Set Theory Notes

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These notes are not necessarily correct, consistent, representative of the course as it stands today, or rigorous. Any result of the above is not the author's fault.

These notes are in progress.

0   Notation

We commonly deal with the following concepts in Set Theory which I will abbreviate as follows for brevity:

Term	Notation
$\{0, 1, 2, \ldots\}$	$\mathbb{N}$

Contents

1 The Fundamentals

1.1 Axiom of Extensionality

For two sets a and b, we have that $a = b$ if and only if for all x we have that:

$$x \in a \Longleftrightarrow x \in b.$$

For two classes A and B, we have that $A = B$ if and only if for all x we have that:

$$x \in a \Longleftrightarrow x \in b.$$

1.2 Axiom of Pair Sets

For any sets x and y, there is a set $z = \{x, y\}$. This is the (unordered) pair set of x and y.

1.3 Axiom of the Powerset

For each set x, there exists a set which is the collection of the subsets of x, the powerset $\mathcal{P}(x)$.

For some set x, we have the powerset defined as follows $\mathcal{P}(x) = \{z : z \subseteq x\}$.

1.4 Axiom of the Empty Set

There exists a set with no members, the empty set $\varnothing$.

We have the empty set defined as follows $\varnothing = \{x : x \neq x\}$.

1.5 Axiom of Subsets

For some set x, we have that $\{y \in x : \Phi(y)\}$ is a set for some well-defined property of sets Φ.

1.6 Axiom of Unions

We have the basic union of two sets x_1 and x_2:

$$x_1 \cup x_2 = \{y \; : \; y \in x_1 \text{ or } y \in x_2\},$$

but for cases where we want to unify the members of the sets in a set X, we define:

$$\bigcup X = \{y \; : \; \exists\, x \in X, y \in x\}.$$

This axiom states that for a set X, $\bigcup X$ is a set.

1.7 Classes

We have that classes are collection of objects, these could also be sets. Classes that are not sets are called proper classes.

1.8 The Set ω

We have the set of natural numbers, $\mathbb{N} = \{0, 1, 2, \ldots\}$, and from this, we define ω:

$$\omega = \{0, 1, 2, \ldots\},$$

where for some n in ω,

$$n = \{0, 1, 2, \ldots, n-1\},$$

with 0_ω being the empty set. We can go beyond this definition, defining:

$$\omega + 1 = \{0, 1, 2, \ldots, \omega\},$$
$$\omega + 2 = \{0, 1, 2, \ldots, \omega, \omega + 1\},$$
$$\ldots$$
$$\omega + n = \{0, 1, 2, \ldots, \omega, \omega + 1, \ldots \omega + n - 1\}.$$

1.9 Russell's Theorem

We have that $R = \{x \; : \; x \notin x\}$ is not a set.

Proof. Suppose we have a set z such that $z = R$, is z in R? If we suppose z is in R, we have that z is not in z by the definition of R (as $z = R$) but z is R so z is not in R, a contradiction. Thus, we have that there is no set z equal to R, so R is not a set but a proper class. $\qquad\square$

1.10   The Universe of Sets

We define the universe of sets as $V = \{x \; : \; x = x\}$. We have that V is a proper class.

Proof. If we suppose V is a set, we apply the axiom of subsets with $\Phi(x) = x \notin x$ and reach a contradiction via Russell's theorem. $\qquad\square$

2 Relations

We will first state the significant properties relations can have. Taking a relation R on X with x, y, z arbitrary in X:

Name	Property
Reflexive	xRx
Irreflexive	$\neg(xRx)$
Symmetric	$xRy \Rightarrow yRx$
Antisymmetric	$[xRy \text{ and } yRx] \Rightarrow [x = y]$
Connected	$[x = y]$ or $[xRy]$ or $[yRx]$
Transitive	$[xRy \text{ and } yRz] \Rightarrow [xRz]$

For example, equivalence relations must satisfy reflexivity, symmetry, and transitivity.

