Linear Algebra 1 (TB2) Notes

paraphrased by Tyler Wright

1 Vector Spaces, Fields, and Maps

1.1 Groups

A group is a *non-empty* set (G) paired with a *binary group operation* $(*)$ denoted by $(G, *)$. The following properties hold for all groups (let $(G, *)$ be a group with elements f, g, h):

- **Associativity**: $f * (g * h) = (f * g) * h$
- **Identity**: $\exists e \in G : e * f = f * e = f$
- **Inverse**: $\exists x \in G : x * f = f * x = e$.

*A note, for a group $(G, *)$ with $g * h = h * g$ for all $g, h \in G$, this group is called **commutative** or **abelian**. However, it should be textitasised that this is **not** a necessary condition for a group.*

1.2 The Invertibility of Matrices

For a matrix $A \in M_{m,n}(\mathbb{F})$, the following are all **equivalent** statements:

- A is **invertible**
- $\det A \neq 0$
- The **rows** of A are **linearly independent**
- The **columns** of A are **linearly independent**
- The **reduced row echelon form** of A is the **identity**
- For all $\mathbf{b} \in \mathbb{F}^n$, $A\mathbf{x} = \mathbf{b}$ has a **unique solution**.

1.3 Fields

A field is a set (F) defined under multiplication and division with the following properties:

- **Associativity** under multiplication and division
- **Commutativity** under multiplication and division
- F contains an **identity** under multiplication and division
- All elements in F contain an **inverse** under addition and multiplication (except 0 under multiplication)
- The defined multiplication is **distributive** across the defined addition.

1.4 Vector Spaces

A group $(V, +_V)$ ($+_V$ denotes addition defined with respect to the set V as it can be ambigious in some cases) is a vector space over the field ($\mathbb{F}$) if the following holds (let $v, w \in V,\ \lambda, \mu \in \mathbb{F}$):

- $(V, +_V)$ is **abelian**
- V is **closed under multiplication** with elements in $\mathbb{F}$
- $\lambda(v +_V w) = \lambda v + \lambda w$
- $(\lambda + \mu)v = \lambda v +_V \mu v$
- $(\lambda\mu)v = \lambda(\mu v)$
- $fv = v$ where f is the **multiplicative identity** of $\mathbb{F}$.

1.5 Subspaces

Let V be a vector space over $\mathbb{F}$, $U \subseteq V$ is a subspace if the following properties hold:

- U is **non-empty**
- U is **closed** under the **addition** defined by V
- U is **closed** under the **multiplication** defined by V.

Some notes on subspaces:

- Subspaces are vector spaces
- The intersection of subspaces is a subspace
- The span of any non-empty subset of a given vector space is a subspace.

1.6 Linear Maps

For V, W vector spaces over $\mathbb{F}$, the map $T : V \to W$ is called linear if the following properties hold (let $u, w \in V,\ \lambda \in \mathbb{F}$):

- $T(u + v) = T(u) + T(v)$
- $T(\lambda u) = \lambda T(u)$.

A note, for a linear map ($T : V \to W$), if $V = W$, T is sometimes referred to as a linear ***operator****. Also, composed linear maps are also linear maps.*

1.7 The Kernel and Image

For a linear map ($T : V \to W$), the kernel is defined as follows:

$$\operatorname{Ker} T = \{v \in V : T(v) = 0\}.$$

The image is defined as follows:

$$\operatorname{Im} T = \{w \in W : \exists v \in V \text{ with } T(v) = w\}.$$

Some notes on linear maps (let $T : V \to W$ be a linear map):

- The kernel and image of T are subspaces of V and W respectively
- For $U \subseteq V$, $T(U)$ is also a subspace (but of W instead of V).

1.8 Bases and Dimension

1.8.1 Definition of linear independence

For V a vector space, with $S \subseteq V$, let $s_1, s_2, ... \in S$,

- S is linearly independent if $\sum_{n=1}^{|S|} \lambda_n s_n = 0 \iff \lambda_i = 0 \ \forall i$
- S is linearly dependent if it's not linearly independent.

