Algorithms Notes

paraphrased by Tyler Wright

*An important note, these notes are absolutely **NOT** guaranteed to be correct, representative of the course, or rigorous. Any result of this is not the author's fault.*

1 Bounding

1.1 Racetrack Principle

For $f, g : \mathbb{N} \to \mathbb{N}$ functions, n, k in $\mathbb{N}$ we have that:

$$\left.\begin{array}{l} f(k) \geq g(k) \\ f'(n) \geq g'(n) \quad (\forall n \geq k) \end{array}\right\} \Rightarrow f(n) \geq g(n) \quad (\forall n \geq k)$$

If a function f is greater than another function g at a value k and has a greater gradient for all values after and including k, f is greater than g for all values after and including k.

1.2 Big O Notation

1.2.1 Definition of the big O notation

For $g : \mathbb{N} \to \mathbb{N}$ a function, $0(g)$ is a set of functions $f : \mathbb{N} \to \mathbb{N}$ such that each for f in $O(g)$:

$$\exists\, c \in \mathbb{R}, n_0 \in \mathbb{N} \text{ such that } \forall n \in N,$$
$$(n \geq n_0) \Rightarrow (0 \leq f(n) \leq cg(n)).$$

1.2.2 The big O notation under multiplication

For $f_1, f_2, g_1, g_2 : \mathbb{N} \to \mathbb{N}$ functions where:

- $f_1 \in O(g_1)$
- $f_2 \in O(g_2)$,

we have that:

- $f_1 + f_2$ is in $O(g_1 + g_2)$
- $f_1 \cdot f_2$ is in $O(g_1 \cdot g_2)$.

1.2.3 Closure of the big O notation

For $g : \mathbb{N} \to \mathbb{N}$ a function, $O(g)$ is closed under addition (this follows from the above).

1.2.4 Polynomials and the big O notation

For $p : \mathbb{N} \to \mathbb{N}$ a polynomial of degree k, p is in $O(n^k)$.

1.3 Θ Notation

1.3.1 Definition of the Θ notation

For $g : \mathbb{N} \to \mathbb{N}$ a function, $\Theta(g)$ is a set of functions $f : \mathbb{N} \to \mathbb{N}$ such that each for f in $\Theta(g)$:

$$\exists\, c_0, c_1 \in \mathbb{R}, n_0 \in \mathbb{N} \text{ such that } \forall n \in N,$$
$$(n \geq n_0) \Rightarrow (0 \leq c_1 g(n) \leq f(n) \leq c_2 g(n)).$$

f is sandwiched by multiples of g.

1.3.2 Equivalency of the Θ notation

For $f, g : \mathbb{N} \to \mathbb{N}$ functions:

$$f \in \Theta(g) \iff g \in \Theta(f).$$

1.3.3 Θ and O notation

For $f, g : \mathbb{N} \to \mathbb{N}$ functions:

$$f \in \Theta(g) \iff f \in O(g).$$

Which also means $g \in O(f)$ by the above equivalency.

1.3.4 Definition of the Ω notation

For $g : \mathbb{N} \to \mathbb{N}$ a function, $\Omega(g)$ is a set of functions $f : \mathbb{N} \to \mathbb{N}$ such that each for f in $\Omega(g)$:

$$\exists\, c \in \mathbb{R}, n_0 \in \mathbb{N} \text{ such that } \forall n \in N,$$
$$(n \geq n_0) \Rightarrow (0 \leq c g(n) \leq f(n)).$$

1.3.5 Equivalency of the Ω notation

For $f, g : \mathbb{N} \to \mathbb{N}$ functions:

$$f \in \Omega(g) \iff g \in O(f).$$