

# Analysis 1B Notes

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*These notes are not necessarily correct, consistent, representative of the course as it stands today, or rigorous. Any result of the above is not the author's fault.*

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# 1 Continuity

## 1.1 Continuous Functions

From Analysis 1A, we have that a function  $f : A \rightarrow \mathbb{R}$  is continuous on  $A$  if:

$$\forall x \in A, \forall \epsilon > 0, \exists \delta > 0 \text{ such that } \forall y \in A, (|y - x| < \delta) \Rightarrow (|f(y) - f(x)| < \epsilon).$$

It's important to note that  $x$  is chosen given before we choose a  $\delta$ . Thus, our choice for  $\delta$  can depend on  $x$  as well as  $\epsilon$ .

**Uniform** continuity requires that  $\delta$  is independent of  $x$ .

*A note, a function being continuous at a value (or set of values for that matter), it equivalent to saying that there exists a limit for the function at that value and that limit is the value of the function applied to that value.*

## 1.2 Uniformly Continuous Functions

Uniform continuity is similar to continuity as we knew it in Analysis 1A. For a function  $f : A \rightarrow \mathbb{R}$ ,  $f$  is uniformly continuous on  $A$  if:

$$\forall \epsilon > 0, \exists \delta > 0 \text{ such that } \forall x, y \in A, (|y - x| < \delta) \Rightarrow (|f(y) - f(x)| < \epsilon).$$

We can see that uniform continuity **implies** continuity but **not** vice versa.

*A note, for uniform continuity, we are saying that given a value  $\epsilon$ , we can always pick a distance ( $\delta$ ) such that if two values are within that distance of each other, the distance between the values after the function is applied to them will be less than  $\epsilon$ . This is essentially testing for divergence to infinity at a value ( $\frac{1}{x}$  is continuous but not uniformly continuous on  $\mathbb{R}_{>0}$ ).*

## 2 Convergence

We have the notion of convergence for sequences of real numbers from Analysis 1A, convergence in this section is similar but specifically for functions.

### 2.1 Pointwise Convergence

A sequence of functions  $(f_n)_{n \in \mathbb{N}}$  from  $A \rightarrow \mathbb{R}$  converges **pointwise** to the function  $f$  on  $A$  if:

$$\lim_{n \rightarrow \infty} (f_n(x)) = f(x). \quad (\forall x \in A)$$

$f$  is called the **pointwise limit** of  $(f_n)_{n \in \mathbb{N}}$ .

*A note, for  $f_n : [0, 1] \rightarrow [0, 1]; x \rightarrow x^n$ ,  $f : [0, 1] \rightarrow [0, 1]; x \rightarrow \delta_1(x)$ ,  $f_n$  converges pointwise to  $f$ .*

### 2.2 Uniform Convergence

A sequence of functions  $(f_n)_{n \in \mathbb{N}}$  from  $A \rightarrow \mathbb{R}$  converges **uniformly** to the function  $f$  on  $A$  if:

$$\forall \epsilon > 0, \exists N \in \mathbb{N} \text{ such that } \forall x \in A, \forall n \in \mathbb{N}, (n \geq N) \Rightarrow (|f(x) - f_n(x)| < \epsilon).$$

*For the same functions outlined in the note under pointwise convergence, we have that  $f_n$  does not converge uniformly to  $f$ . Let  $\epsilon \in (0, 1)$ ,  $x \in [0, 1)$  and suppose  $f_n$  is uniformly convergent to  $f$ ,*

$$\begin{aligned} |f_n(x) - f(x)| &= |x^n| < \epsilon \\ \Rightarrow 0 &\leq x^n < \epsilon < 1 \\ \Rightarrow 0 &\leq x < \epsilon^{\frac{1}{n}} < 1 \\ \Rightarrow \epsilon &= 1 \text{ as } x \in [0, 1). \end{aligned}$$

*This is a contradiction by the definition of  $\epsilon$ . Thus, we have the result.*

### 2.3 Weierstrass' Theorem

For  $a, b \in \mathbb{R}$  with  $a < b$ , if a sequence of continuous functions on  $[a, b]$ ,  $(f_n)_{n \in \mathbb{N}}$  converges uniformly to  $f$  on  $[a, b]$ ,  $f$  is continuous on  $[a, b]$ .

