cantor's theorem,  $|\mathcal{P}(\tau)| > \tau$  which is a cardinal not in our set of cardinals, a contradiction.  $\square$

## 7.21 Unbounded Ordinals

For any set  $x$ , there's an ordinal  $\alpha$  with  $\alpha \not\leq x$ .

*Proof.* We take  $\alpha = |\mathcal{P}(x)|$  and we are done by Cantor's theorem.  $\square$

## 7.22 The Infinite Cardinals

For some ordinal  $\alpha$ , and  $\lambda$  a limit ordinal, we have the infinite cardinals:

$$\begin{aligned}\aleph_0 &= \omega_0 = \omega, \\ \aleph_{\alpha+1} &= \omega_{\alpha+1} = \omega_\alpha^+ = \text{the least ordinal containing } \omega_\alpha \\ \aleph_\lambda &= \omega_\lambda = \sup(\{\omega_\tau : \tau < \lambda\}).\end{aligned}$$

We have a function  $F_\aleph$  from the ordinals to the infinite cardinals defined by:

$$F_\aleph(\alpha) = \omega_\alpha.$$

With for  $\alpha > 1$ ,  $F_\aleph(\alpha)$  is an uncountable cardinal.

## 7.23 The Continuum Hypothesis

The hypothesis states that  $2^{\omega_0} = \omega_1$  and for the general hypothesis, for all ordinals  $\alpha$ ,  $2^{\omega_\alpha} = \omega_{\alpha+1}$ .

With our axioms, we can't prove the specific hypothesis is true or false. They are insufficient to decide this hypothesis.

## 8 Axioms

### 8.1 Axiom of Replacement

For a function  $F$  from  $V$  to itself and a set  $x$ ,  $F''x$  is a set.

### 8.2 Axiom of Choice

For a set of non-empty sets  $\mathcal{G}$ , there is a choice function  $F$  from  $\mathcal{G}$  to  $\bigcup \mathcal{G}$  such that for all  $X$  in  $\mathcal{G}$ ,  $F(X)$  is in  $X$ .

We have that the Axiom of Choice is equivalent to the Well-ordering Principle.

*Proof.* ( $\implies$ ) For an arbitrary set  $Y$ , it is sufficient to show  $Y$  has a well-ordering. We take  $Y \neq \emptyset$  as otherwise  $Y$  is trivially well-ordered. We take  $\mathcal{G} = \{X \subseteq Y : X \neq \emptyset\}$ . By the Axiom of Choice, we have a choice function  $F_0$  for  $\mathcal{G}$ . We take  $u$  to be any set not in  $Y$  ( $Y \neq V$ ) and define  $F$  from  $V$  to  $V$ :

$$F(t) = \begin{cases} F_0(t) & \text{if } t \in \mathcal{G} \\ u & \text{otherwise.} \end{cases}$$

By the recursion theorem, we define  $H_0$  from the ordinals to  $Y \cup \{u\}$ :

$$H_0(\xi) = F(Y \setminus \{H_0(\zeta) : \zeta < \xi\}).$$

We can see that:

$$\begin{aligned} H_0(0) &= F_0(Y) \in Y, \\ H_0(1) &= F_0(Y \setminus \{F_0(Y)\}) \in Y \setminus \{F_0(Y)\}, \\ H_0(n) &= F_0(Y \setminus \{F_k(Y) : k \in [n-1]_0\}) \in Y \setminus \{F_k(Y) : k \in [n-1]_0\}. \end{aligned}$$

So, we can select distinct elements from  $Y$  recursively via our choice function on the subsets of  $Y$ . We want to show that there's some ordinal  $\beta$  such that  $H_0(\beta) = u$ . If we suppose there isn't, then  $H_0$  is injective from the ordinals to  $Y$ , but we know that  $\text{ran}(H_0) \subseteq Y$  is a set. Thus,  $H_0^{-1}$  is a surjection from  $\text{ran}(H_0)$  to the ordinals, which is a contradiction as the ordinals form a proper class. So, we take  $\alpha$  to be the least ordinal such that  $H_0(\alpha) = u$ . We let  $H = H_0 \upharpoonright \alpha$ ,  $H$  is a bijection from  $\alpha$  to  $Y$ , which gives us a well-ordering on  $Y$  via the well-ordering on  $\alpha$ .

( $\impliedby$ ) For  $\mathcal{G}$  any set of non-empty sets, we take  $A = \bigcup \mathcal{G}$ . By the well-ordering principle, there's a well-ordering  $\langle A, R \rangle$ . We can define a choice function as:

$$F(X) = R - \text{least element of } \langle X, R \rangle,$$

as required. □

### 8.3 Chains

Any collection  $\mathcal{G}$  of sets is called a chain if for all  $X$  and  $Y$  in  $\mathcal{G}$ ,  $X \subseteq Y$  or  $Y \subseteq X$ .