A result of linear dependence is that for a linear dependent set S, there exists $s \in S$ such that $\text{span}(S) = \text{span}(S \backslash \{s\})$.

A note, if S is linearly dependent, there's a vector in S such that it can be written as the sum of other vectors in S.

1.8.2 Definition of a basis

For a vector space V, we say $S \subseteq V$ is a basis of V if:

- S spans V
- S is linearly independent.

1.8.3 Properties of bases

Let V be a vector space:

- For $v \in V$, B a basis for V, v can be written uniquely as a linear combination of vectors in B
- V is finitely dimensional if $|B| < \infty$
- If V is finitely dimensional, there must exists a basis of V.

For V a vector space with $S \subseteq V$ a linearly independent set. S can be 'extended' to a basis of V. If S spans V, it's already a basis. If not, we add a vector from $V \setminus \operatorname{span} S$. We can do this iteratively until we have a basis.

1.8.4 Definition of dimension

For a vector space V with a basis B, the order of B is the dimension of V, all bases of V share the same order. This is denoted by $\dim V := |B|$.

1.8.5 Properties of dimension

Let V be a finite dimensional vector space with $U, S \subseteq V$ where U is a subspace:

- S is linearly independent $\Rightarrow |S| < \dim V$
- $\operatorname{span} S = V \Rightarrow |S| \geq dimV$
- $(\operatorname{span} S = V) \wedge (|S| = \dim V) \Rightarrow S$ is a basis of V.
- $\dim U \leq \dim V$
- $\dim U = \dim V \Rightarrow U = V$

1.9 Direct Sums

1.9.1 Definition of a sum

For V a vector space over $\mathbb{F}$ with $U, W \subseteq V$ subspaces, we define their addition as follows:

$$U + W = \{u + w : u \in U, w \in W\}.$$

1.9.2 Definition of a direct sum

For V a vector space over $\mathbb{F}$ with $U, W \subseteq V$ subspaces satisfying $U \cap W = \{0\}$, the addition of U and W $(U + W)$ is called a direct sum denoted by:

$$U \oplus W.$$

So, when subspaces don't intersect, their addition is called a direct sum as they are disjoint.

1.9.3 Decomposition of vector spaces

For V a vector space over $\mathbb{F}$ with $U, W \subseteq V$ subspaces satisfying $U \cap W = \{0\}$, we have that:

$$\forall v \in U \oplus W, v = u + w (\text{for some } u \in U, w \in W).$$

1.9.4 Dimension of direct summed subspaces

For V a vector space over $\mathbb{F}$ with $U, W \subseteq V$ finite dimensional subspaces satisfying $U \cap W = \{0\}$:

$$\dim(U \oplus W) = \dim(U) + \dim(W)$$

1.9.5 Complements of subspaces

For V a finite dimensional vector space over $\mathbb{F}$ with $U \subseteq V$ a subspace, we have that there exists $W \subseteq V$ a subspaces such that:

- $U \cap W = \{0\}$
- $U \oplus W = V$,

this is the complement of U in V.

1.10 The Rank-Nullity Theorem

1.10.1 Definition of rank and nullity

For V, W vector spaces over $\mathbb{F}$ and $T : V \to W$ a linear map, we define:

- **Rank**: $\text{rank}(T) = \dim(\text{Im}(T))$
- **Nullity**: $\text{nullity}(T) = \dim(\text{Ker}(T))$.

