

# Combinatorics Notes

*paraphrased by* Tyler Wright

*An important note, these notes are absolutely **NOT** guaranteed to be correct, representative of the course, or rigorous. Any result of this is not the author's fault.*

# 1 Counting Techniques

## 1.1 The Bijection Rule

For  $n$  in  $\mathbb{N}$ , we define  $[n] := \{1, 2, \dots, n\}$ .

For a given set  $X$ , if there exists a bijective function  $f : [n] \rightarrow X$  for some  $n$  in  $\mathbb{N}$ ,  $X$  has  $n$  elements (or rather  $|X| = n$ ).

This can also be achieved by listing out the elements of  $X = \{x_1, x_2, \dots, x_n\}$  as we can use  $f : [n] \rightarrow X$  where  $i$  maps to  $x_i$ .

## 1.2 The Addition Rule

We can count the amount of elements in a given set  $X$  by splitting  $X$  into disjoint sets, counting them, and adding the results.

For  $n$  in  $\mathbb{N}$ , and  $X_1, \dots, X_n$  pairwise disjoint sets:

$$\left| \bigcup_{i=1}^n X_i \right| = \sum_{i=1}^n |X_i|.$$

*For a set of sets  $A$ , pairwise disjoint means for two given sets in  $A$ , they are either disjoint or equal.*

## 1.3 The Multiplication Rule

If a counting problem can be split into a number of stages, we can use the product of the number of choices at each stage to find the total number of outcomes.

*For example, if we want to find how many three digit numbers there are, we can consider it as choosing three digits. We can choose  $1, 2, \dots, 9$  for the first digit and  $0, 1, \dots, 9$  for the rest so we get  $9 \cdot 10^2$  possibilities.*

## 1.4 Inclusion-Exclusion Principle

For  $n$  in  $\mathbb{N}$ , and  $X_1, \dots, X_n$  sets:

$$\begin{aligned} \left| \bigcup_{i=1}^n X_i \right| &= \sum_{i=1}^n |X_i| \\ &\quad - \sum_{i_1 \neq i_2} |X_{i_1} \cap X_{i_2}| \\ &\quad + \sum_{i_1 \neq i_2 \neq i_3} |X_{i_1} \cap X_{i_2} \cap X_{i_3}| \\ &\quad \dots \end{aligned}$$

*Essentially, this says that the size of the union of some finite number of sets is the sum of their sizes, minus the sum of their **paired** intersections, plus the sum of the intersections of **trios**, etc.*

## 1.5 The Factorial

For  $n$  in  $\mathbb{N}$  we can define the factorial  $n!$ :

$$n! := \begin{cases} 1 & n = 0 \\ \prod_{i=1}^n (i) & \text{otherwise.} \end{cases}$$

For  $k$  in  $\mathbb{N}$  we can further define  $(n)_k$ :

$$(n)_k := \frac{n!}{(n-k)!} = n(n-1)(n-2) \cdots (n-k+1).$$

*This can be thought of as the factorial with  $k$  elements (starting at  $n$ ). So,  $(n)_n = n!$ ,  $(n)_1 = n$ , etc.*

## 1.6 The Binomial Coefficient

For  $n, k$  in  $\mathbb{N}$ , we can define the binomial coefficient:

$$\binom{n}{k} := \frac{n!}{k!(n-k)!} = \frac{(n)_k}{k!}.$$

This is the number of ways of choosing  $k$ -element subsets from an  $n$ -element set.

Furthermore, we have:

$$\binom{n}{k} = \binom{n}{n-k},$$

as choosing  $k$  elements is equivalent to choosing  $n - k$  elements to remove.

There are some notes to be made on the definition:

- $\binom{n}{k} = 0$  if  $k > n$
- $\binom{n}{0} = \binom{n}{n} = 1$
- $\binom{n}{k} \geq 0$

## 1.7 Pascal's Identity

Say we are selecting  $k$  elements from an  $n$ -element set (unordered, without repeats). We will see that there are  $\binom{n}{k}$  possibilities. If we fix an element in the set, we can either include said element in our selection or exclude it giving  $\binom{n-1}{k-1}$  and  $\binom{n-1}{k}$  possibilities respectively. Thus:

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

## 1.8 The Binomial Theorem

By performing induction on Pascal's identity, we can see that for  $a, b$  in  $\mathbb{C}$  and  $n$  in  $\mathbb{N}$ :

$$(a + b)^n = \sum_{i=0}^n \binom{n}{i} a^i b^{n-i}.$$

Setting  $a = b = 1$ , we get  $2^n = \sum_{i=0}^n \binom{n}{i}$ .

