Group Theory Notes

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These notes are not necessarily correct, consistent, representative of the course as it stands today, or rigorous. Any result of the above is not the author's fault.

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Contents

1 The Fundamentals

1.1 Binary Operations

A binary operation on a set X is a map $X \times X \to X$. For a binary operation $*$ on a set X, we say that $*$ is associative if for all x, y, and z in X:

$$x * (y * z) = (x * y) * z.$$

Furthermore, we say e in X is an identity element of $*$ if for all x in X:

$$e * x = x * e,$$

and we say that y in X is the inverse to x if $x * y$ and $y * x$ are both identities of $*$.

1.2 Groups

A group $(G, *)$ is a non-empty set G combined with a binary operation $*$ such that:

- $*$ is associative,
- G contains an identity for $*$,
- for each element in G, there exists some inverse in G with respect to $*$.

1.2.1 Dihedral Groups

The dihedral group D_{2n} is the set of symmetries of the regular n-gon, with a rotation r by $\frac{2\pi}{n}$ radians and a reflection s, $D_{2n} = C_n \cup sC_n$.

1.2.2 Torsion Groups

A group is a torsion group if every element has finite order and torsion-free if every non-identity element has infinite order. The infinite dihedral group is neither a torsion or a torsion-free group.

1.2.3 p-groups

For p a prime, we say that a group G is a p-group if the order of each element of G is a power of p.

1.2.4 Simple Groups

A non-trivial group is simple if its only normal subgroups are itself and the trivial subgroup.

1.3 Set Multiplication (1.5)

For X, Y subsets of a group $(G, *)$, we define:

$$X * Y = \{x * y : x \in X, y \in Y\},$$

the product set of X and Y (which is a subset of G). We have that $*$ is an associative binary operation on $\mathcal{P}(G)$. Additionally, we define:

$$X^{-1} = \{x^{-1} : x \in X\}.$$

However, these definitions do not define a group on $\mathcal{P}(G)$ as an inverse does not necessarily exist for each element, despite the existence of an identity $\{e\}$.

1.4 Properties of Sets

For a group $(G, *)$ with $X \subseteq G$, we have some defined properties:

- X is symmetric if for each x in X, x^{-1} is also in X,
- X is closed under $*$ if for all x, y in X, $x * y$ is in X.

1.5 Subgroups

A subset X of a group $(G, *)$ is a subgroup if and only if $(X, *)$ is a group, denoted by $X \leq G$ (if X is a proper subset, this is denoted by $X < G$).

1.5.1 Centre (1.8)

For a group G, the centre of G is the set of elements that commute with all elements of G, denoted by $Z(G)$:

$$Z(G) = \{z \in G : gz = zg, \forall\, g \in G\}.$$

We have that $Z(G)$ is a subgroup of G.

1.5.2 The Product of Subgroups (1.9)

For H and K subgroups of a group G, HK is a subgroup of G if and only if $HK = KH$.

Proof. ($\Longrightarrow$) We can see that $HK = (HK)^{-1} = K^{-1}H^{-1} = KH$.

($\Longleftarrow$) We have that:

$$\begin{aligned} HK \ni ee &= e, \\ (HK)(HK) &= H(KH)K = H(HK)K = (HH)(KK) = HK, \\ (HK)^{-1} &= K^{-1}H^{-1} = KH = HK, \end{aligned}$$

so $HK \leq G$. □

1.6 The Intersection of Subgroups (1.11)

For a group G with $\mathcal{X}$ a set of subgroups of G:

$$A = \bigcap_{X \in \mathcal{X}} X \leq G.$$

Proof. We have that A must be non-empty as each element of $\mathcal{X}$ must contain e. Taking x and y in A, for each X in $\mathcal{X}$ we know that x and y are also in X. As X is a subgroup, x^{-1} and thus $x^{-1}y$ are in X. As X is arbitrary, $x^{-1}y$ must be in A. Hence, A is a subgroup of G by the subgroup test. □

1.7 The Subgroup Test (1.10)

For a subset X of a group G, X is a subgroup if and only if $X \neq \varnothing$ and $x^{-1}y$ is in X for each x and y in X.

Proof. ($\Longrightarrow$) If $X \leq G$, then e is in X so $X \neq \varnothing$. For x and y in X, x^{-1} is in X, so $x^{-1}y$ is also in X as X is closed.

($\Longleftarrow$) Supposing the latter and taking x and y in X, we have that $x^{-1}x = e$, $x^{-1}e = x^{-1}$, $xy = (x^{-1})^{-1}y$ are all in X. □

1.8 Generated Subgroups (1.12)

For a group G with $X \subseteq G$ non-empty, we define the subgroup generated by X as:

$$\langle X \rangle = \bigcap_{A \leq G, X \subseteq A} A,$$

the intersection of all the subgroups containing X. This can also be called the smallest subgroup containing X. Alternatively, we have that:

$$\langle X \rangle = \Gamma(X) = \{x_1 x_2 \cdots x_n : x_i \in X \cup X^{-1}, m \in \mathbb{N}\}.$$

Proof. We can see that $\Gamma(X) \subseteq \langle X \rangle$ as $\langle X \rangle$ contains X and is a subgroup so it contains all the finite products of elements of $X \cup X^{-1}$. If we can show that $\Gamma(X)$ is a subgroup, then that would mean $\langle X \rangle \subseteq \Gamma(X)$ as $\Gamma(X)$ contains X so would have been included in the intersection used to generate $\langle X \rangle$. We know that $\Gamma(X)$ is non-empty as X is non-empty. We take x and y in $\Gamma(X)$, and some n and m in $\mathbb{N}$ and see that:

$$\begin{aligned} x &= x_1 x_2 \cdots x_n, \\ y &= y_1 y_2 \cdots y_m, \end{aligned}$$

by the definition of $\Gamma(X)$. For each i in $[n]$, we know that x_i^{-1} is in $\Gamma(X)$ as $X^{-1} \subseteq \Gamma(X)$ so:

$$\begin{aligned} x^{-1}y &= (x_1 x_2 \cdots x_n)^{-1} y \\ &= x_n^{-1} x_{n-1}^{-1} \cdots x_1^{-1} y_1 y_2 \cdots y_m, \end{aligned}$$

is in $\Gamma(X)$. Thus, $\Gamma(X)$ is a subgroup, as required. □

1.9 Cyclic Groups (1.13-16)

A group G is cyclic if it is generated by a single element. Elements in G that generate G are called generators. Cyclic groups are abelian, subgroups of cyclic groups are cyclic. For a generator x of a cyclic group G, $|G| = |x|$.

1.10 Order (1.3)

For a group $G = (X, *)$, G has order $|X|$. The order of an element x of X is defined as follows:

$$|x| = \begin{cases} \infty & \text{if } x^n \neq e \text{ for any } n \text{ in } \mathbb{N}, \\ \min\{n \in \mathbb{N} \mid x^n = e\} & \text{otherwise.} \end{cases}$$

Taking x in X:

1. $x^i = x^j$ if and only if $i \equiv j \bmod |x|$,

if x has finite order, then:

2. $x^n = e$ if and only if $|x|$ divides n,
3. $x^n = x^m$ if and only if $|x|$ divides $m - n$,
4. $|x^k| = \frac{n}{\gcd(n,k)}$,

and if x has infinite order, then:

5. $x^n = x^m$ if and only if $n = m$.

Proof. (1) This trivially holds for the identity, we consider $x \neq e$. If $x^i = x^j$ for some $i \not\equiv j \bmod |x|$, we take $i < j$ without loss of generality and see that:

$$x^i = x^j \iff e \equiv x^{j-i},$$

but this contradicts the minimality of $|x|$.

(2) For n in $\mathbb{N}$, we take $n = q|x| + r$ for some q in $\mathbb{Z}$, r in $[|x| - 1]_0$ by the Division Algorithm. Thus:

$$\begin{aligned} x^n &= x^{q|x|}x^r, \\ &= e^q x^r, \\ &= x^r, \end{aligned}$$

and we can see that $x^r = e$ if and only if $r = 0$ as $r < |x|$ and $|x|$ is minimal. Thus, $x^n = e$ if and only if $r = 0$ which occurs if and only if $|x|$ divides n.

(4) Assumed from Problem Sheet 1.

((3) and (5)), We take x to have any order so:

$$x^n = x^m \iff x^{m-n} = e.$$

Thus, if $|x| < \infty$ then $|x|$ divides $m - n$ by (1) and if $|x| = \infty$ then $m - n = 0$ by the definition of order. □

1.11 Cosets (1.18)

For a group G with $H \leq G$ and x in G, the subset xH is a left coset of H in G and similarly, Hx is a right coset. For x and y in G:

1. $G = \bigcup_{x \in G} xH$,
2. $xH = yH$ if and only if x is in yH,
3. either $xH = yH$ or $(xH \cap yH) = \varnothing$,
4. $|xH| = |H|$.

Proof. (1) H contains the identity, so this is trivial.

(2) From the former, xe is in $xH = yH$. From the latter, $x = yh$ for some h in H so $xH = yhH = yH$.

(3) For g in $(xH \cap yH) \neq \varnothing$, $gH = xH = yH$ by (2).

(4) The map from H to xH defined by $h \mapsto xh$ is bijective. □

1.11.1 A Bijection from Left to Right Cosets (1.17)

For a group G with $H \leq G$, the map $xH \mapsto (xH)^{-1} = Hx^{-1}$ is a bijection from the set of left cosets to the set of right cosets.

1.11.2 Index

For a group G with $H \leq G$, the number of distinct left cosets of H in G is called the index of H in G, denoted by $[G : H]$.

1.12 Lagrange's Theorem (1.19)

For a finite group G with $H \leq G$, $|G| = [G : H]|H|$.

Proof. By (1.11), $G = \bigcup_{x \in G} xH$ is the disjoint union of $[G : H]$ left cosets of H, each of order $|H|$. □

1.12.1 Consequences of Lagrange's Theorem (1.20-22)

For a group G:

1. for all x in G, $|x|$ divides $|G|$,
2. if $|G|$ is prime, G is cyclic,
3. for a prime p, and P and Q subgroups of G with order p, $(P \cap Q) = \varnothing$ or $P = Q$,
4. for $K \leq H \leq G$, $[G : K] = [G : H][H : K]$

Proof. (1) $\langle x \rangle \leq G$ of order $|x|$, so $|G| = [G : \langle x \rangle]|x|$.

(2) For all x in G, $|\langle x \rangle|$ must be p or 1. Thus, every non-identity element in G has order p so generates G.

(3) For $g \neq e$ in $(P \cap Q)$, $|g| = p$ so $P = \langle g \rangle = Q$.

(4) Assumed from Problem Sheet 1. □

2 Morphisms

2.1 Homomorphisms

For G and H groups, a homomorphism φ from G to H is a map that for all x and y in G satisfies:

$$\varphi(xy) = \varphi(x)\varphi(y).$$

The image and kernel are defined as:

$$\begin{aligned} \mathrm{Im}(\varphi) &= \{\varphi(g) : g \in G\}, \\ \mathrm{Ker}(\varphi) &= \{g \in G : \varphi(g) = e\}. \end{aligned}$$

2.1.1 Properties of Homomorphisms (2.2-6)

For G and H groups, and φ from G to H a homomorphism, we have that:

1. $\varphi(e) = e$,
2. $\mathrm{Ker}(\varphi)$ is a subgroup of G,
3. $\mathrm{Im}(\varphi)$ is a subgroup of H,
4. φ is injective if and only if $\mathrm{Ker}(\varphi) = \{e\}$,
5. $\varphi(x^{-1}) = \varphi(x)^{-1}$ for every x in G,
6. for $x_1, \ldots, x_n$ in G, $\varphi(x_1 \cdots x_n) = \varphi(x_1) \cdots \varphi(x_n)$,
7. for x in $\varphi(G)$, $\varphi^{-1}(x)$ is a coset in $\mathrm{Ker}(\varphi)$.

These properties lead us to the following:

- for a finitely ordered element g in G, $|\varphi(g)|$ divides $|g|$ by (6),
- if G is a p-group, the image of every homomorphism on G is a p-group also.

We can restrict homomorphisms to subgroups or compose them and the result will be a homomorphism.

2.2 Homomorphisms and Generating Sets (2.7-8)

For G and H groups, a homomorphism φ from G to H, and $X \subseteq G$, we have that $\varphi(\langle X \rangle) = \langle \varphi(X) \rangle$. Furthermore, for another homomorphism ψ from G to H with X a generating set for G, if $\varphi(x) = \psi(x)$ for each x in X, then $\varphi = \psi$.

Proof. We have that:

$$\begin{aligned}
\varphi(\langle X \rangle) &= \{\varphi(x_1 \cdots x_n) : x_1, \cdots, x_n \in (X \cup X^{-1}), n \in \mathbb{N}\} \\
&= \{\varphi(x_1) \cdots \varphi(x_n) : x_1, \cdots, x_n \in (X \cup X^{-1}), n \in \mathbb{N}\} \qquad (2.1.1) \\
&= \{x_1 \cdots x_n : x_1, \cdots, x_n \in (\varphi(X) \cup \varphi(X^{-1})), n \in \mathbb{N}\} \\
&= \{x_1 \cdots x_n : x_1, \cdots, x_n \in (\varphi(X) \cup \varphi(X)^{-1}), n \in \mathbb{N}\} \qquad (2.1.1) \\
&= \langle \varphi(X) \rangle.
\end{aligned}$$

By (2.1.1), $\varphi(x^{-1}) = \psi(x^{-1})$ for every x in X so $\varphi = \psi$ on all members of $X \cup X^{-1}$. But, as every element of G can be written as a finite product of elements in X, $\varphi = \psi$ in general by (2.1.1). □

2.3 Isomorphisms

An isomorphism is a bijective homomorphism. Groups admitting an isomorphism are isomorphic.