1.10.2 The rank-nullity theorem

For V, W finite dimensional vector spaces over $\mathbb{F}$ and $T : V \to W$ a linear map, we can say:

$$\text{rank}(T) + \text{nullity}(T) = \dim(V).$$

1.11 Injectivity and Surjectivity

For V, W vector spaces over $\mathbb{F}$ and $T : V \to W$ a linear map, we can say:

- T injective $\Leftrightarrow$ nullity$(T) = 0$
- T surjective $\Leftrightarrow$ rank$(T) = \dim(W)$
- T injective and $S \subseteq V$ linearly independent $\Rightarrow T(S) \subseteq W$ is linealy independent
- T surjective and $S \subseteq V$ spans $V \Rightarrow T(S)$ spans W
- $\dim(W) > \dim(V) \Rightarrow T$ is not surjective (you can't have surjective maps from 2D to 3D)
- $\dim(W) < \dim(V) \Rightarrow T$ is not injective
- $\dim(W) = \dim(V) \Rightarrow$ means injectivity and surjectivity imply each other (you can't have one without the other).

1.12 Projections

1.12.1 Definition of a projection

For a vector space V, $P : V \to V$ a linear map, we say P is a projection if $P^2 = P$.

1.12.2 Relation to the rank-nullity theorem

For a finite dimensional vector space V, $P : V \to V$ a projection, we have:

$$V = \text{Ker}(P) \oplus \text{Im}(P)$$

1.12.3 The decomposition projection

For V a vector space over $\mathbb{F}$ with $U, W \subseteq V$ subspaces satisfying $U \cap W = \{0\}$, we can define a projection as follows:

$$P(v) = u \text{ where } v = u + w \text{ for some } u \in U, w \in W.$$

1.13 Isomorphisms

1.13.1 Definition of an isomorphism

An isomorphism is a bijective linear map. It's domain and codomain are called isomorphic.

1.13.2 Dimension of the domain and codomain

For two finite dimensional vector spaces V, W:

$$\exists T : V \to W \text{ an isomorphism } \Leftrightarrow \dim(V) = \dim(W)$$

1.14 Change of Bases

1.14.1 Method of changing basis

For V a vector space over $\mathbb{F}$, with $A, B \subseteq V$ bases, we can define a matrix to convert between these bases $C_{AB} = (c_{ij})$:

$$\begin{gathered} C_{AB} \text{ converts from } B \text{ to } A \text{ so we write } A \text{ in terms of } B\text{:} \\ \text{Let } A = \{a_1, a_2, \dots, a_n\} \\ \text{Let } B = \{b_1, b_2, \dots, b_n\} \\ \\ \text{We have:} \\ a_1 = c_{11}b_1 + c_{21}b_2 + \cdots + c_{n1}b_n \\ a_2 = c_{12}b_1 + c_{22}b_2 + \cdots + c_{n2}b_n \\ \cdots \\ a_n = c_{1n}b_1 + c_{2n}b_2 + \cdots + c_{nn}b_n \\ \\ \text{Leading to the matrix (note the transpose):} \\ C_{AB} = \begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n1} & c_{n2} & \cdots & c_{nn} \end{pmatrix} \end{gathered}$$

1.14.2 Properties of the change of basis matrix

For A, B, X bases of a vector space V:

- $C_{AA} = I$ (the identity)
- $C_{AB} = C_{BA}^{-1}$
- $C_{AX}C_{XB} = C_{AB}$

1.14.3 Example of change of basis

$$\text{Take } V = \mathbb{R}^2$$
$$\text{Let } A = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$$
$$\text{Let } B = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\}$$

For C_{AB} we write A in terms of B :

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = (1/2)\begin{pmatrix} 1 \\ 1 \end{pmatrix} + (-1/2)\begin{pmatrix} -1 \\ 1 \end{pmatrix}$$
$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} = (1/2)\begin{pmatrix} 1 \\ 1 \end{pmatrix} + (1/2)\begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

So, after transposing, we get:

$$C_{AB} = \begin{pmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{pmatrix}$$

You can check for yourself that:

$$C_{AB}(b_1) = a_1$$
$$C_{AB}(b_2) = a_2$$

Or rather:

$$C_{AB}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$
$$C_{AB}\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

