Linear Algebra 1 (TB2) Notes

paraphrased by Tyler Wright

*An important note, these notes are absolutely **NOT** guaranteed to be correct, representative of the course, or rigorous. Any result of this is not the author's fault.*

1 Vector Spaces, Fields, and Maps

1.1 Groups

A group is a *non-empty* set (G) paired with a *binary group operation* ($*$) denoted by $(G, *)$. The following properties hold for all groups (let $(G, *)$ be a group with elements f, g, h):

- **Associativity**: $f * (g * h) = (f * g) * h$
- **Identity**: $\exists e \in G : e * f = f * e = f$
- **Inverse**: $\exists x \in G : x * f = f * x = e$.

*A note, for a group $(G, *)$ with $g * h = h * g$ for all $g, h \in G$, this group is called **commutative** or **abelian**. However, it should be textitasised that this is **not** a necessary condition for a group.*

1.2 The Invertibility of Matrices

For a matrix $A \in M_{m,n}(\mathbb{F})$, the following are all **equivalent** statements:

- A is **invertible**
- $\det A = 0$
- The **rows** of A are **linearly independent**
- The **columns** of A are **linearly independent**
- The **reduced row echelon form** of A is the **identity**
- For all $\mathbf{b} \in \mathbb{F}^n$, $A\mathbf{x} = \mathbf{b}$ has a **unique solution**.

1.3 Fields

A field is a set (F) defined under multiplication and division with the following properties:

- **Associativity** under multiplication and division
- **Commutativity** under multiplication and division
- F contains an **identity** under multiplication and division
- All elements in F contain an **inverse** under addition and multiplication (except 0 under multiplication)
- The defined multiplication is **distributive** across the defined addition.

1.4 Vector Spaces

A group $(V, +_V)$ ($+_V$ denotes addition defined with respect to the set V as it can be ambigious in some cases) is a vector space over the field ($\mathbb{F}$) if the following holds (let $v, w \in V$, $\lambda, \mu \in \mathbb{F}$):

- $(V, +_V)$ is **abelian**
- V is **closed under multiplication** with elements in $\mathbb{F}$
- $\lambda(v +_V w) = \lambda v + \lambda w$
- $(\lambda + \mu)v = \lambda v +_V \mu v$
- $(\lambda\mu)v = \lambda(\mu v)$
- $fv = v$ where f is the **multiplicative identity** of $\mathbb{F}$.

1.5 Subspaces

Let V be a vector space over $\mathbb{F}$, $U \subseteq V$ is a subspace if the following properties hold:

- U is **non-empty**
- U is **closed** under the **addition** defined by V
- U is **closed** under the **multiplication** defined by V.

Some notes on subspaces:

- Subspaces are vector spaces
- The intersection of subspaces is a subspace
- The span of any non-empty subset of a given vector space is a subspace.

1.6 Linear Maps

For V, W vector spaces over $\mathbb{F}$, the map $T : V \to W$ is called linear if the following properties hold (let $u, w \in V$, $\lambda \in \mathbb{F}$):

- $T(u + v) = T(u) + T(v)$
- $T(\lambda u) = \lambda T(u)$.

A note, for a linear map ($T : V \to W$), if $V = W$, T is sometimes referred to as a linear ***operator****. Also, composed linear maps are also linear maps.*

1.7 The Kernel and Image

For a linear map $(T : V \to W)$, the kernel is defined as follows:

$$\operatorname{Ker} T = \{v \in V : T(v) = 0\}.$$

The image is defined as follows:

$$\operatorname{Im} T = \{w \in W : \exists v \in V \text{ with } T(v) = w\}.$$

Some notes on linear maps (let $T : V \to W$ be a linear map):

- The kernel and image of T are subspaces of V and W respectively
- For $U \subseteq V$, $T(U)$ is also a subspace (but of W instead of V).

1.8 Bases and Dimension

1.8.1 Definition of linear independence

For V a vector space, with $S \subseteq V$, let $s_1, s_2, ... \in S$,

- S is linearly independent if $\sum_{n=1}^{|S|} \lambda_n s_n = 0 \iff \lambda_i = 0\ \forall i$
- S is linearly dependent if it's not linearly independent.

A result of linear dependence is that for a linear dependent set S, there exists $s \in S$ such that $\operatorname{span}(S) = \operatorname{span}(S \backslash \{s\})$.

A note, if S is linearly dependent, there's a vector in S such that it can be written as the sum of other vectors in S.

1.8.2 Definition of a basis

For a vector space V, we say $S \subseteq V$ is a basis of V if:

- S spans V
- S is linearly independent.

1.8.3 Properties of bases

Let V be a vector space:

- For $v \in V$, B a basis for V, v can be written uniquely as a linear combination of vectors in B
- V is finitely dimensional if $|B| < \infty$
- If V is finitely dimensional, there must exists a basis of V.

For V a vector space with $S \subseteq V$ a linearly independent set. S can be 'extended' to a basis of V. If S spans V, it's already a basis. If not, we add a vector from $V \setminus \operatorname{span} S$. We can do this iteratively until we have a basis.

1.8.4 Definition of dimension

For a vector space V with a basis B, the order of B is the dimension of V, all bases of V share the same order. This is denoted by $\dim V := |B|$.

1.8.5 Properties of dimension

Let V be a finite dimensional vector space with $U, S \subseteq V$ where U is a subspace:

- S is linearly independent $\Rightarrow |S| < \dim V$
- $\operatorname{span} S = V \Rightarrow |S| \geq dimV$
- $(\operatorname{span} S = V) \wedge (|S| = \dim V) \Rightarrow S$ is a basis of V.
- $\dim U \leq \dim V$
- $\dim U = \dim V \Rightarrow U = V$

1.9 Direct Sums