Combinatorics Notes

paraphrased by Tyler Wright

*An important note, these notes are absolutely **NOT** guaranteed to be correct, representative of the course, or rigorous. Any result of this is not the author's fault.*

0 Notation

We commonly deal with the following concepts in Combinatorics which I will abbreviate as follows for brevity:

Term	Notation
The vertex set of a graph G	$V(G)$
The edge set of a graph G	$E(G)$

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1 Counting

1.1 The Multiplication Rule

If a counting problem can be split into a number of stages, we can use the product of the number of choices at each stage to find the total number of outcomes.

1.2 Inclusion-Exclusion Principle

For n in $\mathbb{Z}_{>0}$, and $X_1, \ldots, X_n$ sets:

$$\begin{aligned}\left|\bigcup_{i=1}^{n} X_i\right| &= \sum_{i=1}^{n} |X_i| \\ &- \sum_{i_1 \neq i_2} |X_{i_1} \cap X_{i_2}| \\ &+ \sum_{i_1 \neq i_2 \neq i_3} |X_{i_1} \cap X_{i_2} \cap X i_3| \\ &\ldots.\end{aligned}$$

1.3 The Factorial

For n in $\mathbb{Z}_{\geq 0}$ we can define the factorial $n!$:

$$n! := \begin{cases} 1 & \text{n} = 0 \\ \prod_{i=1}^{n}(i) & \text{otherwise.} \end{cases}$$

For k in $\mathbb{Z}_{>0}$ we can further define $(n)_k$:

$$(n)_k := \frac{n!}{(n-k)!} = n(n-1)(n-2)\cdots(n-k+1).$$

This can be though of as the factorial with k elements (starting at n). So, $(n)_n = n!$, $(n)_1 = n$.

1.4 The Binomial Coefficient

For n, k in $\mathbb{Z}_{\geq 0}$, we can define the binomial coefficient:

$$\binom{n}{k} := \frac{n!}{k!(n-k)!} = \frac{(n)_k}{k!}.$$

Furthermore, we have:

$$\binom{n}{k} = \binom{n}{n-k}.$$

There are some notes to be made on the definition:

- $\binom{n}{k} = 0$ if $k > n$,
- $\binom{n}{k} \geq 0$.

1.4.1 Pascal's Identity

For n, k in $\mathbb{Z}_{\geq 0}$:

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

Proof. Suppose we are making an unordered selection of k elements from n elements without repeats. This gives $\binom{n}{k}$ possibilities. Suppose we take some element in our set of n elements and fix it - there are two cases:

- We include it in our set of k elements, giving $\binom{n-1}{k-1}$ possibilities,
- We exclude it from our set of k elements, giving $\binom{n-1}{k}$ possibilities.

By the addition rule, we get the result as required. □

1.5 The Binomial Theorem

By performing induction on Pascal's identity, we can see that for a, b in $\mathbb{C}$ and n in $\mathbb{Z}_{\geq 0}$:

$$(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^i b^{n-i}.$$

Setting $a = b = 1$, we get $2^n = \sum_{i=0}^{n} \binom{n}{i}$.

1.6 The Pigeonhole Principle

For m, n, k in $\mathbb{Z}_{>0}$, if we have k objects being distributed into n boxes and $n > mk$ then one box must contain at least $k+1$ objects.

1.7 Selection

For this section, we will consider n, k in $\mathbb{Z}_{>0}$.

1.7.1 Ordered Selection with Repeats

As we select, we have n choices, and we select k times. Thus, by the Multiplication Rule, we get n^k outcomes.

1.7.2 Ordered Selection without Repeats

As we select, the amount of choices we have decreases by one each time. We start with n choices and select k times. Thus, by the Multiplication Rule, we get $n(n-1)\cdots(n-k+1) = (n)_k$ outcomes.

1.7.3 Unordered Selection with Repeats

Let the set we are selecting from be $\{x_1, \ldots, x_n\}$. In this case, any solution can be aggregated into a list indicating how many times the i^{th} element was selected (for some i in $[n]$). For example, if we select x_1 three times and x_2 five times, the outcome would be of the form $\{3, 5, \ldots\}$.