For C_{BA} we write B in terms of A :

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} = (1)\begin{pmatrix} 1 \\ 0 \end{pmatrix} + (1)\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$
$$\begin{pmatrix} -1 \\ 1 \end{pmatrix} = (-1)\begin{pmatrix} 1 \\ 0 \end{pmatrix} + (1)\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

So, after transposing, we get:

$$C_{BA} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

You can check for yourself that:

$$C_{BA}(a_1) = b_1$$
$$C_{BA}(a_2) = b_2$$

Or rather:

$$C_{BA}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$
$$C_{BA}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

1.15 Linear Maps and Matrices

1.15.1 Definition of matrices of linear maps

For V, W vector spaces over $\mathbb{F}$ with $\dim(V) = n$ and $\dim(W) = m$ and $T : V \to W$ a linear map. For each choice of basis:

- $B = \{b_1, b_2, \ldots, b_m\} \subseteq V$
- $A = \{a_1, a_2, \ldots, a_n\} \subseteq W$,

we can associate a matrix to T (maps from V to W implying B to A):

$$M_{AB}(T) = (t_{ij}) \in M_{m,n}(\mathbb{F}),$$

with each t_{ij} defined as (write $T(B)$ in terms of A):

$$\begin{aligned}
T(b_1) &= t_{11}a_1 + t_{21}a_2 + \cdots + t_{m1}a_m \\
T(b_2) &= t_{12}a_1 + t_{22}a_2 + \cdots + t_{m2}a_m \\
&\cdots \\
T(b_n) &= t_{1n}a_1 + t_{2n}a_2 + \cdots + t_{mn}a_m.
\end{aligned}$$

Similarly to the change of basis matrices, note the transpose of the values.

1.15.2 Example of matrices of linear maps

Define the following:

$$\begin{aligned}
&B = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\} \subseteq \mathbb{R}^2 \\
&A = \{1\} \subseteq \mathbb{R} \\
&T : \mathbb{R}^2 \to \mathbb{R};\ \begin{pmatrix} x \\ y \end{pmatrix} \mapsto 2x
\end{aligned}$$

Since we are mapping from $\mathbb{R}^2$ to $\mathbb{R}$, our matrix will map from the basis B to the basis A:

$$M_{AB}(T) = (t_{ij}) \in M_{1,2}(\mathbb{R}).$$

So, we write $T(B)$ in terms of A:

$$\left.\begin{aligned}
T\begin{pmatrix} 1 \\ 0 \end{pmatrix} &= 2 = 2(1) \\
T\begin{pmatrix} 0 \\ 1 \end{pmatrix} &= 0 = 0(1)
\end{aligned}\right\} M_{AB}(T) = \begin{pmatrix} 2 & 0 \end{pmatrix}$$

1.15.3 Composition of matrices of linear maps

For U, V, W vector spaces over $\mathbb{F}$, $S : U \to V$, $T : V \to W$ linear maps, let $A \subseteq U$, $B \subseteq V$, and $C \subseteq W$ be bases. We have:

$$M_{CA}(T \circ S) = M_{CB}(T)M_{BA}(S).$$

1.15.4 Change of basis for matrices of linear maps

For V, W vector spaces over $\mathbb{F}$, $T : V \to W$ a linear map, let $A, A' \subseteq V$ and $B, B' \subseteq W$ be bases. We have:

$$M_{B'A'}(T) = C_{B'B}M_{BA}(T)C_{AA'}.$$

1.15.5 Matrices of linear maps and the determinant

For V a vector space with $T : V \to V$ a linear map:

- For any choice of basis B, $\det(M_{BB}(T))$ doesn't change so we define $\det(T) = \det(M_{BB}(T))$
- If V is finite dimensional, T is an isomorphism if $\det(T) \neq 0$.