*Basically, uniform convergence preserves continuity (it also preserves regulation).*

## 2.4 Supremum Norm

### 2.4.1 Definition of the Supremum Norm

For  $a, b \in \mathbb{R}$  with  $a < b$ , let  $f : [a, b] \rightarrow \mathbb{R}$  be a bounded function. The supremum norm of  $f$  on  $[a, b]$  is denoted by  $\|f\|_{[a,b]}$  and is defined by:

$$\|f\|_{[a,b]} := \sup \{|f(x)| : x \in [a, b]\}.$$

*The supremum norm is simply just the furthest distance from zero reached by a function over a closed interval. By definition, it is a real number and  $\exists x \in [a, b]$  such that  $f(x)$  is the supremum norm.*

### 2.4.2 Properties of the Supremum Norm

There are a few key properties of the supremum norm, let  $a$  and  $b$  be as above and let  $\lambda \in \mathbb{R}$ ,  $f, g : [a, b] \rightarrow \mathbb{R}$  be bounded functions:

- $\|f\|_{[a,b]} > 0$
- $\|f\|_{[a,b]} = 0 \Leftrightarrow f = 0$  on  $[a, b]$
- $\|\lambda f\|_{[a,b]} = |\lambda| \|f\|_{[a,b]}$
- $\|f + g\|_{[a,b]} \leq \|f\|_{[a,b]} + \|g\|_{[a,b]}.$

## 2.5 Cauchy Sequences of Functions

For  $a, b \in \mathbb{R}$  with  $a < b$ , denote the set of continuous functions on  $[a, b]$  by  $C([a, b])$ . Let  $(f_n)_{n \in \mathbb{N}}$  be a sequence of functions in  $C([a, b])$ . We say  $(f_n)_{n \in \mathbb{N}}$  is Cauchy if:

$$\forall \epsilon > 0, \exists N \in \mathbb{N} \text{ such that } \forall m, n \in \mathbb{N}, (m, n \geq N) \Rightarrow (\|f_n - f_m\|_{[a,b]} < \epsilon).$$

*This obviously bears an extreme resemblance to the Cauchy sequences of Analysis 1A. Just replacing the sequences of reals with sequences of functions and the modulus with the supremum norm.*

For each continuous function, there exists a Cauchy sequence such that the sequence converges uniformly to said function.

## 3 Integration

### 3.1 Step Functions

For  $a, b \in \mathbb{R}$  with  $a < b$ , a partition of the interval  $[a, b]$  is a set  $P$  of the form:

$$P = \{x_0, x_1, \dots, x_n\} \text{ (for some } n \in \mathbb{N}\text{)} \\ \text{where } a = x_0 < x_1 < \dots < x_n = b.$$

We say a function  $\psi : [a, b] \rightarrow \mathbb{R}$  is a step function if there exists a partition  $P = \{x_0, \dots, x_n\}$  and a set of constants in  $\mathbb{R}$  ( $\{c_0, c_1, \dots, c_n\}$ ) such that:

$$\psi(x) = c_i \text{ } (\forall x \in (x_{i-1}, x_i)).$$

In this case,  $P$  and  $\psi$  are adapted to each other.

$S([a, b])$  is the set of step functions over  $[a, b]$ .

### 3.2 Integration of Step Functions

#### 3.2.1 Definition of integration on step functions

The integral of the step function is simple:

$$\int_a^b \psi(x) dx := \sum_{i=1}^n c_i (x_i - x_{i-1}).$$

As long as the partition is adapted to  $\psi$ , the integral doesn't change.

#### 3.2.2 Properties of integration on step functions

Here are some properties of the integration of step functions, let  $\phi, \psi$  be step functions over  $[a, b]$ ,  $y \in \mathbb{R}$  with  $a < y < b$ ,  $\alpha, \beta \in \mathbb{R}$ :

- **Linearity:**  $\int_a^b \alpha\psi(x) + \beta\phi(x) dx = \alpha \int_a^b \psi(x) dx + \beta \int_a^b \phi(x) dx$
- **Monotonicity:**  $(\psi(x) \leq \phi(x) (\forall x \in [a, b])) \Rightarrow (\int_a^b \psi(x) dx \leq \int_a^b \phi(x) dx)$
- **Continuity:**  $|\int_a^b \psi(x) dx| \leq (b - a) \|\psi(x)\|_{[a, b]}$
- **Additivity:**  $\int_a^b \psi(x) dx = \int_a^y \psi(x) dx + \int_y^b \psi(x) dx$



### 3.3 Regulated Functions

#### 3.3.1 Definition of left and right limits

Let  $A \subseteq \mathbb{R}$  and  $f : A \rightarrow \mathbb{R}$ . For some  $\epsilon > 0$ ,  $a \in A$ , and  $\alpha \in \mathbb{R}$ :

1.  $f$  has a **right limit** of  $\alpha$  at  $a$  if:  
 $\exists \delta > 0$  such that  $(0 < x - a < \delta) \Rightarrow (|f(x) - \alpha| < \epsilon)$
2.  $f$  has a **left limit** of  $\alpha$  at  $a$  if:  
 $\exists \delta > 0$  such that  $(0 < a - x < \delta) \Rightarrow (|f(x) - \alpha| < \epsilon)$ .

