Linear Algebra 1 (TB2) Notes

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*An important note, these notes are absolutely **NOT** guaranteed to be correct, representative of the course, or rigorous. Any result of this is not the author's fault.*

1 Vector Spaces, Fields, and Maps

1.1 Groups

A group is a *non-empty* set (G) paired with a *binary group operation* ($*$) denoted by $(G, *)$. The following properties hold for all groups (let $(G, *)$ be a group with elements f, g, h):

- **Associativity**: $f * (g * h) = (f * g) * h$
- **Identity**: $\exists e \in G : e * f = f * e = f$
- **Inverse**: $\exists x \in G : x * f = f * x = e$.

*A note, for a group $(G, *)$ with $g * h = h * g$ for all $g, h \in G$, this group is called **commutative** or **abelian**. However, it should be textitasised that this is **not** a necessary condition for a group.*

1.2 The Invertibility of Matrices

For a matrix $A \in M_{m,n}(\mathbb{F})$, the following are all **equivalent** statements:

- A is **invertible**
- $\det A = 0$
- The **rows** of A are **linearly independent**
- The **columns** of A are **linearly independent**
- The **reduced row echelon form** of A is the **identity**
- For all $\mathbf{b} \in \mathbb{F}^n$, $A\mathbf{x} = \mathbf{b}$ has a **unique solution**.

1.3 Fields

A field is a set (F) defined under multiplication and division with the following properties:

- **Associativity** under multiplication and division
- **Commutativity** under multiplication and division
- F contains an **identity** under multiplication and division
- All elements in F contain an **inverse** under addition and multiplication (except 0 under multiplication)
- The defined multiplication is **distributive** across the defined addition.

1.4 Vector Spaces

A group $(V, +_V)$ ($+_V$ denotes addition defined with respect to the set V as it can be ambigious in some cases) is a vector space over the field ($\mathbb{F}$) if the following holds (let $v, w \in V$, $\lambda, \mu \in \mathbb{F}$):

- $(V, +_V)$ is **abelian**
- V is **closed under multiplication** with elements in $\mathbb{F}$
- $\lambda(v +_V w) = \lambda v + \lambda w$
- $(\lambda + \mu)v = \lambda v +_V \mu v$
- $(\lambda\mu)v = \lambda(\mu v)$
- $fv = v$ where f is the **multiplicative identity** of $\mathbb{F}$.

1.5 Subspaces

Let V be a vector space over $\mathbb{F}$, $U \subseteq V$ is a subspace if the following properties hold:

- U is **non-empty**
- U is **closed** under the **addition** defined by V
- U is **closed** under the **multiplication** defined by V.

Some notes on subspaces:

- Subspaces are vector spaces
- The intersection of subspaces is a subspace
- The span of any non-empty subset of a given vector space is a subspace.

1.6 Linear Maps

For V, W vector spaces over $\mathbb{F}$, the map $T : V \to W$ is called linear if the following properties hold (let $u, w \in V$, $\lambda \in \mathbb{F}$):

- $T(u + v) = T(u) + T(v)$
- $T(\lambda u) = \lambda T(u)$.

A note, for a linear map ($T : V \to W$), if $V = W$, T is sometimes referred to as a linear ***operator****. Also, composed linear maps are also linear maps.*

1.7 The Kernel and Image

For a linear map ($T : V \to W$), the kernel is defined as follows:

$$\operatorname{Ker} T = \{v \in V : T(v) = 0\}.$$

The image is defined as follows:

$$\operatorname{Im} T = \{w \in W : \exists v \in V \text{ with } T(v) = w\}.$$

Some notes on linear maps (let $T : V \to W$ be a linear map):

- The kernel and image of T are subspaces of V and W respectively
- For $U \subseteq V$, $T(U)$ is also a subspace (but of W instead of V).

1.8 Bases and Dimension

1.8.1 Definition of linear independence

For V a vector space, with $S \subseteq V$, let $s_1, s_2, ... \in S$,

- S is linearly independent if $\sum_{n=1}^{|S|} \lambda_n s_n = 0 \iff \lambda_i = 0 \ \forall i$
- S is linearly dependent if it's not linearly independent.