It can be seen that for each of these solutions, the sum of the elements in the set must equal k. We can construct a solution by starting with a set of all zeroes $\{0, 0, 0, \ldots\}$ and distributing k into the set. For example, for $n = 4$ and $k = 3$ the following are solutions:

$$\begin{aligned}
\{1,1,1,0\} &\text{ as } 1+1+1+0 = 3 = k, \\
\{0,2,0,1\} &\text{ as } 0+2+0+1 = 3 = k, \\
\{3,0,0,0\} &\text{ as } 3+0+0+0 = 3 = k.
\end{aligned}$$

These solutions correspond to $\{x_1, x_2, x_3\}, \{x_2, x_2, x_4\}, \{x_1, x_1, x_1\}$ respectively.

This distribution of k can be thought of as seperating k into n groups. For example, the solution $\{1, 1, 0, 1\}$ corresponds to:

$$\bullet | \bullet || \bullet \,.$$

The dots and dividers are identical respectively, and we have a total of k dots plus $n-1$ dividers equalling $k+n-1$ elements. We can choose where to place the dividers beforehand and then fill in the dots, thus we have:

$$\binom{n-1+k}{n-1},$$

choices.

1.7.4 Unordered Selection without Repeats

This is identical to to the ordered case but we divide by the number of permutations of the solutions as order does not matter. Thus, we get:

$$\frac{(n)_k}{k!} = \binom{n}{k}.$$

2 Generating Functions

For a sequence $(a_n)_{n\geq 0}$, we can associate a power series:

$$f(x) = \sum_{k=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots .$$

We say $f(x)$ is the generating function of (a_n), or write:

$$\begin{aligned} a_0, a_1, a_2, \ldots &\leftrightarrows a_0 + a_1 x + a_2 x^2 + \cdots \\ (a_n)_{n\geq 0} &\leftrightarrows f(x). \end{aligned}$$

Note, however, that this doesn't imply that the series is convergent.

2.1 Generating Functions of Finite Sequences

For finite sequences (or rather, sequences with finitely many non-zero terms), we have that their generating functions can be written as polynomials.

2.2 Useful Generating Functions

The following generating functions are useful to know:

$$\begin{aligned} 1, 1, 1, \ldots &\leftrightarrows 1 + x + x^2 + \cdots = \frac{1}{1-x} \\ 1, -1, 1, -1, \ldots &\leftrightarrows 1 - x + x^2 - x^3 + \cdots = \frac{1}{1+x} \\ \left(\binom{n}{k}\right)_{k\geq 0} &\leftrightarrows (1+x)^n \\ \left(\binom{n-1+k}{n-1}\right)_{k\geq 0} &\leftrightarrows \frac{1}{(1-x)^n} \\ 1, 2, 3, \ldots &\leftrightarrows 1 + 2x + 3x^2 + \cdots = \frac{1}{(1-x)^2} \\ 1, 4, 9, \ldots &\leftrightarrows 1 + 2x + 3x^2 + \cdots = \frac{1+x}{(1-x)^3} \\ 1, 0, 1, 0, 1, \ldots &\leftrightarrows 1 + x^2 + x^4 + \cdots = \frac{1}{1-x^2} \\ 1, \underbrace{0, \ldots, 0}_{n \text{ zeroes}}, 1, \ldots &\leftrightarrows 1 + x^n + x^{2n} + \cdots = \frac{1}{1-x^{n+1}}, \end{aligned}$$

2.3 The Scaling Rule

For a sequence $(a_n)_{n\geq 0}$ with an associated generating function $f(x)$ and c in $\mathbb{R}$:

$$(ca_n)_{n\geq 0} \leftrightarrows cf(x).$$

2.4 The Addition Rule

For the sequences $(a_n)_{n\geq 0}$, $(b_m)_{m\geq 0}$ with the associated generating functions $f(x), g(x)$ respectively:

$$(a+b)_{n\geq 0} \leftrightarrows f(x) + g(x).$$

2.5 The Right-Shift Rule

For a sequence $(a_n)_{n\geq 0}$ with an associated generating function $f(x)$, we can add k in $\mathbb{Z}_{\geq 0}$ leading zeroes by multiplying the sequence by x_k:

$$\underbrace{0, \ldots, 0}_{k \text{ zeroes}}, a_0, a_1, \ldots \leftrightarrows x^k f(x).$$