We can denote right limits by:  $\lim_{x \downarrow a} f(x) = \alpha$ . Similarly for left limits:  $\lim_{x \uparrow a} f(x) = \alpha$ .

*There is a sequential definition too, for any sequence  $(x_n)_{n \in \mathbb{N}}$  that satisfies  $x_n > a$  for all  $n \in \mathbb{N}$  and  $\lim_{n \rightarrow \infty} x_n = a$ , if  $f$  has a right limit,  $\lim_{n \rightarrow \infty} f(x_n) = \alpha$ . There is a similar definition for left limits.*

#### 3.3.2 Definition of a regulated function

A function  $f : [a, b] \rightarrow \mathbb{R}$  is regulated if:

- $f$  has a left limit on all values in  $(a, b]$
- $f$  has a right limit on all values in  $[a, b)$ .

*All continuous functions are regulated. All increasing and decreasing functions are regulated.*

#### 3.3.3 Properties of regulated functions

Let  $R([a, b])$  be the set of functions regulated over  $[a, b]$ . We have that  $R([a, b])$  is closed under:

- Scalar multiplication (over  $\mathbb{R}$ )
- Addition
- Multiplication
- Division (if the divisor is greater than zero over  $[a, b]$ )
- Composition
- The modulus.

*Uniform convergence preserves regulation. Also, all step functions are regulated.*

For  $f$  a regulated function over  $[a, b]$ , we have that:

$$\forall \epsilon > 0, \exists \psi \in S([a, b]) \text{ such that } \|\psi - f\| < \epsilon.$$

*Basically, for any regulated function we can always choose an arbitrarily accurate approximation that is a step function.*

### 3.4 Integration of Regulated Functions

#### 3.4.1 Definition of integration on regulated functions

For a function  $f \in R([a, b])$ , say we have two sequences of step functions,  $(\psi_n)_{n \in \mathbb{N}}$  and  $(\phi_n)_{n \in \mathbb{N}}$ :

- $(\psi_n)_{n \in \mathbb{N}}$  is uniformly convergent to  $f \Rightarrow (\int_a^b \psi_n(x) dx)_{n \in \mathbb{N}}$  is convergent
- $(\psi_n)_{n \in \mathbb{N}}$  and  $(\phi_n)_{n \in \mathbb{N}}$  are uniformly convergent to  $f \Rightarrow \lim_{n \rightarrow \infty} (\int_a^b \psi_n(x) dx) = \lim_{n \rightarrow \infty} (\int_a^b \phi_n(x) dx)$ .

*Basically, we have that no matter what step function we choose to approximate our function, the value of the integral will tend to the same value.*

We define the integral of a regulated function  $f \in R([a, b])$  by choosing a sequence of step functions  $(\psi_n)_{n \in \mathbb{N}}$  such that they converge uniformly to  $f$ :

$$\int_a^b f(x) dx := \lim_{n \rightarrow \infty} \int_a^b \psi_n(x) dx.$$

#### 3.4.2 Properties of integration on regulated functions

The **linearity**, **continuity**, and **additivity** properties hold similarly to the properties of step functions. The **monotonicity** property holds also but the stated definition varies slightly:

- **Monotonicity:** For  $f \in R([a, b])$  with  $f(x) \geq 0$  for  $x \in [a, b]$ , we have that  $\int_a^b f(x) dx \geq 0$ .

Some small notes on regulated functions, let  $f \in R([a, b])$ :

- $|\int_a^b f(x) dx| \leq \int_a^b |f(x)| dx$
- For  $(f_n)_{n \in \mathbb{N}}$  uniformly convergent to  $f$ ,  $\lim_{n \rightarrow \infty} \int_a^b f_n(x) dx = \int_a^b f(x) dx$ .