2.4 Conjugation

For a group G containing some x, y, and g, $x^g = g^{-1}xg$ is the conjugation of x by g, similarly defined for sets. Also, x and y are said to be conjugate if there exists some h in G such that $x = y^h$.

2.4.1 Conjugations on Subgroups (2.10)

For a group G with $H \leq G$ and g in G, H^g is a subgroup of G and $H^g \cong H$.

Proof. By (2.5.1), conjugation is an isomorphism. □

2.5 Automorphisms

An automorphism is an isomorphism from a group to itself. The set of all automorphisms on a group G is denoted by $\mathrm{Aut}(G)$ which is a group under composition.

2.5.1 Inner Automorphisms (2.9)

For a group G, we have that φ from G to G defined for some g in G as $x \mapsto g^{-1}xg$ is an automorphism. Any automorphism of this form is called an inner automorphism.

Proof. For any x and y in G, $\varphi(xy) = g^{-1}xyg = g^{-1}xgg^{-1}g = \varphi(x)\varphi(y)$ so φ is a homomorphism. We have that $g^{-1}xg = e$ implies that $x = gg^{-1} = e$ so $\text{Ker}(\varphi) = \{e\}$. Finally, we see that $x = g^{-1}(gxg^{-1})g$ so φ is surjective as x is arbitrary in G. Thus, φ is an automorphism. □

3 Normal, Characteristic, and Quotient Groups

For a group G, a subgroup H of G is normal if for each g in G, $gH = Hg$. This is denoted by $H \trianglelefteq G$.

We say H is a characteristic subgroup if for every φ in $\mathrm{Aut}(G)$, $\varphi(H) = H$ (denoted by $H \underset{\text{char}}{\trianglelefteq} G$). We know characteristic subgroups are normal as $\mathrm{Aut}(G)$ contains inner automorphisms.

3.1 Properties of Normal Subgroups (2.14-17)

For a group G, the set of normal subgroups on G is closed under set multiplication and intersection. For G and H groups with φ from G to H a homomorphism, we have that:

1. if $K \leq G$ then $\varphi(K) \leq H$,
2. if $K \trianglelefteq G$ then $\varphi(K) \trianglelefteq \varphi(G)$,
3. if $K \leq H$ then $\varphi^{-1}(K) \leq G$,
4. if $K \trianglelefteq H$ then $\varphi^{-1}(K) \trianglelefteq G$.

Using $K = \{e\}$ in (4), we can see that $\mathrm{Ker}(\varphi) \trianglelefteq G$.

Proof. For P and Q normal subgroups of G, $PQ = QP$ by normality so $PQ \leq G$ by (1.5.2). We know that PQ is normal as for all g in G, $g^{-1}PQg = g^{-1}Pgg^{-1}Qg = HK$ by the normality of P and Q. Then, we know that the intersection of subgroups is a subgroup by (1.6), for a set of normal subgroups of $\mathcal{A} \subseteq \mathcal{P}(G)$ and g in G:

$$\left(\bigcap_{A\in\mathcal{A}} A\right)^g = \bigcap_{A\in\mathcal{A}} A^g = \bigcap_{A\in\mathcal{A}} A,$$

so this intersection is normal.

(1) For $\varphi(x)$ and $\varphi(y)$ in $\varphi(K)$, $\varphi(x)^{-1}\varphi(y) = \varphi(x^{-1}y)$ which is in $\varphi(K)$ as K is a subgroup. The result follows from the subgroup test.

(2) For every g in G, we have that $\varphi(g)^{-1}\varphi(K)\varphi(g) = \varphi(K^g) = \varphi(K)$.

(3) For x and y in $\varphi^{-1}(K)$, $\varphi(x^{-1}y) = \varphi(x)^{-1}\varphi(y)$ is in K so $x^{-1}y$ is in $\varphi^{-1}(K)$. The result follows from the subgroup test.

(4) For g in G, $\varphi(g^{-1}\varphi^{-1}(K)g) = \varphi(g)^{-1}K\varphi(g) = K$ so $g^{-1}\varphi^{-1}(K)g \subseteq \varphi^{-1}(K)$. The result follows from (3.2). □

3.2 A Test for Normal and Characteristic Subgroups (2.12)

Let G be a group with $H \leq G$:

1. if for every g in G, $H^g \subseteq H$ then $H \trianglelefteq G$,
2. if for every φ in $\mathrm{Aut}(G)$, $\varphi(H) \subseteq H$ then $H \underset{\text{char}}{\trianglelefteq} G$.

Proof. (2) We suppose that $\varphi(H) \subseteq H$ for each φ in $\mathrm{Aut}(G)$. For φ in $\mathrm{Aut}(G)$, φ^{-1} is also in $\mathrm{Aut}(G)$. We have that $\varphi^{-1}(H) \subseteq H$ by our assumption, applying φ to both sides, we see that $H \subseteq \varphi(H)$ so $H = \varphi(H)$ as required.

(1) We can perform the same argument as (2) as conjugation is an inner automorphism. □

3.3 Characteristic Subgroups of Normal Subgroups

For $C \underset{\text{char}}{\trianglelefteq} N \trianglelefteq G$, we have that $C \trianglelefteq G$.

Proof. Assumed from Problem Sheet 2. □

3.4 Normal Subgroups of Index 2 (2.11)

For a group G with $H \leq G$ such that $[G : H] = 2$, $H \trianglelefteq G$.

Proof. For x in H, $xH = H = Hx$. For x in $G \setminus H$, $xH \neq H$ by (1.11). Thus, xH and H are disjoint cosets of H and as $[G : H] = 2$, $G = H \cup xH$ is a disjoint union. We can apply the same argument to the right coset and deduce that $xH = Hx$ as required. □

3.5 Properties of the Centre (2.13)

For a group G, $Z(G)$ is a characteristic subgroup of G and every subgroup of $Z(G)$ is normal.

Proof. We note that $Z(G) \leq G$ by (1.5.1). For φ in $\mathrm{Aut}(G)$ and z in $Z(G)$, we take some g in G, so $zg = gz$ and thus $\varphi(z)\varphi(g) = \varphi(g)\varphi(z)$ as φ is a homomorphism. Thus, as g was arbitrary and φ is surjective, $\varphi(z)$ must be in $Z(G)$. Since z was arbitrary, $Z(G)$ is a characteristic subgroup by (3.2). Every subgroup of $Z(G)$ contains only elements that commute with all elements of G, so must be normal. □

3.6 Quotient Groups (2.18)

For a group G with $H \trianglelefteq G$, the quotient of G by H, G/H, is a group under set multiplication and for every a and b in G satisfies $(aH)(bH) = (ab)H$. Furthermore, the map π from G to G/H defined by $g \mapsto gH$ is a surjective homomorphism with kernel H, called the quotient homomorphism.

Proof. We know set multiplication is associative so for a and b in G, we see that:

$$\begin{aligned}(aH)(bH) &= aHbH \\ &= (ab)(HH) && (H \trianglelefteq G) \\ &= (ab)H.\end{aligned}$$

Thus, π is a homomorphism, G/H is closed under this operation, eH is the identity, and for g in G, the inverse of gH is $g^{-1}H$. So, G/H is a group under set multiplication. We have that π is trivially surjective and for g in $\mathrm{Ker}(\pi)$, $\varphi(g) = gH = H$ which means that g is in H by (1.11). □

4 The Morphism Theorems

4.1 The Homomorphism Theorem (2.19-21)

For G and H groups with φ from G to H a homomorphism, we take π from G to $G/\operatorname{Ker}(\varphi)$ to be the quotient homomorphism. There exists an isomorphism ψ from $G/\operatorname{Ker}(\varphi)$ to $\operatorname{Im}(\varphi)$ such that $\varphi = \psi \circ \pi$. This shows that:

- every subset of a group is a normal subgroup if and only if it is the kernel of some homomorphism,
- if φ is injective, $G \cong \operatorname{Im}(\varphi)$.

Proof. We set $I = \operatorname{Im}(\varphi)$ and $K = \operatorname{Ker}(\varphi)$, and define ψ from G/K to I by $gK \mapsto \varphi(g)$. We take g and h in G. We consider:

$$\begin{aligned} gK = hK &\iff g^{-1}h \in K \\ &\iff \varphi(g^{-1}h) = e \\ &\iff \varphi(g)^{-1}\varphi(h) = e \\ &\iff \varphi(g) = \varphi(h). \end{aligned}$$

So, ψ is well-defined and injective. We also have that ψ is trivially surjective and $(\psi \circ \pi)(g) = \psi(gK) = \varphi(g)$. Now, we consider:

$$\begin{aligned} \psi(gKhK) &= \psi(ghK) \\ &= \psi(\pi(gh)) \\ &= \varphi(gh) \\ &= \varphi(g)\varphi(h) \\ &= \psi(gK)\psi(hK), \end{aligned}$$

so ψ is a homomorphism. □

4.2 The First Isomorphism Theorem (2.22)

For a group G with $H \leq G$, $N \trianglelefteq G$, and π from G to G/N the quotient homomorphism:

1. $H \cap N \trianglelefteq H$,
2. $H/(H \cap N) \cong \pi(H)$.

Proof. We write $\pi|_H$ for the restriction of π to H and note that:

$$\begin{aligned} \mathrm{Im}(\pi|_H) &= \pi(H), \\ \mathrm{Ker}(\pi|_H) &= (H \cap \mathrm{Ker}(\pi)) = (H \cap N). \end{aligned}$$

As the kernel of a homomorphism is a normal subgroup in the domain group (3.1), $(H \cap N) \trianglelefteq H$. The Homomorphism Theorem implies that $H/(H \cap N) \cong \pi(H)$. □

4.3 Normal Subgroup Products (2.23)

For a group G with $H \leq G$, $N \trianglelefteq G$, and π from G to G/N the quotient homomorphism, we have that $HN \leq G$ and $\pi(H) = HN/N$.

Proof. We know that $HN \leq G$ if and only if $HN = NH$ by (1.5.2) which is implied by the normality of N. We consider the group:

$$\begin{aligned} HN/N &= \{hnN : h \in H, n \in N\} \\ &= \{hN : h \in H\} \\ &= \pi(H), \end{aligned}$$

as required. □

4.4 The Order of Normal Subgroup Products (2.24)

Let G be a group with $N \trianglelefteq G$, and $H \leq G$. If HN is finite, then:

$$|HN| = \frac{|H||N|}{|H \cap N|}.$$

Proof. We can see that:

$$\begin{aligned} \frac{|HN|}{|N|} &= [HN : N] && \text{(Lagrange's Theorem)} \\ &= |\pi(H)| && (4.3) \\ &= [H : H \cap N] && \text{(First Isomorphism Theorem)} \\ &= \frac{|H|}{|H \cap N|}, && \text{(Lagrange's Theorem)} \end{aligned}$$

as required. □

4.5 The Second Isomorphism Theorem (2.25)

For a group G with $N \leq H \leq G$, and N and $H \trianglelefteq G$, we have that $H/N \trianglelefteq G/N$ and $(G/N)/(H/N) \cong G/H$.

Proof. We take φ from G/N to G/H to be defined by $gN \mapsto gH$. We have that:

$$aN = bN \Longrightarrow ab^{-1} \in N \subseteq H \Longrightarrow aH = bH,$$

so φ is well-defined. We have that φ is a homomorphism because:

$$\varphi(aNbN) = \varphi(abN) = abH = aHbH = \varphi(aN)\varphi(bN),$$

and is trivially surjective as:

$$\mathrm{Ker}(\varphi) = \{gN : gH = eH\} = \{gN : g \in H\} = H/N.$$

Thus, $H/N \trianglelefteq G/N$ by (3.1) and $(G/N)/(H/N) \cong G/H$ by the Homomorphism Theorem. □

4.6 The Correspondence Theorem (2.26)

For a group G with $N \trianglelefteq G$, and π from G to G/N the quotient homomorphism, we have that:

1. If $K \subseteq G/N$ then:
   (a) $K \leq G/N$ if and only if $K = H/N$ for some $H \leq G$ containing N,
   (b) $K \trianglelefteq G/N$ if and only if $K = H/N$ for some $H \trianglelefteq G$ containing N,
2. If $N \subseteq H \subseteq G$ then:
   (a) $H \leq G$ if and only if $H = \pi^{-1}(K)$ for some $K \leq G/N$,
   (b) $H \trianglelefteq G$ if and only if $H = \pi^{-1}(K)$ for some $K \trianglelefteq G/N$.

Proof. We can show the ($\Longleftarrow$) directions by (3.1) applied with π.

(1)(a) We take $H = \pi^{-1}(K)$ and thus by the ($\Longleftarrow$) direction of (2) we have that $H \leq G$ and thus $K = \pi(H) = H/N$.

(1)(b) For this case, it is sufficient to show that given $K \trianglelefteq G/N$, $\pi^{-1}(K) \trianglelefteq G$. But, this is already shown in the ($\Longleftarrow$) direction of (2)(b).

(2) We have that $N \leq H \leq G$ so H is a union of cosets of N so $H = \pi^{-1}(\pi(H))$. We apply (3.1) again with π to get the result. □

5 Commutators

For x and y in a group G, we define the commutator of x and y as $[x, y] = x^{-1}y^{-1}xy$. This can be interpreted as the 'cost' of commuting x and y as $xy = yx[x, y]$.