A result of linear dependence is that for a linear dependent set S, there exists $s \in S$ such that $\operatorname{span}(S) = \operatorname{span}(S \backslash \{s\})$.

A note, if S is linearly dependent, there's a vector in S such that it can be written as the sum of other vectors in S.

1.8.2 Definition of a basis

For a vector space V, we say $S \subseteq V$ is a basis of V if:

- S spans V
- S is linearly independent.

1.8.3 Properties of bases

Let V be a vector space:

- For $v \in V$, B a basis for V, v can be written uniquely as a linear combination of vectors in B
- V is finitely dimensional if $|B| < \infty$
- If V is finitely dimensional, there must exists a basis of V.

For V a vector space with $S \subseteq V$ a linearly independent set. S can be 'extended' to a basis of V. If S spans V, it's already a basis. If not, we add a vector from $V \setminus \operatorname{span} S$. We can do this iteratively until we have a basis.

1.8.4 Definition of dimension

For a vector space V with a basis B, the order of B is the dimension of V, all bases of V share the same order. This is denoted by $\dim V := |B|$.

1.8.5 Properties of dimension

Let V be a finite dimensional vector space with $U, S \subseteq V$ where U is a subspace:

- S is linearly independent $\Rightarrow |S| < \dim V$
- $\operatorname{span} S = V \Rightarrow |S| \geq dimV$
- $(\operatorname{span} S = V) \wedge (|S| = \dim V) \Rightarrow S$ is a basis of V.
- $\dim U \leq \dim V$
- $\dim U = \dim V \Rightarrow U = V$

1.9 Direct Sums

1.9.1 Definition of a sum

For V a vector space over $\mathbb{F}$ with $U, W \subseteq V$ subspaces, we define their addition as follows:

$$U + W = \{u + w : u \in U, w \in W\}.$$

1.9.2 Definition of a direct sum

For V a vector space over $\mathbb{F}$ with $U, W \subseteq V$ subspaces satisfying $U \cap W = \{0\}$, the addition of U and W $(U + W)$ is called a direct sum denoted by:

$$U \oplus W.$$

So, when subspaces don't intersect, their addition is called a direct sum as they are disjoint.

1.9.3 Decomposition of vector spaces

For V a vector space over $\mathbb{F}$ with $U, W \subseteq V$ subspaces satisfying $U \cap W = \{0\}$, we have that:

$$\forall v \in U \oplus W, v = u + w (\text{for some } u \in U, w \in W).$$

1.9.4 Dimension of direct summed subspaces

For V a vector space over $\mathbb{F}$ with $U, W \subseteq V$ finite dimensional subspaces satisfying $U \cap W = \{0\}$:

$$\dim(U \oplus W) = \dim(U) + \dim(W)$$

1.9.5 Complements of subspaces

For V a finite dimensional vector space over $\mathbb{F}$ with $U \subseteq V$ a subspace, we have that there exists $W \subseteq V$ a subspaces such that:

- $U \cap W = \{0\}$
- $U \oplus W = V$,

this is the complement of U in V.

1.10 The Rank-Nullity Theorem

1.10.1 Definition of rank and nullity

For V, W vector spaces over $\mathbb{F}$ and $T : V \to W$ a linear map, we define:

- **Rank**: $\operatorname{rank}(T) = \dim(\operatorname{Im}(T))$
- **Nullity**: $\operatorname{nullity}(T) = \dim(\operatorname{Ker}(T))$.