2 Eigenvalues and Eigenvectors

2.1 Definition of an Eigenvalue and Eigenvector

For a vector space V over $\mathbb{F}$ and $T : V \to V$ a linear map, if we have v in V such that $v \neq 0$ and $T(v) = \lambda v$ we say v is an eigenvector with eigenvalue λ.

2.2 Eigenvector Bases and Matrices of Linear Maps

For a vector space V over $\mathbb{F}$ with dimension n and $T : V \to V$ a linear map, if there exists $B = \{v_1, \dots, v_n\}$ a basis for V of eigenvectors of T with eigenvalues $\{\lambda_1, \dots, \lambda_n\}$ then:

$$M_{BB}(T) = \mathrm{diag}(\lambda_1, \dots, \lambda_n).$$

2.3 Linear Independence of Eigenvectors

If we have two eigenvectors with different eigenvalues, they are linearly independent.

2.4 Characteristic Polynomials

2.4.1 Definition of a characteristic polynomial

For a vector space V with $T : V \to V$ a linear map, we define the characteristic polynomial P as a polynomial such that:

$$P(\lambda) = 0 \Rightarrow \lambda \text{ is an eigenvalue of } T.$$

The set of eigenvalues of T (and, equivalently, the set of roots of P) is called the spectrum of T ($\mathrm{spec}(T)$).

2.4.2 Derivation of the characteristic polynomial

For a finite dimensional vector space V over $\mathbb{F}$ with $T : V \to V$ a linear map, let λ be in $\mathbb{F}$:

$$\lambda \text{ is an eigenvalue of } T$$
$$\iff$$
$$\det(T - \lambda I) = 0.$$

So, we can define the characteristic polynomial P as follows:

$$P(\lambda) := \det(T - \lambda I).$$

2.4.3 Eigenspaces

For V a vector space over $\mathbb{F}$, an eigenspace for an eigenvalue λ in $\mathbb{F}$ of a linear map $T : V \to V$ is defined as:

$$V_\lambda := \mathrm{Ker}(T - \lambda I).$$

For $\mathrm{spec}(T) = \{\lambda_1, \ldots, \lambda_n\}$, we have that:

$$V = V_{\lambda_1} \oplus \cdots \oplus V_{\lambda_n}$$
$$\iff$$
$$T \text{ has a basis of eigenvectors.}$$

2.4.4 Calculating characteristic polynomials

In general, with a n-dimensional vector space V over $\mathbb{F}$, $T : V \to V$ a linear map, the characteristic polynomial P can be written as:

$$P(\lambda) = (-1)^n \lambda^n + (-1)^{n-1} \operatorname{trace}(T) \lambda^{n-1} + \cdots + \det(T).$$

This is very simple in the 2×2 case:

$$P(\lambda) = \lambda^2 - \operatorname{trace}(T)\lambda + \det(T).$$

It gets more complicated in the 3×3 case, consider M:

$$M = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}.$$

We calculate a value μ:

$$\mu = \begin{pmatrix} a & b \\ d & e \end{pmatrix} + \begin{pmatrix} e & f \\ h & i \end{pmatrix} + \begin{pmatrix} a & c \\ g & i \end{pmatrix}.$$

And we get the result:

$$P(\lambda) = -\lambda^3 + \operatorname{trace}(T)\lambda^2 - \mu\lambda + \det(T).$$

Similar calculation for 4×4 matrices and upwards become increasingly more complex.

The calculation of μ may seem daunting at first but it can be easily remembered as the 2×2 determinants of the top left, bottom right, and corners.

2.4.5 Roots of characteristic polynomials

We have that for a map T with spectrum $\operatorname{spec}(T) = \{\lambda_1, \dots, \lambda_k\}$, we can write the characteristic polynomial as follows:

$$\begin{aligned} P(\lambda) &= a_n \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_0 \\ &= (\lambda - \lambda_1)^{m_1} (\lambda - \lambda_2)^{m_2} \cdots (\lambda - \lambda_k)^{m_k} \end{aligned}$$

The m_i values are called the multiplicity of the roots. If we are taking complex roots the sum of the m_i values is equal to n. This is because there's always n complex roots (up to multiplicity) but not always a similar amount of real roots. This also means that there can never be more eigenvalues than the dimension of the vector space.

