Analysis 1 (TB2) Notes

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An important note, these notes are absolutely ***NOT*** *guaranteed to be correct, representative of the course, or rigorous. Any result of this is not the author's fault.*

1 Continuity

1.1 Continuous Functions

From Analysis 1A, we have that a function $f : A \to \mathbb{R}$ is continuous on A if:

$$\forall x \in A, \forall \epsilon > 0, \exists\, \delta > 0 \text{ such that } \forall y \in A, (|y - x| < \delta) \Rightarrow (|f(y) - f(x)| < \epsilon).$$

It's important to note that x is chosen given before we choose a δ. Thus, our choice for δ can depend on x as well as ϵ.

Uniform continuity requires that δ is independent of x.

A note, a function being continuous at a value (or set of values for that matter), it equivalent to saying that there exists a limit for the function at that value and that limit is the value of the function applied to that value.

1.2 Uniformly Continuous Functions

Uniform continuity is similar to continuity as we knew it in Analysis 1A. For a function $f : A \to \mathbb{R}$, f is uniformly continuous on A if:

$$\forall \epsilon > 0, \exists\, \delta > 0 \text{ such that } \forall x, y \in A, (|y - x| < \delta) \Rightarrow (|f(y) - f(x)| < \epsilon).$$

We can see that uniform continuity **implies** continuity but **not** vice versa.

A note, for uniform continuity, we are saying that given a value ϵ, we can always pick a distance (δ) such that if two values are within that distance of each other, the distance between the values after the function is applied to them will be less than ϵ. This is essentially testing for divergence to infinity at a value ($\frac{1}{x}$ is continuous but not uniformly continuous on $\mathbb{R}_{>0}$).

2 Convergence

We have the notion of convergence for sequences of real numbers from Analysis 1A, convergence in this section is similar but specifically for functions.

2.1 Pointwise Convergence

A sequence of functions $(f_n)_{n\in\mathbb{N}}$ from $A \to \mathbb{R}$ converges **pointwise** to the function f on A if:

$$\forall x \in A, \lim_{n\to\infty}(f_n(x)) = f(x).$$

f is called the **pointwise limit** of $(f_n)_{n\in\mathbb{N}}$.

A note, for $f_n : [0,1] \to [0,1]; x \to x^n$, $f : [0,1] \to [0,1]; x \to \delta_1(x)$, f_n converges pointwise to f.

2.2 Uniform Convergence

A sequence of functions $(f_n)_{n\in\mathbb{N}}$ from $A \to \mathbb{R}$ converges **uniformly** to the function f on A if:

$$\forall \epsilon > 0, \exists\, N \in \mathbb{N} \text{ such that } \forall x \in A, \forall n \in \mathbb{N}, (n \geq N) \Rightarrow (|f(x) - f_n(x)| < \epsilon).$$

For the same functions outlined in the note under pointwise convergence, we have that f_n does not converge uniformly to f. Let $\epsilon \in (0,1)$, $x \in [0,1)$ and suppose f_n is uniformly convergent to f,

$$\begin{aligned}
&|f_n(x) - f(x)| = |x^n| < \epsilon \\
&\Rightarrow 0 \leq x^n < \epsilon < 1 \\
&\Rightarrow 0 \leq x < \epsilon^{\frac{1}{n}} < 1 \\
&\Rightarrow \epsilon = 1 \text{ as } x \in [0,1).
\end{aligned}$$

This is a contradiction by the definition of ϵ. Thus, we have the result.

2.3 Weierstrass' Theorem

For $a, b \in \mathbb{R}$ with $a < b$, if a sequence of continuous functions on $[a,b]$, $(f_n)_{n\in\mathbb{N}}$ converges uniformly to f on $[a,b]$, f is continuous on $[a,b]$.

Basically, uniform convergence preserves continuity (it also preserves regulation).

