Combinatorics Notes

paraphrased by Tyler Wright

*An important note, these notes are absolutely **NOT** guaranteed to be correct, representative of the course, or rigorous. Any result of this is not the author's fault.*

0 Notation

We commonly deal with the following concepts in Combinatorics which I will abbreviate as follows for brevity:

Term	Notation
The vertex set of a graph G	$V(G)$
The edge set of a graph G	$E(G)$

Contents

1 Counting

1.1 The Multiplication Rule

If a counting problem can be split into a number of stages, we can use the product of the number of choices at each stage to find the total number of outcomes.

1.2 Inclusion-exclusion Principle

For n in $\mathbb{Z}_{>0}$, and $X_1, \ldots, X_n$ sets:

$$\begin{aligned}\left|\bigcup_{i=1}^{n} X_i\right| &= \sum_{i=1}^{n} |X_i| \\ &- \sum_{i_1 \neq i_2} |X_{i_1} \cap X_{i_2}| \\ &+ \sum_{i_1 \neq i_2 \neq i_3} |X_{i_1} \cap X_{i_2} \cap X i_3| \\ &\ldots .\end{aligned}$$

1.3 The Factorial

For n in $\mathbb{Z}_{\geq 0}$ we can define the factorial $n!$:

$$n! := \begin{cases} 1 & \text{n} = 0 \\ \prod_{i=1}^{n}(i) & \text{otherwise.} \end{cases}$$

For k in $\mathbb{Z}_{>0}$ we can further define $(n)_k$:

$$(n)_k := \frac{n!}{(n-k)!} = n(n-1)(n-2)\cdots(n-k+1).$$

This can be though of as the factorial with k elements (starting at n). So, $(n)_n = n!$, $(n)_1 = n$.

1.4 The Binomial Coefficient

For n, k in $\mathbb{Z}_{\geq 0}$, we can define the binomial coefficient:

$$\binom{n}{k} := \frac{n!}{k!(n-k)!} = \frac{(n)_k}{k!}.$$

Furthermore, we have:

$$\binom{n}{k} = \binom{n}{n-k}.$$

There are some notes to be made on the definition:

- $\binom{n}{k} = 0$ if $k > n$,
- $\binom{n}{k} \geq 0$.

1.4.1 Pascal's Identity

For n, k in $\mathbb{Z}_{\geq 0}$:

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

Proof. Suppose we are making an unordered selection of k elements from n elements without repeats. This gives $\binom{n}{k}$ possibilities. Suppose we take some element in our set of n elements and fix it - there are two cases:

- We include it in our set of k elements, giving $\binom{n-1}{k-1}$ possibilities,
- We exclude it from our set of k elements, giving $\binom{n-1}{k}$ possibilities.

By the addition rule, we get the result as required. □

1.5 The Binomial Theorem

By performing induction on Pascal's identity, we can see that for a, b in $\mathbb{C}$ and n in $\mathbb{Z}_{\geq 0}$:

$$(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^i b^{n-i}.$$

Setting $a = b = 1$, we get $2^n = \sum_{i=0}^{n} \binom{n}{i}$.

1.6 The Pigeonhole Principle

For m, n, k in $\mathbb{Z}_{>0}$, if we have k objects being distributed into n boxes and $n > mk$ then one box must contain at least $k + 1$ objects.

1.7 Selection

For this section, we will consider n, k in $\mathbb{Z}_{>0}$.

1.7.1 Ordered Selection with Repeats

As we select, we have n choices, and we select k times. Thus, by the Multiplication Rule, we get n^k outcomes.

1.7.2 Ordered Selection without Repeats

As we select, the amount of choices we have decreases by one each time. We start with n choices and select k times. Thus, by the Multiplication Rule, we get $n(n-1)\cdots(n-k+1) = (n)_k$ outcomes.

1.7.3 Unordered Selection with Repeats

Let the set we are selecting from be $\{x_1, \ldots, x_n\}$. In this case, any solution can be aggregated into a list indicating how many times the i^{th} element was selected (for some i in $[n]$). For example, if we select x_1 three times and x_2 five times, the outcome would be of the form $\{3, 5, \ldots\}$.

It can be seen that for each of these solutions, the sum of the elements in the set must equal k. We can construct a solution by starting with a set of all zeroes $\{0, 0, 0, \ldots\}$ and distributing k into the set. For example, for $n = 4$ and $k = 3$ the following are solutions:

$$\begin{aligned}
\{1, 1, 1, 0\} &\text{ as } 1 + 1 + 1 + 0 = 3 = k, \\
\{0, 2, 0, 1\} &\text{ as } 0 + 2 + 0 + 1 = 3 = k, \\
\{3, 0, 0, 0\} &\text{ as } 3 + 0 + 0 + 0 = 3 = k.
\end{aligned}$$

These solutions correspond to $\{x_1, x_2, x_3\}, \{x_2, x_2, x_4\}, \{x_1, x_1, x_1\}$ respectively.

This distribution of k can be thought of as seperating k into n groups. For example, the solution $\{1, 1, 0, 1\}$ corresponds to:

$$\bullet | \bullet || \bullet .$$

The dots and dividers are identical respectively, and we have a total of k dots plus $n - 1$ dividers equalling $k + n - 1$ elements. We can choose where to place the dividers beforehand and then fill in the dots, thus we have:

$$\binom{n-1+k}{n-1},$$

choices.

1.7.4 Unordered Selection without Repeats

This is identical to to the ordered case but we divide by the number of permutations of the solutions as order does not matter. Thus, we get:

$$\frac{(n)_k}{k!} = \binom{n}{k}.$$

2 Generating Functions

For a sequence $(a_n)_{n\geq 0}$, we can associate a power series:

$$f(x) = \sum_{k=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots .$$

We say $f(x)$ is the generating function of (a_n), or write:

$$\begin{aligned} a_0, a_1, a_2, \ldots &\leftrightarrows a_0 + a_1 x + a_2 x^2 + \cdots \\ (a_n)_{n\geq 0} &\leftrightarrows f(x). \end{aligned}$$

Note, however, that this doesn't imply that the series is convergent.

2.1 Generating Functions of Finite Sequences

For finite sequences (or rather, sequences with finitely many non-zero terms), we have that their generating functions can be written as polynomials.