5.1 Commutators under Homomorphisms

For x and y in a group G with a homomorphism φ from G to H, we have that:

$$\varphi([x, y]) = [\varphi(x), \varphi(y)].$$

Proof. Trivial from the definitions. □

5.2 Commutator Subgroups

For a group G with H and $K \leq G$, we define the subgroup $[H, K]$ by:

$$[H, K] = \langle [h, k] : h \in H, k \in K \rangle.$$

The subgroup $[G, G]$ is the commutator subgroup of G. If G is abelian, $[G, G] = \{e\}$.

5.3 Commutator of Normal Subgroups (2.32)

For a group G with H and $K \trianglelefteq G$, $[H, K] \subseteq (H \cap K)$.

Proof. For h in H and k in K, $[h, k] = h^{-1}k^{-1}hk$ so:

- $h^{-1}k^{-1}h$ is in $h^{-1}Kh = K$,
- $k^{-1}hk$ is in $k^{-1}Hk = H$.

Hence, $[h, k]$ is in $(H \cap K)$. □

5.4 Commutators of Characteristic Subgroups (2.27)

For a group G with H and $K \underset{\text{char}}{\trianglelefteq} G$, $[H, K] \underset{\text{char}}{\trianglelefteq} G$. Thus, $[G, G] \underset{\text{char}}{\trianglelefteq} G$.

Proof. For φ in $\mathrm{Aut}(G)$:

$$\begin{aligned}
\varphi([H, K]) &= \varphi(\langle [h, k] : h \in H, k \in K \rangle) \\
&= \langle \varphi([h, k]) : h \in H, k \in K \rangle && (2.2) \\
&= \langle [\varphi(h), \varphi(k)] : h \in H, k \in K \rangle && (5.1) \\
&= \langle [h, k] : h \in H, k \in K \rangle && (H \text{ and } K \underset{\text{char}}{\trianglelefteq} G) \\
&= [H, K],
\end{aligned}$$

as required. □

5.5 Abelian Quotients (2.28)

For a group G with $H \trianglelefteq G$, G/H is abelian if and only if $[G,G] \leq H$. Furthermore, a quotient of G is abelian if and only if it is isomorphic to a quotient of $G/[G,G]$.

Proof. We take π from G to G/H to be the quotient homomorphism.

($\Longrightarrow$) We take x and y in G, we have that $\pi([x,y]) = [\pi(x), \pi(y)] = eH$, thus $[x,y]$ is in H. Thus, as x and y are arbitary, $[G,G] \subseteq H$.

($\Longleftarrow$) For every xH and yH in G/H, we have that:

$$\begin{aligned}[xH, yH] &= (x^{-1}H)(y^{-1}H)(xH)(yH) \\ &= [x,y]H \\ &= H.\end{aligned}$$

Thus, G/H is abelian. So, G/H is abelian if and only if $[G,G] \leq H$ which is true if and only if G/H is isomorphic to a quotient of $G/[G,G]$ as we have that:

$$(G/[G,G])/(H/[G,G]) \cong G/H,$$

by the Second Isomorphism Theorem. □

5.5.1 Quotients of Abelian Groups (2.29)

Every quotient of an abelian group is abelian.

Proof. If G is abelian then $[G,G] = \{e\}$. So, for each $H \underset{\text{char}}{\trianglelefteq} G$ we have $[G,G] \leq H$ and so G/H is abelian by (5.5). □

5.6 The Abelianisation

For a group G, the abelianisation of G is the quotient group $G/[G,G]$. This group is always abelian and is the largest possible abelian quotient of G.

6 Direct Products

6.1 Outer Direct Product

For groups $G_1, \ldots, G_n$, we set:

$$G_1 \times \cdots \times G_n = \{(a_1, \ldots, a_n) : a_i \in G_i, i \in [n]\},$$

this is a group under component-wise group operations.

6.1.1 Properties of the Outer Direct Product (1.23)

For groups $G_1, \ldots, G_n$, with $G = \prod_{i \in [n]} G_i$:

- $|G| = \prod_{i \in [n]} |G_i|$,
- $Z(G) = \prod_{i \in [n]} Z(G_i)$,
- if G is cyclic, for each i in $[n]$, G_i is cyclic,
- for all σ in S_n, $G \cong \prod_{i \in [n]} G_{\sigma(i)}$,
- for the integers $1 \leq n_1 < n_1 < \cdots < n_r < n$:

$$G \cong (G_1 \times \cdots \times G_{n_1}) \times (G_{n_1+1} \times \cdots \times G_{n_2}) \times \cdots \times (G_{n_r+1} \times \cdots \times G_n),$$

- for $H_1, \ldots, H_n$ groups with $G_i \cong H_i$ for each i in $[n]$ $G \cong \prod_{i \in [n]} H_i$,
- for $x = (x_1, \ldots, x_n)$ in G, if $|x_i| = \infty$ for some i in $[n]$ then $|x| = \infty$, otherwise, $|x| = \text{lcm}(\{|x_1|, \ldots, |x_n|\})$.

6.2 Inner Direct Product

For a group G with $H_1, \ldots, H_n \trianglelefteq G$. We say G is the inner direct product of $H_1, \ldots, H_n$ if:

- $G = H_1 \times \cdots \times H_n$,
- $H_i \cap (H_1 \cdots H_{i-1} H_{i+1} \cdots H_n) = \{e\}$ for all i in $[n]$.

6.3 Component Groups (2.30)

For $G = G_1 \times \cdots \times G_n$ an outer direct product, for each i in $[n]$, we set:

$$\widehat{G_i} = \{(e, \ldots, e, g_i, e, \ldots, e) : g_i \in G_i\}.$$

We have that:

1. for each i in $[n]$, $\widehat{G_i} \trianglelefteq G$,
2. for each i in $[n]$, $\widehat{G_i} \cong G_i$,
3. G is the inner direct product of $\widehat{G_1}, \ldots, \widehat{G_n}$.

Proof. (1) We can see that:

$$\psi((g_1, \ldots, g_n)) = (g_1, \ldots, g_{i-1}, e, g_{i+1}, \ldots g_n),$$

is a homomorphism with kernel $\widehat{G_i}$ so, $\widehat{G_i} \trianglelefteq G$ by (3.1).

(2) We can see that:

$$\varphi_i((e, \ldots, e, g_i, e, \ldots, e)) = g_i,$$

is an isomorphism so, $\widehat{G_i} \cong G_i$.

(3) We have that $G = \widehat{G_1} \cdots \widehat{G_n}$ as:

$$(g_1, \ldots, g_n) = (g_1, e, \ldots, e)(e, g_2, e, \ldots, e) \cdots (e, \ldots, e, g_n).$$

Furthermore, taking i in $[n]$ and $G' = \widehat{G_1} \cdots \widehat{G_{i-1}}\widehat{G_{i+1}} \cdots G_n$, we have that $(\widehat{G_i} \cap G') = \{e\}$ as the elements of G' are of the form $(g_1, \ldots, g_{i-1}, e, g_{i+1}, \ldots, g_n)$ whereas elements of $\widehat{G_i}$ are of the form $(e, \ldots, e, g_i, e, \ldots, e)$. Thus, the only element in common is e. This is sufficient to prove the result as i was chosen arbitrarily. □

6.4 Commuting Normal Elements of Inner Direct Products (2.33)

For a group G with $H_1, \ldots, H_k \trianglelefteq G$ such that $G = H_1 \cdots H_k$ is an inner direct product, and i and j in $[k]$ whenever $i \neq j$, the elements of H_i commute with the elements of H_j.

Proof. As G is an inner direct product, $(H_i \cap H_j) = \{e\}$. Thus, $[H_i, H_j] = \{e\}$ by (5.3). □

6.5 Isomorphism between Products (2.31)

For a group G such that it is the inner direct product of subgroups $H_1, \ldots, H_n$ of G, $G \cong H_1 \times \cdots \times H_n = H$.

Proof. We define φ from H to G by:

$$\varphi((h_1, \ldots, h_n)) = h_1 \cdots h_n,$$

which is a homomorphism by (6.4) and is surjective as $G = H_1 \cdots H_n$. Taking $(h_1, \ldots, h_n)$ in $\mathrm{Ker}(\varphi)$ and some i in $[n]$:

$$\begin{aligned} h_1 \cdots h_n = e &\implies h_i^{-1} = h_1 \cdots h_{i-1} h_{i+1} \cdots h_n \\ &\implies h_i^{-1} \in H_i \cap (H_1 \cdots H_{i-1} H_{i+1} \cdots H_n) \\ &\implies h_i^{-1} = e. \end{aligned}$$

Thus, as i was chosen arbitrarily, $(h_1, \ldots, h_n) = (e, \ldots, e)$. So, φ is an isomorphism. □

6.6 Criteria for Inner Direct Products

6.6.1 By Unique Compositions (2.34)

For a group G with $H_1, \ldots, H_n$ normal subgroups of G, G is an inner direct product of $H_1, \ldots, H_n$ if and only if for all g in G, there exists unique h_i in each H_i such that $g = \prod_i h_i$.

Proof. ($\implies$) We have $g = \prod_{i \in [n]} h_i$ for some h_i in each H_i by the definition of the inner direct product, so it suffices to show this product is unique. We suppose that:

$$g = \prod_{i \in [n]} k_i = \prod_{i \in [n]} h_i,$$

for some k_i and h_i in each H_i. For i in $[n]$, it must be that $k_i = h_i$ as:

$$\begin{aligned} e &= g^{-1} g \\ &= h_n^{-1} \cdots h_1^{-1} k_1 \cdots k_n \\ &= h_1^{-1} k_1 \cdots h_n^{-1} k_n && (6.4) \\ &= h_i^{-1} k_i h_1^{-1} k_1 \cdots h_{i-1}^{-1} k_{i-1} h_{i+1}^{-1} k_{i+1} \cdots h_n^{-1} k_n, && (6.4) \\ k_i^{-1} h_i &= h_1^{-1} k_1 \cdots h_{i-1}^{-1} k_{i-1} h_{i+1}^{-1} k_{i+1} \cdots h_n^{-1} k_n \\ &\in H_i \cap (H_1 \cdots H_{i-1} H_{i+1} \cdots H_n) = \{e\}. \end{aligned}$$

($\Longleftarrow$) We trivially have $G = H_1 \cdots H_n$, so it suffices to show that for each i in $[n]$:

$$\mathcal{H}_i = H_i \cap (H_1 \cdots H_{i-1} H_{i+1} \cdots H_n) = \{e\}.$$

Taking i in $[n]$ and x in $\mathcal{H}_i$, we have that:

$$h_i = x = h_1 \cdots h_{i-1} h_{i+1} \cdots h_n,$$

for some h_i in each H_i. But by the uniqueness of the composition of x, this means that $x = e$ as required. □

6.6.2 By the Size (2.35)

For a finite group G with $H_1, \ldots, H_n \trianglelefteq G$ such that $G = H_1 \cdots H_n$. G is an inner direct product if and only if $|G| = \prod_{i \in [n]} |H_i|$.

Proof. ($\Longrightarrow$) Follows trivially from the definitions.

($\Longleftarrow$) As $|G| = \prod_i |H_i|$, each product of elements $h_1 \cdots h_n$ in $H_1 \cdots H_n$ is distinct. By (6.6.1), G is an inner direct product. □

7 Finitely Generated Abelian Groups

We will write $\mathbb{Z}^n = \{(m_1, \ldots, m_n) : m_1, \ldots, m_n \in \mathbb{Z}\}$ and $e_i = (0, \ldots, 1, \ldots, 0)$ in $\mathbb{Z}^n$ with 1 in the i^{th} entry. These are the standard generators for $\mathbb{Z}^n$.

For some n in $\mathbb{N}$, we write $\mathbb{Z}_n$ to be the integers modulo n which is a group under addition. Additionally, $n\mathbb{Z}$ is a subgroup of $\mathbb{Z}$ and $\mathbb{Z}_n \cong \mathbb{Z}/n\mathbb{Z}$.

7.1 Classification of Cyclic Groups (3.1)

For a cyclic group G, if $|G| = n$ finite, we have that $G \cong \mathbb{Z}_n$. Otherwise, $G \cong \mathbb{Z}$.

Proof. For a generator of G, x, We take φ from $\mathbb{Z}$ to G to be defined by $\varphi(m) = x^m$. Trivially, φ is a surjective homomorphism. If $|x| = \infty$ then $\text{Ker}(\varphi) = \{0\}$, otherwise, $\text{Ker}(\varphi) = |x|\mathbb{Z}$. By the Homomorphism Theorem:

$$G = \text{Im}(\varphi) \cong \mathbb{Z}/\text{ Ker}(\varphi).$$

The result follows as $\mathbb{Z}/\text{ Ker}(\varphi) = \mathbb{Z}$ if $|x| = \infty$ and $\mathbb{Z}_{|x|}$ otherwise. □

7.2 Torsion Subgroup and p-components (3.3)

For an abelian group G with $T \subseteq G$ the set of elements in G of finite order and a prime p with $G_p \subseteq T$ the set of elements in T of with order equal to a power of p. We have that $G_p \leq T \leq G$ and G/T is torsion-free. We have that T called the torsion subgroup of G and G_p called the p-primary component of G.

Proof. We suppose x and y are in T with $|x| = k$, $|y| = m$. We know that $km(x - y) = 0$ so $|x - y| \leq km < \infty$, thus $(x - y)$ is in T so T is a subgroup of G by the subgroup test.

Furthermore, $|x - y|$ must divide km so if x and y are in G_p then km is a power of p. Thus, $|x - y|$ is also a power of p so $(x - y)$ is in G_p. Again, by the subgroup test, G_p is a subgroup of T.