2.6 The Differentiation Rule

For a sequence $(a_n)_{n\geq 0}$ with an associated generating function $f(x)$, we have that:

$$a_1, 2a_2, 3a_3, \ldots \leftrightarrows \frac{d}{dx} f(x).$$

2.7 The Convolution Rule

For the sequences $(a_n)_{n\geq 0}, (b_m)_{m\geq 0}$ with associated generating functions $f(x), g(x)$ respectively. We have that:

$$c_0, c_1, c_2, \ldots \leftrightarrows f(x) \cdot g(x),$$

where:

$$c_n := \sum_{i=0}^{n} a_i b_{n-i} = a_0 b_n + a_1 b_{n-1} + \cdots + a_{n-1} b_1 + a_n b_0.$$

2.8 The Negative Binomial Theorem

For all n in $\mathbb{Z}_{>0}$, we have that:

$$(1+x)^{-n} = \sum_{k=0}^{\infty} (-1)^k \binom{n+k-1}{n-1} x^k.$$

3 Combinatorial Designs

3.1 Set Systems

For V a finite set, we let B be a collection of subsets of V. We call the pair (V, B) a set system with ground set V.

3.1.1 k-uniformity of Set Systems

For a set system (V, B), if for all elements in B, each element has the same cardinality k, we have that (V, B) is k-uniform.

3.2 Block Designs

For v, k, t, λ integers, we suppose:

$$v > k \geq t \geq 1, \qquad \lambda \geq 1.$$

A block design of type:

$$t - (v, k, \lambda),$$

is a set system (V, B) with the following properties:

- V has size v,
- (V, B) is k-uniform,
- Each t-element subset of V is contained in exactly λ 'blocks' (elements of B).

3.2.1 The Quantity of Blocks in a Block Design

For a block design of type $t - (v, k, \lambda)$, we have that the number of blocks b can be derived as follows:

$$b = \frac{\lambda \binom{v}{t}}{\binom{k}{t}}.$$

Proof. We show this by double counting. Take $(V, \mathcal{B})$ to be the associated set system. We consider N the number of pairs (T, B) where T is some t-element subset of V and B is a block containing all of T.

If we consider the choices of T first:

$$N = \binom{v}{t} \cdot \lambda,$$

and if we consider B first:

$$N = b \cdot \binom{k}{t}.$$

By some simple rearranging, we get the result. $\square$

3.3 The Replication Number

In a block design of type $2-(v, k, \lambda)$, every element lies in exactly r blocks where:

$$r(k-1) = \lambda(v-1), \qquad bk = vr.$$

r is the replication number.

Proof. We show this by double counting. Take $(V, \mathcal{B})$ to be the associated set system. We fix v in V and consider N the number of pairs (T, B) where T is some 2-element subset (containing v) of V and B is a block containing all of T. If we consider the choices of T first:

$$N = (v-1) \cdot \lambda,$$

and if we consider B first:

$$N = r(k-1),$$

as there are r blocks containing v and $k-1$ other elements in each block that can form a 2-element subsets with v. If T is instead a 1-element subsets:

$$N = bk,$$

as there are b blocks each with k elements, or:

$$N = vr,$$

because each element appears in r blocks. $\square$

3.4 Fisher's Inequality

For (V, B) a block design of type $2-(v, k, \lambda)$ with $v > k$, we have that:

$$|B| \geq |V|.$$

3.4.1 Incidence Matrices

For a set system (V, B) with $|V| = v$ and $|B| = b$ we define the incidence matrix A as a matrix in $M_{v,b}$ where $A = (a_{ij})$ and:

$$a_{ij} = \begin{cases} 1 & \text{if element } i \text{ is in block } j \\ 0 & \text{otherwise.} \end{cases}$$

4 Graph Theory

4.1 Graphs

A graph G is a set system (V, E) where the elements of E have size 2. Some definitions and facts follow from the definition:

- The elements of V are **vertices**,
- The elements of E are called **edges**,
- The size of V is often called the **order** of G,
- G is a 2-uniform set with ground set V,
- u, v in V are adjacent if $\{u, v\}$ is in E.