2.2 Useful Generating Functions

The following generating functions are useful to know:

$$\begin{aligned} 1, 1, 1, \ldots &\leftrightarrows 1 + x + x^2 + \cdots = \frac{1}{1-x} \\ 1, -1, 1, -1, \ldots &\leftrightarrows 1 - x + x^2 - x^3 + \cdots = \frac{1}{1+x} \\ \left(\binom{n}{k}\right)_{k\geq 0} &\leftrightarrows (1+x)^n \\ \left(\binom{n-1+k}{n-1}\right)_{k\geq 0} &\leftrightarrows \frac{1}{(1-x)^n} \\ 1, 2, 3, \ldots &\leftrightarrows 1 + 2x + 3x^2 + \cdots = \frac{1}{(1-x)^2} \\ 1, 4, 9, \ldots &\leftrightarrows 1 + 2x + 3x^2 + \cdots = \frac{1+x}{(1-x)^3} \\ 1, 0, 1, 0, 1, \ldots &\leftrightarrows 1 + x^2 + x^4 + \cdots = \frac{1}{1-x^2} \\ 1, \underbrace{0, \ldots, 0}_{n \text{ zeroes}}, 1, \ldots &\leftrightarrows 1 + x^n + x^{2n} + \cdots = \frac{1}{1-x^{n+1}}, \end{aligned}$$

2.3 The Scaling Rule

For a sequence $(a_n)_{n\geq 0}$ with an associated generating function $f(x)$ and c in $\mathbb{R}$:

$$(ca_n)_{n\geq 0} \leftrightarrows cf(x).$$

2.4 The Addition Rule

For the sequences $(a_n)_{n\geq 0}$, $(b_m)_{m\geq 0}$ with the associated generating functions $f(x), g(x)$ respectively:

$$(a+b)_{n\geq 0} \leftrightarrows f(x) + g(x).$$

2.5 The Right-shift Rule

For a sequence $(a_n)_{n\geq 0}$ with an associated generating function $f(x)$, we can add k in $\mathbb{Z}_{\geq 0}$ leading zeroes by multiplying the sequence by x_k:

$$\underbrace{0,\ldots,0}_{k \text{ zeroes}}, a_0, a_1, \ldots \leftrightarrows x^k f(x).$$

2.6 The Differentiation Rule

For a sequence $(a_n)_{n\geq 0}$ with an associated generating function $f(x)$, we have that:

$$a_1, 2a_2, 3a_3, \ldots \leftrightarrows \frac{d}{dx} f(x).$$

2.7 The Convolution Rule

For the sequences $(a_n)_{n\geq 0}, (b_m)_{m\geq 0}$ with associated generating functions $f(x), g(x)$ respectively. We have that:

$$c_0, c_1, c_2, \ldots \leftrightarrows f(x) \cdot g(x),$$

where:

$$c_n := \sum_{i=0}^{n} a_i b_{n-i} = a_0 b_n + a_1 b_{n-1} + \cdots + a_{n-1} b_1 + a_n b_0.$$

2.8 The Negative Binomial Theorem

For all n in $\mathbb{Z}_{>0}$, we have that:

$$(1+x)^{-n} = \sum_{k=0}^{\infty} (-1)^k \binom{n+k-1}{n-1} x^k.$$

3 Combinatorial Designs

3.1 Set Systems

For V a finite set, we let B be a collection of subsets of V. We call the pair (V, B) a set system with ground set V.

3.1.1 k-uniformity of Set Systems

For a set system (V, B), if for all elements in B, each element has the same cardinality k, we have that (V, B) is k-uniform.

3.2 Block Designs

For v, k, t, λ integers, we suppose:

$$v > k \geq t \geq 1, \qquad \lambda \geq 1.$$

A block design of type:

$$t - (v, k, \lambda),$$

is a set system (V, B) with the following properties:

- V has size v,
- (V, B) is k-uniform,
- Each t-element subset of V is contained in exactly λ 'blocks' (elements of B).

3.2.1 The Quantity of Blocks in a Block Design

For a block design of type $t - (v, k, \lambda)$, we have that the number of blocks b can be derived as follows:

$$b = \frac{\lambda \binom{v}{t}}{\binom{k}{t}}.$$

Proof. We show this by double counting. Take $(V, \mathcal{B})$ to be the associated set system. We consider N the number of pairs (T, B) where T is some t-element subset of V and B is a block containing all of T.

If we consider the choices of T first:

$$N = \binom{v}{t} \cdot \lambda,$$

and if we consider B first:

$$N = b \cdot \binom{k}{t}.$$

By some simple rearranging, we get the result. □

3.3 The Replication Number

In a block design of type $2-(v,k,\lambda)$, every element lies in exactly r blocks where:

$$r(k-1) = \lambda(v-1), \qquad bk = vr.$$

r is the replication number.

Proof. We show this by double counting. Take $(V, \mathcal{B})$ to be the associated set system. We fix v in V and consider N the number of pairs (T, B) where T is some 2-element subset (containing v) of V and B is a block containing all of T. If we consider the choices of T first:

$$N = (v-1) \cdot \lambda,$$

and if we consider B first:

$$N = r(k-1),$$

as there are r blocks containing v and $k-1$ other elements in each block that can form a 2-element subsets with v. If T is instead a 1-element subsets:

$$N = bk,$$

as there are b blocks each with k elements, or:

$$N = vr,$$

because each element appears in r blocks. □

3.4 Fisher's Inequality

For (V, B) a block design of type $2-(v,k,\lambda)$ with $v > k$, we have that:

$$|B| \geq |V|.$$

3.4.1 Incidence Matrices

For a set system (V, B) with $|V| = v$ and $|B| = b$ we define the incidence matrix A as a matrix in $M_{v,b}$ where $A = (a_{ij})$ and:

$$a_{ij} = \begin{cases} 1 & \text{if element } i \text{ is in block } j \\ 0 & \text{otherwise.} \end{cases}$$

4 Graph Theory

4.1 Graphs

A graph G is a set system (V, E) where the elements of E have size 2. Some definitions and facts follow from the definition:

- The elements of V are **vertices**,
- The elements of E are called **edges**,
- The size of V is often called the **order** of G,
- G is a 2-uniform set with ground set V,
- u, v in V are adjacent if $\{u, v\}$ is in E.

