Group Theory Notes

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These notes are not necessarily correct, consistent, representative of the course as it stands today or, rigorous. Any result of the above is not the author's fault.

0 Notation

We commonly deal with the following concepts in Group Theory which I will abbreviate as follows for brevity:

Term	Notation
Additive identity of set X	0_X
Multiplicative identity of a set X	1_X
For a field F, $(F\backslash\{0_F\}, \times)$	F^*
(invertible $n \times n$ matrices on $F, \times$)	$GL_n(F)$

Contents

1 The Fundamentals

1.1 Binary Operations

A binary operation on a set X is a map $X \times X \to X$.

Take a binary operation $*$ on a set X, we say that $*$ is associative if for all x, y, z in X:

$$x * (y * z) = (x * y) * z.$$

Furthermore, we say e in X is an identity element of $*$ if for all x in X:

$$e * x = x * e,$$

and we say that y in X is the inverse to x if $x * y$ and $y * x$ are both identities of $*$.

1.2 Groups

A group $(G, *)$ is a non-empty set G combined with a binary operation $*$ such that:

- $*$ is associative,
- G contains an identity for $*$,
- for each element in G, there exists some inverse in G with respect to $*$.

1.2.1 Symmetric Groups

For a set X, the set of bijections $X \to X$ is a group under function composition denoted by $\mathrm{Sym}(X)$. We typically write $\mathrm{Sym}(\{1, 2, \ldots, n\})$ as S_n.

1.2.2 Cyclic Groups

If we consider a regular n-gon P_n, we take rotations of $\frac{2\pi}{n}$ radians about the centre to be r and can define:

$$C_n = \{e, r, r^2, \ldots, r^{n-1}\},$$

to be the group of rotational symmetries of P_n, the cyclic group on P_n.

1.2.3 Dihedral Groups

If we consider again, a regular n-gon P_n and take:

$$r = \text{a rotation of } \frac{2\pi}{n} \text{ radians about the centre,}$$
$$s = \text{reflection in some fixed line of symmetry,}$$

then we have that:

$$\mathrm{Sym}(P_n) = \{e, r, r^2, \ldots, r^{n-1}, s, rs, r^2s, \ldots, r^{n-1}s\},$$

called the dihedral group, denoted by D_{2n}.

1.2.4 The Infinite Cyclic/Dihedral Group

A map φ from $\mathbb{Z} \to \mathbb{Z}$ is a symmetry if for some n and m in $\mathbb{Z}$:

$$|\varphi(m) - \varphi(n)| = |m - n|.$$

Taking r to be the symmetry $n \mapsto n + 1$, we can define the infinite cyclic group:

$$C_\infty = \{\ldots, r^{-2}, r^{-1}, e, r, r^2, \ldots\}.$$

Taking s to be the symmetry $n \mapsto -n$, we can define the infinite dihedral group:

$$D_\infty = \{\ldots, r^{-2}, r^{-1}, e, r, r^2, \ldots, r^{-2}s, r^{-1}s, s, rs, r^2s\}.$$