Linear Algebra 2 Notes

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An important note, these notes are absolutely ***NOT*** *guaranteed to be correct, representative of the course, or rigorous. Any result of this is not the author's fault.*

1 Groups, Rings, and Fields

1.1 Definition of a Group

A group is a set G combined with a group operation $\circ : G \times G \to G$ such that:

- For all g, h, j in G, $g(hj) = (gh)j$ (associativity)
- There exists e in G such that $eg = ge = g$ for all g in G
- For all g in G, there exists g^{-1} in G such that $gg^{-1} = g^{-1}g = e$ where e is the identity of G.

1.2 Definition of a Homomorphism

A homomorphism between two groups G, H is a function $f : G \to H$ such that $f(gh) = f(g)f(h)$ for all g, h in G.

1.3 Properties of Homomorphisms

We can derive some properties of homomorphisms, for G, H groups, and $f : G \to H$ a homomorphism:

- The image of the identity in G is the identity in H
- The kernel of f is a subgroup of G
- The image of f is a subgroup of H
- Bijective homomorphisms are isomorphisms.

1.4 Definition of a Ring

A ring with unity is a set R along with an addition map $+$, and a multiplication map $\circ$ where $+, \circ : R \times R \to R$ such that:

- $(R, +)$ is an abelian group (of which the identity is called zero)
- The multiplication operation is associative
- The multiplication operation has a two-sided identity not equal to the zero identity (called one)
- For all a, b, c in R, $a(b + c) = ab + ac$ and $(a + b)c = ac + bc$.

A ring is commutative if the multiplication operation is commutative.

1.5 Definition of a Subring

For the ring $R = (R', +, \circ)$ and S a set, S is a subring of R if $S \subseteq R'$ and $(S, +, \circ)$ is a ring.

1.6 Definition of a Ring Homomorphism

For rings with unity R and S, $f : R \to S$ is a ring homomorphism if for all a, b in R:

$$\begin{aligned} f(a + b) &= f(a) + f(b) \\ f(ab) &= f(a)f(b) \\ f(1_R) &= 1_S \end{aligned}$$

Essentially, this says that f is a homomorphism for the groups formed by R and S under addition and multiplication.

1.7 Definition of a Field

A field $\mathbb{F}$ is a ring with unity with the following properties:

- $(\mathbb{F} \setminus \{0\}, \circ)$ is an abelian group.

1.8 Definition of the Field Characteristic

For a field $\mathbb{F}$, the field characteristic $\text{char}(\mathbb{F})$ is the smallest positive integer n such that:

$$\sum_{i=1}^{n} 1 = 1 + 1 + \ldots + 1 = 0,$$

or zero if no such value n exists.

1.9 Definition of the Algebraic Closure of Fields

A field $\mathbb{F}$ is called algebraically closed if all non-constant polynomials with coefficients in $\mathbb{F}$ also has a root in $\mathbb{F}$.

2 Vector Spaces

2.1 Definition of a Vector Space

A vector space over a field $\mathbb{F}$ is a set V with an addition operation $+ : V \times V \to V$ and a scalar multiplication operations $\circ : \mathbb{F} \times V \to V$ such that for all a, b in $\mathbb{F}$ and v, w in V:

- $(V, +)$ is an abelian group
- $1 \circ v = v$ where 1 is the multiplicative identity of $\mathbb{F}$
- $(ab) \circ v = a \circ (b \circ v)$
- $(a + b) \circ v = a \circ v + b \circ v$
- $a \circ (v + w) = a \circ v + a \circ w$.

2.2 Definition of a Subspace

For V a vector space over the field $\mathbb{F}$ and W a set, W is a subspace of V if it is a subset of V and is a vector space with respect to the addition and scalar multiplication defined by V.

It is sufficient to verify that for any a in $\mathbb{F}$ and v, w in W we have that $a(v + w)$ is in W.

2.3 Definition of a Linear Combination

For a set V with addition operation $+$, a field $\mathbb{F}$ and n in $\mathbb{N}$, a linear combination of $v_1, \ldots, v_n$ in V is:

$$\sum_{i=1}^{n} a_i v_i,$$

for $a_1, \ldots, a_n$ in $\mathbb{F}$.

