Combinatorics Notes

paraphrased by Tyler Wright

An important note, these notes are absolutely ***NOT*** *guaranteed to be correct, representative of the course, or rigorous. Any result of this is not the author's fault.*

1 Counting Techniques

1.1 The Bijection Rule

For n in $\mathbb{N}$, we define $[n] := \{1, 2, \ldots, n\}$.

For a given set X, if there exists a bijective function $f : [n] \to X$ for some n in $\mathbb{N}$, X has n elements (or rather $|X| = n$).

This can also be achieved by listing out the elements of $X = \{x_1, x_2, \ldots, x_n\}$ as we can use $f : [n] \to X$ where i maps to x_i.

1.2 The Addition Rule

We can count the amount of elements in a given set X by splitting X into disjoint sets, counting them, and adding the results.

For n in $\mathbb{N}$, and $X_1, \ldots, X_n$ pairwise disjoint sets:

$$\left| \bigcup_{i=1}^{n} X_i \right| = \sum_{i=1}^{n} |X_i|.$$

For a set of sets A, pairwise disjoint means for two given sets in A, they are either disjoint or equal.

1.3 The Multiplication Rule

If a counting problem can be split into a number of stages, we can use the product of the number of choices at each stage to find the total number of outcomes.

For example, if we want to find how many three digit numbers there are, we can consider it as choosing three digits. We can choose $1, 2, \ldots, 9$ for the first digit and $0, 1, \ldots, 9$ for the rest so we get $9 \cdot 10^2$ possibilities.

1.4 Inclusion-Exclusion Principle

For n in $\mathbb{N}$, and $X_1, \dots, X_n$ sets:

$$\begin{aligned}\left|\bigcup_{i=1}^{n} X_i\right| &= \sum_{i=1}^{n} |X_i| \\ &- \sum_{i_1 \neq i_2} |X_{i_1} \cap X_{i_2}| \\ &+ \sum_{i_1 \neq i_2 \neq i_3} |X_{i_1} \cap X_{i_2} \cap X i_3| \\ &\dots\end{aligned}$$

Essentially, this says that the size of the union of some finite number of sets is the sum of their sizes, minus the sum of their ***paired*** *intersections, plus the sum of the intersections of* ***trios****, etc.*

1.5 The Factorial

For n in $\mathbb{N}$ we can define the factorial $n!$:

$$n! := \begin{cases} 1 & \text{n = 0} \\ \prod_{i=1}^{n}(i) & \text{otherwise.} \end{cases}$$

For k in $\mathbb{N}$ we can further define $(n)_k$:

$$(n)_k := \frac{n!}{(n-k)!} = n(n-1)(n-2)\cdots(n-k+1).$$

This can be though of as the factorial with k *elements (starting at* n*). So,* $(n)_n = n!$*,* $(n)_1 = n$*, etc.*

1.6 The Binomial Coefficient

For n, k in $\mathbb{N}$, we can define the binomial coefficient:

$$\binom{n}{k} := \frac{n!}{k!(n-k)!} = \frac{(n)_k}{k!}.$$

This is the number of ways of choosing k-element subsets from an n-element set. Furthermore, we have:

$$\binom{n}{k} = \binom{n}{n-k},$$

as choosing k elements is equivalent to choosing $n-k$ elements to remove.

There are some notes to be made on the definition:

- $\binom{n}{k} = 0$ if $k > n$
- $\binom{n}{0} = \binom{n}{n} = 1$
- $\binom{n}{k} \geq 0$

1.7 Pascal's Identity

Say we are selecting k elements from an n-element set (unordered, without repeats). We will see that there are $\binom{n}{k}$ possibilities. If we fix an element in the set, we can either include said element in our selection or exclude it giving $\binom{n-1}{k-1}$ and $\binom{n-1}{k}$ possibilities respectively. Thus:

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

1.8 The Binomial Theorem

By performing induction on Pascal's identity, we can see that for a, b in $\mathbb{C}$ and n in $\mathbb{N}$:

$$(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^i b^{n-i}.$$

Setting $a = b = 1$, we get $2^n = \sum_{i=0}^{n} \binom{n}{i}$.

1.9 The Pigeonhole Principle

For m, n, k in $\mathbb{N}$, if we have k objects being distributed into n boxes and $n > mk$ then one box must contain at least $k+1$ objects.

2 Selection

For this section, we will consider n, k in $\mathbb{N}$.

