# Data Structures and Algorithms Notes

*paraphrased by* Tyler Wright

*An important note, these notes are absolutely* ***NOT*** *guaranteed to be correct, representative of the course, or rigorous. Any result of this is not the author's fault.*

# 1 Graph Theory

## 1.1 Definition of a Graph

A graph is a pair of sets $G = (V, E)$, where $V$ is a set of vertices (or nodes) and $E$ is a set of edges (or arcs).

## 1.2 Definition of an Edge

An edge of a graph $G = (V, E)$ is $e = \{u, v\}$ in $E$ where $u$, $v$ are vertices in $V$.

## 1.3 Definition of a Neighbourhood

For a graph $G = (V, E)$ with $v$ in $V$, the neighbourhood of $v$ is the set $V' \subseteq V$ of vertices connected to $v$ by an edge in $E$.

The neighbourhood of $v$ is denoted by $N(v)$.

The neighbourhood of a set of vertices is the union of the neighbourhoods of each vertex.

## 1.4 Definition of Degree

For a graph $G = (V, E)$ with $v$ in $V$, the degree of $v$ is the size of its neighbourhood.

The degree of $v$ is denoted by $d(v)$.

## 1.5 The Handshake Lemma

For a graph $G = (V, E)$, we have that:

$$|E| = \frac{\sum_{v \in V} d(v)}{2}.$$

*This is because each edge visits two vertices, so by counting the degree of each vertex we count each edge exactly twice.*

## 1.6 $k$-regular Graphs

For a graph $G = (V, E)$, we have that $G$ is $k$-regular for some $k$ in $\mathbb{Z}_{>0}$ if for all $v$ in $V$, we have:

$$d(v) = k.$$

We cannot have a $k$-regular graph where $k$ is odd and $|V|$ is odd by the Handshake Lemma.

## 1.7 Isomorphic Graphs

Graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are called isomorphic if there exists a bijection $f : V_1 \to V_2$ such that:

$$\{u, v\} \in E_1 \iff \{f(u), f(v)\} \in E_2.$$

This relationship is denoted by $G_1 \cong G_2$.

## 1.8 Definition of a Subgraph

A graph $G' = (V', E')$ is a subgraph of $G = (V, E)$ if $V' \subseteq V$ and $E' \subseteq E$.

## 1.9 Definition of an Induced Subgraph

An induced subgraph generated from $G = (V, E)$ by $V' \subseteq V$ is the graph $G' = (V', E')$ where:

$$E' = \{\{u, v\} \in E \text{ such that } u, v \in V'\}.$$

*Essentially, you generate an induced subgraph from a subset of the vertices of a graph by selecting edges that join vertices in the subset.*

## 1.10 Walks

### 1.10.1 Definition of a walk

A walk in a graph $G = (V, E)$ is a set of vertices in $V$ connected by edges in $E$. The length of the walk is the number of edges traversed in the walk.

### 1.10.2 Definition of a path

A path is a walk where no vertices are repeated.

### 1.10.3 Definition of an Euler walk

An Euler walk is a walk such that every edge is traversed exactly once. Thus, for a graph $G = (V, E)$, the length is $|E|$.

### 1.10.4 Conditions for an Euler walk

For an Euler walk to be possible on a given graph, all vertices must have an even degree **or** exactly two vertices have odd degree.

If all vertices have even degree we have that the Euler walk is a cycle, if exactly two vertices have odd degree then we have that these vertices are the start and end points of our Euler walk.

## 1.11 Definition of a Connected Graph

A connected graph is a graph where for each pair of vertices, there is a path connecting them.

## 1.12 Definition of a Component

A component of a graph $G = (V, E)$ is a maximal connected induced subgraph of $G$. This means an induced subgraph of $G$ that is connected but is not longer connected if a vertex is removed.

Connected graphs have a single component, the entire graph.

## 1.13 Digraphs

### 1.13.1 Definition of a digraph

A digraph (or directed graph) is a graph where each of the edges has a direction. This direction means the edge can only be traversed in a single direction.

### 1.13.2 The Directed Handshake Lemma

For a digraph $G = (V, E)$, we have that:

$$\sum_{v \in V} d^-(v) = \sum_{v \in V} d^+(v) = |E|.$$

*This is because if we consider the 'tail' of an edge (the vertex it leaves), each edge has exactly one tail.*

### 1.13.3 Definition of a strongly connected digraph

A digraph $G = (V, E)$ is strongly connected if for each $u, v$ in $E$, there exists a path from $u$ to $v$ **and** from $v$ to $u$.