*The first point is similar to the triangle inequality applied to summations. The second was covered similarly but strictly for step functions, not all regulated functions.*

### 3.5 The Mean-Value Theorem of Integration

For  $f \in C([a, b])$ , let  $g \in R([a, b])$  and satisfy the following:

- $g(x) \geq 0$  for  $x \in [a, b]$
- $\int_a^b g(x) dx > 0$

With these assumptions, we have that  $\exists x \in (a, b)$  with:

$$f(x) \int_a^b g(t) dt = \int_a^b f(t)g(t) dt$$

*Note that the function  $f$  is continuous. This is a stronger statement than just saying it's regulated. Also, consider  $g = 1$ :*

$$\begin{aligned} f(x) \int_a^b g(t) dt &= f(x) \int_a^b 1 dt \\ &= f(x)(b - a) \end{aligned} \tag{1}$$

$$\int_a^b f(t)g(t) dt = \int_a^b f(t) dt \tag{2}$$

(1) and (2)  $\Rightarrow$

$$\int_a^b f(t) dt = f(x)(b - a)$$

## 4 Differentiation

### 4.1 Definition of Differentiation

For a function  $f$  defined on a  $\delta$ -neighbourhood of some  $a \in \mathbb{R}$ , we have that  $f$  is differentiable at  $a$  if the following exists in  $\mathbb{R}$ :

$$\lim_{h \rightarrow 0} \left( \frac{f(a + h) - f(a)}{h} \right).$$

*Differentiability at a value  $a$  implies continuity at  $a$ .*

## 4.2 Properties of Differentiation

### 4.2.1 Closure of the set of differentiable functions

Let  $f, g$  be differentiable functions. The set of differentiable functions is closed under:

- **Addition:**  $(f + g)' = f' + g'$
- **Multiplication:**  $(fg)' = f'g + fg'$
- **Division:**  $\frac{f}{g} = \frac{f'g - fg'}{g^2}$  (for  $g$  non-zero)
- **Composition:**  $(f \circ g)' = g'(f' \circ g)$ .

### 4.2.2 The implications of zero derivatives

For a differentiable function  $f$ :

- $f(x_0)$  is a maximum or minimum  $\Rightarrow f'(x_0) = 0$
- $f'(x_0) = 0, f''(x_0) > 0 \Rightarrow f(x_0)$  is a minimum
- $f'(x_0) = 0, f''(x_0) < 0 \Rightarrow f(x_0)$  is a maximum
- $f'(x) = 0 (\forall x \in [a, b]) \Rightarrow f$  is constant on  $[a, b]$ .

### 4.2.3 Rolle's Theorem and the Mean Value Theorem

For  $f$  continuous on  $[a, b]$  and differentiable on  $(a, b)$ ,

$$(f(a) = f(b)) \Rightarrow (\exists x_0 \in (a, b) \text{ such that } f'(x_0) = 0) \quad (\text{Rolle's Theorem})$$

$$\exists x_0 \in (a, b) \text{ such that } f'(x_0) = \frac{f(b) - f(a)}{b - a}. \quad (\text{Mean Value Theorem})$$

*Rolle's Theorem is a special case of the Mean Value Theorem. The Mean Value Theorem says that over an interval, the derivative is equal to the average derivative across the interval at some value.*

### 4.2.4 Cauchy's Mean Value Theorem

For  $f, g$  continuous on  $[a, b]$  and differentiable on  $(a, b)$ ,

$$\exists x_0 \in (a, b) \text{ such that } [(f(b) - f(a))g'(x_0) = (g(b) - g(a))f'(x_0)].$$

### 4.2.5 Other properties of the derivative

For  $f, g$  differentiable functions,

- $(f'(x) = g'(x) (\forall x \in [a, b])) \Rightarrow (f(x) = g(x) + c (c \in \mathbb{R}))$
- $f'(x) > 0 (\forall x \in [a, b]) \Rightarrow f$  is strictly increasing (similarly for strictly decreasing).

## 4.3 L'Hôpital's Rule

For  $f, g$  differentiable functions defined on a  $\delta$ -neighbourhood of  $a \in \mathbb{R}$ , if:

- $g'(x) \neq 0 (\forall x \in (a - \delta, a + \delta) \setminus \{a\})$
- $f(a) = g(a) = 0$
- $\lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$  exists in  $\mathbb{R}$ .

Then,

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}.$$

*So, if we have two functions that equal zero at a value, this rule helps us find the derivative of their quotient as long as the denominator isn't zero nearby.*

## 5 Calculus

### 5.1 Differentiability on a Closed Interval

We say that a function  $f : [a, b] \rightarrow \mathbb{R}$  is differentiable on  $[a, b]$  if it's differentiable on  $(a, b)$  and the left and right derivatives at  $a$  and  $b$  exist in  $\mathbb{R}$ .

