Introduction to Group Theory Notes

paraphrased by Tyler Wright

An important note, these notes are absolutely ***NOT*** *guaranteed to be correct, representative of the course, or rigorous. Any result of this is not the author's fault.*

1 The Basics of Groups

1.1 Binary operations

A binary operation on a set G is a function:

$$* : G \times G \to G.$$

It's just a function that takes two values and gives a single output. Examples are addition, multiplication, and composition.

Such an operation is called **commutative** if:

$$x * y = y * x. \qquad (\forall x, y \in G)$$

1.2 Definition of a Group

A group is a set G paired with a binary operation $*$ such that they satisfy the following:

- **Associativity**: For $x, y, z \in G$, $(x * y) * z = x * (y * z)$
- **Identity**: $\exists\, e \in G$ such that $\forall g \in G$, $e * g = g * e = g$
- **Inverses**: $\forall g \in G$, $\exists\, g^{-1} \in G$ such that $g * g^{-1} = g^{-1} * g = e$.

A group is called commutative or Abelian if all its elements commute with the given operation.

1.3 Consequences of the Definition

1.3.1 Left and right cancellation

We can left and right cancel with inverses:

$$(ax = bx) \Rightarrow (a = b) \qquad (\forall a, b, x \in G)$$
$$(xa = xb) \Rightarrow (a = b). \qquad (\forall a, b, x \in G)$$

However, $ax = xb$ does not imply $a = b$ unless the group is Abelian.

1.3.2 Uniqueness of the identity and inverses

We have uniqueness of certain elements:

- The identity of a group is unique
- The inverse of an element is unique.

1.3.3 Inverse properties

For a group G with elements x, y:

- $(x^{-1})^{-1} = x$
- $(xy)^{-1} = y^{-1}x^{-1}$.

1.3.4 Exponent properties

For a group G with an element x and $m, n \in \mathbb{Z}$:

- $x^{-n} = (x^{-1})^n$
- $(x^n)(x^m) = x^{n+m}$.

However, $(xy)^n$ may not equal $x^n y^n$ unless G is Abelian.

2 Dihedral Groups

2.1 Definition of a Dihedral Group

The dihedral group D_{2n} is the group of symmetries of an n-sided polygon. This group has order $2n$ as is defined as:

$$\begin{aligned} D_{2n} &= \langle a \rangle \cap b\langle a \rangle \\ &= e, a, a^2, \ldots, a^{n-1}, b, ba, ba^2, \ldots, ba^{n-1}. \end{aligned}$$

Where a is a rotation of $\frac{2\pi}{n}$ radians around the centre of the polygon and b is a reflection in the line through vertex 1 and the centre of the polygon.

2.2 Properties of a Dihedral Group

For the dihedral group D_{2n}:

- $a^n = e$
- $b^2 = e$
- $a^n b = ba^{-n}$