Analysis 1 (TB2) Notes

paraphrased by Tyler Wright

An important note, these notes are absolutely ***NOT*** *guaranteed to be correct, representative of the course, or rigorous. Any result of this is not the author's fault.*

1 Continuity

1.1 Continuous Functions

From Analysis 1A, we have that a function $f : A \to \mathbb{R}$ is continuous on A if:

$$\forall x \in A, \forall \epsilon > 0, \exists \delta > 0 \text{ such that } \forall y \in A, (|y - x| < \delta) \Rightarrow (|f(y) - f(x)| < \epsilon).$$

It's important to note that x is chosen given before we choose a δ. Thus, our choice for δ can depend on x as well as ϵ.

Uniform continuity requires that δ is independent of x.

A note, a function being continuous at a value (or set of values for that matter), it equivalent to saying that there exists a limit for the function at that value and that limit is the value of the function applied to that value.

1.2 Uniformly Continuous Functions

Uniform continuity is similar to continuity as we knew it in Analysis 1A. For a function $f : A \to \mathbb{R}$, f is uniformly continuous on A if:

$$\forall \epsilon > 0, \exists \delta > 0 \text{ such that } \forall x, y \in A, (|y - x| < \delta) \Rightarrow (|f(y) - f(x)| < \epsilon).$$

We can see that uniform continuity **implies** continuity but **not** vice versa.

A note, for uniform continuity, we are saying that given a value ϵ, we can always pick a distance (δ) such that if two values are within that distance of each other, the distance between the values after the function is applied to them will be less than ϵ. This is essentially testing for divergence to infinity at a value ($\frac{1}{x}$ is continuous but not uniformly continuous on $\mathbb{R}_{>0}$).

2 Convergence

We have the notion of convergence for sequences of real numbers from Analysis 1A, convergence in this section is similar but specifically for functions.

2.1 Pointwise Convergence

A sequence of functions $(f_n)_{n\in\mathbb{N}}$ from $A \to \mathbb{R}$ converges **pointwise** to the function f on A if:

$$\forall x \in A, \lim_{n\to\infty}(f_n(x)) = f(x).$$

f is called the **pointwise limit** of $(f_n)_{n\in\mathbb{N}}$.

A note, for $f_n : [0,1] \to [0,1]; x \to x^n$, $f : [0,1] \to [0,1]; x \to \delta_1(x)$, f_n converges pointwise to f.

2.2 Uniform Convergence

A sequence of functions $(f_n)_{n\in\mathbb{N}}$ from $A \to \mathbb{R}$ converges **uniformly** to the function f on A if:

$$\forall \epsilon > 0, \exists N \in \mathbb{N} \text{ such that } \forall x \in A, \forall n \in \mathbb{N}, (n \geq N) \Rightarrow (|f(x) - f_n(x)| < \epsilon).$$

For the same functions outlined in the note under pointwise convergence, we have that f_n does not converge uniformly to f. Let $\epsilon \in (0,1)$, $x \in [0,1)$ and suppose f_n is uniformly convergent to f,

$$\begin{aligned}
&|f_n(x) - f(x)| = |x^n| < \epsilon \\
&\Rightarrow 0 \leq x^n < \epsilon < 1 \\
&\Rightarrow 0 \leq x < \epsilon^{\frac{1}{n}} < 1 \\
&\Rightarrow \epsilon = 1 \text{ as } x \in [0,1).
\end{aligned}$$

This is a contradiction by the definition of ϵ. Thus, we have the result.

2.3 Weierstrass' Theorem

For $a, b \in \mathbb{R}$ with $a < b$, if a sequence of continuous functions on $[a,b]$, $(f_n)_{n\in\mathbb{N}}$ converges uniformly to f on $[a,b]$, f is continuous on $[a,b]$.

Basically, uniform convergence preserves continuity.

2.4 Supremum Norm

For $a, b \in \mathbb{R}$ with $a < b$, let $f : [a, b] \to \mathbb{R}$ be a bounded function. The supremum norm of f on $[a, b]$ is denoted by $||f||_{[a,b]}$ and is defined by:

$$||f||_{[a,b]} := \sup\{\|f(x)| : x \in [a, b]\}.$$

The supremum norm is simply just the furthest distance from zero reached by a function over a closed interval. By definition, it is a real number and $\exists x \in [a, b]$ such that $f(x)$ is the supremum norm.

There are a few key properties of the supremum norm, let a and b be as above and let $\lambda \in \mathbb{R}$, $f, g : [a, b] \to \mathbb{R}$ be bounded functions:

- $||f||_{[a,b]} > 0$
- $||f||_{[a,b]} = 0 \Leftrightarrow f = 0$ on $[a, b]$
- $||\lambda f||_{[a,b]} = |\lambda|||f||_{[a,b]}$
- $||f + g||_{[a,b]} = ||f||_{[a,b]} + ||g||_{[a,b]}$.

2.5 Cauchy Sequences of Functions

For $a, b \in \mathbb{R}$ with $a < b$, denote the set of continuous functions on $[a, b]$ by $C([a, b])$. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of functions in $C([a, b])$. We say $(f_n)_{n\in\mathbb{N}}$ is Cauchy if:

$$\forall \epsilon > 0, \exists N \in \mathbb{N} \text{ such that } \forall m, n \in \mathbb{N}, (m, n \geq N) \Rightarrow (||f_n - f_m||_{[a,b]} < \epsilon).$$

This obviously bears an extreme resemblance to the Cauchy sequences of Analysis 1A. Just replacing the sequences of reals with sequences of functions and the modulus with the supremum norm.

For each continuous function, there exists a Cauchy sequence such that the sequence converges uniformly to said function.