2.4 Supremum Norm

2.4.1 Definition of the Supremum Norm

For $a, b \in \mathbb{R}$ with $a < b$, let $f : [a, b] \to \mathbb{R}$ be a bounded function. The supremum norm of f on $[a, b]$ is denoted by $\|f\|_{[a,b]}$ and is defined by:

$$\|f\|_{[a,b]} := \sup \{\|f(x)| : x \in [a, b]\}.$$

The supremum norm is simply just the furthest distance from zero reached by a function over a closed interval. By definition, it is a real number and $\exists\, x \in [a, b]$ such that $f(x)$ is the supremum norm.

2.4.2 Properties of the Supremum Norm

There are a few key properties of the supremum norm, let a and b be as above and let $\lambda \in \mathbb{R}$, $f, g : [a, b] \to \mathbb{R}$ be bounded functions:

- $\|f\|_{[a,b]} > 0$
- $\|f\|_{[a,b]} = 0 \Leftrightarrow f = 0$ on $[a, b]$
- $\|\lambda f\|_{[a,b]} = |\lambda|\|f\|_{[a,b]}$
- $\|f + g\|_{[a,b]} = \|f\|_{[a,b]} + \|g\|_{[a,b]}$.

2.5 Cauchy Sequences of Functions

For $a, b \in \mathbb{R}$ with $a < b$, denote the set of continuous functions on $[a, b]$ by $C([a, b])$. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of functions in $C([a, b])$. We say $(f_n)_{n\in\mathbb{N}}$ is Cauchy if:

$$\forall \epsilon > 0, \exists\, N \in \mathbb{N} \text{ such that } \forall m, n \in \mathbb{N}, (m, n \geq N) \Rightarrow (\|f_n - f_m\|_{[a,b]} < \epsilon).$$

This obviously bears an extreme resemblance to the Cauchy sequences of Analysis 1A. Just replacing the sequences of reals with sequences of functions and the modulus with the supremum norm.

For each continuous function, there exists a Cauchy sequence such that the sequence converges uniformly to said function.

3 Integration

3.1 Step Functions

For $a, b \in \mathbb{R}$ with $a < b$, a partition of the interval $[a, b]$ is a set P of the form:

$$P = \{x_0, x_1, \ldots, x_n\} \text{ (for some } n \in \mathbb{N})$$
$$\text{where } a = x_0 < x_1 < \ldots < x_n = b.$$

We say a function $\psi : [a, b] \to \mathbb{R}$ is a step function if there exists a partition $P = \{x_0, \ldots, x_n\}$ and a set of constants in $\mathbb{R}$ ($\{c_0, c_1, \ldots, c_n\}$) such that:

$$\psi(x) = c_i \, (\forall x \in (x_{i-1}, x_i)).$$

In this case, P and ψ are adapted to each other.

$S[a, b]$ is the set of step functions over $[a, b]$.

3.2 Integration of Step Functions

3.2.1 Definition of integration on step functions

The integral of the step function is simple:

$$\int_a^b \psi(x)\, dx := \sum_{i=1}^{n} c_i(x_i - x_{i-1}).$$

As long as the partition is adapted to ψ, the integral doesn't change.

3.2.2 Properties of integration on step functions

Here are some properties of the integration of step functions, let ϕ, ψ be step functions over $[a, b]$, $y \in \mathbb{R}$ with $a < y < b$, $\alpha, \beta \in \mathbb{R}$:

- **Linearity**: $\int_a^b \alpha\psi(x) + \beta\phi(x)\, dx = \alpha \int_a^b \psi(x)\, dx + \beta \int_a^b \phi(x)\, dx$
- **Monotonicity**: $(\psi(x) \leq \phi(x)(\forall x \in [a, b])) \Rightarrow (\int_a^b \psi(x)\, dx \leq \int_a^b \phi(x)\, dx)$
- **Continuity**: $|\int_a^b \psi(x)\, dx| \leq (b - a)\|\psi(x)\|_{[a,b]}$
- **Additivity**: $\int_a^b \psi(x)\, dx = \int_a^y \psi(x)\, dx + \int_y^b \psi(x)\, dx$

3.3 Regulated Functions

3.3.1 Definition of left and right limits

Let $A \subseteq \mathbb{R}$ and $f : A \to \mathbb{R}$. For some $\epsilon > 0$, $a \in A$, and $\alpha \in \mathbb{R}$:

1. f has a **right limit** of α at a if:
   $\exists\, \delta > 0$ such that $(0 < x - a < \delta) \Rightarrow (|f(x) - \alpha| < \epsilon)$
2. f has a **left limit** of α at a if:
   $\exists\, \delta > 0$ such that $(0 < a - x < \delta) \Rightarrow (|f(x) - \alpha| < \epsilon)$.