## 1.9 The Pigeonhole Principle

For  $m, n, k$  in  $\mathbb{N}$ , if we have  $k$  objects being distributed into  $n$  boxes and  $n < mk$  then one box must contain at least  $k + 1$  objects.

## 2 Selection

For this section, we will consider  $n, k$  in  $\mathbb{N}$ .

### 2.1 Ordered Selection with Repeats

As we select, we have  $n$  choices, and we select  $k$  times. Thus, by the Multiplication Rule, we get  $n^k$  outcomes.

### 2.2 Ordered Selection without Repeats

As we select, the amount of choices we have decreases by one each time. We start with  $n$  choices and select  $k$  times. Thus, by the Multiplication Rule, we get  $n(n-1)\cdots(n-k+1) = (n)_k$  outcomes.

### 2.3 Unordered Selection with Repeats

Let the set we are selecting from be  $\{x_1, \dots, x_n\}$ . In this case, any solution can be aggregated into a list indicating how many times the  $i^{\text{th}}$  element was selected (for some  $i$  in  $[n]$ ). For example, if we select  $x_1$  three times and  $x_2$  five times, the outcome would be of the form  $\{3, 5, \dots\}$ .

It can be seen that for each of these solutions, the sum of the elements in the set must equal  $k$ . We can construct a solution by starting with a set of all zeroes  $\{0, 0, 0, \dots\}$  and distributing  $k$  into the set. For example, for  $n = 4$  and  $k = 3$  the following are solutions:

$$\begin{aligned} \{1, 1, 1, 0\} &\text{ as } 1 + 1 + 1 + 0 = 3 = k, \\ \{0, 2, 0, 1\} &\text{ as } 0 + 2 + 0 + 1 = 3 = k, \\ \{3, 0, 0, 0\} &\text{ as } 3 + 0 + 0 + 0 = 3 = k. \end{aligned}$$

These solutions correspond to  $\{x_1, x_2, x_3\}$ ,  $\{x_2, x_2, x_4\}$ ,  $\{x_1, x_1, x_1\}$  respectively.

This distribution of  $k$  can be thought of as separating  $k$  into  $n$  groups. For example, the solution  $\{1, 1, 0, 1\}$  corresponds to:

$$\bullet | \bullet || \bullet .$$

The dots and dividers are identical respectively, and we have a total of  $k$  dots plus  $n - 1$  dividers equalling  $k + n - 1$  elements. We can choose where to place the dividers beforehand and then fill in the dots, thus we have:

$$\binom{k + n - 1}{n - 1}$$

choices.

## 2.4 Unordered Selection without Repeats

This is identical to the ordered case but we divide by the number of permutations of the solutions as order does not matter. Thus, we get:

$$\frac{(n)_k}{k!} = \binom{n}{k}.$$

# 3 Generating Functions

## 3.1 Definition of a Generating Function

For a sequence  $(a_n)_{n \geq 0}$ , we can associate a **formal power series**:

$$f(x) = \sum_{k=0}^{\infty} a_k x^k = a_0 + a_1 x + a_2 x^2 + \cdots.$$

We say  $f(x)$  is the generating function of  $(a_n)$ , or write:

$$\begin{aligned} a_0, a_1, a_2, \dots &\leftrightarrow a_0 + a_1 x + a_2 x^2 + \cdots \\ (a_n)_{n \geq 0} &\leftrightarrow f(x). \end{aligned}$$

Note, however, that this doesn't imply that the series is convergent.

## 3.2 Generating Functions of Finite Sequences

For finite sequences (or rather, sequences with finitely many non-zero terms), we have that their generating functions can be written as polynomials.