If we suppose that for some z in G, $z + T$ has finite order, there must exist some m in $\mathbb{N}$ with $mx + T = T$, so mx is in T. By the definition of T, there exists some n in $\mathbb{N}$ with $nmx = 0$. This means x has finite order so is in T. Thus, G/T is torsion-free. □

7.3 The Primary Decomposition Theorem (3.4)

For a finite abelian group G, we take $p_1, \ldots, p_k$ to be the prime factors of $|G|$. We have that $G = G_{p_1} \oplus \cdots \oplus G_{p_k}$.

Proof. We take x in G, by Lagrange's Theorem, $|x|$ divides $|G|$ so $|x| = p_1^{l_1} \cdots p_k^{l_k}$ for some $l_1, \ldots, l_k$ in $\mathbb{N}_0$. For each i in $[k]$, we set:

$$n_i = \prod_{j \in [k] \setminus \{i\}} p_j^{l_j},$$

and note that $|n_i x| = p_i^{l_i}$ so $n_i x$ is in G_{p_i}. It must be that $\gcd(n_1, \ldots, n_k) = 1$ as they are powers of distinct primes, so by the Euclidean Algorithm there exists $m_1, \ldots, m_k$ such that $m_1 n_1 + \cdots + m_k n_k = 1$. Thus:

$$\begin{aligned} x &= \left(\sum_{i=1}^{k} m_i n_i \right) \cdot x \\ &= \sum_{i=1}^{k} m_i (n_i x) \in \sum_{i=1}^{k} G_{p_i}. \end{aligned}$$

Thus, $G = G_{p_1} + \cdots + G_{p_k}$. Now, for each i in $[k]$, we consider x_i and x_i' in G_{p_i} such that $\sum_{i \in [k]} x_i = \sum_{i \in [k]} x_i'$. We write $y_i = x_i - x_i'$ so that $\sum_{i \in [k]} y_i = 0$, take d_i such that $|y_i| = p_i^{d_i}$, and set:

$$r_i = \prod_{j \in [k] \setminus \{i\}} |y_i| = \prod_{j \in [k] \setminus \{i\}} p_j^{d_j}.$$

As $|y_i|$ divides $|r_j|$ for all $j \in [k] \setminus \{i\}$, we know that $r_i y_j = 0$. This implies that $r_i y_i = 0$ as $\sum_{i=1}^{k} y_i = 0$.

Moreover, as r_i and p_i are coprime by definition, the Euclidean Algorithm implies that there exists a and b in $\mathbb{Z}$ such that:

$$\begin{aligned} a r_i + b p_i^{d_i} = 1 &\implies y_i = (a r_i + b p_i^{d_i}) y_i \\ &\implies y_i = a r_i y_i + b p_i^{d_i} y_i \\ &\implies y_i = 0 + 0 = 0, \end{aligned}$$

so $x_i = x_i'$ for each i in $[k]$. Thus, the composition of each element of G in our sum is unique, implying G is the direct sum of $G_{p_1}, \ldots, G_{p_k}$ by (6.6.1). □

7.4 Order of Finitely Generated Abelian Torsion Groups (3.5)

A finitely generated torsion group is finite.

Proof. For $x_1, \ldots, x_n$ the finite generating set of an abelian torsion group G:

$$G = \{k_1x_1 + \cdots + k_nx_n : 0 \leq k_i < |x_i|\},$$

which is finite since $|x_i| < \infty$ for all i in $[n]$. □

7.5 Order of Powers of Elements in p-groups (3.6)

For a prime p and a p-group G, we take g in G and set k in $\mathbb{N}$ to np^r with n and p coprime, and r in $\mathbb{N}_0$. If $p^r \leq |g|$ then $|g^k| = \frac{|g|}{p^r}$.

Proof. We know that $|g| = p^m$ for some m as G is a p-group. For d in $\mathbb{N}$:

$$\begin{aligned}
(g^k)^d = e &\iff g^{dnp^r} = e \\
&\iff p^m \text{ divides } dnp^r \\
&\iff p^m \text{ divides } dp^r && (n \text{ and } p \text{ coprime}) \\
&\iff p^{m-r} \text{ divides } d.
\end{aligned}$$

Thus, $|g^k| = p^{m-r} = \frac{|g|}{p^r}$ as required. □

7.6 Elements with Coset of Maximal Cyclic Subgroup Order (3.7)

For some prime p and a finite abelian p-group G, we take g in G to have maximum order. For every x in G, there exists y in $x + \langle g \rangle$ such that the order of y in G is equal to the order of $x + \langle g \rangle$ in $G/\langle g \rangle$.

Proof. We write $|x + \langle g \rangle| = p^m$ for some m, noting that $p^m x$ is in $\langle g \rangle$ so $p^m x = kg$ for some k in $\mathbb{N}_0$. We write $k = np^r$ with n and p coprime. If $p^r = 0$ or $p^r > |g|$ then $kg = 0$ so $|x| = p^m = |x + \langle g \rangle|$. Otherwise, by (7.5), $|kg| = \frac{|g|}{p^r}$ and $|p^m x| = \frac{|x|}{p^m}$ as p^m is minimal so $p^m \leq |x|$. Thus, $\frac{|g|}{p^r} = \frac{|x|}{p^m}$ as $p^m x = kg$. The maximality of g implies that $|g| \geq |x|$ so $r \geq m$ and thus p^m divides k. We define:

$$y = x - \frac{k}{p^m} g.$$

Hence, $p^m y = p^m x - kg = 0$ so $|y|$ divides p^m. But, as y is in $x + \langle g \rangle$, (2.1.1) applied to the quotient homomorphism from G to $G/\langle g \rangle$ implies that p^m divides $|y|$ so $|y| = p^m$ as required. □

7.7 Decomposition of Finite Abelian p-groups (3.8)

For a finite abelian p-group G with p prime, there exists a k in $\mathbb{N}_0$ and $m_1, \ldots, m_k$ in $\mathbb{N}$ such that $G \cong \mathbb{Z}_{p^{m_1}} \oplus \cdots \oplus \mathbb{Z}_{p^{m_k}}$.

Proof. It is sufficient to show that for $x_1, \ldots, x_k$ in G, G is the inner direct sum:

$$G = \langle x_1 \rangle \oplus \cdots \oplus \langle x_k \rangle. \tag{$*$}$$

If $G = \{0\}$ then this is trivial so we assume $|G| > 1$. By strong induction, we assume every group of order lesser to that of G can be written in the form shown in $(*)$.

We take g in G to have maximum order, $g \neq e$ as our group is non-trivial so $|G/\langle g \rangle| < |G|$ so by induction, there exists $x_1, \ldots, x_k$ in G such that:

$$G/\langle g \rangle = \langle x_1 + \langle g \rangle \rangle \oplus \cdots \oplus \langle x_k + \langle g \rangle \rangle. \tag{$\bullet$}$$

By (7.6), we can assume that $|x_i| = |x_i + \langle g \rangle|$, so:

$$\begin{aligned} |G/\langle g \rangle| &= |\langle x_1 + \langle g \rangle \rangle| \cdots |\langle x_k + \langle g \rangle \rangle| \\ &= |x_1| \cdots |x_k|, \end{aligned}$$

which combined with Lagrange's Theorem means that:

$$\begin{aligned} |G| &= [G : \langle g \rangle] \cdot |g| \\ &= |G/\langle g \rangle| \cdot |g| \\ &= |x_1| \cdots |x_k| \cdot |g|. \end{aligned}$$

We want to show that $G = \langle x_1 \rangle + \cdots + \langle x_k \rangle + \langle g \rangle$ and for all h in G, $h = ng + \sum_{i=1}^{k} l_i x_i$ for some $l_1, \ldots, l_k, n$ in $\mathbb{N}_0$. By $(\bullet)$ we know that:

$$\begin{aligned} h + \langle g \rangle = (l_1 x_1 + \cdots + l_k x_k) + \langle g \rangle &\implies h \in (l_1 x_1 + \cdots + l_k x_k) + \langle g \rangle \\ &\implies h = l_1 x_1 + \cdots + l_k x_k + ng. \end{aligned} \tag{1.11}$$

As we have that G is a sum of $\langle x_1 \rangle, \ldots, \langle x_k \rangle, \langle g \rangle$ and its size is a product of the size of these groups, G is an inner direct product of said elements by (6.6.2). □

7.8 Homomorphism from $\mathbb{Z}^n$ to Group Subsets (3.10)

For n in $\mathbb{N}$ and an abelian group G, for every $g_1, \ldots, g_n$ in G, there exists a unique homomorphism $\varphi : \mathbb{Z}^n \to G$ satisfying $\varphi(e_i) = g_i$ for all i. In particular, $\varphi((m_1, \ldots, m_1)) = m_1 g_1 + \cdots + m_n g_n$.

Proof. This is trivially a homomorphism and is unique by (2.2). □

7.9 One-way Inverses on Homomorphisms to $\mathbb{Z}^n$ (3.11)

For an abelian group G and $\alpha : G \to \mathbb{Z}^n$ is a surjective homomorphism, there exists an injective homomorphism $\beta : \mathbb{Z}^n \to G$ such that $\alpha \circ \beta = \iota_{\mathbb{Z}^n}$ (the identity on $\mathbb{Z}^n$).

Proof. If $n = 0$, this is trivial. Otherwise, there exists $g_1, \ldots, g_n$ in G such that $\alpha(g_i) = e_i$ for all i as α is surjective. By (7.8), we know that there exists a homomorphism β from $\mathbb{Z}^n$ to G such that $\beta(e_i) = g_i$ for all i. This gives us that $(\alpha \circ \beta)(e_i) = e_i$ which completely defines $(\alpha \circ \beta)$ by (2.2). Thus, $(\alpha \circ \beta) = \iota_{\mathbb{Z}^n}$. We can see that:

$$\begin{aligned} \mathrm{Ker}(\beta) &\subseteq \mathrm{Ker}(\alpha \circ \beta) \\ &= \mathrm{Ker}(\iota_{\mathbb{Z}^n}) \\ &= \{0\}. \end{aligned} \tag{2.1.1}$$

Thus, $\mathrm{Ker}(\beta) = \{0\}$ so β is injective as required. □

7.10 Abelian Groups with $\mathbb{Z}^n$ Quotients (3.9)

For an abelian group G with $H \leq G$ satisfying $G/H \cong \mathbb{Z}^n$ for some n in $\mathbb{N}_0$, we have that $G = H \oplus K$ for some $K \leq G$ satisfying $K \cong \mathbb{Z}^n$.

Proof. We take π from G to G/H to be the quotient homomorphism and ψ from G/H to $\mathbb{Z}^n$ to be an isomorphism. We set $\alpha = (\psi \circ \pi)$ which is a surjective homomorphism from G to $\mathbb{Z}^n$. By (7.9), we have β from $\mathbb{Z}^n$ to G an injective homomorphism with $(\alpha \circ \beta) = \iota_{\mathbb{Z}^n}$. We note that $H = \mathrm{Ker}(\alpha) \leq G$ and set $K = \beta(\mathbb{Z}^n) \leq G$. As β is injective, $K \cong \mathbb{Z}^n$. For some g in G:

$$\begin{aligned} \alpha(g - (\beta \circ \alpha)(g)) &= \alpha(g) - \alpha((\beta \circ \alpha)(g)) \\ &= \alpha(g) - ((\alpha \circ \beta) \circ \alpha)(g) \\ &= \alpha(g) - \alpha(g) \\ &= 0. \end{aligned}$$

Therefore, $(g - (\beta \circ \alpha)(g))$ is in $\mathrm{Ker}(\alpha) = H$ so g is in $(\beta \circ \alpha)(g) + H$. In particular, g is in $K + H = H + K$. As $(\alpha \circ \beta) = \iota_{\mathbb{Z}^n}$, $(\mathrm{Ker}(\alpha) \cap \beta(\mathbb{Z}^n)) = \{0\}$ which means $(H \cap K) = \{0\}$. Thus, $G = H \oplus K$ as required. □

7.11 $\mathbb{Z}^n$ Subgroups of Finitely Generated Groups (3.12)

For a finitely generated abelian group G with $H \leq G$ satisfying $G/H \cong \mathbb{Z}^n$ for some n in $\mathbb{N}_0$, H is finitely generated.

Proof. We know that $G \cong H \oplus \mathbb{Z}^n$ by (7.10). The projection π from $H \oplus \mathbb{Z}^n$ onto H defined by $(h, z) \mapsto h$ is a homomorphism. Since $H \oplus \mathbb{Z}^n$ is finitely generated, H is finitely generated by these generators under π. □

7.12 Fundamental Theorem of Finitely Generated Torsion-free Abelian Groups (3.13)

For n in $\mathbb{N}$ and G a finitely generated torsion-free abelian group generated by at most n elements, $G \cong \mathbb{Z}^k$ for some $k \leq n$.

Proof. We take $\{g_1, \ldots, g_n\}$ to be a generating set of G. If $n = 1$, G is cyclic and has infinite order so $G \cong \mathbb{Z}$. Otherwise, we proceed by induction, set:

$$H = \{x \in G : \exists\, m \in \mathbb{N} \text{ such that } mx \in \langle g_n \rangle\},$$

and observe that H is a subgroup via the subgroup test. We consider the quotient G/H and the quotient homomorphism π from G to G/H. We know that G/H is torsion-free as:

$$\begin{aligned} k\pi(x) = 0 &\implies \pi(kx) = 0 \\ &\implies kx \in H \\ &\implies lkx \in \langle g_n \rangle \text{ for some } l \in \mathbb{N} \\ &\implies x \in H \\ &\implies \pi(x) = 0. \end{aligned}$$

Thus, 0 is the only element of finite order in G/H as π is surjective. Clearly g_n is in H, so G/H is generated by $\{\pi(g_1), \ldots, \pi(g_{n-1})\}$. By induction, $G/H \cong \mathbb{Z}^k$ for some $k < n$. By (7.10), $G \cong H \oplus \mathbb{Z}^k$ so it's sufficient to show that $H \cong \{0\}$ or $\mathbb{Z}$.