4.1.1 Graph Isomorphisms

For two graphs $G_1 = (V_1, E_1)$, $G_2 = (V_2, E_2)$, we say that G_1 and G_2 are isomorphic ($G_1 \cong G_2$) if there exists a bijection $\phi : V_1 \to V_2$ such that for each pair of vertices u, v in V we have that:

$$\{u, v\} \in E_1 \iff \{\phi(u), \phi(v)\} \in E_2.$$

4.2 Neighbourhood and Degree

For a graph $G = (V, E)$ the neighbourhood of v in V is the set of all adjacent vertices (denoted by $N_G(v)$). The neighbourhood of a set S is simply the union of the neighbourhoods of the elements of S (minus the vertices in S). The degree of v is simply the size of $N_G(v)$ denoted by $\deg(v)$.

4.2.1 Minimum and Maximum Degree

For a graph $G = (V, E)$ we have that the following to represent minimum and maximum degree:

$$\delta(G) := \min\{\deg(v) : v \in V\}$$
$$\Delta(G) := \max\{\deg(v) : v \in V\}.$$

4.3 The Handshake Lemma

For a graph $G = (V, E)$, we have that:

$$|E| = \frac{\sum_{v \in V} \deg(v)}{2},$$

as each edge visits two vertices, contributing twice to the sum of the degrees of a graph.

4.4 Subgraphs

A graph $G' = (V', E')$ is a subgraph of $G = (V, E)$ if $V' \subseteq V$ and $E' \subseteq E$ such that for all e in E' we have that $e \subseteq V'$.

4.4.1 Induced Subgraphs

An induced subgraph generated of $G = (V, E)$ is a subgraph $G' = (V', E')$ where:

$$E' = \{\{u, v\} \in E \text{ such that } u, v \in V'\}.$$

Induced subgraphs are generated from a subset of the vertices of a graph by selecting all the edges that are subsets of our chosen vertex set.

4.5 Complements of Graphs

For a graph $G = (V, E)$, we have that $\bar{G} = (V, \bar{E})$ is the complement of G where $\bar{E} = \{\{u, v\} : u, v \in V\} \setminus E$.

4.6 Walks

A walk of length is a set of vertices connected by edges. Its length is the number of edges it traverses.

A walk is closed if its first and last vertex are identical.

4.6.1 Types of Walks

Name	Closed?	Repeats vertices?	Repeats edges?
Walk	Not necessarily	Can	Can
Trail	Not necessarily	Can	Cannot
Paths	Not necessarily	Cannot	Cannot
Circuit	Yes	Can	Cannot
Cycles	Yes	Cannot	Cannot

4.6.2 Walks in Paths and Paths in Walks

Consider $G = (V, E)$ with u, v in V, we have that:

$$\text{There's a walk between } u \text{ and } v \iff \text{There's a path between } u \text{ and } v.$$

Thus, where there's a cycle, there's a circuit and vice-versa. Additionally, if G admits an odd circuit, there's also an odd cycle (and the converse holds too).

4.7 Connected Graphs

A graph is connected if there exists a path between any two vertices in the graph.

4.7.1 Connected Components

A component of a graph G is a maximally connected subgraph of G.

4.8 Euler Circuits

An Euler circuit is a circuit in which each edge in a graph is traversed exactly once. As a consequence, each vertex is travelled at least once. Graphs with Euler circuits are said to be Eulerian.

4.8.1 Partitioning Even Regular Graphs

For a graph $G = (V, E)$, if each vertex has even degree, we can partition its edge set into disjoint subsets $E_1, \dots, E_s$ such that for each i in $[s]$, E_i is the edge set of a cycle.