### 3.3 The Scaling Rule

For a sequence  $(a_n)_{n \geq 0}$  with an associated generating function  $f(x)$  and  $c$  in  $\mathbb{R}$ :

$$(ca_n)_{n \geq 0} \Leftrightarrow cf(x).$$

### 3.4 The Addition Rule

For the sequences  $(a_n)_{n \geq 0}$ ,  $(b_m)_{m \geq 0}$  with the associated generating functions  $f(x)$ ,  $g(x)$  respectively:

$$(a + b)_{n \geq 0} \Leftrightarrow f(x) + g(x).$$

### 3.5 The Right-Shift Rule

For a sequence  $(a_n)_{n \geq 0}$  with an associated generating function  $f(x)$ , we can add  $k$  in  $\mathbb{N}$  leading zeroes by multiplying the sequence by  $x_k$ :

$$0, \dots, 0, a_0, a_1, \dots \Leftrightarrow x^k f(x).$$

### 3.6 The Differentiation Rule

For a sequence  $(a_n)_{n \geq 0}$  with an associated generating function  $f(x)$ , we have that:

$$a_1, 2a_2, 3a_3, \dots \Leftrightarrow \frac{d}{dx} f(x).$$

*So, each element in the sequence is multiplied by its index and left-shifted by one, with the farthest left term (the constant) removed.*

### 3.7 The Convolution Rule

For the sequences  $(a_n)_{n \geq 0}$ ,  $(b_m)_{m \geq 0}$  with associated generating functions  $f(x)$ ,  $g(x)$  respectively. We have that:

$$c_0, c_1, c_2, \dots \Leftrightarrow f(x) \cdot g(x),$$

where:

$$c_n := \sum_{i=0}^n a_i b_{n-i} = a_0 b_n + a_1 b_{n-1} + \dots + a_{n-1} b_1 + a_n b_0.$$

### 3.8 The Negative Binomial Theorem

For all  $n$  in  $\mathbb{N}$ , we have that:

$$(1+x)^{-n} = \sum_{k=0}^{\infty} (-1)^k \binom{n+k-1}{n-1} x^k.$$

## 4 Combinatorial Designs

### 4.1 Definition of a Set System

For  $V$  a finite set, we let  $B$  be a collection of subsets of  $V$ . We call the pair  $(V, B)$  a set system with **ground set**  $V$ .

If for all elements in  $B$ , each element has the same cardinality  $k$ , we have that  $(V, B)$  is  **$k$ -uniform**.

*We have that  $B \subseteq \mathcal{P}(V)$  (that is, the powerset of  $V$ ).*

### 4.2 Definition of Block Design

For  $v, k, t, \lambda$  integers, we suppose:

$$v > k \geq t \geq 1, \quad \lambda \geq 1.$$

A block design of type:

$$t - (v, k, \lambda),$$

is a set system  $(V, B)$  with the following properties:

- $V$  has size  $v$
- $(V, B)$  is  $k$ -uniform
- Each  $t$ -element subset of  $V$  is contained in exactly  $\lambda$  'blocks' (elements of  $B$ ).



### 4.3 The Quantity of Blocks in a Block Design

For a block design of type  $t - (v, k, \lambda)$ , we have that the number of blocks  $b$  can be derived as follows:

$$b = \frac{\lambda \binom{v}{t}}{\binom{k}{t}}.$$

### 4.4 Definition of the Replication Number

In a block design of type  $2 - (v, k, \lambda)$ , every element lies in exactly  $r$  blocks where:

$$r(k-1) = \lambda(v-1), \quad bk = vr.$$

$r$  is the replication number.

### 4.5 Fisher's Inequality

For  $(V, B)$  a block design of type  $2 - (v, k, \lambda)$  with  $v > k$ , we have that:

$$|B| \geq |V|.$$

### 4.6 Definition of an Incidence Matrix

For a set system  $(V, B)$  with  $|V| = v$  and  $|B| = b$  we define the incidence matrix  $A$  as a matrix in  $M_{v,b}$  where  $A = (a_{ij})$  and:

$$a_{ij} = \begin{cases} 1 & \text{if element } i \text{ is in block } j \\ 0 & \text{otherwise.} \end{cases}$$

There are some important notes to be made:

- Each column contains  $k$  many '1's
- Each row contains  $r$  (the replication number) many '1's
- Each pair of rows contains  $\lambda$  many '1's in the same column

## 5 The Basics of Graph Theory

### 5.1 Definition of a Graph

A graph  $G$  is a set system  $(V, E)$  where the elements of  $E$  have size 2. Some definitions and facts follow from the definition:

- The elements of  $V$  are **vertices**
- The elements of  $E$  are called **edges**
- The size of  $V$  is often called the **order** of  $G$
- $G$  is a 2-uniform set with ground set  $V$
- $u, v$  in  $V$  are adjacent if  $u, v$  is in  $E$ .