### 1.13.4 Definition of a weakly connected digraph

A digraph $G = (V, E)$ is weakly connected if for each $u, v$ in $E$, there exists a path from $u$ to $v$ **or** from $v$ to $u$.

### 1.13.5 Definition of components of digraphs

A strong component of a digraph is the maximal *strongly* connected induced subgraph.

A weak component of a digraph is the maximal *weakly* connected induced subgraph.

*So, these are induced subgraphs that are strongly/weakly connected but are no longer strongly/weakly connected once a vertex is removed.*

### 1.13.6 Definition of neighbourhoods in digraphs

The neighbourhood of a vertex in a digraph can be considered by looking at the edges *from* the vertex and the edges *to* the vertex.

The in-neighbourhood of a vertex $v$ are the edges that enter $v$. The out-neighbourhood of a vertex $v$ are the edges that exit $v$. These are denoted by $N^-(v)$ and $N^+(v)$ respectively.

### 1.13.7 Definition of degrees in digraphs

For a vertex $v$, the in-degree of the vertex $d^-(v)$ is the size of the in-neighbourhood and the out-degree of the vertex $d^+(v)$ is the size of the out-neighbourhood.

It can be seen that the degree of a given vertex is the sum of its in and out degree (in a digraph).

### 1.13.8 Conditions for an Euler walk in a digraph

For an Euler walk to be possible on a given digraph, we have two cases, either:

- the digraph is strongly connected and every vertex has equal in and out degrees, or
- one vertex has an in-degree one greater than its out-degree, another has an out-degree one greater than its in-degree, and all remaining vertices have equal in and out degrees.

In the first case we have that the Euler walk is a cycle, in the second we have that the special vertices are the start and end points of our Euler walk.

### 1.13.9 Cycles

### 1.13.10 Definition of a cycle

A cycle is a walk where the first and last vertices are the same and each vertex appears at most once (barring the first and last vertex).

### 1.13.11 Definition of a Hamiltonian cycle

A Hamiltonian cycle is a cycle where each vertex is visited.

### 1.13.12 Conditions for a Hamiltonian cycle

Whilst the conditions necessary for a Hamiltonian cycle in general are unknown, by Dirac's theorem, we know that for a graph with $n$ vertices, if every vertex has degree $\frac{n}{2}$ or greater then a Hamiltonian cycle exists.

## 1.14 Trees

### 1.14.1 Definition of a forest

A forest is a graph with no cycles.

### 1.14.2 Definition of a tree

A tree is a connected forest (or a connected graph with no cycles).

### 1.14.3 Path uniqueness of trees

For a tree $T = (V, E)$, we have that for any $u, v$ in $V$, there exists a unique path from $u$ to $v$.

*To prove this, suppose there are two unique paths between $u$ and $v$. These paths must diverge and if we connect them, they form a cycle which contradicts the definition of a tree.*

### 1.14.4 The magnitude of edges in trees

For a tree $T = (V, E)$, we have that $|E| = |V| - 1$.

### 1.14.5 Rooted trees

For a tree $T = (V, E)$, we can root $T$ with some $r$ in $V$. For $v$ in $V \setminus r$, we define $P_v$ to be the path from $r$ to $v$, we then direct the edges from $r$ to $v$ for each $P_v$.

For $u, v$ in $V \setminus \{r\}$, we say that:

- $u$ is an **ancestor** of $v$ if $u$ lies on $P_v$
- $u$ is the **parent** of $v$ if $u$ is in the in-neighbourhood of $v$
- $v$ is a **leaf** if it has degree 1
- $L_0 = \{r\}$ and $L_n = \{v : |P_v| = n\}$ are the **levels** of $T$
- The **depth** of a tree is the greatest $n$ where $L_n$ is non-empty.

### 1.14.6 Lower bound on the amount of leaves in a tree

For a tree with $T = (V, E)$, if $V > 1$, there must be at least 2 leaves.

### 1.14.7 Equivalent statements to the tree definition

For a graph $T = (V, E)$, we have that the following are equivalent:

- $T$ is a tree
- $T$ is connected and has no cycles
- $|E| = n - 1$ and T is connected
- $|E| = n - 1$ and T has no cycles
- T has a unique path between any two vertices

## 1.15 Bipartitions

### 1.15.1 Definition of a bipartite graph

For $G = (V, E)$, we have that $G$ is bipartite if there exists $A \subset V$, $B \subset V$ such that $A$ and $B$ are disjoint and the induced subgraphs of $A$ and $B$ have no edges. $A$ and $B$ are bipartitions of $G$.