Proof. Supposing each v in V has even degree, if $E = \emptyset$ then the statement holds trivially. Suppose E is non-empty and the statement holds for all graphs with strictly fewer edges. We pick v in V and generate a path P (starting at v) by checking if the current end of our path has an edge connecting to some v' in P. If it does, we have cycle. If not, there will always be an edge to choose as we entered the vertex and it has even degree (so there must be another edge to leave it). As the edge set is finite, this process must end, giving us a circuit (so a cycle). As we have generated a cycle C, we create $G' = (V, E \setminus E(C))$. But now $|E(G')| < |E|$ so we can split its edge set into disjoint subsets $E_1, \dots, E_s$ satisfying the statement. Thus, $E_1, \dots, E_s, E(C)$ satisfies the statement for G. $\square$

4.8.2 Conditions for an Euler Circuit

An Euler circuit in a connected graph $G = (V, E)$ exists if and only if each vertex in V has even degree.

Proof. If G has an Euler circuit, the circuit must enter and exit each v in V an even number of times. Thus, the degree of each vertex is even. If each v in V has even degree, consider (4.8.1), partitioning E into disjoint subsets $E_1, \ldots, E_s$ all edge sets of cycles. Taking $V(E_i)$ to be the vertex set traversed by E_i for all i in $[s]$, we have that $V(E_1)$ must share a vertex with some $V(E_i)$ for some i in $[s]$ as otherwise this would contradict the connectivity of G. We stitch the edge sets together to form a circuit starting at some intersection of $V(E_1)$ and $V(E_i)$ and traversing all of E_1 then E_i. We repeat this until there is only one edge set which must be our Euler circuit as its edge set is the union of a partition of the edge set. $\square$

4.9 Hamiltonian Cycles

A Hamiltonian cycle is a cycle that visits each vertex exactly once. Graphs with Hamiltonian cycles are said to be Hamiltonian.

4.9.1 Hamiltonian Paths

A Hamiltonian path is a path that visits each vertex exactly once.

4.9.2 Dirac's Theorem

For a graph $G = (V, E)$ where $n = |V| \geq 3$:

$$\delta(G) \geq \frac{n}{2} \Rightarrow G \text{ is Hamiltonian.}$$

Proof. Observe that for some x, y in V if $\{x, y\}$ is not in E, then we have that as $|V \setminus \{x, y\}| = n - 2$ and $|N_G(x)| \geq \frac{n}{2}$, and $|N_G(y)| \geq \frac{n}{2}$:

$$N_G(x) \cap N_G(y) \neq \emptyset,$$

by the Pigeonhole principle. Take $P = (x_1, \ldots, x_k)$ to be the longest path in G. We have that $k \geq 3$ as G is connected on at least 3 vertices. Also, we can assume G has no k-cycle as:

- If $k = n$, we have the desired Hamiltonian cycle,
- If $k < n$, we have a k-cycle in G, but as G is connected we can take some x in $N_G(P)$ and connect it to P to form a path of length $k + 1$ contradicting the maximality of P.

Thus, we have that $\{x_1, x_k\}$ is not in E. Also, we have that for any i in $\{2, \ldots, k-1\}$, we can't have $\{x_1, x_i\}$ and $\{x_{i-1}, x_k\}$ in E as that would form a k-cycle P_k:

$$P_k = (x_1, x_i, \ldots, x_k, x_{i-1}, \ldots, x_2). \tag{1}$$

By the maximality of P:

$$\begin{aligned} N_G(x_1) &\subseteq \{x_2, \ldots, x_{k-1}\} \\ N_G(x_k) &\subseteq \{x_2, \ldots, x_{k-1}\}, \end{aligned}$$

as otherwise we could simply connect the element not in our path to end of P, contradicting the maximality of P. It follows that:

$$N_G(x_1) = \{x \in V : \{x_1, x\} \in E\} \qquad \text{and} \qquad N_G(x_i)^+ = \{x_i : x_{i-1} \in N_G(x_k)\},$$

are disjoint subsets of $\{x_2, \ldots, x_k\}$ by the statement describing (1). But, $\{x_2, \ldots, x_k\}$ is of size $k-1$ and; $N_G(x_1)$ and $N_G(x_1)^+$ both have size at least $\frac{n}{2}$. Thus, a contradiction - G has a Hamiltonian cycle. $\square$