2.1 Ordered Selection with Repeats

As we select, we have n choices, and we select k times. Thus, by the Multiplication Rule, we get n^k outcomes.

2.2 Ordered Selection without Repeats

As we select, the amount of choices we have decreases by one each time. We start with n choices and select k times. Thus, by the Multiplication Rule, we get $n(n-1)\cdots(n-k+1) = (n)_k$ outcomes.

2.3 Unordered Selection with Repeats

Let the set we are selecting from be $\{x_1, \ldots, x_n\}$. In this case, any solution can be aggregated into a list indicating how many times the i^{th} element was selected (for some i in $[n]$). For example, if we select x_1 three times and x_2 five times, the outcome would be of the form $\{3, 5, \ldots\}$.

It can be seen that for each of these solutions, the sum of the elements in the set must equal k. We can construct a solution by starting with a set of all zeroes $\{0, 0, 0, \ldots\}$ and distributing k into the set. For example, for $n = 4$ and $k = 3$ the following are solutions:

$$\{1,1,1,0\} \text{ as } 1+1+1+0 = 3 = k,$$
$$\{0,2,0,1\} \text{ as } 0+2+0+1 = 3 = k,$$
$$\{3,0,0,0\} \text{ as } 3+0+0+0 = 3 = k.$$

These solutions correspond to $\{x_1, x_2, x_3\}, \{x_2, x_2, x_4\}, \{x_1, x_1, x_1\}$ respectively.

This distribution of k can be thought of as seperating k into n groups. For example, the solution $\{1, 1, 0, 1\}$ corresponds to:

$$\bullet | \bullet || \bullet .$$

The dots and dividers are identical respectively, and we have a total of k dots plus $n-1$ dividers equalling $k+n-1$ elements. We can choose where to place the dividers beforehand and then fill in the dots, thus we have:

$$\binom{k+n-1}{n-1}$$

choices.

2.4 Unordered Selection without Repeats

This is identical to to the ordered case but we divide by the number of permutations of the solutions as order does not matter. Thus, we get:

$$\frac{(n)_k}{k!} = \binom{n}{k}.$$

3 Generating Functions

3.1 Definition of a Generating Function

For a sequence $(a_n)_{n\geq 0}$, we can associate a **formal power series**:

$$f(x) = \sum_{k=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots .$$

We say $f(x)$ is the generating function of (a_n), or write:

$$\begin{aligned} a_0, a_1, a_2, \ldots &\leftrightarrows a_0 + a_1 x + a_2 x^2 + \cdots \\ (a_n)_{n\geq 0} &\leftrightarrows f(x). \end{aligned}$$

Note, however, that this doesn't imply that the series is convergent.

3.2 Generating Functions of Finite Sequences

For finite sequences (or rather, sequences with finitely many non-zero terms), we have that their generating functions can be written as polynomials.

3.3 The Scaling Rule

For a sequence $(a_n)_{n\geq 0}$ with an associated generating function $f(x)$ and c in $\mathbb{R}$:

$$(ca_n)_{n\geq 0} \leftrightarrows cf(x).$$

3.4 The Addition Rule

For the sequences $(a_n)_{n\geq 0}$, $(b_m)_{m\geq 0}$ with the associated generating functions $f(x), g(x)$ respectively:

$$(a+b)_{n\geq 0} \leftrightarrows f(x) + g(x).$$

3.5 The Right-Shift Rule

For a sequence $(a_n)_{n\geq 0}$ with an associated generating function $f(x)$, we can add k in $\mathbb{N}$ leading zeroes by multiplying the sequence by x_k:

$$0, \ldots, 0, a_0, a_1, \ldots \leftrightarrows x^k f(x).$$

3.6 The Differentiation Rule

For a sequence $(a_n)_{n\geq 0}$ with an associated generating function $f(x)$, we have that:

$$a_1, 2a_2, 3a_3, \ldots \leftrightarrows \frac{d}{dx} f(x).$$

So, each element in the sequence is multiplied by its index and left-shifted by one, with the farthest left term (the constant) removed.