2.4.6 The characteristic polynomial and matrix properties

In general, with a n-dimensional vector space V over $\mathbb{F}$, $T : V \to V$ a linear map, the characteristic polynomial P can be written as:

$$P(\lambda) = (-1)^n \lambda^n + (-1)^{n-1} \operatorname{trace}(T) \lambda^{n-1} + \cdots + \det(T).$$

So, by the properties of polynomials, we know that:

$$\begin{aligned} \det(T) &= \text{ product of the roots of } P \\ \operatorname{trace}(T) &= \text{ sum of the roots of } P. \end{aligned}$$

As it's the characteristic polynomial, the roots are just the eigenvalues. Let $\operatorname{spec}(T) = \{\lambda_1, \ldots, \lambda_n\}$ (not necessarily distinct):

$$\begin{aligned} \det(T) &= \prod_{i=1}^{k} \lambda_k \\ \operatorname{trace}(T) &= \sum_{i=1}^{k} \lambda_k \end{aligned}$$

3 Inner Products

3.1 Definition of an Inner Product

For V a vector space over $\mathbb{C}$, an inner product on V is a map $\langle , \rangle : V \times V \to \mathbb{C}$ with the following properties:

- $\langle v, v \rangle \geq 0$
- $\langle v, v \rangle = 0 \Leftrightarrow v = 0$
- $\langle v, w \rangle = \overline{\langle w, v \rangle}$
- $\langle v, u + w \rangle = \langle v, u \rangle + \langle v, w \rangle$
- $\langle v, \lambda w \rangle = \lambda \langle v, w \rangle \Rightarrow \langle \lambda v, w \rangle = \overline{\lambda} \langle v, w \rangle$.

  When our values are real, we can just remove the conjugate bar.

3.2 Inner Product Spaces

A vector space paired with an inner product is called an inner product space. If it's over the real/complex numbers we call it a real/complex inner product space (respectively).

3.3 The Norm

For an inner product space $(V, \langle , \rangle)$, we define the norm by:

$$\|v\| := \sqrt{\langle v, v\rangle}$$

3.4 Properties of the Norm

For an inner product space $(V, \langle , \rangle)$:

- $|\langle v, w\rangle| \leq \|v\|\|w\|$
- $\|v\| = 0 \Leftrightarrow v = 0$
- $\|\lambda v\| = |\lambda|\|v\|$
- $\|v + w\| \leq \|v\| + \|w\|$.

3.5 Matrix Derivation from the Inner Product

For an inner product space $(V, \langle , \rangle)$, where $\dim(V) = n$ with $T : V \to V$ a linear map. If we have an orthonormal basis $B = \{v_1, \ldots, v_n\}$ we have:

$$M_{BB}(T) = (a_{ij}) = (\langle v_i, T(v_j)\rangle)$$

4 Orthogonality

4.1 Definition of Orthogonal Vectors and Subspaces

For an inner product space $(V, \langle , \rangle)$:

- v, w in V are orthogonal $\Leftrightarrow \langle v, w\rangle = 0$
- U, W subspaces of V are orthogonal $\Leftrightarrow$ u and w are orthogonal for all u in U, w in W.

4.2 Orthogonal Complements

For an inner product space $(V, \langle , \rangle)$ with $U \subseteq V$ a subspace, we define the orthogonal complement $U^\perp$ as follows:

$$U^\perp := \{v \in V : \langle v, u\rangle = 0, \forall u \in U\}.$$

A consequence of the definition is that $V = U \oplus U^\perp$.