We consider $H/\langle g_n \rangle$ which is finitely generated by (2.2) and a torsion group as for all h in H, there's some l such that $lh + \langle g_n \rangle = \langle g_n \rangle$. By (7.4), $H/\langle g_n \rangle$ is finite, we write m for its order. Thus, for all h in H, $mh + \langle g_n \rangle = \langle g_n \rangle$ as $|h|$ must divide m by Lagrange's Theorem so mh is in $\langle g_n \rangle$. We define φ from H to $\langle g_n \rangle$ by $\varphi(h) = mh$ which is an injective homomorphism as $H \leq G$ is torsion-free so $mh = 0$ implies that $h = 0$. Hence, $H \cong \varphi(H) \leq \langle g_n \rangle$, so H is cyclic. Thus, $H \cong \mathbb{Z}$ because H has infinite order and is cyclic. □

7.13 Fundamental Theorem of Finitely Generated Abelian Groups (3.2)

For a finitely generated abelian group G, there exists non-negative integers n and k, primes $p_1, \ldots, p_k$, and natural numbers $n_1, \ldots, n_k$ such that:

$$G \cong \mathbb{Z}_{p_1^{m_1}} \oplus \cdots \oplus \mathbb{Z}_{p_k^{m_k}} \oplus \mathbb{Z}^n$$

Proof. We take $T \leq G$ to be the torsion subgroup. As G is finitely generated, G/T is by (2.2), and $G/T \cong \mathbb{Z}^n$ for some n by (7.12). Then, (7.10) gives us that $G \cong T \oplus \mathbb{Z}^n$ and (7.4 and 7.11) implies that T is finite. By (7.3), there are finitely many primes $p_1, \ldots, p_m$ such that $G_{p_i} \neq \{0\}$, each G_{p_i} is finite, and $T = G_{p_1} \oplus \cdots \oplus G_{p_m}$. Then, (7.7) and $G \cong T \oplus \mathbb{Z}^n$ gives us that $G_{p_i} = \mathbb{Z}_{p_i^{m_1}} \oplus \cdots \oplus \mathbb{Z}_{p_i^{m_d}}$ for each i in $[m]$, leading to the result. □

8 Symmetric Groups

For a set X, a permutation of X is a bijection from X to X, the set of all permutations of X forms a group under composition which is denoted by $\mathrm{Sym}(X)$. For n in $\mathbb{N}$, we write $\mathrm{Sym}([n])$ as S_n. Note that $|\mathrm{Sym}(X)| = |X|!$.

Proof. We prove this by considering the number of bijections between sets X and Y of size n. For $n = 1$, there's only one bijection from X to Y. For $n > 1$, some x_0 in X and y_0 in Y:

$$\begin{aligned}
|\{\text{bijections from } X \text{ to } Y\}| &= \sum_{y \in Y} |\{\text{bijections from } X \text{ to } Y : x_0 \mapsto y\}| \\
&= \sum_{y \in Y} |\{\text{bijections from } X \setminus \{x_0\} \text{ to } Y \setminus \{y_0\}\}| \\
&= \sum_{y \in Y} (n-1)! \\
&= n \cdot (n-1)! \\
&= n!,
\end{aligned}$$

proving the result by induction. □

8.1 Cycles

For k in $\mathbb{N}$, a permutation f in S_n is called k-cycle if there are k distinct $i_1, \ldots, i_k$ in $[n]$ such that:

$$f(i_j) = \begin{cases} i_{j+1} & j \in [k-1] \\ i_1 & j = k, \end{cases}$$

in which case, we write $f = (i_1, \ldots, i_k)$. A 2-cycle is called a transposition and cycles $(i_1, \ldots, i_k)$, $(j_1, \ldots, j_l)$ are disjoint if $\{i_1, \ldots, i_k\} \cap \{j_1, \ldots, j_l\} = \varnothing$. Furthermore:

- a k-cycle has order k,
- $(i_1, \ldots, i_k) = (i_2, \ldots, i_k, i_1)$,
- $(i_1, \ldots, i_k)^{-1} = (i_k, \ldots, vi_1)$,
- $(i_1, \ldots, i_k) = (i_1, i_2)(i_2, i_3) \cdots (i_{k-1}, i_k)$,
- disjoint cycles commute.

8.2 Permutations as Disjoint Cycles (4.1)

For n in $\mathbb{N}$, each element of S_n can be written as a product of disjoint cycles with lengths summing to n which is unique up to reordering. Every element can also be written as a product of transpositions. From this, we can see that S_n is generated by the set of transpositions on 1, $\{(1,2),(1,3),\ldots,(1,n)\}$ as $(1,i)(1,j)(1,i)=(i,j)$.

8.3 Cycle Type (4.2)

For f in S_n written as a product of disjoint cycles with lengths summing to n, we take $l_1,\ldots,l_k$ be the lengths of these cycles in descending order. The k-tuple $(l_1,\ldots,l_k)$ is the cycle type of f. From this, we can see that $|f|=\operatorname{lcm}(l_1,\ldots,l_k)$.

8.4 Conjugacy in S_n (4.4)

For all g in S_n with $i_1,\ldots,i_k$ distinct elements of $[n]$:

$$g(i_1,\ldots,i_k)g^{-1}=(g(i_1),\ldots,g(i_k)).$$

Proof. For $k=1$, $(i_1)=e$ so $g(i_1)g^{-1}=gg^{-1}=e=(g(i_1))$. For $k=2$ we take (i,j) in S_n and r in $[n]$, then:

$$(g(i,j)g^{-1})(r)=\begin{cases} r & \text{if } g^{-1}(r)\notin\{i,j\} \\ g(i) & \text{if } g^{-1}(r)=j \\ g(j) & \text{if } g^{-1}(r)=i. \end{cases}$$

Thus, $g(i_1,i_2)g^{-1}=(g(i_1),g(i_2))$. For $k>2$, $(i_1,\ldots,i_k)=(i_1,i_2)(i_2,i_3)\cdots(i_{k-1},i_k)$ so:

$$\begin{aligned} g(i_1,\ldots,i_k)g^{-1} &= g(i_1,i_2)\cdots(i_{k-1},i_k)g^{-1} \\ &= g(i_1,i_2)g^{-1}g\cdots g^{-1}g(i_{k-1},i_k)g^{-1} \\ &= (g(i_1),g(i_2))\cdots(g(i_{k-1}),g(i_k)) \\ &= (g(i_1),\ldots,g(i_k)), \end{aligned}$$

as required. □

8.5 Conjugacy and Cycle Type (4.3)

We have that x and y in S_n are conjugate if and only if they have the same cycle type.

Proof. We take the cycle type of x be $(l_1, \ldots, l_k)$ so:

$$x = (a_1^{(1)}, \ldots, a_{l_1}^{(1)}) \cdots (a_1^{(k)}, \ldots, a_{l_k}^{(k)}),$$

with each value in $[n]$ corresponding uniquely to some $a_j^{(r)}$. For g in S_n:

$$\begin{aligned} gxg^{-1} &= g(a_1^{(1)}, \ldots, a_{l_1}^{(1)})g^{-1}g \cdots g^{-1}g(a_1^{(k)}, \ldots, a_{l_k}^{(k)})g^{-1} \\ &= (g\big(a_1^{(1)}\big), \ldots, g\big(a_{l_1}^{(1)}\big)) \cdots (g\big(a_1^{(k)}\big), \ldots, g\big(a_{l_k}^{(k)}\big)), \qquad (*) \end{aligned}$$

these are still disjoint cycles creating an element of cycle type $(l_1, \ldots, l_k)$ so, all conjugates of x have the same cycle type as x. For any y in S_n with cycle type equal to $(l_1, \ldots, l_k)$:

$$y = (b_1^{(1)}, \ldots, b_{l_1}^{(1)}) \cdots (b_1^{(k)}, \ldots, b_{l_k}^{(k)}),$$

with each value in $[n]$ corresponding uniquely to some $b_j^{(r)}$. We define g in S_n by $g\big(a_i^{(j)}\big) = b_i^{(j)}$ and see that $gxg^{-1} = y$ by $(*)$. □

8.6 Parity of Transposition Representations (4.5)

For x in S_n with $x = t_1 \cdots t_r = s_1 \cdots s_k$ where each t_i and s_i is a transposition, $r \equiv k \bmod 2$.

8.7 Signature (4.6)

For x in S_n with $x = t_1 \cdots t_r$ and each t_i a transposition, the signature of x is defined as:

$$\varepsilon(x) = \begin{cases} 1 & r \equiv 0 \bmod 2 \\ -1 & \text{otherwise.} \end{cases}$$

We have that ε is a homomorphism from S_n to $(\{-1, 1\}, \times)$.

Proof. For x and y in S_n with $x = x_1 \cdots x_r$ and $y = y_1 \cdots y_s$ where each x_i and y_j is a transposition:

$$\begin{aligned} \varepsilon(xy) &= \varepsilon(x_1 \cdots x_r y_1 \cdots y_s) \\ &= (-1)^{r+s} \\ &= \varepsilon(x)\varepsilon(y), \end{aligned}$$

as required. □

8.8 Alternating Groups (4.7)

We define the alternating group A_n to be the set of even permutations in S_n. We have that $A_n \trianglelefteq S_n$.

Proof. The result follows from (3.1) and $A_n = \mathrm{Ker}(\varepsilon)$. □

8.9 Subgroups of Index 2 in S_n (4.8)

For $n > 1$, $H \leq S_n$ has index 2 if and only if $H = A_n$.

Proof. ($\Longrightarrow$) We know that $H \trianglelefteq S_n$ by (3.4) so we consider S_n/H which must be isomorphic to C_2 and thus $(\{-1, 1\}, \times)$ by (1.12.1). Hence, there is a surjective homomorphism π from S_n to $(\{-1, 1\}, \times)$ with kernel H. For t_1 and t_2 transpositions, there exists g such that $t_1 = g^{-1}t_2g$ by (8.5) so:

$$\begin{aligned} \pi(t_1) &= \pi(g)^{-1}\pi(t_2)\pi(g) \\ &= \pi(t_2)\pi(g)^{-1}\pi(g) && ((\{-1, 1\}, \times) \text{ is abelian}) \\ &= \pi(t_2), \end{aligned}$$

meaning π takes the same value k on all transpositions. The set of transpositions T generates S_n so $\pi(T)$ generates $(\{-1, 1\}, \times)$ but $\pi(T) = \{k\}$ so $k = -1$. Thus, for $x = x_1 \cdots x_r$ a product of transpositions, $\pi(x) = (-1)^r = \varepsilon(x)$ so $\pi = \varepsilon$. As such, $H = \mathrm{Ker}(\pi) = \mathrm{Ker}(\varepsilon) = A_n$.

($\Longleftarrow$) By the Homomorphism Theorem, $\mathrm{Im}(\varepsilon) \cong S_n/\mathrm{Ker}(\varepsilon) = S_n/A_n$. Thus, $[S_n : A_n] = 2$. □

8.10 Generating Alternating Groups by 3-Cycles (4.9)

For n in $\mathbb{N}$, A_n is generated by its subset of 3-cycles.

Proof. Each element of A_n is a product of an even number of transpositions, so a product of permutations of the form $(i, j)(k, l)$. It suffices to show that these permutations must be 3-cycles.

Case 1 If $\{i, j\} = \{k, l\}$, as $(i, j) = (j, i)$, $(i, j)(k, l) = e$, a product of zero 3-cycles.

Case 2 If $|\{i, j\} \cap \{k, l\}| = 1$, we take $j = k$ without loss of generality so:

$$(i, j)(k, l) = (i, j)(j, l) = (i, j, l),$$

a 3-cycle.

Case 3 If i, j, k, and l are all distinct then:

$$(i,j)(k,l) = (i,j)(j,k)(j,k)(k,l) = (i,j,k)(j,k,l),$$

a product of two 3-cycles. □

9 Group Actions

For a group G and a non-empty set X, an action of G on X is a homomorphism φ from G to $\mathrm{Sym}(X)$. We say that:

- the action is faithful if φ is injective,
- the action is transitive if for all x, y in X, there exists g in G such that $\varphi(g)(x) = y$.

9.1 The Orbit and Stabiliser

For an action φ on a group G and a set X, for each x in X:

$$\mathrm{Orb}_G(x) = \{\varphi(g)(x) : g \in G\},$$
$$\mathrm{Stab}_G(x) = \{g \in G : \varphi(g)(x) = x\},$$

are the orbit and stabiliser of x, respectively.

9.2 The Orbit-Stabiliser Theorem (5.1)

For an action φ on a group G and a set X with x in X, $\mathrm{Stab}_G(x)$ is a subgroup of G and there is a well-defined bijection ψ from $\mathrm{Orb}_G(x)$ to $G/\mathrm{Stab}_G(x)$ defined by:

$$\psi(\varphi(g)(x)) = g\,\mathrm{Stab}_G(x).$$

If G is finite, $|G| = |\mathrm{Orb}_G(x)| \cdot |\mathrm{Stab}_G(x)|$.