### 8.4 Zorn's Lemma

For a set  $\mathcal{F}$  such that for every chain  $\mathcal{G} \subseteq \mathcal{F}$ ,  $\bigcup \mathcal{G}$  is in  $\mathcal{F}$ , we have that  $\mathcal{F}$  contains a maximal element  $Y$  where for all  $Z$  in  $\mathcal{F}$ :

$$Y \subseteq Z \implies Y = Z.$$

We have that this lemma is equivalent to the Axiom of Choice and the Well-ordering principle.

*Proof.* (ZL  $\implies$  AC) For a collection of non-empty sets  $\mathcal{G}$ , we want a choice function for  $\mathcal{G}$ . We define  $\mathcal{F}$  to be the set of all choice function that exist for subsets of  $\mathcal{G}$ , that is, for  $f$  in  $\mathcal{F}$ :

$$\text{dom}(f) \subseteq \mathcal{G} \text{ and } \forall x \in \text{dom}(f), f(x) \in x.$$

We know that  $\mathcal{F}$  is non-empty as for some  $x$  in  $\mathcal{G}$ ,  $x$  is non-empty so we choose any  $u$  in  $x$  and thus  $\{\langle x, u \rangle\}$  is in  $\mathcal{F}$ . For any chain  $\mathcal{H}$  in  $\mathcal{F}$ ,  $\mathcal{H}$  is a chain of partial choice functions on subsets of  $\mathcal{G}$ . We take  $h = \bigcup \mathcal{H}$ , so  $h$  is a function with  $\text{dom}(h) = \bigcup \{\text{dom}(f) : f \in \mathcal{H}\} \subseteq \mathcal{G}$ . Thus,  $h$  is a choice function so is in  $\mathcal{F}$ .

By Zorn's Lemma, there's a maximal  $m$  in  $\mathcal{F}$  and we want to show that  $m$  is a choice function for  $\mathcal{G}$ . We know  $m$  must be a partial choice function so it's sufficient to show that  $\text{dom}(m) = \mathcal{G}$ . We suppose that  $\text{dom}(m) \neq \mathcal{G}$ , and take  $x$  in  $\mathcal{G} \setminus \text{dom}(m)$  which must be non-empty as it is in  $\mathcal{G}$ . For  $u$  in  $x$ ,  $m \cup \{\langle u, x \rangle\}$  is a partial choice function in  $\mathcal{F}$  with domain  $\text{dom}(m) \cup \{u\}$  so  $m \subset m \cup \{\langle u, x \rangle\}$ . This is a contradiction of the maximality of  $m$ , so  $m$  is a choice function for  $\mathcal{G}$ .

(WP  $\implies$  ZL) We take  $\mathcal{F}$  to be a set such that for every chain  $\mathcal{G} \subseteq \mathcal{F}$  we have that  $\bigcup \mathcal{G}$  is in  $\mathcal{F}$ . By the Well-ordering Principle,  $\mathcal{F}$  can be well-ordered by some relation  $R$ , we take an ordinal  $\alpha$  such that  $\langle \alpha, \in \rangle \cong \langle \mathcal{F}, R \rangle$  for some order isomorphism  $k$ . By recursion on the ordinals  $\beta < \alpha$ , we define a maximal chain  $\mathcal{H}$  of  $\mathcal{F}$ . We start by putting  $k(0)$  into  $\mathcal{H}$ , if  $k(0) \subset k(1)$  then we add  $k(1)$  too, if not, we move on, adding  $k(\beta)$  if it contains the current maximal element of  $\mathcal{H}$ . This clearly forms a chain, and we will show that  $Y = \bigcup \mathcal{H}$  is a maximal element of  $\mathcal{F}$ . By the definition of  $\mathcal{F}$ , as  $\mathcal{H}$  is a chain,  $Y$  is in  $\mathcal{F}$ . If we suppose that is some  $Z$  in  $\mathcal{F}$  with  $Y \subseteq Z$ , then  $k(\gamma) \subseteq Z$  for any  $\gamma$  such that  $k(\gamma)$  is in  $\mathcal{H}$ . As  $Y$  is in  $\mathcal{F}$ , for some  $\delta < \alpha$ ,  $Z = k(\delta)$ . But, by the definition of our recursion, at the stage  $\gamma$ , we decided that  $Z$  should be added to  $\mathcal{H}$  so  $Z \subseteq \bigcup \mathcal{H} = Y$  and as such,  $Z = Y$  as required.  $\square$



