# Group Theory Notes

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*These notes are not necessarily correct, consistent, representative of the course as it stands today or, rigorous. Any result of the above is not the author's fault.*

## 0 Notation

We commonly deal with the following concepts in Group Theory which I will abbreviate as follows for brevity:

| Term | Notation |
|---:|:---:|
| Additive identity of set $X$ | $0_X$ |
| Multiplicative identity of a set $X$ | $1_X$ |
| For a field $F$, $(F \backslash \{0_F\}, \times)$ | $F^*$ |
| (invertible $n \times n$ matrices on $F, \times$) | $GL_n(F)$ |

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# 1 The Fundamentals

## 1.1 Binary Operations

A binary operation on a set $X$ is a map $X \times X \to X$.

Take a binary operation $*$ on a set $X$, we say that $*$ is associative if for all $x, y, z$ in $X$:

$$x * (y * z) = (x * y) * z.$$

Furthermore, we say $e$ in $X$ is an identity element of $*$ if for all $x$ in $X$:

$$e * x = x * e,$$

and we say that $y$ in $X$ is the inverse to $x$ if $x * y$ and $y * x$ are both identities of $*$.

## 1.2 Groups

A group $(G, *)$ is a non-empty set $G$ combined with a binary operation $*$ such that:

- $*$ is associative,
- $G$ contains an identity for $*$,
- for each element in $G$, there exists some inverse in $G$ with respect to $*$.

### 1.2.1 Symmetric Groups

For a set $X$, the set of bijections $X \to X$ is a group under function composition denoted by $\mathrm{Sym}(X)$. We typically write $\mathrm{Sym}(\{1, 2, \ldots, n\})$ as $S_n$.

### 1.2.2 Cyclic Groups

If we consider a regular $n$-gon $P_n$, we take rotations of $\frac{2\pi}{n}$ radians about the centre to be $r$ and can define:

$$C_n = \{e, r, r^2, \ldots, r^{n-1}\},$$

to be the group of rotational symmetries of $P_n$, the cyclic group on $P_n$.

### 1.2.3 Dihedral Groups

If we consider again, a regular $n$-gon $P_n$ and take:

$$\begin{aligned} r &= \text{a rotation of } \frac{2\pi}{n} \text{ radians about the centre,} \\ s &= \text{reflection in some fixed line of symmetry,} \end{aligned}$$

then we have that:

$$\mathrm{Sym}(P_n) = \{e, r, r^2, \ldots, r^{n-1}, s, rs, r^2s, \ldots, r^{n-1}s\},$$

called the dihedral group, denoted by $D_{2n}$.

### 1.2.4 The Infinite Cyclic/Dihedral Group

A map $\varphi$ from $\mathbb{Z} \to \mathbb{Z}$ is a symmetry if for some $n$ and $m$ in $\mathbb{Z}$:

$$|\varphi(m) - \varphi(n)| = |m - n|.$$

Taking $r$ to be the symmetry $n \mapsto n + 1$, we can define the infinite cyclic group:

$$C_\infty = \{\ldots, r^{-2}, r^{-1}, e, r, r^2, \ldots\}.$$

Taking $s$ to be the symmetry $n \mapsto -n$, we can define the infinite dihedral group:

$$D_\infty = \{\ldots, r^{-2}, r^{-1}, e, r, r^2, \ldots, r^{-2}s, r^{-1}s, s, rs, r^2s\}.$$