1.10.2 The rank-nullity theorem

For V, W finite dimensional vector spaces over $\mathbb{F}$ and $T : V \to W$ a linear map, we can say:

$$\operatorname{rank}(T) + \operatorname{nullity}(T) = \dim(V).$$

1.11 Injectivity and Surjectivity

For V, W vector spaces over $\mathbb{F}$ and $T : V \to W$ a linear map, we can say:

- T injective $\Leftrightarrow \operatorname{nullity}(T) = 0$
- T surjective $\Leftrightarrow \operatorname{rank}(T) = \dim(W)$
- T injective and $S \subseteq V$ linearly independent $\Rightarrow T(S) \subseteq W$ is linealy independent
- T surjective and $S \subseteq V$ spans $V \Rightarrow T(S)$ spans W
- $\dim(W) > \dim(V) \Rightarrow T$ is not surjective (you can't have surjective maps from 2D to 3D)
- $\dim(W) < \dim(V) \Rightarrow T$ is not injective
- $\dim(W) = \dim(V) \Rightarrow$ means injectivity and surjectivity imply each other (you can't have one without the other).

1.12 Projections

1.12.1 Definition of a projection

For a vector space V, $P : V \to V$ a linear map, we say P is a projection if $P^2 = P$.

1.12.2 Relation to the rank-nullity theorem

For a finite dimensional vector space V, $P : V \to V$ a projection, we have:

$$V = \operatorname{Ker}(P) \oplus \operatorname{Im}(P)$$

1.12.3 The decomposition projection

For V a vector space over $\mathbb{F}$ with $U, W \subseteq V$ subspaces satisfying $U \cap W = \{0\}$, we can define a projection as follows:

$$P(v) = u \text{ where } v = u + w \text{ for some } u \in U, w \in W.$$

1.13 Isomorphisms

1.13.1 Definition of an isomorphism

An isomorphism is a bijective linear map. It's domain and codomain are called isomorphic.

1.13.2 Dimension of the domain and codomain

For two finite dimensional vector spaces V, W:

$$\exists T : V \to W \text{ an isomorphism } \Leftrightarrow \dim(V) = \dim(W)$$

1.14 Change of Bases

1.14.1 Method of changing basis

For V a vector space over $\mathbb{F}$, with $A, B \subseteq V$ bases, we can define a matrix to convert between these bases $C_{AB} = (c_{ij})$:

$$\begin{gathered}
C_{AB} \text{ converts from } B \text{ to } A \text{ so we write } A \text{ in terms of } B\text{:} \\
\text{Let } A = \{a_1, a_2, \ldots, a_n\} \\
\text{Let } B = \{b_1, b_2, \ldots, b_n\} \\
\\
\text{We have:} \\
a_1 = c_{11}b_1 + c_{21}b_2 + \cdots + c_{n1}b_n \\
a_2 = c_{12}b_1 + c_{22}b_2 + \cdots + c_{n2}b_n \\
\cdots \\
a_n = c_{1n}b_1 + c_{2n}b_2 + \cdots + c_{nn}b_n \\
\\
\text{Leading to the matrix (note the transpose):} \\
C_{AB} = \begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n1} & c_{n2} & \cdots & c_{nn} \end{pmatrix}
\end{gathered}$$

1.14.2 Properties of the change of basis matrix

For A, B, X bases of a vector space V:

- $C_{AA} = I$ (the identity)
- $C_{AB} = C_{BA}^{-1}$
- $C_{AX}C_{XB} = C_{AB}$

1.14.3 Example of change of basis

$$\text{Take } V = \mathbb{R}^2$$
$$\text{Let } A = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$$
$$\text{Let } B = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\}$$

For C_{AB} we write A in terms of B :

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = (1/2) \begin{pmatrix} 1 \\ 1 \end{pmatrix} + (-1/2) \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$
$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} = (1/2) \begin{pmatrix} 1 \\ 1 \end{pmatrix} + (1/2) \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

So, after transposing, we get:

$$C_{AB} = \begin{pmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{pmatrix}$$

You can check for yourself that:

$$C_{AB}(b_1) = a_1$$
$$C_{AB}(b_2) = a_2$$

Or rather:

$$C_{AB} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$
$$C_{AB} \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

For C_{BA} we write B in terms of A :

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} = (1) \begin{pmatrix} 1 \\ 0 \end{pmatrix} + (1) \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$
$$\begin{pmatrix} -1 \\ 1 \end{pmatrix} = (-1) \begin{pmatrix} 1 \\ 0 \end{pmatrix} + (1) \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