4.3 Pythagoras' Theorem

For an inner product space $(V, \langle , \rangle)$ with v, u in V such that $\langle v, u\rangle = 0$:

$$\|v+u\|^2 = \|v\|^2 + \|u\|^2$$
or
$$\langle v+u, v+u\rangle = \langle v, v\rangle + \langle u, u\rangle$$

4.4 Orthonormal Bases

4.4.1 Definition of an orthonormal basis

For an inner product space $(V, \langle , \rangle)$ with a basis $B = \{v_1, v_2, \ldots, v_n\}$. B is an orthonormal basis if:

$$\langle v_i, v_j\rangle = \delta_{ij}.$$

4.4.2 Existence of an orthonormal basis

All finite dimensional inner product spaces have an orthonormal basis.

4.4.3 Properties of orthonormal bases

For an inner product space $(V, \langle , \rangle)$ with an orthonormal basis $B = \{v_1, v_2, \ldots, v_n\}$ and v, w in V:

- $v = \sum_{i=1}^{n} \langle v_i, v\rangle v_i$
- $\langle v, w\rangle = \sum_{i=1}^{n} \overline{\langle v_i, v\rangle} \langle v_i, w\rangle$
- $\langle v, v\rangle = \sum_{i=1}^{n} |\langle v_i, v\rangle|^2$.

4.4.4 Orthogonal projections

For an inner product space $(V, \langle , \rangle)$ a linear map $P : V \to V$ is called an orthogonal projection if:

- $P^2 = P$
- $\langle Pv, w\rangle = \langle v, Pw\rangle$.

4.4.5 Constructing an orthogonal projection

For an inner product space $(V, \langle , \rangle)$ with a subspace $U \subseteq V$ with an orthonormal basis $B = \{u_1, u_2, \ldots, u_k\}$, we have an orthogonal projection:

$$P : V \to V$$
$$P(v) := \sum_{i=1}^{k} \langle u_i, v \rangle u_i$$

So, we can write vectors as a linear combination of basis vectors, if we do that and then remove some of those terms we get this projection. We are writing vectors in terms of a subset of basis vectors.

4.4.6 Properties of orthogonal projections

For an inner product space $(V, \langle , \rangle)$ with an orthogonal projection P, we have:

- $(I - P)$ is also an orthogonal projection
- $V = \mathrm{Ker}(P) \oplus \mathrm{Im}(P)$
- $\mathrm{Ker}(P) = \mathrm{Im}(P)^{\perp}$
- $\|v - w\| \geq \|v - Pv\|$ $(w \in \mathrm{Im}(P))$

5 Adjoint Operators

5.1 Definition of an Adjoint Operator

For an inner product space $(V, \langle , \rangle)$ with $T : V \to V$ a linear map. We define the adjoint operator T^* by the relation:

$$\langle T^* v, w \rangle = \langle v, Tw \rangle.$$

5.2 Adjoint Matrices

For an inner product space $(V, \langle , \rangle)$ with $T : V \to V$ a linear map with associated matrix $M_{BB}(T) = (a_{ij})$, the adjoint has associated adjoint matrix:

$$M_{BB}(T^*) = (\overline{a_{ji}}).$$

Take care to notice the transposition.

5.3 Properties of Adjoint Operators

For an inner product space $(V, \langle , \rangle)$ with $S, T : V \to V$ linear maps:

- $(S + T)^* = S^* + T^*$
- $(ST)^* = T^* S^*$
- $(T^*)^* = T$
- For T invertible, $(T^*)^{-1} = (T^{-1})^*$.

5.4 Types of Operators

5.4.1 Normal, unitary, and self-adjoint

For an inner product space $(V, \langle , \rangle)$ with $T : V \to V$ a linear map. We say T has:

- The **Normal** property if $T^* T = TT^*$
- The **Unitary** property if $T^* T = I$
- The **Self-adjoint** (hermitian) property if $T* = T$

All unitary operators are normal. All self-adjoint operators are normal.