4.1.1 Graph Isomorphisms

For two graphs $G_1 = (V_1, E_1)$, $G_2 = (V_2, E_2)$, we say that G_1 and G_2 are isomorphic ($G_1 \cong G_2$) if there exists a bijection $\phi : V_1 \to V_2$ such that for each pair of vertices u, v in V we have that:

$$\{u, v\} \in E_1 \iff \{\phi(u), \phi(v)\} \in E_2.$$

4.2 Neighbourhood and Degree

For a graph $G = (V, E)$ the neighbourhood of v in V is the set of all adjacent vertices (denoted by $N_G(v)$). The neighbourhood of a set S is simply the union of the neighbourhoods of the elements of S (minus the vertices in S). The degree of v is simply the size of $N_G(v)$ denoted by $\deg(v)$.

4.2.1 Minimum and Maximum Degree

For a graph $G = (V, E)$ we have that the following to represent minimum and maximum degree:

$$\delta(G) := \min\{\deg(v) : v \in V\}$$
$$\Delta(G) := \max\{\deg(v) : v \in V\}.$$

4.3 The Handshake Lemma

For a graph $G = (V, E)$, we have that:

$$|E| = \frac{\sum_{v \in V} \deg(v)}{2},$$

as each edge visits two vertices, contributing twice to the sum of the degrees of a graph.

4.4 Subgraphs

A graph $G' = (V', E')$ is a subgraph of $G = (V, E)$ if $V' \subseteq V$ and $E' \subseteq E$ such that for all e in E' we have that $e \subseteq V'$.

4.4.1 Induced Subgraphs

An induced subgraph generated of $G = (V, E)$ is a subgraph $G' = (V', E')$ where:

$$E' = \{\{u, v\} \in E \text{ such that } u, v \in V'\}.$$

Induced subgraphs are generated from a subset of the vertices of a graph by selecting all the edges that are subsets of our chosen vertex set.

4.5 Complements of Graphs

For a graph $G = (V, E)$, we have that $\bar{G} = (V, \bar{E})$ is the complement of G where $\bar{E} = \{\{u, v\} : u, v \in V\} \setminus E$.

4.6 Walks

A walk of length is a set of vertices connected by edges. Its length is the number of edges it traverses.

A walk is closed if its first and last vertex are identical.

4.6.1 Types of Walks

Name	Closed?	Repeats vertices?	Repeats edges?
Walk	Not necessarily	Can	Can
Trail	Not necessarily	Can	Cannot
Paths	Not necessarily	Cannot	Cannot
Circuit	Yes	Can	Cannot
Cycles	Yes	Cannot	Cannot

4.6.2 Walks in Paths and Paths in Walks

Consider $G = (V, E)$ with u, v in V, we have that:

$$\text{There's a walk between } u \text{ and } v \iff \text{There's a path between } u \text{ and } v.$$

Thus, where there's a cycle, there's a circuit and vice-versa.

4.6.3 Odd Cycles in Odd Circuits

If G a graph admits an odd circuit, there's also an odd cycle in G.

Proof. Take $C = (x_1, \ldots, x_n)$ to be an odd circuit in G. Take i, j in $[n]$ such that $i < j$ and $x_i = x_j$ (if no such i, j exist then C is an odd cycle). As:

$$\begin{aligned} C_1 &= (x_i, x_{i+1}, \ldots, x_j) \\ C_2 &= (x_j, x_{j+1}, \ldots, x_n, x_1, \ldots, x_i), \end{aligned}$$

partition C, their lengths must sum to the length of C, which is odd. Thus, the length of C_1 or C_2 must be odd. Supposing without loss of generality that C_1 is of odd length, if C_1 is a cycle, we are done. If not, we repeat the process on C_1. The length of the circuit we are considering is strictly decreasing by this process, so this must terminate. □

4.7 Connected Graphs

A graph is connected if there exists a path between any two vertices in the graph.

4.7.1 Connected Components

A component of a graph G is a maximally connected subgraph of G.

4.8 Euler Circuits

An Euler circuit is a circuit in which each edge in a graph is traversed exactly once. As a consequence, each vertex is travelled at least once. Graphs with Euler circuits are said to be Eulerian.

4.8.1 Partitioning Even-regular Graphs

For a graph $G = (V, E)$, if each vertex has even degree, we can partition its edge set into disjoint subsets $E_1, \dots, E_s$ such that for each i in $[s]$, E_i is the edge set of a cycle.

Proof. Supposing each v in V has even degree, if $E = \emptyset$ then the statement holds trivially. Suppose E is non-empty and the statement holds for all graphs with strictly fewer edges. We pick v in V and generate a path P (starting at v) by checking if the current end of our path has an edge connecting to some v' in P. If it does, we have cycle. If not, there will always be an edge to choose as we entered the vertex and it has even degree (so there must be another edge to leave it). As the edge set is finite, this process must end, giving us a circuit (so a cycle). As we have generated a cycle C, we create $G' = (V, E \setminus E(C))$. But now $|E(G')| < |E|$ so we can split its edge set into disjoint subsets $E_1, \dots, E_s$ satisfying the statement. Thus, $E_1, \dots, E_s, E(C)$ satisfies the statement for G. $\square$

4.8.2 Conditions for an Euler Circuit

An Euler circuit in a connected graph $G = (V, E)$ exists if and only if each vertex in V has even degree.

Proof. If G has an Euler circuit, the circuit must enter and exit each v in V an even number of times. Thus, the degree of each vertex is even. If each v in V has even degree, consider (4.8.1), partitioning E into disjoint subsets $E_1, \dots, E_s$ all edge sets of cycles. Taking $V(E_i)$ to be the vertex set traversed by E_i for all i in $[s]$, we have that $V(E_1)$ must share a vertex with some $V(E_i)$ for some i in $[s]$ as otherwise this would contradict the connectivity of G. We stitch the edge sets together to form a circuit starting at some intersection of $V(E_1)$ and $V(E_i)$ and traversing all of E_1 then E_i. We repeat this until there is only one edge set which must be our Euler circuit as its edge set is the union of a partition of the edge set. $\square$

4.9 Hamiltonian Cycles

A Hamiltonian cycle is a cycle that visits each vertex exactly once. Graphs with Hamiltonian cycles are said to be Hamiltonian.