Saying $G$ is bipartite is equivalent to saying $G$ has no cycles of odd length.

### 1.15.2 Definition of a matching

A matching in a graph is a set of disjoint edges.

A matching is **perfect** if each vertex is contained in some matching edge.

### 1.15.3 Definition of a semi-matching

For $k$ in $\mathbb{Z}_{>0}$, a $k$ to 1 semi-matching in a bipartite graph $G$ with a bipartition $\{A, B\}$ is a subgraph of $G$ where each vertex in $A$ has degree at most $k$ and each vertex in $B$ has degree at most 1.

### 1.15.4 Definition of an augmenting path

Given a matching $M$ in a bipartite graph $G = (V, E)$, an augmenting path is a set of vertices in $V$ connected by edges $e_i$ in $E$ such that:

$$e_i \text{ is } \begin{cases} \text{in } M & \text{for } i \text{ odd} \\ \text{not in } M & \text{for } i \text{ even.} \end{cases}$$

With the condition that the first and last vertices in the path are not in the matching.

### 1.15.5 Hall's Theorem

For a bipartite graph $G = (V, E)$ with the bipartition $(A, B)$ has a perfect matching if and only if $|A| = |B|$ and for all $X \subseteq A$, $|N(X)| \geq |X|$.

# 2 Types of Algorithms

## 2.1 Greedy Algorithms

These types of algorithms start with a trivial solution and iteratively optimise their solution based on the information available at the time. They do not retroactively change the solution based on new data, only add to it.

# 3 Data Structures

## 3.1 Stacks

A stack is a list of variables. It supports three operations:

| Name | Description | Worst case runtime |
|---|---|---|
| `create()` | Creates a new stack | $O(1)$ |
| `push(x)` | Adds `x` to the end of the stack | $O(1)$ |
| `pop()` | Removes and returns the last element of the stack | $O(1)$ |

## 3.2 Queues

A queue is a list of variables. It supports three operations:

| Name | Description | Worst case runtime |
|---|---|---|
| `create()` | Creates a new queue | $O(1)$ |
| `add(x)` | Adds `x` to the end of the queue | $O(1)$ |
| `serve()` | Removes and returns the first element of the queue | $O(1)$ |

## 3.3 Linked List

A linked list is a list of variables represented by nodes which point to the next and previous element in the list (null if one does not exist). It supports four operations:

| Name | Description | Worst case runtime |
|---|---|---|
| `create()` | Creates a new linked list | $O(1)$ |
| `insert(x, i)` | Inserts `x` after node `i` | $O(1)$ |
| `delete(i)` | Removes node `i` | $O(1)$ |
| `lookup(i)` | Returns node `i` | $O(1)$ |

## 3.4 Arrays

An array is a list of variables of fixed length. It supports three operations:

| Name | Description | Worst case runtime |
|---|---|---|
| `create(n)` | Creates a new array of size `n` | $O(1)$ |
| `update(x, i)` | Overwrites the data at position `i` with `x` | $O(1)$ |
| `lookup(i)` | Returns the value at `i` | $O(1)$ |

## 3.5 Hash Tables

A hash table is an array of linked lists storing key-value pairs. We use a **hash function** to map data to a linked list. As we are using linked lists, if multiple keys map to the same index, we can just add them to the list - and when looking up data, we can find the right list with the hash function and then match our key.

It supports four operations:

| Name | Description | Average runtime |
|---|---|---|
| `create(n)` | Creates a `n` sized array of linked lists and chooses a hash function `h` | $O(1)$ |
| `insert(k, v)` | Inserts the pair (`k, v`), if $\frac{n}{2}$ pairs are stored, we create a hash table of double the size and copy the contents into it | $O(1)$ |
| `delete(k)` | Deletes the pair corresponding to the key `k` | $O(1)$ |
| `lookup(k)` | Returns the pair corresponding to the key `k` | $O(1)$ |

### 3.5.1 Markov's Inequality

For $X \geq 0$ a random variable with mean $\mu$, for all $t$ in $\mathbb{R}_{\geq 0}$:

$$\mathbb{P}(X \geq t) \leq \frac{\mu}{t}.$$

So, if $X$ is the expected time it takes for an algorithm to terminate, we can say how likely it is for an algorithm to terminate based on our prediction.

# 4 Fast Fourier Transforms

## 4.1 Polynomials

### 4.1.1 Definition of a Polynomial

A polynomial of degree $n$ in $\mathbb{Z}_{\geq 0}$ is a function $A$:

$$A(x) = \sum_{i=0}^{n} a_i x^i,$$

where $a_i$ are the coefficients of $A$. We say for $k > n$, $k$ is a degree-bound of $A$. We can represent this by listing the coefficients, called the **coefficient representation**.