We can denote right limits by: $\lim_{x \downarrow a} f(x) = \alpha$. Similarly for left limits: $\lim_{x \uparrow a} f(x) = \alpha$.

There is a sequential definition too, for any sequence $(x_n)_{n \in \mathbb{N}}$ that satisfies $x_n > a$ for all $n \in \mathbb{N}$ and $\lim_{n \to \infty} x_n = a$, if f has a right limit, $\lim_{n \to \infty} f(x_n) = \alpha$. There is a similar definition for left limits.

3.3.2 Definition of a regulated function

A function $f : [a, b] \to \mathbb{R}$ is regulated if:

- f has a left limit on all values in $(a, b]$
- f has a right limit on all values in $[a, b)$.

All continuous functions are regulated. All increasing and decreasing functions are regulated.

3.3.3 Properties of regulated functions

Let $R([a, b])$ be the set of functions regulated over $[a, b]$. We have that $R([a, b])$ is closed under:

- Scalar multiplication (over $\mathbb{R}$)
- Addition
- Multiplication
- Division (if the divisor is greater than zero over $[a, b]$)
- Composition
- The modulus.

Uniform convergence preserves regulation. Also, all step functions are regulated.

For f a regulated function over $[a, b]$, we have that:

$$\forall \epsilon > 0, \exists\, \psi \in S([a,b]) \text{ such that } \|\psi - f\| < \epsilon.$$

Basically, for any regulated function we can always choose an arbitrarily accurate approximation that is a step function.

3.4 Integeration of Regulated Functions

3.4.1 Definition of integration on regulated functions

For a function $f \in R([a,b])$, say we have two sequences of step functions, $(\psi_n)_{n\in\mathbb{N}}$ and $(\phi_n)_{n\in\mathbb{N}}$:

- $(\psi_n)_{n\in\mathbb{N}}$ is uniformly convergent to $f \Rightarrow (\int_a^b \psi_n(x)\,dx)_{n\in\mathbb{N}}$ is convergent
- $(\psi_n)_{n\in\mathbb{N}}$ and $(\phi_n)_{n\in\mathbb{N}}$ are uniformly convergent to $f \Rightarrow$ $\lim_{n\to\infty}(\int_a^b \psi_n(x)\,dx) = \lim_{n\to\infty}(\int_a^b \phi_n(x)\,dx)$.

Basically, we have that no matter what step function we choose to approximate our function, the value of the integral will tend to the same value.

We define the integral of a regulated function $f \in R([a,b])$ by choosing a sequence of step functions $(\psi_n)_{n\in\mathbb{N}}$ such that they converge uniformly to f:

$$\int_a^b f(x)\,dx := \lim_{n\to\infty} \int_a^b \psi_n(x)\,dx.$$

3.4.2 Properties of integration on regulated functions

The **linearity**, **continuity**, and **additivity** properties hold similarly to the properties of step functions. The **monotonicity** property holds also but the stated definition varies slightly:

- **Monotonicity**: For $f \in R([a,b])$ with $f(x) \geq 0$ for $x \in [a,b]$, we have that $\int_a^b f(x)\,dx \geq 0$.

Some small notes on regulated functions, let $f \in R([a,b])$:

- $|\int_a^b f(x)\,dx| \leq \int_a^b |f(x)|\,dx$
- For $(f_n)_{n\in\mathbb{N}}$ uniformly convergent to f, $\lim_{n\to\infty} \int_a^b f_n(x)\,dx = \int_a^b f(x)\,dx$.

The first point is similar to the triangle inequality applied to summations. The second was covered similarly but strictly for step functions, not all regulated functions.