4.9.1 Hamiltonian Paths

A Hamiltonian path is a path that visits each vertex exactly once.

4.9.2 Dirac's Theorem

For a graph $G = (V, E)$ where $n = |V| \geq 3$:

$$\delta(G) \geq \frac{n}{2} \Rightarrow G \text{ is Hamiltonian.}$$

Proof. Observe that for some x, y in V if $\{x, y\}$ is not in E, then we have that as $|V \setminus \{x, y\}| = n - 2$ and $|N_G(x)| \geq \frac{n}{2}$, and $|N_G(y)| \geq \frac{n}{2}$:

$$N_G(x) \cap N_G(y) \neq \emptyset,$$

by the Pigeonhole principle. Take $P = (x_1, \ldots, x_k)$ to be the longest path in G. We have that $k \geq 3$ as G is connected on at least 3 vertices. Also, we can assume G has no k-cycle as:

- If $k = n$, we have the desired Hamiltonian cycle,
- If $k < n$, we have a k-cycle in G, but as G is connected we can take some x in $N_G(P)$ and connect it to P to form a path of length $k + 1$ contradicting the maximality of P.

Thus, we have that $\{x_1, x_k\}$ is not in E. Also, we have that for any i in $\{2, \ldots, k-1\}$, we can't have $\{x_1, x_i\}$ and $\{x_{i-1}, x_k\}$ in E as that would form a k-cycle P_k:

$$P_k = (x_1, x_i, \ldots, x_k, x_{i-1}, \ldots, x_2). \tag{1}$$

By the maximality of P:

$$N_G(x_1) \subseteq \{x_2, \ldots, x_{k-1}\}$$
$$N_G(x_k) \subseteq \{x_2, \ldots, x_{k-1}\},$$

as otherwise we could simply connect the element not in our path to end of P, contradicting the maximality of P. It follows that:

$$N_G(x_1) = \{x \in V : \{x_1, x\} \in E\} \qquad \text{and} \qquad N_G(x_i)^+ = \{x_i : x_{i-1} \in N_G(x_k)\},$$

are disjoint subsets of $\{x_2, \ldots, x_k\}$ by the statement describing (1). But, $\{x_2, \ldots, x_k\}$ is of size $k - 1$ and; $N_G(x_1)$ and $N_G(x_1)^+$ both have size at least $\frac{n}{2}$. Thus, a contradiction - G has a Hamiltonian cycle. □

5 Bipartite Graphs

A graph $G = (V, E)$ is bipartite if V can be partitioned into two vertex sets V_1, V_2 such that each edge connects a vertex from V_1 to a vertex in V_2.

5.1 Characterisation of Bipartite Graphs

A graph is bipartite if and only if it contains no odd cycle.

Proof. Suppose we have a graph $G = (V, E)$ bipartite with bipartition (A, B). Consider a path P in G of odd length. Assume, without loss of generality, that the first vertex in P is in A. Thus, as there are only edges connecting A and B, the second vertex must be in B, the third, in A, etc. We can see that as P is odd, the first and last vertex must both be in A. Thus, P cannot be a cycle as there are no edges connecting two vertices in A.

Suppose we have a graph $G = (V, E)$ that admits no odd cycle. If G is not connected, we just consider the connected components of G as we can union bipartitions to of bipartite graphs to form new bipartite graphs. If G is connected, we define $d : V \times V \to \mathbb{Z}_{\geq 0}$ the function returning the length of the shortest path between two vertices in V (well-defined by the connectivity of G). We take some x_0 in V and define:

$$X = \{x \in V : d(x_0, x) \text{ is even}\}$$
$$Y = \{x \in V : d(x_0, x) \text{ is odd}\}.$$

Suppose two vertices x_1, x_2 in X (or Y) are connected, then the paths connecting x_0 to x_1 and x_2 to x_0 can be connected to form an odd circuit. But, odd circuits must contain odd cycles by (4.6.3) which means there must be an odd cycle in G, a contradiction. Thus, X and Y admit no edges between their respective vertices, so G is bipartite with bipartition (X, Y).

□

5.2 The Handshake Lemma for Bipartite Graphs

We have that for $G = (V, E)$ a bipartite graph with bipartition V_1, V_2:

$$\sum_{v \in V_1} \deg(v) = \sum_{v \in V_2} \deg(v).$$

Proof. Each edge has a vertex in V_1 and V_2. □

5.3 Matchings

For $G = (V, E)$ a bipartite graph with bipartition X, Y, a matching from X to Y is a set of edges $M \subseteq E$ such that $f : X \to Y$ defined by:

$$f(x) := y \qquad \text{where } \{x, y\} \in M,$$

is well-defined and injective.

5.4 Hall's Marriage Theorem

For $G = (V, E)$ a bipartite graph with bipartition X, Y:

$$\begin{gathered} G \text{ has a matching from } X \text{ to } Y \\ \iff \\ \text{For all } S \subseteq X, |N(S)| \geq |S|. \end{gathered}$$

Proof. Suppose G has a matching from X to Y, we have that for each x in X, there must be an edge $\{x, y\}$ for some y in Y. Thus, for each $S \subseteq X$, $|N(S)| \geq |S|$. Suppose for all $S \subseteq X$, $|N(S)| \geq |S|$. If $|X| = 1$, $S = X = \{x\}$ meaning $|N(x)| \geq 1$, so there is some edge from x to an element in y. This edge is our matching. Suppose now that $|X| > 1$ and for all sizes of X in $\{1, \dots, |X| - 1\}$, the statement holds.

Case 1 Suppose for all $S \subseteq X$ we actually have that $|N(S)| > |S|$ (a stronger condition). Choose some x in X and some y in Y connected to X. We form G' by removing x, y and, all their incident edges from G. The vertex class of X in G' is strictly smaller than that of X in G. For each vertex in X in G', we have that at most one edge was removed (the one connected to y). Thus, we have that $|N_{G'}(S)| \geq |S|$ so by the inductive hypothesis, G' has a matching. By adding $\{x, y\}$ to this matching, we get a matching for G.