### 4.1.2 Fast Polynomial Evaluation

We can evaluate polynomials quickly using *Horner's Rule*, for a polynomial $A$ degree $n$:

$$A(x) = a_0 + x(a_1 + x(a_2 + \cdots + x(a_n))))).$$

This can be simplified in the following code:

```
int polynomial(int[] coeffs, int x) {
  int output = 0;
  for (i = n; i >= 0; i--) {
    output = (output * x) + coeffs[i]
  }
  return output;
}
```

We have that this is $O(n)$.

### 4.1.3 Point Intersection with Polynomials

For a given set of points of size $n$, we have that there exists a unique polynomial with degree-bound $n$ such that the polynomial intersects all the given points.

### 4.1.4 Point-Value Representation

We can represent a polynomial by a set of points it intersects like so:

$$\{(x_0, y_0), \ldots, (x_n, y_n)\},$$

for a polynomial degree $n + 1$.

### 4.1.5 Polynomial Addition

For two polynomials $A, B$ with coefficients $a_i, b_i$ and degrees $n, m$ respectively, we have that:

$$(A + B)(x) = \sum_{i=0}^{\max(n,m)} (a_i + b_i)x^i.$$

If $m > n$ or vice versa, we pad out the shorter polynomial with zeroes. We can do this with the point-value representation by adding the '$y$-values'. We have that addition as it's defined here is $O(n)$.

### 4.1.6 Polynomial Multiplication

For two polynomials $A, B$ with coefficients $a_i, b_i$ and degrees $n, m$ respectively, we have that:

$$C(x) = (A \cdot B)(x) = \sum_{i=0}^{k} c_i x_i,$$

where $k = 2 \cdot \max(n, m)$ and:

$$c_i = \sum_{j=0}^{i} a_j b_{j-1}.$$

We can do this with the point-value representation, for:

$$A = \{(x_0, y_0), \ldots, (x_n, y_n)\},$$
$$B = \{(x_{n+1}, z_0), \ldots, (x_{n+m}, z_m)\},$$

We have that:

$$C = A \cdot B = \{(x_0, y_0 \cdot z_0), \ldots, (x_k, y_k \cdot z_k)\}$$

This is much easier, yielding an $O(n)$ algorithm rather than an $O(n^2)$ algorithm.

## 4.2 Fast Fourier Transform

### 4.2.1 Roots of Unity

The idea is that we evaluate a polynomial to perform pointwise multiplication and then interpolate back into a polynomial. We need to evaluate a polynomial of degree $n$ at $n+1$ points to convert it to point-value form. We use the $n+1$ roots of unity:

$$\omega_{n+1}^k = e^{\frac{2\pi i}{n+1}k},$$

for $k$ in $\{0, 1, \ldots n\}$. Therefore considering:

$$y_k = A(\omega_{n+1}^k),$$

for $A$ a polynomial, $k$ as above, and the vector of all ordered $y_k$ being the **Discrete Fourier Transform (DFT)** of the coefficient vector of $A$.

**Cancellation Lemma:** we have that $\omega_{dn}^{dk} = \omega_n^k$.

**Halving Lemma:** we have that if $n$ is even, the set of all the squared roots of unity is just the set of roots of unity for $\frac{n}{2}$.

*This is true due to the Cancellation Lemma, we have:*

$$(\omega_{2k}^j)^2 = \omega_{2k}^{2j} = \omega_k^j.$$

### 4.2.2 Method of the Fast Fourier Transform

For a polynomial $A$ degree $n$, we define $A^{[0]}$ and $A^{[1]}$ as:

$$A^{[0]} = a_0 + a_2 x + \cdots + a_{n-2} x^{(n/2)-1}$$
$$A^{[1]} = a_1 + a_3 x + \cdots + a_{n-1} x^{(n/2)-1},$$

so we have that:

$$A(x) = A^{[0]}(x^2) + xA^{[1]}(x^2).$$

So, we can split a DFT computation into two equally sized parts, compute them, and then combine them in linear time.

## 4.3 Polynomial Multiplication

So, the steps are laid out, for polynomials $A, B$ with degree bound $n$, as follows:

- Set the degree of $A$ and $B$ to $2n$, padding with zeroes
- Perform the fast Fourier transform
- Form our point-value representation and multiply pointwise
- Interpolate with the inverse fast Fourier transform.

This process is $O(n \log(n))$.