Linear Algebra 1 (TB2) Notes

paraphrased by Tyler Wright

*An important note, these notes are absolutely **NOT** guaranteed to be correct, representative of the course, or rigorous. Any result of this is not the author's fault.*

1 Vector Spaces, Fields, and Maps

1.1 Groups

A group is a *non-empty* set (G) paired with a *binary group operation* $(*)$ denoted by $(G, *)$. The following properties hold for all groups:

Let $(G, *)$ be a group with elements $f, g, h,$

- **Associativity**: $f * (g * h) = (f * g) * h$

- **Identity**: $\exists e \in G : e * f = f * e = f$

- **Inverse**: $\exists x \in G : x * f = f * x = e$

A note, for a group $(G, *)$ with $g * h = h * g$ for all $g, h \in G$, this group is called **commutative** *or* **abelian**. *However, it should be textitasised that this is* **not** *a necessary condition for a group.*

1.2 A note on the invertibility of matrices

For a matrix $A \in M_{m,n}(\mathbb{F})$, the following are all **equivalent** statements:

- A is **invertible**

- $\det A = 0$

- The **rows** of A are **linearly independent**

- The **columns** of A are **linearly independent**

- The **reduced row echelon form** of A is the **identity**

- For all $\mathbf{b} \in \mathbb{F}^n$, $A\mathbf{x} = \mathbf{b}$ has a **unique solution**

1.3 Fields

A field is a set (F) defined under multiplication and division with the following properties:

- **Associativity** under multiplication and division

- **Commutativity** under multiplication and division

- F contains an **identity** under multiplication and division

- All elements in F contain an **inverse** under addition and multiplication (except 0 under multiplication)

- The defined multiplication is **distributive** across the defined addition

1.4 Vector Spaces

A group $(V, +_V)$ ($+_V$ denotes addition defined with respect to the set V as it can be ambigious in some cases) is a vector space over the field $(\mathbb{F})$ if the following holds:

Let $v, w \in V$, $\lambda, \mu \in \mathbb{F}$,

- $(V, +_V)$ is **abelian**

- V is **closed under multiplication** with elements in $\mathbb{F}$

- $\lambda(v +_V w) = \lambda v + \lambda w$

- $(\lambda + \mu)v = \lambda v +_V \mu v$

- $(\lambda\mu)v = \lambda(\mu v)$

- $fv = v$ where f is the **multiplicative identity** of $\mathbb{F}$

1.5 Subspaces

Let V be a vector space over $\mathbb{F}$, $U \subseteq V$ is a subspace if the following properties hold:

- U is **non-empty**

- U is **closed** under the **addition** defined by V

- U is **closed** under the **multiplication** defined by V

Some notes on subspaces:

- Subspaces are vector spaces

- The intersection of subspaces is a subspace

- The span of any non-empty subset of a given vector space is a subspace

1.6 Linear Maps

For V, W vector spaces over $\mathbb{F}$, the map $T : V \to W$ is called linear if the following properties hold:

Let $u, w \in V$, $\lambda \in \mathbb{F}$,

- $T(u + v) = T(u) + T(v)$

- $T(\lambda u) = \lambda T(u)$

*A note, for a linear map ($T : V \to W$), if $V = W$, T is sometimes referred to as a linear **operator**. Also, composed linear maps are also linear maps*

1.7 The Kernel and Image

For a linear map ($T : V \to W$), the kernel is defined as follows:

$$\operatorname{Ker} T = \{v \in V : T(v) = 0\}$$

The image is defined as follows:

$$\operatorname{Im} T = \{w \in W : \exists v \in V \text{ with } T(v) = w\}$$

Some notes on linear maps (let $T : V \to W$ be a linear map):

- The kernel and image of T are subspaces of V and W respectively

- For $U \subseteq V$, $T(U)$ is also a subspace (but of W instead of V)

1.8 Bases and Dimension

For V a vector space, with $S \subseteq V$,

Let $s_1, s_2, ... \in S$,

- S is linearly independent if $\sum_{n=1}^{|S|} \lambda_n s_n = 0 \iff \lambda_i = 0 \; \forall i$

- S is linearly dependent if it's not linearly independent

A note, if S is linearly dependent, there's a vector in S such that it can be written as the sum of other vectors in S.

A result of linear dependence is that for a linear dependent set S, there exists $s \in S$ such that $\text{span}(S) = \text{span}(S \backslash \{s\})$