Case 2 Suppose there is some $S \subseteq X$ such that $|N(S)| = |S|$. Consider the graph $G_S = (S \cup N_G(S), E_S)$ the induced subgraph on $S \cup N_G(S)$. By the inductive hypothesis, we have a matching in G_S. Consider $G'_S = (V \setminus V(G_S), E'_S)$ the induced subgraph on $V \setminus V(G_S)$. Observe for each $T \subseteq V(G'_S)$, $|N_G(T \cup S)| \geq |T| + |S|$, thus:

$$|N_{G'_S}(T)| = |N_G(T \cup S) \setminus N_G(S)| \geq |T|.$$

Thus, by the inductive hypothesis, we have a matching from S to $N_G(S)$ and from $X \setminus S$ to $Y \setminus N_G(S)$. Combining these gives a matching in G. □

5.4.1 Degree Constrained Hall's Theorem

For $G = (V, E)$ a bipartite graph with bipartition X, Y:

$$\min\{\deg(x) : x \in X\} \geq \max\{\deg(y) : y \in Y\}.$$

then G has a matching from X to Y (but not necessarily the converse).

Proof. We prove this by double counting. Given $S \subseteq X$, we try to determine the number of edges $|E_S|$ from S to $N(S)$. We know that:

$$|E_S| = \sum_{s \in S} \deg(s) \geq |S| \cdot \min\{\deg(s) : s \in S\}.$$

We can similarly bound it above by:

$$|E_S| = \sum_{s \in N(S)} \deg(s) \leq |N(S)| \cdot \max\{\deg(s) : s \in N(S)\}.$$

Giving us that:

$$|S| \cdot \min\{\deg(s) : s \in S\} \leq |N(S)| \cdot \max\{\deg(s) : s \in N(S)\}. \tag{2}$$

But, by the properties of G:

$$\max\{\deg(s) : s \in N(S)\} \leq \max\{\deg(y) : y \in Y\} \leq \min\{\deg(x) : x \in X\},$$

and we can also see that:

$$\min\{\deg(s) : s \in S\} \geq \min\{\deg(x) : x \in X\}$$

as $S \subseteq X$. So, the following:

$$|S| \cdot \min\{\deg(x) : x \in X\} \leq |N(S)| \cdot \min\{\deg(x) : x \in X\},$$

from (2) implies that $|N(S)| \geq |S|$. As required by (5.4). $\square$

6 Trees and Forests

A graph is a forest if it is acyclic. A tree is a connected forest.

6.1 Leaves

For a tree $T = (V, E)$, for a vertex v in V, v is a leaf if $\deg(v) = 1$.

6.1.1 Existence of Leaves

If $|V| \geq 2$, we have that T has at least two leaves.

Proof. Take a maximal path $P = (x_1, \ldots, x_k)$ in T. Thus, $N(x_1) \subseteq P$ and $N(x_k) \subseteq P$ as otherwise we could add another vertex to our path. But, if x_1 connects to any vertex in $\{x_3, \ldots, x_k\}$ (and similarly for x_k and $\{x_1, \ldots, x_{k-2}\}$) then we have a cycle. So, $N(x_1) = \{x_2\}$ (and $N(x_k) = \{x_{k-1}\}$) so we have two leaves. □

6.2 Characterisation of Trees

We have that for a graph $G = (V, E)$, the following are equivalent:

- G is a tree,
- G is maximally acyclic (G is acyclic and the addition of any edge forms a cycle),
- G is minimally connected (G is connected and the removal of any edge disconnects it),
- G is connected and $|E| = |V| - 1$,
- G is acyclic and $|E| = |V| - 1$,
- Any two vertices in G are connected by a unique path.

6.3 Minimum Spanning Trees

In a graph $G = (V, E)$, a spanning tree $T = (V, E_T)$ is a tree with $E_T \subseteq E$.

A spanning tree on G is minimal if there is no other spanning tree on G with a lower weight.

6.3.1 Existence of Spanning Trees

We have that there is a spanning tree in a graph G if and only if G is connected.

6.3.2 Kruskal's algorithm

For a connected graph $G = (V, E)$ and weight function $w : E \to \mathbb{R}$, we have the following steps to the algorithm:

1. Generate a graph $T = (V, \emptyset)$,
2. Sort the edges in E with respect to w,
3. For each edge $\{u, v\}$ (in increasing order of weights):
   - If u or v are not connected in T, add (u, v) to T,
   - Stop if there are $|V| - 1$ edges in T or if we have run out of edges.

This solves the minimum spanning tree problem. If G is not connected, we get that T has $n - k$ components.

Proof. Take n to be the order of V and T to be the spanning tree output by Kruskal's. Take T' to be any other spanning tree of G. Take $E(T) = \{e_1, \ldots, e_{n-1}\}$ and $E(T') = \{e'_1, \ldots, e'_{n-1}\}$ such that:

$$\begin{aligned} w(e_1) &\leq \cdots \leq w_{e_{n-1}} \\ w(e'_1) &\leq \cdots \leq w_{e'_{n-1}}. \end{aligned}$$

We want to show for each i in $[n-1]$ that:

$$w(e_i) \leq w_(e'_i) \tag{3}$$

Suppose that (3) is not true, so there is some i in $[n-1]$ such that $w(e_i) > w(e'_i)$. Take i to be the least such value that $w(e_i) > w(e'_i)$ and consider $S = \{e_1, \ldots, e_{i-1}\}$ and $S' = \{e'_1, \ldots, e'_i\}$. We have that (V, S) and (V, S') are acyclic as they are subgraphs of trees, so if we can find some e in S' such that $(V, S \cup \{e\})$ is acyclic then we would reach a contradiction as:

$$\begin{aligned} w(e) &\leq w(e'_i) && \text{(as } e \text{ is already in } S'\text{)} \\ &< w(e_i). \end{aligned}$$

Meaning that e has already been considered by our algorithm when building T and should have been added if it didn't form a cycle.

We consider $V_1, \dots, V_k$ the vertex sets of the components of (V, S). Thus, for each j in $[k]$:

$$|S \cap \{\{x, y\} : x, y \in V_j\}| = |V_j| - 1,$$

and summing over all j, we get that $|S| = n - k$. However, (V, S') contains no cycles so we have for each j in $[k]$:

$$|S' \cap \{\{x, y\} : x, y \in V_j\}| \leq |V_j| - 1,$$

and summing over all j, we get that $|S| \leq n - k$. But:

$$|S'| = i = |S| + 1 = n - k + 1,$$

which means there must be an edge connecting distinct components of (V, S). This means $(V, S \cup \{e\})$ is acyclic (e being the edge connecting the components) as required. □

7 Cliques and Independent Sets

A clique of size n in a graph G is a subgraph G' of G isomorphic to K_n. We often call the clique on three vertices (K_3) a triangle. A graph G contains a triangle if it has a 3-clique.