### 5.2 The Fundamental Theorem of Calculus

Let  $f : [a, b] \rightarrow \mathbb{R}$  be a continuous function, we have the following is differentiable on  $[a, b]$  and  $F' = f$  on  $[a, b]$ :

$$F(x) := \int_a^x f(t) dt.$$

A result is that, for  $G$  differentiable on  $[a, b]$  with  $G' = f$ :

$$\int_a^b f(x) dx = G(b) - G(a).$$

## 5.3 Integration by Parts

For  $f, g : [a, b] \rightarrow \mathbb{R}$  continuously differentiable (differentiable with their derivatives being continuous):

$$\int_a^b f'(x)g(x) dx = [f(x)g(x)]_a^b - \int_a^b f(x)g'(x) dx.$$

## 5.4 Integration by Substitution

Let  $f : [a, b] \rightarrow \mathbb{R}$  be a continuous function, suppose  $\phi : [c, d] \rightarrow [a, b]$  is continuously differentiable (differentiable with their derivatives being continuous) and bijective. Then we have:

$$\int_a^b f(x) dx = \int_c^d f(\phi(t))\phi'(t) dt$$

## 5.5 Taylor's Theorem

### 5.5.1 Polynomial coefficients

If for  $p : \mathbb{R} \rightarrow \mathbb{R}$  a polynomial function with degree  $n$  and coefficients  $a_0, a_1, \dots, a_n$  we have:

$$p(x) = \sum_{i=0}^n a_i x^i,$$

and we have that:

$$a_k = \frac{1}{k!} f^{(k)}(0). \quad (k \in \{0, 1, \dots, n\})$$

### 5.5.2 Taylor Polynomials

For  $a \in \mathbb{R}$ , suppose the function  $f$  is  $n$  times differentiable on a  $\delta$ -neighbourhood of  $a$ , the Taylor polynomial for  $f$  degree  $n$  around  $a$  is defined by:

$$\begin{aligned} T_n(a, x) &:= f(a) + f'(a)(x - a) + \dots + \frac{f^{(n)}(a)}{n!}(x - a)^n & (x \in \mathbb{R}) \\ &:= \sum_{i=0}^n \frac{f^{(i)}(a)}{i!}(x - a)^i. \end{aligned}$$

### 5.5.3 Taylor's Theorem

For  $a \in \mathbb{R}$ , suppose the function  $f$  is  $n$  times differentiable on a  $\delta$ -neighbourhood of  $a$ . For some  $c$  between  $a$  and  $x$ :

$$f(x) = f(a) + f'(a)(x - a) + \cdots + \frac{f^{(n-1)}(a)}{(n-1)!}(x - a)^{n-1} + R_n$$
$$R_n = \frac{f^{(n)}(c)}{n!}(x - a)^n. \quad (\text{Lagrange Form})$$

## 6 Series

### 6.1 The Limit Superior and Limit Inferior

#### 6.1.1 Definition of the limit superior and limit inferior

The limit inferior and superior gives bounds on the limit of a subsequence. This can be used in conjunction with the Bolzano-Weierstrass Theorem (bounded sequences have convergent subsequences). For a sequence  $(a_n)_{n \in \mathbb{N}}$ , we have the following definitions:

- **Limit superior:**  $\limsup_{n \rightarrow \infty}(a_n) := \lim_{n \rightarrow \infty}(\sup\{a_k : k \geq n\})$
- **Limit inferior:**  $\liminf_{n \rightarrow \infty}(a_n) := \lim_{n \rightarrow \infty}(\inf\{a_k : k \geq n\})$ .

#### 6.1.2 Properties of the $\limsup$ and $\liminf$

For some sequence  $(a_n)_{n \in \mathbb{N}}$ , some direct consequences of the definition are:

- $a_n$  bounded above  $\Rightarrow \limsup_{n \rightarrow \infty}(a_n) \in \mathbb{R}$
- $a_n$  bounded below  $\Rightarrow \liminf_{n \rightarrow \infty}(a_n) \in \mathbb{R}$
- $a_n$  not bounded above  $\Rightarrow \limsup_{n \rightarrow \infty}(a_n) = \infty$
- $a_n$  not bounded below  $\Rightarrow \liminf_{n \rightarrow \infty}(a_n) = -\infty$
- If the  $\limsup$  or  $\liminf$  exists in  $\mathbb{R}$ , then there exists a subsequence such that the limit of the sequence is the  $\limsup$  or  $\liminf$  respectively
- A sequence is convergent if and only if its  $\limsup$  and  $\liminf$  exist in  $\mathbb{R}$  and are equal.