2.1 Partial Orderings

We say that a relation $\prec$ on a set X is a (strict) partial ordering if it is irreflexive and transitive.

Similarly, we say that a relation $\preceq$ on a set X is a non-strict partial ordering if it is reflexive, antisymmetric, and transitive.

A partial ordering $(X, \prec)$ is wellfounded if for any non-empty subset Y of X, Y has a least element under $\prec$.

2.2 Bounding

For a partially ordered set $(X, \prec)$:

- x_0 in X is the minimum of X if for all x in X, $x_0 \preceq x$,

- x' in X is minimal in X if for all x in X, $\neg(x \prec x')$,

- x_1 in X is the maximum of X if for all x in X, $x \preceq x_1$,

- x' in X is maximal in X if for all x in X, $\neg(x' \prec x)$.

Taking a non-empty subset Y of X, we consider the subordering $(Y, \prec)$ and for some α in X we say:

- α is a lower bound for Y if for all y in Y, $\alpha \prec y$,

- α is the infimum of Y if it's a lower bound and for all lower bounds λ of Y, $\alpha \preceq \lambda$,

- α is an upper bound for Y if for all y in Y, $y \prec \alpha$,

- α is the supremum of Y if it's an upper bound and for all upper bounds τ of Y, $\tau \preceq \alpha$.

2.3 Order Preserving Maps

We say that $f : (X, \prec_1) \to (Y, \prec_2)$ is an order preserving map if for each x_1, x_2 in X:

$$x_1 \prec_1 x_2 \implies f(x_1) \prec_2 f(x_2).$$

Two orderings are (order) isomorphic if there is a bijective order preserving map between them.

2.4 Representation Theorem for Partially Ordered Sets

For a partially ordered set $(X, \prec)$, there is a set $Y \subseteq \mathcal{P}(X)$ which is such that $(X, \preceq)$ is order isomorphic to $(Y, \subseteq)$.

Proof. For some x in X, we set $X^x = \{x' \in X : x' \preceq x\}$, the set of elements preceding or equal to x. For x, y in X, $x \neq y$ implies that $X^x \neq X^y$ as these sets contain x and y (resp.) so $x \mapsto X^x$ is injective. This map is surjective trivially (mapping from X to $\{X^x : x \in X\}$). We have that:

$$x \preceq y \iff X^x \subseteq X^y,$$

by our definition. Thus, $x \mapsto X^x$ is an order isomorphism. $\square$

2.5 Total Orderings

A relation $\prec$ on a set X is a (strict) total ordering if it is a connected strict partial ordering.

Similarly, we say that a relation $\preceq$ on a set X is a non-strict total ordering if it is a connected non-strict partial ordering.

2.6 Well-orderings

A relation $\prec$ on a set X is a well-ordering if it is a strict total ordering and for any non-empty subset Y of X, Y has a least element under $\prec$. We denote this with $(X, \prec) \in WO$.

2.7 Ordered Pairs

For x, y sets, the ordered pair of x and y is the set:

$$\langle x, y \rangle = \{\{x\}, \{x, y\}\}.$$

2.7.1 Uniqueness of Ordered Pairs

For x, y, u, v sets, we have that:

$$\langle x, y \rangle = \langle u, v \rangle \iff (x = u) \text{ and } (y = v).$$

Proof. Suppose the former, if $x = y$ then $\langle x, y \rangle = \{\{x\}, \{x, x\}\} = \{\{x\}\}$. Thus, $\langle u, v \rangle = \{\{u\}\}$ as it is equal to $\langle x, y \rangle$ which has one element, hence $u = v$. By the Axiom of Extensionality, we have that $x = u$ and so $y = x = u = v$.

If $x \neq y$, then $\langle x, y \rangle$ and $\langle u, v \rangle$ both have the same two elements by our assumption (so $u \neq v$). We cannot have $\{x\} = \{u, v\}$ so $\{x\} = \{u\}$ which means $x = u$ by the Axiom of Extensionality. Thus, $\{u, v\} = \{x, y\} = \{u, y\}$ so $y = v$.