7.1 Mantel's Theorem

For $G = (V, E)$ a graph of order n that contains no triangles, we have that:

$$|E| \leq \left\lfloor \frac{n^2}{4} \right\rfloor = \left\lfloor \left(\frac{n}{2}\right)^2 \right\rfloor .$$

Proof. Suppose $n = 1$, $|E| = 0$ so:

$$0 = |E| \leq \left\lfloor \frac{n^2}{4} \right\rfloor = 0.$$

Suppose $n = 2$, $|E| = 1$ so:

$$1 = |E| \leq \left\lfloor \frac{n^2}{4} \right\rfloor = 1.$$

Suppose $n > 2$ and for any graph of strictly smaller order, we have that the statement holds. Let x, y be vertices in V joined by an edge. We observe that:

$$\deg(x) + \deg(y) \leq n, \tag{4}$$

otherwise, the intersection of their neighbourhood is non-empty giving us a triangle in G. Let $G' = (V \setminus \{x, y\}, E')$ be the induced subgraph on $V \setminus \{x, y\}$. We can see that G' must not contain a triangle otherwise G would contain said triangle. So, we apply the inductive hypothesis to see that:

$$|E'| \leq \frac{(n-2)^2}{4} = \frac{n^2}{4} - n + 1 \tag{5}$$

By applying (4) and the above we see that:

$$\begin{aligned} |E| &\leq |E'| + \overbrace{\deg(x) + \deg(y) - 1}^{\text{edges removed by } G'} \\ &\leq \frac{(n-2)^2}{4} + n - 1 \\ &= \frac{n^2}{4}. \end{aligned}$$

As required. $\square$

7.1.1 Mantel's Theorem on Equality

For $G = (V, E)$ a graph of order n that contains no triangles, we have that:

$$\left[|E| = \left\lfloor \frac{n^2}{4} \right\rfloor\right] \Rightarrow \left[G \cong K_{k,n-k} \text{ where } k = \left\lfloor \frac{n}{2} \right\rfloor\right].$$

Proof. Suppose $n = 1$, $|E| = 0$ so $G \cong K_{0,1}$. Suppose $n = 2$, $|E| = 1$ so $G \cong K_{1,1}$. Suppose $n > 2$ and for any graph of strictly smaller order, we have that the statement holds. Let x, y be vertices in V joined by an edge. We observe that:

$$\deg(x) + \deg(y) \leq n, \tag{6}$$

otherwise, the intersection of their neighbourhood is non-empty giving us a triangle in G. Let $G' = (V \setminus \{x, y\}, E')$ be the induced subgraph on $V \setminus \{x, y\}$. We can see that G' must not contain a triangle otherwise G would contain said triangle. We can see that G' has at most $n - 1$ less edges than G by (6) but Mantel's Theorem says:

$$|E'| \leq \frac{(n-2)^2}{4} = \frac{n^2}{4} - n + 1, \tag{7}$$

meaning G' removes at least $n - 1$ edges from G. This means that the relations in (6) and (7) are actually equality. So, by the inductive hypothesis, G' is isomorphic to $K_{\lfloor (n-2)/2 \rfloor, (n-2) - \lfloor (n-2)/2 \rfloor}$. Taking G' to have bipartition (X, Y), we try adding x and y back to G' to get G. We see that because:

$$\deg(x) + \deg(y) = n,$$

and any edges from x or y can't connected two vertices in X or Y (or that would form a triangle), the only possibility is that x is connected to all of Y and y is connected to all of X (or vice versa) as required. $\square$

8 Planar Graphs

8.1 Arcs

An arc is a subset of $\mathbb{R}^2$ of the type $\sigma : [0,1] \to \mathbb{R}^2$ where σ is an injective, continuous map and $\sigma(0), \sigma(1)$ are the endpoints of the arc. Injectivity here ensures the arc does not cross itself.

8.2 Drawings

For a graph $G = (V, E)$, drawing it is equivalent to assigning:

- A point p in $\mathbb{R}^2$ for each v in V (such that the map from vertices to points is injective),

- An arc σ for each $e = \{x, y\}$ in E (such that σ intersects exactly two points, the points corresponding to x and y).

8.3 Planar Drawings and Graphs

A drawing with a set of arcs A is planar if for each σ_1, σ_2 in A, we have that σ_1, σ_2 either intersect at their endpoints or not at all. A graph is planar if it admits at least one planar drawing.

8.3.1 Examples of Non-planar Subgraphs

We have that K_5 and $K_{3,3}$ are not planar.

8.3.2 Consequences of Non-planar Subgraphs

If a graph has a non-planar subgraph, it can't be planar.

8.4 Jordan Curves

An arc in the plane whose endpoints are equal is called a Jordan curve.

8.4.1 Jordan Curve Theorem

For any Jordan curve C, C divides the plane into exactly two connected regions called the 'interior' and 'exterior'. The curve is the boundary of these regions.

8.5 Faces

For a planar graph G, a face of a drawing of G is a connected region bound by the Jordan curves formed by the arcs of the drawing. The region going off to infinity is the outer face and the rest are inner faces.