### 6.1.3 Alternate definition of the $\limsup$ and $\liminf$

We have that for some sequence  $(a_n)_{n \in \mathbb{N}}$ , let  $a \in \mathbb{R}$ ,  $\epsilon > 0$ :

- $\limsup_{n \rightarrow \infty} (a_n) = a \Leftrightarrow$ 
  - $\exists N \in \mathbb{N}$  such that  $a_n < a + \epsilon$  for  $n \geq N$
  - $a_m > a - \epsilon$  for infinitely many  $m \in \mathbb{N}$
- $\liminf_{n \rightarrow \infty} (a_n) = a \Leftrightarrow$ 
  - $\exists N \in \mathbb{N}$  such that  $a_n > a - \epsilon$  for  $n \geq N$
  - $a_m < a + \epsilon$  for infinitely many  $m \in \mathbb{N}$ .

## 6.2 Subsequential Limits

A subsequential limit of a sequence  $(a_n)_{n \in \mathbb{N}}$  is a value  $a \in \mathbb{R}$  such that there exists a subsequence  $(a_{n_k})_{k \in \mathbb{N}}$  where:

$$\lim_{k \rightarrow \infty} (a_{n_k}) = a.$$

If we consider the set of all subsequential limits  $S$ , if the  $\limsup$  or  $\liminf$  of  $a_n$  exist in  $\mathbb{R}$  we have:

- $\limsup_{n \rightarrow \infty} (a_n) = \max(S)$
- $\liminf_{n \rightarrow \infty} (a_n) = \min(S),$

respectively.

## 6.3 Types of Convergence

### 6.3.1 Absolute Convergence

For a series  $\sum_{n=1}^{\infty} (a_n)$ , we have that it is absolutely convergent if  $\sum_{n=1}^{\infty} (|a_n|)$  is convergent.

We have that all absolutely convergent series are convergent.

### 6.3.2 Conditional Convergence

For a series  $\sum_{n=1}^{\infty} (a_n)$ , we have that it is conditionally convergent if it's convergent but not absolutely convergent.



## 6.4 Limit Theorems for Sequences of Functions

### 6.4.1 Uniform convergence under integration

For  $(f_n)_{n \in \mathbb{N}}$  a sequence of regulated functions on  $[a, b]$ , assume  $f_n$  converges uniformly on  $[a, b]$  to some function  $f$ . Let:

$$F_n(x) := \int_a^x f_n(t) dt$$
$$F(x) := \int_a^x f(t) dt.$$

We have that  $(F_n)_{n \in \mathbb{N}}$  converges uniformly to  $F$  on  $[a, b]$ .

*So, we have that uniform convergence is preserved under integration.*

### 6.4.2 Uniform convergence under differentiation

Let  $(f_n)_{n \in \mathbb{N}}$  be a sequence of continuously differentiable (differentiable with their derivatives being continuous) functions on  $[a, b]$ . Assume:

- $(f'_n)_{n \in \mathbb{N}}$  converges uniformly to a function  $g$
- $(f_n(a))_{n \in \mathbb{N}}$  converges in  $\mathbb{R}$ .

Then we have that  $(f_n)_{n \in \mathbb{N}}$  converges uniformly to a function  $f$  such that  $f' = g$  on  $[a, b]$ .

### 6.4.3 Uniform convergence of series

For  $(g_n)_{n \in \mathbb{N}}$  a sequence of real valued functions defined on  $[a, b]$ , define:

$$f := \sum_{k=1}^{\infty} g_k$$
$$f_n := \sum_{k=1}^n g_k.$$

We say  $f$  converges uniformly if  $f_n$  converges pointwise and uniformly to  $f$  on  $[a, b]$ .

## 6.5 Tests for Series Convergence

### 6.5.1 Root Test

For a series  $\sum_{n=1}^{\infty}(a_n)$ , where each  $a_k$  ( $k \in \mathbb{N}$ ) is non-negative, set:

$$\lambda := \limsup_{n \rightarrow \infty} (a_n^{\frac{1}{n}}).$$

We have that:

- $\lambda < 1 \Rightarrow$  convergence
- $\lambda > 1 \Rightarrow$  divergence
- $\lambda = 1 \Rightarrow$  may be convergent or divergent

### 6.5.2 Alternating Series Test

For a series  $\sum_{n=1}^{\infty}(a_n)$ , where each  $a_k$  ( $k \in \mathbb{N}$ ) is positive and  $a_n$  is decreasing with limit 0, we have that the following is convergent:

$$\sum_{n=1}^{\infty} (-1)^n a_n.$$

### 6.5.3 Weierstrass M-Test

For  $(g_n)_{n \in \mathbb{N}}$  a sequence of continuous functions on  $[a, b]$ , assume that for each  $k \in \mathbb{N}$  there exists  $M_k$  such that  $\|g_k\| \leq M_k$  and  $\sum_{k=1}^{\infty} M_k < \infty$ . Then the following converges uniformly on  $[a, b]$ :

$$\sum_{k=1}^{\infty} g_k.$$

*This is saying if we can find a sequence that is convergent as a series and bounds our sequence of functions then the series of our functions is uniformly convergent. Note that  $M_k$  can never be negative as the supremum norm of any function is non-negative.*