### 5.2 Graph Isomorphisms

For two graphs  $G_1 = (V_1, E_1)$ ,  $G_2 = (V_2, E_2)$ , we say that  $G_1$  and  $G_2$  are isomorphic ( $G_1 \cong G_2$ ) if there exists a bijection  $\phi : V_1 \rightarrow V_2$  such that for each pair of vertices  $u, v$  in  $V$  we have that:

$$\{u, v\} \in E_1 \iff \{\phi(u), \phi(v)\} \in E_2.$$

### 5.3 Definition of Neighbourhood and Degree

For a graph  $G = (V, E)$  the **neighbourhood** of  $v$  in  $V$  is the set of all adjacent vertices (denoted by  $N_G(v)$ ). The neighbourhood of a set  $S$  is simply the union of the neighbourhoods of the elements of  $S$ . The **degree** is simply the size of  $N_G(v)$  denoted by  $\deg(v)$ .

### 5.4 Notation for Minimum and Maximum Degree

For a graph  $G = (V, E)$  we have that the following to represent minimum and maximum degree:

$$\begin{aligned}\delta(G) &:= \min\{\deg(v) : v \in V\} \\ \Delta(G) &:= \max\{\deg(v) : v \in V\}.\end{aligned}$$

### 5.5 Definition of Degree Sequence

For a graph  $G = (V, E)$  a graph with  $V = \{x_1, \dots, x_n\}$ , where  $V$  is ordered such that  $i \geq j$  implies  $\deg(x_i) \geq \deg(x_j)$ . The sequence  $(d_k)_{k \in [n]}$  is defined as follows:  $d_i = \deg(x_i)$ .

## 5.6 The Handshake Lemma

For a graph  $G = (V, E)$ , we have that:

$$|E| = \frac{\sum_{v \in V} \deg(v)}{2}.$$

*This is because each edge visits two vertices, so by counting the degree of each vertex we count each edge exactly twice.*

## 5.7 Subgraphs

### 5.7.1 Definition of a subgraph

A graph  $G' = (V', E')$  is a subgraph of  $G = (V, E)$  if  $V' \subseteq V$  and  $E' \subseteq E$  such that for all  $e$  in  $E'$  we have that  $e \subseteq V'$ .

### 5.7.2 Definition of an induced subgraph

An induced subgraph generated of  $G = (V, E)$  is a subgraph  $G' = (V', E')$  where:

$$E' = \{\{u, v\} \in E \text{ such that } u, v \in V'\}.$$

*Essentially, you generate an induced subgraph from a subset of the vertices of a graph by selecting edges that join vertices in the subset.*

## 5.8 Walks

### 5.8.1 Definition of a walk

We have that a walk of length  $n$ , is a set of  $n + 1$  vertices connected by  $n$  edges.

### 5.8.2 Definition of a trail

A trail is a walk where no edges are repeated.

### 5.8.3 Definition of a path

A path is a walk where no vertices are repeated (barring the last one).

### 5.8.4 Definition of a circuit

A circuit is a walk where the first and last vertices are identical.

### 5.8.5 Definition of a cycle

A cycle is a path where the first and last vertices are identical.

### 5.8.6 Equivalence of walks and paths

If for some graph  $G = (V, E)$  with  $u, v$  in  $V$ , we have that:

There's a walk between  $u$  and  $v \iff$  There's a path between  $u$  and  $v$ .

Thus, where there's a cycle, there's a circuit.

If we have that a graph  $G$  has an odd circuit, there's also an odd cycle (and the converse holds too).

### 5.8.7 Definition of connected graph

A graph is connected if there exists a path (or walk) between any two vertices in the graph.

## 5.9 Definition of a Component

A component of a graph  $G$  is a maximal connected induced subgraph of  $G$ . This means an induced subgraph of  $G$  that is connected but is not longer connected if a vertex is removed.

