Combinatorics Notes

paraphrased by Tyler Wright

An important note, these notes are absolutely ***NOT*** *guaranteed to be correct, representative of the course, or rigorous. Any result of this is not the author's fault.*

1 Counting Techniques

1.1 The Bijection Rule

For n in $\mathbb{N}$, we define $[n] := \{1, 2, \ldots, n\}$.

For a given set X, if there exists a bijective function $f : [n] \to X$ for some n in $\mathbb{N}$, X has n elements (or rather $|X| = n$).

This can also be achieved by listing out the elements of $X = \{x_1, x_2, \ldots, x_n\}$ as we can use $f : [n] \to X$ where i maps to x_i.

1.2 The Addition Rule

We can count the amount of elements in a given set X by splitting X into disjoint sets, counting them, and adding the results.

For n in $\mathbb{N}$, and $X_1, \ldots, X_n$ pairwise disjoint sets:

$$\left| \bigcup_{i=1}^{n} X_i \right| = \sum_{i=1}^{n} |X_i|.$$

For a set of sets A, pairwise disjoint means for two given sets in A, they are either disjoint or equal.

1.3 The Multiplication Rule

If a counting problem can be split into a number of stages, we can use the product of the number of choices at each stage to find the total number of outcomes.

For example, if we want to find how many three digit numbers there are, we can consider it as choosing three digits. We can choose $1, 2, \ldots, 9$ *for the first digit and* $0, 1, \ldots, 9$ *for the rest so we get* $9 \cdot 10^2$ *possibilities.*

1.4 Inclusion-Exclusion Principle

For n in $\mathbb{N}$, and $X_1, \ldots, X_n$ sets:

$$\begin{aligned}\left|\bigcup_{i=1}^{n} X_i\right| &= \sum_{i=1}^{n} |X_i| \\ &- \sum_{i_1 \neq i_2} |X_{i_1} \cap X_{i_2}| \\ &+ \sum_{i_1 \neq i_2 \neq i_3} |X_{i_1} \cap X_{i_2} \cap X i_3| \\ &\ldots\end{aligned}$$

Essentially, this says that the size of the union of some finite number of sets is the sum of their sizes, minus the sum of their ***paired*** *intersections, plus the sum of the intersections of* ***trios****, etc.*

1.5 The Factorial

For n in $\mathbb{N}$ we can define the factorial $n!$:

$$n! = \begin{cases} 1 & \text{n = 0} \\ \prod_{i=1}^{n}(i) & \text{otherwise.} \end{cases}$$

For k in $\mathbb{N}$ we can further define $(n)_k$:

$$(n)_k = \frac{n!}{(n-k)!} = n(n-1)(n-2)\cdots(n-k+1).$$

This can be though of as the factorial with k elements (starting at n). So, $(n)_n = n!$, $(n)_1 = n$, etc.

1.6 The Binomial Coefficient

For n, k in $\mathbb{N}$, we can define the binomial coefficient:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{(n)_k}{k!}.$$

This is the number of ways of choosing k-element subsets from an n-element set. Furthermore, we have:

$$\binom{n}{k} = \binom{n}{n-k},$$

as choosing k elements is equivalent to choosing $n-k$ elements to remove.

There are some notes to be made on the definition:

- $\binom{n}{k} = 0$ if $k > n$
- $\binom{n}{0} = \binom{n}{n} = 1$
- $\binom{n}{k} \geq 0$

1.7 Pascal's Identity

Say we are selecting k elements from an n-element set (unordered, without repeats). We will see that there are $\binom{n}{k}$ possibilities. If we fix an element in the set, we can either include said element in our selection or exclude it giving $\binom{n-1}{k-1}$ and $\binom{n-1}{k}$ possibilities respectively. Thus:

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

1.8 The Binomial Theorem

By performing induction on Pascal's identity, we can see that for a, b in $\mathbb{C}$ and n in $\mathbb{N}$:

$$(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^i b^{n-i}.$$

Setting $a = b = 1$, we get $2^n = \sum_{i=0}^{n} \binom{n}{i}$.

1.9 The Pigeonhole Principle

For m, n, k in $\mathbb{N}$, if we have k objects being distributed into n boxes and $n > mk$ then one box must contain at least $k+1$ objects.

2 Selection

For this section, we will consider n, k in $\mathbb{N}$.

2.1 Ordered Selection with Repeats

As we select, we have n choices, and we select k times. Thus, by the Multiplication Rule, we get n^k outcomes.