2.4 Definition of the Span

For a set V with addition operation $+$ and a field $\mathbb{F}$, the span of $W \subseteq V$ is the set of all the linear combinations of the values in W. Denoted by $\text{span}(W)$.

2.5 Definition of Linear Independence

For a vector space V and $W \subseteq V$, we say W is linearly dependent if there exists a non-trivial linear combination of all the vectors in W equal to zero (and linearly independent otherwise).

2.6 Properties of Linear Independence

For a vector space V with $W \subseteq V$:

- $0 \in W \Rightarrow W$ is linearly independent
- W linearly independent $\Rightarrow$ any $X \subseteq W$ is linearly independent
- If there's a linearly dependent subset of W, then W is linearly dependent.

2.7 Definition of a Basis

For a vector space V with $W \subseteq V$, if W is linearly independent and $\text{span}(W) = V$, we say that W is a basis of V.

Saying W is a basis is equivalent to saying that each vector in V can be **uniquely** written as a linear combination of vectors in W.

Additionally, for finite vector spaces, we have that all bases have the same amount of elements.

2.8 Definition of Dimension

For non-infinite bases, we say that the value of the basis is the dimension of the vector space it is a member of. Vector spaces with such bases are called finite-dimensional and all other vector spaces are infinite-dimensional.

By convention, for a vector space V, $\dim(\{0_V\}) = 0$.

2.9 Isomorphisms from Dimension

For V, W finite-dimensional vector spaces over $\mathbb{F}$ with $\dim(V) = \dim(W)$, then $V \cong W$.

If we set $n = \dim(V)$, we have that $V \cong \mathbb{F}^n$.

Such an isomorphism can be found by mapping a vector in terms of some chosen basis vectors ($v = a_1v_1 + a_2v_2 + \cdots + a_nv_n$) to the coefficients $(a_1, a_2, \ldots, a_n)$.

3 Linear Maps

3.1 Definition of a Linear Map

Let V, W be vector spaces over a field $\mathbb{F}$, we have that $f : V \to W$ is a linear map if for all a, b in $\mathbb{F}$ and u, v in V:

$$f(au + bv) = af(u) + bf(v).$$

A bijective linear map is called an isomorphism. If $f : V \to W$ is an isomorphism, we say that V and W are isomorphic, denoted by $V \cong W$.

3.2 The Kernel of Linear Maps

Let V, W be vector spaces over a field $\mathbb{F}$, and $f : V \to W$ be a linear map. We define the kernel of f as:

$$\mathrm{Ker}(f) = \{v \in V : f(v) = 0_{\mathbb{F}}\}.$$

Saying $\mathrm{Ker}(f)$ is $\{0_{\mathbb{F}}\}$ is equivalent to saying f is injective.

3.3 The Image of Linear Maps

Let V, W be vector spaces over a field $\mathbb{F}$, and $f : V \to W$ be a linear map. We define the image of f as:

$$\mathrm{Im}(f) = \{w \in W : \exists\, v \in V \text{ with } f(v) = w\}.$$

Saying $\mathrm{Im}(f)$ is W is equivalent to saying f is surjective.

3.4 The Inverse of Linear Maps

For a bijective linear map f, the inverse of f is also linear.

3.5 Properties of the Set of Linear Maps

For V, W vector spaces over a field $\mathbb{F}$, we define $\mathcal{L}(V, W)$ to be the set of all linear maps from V to W.

3.6 The Rank-Nullity Theorem

For V, W finite-dimensional vector spaces and $f : V \to W$ a linear map, we have that:

$$\dim(V) = \dim(\mathrm{Ker}(f)) + \dim(\mathrm{Im}(f)).$$

Thus, for a linear map $f : V \to V$, if f is injective or surjective then it's an isomorphism.

4 Matrices

4.1 Definition of a Matrix

For m, n in $\mathbb{Z}_{>0}$ and $\mathbb{F}$ a field. An $m \times n$ matrix with entries in $\mathbb{F}$ is a map $M : [m] \times [n] \to \mathbb{F}$, more commonly written as $M = (a_{ij})$ representing the rectangular array of values held by M.

The set of all $m \times n$ matrices over $\mathbb{F}$ is denoted by $M_{m\times n}(\mathbb{F})$.

