Set Theory Notes

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These notes are not necessarily correct, consistent, representative of the course as it stands today or, rigorous. Any result of the above is not the author's fault.

0 Notation

We commonly deal with the following concepts in Set Theory which I will abbreviate as follows for brevity:

Term	Notation
$\{0, 1, 2, \ldots\}$	$\mathbb{N}$

Contents

1 The Fundamentals

1.1 Axiom of Extensionality

For two sets a and b, we have that $a = b$ if and only if for all x we have that:

$$x \in a \Longleftrightarrow x \in b.$$

For two classes A and B, we have that $A = B$ if and only if for all x we have that:

$$x \in a \Longleftrightarrow x \in b.$$

1.2 Axiom of Pair Sets

For any sets x and y, there is a set $z = \{x, y\}$. This is the (unordered) pair set of x and y.

1.3 Axiom of the Powerset

For each set x, there exists a set which is the collection of the subsets of x, the powerset $\mathcal{P}(x)$.

For some set x, we have the powerset defined as follows $\mathcal{P}(x) = \{z \,|\, z \subseteq x\}$.

1.4 Axiom of the Empty Set

There exists a set with no members, the empty set $\varnothing$.

We have the empty set defined as follows $\varnothing = \{x \,|\, x \neq x\}$.

1.5 Axiom of Subsets

For some set x, we have that $\{y \in x \,|\, \Phi(y)\}$ is a set for some well-defined property of sets Φ.

1.6 Axiom of Unions

We have the basic union of two sets x_1 and x_2:

$$x_1 \cup x_2 = \{y \,|\, y \in x_1 \text{ or } y \in x_2\},$$

but for cases where we want to unify the members of the sets in a set X, we define:

$$\bigcup X = \{y \,|\, \exists\, x \in X, y \in x\}.$$

This axiom states that for a set X, $\bigcup X$ is a set.

1.7 Classes

We have that classes are collection of objects, these could also be sets. Classes that are not sets are called proper classes.

1.8 The Set ω

We have the set of natural numbers, $\mathbb{N} = \{0, 1, 2, \ldots\}$, and from this, we define ω:

$$\omega = \{0, 1, 2, \ldots\},$$

where for some n in ω,

$$n = \{0, 1, 2, \ldots, n-1\},$$

with 0_ω being the empty set. We can go beyond this definition, defining:

$$\begin{aligned}
\omega + 1 &= \{0, 1, 2, \ldots, \omega\}, \\
\omega + 2 &= \{0, 1, 2, \ldots, \omega, \omega + 1\}, \\
&\ldots \\
\omega + n &= \{0, 1, 2, \ldots, \omega, \omega + 1, \ldots \omega + n - 1\}.
\end{aligned}$$

1.9 Russell's Theorem

We have that $R = \{x \,|\, x \notin x\}$ is not a set.

Proof. Suppose we have a set z such that $z = R$, is z in R? If we suppose z is in R, we have that z is not in z by the definition of R (as $z = R$) but z is R so z is not in R, a contradiction. Thus, we have that there is no set z equal to R, so R is not a set but a proper class. $\square$

1.10 The Universe of Sets

We define the universe of sets as $V = \{x \mid x = x\}$. We have that V is a proper class.

Proof. If we suppose V is a set, we apply the axiom of subsets with $\Phi(x) = x \notin x$ and reach a contradiction via Russell's theorem. $\square$

2 Relations

We will first state the significant properties relations can have. Taking a relation R on X with x, y, z arbitrary in X:

Name	**Property**
Reflexive	xRx
Irreflexive	$\neg(xRx)$
Symmetric	$xRy \Rightarrow yRx$
Antisymmetric	$[xRy$ and $yRx] \Rightarrow [x = y]$
Connected	$[x = y]$ or $[xRy]$ or $[yRx]$
Transitive	$[xRy$ and $yRz] \Rightarrow [xRz]$

For example, equivalence relations must satisfy reflexivity, symmetry, and transitivity.

2.1 Partial Orderings

We say that a relation $\prec$ on a set X is a (strict) partial ordering if it is irreflexive and transitive.

Similarly, we say that a relation $\preceq$ on a set X is a non-strict partial ordering if it is reflexive, antisymmetric, and transitive.

A partial ordering $(X, \prec)$ is wellfounded if for any non-empty subset Y of X, Y has a least element under $\prec$.

2.2 Bounding

For a partially ordered set $(X, \prec)$:

- x_0 in X is the minimum of X if for all x in X, $x_0 \preceq x$,
- x' in X is minimal in X if for all x in X, $\neg(x \prec x')$,
- x_1 in X is the maximum of X if for all x in X, $x \preceq x_1$,
- x' in X is maximal in X if for all x in X, $\neg(x' \prec x)$.

Taking a non-empty subset Y of X, we consider the subordering $(Y, \prec)$ and for some α in X we say:

- α is a lower bound for Y if for all y in Y, $\alpha \prec y$,
- α is the infimum of Y if it's a lower bound and for all lower bounds λ of Y, $\alpha \preceq \lambda$,
- α is an upper bound for Y if for all y in Y, $y \prec \alpha$,
- α is the supremum of Y if it's an upper bound and for all upper bounds τ of Y, $\tau \preceq \alpha$.

2.3 Order Preserving Maps

We say that $f : (X, \prec_1) \to (Y, \prec_2)$ is an order preserving map if for each x_1, x_2 in X:

$$x_1 \prec_1 x_2 \Longrightarrow f(x_1) \prec_2 f(x_2).$$

Two orderings are (order) isomorphic if there is a bijective order preserving map between them.

2.4 Representation Theorem for Partially Ordered Sets

For a partially ordered set $(X, \prec)$, there is a set $Y \subseteq \mathcal{P}(X)$ which is such that $(X, \preceq)$ is order isomorphic to $(Y, \subseteq)$.

Proof. For some x in X, we set $X^x = \{x' \in X : x' \preceq x\}$, the set of elements preceding or equal to x. For x, y in X, $x \neq y$ implies that $X^x \neq X^y$ as these sets contain x and y (resp.) so $x \mapsto X^x$ is bijective. We have that:

$$x \preceq y \Longleftrightarrow X^x \subseteq X^y,$$

by our definition. Thus, $x \mapsto X^x$ is an order isomorphism. □

2.5 Total Orderings

A relation $\prec$ on a set X is a (strict) total ordering if it is a connected strict partial ordering.

Similarly, we say that a relation $\preceq$ on a set X is a non-strict total ordering if it is a connected non-strict partial ordering.

2.6 Well-orderings

A relation $\prec$ on a set X is a well-ordering if it is a strict total ordering and for any non-empty subset Y of X, Y has a least element under $\prec$. We denote this with $(X, \prec) \in WO$.