Suppose the latter, then the former holds trivially. $\qquad\square$

2.7.2 The Ordered k-tuple

We define the k-tuple inductively. The 2-tuple is already defined. We define the 3-tuple:

$$\langle x_1, x_2, x_3 \rangle = \langle \langle x_1, x_2 \rangle, x_3 \rangle,$$

and for k in $\{3, 4, \dots\}$:

$$\langle x_1, x_2, \dots, x_k \rangle = \langle \langle x_1, x_2, \dots, x_{k-1} \rangle, x_k \rangle.$$

2.7.3 The Product of Sets

For A, B sets, we define:

$$A \times B = \{\langle a, b \rangle : a \in A, b \in B\}.$$

Similarly to k-tuples, for $A_1, A_2, \ldots, A_k$ sets, we have $A_1 \times A_2$ defined, so we define:

$$A_1 \times A_2 \times \cdots \times A_k = (A_1 \times A_2 \times \cdots \times A_{k-1}) \times A_k,$$

defining the k-product for k in $\{2, 3, \ldots\}$. This is not associative.

2.8 Binary Relations

A binary relation R is a class of ordered pairs. We write $R^{-1} = \{\langle y, x \rangle : \langle x, y \rangle \in R\}$.

2.8.1 Domain and Range

For a relation R, we define:

$$\begin{aligned}
\text{dom}(R) &= \{x : \exists\, y \text{ where } \langle x, y \rangle \in R\}, \\
\text{ran}(R) &= \{y : \exists\, x \text{ where } \langle x, y \rangle \in R\}, \\
\text{Field}(R) &= \text{dom}(R) \cup \text{ran}(R).
\end{aligned}$$

2.9 Functions

A relation F is a function if for all x in $\text{dom}(F)$, there is a unique y in $\text{ran}(F)$ with $\langle x, y \rangle$ in F.

If F is a function, it is injective if and only if for all x, x':

$$(\langle x, y \rangle \in F \text{ and } \langle x', y \rangle \in F) \Rightarrow (x = x').$$

2.9.1 Ranges and Restrictions

For $F : X \to Y$:

- $F''A = \{y \in Y : \exists\, x \in A \text{ such that } F(x) = y\}$ the range of F on A,

- $F \upharpoonright A = \{\langle x, y \rangle \in F : x \in A\}$ the restriction of F to A.

We can see that $F''A = \text{ran}(F \upharpoonright A)$.

2.9.2   The Set of Functions

For X, Y sets, we have that ${}^{X}Y = \{F : F : X \to Y\}$.

2.9.3   Indexed Cartesian Products

For a set I with each i in I corresponding to a non-empty set A_i:

$$\prod_{i \in I} A_i = \{\text{functions } f : \text{dom}(f) = I \text{ and } f(i) \in A_i \text{ for all } i \in I\}.$$

3 Transitive Sets

A set x is transitive if and only if for all y in x, $y \subseteq x$. This can be abbreviated to $\cup x \subseteq x$.

3.1 The Successor Function

For a set x, $S(x) = x \cup \{x\}$ is the successor of x. $S(x) = x$ is equivalent to saying x is transitive.

3.2 Transitive Closure

For a set x, to find a superset of x which is transitive, the transitive closure TC of x, we recurse:

$$\bigcup^{0} x = x,$$
$$\bigcup^{n+1} x = \bigcup \left(\bigcup^{n} x \right),$$

which we can write as:

$$TC(x) = \bigcup \left\{ \bigcup^{n} x : n \in \mathbb{N} \right\}.$$

The transitive closure is always transitive.

3.2.1 Properties of Transitive Closure

For a set x:

- $x \subseteq TC(x)$,

- If t is transitive and $x \subseteq t$ then $TC(x) \subseteq t$. $TC(x)$ is the smallest transitive set containing x,

- By the above, $TC(x) = x$ if and only if x is transitive.