Proof. We want to show that $\mathrm{Stab}_G(x) \leq G$. As φ is a homomorphism, $\varphi(e) = e$, so e is in $\mathrm{Stab}_G(x)$. For g and h in $\mathrm{Stab}_G(x)$:

$$\begin{aligned}\varphi(gh)(x) &= (\varphi(g) \circ \varphi(h))(x)\\ &= \varphi(g)(x)\\ &= x,\end{aligned}$$

so $\mathrm{Stab}_G(x)$ is closed. For inverses, we see that:

$$\begin{aligned}(\varphi(g^{-1}) \circ \varphi(g))(x) = x &\iff \varphi(g^{-1})(x) = x\\ &\iff g^{-1} \in \mathrm{Stab}_G(x).\end{aligned}$$

So, $\mathrm{Stab}_G(x) \leq G$. We know that ψ is well-defined and injective as:

$$\begin{aligned}\varphi(g)(x) = \varphi(h)(x) &\iff \varphi(h^{-1}g)(x) = x\\ &\iff h^{-1}g \in \mathrm{Stab}_G(x)\\ &\iff g \in h\,\mathrm{Stab}_G(x)\\ &\iff g\,\mathrm{Stab}_G(x) = h\,\mathrm{Stab}_G(x).\end{aligned}$$

As ψ is trivially surjective, it is a bijection as required. □

9.3 Relation via the Orbit (5.2)

For an action φ on a group G and a set X, we define an equivalence relation on X by $x \sim y$ if y is in $\mathrm{Orb}_G(x)$. The orbits of elements x in G are the equivalence classes of this relation, so they partition X.

Proof. We consider the conditions for equivalence relations:

Reflexivity For all x in X, we have that $\varphi(e)(x) = x$ so $x \sim x$.

Symmetry If $\varphi(g)(x) = y$ then $\varphi(g^{-1})(y) = x$.

Transitivity If $x \sim y \sim z$ then there exists g such that $y = \varphi(g)(x)$ and h such that $z = \varphi(h)(y)$. Thus, $z = \varphi(hg)(x)$ so $x \sim z$. □

9.4 Fixed Points (5.3)

For an action φ on a group G and a set X, x in X is a fixed point for φ if $\mathrm{Orb}_G(x) = \{x\}$. We write $\mathrm{Fix}_G(X)$ for the set of fixed points of φ. We write $\mathcal{O}_G(X)$ for the set of orbits of X under φ. For each orbit O in $\mathcal{O}_G(X)$, we pick can arbitrary element $x_O \in O$ and see that for X finite:

$$|X| = |\mathrm{Fix}_G(X)| + \sum_{O \in \mathcal{O}_G(X),\, |O|>1} [G : \mathrm{Stab}_G(x_O)].$$

Proof. We first note that the fixed points of φ are just the singleton orbits in $\mathcal{O}_G(X)$. Thus:

$$\begin{aligned} \mathcal{O}_G(X) &= \mathrm{Fix}_G(X) \cup \{O \in \mathcal{O}_G(X) : |O| > 1\} \\ &= \mathrm{Fix}_G(X) \cup \mathcal{O}_G^{(1)}(X), \end{aligned}$$

is a disjoint union. We have that:

$$\begin{aligned} |X| &= \sum_{O \in \mathcal{O}_G(X)} |O| && (9.3) \\ &= |\mathrm{Fix}_G(X)| + \sum_{O \in \mathcal{O}_G^{(1)}(X)} |O| \\ &= |\mathrm{Fix}_G(X)| + \sum_{O \in \mathcal{O}_G^{(1)}(X)} |G/\,\mathrm{Stab}_G(x_O)| && (9.2) \\ &= |\mathrm{Fix}_G(X)| + \sum_{O \in \mathcal{O}_G^{(1)}(X)} [G : \mathrm{Stab}_G(x_O)]. \end{aligned}$$

□

9.5 The Conjugation Action

For a group G acting on itself via conjugacy ($\varphi(g)(x) = gxg^{-1}$), where φ is this action and x in G, the conjugacy class of x, denoted by x^G, is defined by:

$$\mathrm{Orb}_G(x) = x^G = \{gxg^{-1} : g \in G\}.$$

The centraliser of x is defined by:

$$\mathrm{Stab}_G(x) = C_G(x) = \{g \in G : gxg^{-1} = x\}.$$

For $H \leq G$, the normaliser of H in G is defined by:

$$N_G(H) = \{g \in G : gHg^{-1} = H\}.$$

We note that this is also the stabiliser of H under the conjugation action of G onto the set of subgroups of G.

9.6 Partitioning on Conjugacy Classes (5.4)

For a group G, the conjugacy classes of G partition G.

Proof. The conjugacy classes are the orbits of this action leading to the result by (9.3). □

9.7 The Orbit-Stabiliser Theorem for Conjugation (5.5)

For a group G with x in G where we take φ to be the conjugacy action on G, we have that $\mathrm{Stab}_G(x) = C_G(x) \leq G$ and there exists a well-defined bijection ψ from $\mathrm{Orb}_G(x) = x^G$ to $G/C_G(x)$ defined by:

$$\psi(\varphi(g)(x)) = \psi(gxg^{-1}) = gC_G(x).$$

If G is finite, $|G| = |x^G| \cdot |C_G(x)|$. If we apply this to the conjugation action of G onto the set of its subgroups, we get that:

$$|\{K \leq G : K \text{ is conjugate to } H\}| = |G/N_G(H)| = [G : N_G(H)].$$

Proof. This follows directly from (9.2). □

9.8 The Class Equation (5.6)

For a finite group G, we write $\mathcal{C}$ for the set of conjugacy classes of G, for each conjugacy class C, we can pick an arbitrary element g_C and see that:

$$|G| = |Z(G)| + \sum_{C \in \mathcal{C}(G), |C|>1} [G : C_G(g_C)].$$

Proof. This follows directly from (9.4) as for g in G and z in $Z(G)$, $g^{-1}zg = z$. □

10 Sylow's Theorems

10.1 Cauchy's Theorem (6.1-2)

For a finite group G and a prime p such that p divides $|G|$, G contains an element of order p.

Proof. We first prove the theorem for abelian groups, then for all groups.

Abelian Case Suppose G is abelian. If $|G| = p$, then G is cyclic with a generator of order p. So, we consider $|G| > p$ and proceed by induction on $|G|$. We take g in $G \setminus \{e\}$, if p divides $|g|$, we take $g^{\frac{|g|}{p}}$ and we are done. Otherwise, by Lagrange's theorem, $|G| = |g| \cdot [G : \langle g \rangle]$ so p divides $[G : \langle g \rangle]$. We have that $G/\langle g \rangle$ is an abelian group by (5.5.1) and has order strictly less than $|G|$, so by induction, it contains an element of order p, $h\langle g \rangle$. We write $n = |h|$, we have that:

$$(h\langle g \rangle)^n = h^n \langle g \rangle = e\langle g \rangle = \langle g \rangle,$$

so p divides n. Thus, $h^{\frac{n}{p}}$ has order p in G as required.

General Case We remove our supposition that G is abelian. As before, if $|G| = p$, then G is cyclic with a generator of order p. So, we consider $|G| > p$ and proceed by induction on $|G|$. If p divides $|Z(G)|$, as $Z(G)$ is abelian, we are done by the first case. Otherwise, we consider the class equation:

$$|G| = |Z(G)| + \sum_{C \in \mathcal{C}(G),\, |C|>1} [G : C_G(g_C)].$$

As p divides $|G|$ but not $|Z(G)|$, there is some term of the summation that is not divisible by p. Thus, there exists g in G such that $g \in C$ where C is a conjugacy class of size at least 2 and $[G : C_G(g)]$ is not divisible by p. We have that Lagrange's Theorem implies that $|C_G(g)|$ is divisible by p as:

$$|G| = |C_G(g)|[G : C_G(g)].$$

Since $|C| \geq 2$, g is not central in G, so $C_G(g) \neq G$. By induction, $C_G(g)$ contains an element of order p. Hence, G does. □

10.2 Order of p-groups (6.3)

For a prime p and a finite group G, G is a p-group if and only if $|G| = p^m$ for some m in $\mathbb{N}$.

Proof. If $|G| = p^m$ for some m in $\mathbb{N}$ then every element has order dividing p^m by Lagrange's Theorem. As such, G is a p-group. Conversely, if $|G|$ is divisible by some prime $q \neq p$, then Cauchy's Theorem implies that G has an element of order q, which is not a power of p. □

10.3 Sylow's First Theorem (6.4)

We consider a prime p and a finite group G with $|G| = p^r m$ for some r in $\mathbb{N}_0$ and some m in $\mathbb{N}$ such that p does not divide m. We have that for every k in $\mathbb{N}_0$, there exists a subgroup of G of order p^k if and only if $k \leq r$.

Proof. We note that when $k > r$, p^k cannot divide $p^r m$ so, by Lagrange's Theorem, there's no subgroup of size p^k. Thus, we consider k in $[r]_0$. The theorem is trivial when $G = \{e\}$, so we assume $|G| > 1$ and proceed by induction on $|G|$.

Case 1 We suppose that p divides $|Z(G)|$. Cauchy's Theorem implies that there is a central element x of order p, so $\langle x \rangle \trianglelefteq Z(G) \underset{\text{char}}{\trianglelefteq} G$ (by (3.5)). We consider $G/\langle x \rangle$ which has size $p^{r-1}m$ so by induction has subgroups of order p^k for k in $[r-1]_0$ denoted by $H_0, \ldots, H_{r-1}$ with each i in $[r-1]_0$ yielding $|H_i| = p^i$.

We take π from G to $G/\langle x \rangle$ to be the quotient homomorphism and note that by the Correspondence Theorem, for all i in $[r-1]_0$, $\pi^{-1}(H_i) \leq G$ and so:

$$|\pi^{-1}(H_i)| = |\langle x \rangle| \cdot |H_i| = p|H_i| = p^{i+1}.$$

Thus, since we have the trivial subgroup of order 1, we have subgroups of order $1, p, \ldots, p^r$ as required.

Case 2 We suppose that p does not divide $|Z(G)|$. We take $\mathcal{C}$ to be the set of conjugacy classes in G and for each c in $\mathcal{C}$, we pick an element g_C in C and use the class equation:

$$|G| = |Z(G)| + \sum_{C \in \mathcal{C}, |C| \geq 2} [G : C_G(g_C)].$$

Thus, there must be some g not in $Z(G)$ such that $[G : C_G(g)]$ is not divisible by p so:

$$\frac{|G|}{|C_G(g)|},$$

is not divisible by p. However, since $|G|$ is divisible by p^r, $|C_G(g)|$ must be also. As g is not in $Z(G)$, $C_G(g) \neq G$ so $|C_G(g)| < |G|$. By induction, $C_G(g)$ contains subgroups of order $1, p, \ldots, p^r$ which are also subgroups of G. □

10.4 Sylow Subgroups

For a prime p and a group G, a p-subgroup $H \leq G$ is a Sylow p-subgroup if it is not a subgroup of any other p-subgroup of G. We write $\mathrm{Syl}_p(G)$ for the set of these subgroups and $n_p(G)$ for the quantity of them.

10.5 Closure of p-groups under Conjugacy (6.5)

For a prime p and a group G with $H \leq G$ a p-group, for every g in G, H^g is also a p-group. If H is a Sylow p-group, so is H^g.

Proof. As conjugacy is an automorphism, H^g is another p-group. If H is a Sylow p-subgroup and H^g is not, then H^g must be a proper subgroup of some other p-group $K \leq G$. However, K^g should be another p-subgroup but:

$$H = g^{-1}(gHg^{-1})g < g^{-1}Kg, \tag{2.5.1}$$

which contradicts the fact that H is a Sylow p-group. □

10.6 Sylow's Second Theorem (6.6)

For a prime p and a finite group G, the Sylow p-groups of G are all conjugate to each other. Thus, the conjugation action of G on the Sylow p-subgroups gives us that $|\mathrm{Orb}_G(P)| = n_p(G)$. So, by the Orbit-Stabiliser Theorem:

$$|\mathrm{Stab}_G(P)| = \frac{|G|}{n_p(G)}.$$

Proof. We write $|G| = p^r m$ with p not dividing m. By Sylow's First Theorem, we have that there exists a Sylow p-subgroup $P \leq G$ with $|P| = p^r$. We will show P is conjugate to an arbitrary Sylow p-subgroup H. We take H to act on G/P by $\varphi(h)(gP) = (hg)P$ and $\mathcal{O}$ to be the set of orbits of this action. By (9.3), the orbits partition G/P so $m = |G/P| = [G : P] = \sum_{O \in \mathcal{O}} |O|$. But, m cannot be divisible by p so there must be some orbit O with $|O|$ not divisible by p. The Orbit-Stabiliser Theorem gives us that:

$$|H| = |O| \cdot |\mathrm{Stab}_H(x)|,$$

for some x in G/P, so $|O|$ divides $|H|$. Since H is a p-group, $|O|$ must be a power of p. Thus, $|O| = 1$ and as such, the action of H on G/P has a fixed point, for some g in G and for all h in H:

$$\begin{aligned} HgP = gP &\iff g^{-1}HgP = P \\ &\iff g^{-1}Hg \subseteq P. \end{aligned}$$

By (10.5), $g^{-1}Hg = P$. □

10.7 Order of Sylow Subgroups (6.7)

For a prime p and a finite group G with $|G| = p^r m$ where r is in $\mathbb{N}_0$, m is in $\mathbb{N}$, and p doesn't divide m, every Sylow p-subgroup of G has order p^r.

Proof. This is direct from Sylow's First and Second Theorem. □

10.8 The Quantity of Sylow Subgroups (6.8)

For a finite group G and $P \leq G$ a Sylow p-subgroup, $n_p(G) = [G : N_G(P)]$. In particular, $P \trianglelefteq G$ if and only if P is the unique Sylow p-subgroup of G.