#### 6.5.4 Simpler Forms of the M-Test

For  $(g_n)_{n \in \mathbb{N}}$  a sequence of continuously differentiable (differentiable with their derivatives being continuous) functions on  $[a, b]$ . For:

$$f := \sum_{n=1}^{\infty} g_n \Rightarrow f' := \sum_{n=1}^{\infty} g'_n,$$

we have that the sequence of partial sums converges uniformly on  $[a, b]$  to  $f$  and  $f$  is continuously differentiable on  $[a, b]$  if:

$$\sum_{n=1}^{\infty} [\|g_n\| + \|g'_n\|] < \infty.$$

Additionally, for  $(g_n)_{n \in \mathbb{N}}$  a sequence of regulated functions, we define:

$$f := \sum_{n=1}^{\infty} g_n.$$

So, if:

$$\sum_{n=1}^{\infty} \|g_n\| < \infty,$$

then  $f$  converges uniformly on  $[a, b]$  and is regulated.

## 7 Power Series

### 7.1 Definition of Radius of Convergence

The radius of convergence  $R$  for a power series:

$$\mathcal{P} = \sum_{n=0}^{\infty} a_n x^n,$$

is defined as:

$$R := \begin{cases} \infty & \text{if } \mathcal{P} \text{ converges for all } x \in \mathbb{R} \\ r & \text{if } \mathcal{P} \text{ converges if and only if } |x| < r \in \mathbb{R} \\ 0 & \text{if } \mathcal{P} \text{ diverges for all } x \in \mathbb{R} \setminus \{0\}. \end{cases}$$

## 7.2 Tests for the Radius of Convergence

### 7.2.1 Cauchy-Hadamard Theorem

Let  $(a_n)_{n \in \mathbb{N} \cup \{0\}}$  be a sequence of real numbers and define:

$$\alpha := \limsup_{n \rightarrow \infty} |a_n|^{\frac{1}{n}}$$
$$R := \begin{cases} \infty & \text{for } \alpha = 0 \\ 0 & \text{for } \alpha = \infty \\ \frac{1}{\alpha} & \text{otherwise.} \end{cases}$$

We have that the series  $\sum_{n=0}^{\infty} a_n x^n$  has radius of convergence  $R$ .

### 7.2.2 Ratio Test

Let  $(a_n)_{n \in \mathbb{N} \cup \{0\}}$  be a sequence of real numbers such that for:

$$\beta := \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right|,$$

if we have that  $\beta$  exists in  $\mathbb{R}$ ,  $\sum_{n=0}^{\infty} a_n x^n$  has radius of convergence  $R$  defined by:

$$R := \begin{cases} \infty & \text{for } \beta = 0 \\ 0 & \text{for } \beta = \infty \\ \frac{1}{\beta} & \text{otherwise.} \end{cases}$$

## 7.3 Consequences of the Radius of Convergence

### 7.3.1 Preservation of the Radius of Convergence

For a power series  $\sum_{n=0}^{\infty} a_n x^n$  with radius of convergence  $R$ , the following have the same radius of convergence:

$$\sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$\sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}.$$

*The radius of convergence is preserved by integration and differentiation.*

Furthermore, if we have the power series  $\sum_{n=0}^{\infty} a_n x^n$  with radius of convergence  $R$ , then we can define:

$$f : (-R, R) \rightarrow \mathbb{R}; f(x) = \sum_{n=0}^{\infty} a_n x^n.$$

We have that  $f$  is continuously differentiable on  $(-R, R)$  with:

$$f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$\int_0^x f(t) dt = \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1},$$

defined on  $(-R, R)$ .

## 8 Elementary Functions

### 8.1 The Exponential and Logarithm

#### 8.1.1 Definition of the Exponential

We define the exponential function  $\exp : \mathbb{R} \rightarrow \mathbb{R}$  by:

$$\exp(x) := \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

#### 8.1.2 Properties of the exponential

Let  $f$  be the exponential function:

- $f$  is differentiable
- $f = f'$
- $f(0) = 1$
- $f(x)f(y) = f(x+y)$
- $f > 0$
- $f$  is strictly increasing
- $f \rightarrow \infty$  as  $x \rightarrow \infty$
- $f \rightarrow 0$  as  $x \rightarrow -\infty$
- The range of  $f$  is  $(0, \infty)$
- $\frac{f(x)}{x^k} \rightarrow \infty$  as  $x \rightarrow \infty \forall k \in \mathbb{N}$

#### 8.1.3 Euler's number

We call Euler's number  $e := \exp(1)$ . We have that for  $x \in \mathbb{R}$ :

$$\exp(x) = e^x.$$

### 8.1.4 Definition of the natural logarithm

We define the natural logarithm as the inverse of the exponential.