5.4.2 Real associated matrices

For an inner product space $(V, \langle , \rangle)$ with $T : V \to V$ a linear map with associated matrix M. If M is real $M^* = M^t$ (as it's the transpose and conjugate) this leads to the following results:

- Self-adjoint $\Leftrightarrow M = M^t$ (symmetric)
- Unitary $\Leftrightarrow M^{-1} = M^t$ (orthogonal)

5.4.3 Eigenvalues of self-adjoint operators

For an inner product space with a linear, self-adjoint map T, we have that the eigenvalues of T are all real.

5.4.4 Column vectors of unitary matrices

For M in $M_n(\mathbb{C})$, M is unitary if and only if the columns vectors of M form an orthonormal basis.

5.4.5 Properties of unitary operators

For an inner product space $(V, \langle, \rangle)$ with $S, T : V \to V$ linear, unitary maps. We have:

- T^{-1}, T^*, ST unitary
- $\|Tv\| = \|v\|$
- All eigenvalues of T have modulus 1

5.4.6 Eigenvalues of normal operators

For an inner product space $(V, \langle, \rangle)$ with $T : V \to V$ a linear, normal map. If we have v an eigenvector of T with eigenvalue λ then v is an eigenvector of T^* with eigenvalue $\overline{\lambda}$.

5.4.7 Eigenvectors of normal operators

We had a result previously that stated that eigenvectors with different eigenvalues were linearly independent. We now say for an inner product space with a linear, normal map T, we have that for two eigenvectors of T, they are orthogonal if their eigenvalues differ.

5.4.8 Decomposition via normal operators

For a finite dimensional, complex inner product space $(V, \langle, \rangle)$ with $T : V \to V$ a linear, normal map with $\text{spec}(T) = \{\lambda_1, \ldots, \lambda_k\}$:

$$V = V_{\lambda_1} \oplus \cdots \oplus V_{\lambda_k}$$

So, if we have a finite dimensional, complex inner product space with a normal operator, we can write the space as a direct sum of eigenspaces. Thus, the space has a basis of eigenvectors. Thus, the associated matrix to the operator is diagonalisable.

5.4.9 Diagonalisation via normal operators

For M in $M_n(\mathbb{C})$ a normal matrix, we have that there exists a unitary matrix D in $M_n(\mathbb{C})$ such that the columns of D are the eigenvectors of M and:

$$D^*MD = \text{diag}(\lambda_1, \ldots, \lambda_n),$$

where the diagonal matrix is formed by $\text{spec}(M) = \{\lambda_1, \ldots, \lambda_n\}$.

So, all normal matrices are diagonalisable: self-adjoint (hermitian), unitary, real symmetric.

6 Real Matrices

6.1 Motivation

We have worked with complex matrices of size n as they always have n eigenvalues. If matrix is self-adjoint however, then all the eigenvalues are real. So, it would make sense to consider real, self-adjoint matrices (also known as real symmetric matrices).

6.2 Orthogonal Matrices

6.2.1 Definition of an orthogonal matrix

Unitary and real matrices are called orthogonal. Additionally, we have O in $M_n(\mathbb{R})$ orthogonal if the columns form an orthonormal basis.

6.2.2 Closure of orthogonal matrices

For O_1, O_2 in $M_n(\mathbb{R})$ orthogonal matrices, O_1O_2 and $O_1^{-1} = O_1^t$ are orthogonal.

6.3 Diagonalisation of Real Symmetric Matrices

For M in $M_n(\mathbb{R})$ a symmetric, real matrix, we have that there exists a matrix O in $M_n(\mathbb{R})$ orthogonal and:

$$O^t M O = \mathrm{diag}(\lambda_1, \ldots, \lambda_n),$$

where the diagonal matrix is formed by $\mathrm{spec}(M) = \{\lambda_1, \ldots, \lambda_n\}$.