Proof. By Sylow's Second Theorem:

$$\begin{aligned} n_p(G) &= |\{H \leq G : H \text{ is conjugate to } P\}| \\ &= [G : N_G(P)]. \end{aligned} \tag{9.7}$$

If $n_p(G) = 1$, then $P^g = P$ for all g in G by Sylow's Second Theorem. □

10.9 Sylow Subgroups of Abelian Groups (6.9)

For a finite abelian group G, $n_p(G) = 1$ for all primes p.

Proof. As G is abelian, $N_G(P) = G$ so we have that $n_p(G) = [G : N_G(P)] = 1$. □

10.10 Fixed Point of Conjugation on Sylow Subgroups (6.11)

For a finite group G and a Sylow p-subgroup P where P acts on $\mathrm{Syl}_p(G)$ by conjugation via φ, we have that $\mathrm{Fix}_P(\mathrm{Syl}_p(G)) = \{P\}$.

Proof. We know that P is in $\mathrm{Fix}_P(\mathrm{Syl}_p(G))$ as $gPg^{-1} = P$ for any g in P. For Q in $\mathrm{Fix}_P(\mathrm{Syl}_p(G))$, by definition, $gQg^{-1} = Q$ for all g in P so, $P \subseteq N_G(Q)$. As we know $Q \trianglelefteq N_G(Q)$, (4.3) shows that $PQ \leq G$, and by (4.4), $|PQ|$ divides $|P||Q|$. But, as P and Q are p-groups, they must have an order that is a power of p. Thus, $|PQ|$ is also a power of p so PQ is a p-group. However, since P and Q are both Sylow p-subgroups in PQ, $P = PQ = Q$, as required. □

10.11 Sylow's Third Theorem (6.10)

For a prime p and a finite group G with $|G| = p^r m$ for some prime p that doesn't divide m, $n_p(G)$ divides m and $n_p(G) \equiv 1 \bmod p$.

Proof. We take P to be a Sylow p-subgroup with P acting on $\mathrm{Syl}_p(G)$ by conjugation. By Lagrange's Theorem, p divides $[P : \mathrm{Stab}_P(Q)]$ for every Q in $\mathrm{Syl}_p(G)$ that is not in $\mathrm{Fix}_P(G)$. So, we have that:

$$\begin{aligned} n_p(G) &\equiv |\mathrm{Fix}_P(\mathrm{Syl}_p(G))| \bmod p && (9.4)\\ &\equiv 1. && (10.10) \end{aligned}$$

By (10.8) and Lagrange's Theorem, $n_p(G)$ must divide $|G|$ and as $n_p(G) \equiv 1 \bmod p$, $n_p(G)$ is not divisible by p. Thus, $n_p(G)$ divides m. □

11 Finite Simple Groups

11.1 Classification of Abelian Simple Groups (7.2)

For an abelian group G, G is simple if and only if $G \cong \mathbb{Z}_p$ for some prime p.

Proof. ($\Longrightarrow$) For some non-identity element x in G, $\langle x \rangle \trianglelefteq G$ so $\langle x \rangle = G$ as G is simple. As such, G is cyclic. If G is infinite, $\langle x^2 \rangle$ is a non-trivial proper normal subgroup of G, a contradiction of the simplicity of G. If $|G|$ is not prime, $|G| = mn$ for some m and n in $\mathbb{N}_{>1}$. Then $\langle x^m \rangle$ is, again, a non-trivial proper normal subgroup of G. As such, G is a finite cyclic group of prime order, so $G \cong \mathbb{Z}_p$ for some prime p.

($\Longleftarrow$) By Lagrange's Theorem, $\mathbb{Z}_p$ has no non-trivial proper subgroups. □

11.2 Bound on the Order of Centres of Finite p-groups (7.3)

For a prime p and G a non-trivial finite p-group, $|Z(G)| \geq p$.

Proof. By (10.2), $|G| = p^m$ for some m in $\mathbb{Z}$. For some g in G, if the conjugacy class of g contains more than one element, then $C_G(g) \neq g$ so $[G : C_G(g)] > 1$. By Lagrange's Theorem, $[G : C_G(g)]$ must be a multiple of p. Since $|G|$ is also a multiple of p, $|Z(G)|$ must be too. As $Z(G)$ contains the identity, $|Z(G)| \geq p$. □

11.3 Existence of Non-abelian Finite Simple p-groups (7.4)

There are no non-abelian finite simple p-groups.

Proof. The centre of a finite simple p-group G has size at least p by (11.2), so for G to be simple, $Z(G) = G$ so G is abelian. □

11.4 Classification of Simple p-groups (7.5)

For a prime p and a finite simple p-group G, G is simple if and only if $G \cong \mathbb{Z}_p$.

Proof. By (11.3), G is abelian. By (11.1), we have the result. □

11.5 Bound on the Quantity of Sylow p-subgroups in Non-abelian Finite Simple Groups (7.6)

For a non-abelian finite simple group G and a prime p dividing $|G|$, $n_p(G) > 1$.

Proof. Sylow's First Theorem implies that G has at least one non-trivial Sylow p-subgroup P. By (11.4), there are no non-abelian finite simple p-groups so P is a non-trivial proper subgroup of G. As G is simple, $P \not\trianglelefteq G$ so there exists some conjugation of P not equal to P which would also be a Sylow p-subgroup. Thus, $n_p(G) > 1$. □

11.6 Simple Groups of Order 56 (7.7)

There are no simple groups of order 56.

Proof. We appeal to the contrary and take G to be a simple group of order $56 = 7 \cdot 2^3$. We know that G is not abelian by (11.1). We know that $n_7(G) > 1$ by (11.5) and by Sylow's Third Theorem, $n_7(G) \equiv 1 \bmod 7$ and $n_7(G)$ divides 8. Thus, $n_7(G)$ must be 8.

By Cauchy's Theorem, every Sylow 7-subgroup has size 7, so must be isomorphic to C_7. As these subgroups are distinct, their intersection must be $\{e\}$. This gives us $48 = 7 \cdot 6$ distinct elements of order 7 in G. This leaves 8 elements not of order 7, which must form a Sylow 2-subgroup of order 8 by Sylow's First Theorem. This accounts for all 56 elements of G, there can be no other Sylow 2-subgroups, contradicting (11.5). □

11.7 Simple Groups of Order consisting of 2 or 3 Factors (7.8)

For p, q, and r primes, there are no finite simple groups of order pq or pqr.

Proof. We suppose that G is a finite simple group, we note that $|G|$ is pq or pqr, G cannot be abelian by (11.1)

Case 1 We suppose that $|G| = pq$. By (11.4), $p \neq q$. Sylow's Third Theorem implies that $n_p(G)$ divides q and $n_q(G)$ divides p but this means:

$$n_p(G) \in \{1, q\},$$
$$n_q(G) \in \{1, p\}.$$

But, by (11.5), $n_p(G)$ and $n_q(G)$ must be greater than 1, so $n_p(G) = q$ and $n_q(G) = p$. Again, by Sylow's Third Theorem, we have that:

$$p \equiv 1 \bmod q,$$
$$q \equiv 1 \bmod p.$$

But, if we suppose that $q < p$, then $q \equiv q \bmod p$ and similarly for $q > p$. This is a contradiction.

Case 2 We suppose that $|G| = pqr$. By (11.4), we have that either $pqr = p^2q$ with p and q distinct (**2a**) or p, q, and r are all distinct (**2b**).

Case 2a Sylow's Third Theorem implies that:

$$n_p(G) \in \{1, q\},$$
$$n_q(G) \in \{1, p, p^2\},$$

and with (11.5), they both must be greater than 1. If $n_q(G) = p$, we have a contradiction by the reasoning in **Case 1**. So, it must be that $n_q(G) = p^2$. Hence, we have p^2 distinct subgroups of order q which admit $q-1$ unique elements of order q. Thus, there are $p^2(q-1) = p^2q - p^2 = |G| - p^2$ elements of order q in G. This leaves p^2 elements not of order q, which must form a unique Sylow p-subgroup. But, we know that $n_p(G) > 1$, so this is a contradiction.

Case 2b We suppose that $p < q < r$ without loss of generality. Sylow's Third Theorem implies that $n_r(G)$ divides pq and is congruent to 1 mod r combined with (11.5), so $n_r(G)$ is in $\{p, q, pq\}$. But, as $r > q > p$, $n_r(G)$ must be equal to pq as otherwise:

$$n_r(G) = p \not\equiv 1 \bmod r,$$
$$n_r(G) = q \not\equiv 1 \bmod r.$$

By a similar argument, $n_q(G)$ is in $\{r, pr\}$ and $n_p(G)$ is in $\{q, r, qr\}$. Thus, in G, there are:

$$pq(r-1) \text{ elements of order } r,$$
$$\text{at least } r(q-1) \text{ elements of order } q,$$
$$\text{at least } q(p-1) \text{ elements of order } p.$$

This accounts for:

$$\begin{aligned} pq(r-1) + r(q-1) + q(p-1) &= pqr - pq + rq - r + qp - q \\ &= pqr + (rq - r - q), \end{aligned}$$

elements in G, but this is greater than $pqr = |G|$, a contradiction. □

11.8 Simplicity of the First Alternating Groups (7.10)

We have that $A_1 = A_2 = \{e\}$, $A_3 \cong C_3$ is simple, and A_4 is not simple.

Proof. We can see that $S_1 = \{e\}$ and $S_2 = \{e, (1,2)\}$, so $A_1 = A_2 = \{e\}$. Also, $A_3 = \{e, (1,2,3), (1,3,2)\} \cong C_3$, and A_4 has a normal subgroup:

$$\{e, (1,2)(3,4), (1,3)(2,4), (1,4)(2,3)\},$$

as required. □

11.9 Conjugacy of 3-cycles in Alternating Groups (7.11)

For $n \geq 5$, all the 3-cycles in A_n are conjugate.

Proof. By (8.4), for i, j, and k arbitrary in $[n]$, we have that if there's some g in A_n such that:

$$g(1) = i, \qquad g(2) = j, \qquad g(3) = k, \tag{$*$}$$

then $g(1,2,3)g^{-1} = (i,j,k)$. Thus, it's sufficient to show such g exists in A_n. We take g_0 to be the element of S_n with property $(*)$. We suppose that g_0 is not in A_n, so is a composition of an odd number of transpositions. As such, $(4,5)g_0$ is in A_n and adheres to the property $(*)$ as required. □

11.10 Simple Alternating Groups (7.9)

The alternating group A_n is simple for $n = 3$ and $n \geq 5$.

Proof. For $n < 5$, we have (11.8). We suppose $n \geq 5$ and take $N \trianglelefteq A_n$ with $N \neq \{e\}$ and $a \neq e$ in N. We note that it is sufficient to show that N contains a 3-cycle as, by definition, it would then contain every conjugate of that 3-cycle so with (11.9 N would equal A_n, showing there are no proper non-trivial normal subgroups of A_n. We write a as a product of disjoint cycles $a_1, \ldots, a_t$, assuming each a_i is not a 1-cycle:

$$a = a_1 \cdots a_t. \tag{$*$}$$

For all b in A_n, as a^{-1} is also in N and N is normal, we have that $aba^{-1}b^{-1}$ is also in N.

Case 1 We suppose $(*)$ contains an r-cycle with $r \geq 4$, without loss of generality we set $a_1 = (i_1, \ldots, i_r)$ and then take $b = (i_1, i_2, i_3)$ in A_n. We know that:

$$\begin{aligned} aba^{-1}b^{-1} &= (a(i_1), a(i_2), a(i_3))(i_3, i_2, i_1) \\ &= (i_2, i_3, i_4)(i_3, i_2, i_1) \\ &= (i_2, i_4, i_2), \end{aligned}$$

is a 3-cycle.

Case 2 We suppose $(*)$ contains at least two 3-cycles, (i_1, i_2, i_3) and (i_4, i_5, i_6), we take $b = (i_1, i_2, i_4)$ in A_n. We know that:

$$\begin{aligned} aba^{-1}b^{-1} &= (a(i_1), a(i_2), a(i_4))(i_4, i_2, i_1) \\ &= (i_2, i_3, i_5)(i_4, i_2, i_1) \\ &= (i_1, i_4, i_3, i_5, i_2), \end{aligned}$$

is a 5-cycle. This induces a 3-cycle in N by **Case 1**.

Case 3 We suppose $(*)$ contains exactly one 3-cycle (i_1, i_2, i_3) and at least one transposition (i_4, i_5). We take $b = (i_1, i_2, i_4)$ in A_n. We know that:

$$\begin{aligned} aba^{-1}b^{-1} &= (a(i_1), a(i_2), a(i_4))(i_4, i_2, i_1) \\ &= (i_1, i_4, i_3, i_5, i_2), \end{aligned}$$

inducing a 3-cycle in N by **Case 2**.

Case 4 We suppose that $(*)$ contains only transpositions. As such, t must be even as at least 2, we take (i_1, i_2) and (i_3, i_4) to be two of these transpositions. As $n \geq 5$, there's some i_5 in $[n] \setminus \{i_1, \ldots, i_4\}$. We take $b = (i_1, i_3, i_5)$. We know that:

$$\begin{aligned} aba^{-1}b^{-1} &= (a(i_1), a(i_3), a(i_5))(i_5, i_3, i_1) \\ &= (i_2, i_4, a(i_5))(i_5, i_3, i_1) \\ &= \begin{cases} (i_1, i_2, i_4, i_5, i_3) & a(i_5) = i_5 \\ (i_1, i_2, i_6)(i_5, i_3, i_1) & \text{otherwise.} \end{cases} \end{aligned}$$

In the former case, we use **Case 2**. In the latter case, i_6 is in $[n] \setminus \{i_1, \ldots, i_5\}$ (as a is formed by disjoint cycles) so we have two disjoint 3-cycles, which we use **Case 2** on. □

11.11 Faithful Non-trivial Actions on Simple Groups (7.13)

For a simple group G and a non-empty set X with φ from G to $\mathrm{Sym}(X)$ a non-trivial action, φ is faithful and G is isomorphic to a subgroup of $\mathrm{Sym}(X)$.