3.7 The Convolution Rule

For the sequences $(a_n)_{n\geq 0}, (b_m)_{m\geq 0}$ with associated generating functions $f(x), g(x)$ respectively. We have that:

$$c_0, c_1, c_2, \ldots \leftrightarrows f(x) \cdot g(x),$$

where:

$$c_n := \sum_{i=0}^{n} a_i b_{n-i} = a_0 b_n + a_1 b_{n-1} + \cdots + a_{n-1} b_1 + a_n b_0.$$

3.8 The Negative Binomial Theorem

For all n in $\mathbb{N}$, we have that:

$$(1+x)^{-n} = \sum_{k=0}^{\infty} (-1)^k \binom{n+k-1}{n-1} x^k.$$

4 Combinatorial Designs

4.1 Definition of a Set System

For V a finite set, we let B be a collection of subsets of V. We call the pair (V, B) a set system with **ground set** V.

If for all elements in B, each element has the same cardinality k, we have that (V, B) is **k-uniform**.

We have that $B \subseteq \mathcal{P}(V)$ (that is, the powerset of V).

4.2 Definition of Block Design

For v, k, t, λ integers, we suppose:

$$v > k \geq t \geq 1, \qquad \lambda \geq 1.$$

A block design of type:

$$t - (v, k, \lambda),$$

is a set system (V, B) with the following properties:

- V has size v
- (V, B) is k-uniform
- Each t-element subset of V is contained in exactly λ 'blocks' (elements of B).

4.3 The Quantity of Blocks in a Block Design

For a block design of type $t - (v, k, \lambda)$, we have that the number of blocks b can be derived as follows:

$$b = \frac{\lambda\binom{v}{t}}{\binom{k}{t}}.$$

4.4 Definition of the Replication Number

In a block design of type $2 - (v, k, \lambda)$, every element lies in exactly r blocks where:

$$r(k-1) = \lambda(v-1), \qquad bk = vr.$$

r is the replication number.

4.5 Fisher's Inequality

For (V, B) a block design of type $2-(v, k, \lambda)$ with $v > k$, we have that:

$$|B| \geq |V|.$$

4.6 Definition of an Incidence Matrix

For a set system (V, B) with $|V| = v$ and $|B| = b$ we define the incidence matrix A as a matrix in $M_{v,b}$ where $A = (a_{ij})$ and:

$$a_{ij} = \begin{cases} 1 & \text{if element } i \text{ is in block } j \\ 0 & \text{otherwise.} \end{cases}$$

There are some important notes to be made:

- Each column contains k many '1's
- Each row contains r (the replication number) many '1's
- Each pair of rows contains λ many '1's in the same column

5 The Basics of Graph Theory

5.1 Definition of a Graph

A graph G is a set system (V, E) where the elements of E have size 2. Some definitions and facts follow from the definition:

- The elements of V are **vertices**
- The elements of E are called **edges**
- The size of V is often called the **order** of G
- G is a 2-uniform set with ground set V
- u, v in V are adjacent if u, v is in E.

5.2 Graph Isomorphisms

For two graphs $G_1 = (V_1, E_1)$, $G_2 = (V_2, E_2)$, we say that G_1 and G_2 are isomorphic ($G_1 \cong G_2$) if there exists a bijection $\phi : V_1 \to V_2$ such that for each pair of vertices u, v in V we have that:

$$\{u, v\} \in E_1 \Longleftrightarrow \{\phi(u), \phi(v)\} \in E_2.$$

5.3 Definition of Neighbourhood and Degree

For a graph $G = (V, E)$ the **neighbourhood** of v in V is the set of all adjacent vertices (denoted by $N_G(v)$). The neighbourhood of a set S is simply the union of the neighbourhoods of the elements of S. The **degree** is simply the size of $N_G(v)$ denoted by $\deg(v)$.

5.4 Notation for Minimum and Maximum Degree

For a graph $G = (V, E)$ we have that the following to represent minimum and maximum degree:

$$\delta(G) := \min\{\deg(v) : v \in V\}$$
$$\Delta(G) := \max\{\deg(v) : v \in V\}$$

5.5 Definition of Degree Sequence

For a graph $G = (V, E)$ a graph with $V = \{x_1, \dots, x_n\}$, where V is ordered such that $i \geq j$ implies $\deg(x_i) \geq \deg(x_j)$. The sequence $(d_k)_{k\in[n]}$ is defined as follows:

$$d_i = \deg(x_i).$$

5.6 The Handshake Lemma

For a graph $G = (V, E)$, we have that:

$$|E| = \frac{\sum_{v\in V} \deg(v)}{2}.$$

This is because each edge visits two vertices, so by counting the degree of each vertex we count each edge exactly twice.