4.2 Types of Matrix

For m, n in $\mathbb{Z}_{>0}$ and $\mathbb{F}$ a field, let M be in $M_{m\times n}(\mathbb{F})$. We have the following types of matrix:

- **Square**: where $m = n$
- **Upper Triangular**: if $a_{ij} = 0$ for $i > j$
- **Lower Triangular**: if $a_{ij} = 0$ for $i < j$
- **Diagonal**: if $a_{ij} = 0$ for $i \neq j$
- **Symmetric**: if $a_{ij} = a_{ji}$
- **Anti-symmetric**: if $a_{ij} = -a_{ji}$.

4.3 Properties of the Space of Matrices

For m, n in $\mathbb{Z}_{>0}$ and $\mathbb{F}$ a field, we have that $M_{m\times n}(\mathbb{F})$ is a vector space over $\mathbb{F}$ where matrices are added and multiplied by scalars component-wise. So, for $M_1 = (a_{ij}), M_2 = (b_{ij})$ in $M_{m\times n}$ and c in $\mathbb{F}$ we have:

$$cM_1 = (ca_{ij})$$
$$M_1 + M_2 = (a_{ij} + b_{ij}).$$

Additionally, the zero vector is $M_0 = (0)$ and the vector space has a basis consisting of M_{ij} where all entries are zero except the $(i, j)^{\text{th}}$ entry. This leads to the conclusion that the dimension is mn and thus that $M_{m\times n} \cong \mathbb{F}^{mn}$.

4.4 Matrix Multiplication

For a, b, c in $\mathbb{Z}_{>0}$ and a field $\mathbb{F}$, we can define the multiplication of the two matrices $X = (x_{ij})$ in $M_{a\times b}$ and $Y = (y_{ij})$ in $M_{b\times c}$ as follows:

$$XY = (\sum_{k=1}^{b} x_{ik}y_{kj}).$$

This operation is not commutative in general but is associative.

For A, B in M_n, we have that AB is also in M_n. This, along with matrix addition, makes M_n a ring with unity with multiplicative identity $I_n = (\delta_{ij})$. However, there exists A, B in M_n such that $AB = 0$ so, M_n is not a field.

4.5 Matrices of Linear Maps

For V, W vector spaces over a field $\mathbb{F}$, for some m, n in $\mathbb{Z}_{>0}$ we have $A = \{v_1, \ldots, v_n\}$, $B = \{w_1, \ldots, w_n\}$ bases for V and W respectively. Given f in $\mathcal{L}(V, W)$, the matrix associated to f (with respect to the bases A and B) is the $m \times n$ matrix:

$$M_{BA}(f) = (a_{ij}),$$

where we define a_{ij} by:

$$f(v_j) = \sum_{i=1}^{m} a_{ij}w_i,$$

for each j in $[n]$.

4.6 Matrices of Composed Linear Maps

For U, V, W vector spaces over a field $\mathbb{F}$, for some l, m, n in $\mathbb{Z}_{>0}$ we have $A = \{u_1, \ldots, u_n\}$, $B = \{v_1, \ldots, v_n\}$, $C = \{w_1, \ldots, w_n\}$ bases for U, V, W respectively. Given g, f in $\mathcal{L}(V, W)$, we have:

$$M_{CA}(g \circ f) = M_{CB}(g)M_{BA}(f).$$

4.7 Transition Matrices

For a finite-dimensional vector space V, with an identity I and bases A, A', we call $M_{A'A}(I) = C_{A'A}$ a transition matrix.

We have that $C_{A'A}$ is invertible and $C^{-1}_{A'A} = C_{AA'}$.

Essentially, the transition matrix transforms between bases.

4.8 Matrix Transitions

For a finite-dimensional vector space V, with $f : V \to V$ a linear operator, and bases A, B:

$$\begin{aligned} M_{BB}(f) &= C^{-1}_{AB}M_{AA}(f)C_{AB} \\ &= C_{BA}M_{AA}(f)C_{AB}. \end{aligned}$$

4.9 Similar Matrices

For matrices A', A, we say that A' and A are similar if there exists an invertible matrix C such that:

$$A' = C^{-1}AC.$$

This is denoted by $A' \sim A$. Similarity forms an equivalence relation on the space of square matrices.

If we have $A \sim A'$ and A represents some linear operator f for some basis B, then we have that for some basis B', f has matrix A'.