Proof. We have that $\mathrm{Ker}(\varphi) \neq G$ as the action is non-trivial, and as $\mathrm{Ker}(\varphi) \trianglelefteq G$ and G is simple, $\mathrm{Ker}(\varphi) = \{e\}$. Thus, φ is faithful. The Homomorphism Theorem implies that $\varphi(G)$ is isomorphic to some subgroup of $\mathrm{Sym}(X)$. □

11.12 Alternating Subgroups of Index n (7.12)

For $n \geq 5$, if $H \leq A_n$ has index n, then $H \cong A_{n-1}$.

Proof. We take φ from A_n to $\mathrm{Sym}(A_n/H)$ to be the left multiplication action. This action is transitive, so non-trivial and we know that A_n is simple by (11.10) so it must be a faithful action by (11.11). We take ψ from H to $\mathrm{Sym}(A_n/H)$ to be the restriction of φ, noting that H is a fixed point for ψ. We define an action on $X = (A_n/H) \setminus \{H\}$ as ψ' from H to $\mathrm{Sym}(X)$ as the restriction of ψ.

We want to show that ψ' is faithful, we take h in H with $h \neq e$ so $\psi(h)(xH) \neq xH$ for some x in A_n (as ψ is faithful). But, since $\psi(h)(H) = H$ for all h in H, $xH \neq H$. As such, $\psi'(h)(xH) \neq xH$ so ψ' is faithful. But, ψ' acts on X of size $n-1$, so with the Homomorphism Theorem, we have that $H \cong \psi'(H) \leq \mathrm{Sym}(X) \cong S_{n-1}$. By Lagrange's Theorem:

$$\begin{aligned} |A_n| = |H| \cdot [A_n : H] &\iff \frac{n!}{2} = n|H| \\ &\iff |H| = \frac{(n-1)!}{2}, \end{aligned}$$

and the only subgroup of S_{n-1} of index 2 is A_{n-1} by (8.9). □

11.13 Simple Groups of Order 60 (7.14)

All simple groups of order 60 are isomorphic to A_5.

Proof. We take G to be a simple group of order 60. We know that G is not abelian as 60 is not prime (by (11.1)). We then use (11.5) to show that $n_p(G) > 1$ for all primes dividing 60. Sylow's Third Theorem implies that $n_5(G) \equiv 1 \bmod 5$ and divides 12 so $n_5(G) = 6$. Sylow's Second Theorem implies that G acts transitively by conjugation on $\mathrm{Syl}_5(G)$ and by (11.11), this action is faithful and G is isomorphic to a subgroup of $\mathrm{Sym}(\mathrm{Syl}_5(G))$. As $n_5(G) = 6$, $\mathrm{Sym}(\mathrm{Syl}_5(G)) \cong S_6$ so G is isomorphic to some subgroup $G' \leq S_6$.

We want to show that $(G' \cap A_6) = G'$, we let π from S_6 to S_6/A_6 be the quotient homomorphism. The First Isomorphism Theorem implies that $\pi(G') \cong G'/(A_6 \cap G')$ but as G' is simple, $G'/(A_6 \cap G')$ must have order 1 or 60. However, S_6/A_6 has order 2, so $|G'/A_6 \cap G'| = 1$. As such, $(A_6 \cap G') = G'$. In particular, $G' \leq A_6$. As $|A_6| = 360$ and $|G'| = 60$, it must be that $G' \cong A_5$ by (11.12). □

11.14 The Smallest Non-abelian Finite Simple Group (7.15)

The smallest non-abelian finite simple group has order 60.

Proof. We take the smallest non-abelian finite simple group to be G. By (11.13), $|G| \leq 60$ as A_5 is a non-abelian finite simple group. By (11.4), $|G|$ can't be a prime power, by (11.6), $|G| \neq 56$, and by (11.7), $|G|$ can't be the product of two or three primes. Thus:

$$|G| \in \{24, 36, 40, 48, 54, 60\}.$$

We reason on a case-by-case basis:

- By Sylow's Third Theorem and (11.5), if $|G| = 24$ or 48 then $n_2(G) = 3$. However, Sylow's Second Theorem and (11.11) implies that G is isomorphic to a subgroup of S_3 which is impossible as $|S_3| = 6$,
- By Sylow's Third Theorem and (11.5), if $|G| = 36$ then $n_3(G) = 4$. However, Sylow's Second Theorem and (11.11) implies that G is isomorphic to a subgroup of S_4 which is impossible as $|S_4| = 24$,
- By Sylow's Third Theorem, if $|G| = 40$ then $n_5(G) = 1$, contradicting (11.11),
- By Sylow's Third Theorem, if $|G| = 54$ then $n_3(G) = 1$, contradicting (11.11).

Thus, $|G| = 60$. □

12 Soluble and Nilpotent Groups

12.1 Normal and Subnormal Series

For a group G, a subnormal series of G is a finite sequence:

$$\{e\} = G_0 \trianglelefteq G_1 \trianglelefteq \cdots \trianglelefteq G_n = G.$$

If each G_i is such that $G_i \trianglelefteq G$ then this is a normal series. The length of the series is n and we call each G_{i+1}/G_i a factor of the series.

12.2 Soluble Groups

A group is soluble if it has a subnormal series in which every factor is abelian. The length of the shortest such series is the derived length.

12.3 Insolubility of Non-abelian Simple Groups (8.1)

For a non-abelian simple group G, G is not soluble.

Proof. If we suppose G is soluble and take a subnormal series $G_0, G_1, \ldots, G_n$ for G and m to be maximal such that $G_m \neq G$, we see that $G_m \trianglelefteq G_{m+1} = G$ so $G_m = \{e\}$ as G is simple. As such, $G/\{e\} \cong G$ is a non-abelian factor of G, contradicting the solubility of G. □

12.4 Derived Series

For a group G, the derived series of G is:

$$\cdots \leq G^{(1)} \leq G^{(0)} = G,$$

where for each i in $\mathbb{N}$, we define:

$$\begin{aligned} G^{(0)} &= G, \\ G^{(i+1)} &= [G^{(i)}, G^{(i)}]. \end{aligned}$$

We say that $G^{(n)}$ is the n^{th} derived subgroup of G.

12.5 Derived and Subnormal Series (8.3)

For a group G, G is soluble of derived length at most n if and only if $G^{(n)} = \{e\}$.

Proof. ($\Longrightarrow$) We have $\{e\} = G_0 \trianglelefteq G_1 \trianglelefteq \cdots \trianglelefteq G_n = G$ a subnormal series for G with abelian factors. We want to show that $G^{(i)} \leq G_{n-i}$ for each i in $[n]$. This shows that $G^{(n)} \leq G_0 = \{e\}$ and hence $G^{(n)} = \{e\}$ as required. If $i = 0$, then this is true by definition. We proceed by induction with $i > 0$, by our hypothesis, we have that $G^{(i-1)} \leq G_{n-i+1}$. Since G_{n-i+1}/G^{n-i} is abelian by assumption, (5.5) implies that $[G_{n-i+1}, G_{n-i+1}] \subseteq G_{n-i}$ so we have that $G^{(i)} = [G^{(i-1)}, G^{(i-1)}] \leq G_{n-i}$.

($\Longleftarrow$) We know that $\{e\} = G^{(n)} \trianglelefteq G^{(n-1)} \trianglelefteq \cdots \trianglelefteq G^{(0)} = G$ is a subnormal series for G by (5.4) and the factors are abelian by (5.5). Thus, G is soluble of derived length at most n. □

12.6 Derived Groups under Homomorphisms (8.4)

For G and H groups with a homomorphism φ from G to H, we have that $\varphi(G^{(k)}) = \varphi(G)^{(k)}$ for all k in $\mathbb{Z}_{\geq 0}$.

Proof. We have the $k = 0$ case by definition, we proceed by induction assuming $k > 0$:

$$\begin{aligned}
\varphi(G)^{(k)} &= [\varphi(G)^{(k-1)}, \varphi(G)^{(k-1)}] \\
&= \langle [x, y] : x, y \in \varphi(G)^{(k-1)} \rangle \\
&= \langle [x, y] : x, y \in \varphi(G^{(k-1)}) \rangle && \text{(IH)} \\
&= \langle [\varphi(g), \varphi(h)] : g, h \in G^{(k-1)} \rangle \\
&= \langle \varphi([g, h]) : g, h \in G^{(k-1)} \rangle && (5.1) \\
&= \varphi(\langle [g, h] : g, h \in G^{(k-1)} \rangle) && (2.2) \\
&= \varphi(G^{(k)}),
\end{aligned}$$

as required. □

12.7 Solubility of Subgroups and Quotients (8.5)

For a soluble group G of derived length at most n, every subgroup and quotient of G is soluble of derived length at most n.

Proof. By (12.5), $G^{(n)} = \{e\}$ so for $H \leq G$ then $H^{(n)} \leq G^{(n)} = \{e\}$ so H is soluble of derived length at most n by (12.5). For $N \trianglelefteq G$ with π from G to G/N the quotient homomorphism, (12.6) implies that:

$$(G/N)^{(n)} = \varphi(G)^{(n)} = \pi(G^{(n)}) = \pi(\{e\}) = N. \tag{8.4}$$

Thus, by (12.5), G/N is soluble of derived length at most n. □

12.8 Commutator of Symmetric Groups (8.7)

For n in $\mathbb{N}$, $[S_n, S_n] = A_n$.

Proof. The cases for $n < 3$ are trivial as $A_n = \{e\}$, so we consider $n \geq 3$. As $S_n/A_n \cong C_2$ we have $[S_n, S_n] \leq A_n$ by (5.5). Thus, it is sufficient to show that $A_n \leq [S_n, S_n]$. For a 3-cycle (x, y, z), we know that:

$$(x, y, z) = [(x, y), (y, z)] \in [S_n, S_n],$$

and as the 3-cycles generate A_n and (x, y, z) was arbitrary, we are done. □

12.9 Insolubility of Symmetric Groups (8.6)

For n in $\mathbb{N}$, S_n is soluble if and only if $n \leq 4$.

Proof. As $S_1 \leq S_2 \leq S_3 \leq S_4$, (12.7) shows that the cases for $n \leq 4$ all follow from the case $n = 4$ where:

$$\{e\} \trianglelefteq \{e, (1,2)(3,4), (1,3)(2,4), (1,4)(2,3)\} \trianglelefteq A_4 \trianglelefteq S_4,$$

has abelian factors so S_4 is soluble. For $n > 4$, A_n is a non-abelian simple group so by (12.3), it is not soluble. So, by (12.7), S_n is not soluble. □

12.10 Central Series

For a group G, we define a finite series $\{e\} = Z_0 \leq \cdots \leq Z_k = G$ of subgroups of G to be central if for every i in $[k]$, we have that $[G, Z_{i+1}] \leq Z_i$. The length of the series is k.

12.11 Nilpotent Groups

A group that admits a central series is nilpotent. The length of the shortest such series is called the step of G.

12.12 Lower Central Series

For a group G, the lower central series of G is:

$$\cdots \leq G_2 \leq G_1 = G,$$

where for each i in $\mathbb{N}$, we define:

$$G_1 = G,$$
$$G_{i+1} = [G, G_i].$$

12.13 Lower Central and Central Series (8.8)

For a group G with a lower central series $\cdots \leq G_2 \leq G_1 = G$, G is nilpotent of step at most s if and only if $G_{s+1} = \{e\}$.

Proof. ($\Longrightarrow$) As G is nilpotent of step at most s, it has a central series:

$$\{e\} = Z_0 \leq \cdots \leq Z_s = G.$$

We want to show that $G_i \leq Z_{s+1-i}$ for all i in $[s]$ as this shows that $G_{s+1} = \{e\}$. If $i = 1$, then this is true by definition. We proceed by induction with $i > 0$:

$$\begin{aligned} G_i &= [G, G_{i-1}] \\ &\leq [G, Z_{s+2-i}] && \text{(IH)} \\ &\leq Z_{s+1-i}. \end{aligned}$$

($\Longleftarrow$) We have that:

$$\{e\} = G_{s+1} \leq \cdots \leq G_2 \leq G_1 = G,$$

is a central series of length s as for all k in $[s]$, $[G, G_k] = [G, G_{k+1}]$ by definition. □

12.14 The Normaliser Condition (8.9)

For a nilpotent group G with $H < G$, we have that $H \neq N_G(H)$.

Proof. We take $\{e\} = Z_0 \leq \cdots \leq Z_k = G$ to be a central series for G and set $n = \max(\{m \in [k]_0 : Z_m \leq H\})$. It must be that $n < k$ as $H \neq G$, so we consider some z in Z_{n+1} and h in H. By definition, $h^{-1}z^{-1}hz = [h, z]$ is in Z_n so $z^{-1}hz$ is in hZ_n. But, as $Z_n \leq H$, $hZ_n \subseteq H$. Since h was arbitrary, $z^{-1}Hz = H$ so z is in $N_G(H)$ and thus, as z was arbitrary, $Z_{n+1} \leq N_G(H)$. By the maximality of n, $H \not\leq Z_{n+1} \leq N_G(H)$ so $N_G(H) \neq H$. □