8.5.1 Euler's Formula

For a connected graph $G = (V, E)$ where F is the set of faces of a given drawing of G, we have that:

$$|V| - |E| + |F| = 2.$$

Proof. Suppose $|E| = 0$, then $|V| = 1$ and the number of faces is clearly one. Thus:

$$|V| - |E| + |F| = 1 - 0 + 1 = 2,$$

as required. Suppose $|E| \geq 1$ and that Euler's formula holds for all graphs on strictly fewer edges. If G contains no cycles, it is a tree so we know that $|V| = |E| - 1$. As there are no cycles, there are no Jordan curves in our drawing. Thus, we know that $|F| = 1$:

$$|V| - |E| + |F| = |V| - (|V| - 1) + 1 = 2.$$

If we now consider the case where G contains at least one cycle, we fix some e in E in the edge set of the cycle and consider $G' = (V, E \setminus \{e\})$. As e was part of a cycle, G' is connected. So, we apply our inductive hypothesis to see that where F' is the set of faces of the drawing of G' (the drawing for G with the arc corresponding to e removed):

$$\begin{aligned} |V| - |E \setminus \{e\}| + |F'| &= |V| - (|E| - 1) + |F'| \\ &= 2. \end{aligned}$$

We can see that e was adjacent to two distinct faces of the drawing of G. Thus, upon its removal, these two faces became one. So, $|F'| = |F| - 1$ and thus:

$$\begin{aligned} |V| - |E| + |F| &= |V| - |E| + |F'| + 1 \\ &= |V| - (|E| - 1) + |F'| \\ &= 2. \end{aligned}$$

as required. $\square$

8.6 Edge Bound on Planar Graphs

For $G = (V, E)$ a planar graph on at least three vertices:

$$|E| \leq 3(|V| - 2).$$

Proof. Consider a planar drawing on G with a set of faces F. By Euler's formula, we know that:

$$|V| - |E| + |F| = 2.$$

We consider the number N of pairs (e, f) where e is an edge and f is a face bordered by the arc corresponding to e. Since each e in E borders on two faces, $N \leq 2|E|$. Since each face is formed by a cycle which has at least 3 edges, $N \geq 3|F|$. Thus, $2|E| \geq 3|F|$. Combining this with Euler's formula we get that:

$$6 = 3|V| - 3|E| + 3|F| \leq 3|V| - |E|.$$

The result follows. □

9 Graph Colouring

9.1 k-colouring

A k-colouring of a graph $G = (V, E)$ is an assignment of $[k]$ to V performed by $c : V \to [k]$ such that for u, v adjacent vertices in V, $c(u) \neq c(v)$. A graph is k colourable if a k-colouring exists for it.

9.2 Chromatic Number

The chromatic number of a graph G denoted by $\chi(G)$ is the smallest k such that G is k colourable.

9.2.1 Bound on Chromatic Number

For a graph $G = (V, E)$, we have that for some k in $\mathbb{Z}$, if $\Delta(G) \leq k$ then $\chi(G) \leq k+1$.

Proof. Suppose $|V| = 1$, then $\Delta(G) = 0$ and we can see that G is 1-colourable. Suppose $|V| > 1$ and the statement holds for all graphs on strictly fewer vertices. Pick some x in V, let $G' = (V \setminus \{x\}, E')$ be the induced graph on $V \setminus \{x\}$. We have that $\Delta(G') \leq \Delta(G)$ as we only removed edges to form G' from G. Thus, G' is at least $(k+1)$-colourable by the inductive hypothesis. We take some $(k+1)$-colouring of G' and we apply it to G, leaving all but x coloured. As $\deg(x) \leq k$, even if each of the vertices connected to x have distinct colours, there will still be the $(k+1)^{\text{th}}$ colour for x to use. Thus, G is $(k+1)$-colourable as required. $\square$

9.3 Maps

A map is a graph derived from some traditional map where regions correspond to faces, points where at least three regions border each other are vertices and, the border between exactly two regions are edges. We assume these regions are connected and they do not touch solely at a point (or several points).

9.3.1 Dual Graphs

Given a planar graph $G = (V, E)$ and a fixed planar drawing of G, the dual graph $G^* = (V^*, E^*)$ relative to this drawing is a planar graph obtained by assigning a vertex to each face and connecting these vertices by and edge if their corresponding faces border.

These are key in considering colourings of maps as we can consider the coloring of faces of a given graph.

9.3.2 k-colourability of the Dual Graph

We have that for a graph G, G is k-colourable if and only if G^* is k-colourable.

9.3.3 Five Colour Theorem

Every map with corresponding graph $G = (V, E)$ can be coloured with five colours, that is $\chi(G) \leq 5$.

Proof. Suppose $|V| \leq 5$, then we just assign each vertex a unique colour and we are done. Suppose $|V| > 5$ and for all planar graphs on strictly fewer vertices, we have that said graph is 5-colourable. We know that G has a vertex of degree at most five as if all v in V were such that $\deg(v) \geq 6$ then:

$$|E| = \frac{1}{2}\sum_{v \in V} \deg(v) \geq 3|V|,$$

but $|E| \leq 3|V| - 6$ so:

$$3|V| \leq |E| \leq 3|V| - 6,$$

which is impossible. So, taking v in V to have degree at most five, we split our argument into two cases.

Case 1 Suppose $\deg(v) < 5$. We consider $G' = (V \setminus \{v\}, E')$ the induced graph on $V \setminus \{v\}$. By the inductive hypothesis, we have a 5-colouring of G'. Applying this to G, we leave v yet to be coloured. As v has four or fewer neighbours, we can choose a remaining colour from our colour set and colour v.

Case 2 Suppose $\deg(v) = 5$. We fix a planar drawing of G and label the neighbours of v with t, u, x, y, z (clockwise about v with respect to our fixed drawing). By the inductive hypothesis, we have a 5-colouring on the induced subgraph on $V \setminus \{v\}$ (similarly to **Case 1**). Applying this colouring to G, we have yet to colour v. If t, u, x, y, z use at most four colours, we can use the remaining colour in our colour set to colour v. Otherwise, we consider:

$$C : N(v) \to \{c_1, c_2, c_3, c_4, c_5\},$$

where $\{c_1, \ldots, c_5\}$ is the colour set and for each v' in $N(v)$, $C(v')$ maps to the colour of v' in our colouring.

Case 2a Suppose there is no path connecting x and z in G without traversing v. Take V_x to be the set of vertices with a path P leading to x such that:

- For each vertex in P, it is either $C(x)$ or $C(z)$ coloured,
- P does not include v.

We can see that z is not in V_x as we cannot traverse v. We define a new colouring on G where each v' in V_x has its colour changed to $C(x)$ if it was $C(z)$ coloured and vice versa. This must be a valid colouring because if there was a conflict, then that would mean that some $C(x)$ coloured vertex v_x was connected to a $C(z)$ coloured vertex v_z but exactly one of v_x and v_z was not in V_x. But, this violates the definition of V_x as they are connected so the path to one of them defines the path to the other. With this new colouring, $C(x) = C(z)$ so we have a spare colour for v.