5 Eigenvectors and Eigenvalues

5.1 Definition of an Eigenvectors and Eigenvalues

For a vector space V over $\mathbb{F}$ with $f : V \to V$ a linear operator, a non-zero vector v in V is an eigenvector if $f(v) = \lambda v$ for some λ in $\mathbb{F}$ which is called the eigenvalue corresponding to v.

5.2 Definition of an Eigenspace

For a vector space V over $\mathbb{F}$ with $f : V \to V$ a linear operator and some eigenvalue λ, we define the eigenspace of λ as the set of eigenvectors with eigenvalue λ.

This is denoted by $E(\lambda)$ and $E(\lambda) \cup \{0_V\}$ forms a subspace of V. The dimension of $E(\lambda)$ is the geometric multiplicity of λ.

6 Direct Sums and Projections

6.1 Definition of a Direct Sum

For V, W vector spaces, we define the direct product of V and W as:

$$V \oplus W = \{(v, w) : v \in V, w \in W\},$$

with addition and scalar multiplication defined coordinate-wise and zero vector $(0_V, 0_W)$.

6.2 The Equivalence of Direct Sums

For $V, W \subseteq U$, we have that the following are equivalent:

- $U = V \oplus W$
- Each element in U can be written uniquely as the sum of elements in V and W
- The map $f : V \oplus W \to U; (v, w) \mapsto v + w$ is isomorphism.

6.3 The Addition Map for Direct Sums

For V, W subspaces of a vector space U, and $f : V \oplus W \to U$ defined by:

$$f((v, w)) = v + w,$$

we have that:

- f is linear
- f is injective if and only if $V \cap W = \{0\}$
- f is surjective if and only if $V \cup W$ spans U.

6.4 Projections

For V, W subspaces of U with $U = V \oplus W$, the projection **onto** V along W is the linear operator $P_{V,W} : U \to U$ where:

$$P_{V,W}(u) = v,$$

where $u = v + w$ for some unique v in V and w in W.

We have that for a linear operator P, P is a projection if and only if $P \circ P = P$.

6.5 f-invariance

For a vector space V with $U \subseteq V$ a subspace and $f : U \to U$ a linear operator, we have that U is f-invariant if for all u in U we have $f(u)$ in U.

The eigenspaces of f are examples of f-invariant spaces.

6.6 Matrices of Linear Maps (using f-invariance)

For $U, W \subseteq V$ subspaces of the vector space V such that $V = U \oplus W$, let B_U, B_W be finite bases of U and V respectively. If we have a linear operator $f : V \to V$ such that U and W are f-invariant, we have that the matrix with respect to the basis $B = B_U \cup B_W$ of f has the following block form:

$$M_{BB}(f) = \begin{pmatrix} M_{B_V B_V}(f) & 0 \\ 0 & M_{B_W B_W}(f) \end{pmatrix}.$$

7 Quotient Spaces

7.1 Definition of a Quotient Space

For a vector space V with $W \subseteq V$ a subspace. We define an equivalence relation on V by declaring:

$$v_1 \sim v_2 \text{ if } v_1 - v_2 \in W.$$

The set of equivalence classes is called the quotient of V by W and is denoted by V/W. For some v in V, we denote the class containing v by $v + W$ (similarly to cosets in Introduction to Group Theory). So, we have:

$$V/W = \{v + W : v \in V\},$$

with addition and multiplication defined for v_1, v_2 in V and a in the field:

$$(v_1 + W) + (v_2 + W) = (v_1 + v_2) + W$$
$$a(v_1 + W) = av_1 + W.$$

7.2 Linear Map to the Quotient Space

For a vector space V with $W \subseteq V$ a subspace, we can define $\pi : V \to V/W$ for some v in V by $f(v) = v + W$. We have that π is linear and its kernel is W.

7.3 Isomorphisms formed by Linear Maps

For V, W vector spaces and $f : V \to W$ a linear map, we have an isomorphism $\mathrm{Im}(f) \cong V/\mathrm{Ker}(f)$.

7.4 Existence of a Linear Operator on the Quotient Space

For a vector space V with $W \subseteq V$ a subspace and a linear operator $f : V \to V$, there exists a well-defined operator $\bar{f} : V/W \to V/W$ if and only if W if f-invariant. We call this the induced operator on V/W.