## 9 The Universe of Sets

### 9.1 The Well-founded Hierarchy of Sets

For a limit ordinal  $\lambda$ , we define the function  $V_\alpha$  by transfinite recursion:

$$\begin{aligned} V_0 &= \emptyset, \\ V_{\alpha+1} &= \mathcal{P}(V_\alpha), \\ V_\lambda &= \bigcup_{\alpha < \lambda} \mathcal{P}(V_\alpha), \\ V &= \bigcup_{\alpha \in \text{On}} \mathcal{P}(V_\alpha). \end{aligned}$$

### 9.2 Transitivity of $V_\alpha$

For any  $\alpha$ , we have that  $V_\alpha$  is transitive and for all  $\beta < \alpha$ ,  $V_\beta$  is in  $V_\alpha$ .

*Proof.* We proceed by induction, for  $\alpha = 0$ ,  $V_0 = \emptyset$  which trivially satisfies both statements. For  $\alpha = \beta + 1$ , we use the fact that if  $\beta$  is transitive, then  $\mathcal{P}(\beta)$  is also. By the inductive hypothesis,  $V_\alpha = \mathcal{P}(V_\beta)$  is transitive. As  $V_\beta$  is in  $\mathcal{P}(V_\beta)$ , we have that  $V_\beta$  is in  $V_\alpha$  and if  $\beta' < \beta$  then by the inductive hypothesis,  $V_{\beta'}$  is in  $V_\beta$  and hence  $V_{\beta'}$  is in  $V_\alpha$  by transitivity. For  $\alpha$  a limit ordinal,  $V_\alpha = \bigcup_{\beta < \alpha} V_\beta$  is transitive by the inductive hypothesis. For  $\beta < \alpha$ , it must be that  $V_\beta$  is in  $V_\alpha$  by the definition and transitivity of  $V_\alpha$ .  $\square$

### 9.3 Axiom of Foundation

Every set  $x$  is well-founded, so if  $x$  is non-empty, there exists some  $y$  in  $x$  such that  $x \cap y = \emptyset$ .

This is equivalent to saying there exists some  $\alpha$  such that  $x$  is in  $V_\alpha$ .

*Proof.* For a set  $x$ , we set  $T = TC(X)$ . If  $T \subseteq V$  then for some ordinal  $\alpha$ ,  $T \subseteq V_\alpha$  as  $\rho''T$  is a set of ordinals by the Axiom of Replacement. So, for some  $\alpha$ ,  $\rho''T \subseteq \alpha$  so  $T \subseteq V_\alpha$ . So, we are done for this case as  $x \subseteq T \subseteq V_\alpha$  so  $x \in V_{\alpha+1}$ .

If we suppose that  $T \setminus V \neq \emptyset$  and take  $y$  in  $T \setminus V$  such that  $(T \setminus V) \cap y = \emptyset$  by the Axiom of Foundation, then for any  $z$  in  $y$ , as  $z$  must be in  $T$  by the properties of  $TC$ . Also,  $z$  must be in  $V$  as  $(T \setminus V) \cap y = \emptyset$ . Hence,  $y \subseteq V$ . But, as in the first case,  $\rho''y$  is a set of ordinals, with some strict upper bound  $\beta$ . As such,  $y \subseteq V_\beta$  which implies  $y$  is in  $V_{\beta+1}$  which is a contradiction of the definition of  $y$ .  $\square$

## 9.4 The Rank Function

For any  $x$  in  $V$ ,  $\rho(x)$  is the least  $\tau$  such that  $x \subseteq V_\tau$  (or rather,  $x$  is in  $V_{\tau+1}$ ). We have that:

1.  $\rho(\{x\}) = \rho(x) + 1$ ,
2.  $\rho(\{x, y\}) = \max(\{\rho(x), \rho(y)\}) + 1$ ,
3.  $\rho(\langle x, y \rangle) = \max(\{\rho(x), \rho(y)\}) + 2$ ,
4.  $V_\alpha = \{x \in V : \rho(x) < \alpha\}$ ,
5. For  $x$  in  $V$ , and all  $y$  in  $x$ ,  $y$  is in  $V$  and  $\rho(y) < \rho(x)$ ,
6. For  $x$  in  $V$ ,  $\rho(x) = \sup(\{\rho(y) + 1 : y \in x\}) = \sup^+(\{\rho(y) : y \in x\})$ .

So, the relation on sets:

$$xRy \iff \rho(x) < \rho(y),$$

is a strict partial order that is well-founded, meaning there is a  $R$ -least element of every non-empty  $X \subseteq V$ .