2.2 Ordered Selection without Repeats

As we select, the amount of choices we have decreases by one each time. We start with n choices and select k times. Thus, by the Multiplication Rule, we get $n(n-1)\cdots(n-k+1) = (n)_k$ outcomes.

2.3 Unordered Selection with Repeats

Let the set we are selecting from be $\{x_1, \ldots, x_n\}$. In this case, any solution can be aggregated into a list indicating how many times the i^{th} element was selected (for some i in $[n]$). For example, if we select x_1 three times and x_2 five times, the outcome would be of the form $\{3, 5, \ldots\}$.

It can be seen that for each of these solutions, the sum of the elements in the set must equal k. We can construct a solution by starting with a set of all zeroes $\{0, 0, 0, \ldots\}$ and distributing k into the set. For example, for $n = 4$ and $k = 3$ the following are solutions:

$$\{1,1,1,0\} \text{ as } 1+1+1+0 = 3 = k,$$
$$\{0,2,0,1\} \text{ as } 0+2+0+1 = 3 = k,$$
$$\{3,0,0,0\} \text{ as } 3+0+0+0 = 3 = k.$$

These solutions correspond to $\{x_1, x_2, x_3\}, \{x_2, x_2, x_4\}, \{x_1, x_1, x_1\}$ respectively.

This distribution of k can be thought of as seperating k into n groups. For example, the solution $\{1, 1, 0, 1\}$ corresponds to:

$$\bullet | \bullet || \bullet.$$

The dots and dividers are identical respectively, and we have a total of k dots plus $n-1$ dividers equalling $k+n-1$ elements. We can choose where to place the dividers beforehand and then fill in the dots, thus we have:

$$\binom{k+n-1}{n-1}$$

choices.

2.4 Unordered Selection without Repeats

This is identical to to the ordered case but we divide by the number of permutations of the solutions as order does not matter. Thus, we get:

$$\frac{(n)_k}{k!} = \binom{n}{k}.$$

3 Generating Functions

3.1 Definition of a Generating Function

For a sequence $(a_n)_{n\geq 0}$, we can associate a **formal power series**:

$$f(x) = \sum_{k=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots .$$

We say $f(x)$ is the generating function of (a_n), or write:

$$\begin{aligned} a_0, a_1, a_2, \ldots &\leftrightarrows a_0 + a_1 x + a_2 x^2 + \cdots \\ (a_n)_{n\geq 0} &\leftrightarrows f(x). \end{aligned}$$

Note, however, that this doesn't imply that the series is convergent.

3.2 Generating Functions of Finite Sequences

For finite sequences (or rather, sequences with finitely many non-zero terms), we have that their generating functions can be written as polynomials.

3.3 The Scaling Rule

For a sequence $(a_n)_{n\geq 0}$ with an associated generating function $f(x)$ and c in $\mathbb{R}$:

$$(ca_n)_{n\geq 0} \leftrightarrows cf(x).$$

3.4 The Addition Rule

For the sequences $(a_n)_{n\geq 0}$, $(b_m)_{m\geq 0}$ with the associated generating functions $f(x), g(x)$ respectively:

$$(a+b)_{n\geq 0} \leftrightarrows f(x) + g(x).$$

3.5 The Right-Shift Rule

For a sequence $(a_n)_{n\geq 0}$ with an associated generating function $f(x)$, we can add k in $\mathbb{N}$ leading zeroes by multiplying the sequence by x_k:

$$0, \ldots, 0, a_0, a_1, \ldots \leftrightarrows x^k f(x).$$

3.6 The Differentiation Rule

For a sequence $(a_n)_{n\geq 0}$ with an associated generating function $f(x)$, we have that:

$$a_1, 2a_2, 3a_3, \ldots \leftrightarrows \frac{d}{dx} f(x).$$

So, each element in the sequence is multiplied by its index and left-shifted by one, with the farthest left term (the constant) removed.

3.7 The Convolution Rule

For the sequences $(a_n)_{n\geq 0}, (b_m)_{m\geq 0}$ with associated generating functions $f(x), g(x)$ respectively. We have that:

$$c_0, c_1, c_2, \ldots \leftrightarrows f(x) \cdot g(x),$$

where:

$$c_n := \sum_{i=0}^{n} a_i b_{n-i} = a_0 b_n + a_1 b_{n-1} + \cdots + a_{n-1} b_1 + a_n b_0.$$