### 8.1.5 Properties of the natural logarithm

Let  $f$  be the natural logarithm:

- $f$  is differentiable
- $f(1) = 0$
- $f$  is increasing
- $f'(x) = 1/x$
- $f(xy) = f(x) + f(y)$
- $f \rightarrow \infty$  as  $x \rightarrow \infty$
- $f(x/y) = f(x) - f(y)$
- $f \rightarrow -\infty$  as  $x \downarrow 0$
- $f(1/x) = -f(x)$
- $\frac{f(x)}{x^k} \rightarrow 0$  as  $x \rightarrow \infty \forall k \in \mathbb{N}$ .

### 8.1.6 The natural logarithm as a power series

For  $x \in (-1, 1)$ :

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}.$$

### 8.1.7 Exponentials

For  $a > 0$ ,  $x \in \mathbb{R}$  we can write:

$$a^x := e^{x \ln(a)}.$$

## 8.2 Trigonometric Functions

### 8.2.1 Definition of sin and cos

We can define the sin and cos functions as follows:

$$\begin{aligned}\sin(x) &:= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots \\ \cos(x) &:= 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \dots\end{aligned}$$

### 8.2.2 Differentiability of sin and cos

These are both differentiable with:

$$\begin{aligned}(\sin)' &= \cos \\ (\cos)' &= -\sin.\end{aligned}$$

### 8.2.3 The Pythagorean Identity

We have that  $\sin^2(x) + \cos^2(x) = 1$ .

## 9 Improper Integrals

### 9.1 Improper Integrals of the First Kind

For  $a \in \mathbb{R}$ ,  $f : [a, \infty) \rightarrow \mathbb{R}$  regulated on  $[a, b]$  for all  $b \geq a$ . We define the improper integral of the first kind as:

$$\int_a^\infty f(x) dx := \lim_{b \rightarrow \infty} \int_a^b f(x) dx.$$

Similarly, we define:

$$\begin{aligned}\int_{-\infty}^b f(x) dx &:= \lim_{a \rightarrow -\infty} \int_a^b f(x) dx \\ \int_{-\infty}^\infty f(x) dx &:= \lim_{a \rightarrow -\infty} \int_a^0 f(x) dx + \lim_{b \rightarrow \infty} \int_0^b f(x) dx.\end{aligned}$$

### 9.2 Convergence of Functions

For  $f : [a, \infty) \rightarrow \mathbb{R}$  continuous:

$$\begin{aligned}\lim_{x \rightarrow \infty} f(x) &\in \mathbb{R} \\ &\Leftrightarrow \\ \forall \epsilon > 0, \exists R > a \text{ such that } \forall x, y > R, [|f(x) - f(y)| < \epsilon].\end{aligned}$$

### 9.3 Sandwiched Improper Integrals

For  $a \in \mathbb{R}$  with  $g : [a, \infty) \rightarrow \mathbb{R}$  non-negative with  $\int_a^\infty g(x) dx$  convergent. So, for  $f : [a, \infty) \rightarrow \mathbb{R}$  with  $|f(x)| < g(x)$  for each  $x$  in  $[a, \infty)$ :

$$\int_a^\infty f(x) dx \in \mathbb{R}.$$

### 9.4 Integral Test

For  $f : [1, \infty) \rightarrow (0, \infty)$  a positive decreasing function, for any  $N \in \mathbb{N}$ :

$$\sum_{n=2}^N f(n) \leq \int_1^N f(x) dx \leq \sum_{n=1}^{N-1} f(n).$$

*This link may help with understanding: [desmos.com/calculator/me8mtv1slr](https://www.desmos.com/calculator/me8mtv1slr).*

### 9.5 Improper Integrals of the Second Kind

For  $f : (a, b] \rightarrow \mathbb{R}$  regulated on  $[a + \delta, b]$  for each  $\delta \geq 0$ . We define the improper integral of the second kind as:

$$\int_a^b f(x) dx := \lim_{\delta \downarrow 0} \int_{a+\delta}^b f(x) dx$$

Like the integral of the first kind, this has similar definitions for the upper and both endpoints being unbounded.