*Proof.* (1) We take  $\rho(x) = \alpha$ , so  $x \in V_{\alpha+1}$  and as such,  $\{x\} \subseteq V_{\alpha+1}$ . So,  $\rho(\{x\}) \leq \alpha + 1$ . But,  $\rho(\{x\}) \not\leq \alpha + 1$  since this would imply that  $\rho(x) < \rho(\{x\}) < \alpha + 1$  which is a contradiction.

(2) We take  $\max(\{\rho(x), \rho(y)\}) = \alpha$ , so we have that  $\alpha$  is minimal such that  $x \subseteq V_\alpha$  and  $y \subseteq V_\alpha$ . As such, by the working in (1), we have that  $\rho(\{x, y\}) = \alpha + 1$ .

(3) As  $\langle x, y \rangle = \{x, \{x, y\}\}$ , we apply (1) and (2) to get the result.

(4) For  $x$  in  $V$ , then  $\rho(x) < \alpha$  is equivalent to saying that there is some  $\beta < \alpha$  such that  $x \subseteq V_\beta$  or rather  $x$  is in  $V_{\beta+1}$ . This is then equivalent to saying  $x$  is in  $V_\alpha$  as  $V_{\beta+1} \subseteq V_\alpha$  by the transitivity of  $V$ .

(5) We take  $\rho(x) = \alpha$  so  $x \subseteq V_\alpha$  and as such  $y$  in  $x$  must be in  $V_\alpha$  so  $\rho(y) < \alpha$ .

(6) We take  $\alpha = \sup^+(\{\rho(y) : y \in x\})$  and  $y$  in  $x$ . By (5),  $\rho(y) < \rho(y) + 1 \leq \rho(x)$  so,  $\alpha \leq \rho(x)$ . By (4),  $\rho(y) < \rho(y) + 1 \leq \alpha$  so,  $y$  is in  $V_\alpha$ . Thus,  $x \subseteq V_\alpha$  so  $\rho(x) \leq \alpha$ .  $\square$

## 9.5 Rank and Ordinals

For an ordinal  $\alpha$ ,  $\rho(\alpha) = \alpha$  and  $(\text{On} \cap V_\alpha) = \alpha$ .

*Proof.* The result is trivial for  $\alpha = 0$ , so we proceed by induction with  $\alpha > 0$ . By (9.4(6)):

$$\begin{aligned}\rho(\alpha) &= \sup^+(\{\rho(\beta) : \beta < \alpha\}) \\ &= \sup^+(\{\beta : \beta < \alpha\}) \\ &= \alpha.\end{aligned}\tag{IH}$$

From this, we know that  $\alpha \subseteq (\text{On} \cap V_\alpha)$ . We take  $\beta$  in  $(\text{On} \cap V_\alpha)$  so  $\beta = \rho(\beta) < \alpha$ . Thus,  $(\text{On} \cap V_\alpha) \subseteq \alpha$  and as such,  $\alpha = (\text{On} \cap V_\alpha)$ .  $\square$

## 9.6 Principle of $\in$ -induction

For a well-defined and definite property of sets  $\Phi$ :

$$[\forall z \in y, \Phi(z) \implies \Phi(y)] \implies \Phi(y), \tag{*}$$

and if  $x$  is a transitive set, we have  $(*)$  for all  $y$  in  $x$ .

*Proof.* For a transitive set  $x$ , we take  $Z = \{y \in x : \neg\Phi(y)\}$ , supposing  $Z \neq \emptyset$ . By the Axiom of Foundation, we have  $y_0$  in  $Z$  such that  $y_0$  is  $\in$ -minimal (meaning  $(y_0 \cap Z) = \emptyset$ ). For any  $u$  in  $y_0$ ,  $u$  must be in  $x$  as  $x$  is transitive. By the minimality of  $y_0$ , it must be that  $\Phi(u)$  holds as otherwise  $(y_0 \cap Z)$  would contain  $u$ . As such, assuming the antecedent, we get  $\Phi(y_0)$  which is a contradiction of the membership of  $y_0$  in  $Z$ . For the case on classes, we just take  $Z = \{y : \neg\Phi(y)\}$  and use the same argument.  $\square$

## 9.7 Theorem of $\in$ -recursion

For a function  $G$  from  $V$  to  $V$ , there is exactly one function  $H$  from  $V$  to  $V$  such that for all  $x$ :

$$H(x) = G(H \upharpoonright x) = G(\{\langle y, H(y) \rangle : y \in x\}).$$

*Proof.* The proof operates similarly to that on ordinals, but is omitted.  $\square$