5.7 Subgraphs

5.7.1 Definition of a subgraph

A graph $G' = (V', E')$ is a subgraph of $G = (V, E)$ if $V' \subseteq V$ and $E' \subseteq E$ such that for all e in E' we have that $e \subseteq V'$.

5.7.2 Definition of an induced subgraph

An induced subgraph generated of $G = (V, E)$ is a subgraph $G' = (V', E')$ where:

$$E' = \{\{u, v\} \in E \text{ such that } u, v \in V'\}.$$

Essentially, you generate an induced subgraph from a subset of the vertices of a graph by selecting edges that join vertices in the subset.

5.8 Walks

5.8.1 Definition of a walk

We have that a walk of length n, is a set of $n+1$ vertices connected by n edges.

5.8.2 Definition of a trail

A trail is a walk where no edges are repeated.

5.8.3 Definition of a path

A path is a walk where no vertices are repeated (barring the last one).

5.8.4 Definition of a circuit

A circuit is a walk where the first and last vertices are identical.

5.8.5 Definition of a cycle

A cycle is a path where the first and last vertices are identical.

5.8.6 Equivalence of walks and paths

If for some graph $G = (V, E)$ with u, v in V, we have that:

$$\text{There's a walk between } u \text{ and } v \iff \text{There's a path between } u \text{ and } v.$$

Thus, where there's a cycle, there's a circuit.

If we have that a graph G has an odd circuit, there's also an odd cycle (and the converse holds too).

5.8.7 Definition of connected graph

A graph is connected if there exists a path (or walk) between any two vertices in the graph.

5.9 Definition of a Component

A component of a graph G is a maximal connected induced subgraph of G. This means an induced subgraph of G that is connected but is not longer connected if a vertex is removed.

6 Euler Circuits

6.1 Definition of an Euler Circuit

An Euler circuit is a circuit in which each edge in a graph is traversed exactly once (or a trail which traverses every edge). As a consequence, each vertex is travelled at least once.

Graphs with Euler circuits are said to be **Eulerian**.

6.2 Conditions for an Euler Circuit

An Euler circuit in a graph G exists if and only if G is connected and each vertex in G has even degree.

7 Hamiltonian Cycles

7.1 Definition of a Hamiltonian Cycle

For a graph $G = (V, E)$ where $|V| = n$, a Hamiltonian cycle in G is a cycle of length n, meaning it visits each vertex exactly once.

Graphs with Hamiltonian cycles are said to be **Hamiltonian**.

7.2 Definition of a Hamiltonian Path

For a graph $G = (V, E)$ where $|V| = n$, a Hamiltonian path is a path of length $n-1$, meaning it visits each vertex at least once.

7.3 Dirac's Theorem

For a graph $G = (V, E)$ where $|V| \geq 3$:

$$\delta(G) \geq \frac{n}{2} \Rightarrow G \text{ is Hamiltonian.}$$

8 Bipartite Graphs

8.1 Definition of a Bipartite Graph

A graph $G = (V, E)$ is bipartite if V can be partitioned into two vertex sets V_1, V_2 such that each edge connects a vertex from V_1 to a vertex in V_2.

8.2 Characterisation of Bipartite Graphs

A graph is bipartite if and only if it contains no odd cycle.

8.3 The Handshake Lemma for Bipartite Graphs

We have that for $G = (V, E)$ a bipartite graph with bipartition V_1, V_2:

$$\sum_{v \in V_1} \deg(v) = \sum_{v \in V_2} \deg(v).$$

8.4 Hall's Marriage Problem

8.4.1 Definition of a Matching

For $G = (V, E)$ a bipartite graph with bipartition X, Y, a matching from X to Y is a set of edges:

$$M = \{(x, y) : x \in X, y \in Y\},$$

such that $f : X \to Y$ defined by:

$$f(x) := y \qquad \text{where } (x, y) \in M,$$

is injective.

In other words, $|M| = |X|$ *and each* y *in* Y *appears in at most one edge in* M*.*

8.4.2 Hall's Marriage Theorem

For $G = (V, E)$ a bipartite graph with bipartition X, Y:

$$\begin{gathered} G \text{ has a matching from } X \text{ to } Y \\ \iff \\ \text{For all } S \subseteq X, |N(S)| \geq |S|. \end{gathered}$$

We also have that if:

$$\min_{x \in X}\big[\deg(x)\big] \geq \min_{y \in Y}\big[\deg(y)\big],$$

then G has a matching from X to Y.