So, after transposing, we get:

$$C_{BA} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

You can check for yourself that:

$$C_{BA}(a_1) = b_1$$
$$C_{BA}(a_2) = b_2$$

Or rather:

$$C_{BA} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$
$$C_{BA} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

1.15 Linear Maps and Matrices

1.15.1 Definition of matrices of linear maps

For V, W vector spaces over $\mathbb{F}$ with $\dim(V) = n$ and $\dim(W) = m$ and $T : V \to W$ a linear map. For each choice of basis:

- $A = \{a_1, a_2, \dots, a_n\} \subseteq V$
- $B = \{b_1, b_2, \dots, b_m\} \subseteq W$,

we can associate a matrix to T:

$$M_{BA}(T) = (t_{ij}) \in M_{m,n}(\mathbb{F}),$$

with each t_{ij} defined as:

$$\begin{aligned}
T(a_1) &= t_{11}b_1 + t_{21}b_2 + \cdots + t_{m1}b_m \\
T(a_2) &= t_{12}b_1 + t_{22}b_2 + \cdots + t_{m2}b_m \\
&\cdots \\
T(a_n) &= t_{1n}b_1 + t_{2n}b_2 + \cdots + t_{mn}b_m.
\end{aligned}$$

Similarly to the change of basis matrices, note the transpose of the values.

1.15.2 Example of matrices of linear maps

Define the following:

$$\begin{aligned}
&A = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\} \subseteq \mathbb{R}^2 \\
&B = \{1\} \subseteq \mathbb{R} \\
&T : \mathbb{R}^2 \to \mathbb{R};\ \begin{pmatrix} x \\ y \end{pmatrix} \mapsto 2x
\end{aligned}$$

Since we are mapping from $\mathbb{R}^2$ to $\mathbb{R}$, our matrix will map from the basis A to the basis B:

$$M_{BA}(T) = (t_{ij}) \in M_{1,2}(\mathbb{R}).$$

So, we write $T(A)$ in terms of B:

$$\left.\begin{aligned}
T\begin{pmatrix} 1 \\ 0 \end{pmatrix} &= 2 = 2(1) \\
T\begin{pmatrix} 0 \\ 1 \end{pmatrix} &= 0 = 0(1)
\end{aligned}\right\} M_{BA}(T) = \begin{pmatrix} 2 & 0 \end{pmatrix}$$

1.15.3 Composition of matrices of linear maps

For U, V, W vector spaces over $\mathbb{F}$, $S : U \to V$, $T : V \to W$ linear maps, let $A \subseteq U$, $B \subseteq V$, and $C \subseteq W$ be bases. We have:

$$M_{CA}(T \circ S) = M_{CB}(T) M_{BA}(S).$$

1.15.4 Change of basis for matrices of linear maps

For V, W vector spaces over $\mathbb{F}$, $T : V \to W$ a linear map, let $A, A' \subseteq V$ and $B, B' \subseteq W$ be bases. We have:

$$M_{B'A'}(T) = C_{B'B} M_{BA}(T) C_{AA'}.$$

1.15.5 Matrices of linear maps and the determinant

For V a vector space with $T : V \to V$ a linear map:

- For any choice of basis B, $\det(M_{BB}(T))$ doesn't change so we define $\det(T) = \det(M_{BB}(T))$
- If V is finite dimensional, T is an isomorphism if $\det(T) \neq 0$.

2 Eigenvalues and Eigenvectors

2.1 Definition of an Eigenvalue and Eigenvector

For a vector space V over $\mathbb{F}$ and $T : V \to V$ a linear map, if we have v in V such that $v \neq 0$ and $T(v) = \lambda v$ we say v is an eigenvector with eigenvalue λ.

2.2 Eigenvector Bases and Matrices of Linear Maps

For a vector space V over $\mathbb{F}$ with dimension n and $T : V \to V$ a linear map, if there exists $B = \{v_1, \dots, v_n\}$ a basis for V of eigenvectors of T with eigenvalues $\{\lambda_1, \dots, \lambda_n\}$ then:

$$M_{BB}(T) = \mathrm{diag}(\lambda_1, \dots, \lambda_n).$$