Case 2b Suppose there is a path P_{xz} connecting x and z in G without traversing v. This path combined with $\{x, v\}$ and $\{v, z\}$ forms a cycle and hence a Jordan curve enveloping either y or; t and u. Suppose there is a path connecting y and t, then it must use a vertex in the cycle formed by P_{xz} as one of y or t are on the interior of the Jordan curve formed by it. Thus, taking V_y to be the set of vertices with a path P leading to y such that:

- For each vertex in P, it is either $C(y)$ or $C(t)$ coloured,
- P does not include v.

We must have an analogous situation to **Case 2a** as there is no path from y to t solely using vertices in V_y. This ends the proof. $\square$

10 Ramsey Numbers

For some s in $\mathbb{Z}_{>1}$, we let $r(s)$ be the smallest n in $\mathbb{N}$ such that whenever the edges of K_n are 2-coloured, there exists a monochromatic K_s. We have that this exists for all s as chosen above (proven in (10.2)).

10.1 Off-diagonal Ramsey Numbers

For some s, t in $\mathbb{Z}_{>1}$, we let $r(s,t)$ be the least n in $\mathbb{N}$ such that whenever the edges of K_n are 2-coloured with colour set $\{A, B\}$, there exists an A-monochromatic K_s or a B-monochromatic K_t. We have that this exists for all s, t as chosen above (proven in (10.2)).

10.1.1 Properties of the Off-diagonal Ramsey Number

We have for all s, t in $\mathbb{Z}_{>1}$:

- $r(s,s) = r(s)$,
- $r(s,t) = r(t,s)$,
- $r(2,s) = s$.

10.2 Ramsey's Theorem

We have that for all s, t in $\mathbb{Z}_{>2}$, $r(s,t)$ exists and:

$$r(s,t) \leq r(s-1,t) + r(s,t-1). \tag{8}$$

Proof. It suffices to show that if $r(s-1,t)$ and $r(s,t-1)$ exist then (8) is true because then we can show the theorem is true by induction on $s+t$. We have $r(2,2) = 2$ so we suppose $r(s-1,t)$ and $r(s,t-1)$ exist for each s,t in $\mathbb{Z}_{>2}$. Set $a = r(s-1,t)$, $b = r(s,t-1)$, and consider a 2-colouring on K_{a+b}. Pick v in $V(K_{a+b})$, we can see that it has degree $a+b-1$ so must have at least a 'red' neighbours or at least b 'blue' neighbours.

Case 1 Suppose v has a 'red' neighbours. We consider K_a the induced subgraph on the 'red' neighbourhood of v which must contain a 'red' K_{s-1} or a 'blue' K_t. In the former case, a 'red' K_s is formed in K_{a+b} as each vertex in the 'red' K_{s-1} is attached by a 'red' edge to v. In the latter case, the 'blue' K_t is 'blue' in K_{a+b} also.

Case 2 The working is analogous to **Case 1**. □

10.3 An Upper Bound on Ramsey Numbers

For all s, t in $\mathbb{Z}_{>1}$, we have that:

$$r(s,t) \leq 2^{s+t},$$

an consequence of this is that:

$$r(s) \leq 4^s.$$

Proof. Suppose $s = 2$, then as $r(2,t) = t$ we can consider the inequality:

$$t \leq 2^{2+t} = 4 \cdot 2^t,$$

which holds for each t as defined in the theorem. Suppose $s, t > 2$ and that the theorem holds for each s', t' in $\mathbb{Z}_{>1}$ such that $s' + t' < s + t$. We know that:

$$r(s,t) \leq r(s, t-1) + r(s-1, t),$$

and by the inductive hypothesis:

$$r(s,t) \leq 2^{s+t-1} + 2^{s+t-1} = 2^{s+t}.$$

□

10.4 The k-colour Ramsey Number

For some k in $\mathbb{Z}_{>0}$, s in $\mathbb{Z}_{>1}$, we let $r_k(s)$ be the smallest n in $\mathbb{N}$ such that whenever the edges of K_n are k-coloured, there exists a monochromatic K_s. We have that this exists for all k, s as chosen above.

Proof. For $k = 1$, $r_k(s) = s$ as it simply asks which is the smallest n such that K_s is isomorphic to a subgraph of K_n. We have already proven the case for $k = 2$ by our work in this chapter.

Take the colour set to be $\{c_1, \ldots, c_k\}$. For any $k > 2$, we want to show that $r_k(s) \leq r(s, r_{k-1}(s)) = n$ by considering a two colouring on K_n with c_1 and $c_2, \ldots, c_k$. By the definition of n, in a given colouring of K_n, we either get a c_1 coloured K_s or a $c_2, \ldots, c_k$ coloured $K_{r_{k-1}(s)}$. In the former case, we are done. In the latter case, we consider just the subgraph isomorphic to $K_{r_{k-1}(s)}$ and by the definition of r_{k-1} we see that it must contain a c_i coloured K_s for some i in $\{2, \ldots, k\}$ as required. □

10.5 Infinite Ramsey

For a set A and k in $\mathbb{Z}_{>0}$, we have that:

$$A^{(k)} = \{\{a, b\} : a, b \in A, a \neq b\},$$

the set of subsets of A of size two not containing duplicates.

Let $\mathbb{N}^{(2)}$ be 2-coloured, we have that there exists an infinite set $M \subseteq \mathbb{N}$ such that $M^{(2)}$ in $\mathbb{N}^{(2)}$ is monochromatic.

Proof. Take a_1 in $\mathbb{N}$. Since there are infinitely many a in $\mathbb{N}$ such that $\{a_1, a\}$ is in $\mathbb{N}^{(2)}$, there exists an infinite $A_1 \subseteq \mathbb{N} \setminus \{a_1\}$ such that for all b in A_1, we have $\{a_1, b\}$ coloured identically with some colour c_1. We choose a_2 in A_1 and obtain an infinite set $A_2 \subseteq A_1 \setminus \{a_2\}$ such that for all b in A_2, we have $\{a_2, b\}$ coloured identically with some colour c_2. We continue this infinitely obtaining $a_1, a_2, \ldots$ in $\mathbb{N}$ together with the colours $c_1, c_2, \ldots$ such that for i, j in $\mathbb{Z}_{>0}$ with $i < j$, the edge $\{a_i, a_j\}$ has colour c_i. However, since there are only two colours, infinitely many of these colours must agree which gives us our set of values. $\square$