7.5 Matrices formed using Quotient Spaces

Consider a finite-dimension vector space V and $f : V \to V$ a linear operator with W an f-invariant subspace of V. If we have B_W a basis for W, that we extend to a basis B of V and set A:

$$A = \{v + W : v \in B \setminus B_W\},$$

a basis of V/W and we can form a matrix in block form:

$$M_{BB}(f) = \begin{pmatrix} M_{B_W B_W}(f) & * \\ 0 & M_{AA}(\bar{f}) \end{pmatrix},$$

where $\bar{f}$ is the induced operator on V/W and $*$ marks the area of the matrix which we cannot determine.

8 Dual Spaces

8.1 Definition of a Dual Space

For V a vector space over $\mathbb{F}$, we have that the dual space V^* is $\mathcal{L}(V, \mathbb{F})$, the set of linear maps from V to $\mathbb{F}$. We have that addition and scalar multiplication are defined for some v in V, f, g in V^*, and a in $\mathbb{F}$:

$$(f + g)(v) = f(v) + g(v),$$
$$(af)(v) = af(v).$$

8.2 Definition of a Dual Basis

For V a finite-dimensional vector space over $\mathbb{F}$, with $\dim(v) = n$ and a basis $B = \{v_1, \ldots, v_n\}$. We define the dual basis $B^* = \{v_1^*, \ldots, v_n^*\}$ by defining $v_i^* : V \to \mathbb{F}$ as the unique linear map such that:

$$v_i^*(v_j) = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j. \end{cases}$$

Equivalently, for v in V, we have that there's unique $(a_1, \ldots, a_n)$ in $\mathbb{F}$ such that:

$$v = \sum_{i=1}^{n} a_i v_i,$$

so we let v_i be such that:

$$v_i^*(v) = v_i^*\left(\sum_{j=1}^{n} a_j v_j\right) = \sum_{j=1}^{n} a_j v_i^*(v_j).$$

We have that B^* is a basis for V^*. Additionally, we have that V and V^* are isomorphic by the isomorphism mapping v_i to v_i^*.

8.3 Definition of the Annihilator

For V a vector space over $\mathbb{F}$ with $S \subseteq V$, the annihilator of S is the subspace S^0 of V^* where for f in S^0, $S \subseteq \mathrm{Ker}(f)$ (or rather, for all s in S, $f(s) = 0$).

8.4 Properties of the Annihilator

For V a vector space with $U, W \subseteq V$ subspaces, we have that:

- $(U + W)^0 = U^0 \cap W^0$
- $U \subseteq W \Rightarrow W^0 \subseteq U^0$,

and for V finite-dimensional,

- $(U \cap W)^0 = W^0 + U^0$
- $\dim(W) + \dim(W^0) = \dim(V)$.

8.5 Isomorphism to the Double Dual Space

For V a finite-dimensional vector space over $\mathbb{F}$, we have $F : V \to V^{**}$. That is:

$$V^{**} = \mathcal{L}(V^*, \mathbb{F}) = \mathcal{L}(\mathcal{L}(V, \mathbb{F}), \mathbb{F}),$$

so for some v in V we have:

$$F(v) : V^* \to \mathbb{F}.$$

We define F for some f in V^* as follows:

$$F(v)(f) = f(v).$$

We have that F is an isomorphism.

8.6 Definition of the Transpose

For V, W vector spaces with $f : V \to W$ a linear map. We define the transpose as $f^t : W^* \to V^*$ where for g in W^*, v in V:

$$f^t(g) = (g \circ f).$$

So, for some v in V:

$$f^t(g)(v) = (g \circ f)(v) = g(f(v)).$$

8.7 The Transpose and Matrices

If we have V, W finite-dimensional vector spaces over $\mathbb{F}$ with bases $A = \{v_1, \ldots, v_n\}$, $B = \{w_1, \ldots, w_m\}$ and corresponding dual bases $A^* = \{v_1^*, \ldots, v_n^*\}$, $B^* = \{w_1^*, \ldots, w_m^*\}$ respectively, we have that for some linear map $f : V \to W$, and $f^t : W^* \to V^*$ the transpose map:

$$M_{BA}(f) = \left(M_{A^*B^*}(f^t)\right)^t.$$

That is, for a given map, the matrix of transpose map is itself the matrix transpose of the matrix of the map.