3.5 The Mean-Value Theorem of Integeration

For $f \in C([a,b])$, let $g \in R([a,b])$ and satisfy the following:

- $g(x) \geq 0$ for $x \in [a,b]$
- $\int_a^b g(x)\,dx > 0$

With these assumptions, we have that $\exists\, x \in (a,b)$ with:

$$f(x) \int_a^b g(t)\,dt = \int_a^b f(t)g(t)\,dt$$

Note that the function f is continuous. This is a stronger statement than just saying it's regulated. Also, consider $g = 1$:

$$\begin{aligned} f(x) \int_a^b g(t)\,dt &= f(x) \int_a^b 1\,dt \\ &= f(x)(b-a) \end{aligned} \tag{1}$$

$$\int_a^b f(t)g(t)\,dt = \int_a^b f(t)\,dt \tag{2}$$

$$\begin{aligned} &(1) \text{ and } (2) \;\Rightarrow \\ &\int_a^b f(t)\,dt = f(x)(b-a) \end{aligned}$$

4 Differentiation

4.1 Definition of Differentiation

For a function f defined on a δ-neighbourhood of some $a \in \mathbb{R}$, we have that f is differentiable at a if the following exists in $\mathbb{R}$:

$$\lim_{h \to 0} \left(\frac{f(a+h) - f(a)}{h} \right).$$

Differentiability at a value a implies continuity at a.

4.2 Properties of Differentiation

4.2.1 Closure of the set of differentiable functions

Let f, g be differentiable functions. The set of differentiable functions is closed under:

- **Addition**: $(f+g)' = f' + g'$
- **Multiplication**: $(fg)' = f'g + fg'$
- **Division**: $\frac{f}{g} = \frac{f'g - fg'}{g^2}$ (for g non-zero)
- **Composition**: $(f \circ g)' = g'(f' \circ g)$.

4.2.2 The implications of zero derivatives

For a differentiable function f:

- $f(x_0)$ is a maximum or minimum $\Rightarrow f'(x_0) = 0$
- $f'(x) = 0\,(\forall x \in [a,b]) \Rightarrow f$ is constant on $[a,b]$.

4.2.3 Rolle's Theorem and the Mean Value Theorem

For f continuous on $[a,b]$ and differentiable on (a,b),

$$(f(a) = f(b)) \Rightarrow (\exists\, x_0 \in (a,b) \text{ such that } f'(x_0) = 0) \qquad \text{(Rolle's Theorem)}$$
$$\exists\, x_0 \in (a,b) \text{ such that } f'(x_0) = \frac{f(b) - f(a)}{b - a}. \qquad \text{(Mean Value Theorem)}$$

Rolle's Theorem is a special case of the Mean Value Theorem. The Mean Value Theorem says that over an interval, the derivative is equal to the average derivative across the interval at some value.

4.2.4 Cauchy's Mean Value Theorem

For f, g continuous on $[a,b]$ and differentiable on (a,b),

$$\exists\, x_0 \in (a,b) \text{ such that } \left[(f(b) - f(a))g'(x_0) = (g(b) - g(a))f'(x_0)\right].$$

4.2.5 Other properties of the derivative

For f, g differentiable functions,

- $(f'(x) = g'(x)\,(\forall x \in [a,b])) \Rightarrow (f(x) = g(x) + c\,(c \in \mathbb{R}))$
- $f'(x) > 0\,(\forall x \in [a,b]) \Rightarrow f$ is strictly increasing (similarly for strictly decreasing).

4.3 L'Hôpital's Rule

For f, g differentiable functions defined on a δ-neighbourhood of $a \in \mathbb{R}$, if:

- $g'(x) \neq 0\,(\forall x \in (a-\delta, a+\delta)\backslash\{a\})$
- $f(a) = g(a) = 0$
- $\lim_{x\to a} \frac{f'(x)}{g'(x)}$ exists in $\mathbb{R}$.

Then,

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}.$$

So, if we have two functions that equal zero at a value, this rule helps us find the derivative of the their quotient as long as the denominator isn't zero nearby.

5 Calculus