## 6 Euler Circuits

### 6.1 Definition of an Euler Circuit

An Euler circuit is a circuit in which each edge in a graph is traversed exactly once (or a trail which traverses every edge). As a consequence, each vertex is travelled at least once.

Graphs with Euler circuits are said to be **Eulerian**.

### 6.2 Conditions for an Euler Circuit

An Euler circuit in a graph  $G$  exists if and only if  $G$  is connected and each vertex in  $G$  has even degree.

## 7 Hamiltonian Cycles

### 7.1 Definition of a Hamiltonian Cycle

For a graph  $G = (V, E)$  where  $|V| = n$ , a Hamiltonian cycle in  $G$  is a cycle of length  $n$ , meaning it visits each vertex exactly once.

Graphs with Hamiltonian cycles are said to be **Hamiltonian**.

### 7.2 Definition of a Hamiltonian Path

For a graph  $G = (V, E)$  where  $|V| = n$ , a Hamiltonian path is a path of length  $n - 1$ , meaning it visits each vertex at least once.

### 7.3 Dirac's Theorem

For a graph  $G = (V, E)$  where  $|V| \geq 3$ :

$$\delta(G) \geq \frac{n}{2} \Rightarrow G \text{ is Hamiltonian.}$$

## 8 Bipartite Graphs

### 8.1 Definition of a Bipartite Graph

A graph  $G = (V, E)$  is bipartite if  $V$  can be partitioned into two vertex sets  $V_1, V_2$  such that each edge connects a vertex from  $V_1$  to a vertex in  $V_2$ .

### 8.2 Characterisation of Bipartite Graphs

A graph is bipartite if and only if it contains no odd cycle.

### 8.3 The Handshake Lemma for Bipartite Graphs

We have that for  $G = (V, E)$  a bipartite graph with bipartition  $V_1, V_2$ :

$$\sum_{v \in V_1} \deg(v) = \sum_{v \in V_2} \deg(v).$$

## 8.4 Hall's Marriage Problem

### 8.4.1 Definition of a Matching

For  $G = (V, E)$  a bipartite graph with bipartition  $X, Y$ , a matching from  $X$  to  $Y$  is a set of edges:

$$M = \{(x, y) : x \in X, y \in Y\},$$

such that  $f : X \rightarrow Y$  defined by:

$$f(x) := y \quad \text{where } (x, y) \in M,$$

is injective.

*In other words,  $|M| = |X|$  and each  $y$  in  $Y$  appears in at most one edge in  $M$ .*

### 8.4.2 Hall's Marriage Theorem

For  $G = (V, E)$  a bipartite graph with bipartition  $X, Y$ :

$G$  has a matching from  $X$  to  $Y$

$$\iff$$

For all  $S \subseteq X$ ,  $|N(S)| \geq |S|$ .

We also have that if:

$$\min_{x \in X} [\deg(x)] \geq \max_{y \in Y} [\deg(y)],$$

then  $G$  has a matching from  $X$  to  $Y$ .

## 9 Trees and Forests

### 9.1 Definition of a Forest

A graph  $F = (V, E)$  is a forest if it has no cycles (is **acyclic**).

### 9.2 Definition of a Tree

A graph is a tree if it is a forest and is connected.

### 9.3 Definition of a Leaf

For a vertex  $v$  in a tree,  $v$  is a leaf if it has degree one.

### 9.4 Existence of Leaves

For a tree  $T$  of order 2 or more, we have that  $T$  has a leaf.

### 9.5 Characterisation of Trees

We have that for a graph  $G = (V, E)$ , the following is equivalent:

- $G$  is a tree
- $G$  is maximally acyclic ( $G$  is acyclic and the addition of any edge forms a cycle)
- $G$  is minimally connected ( $G$  is connected and the removal of any edge disconnects it)
- $G$  is connected and  $|E| = |V| - 1$
- $G$  is acyclic and  $|E| = |V| - 1$
- Any two vertices in  $G$  are connected by a unique path.

### 9.6 Minimum Spanning Trees

In a connected, undirected graph  $G = (V, E)$ , we have that a spanning tree  $T = (V, E')$  of  $G$  is a subgraph of  $G$  where  $T$  is a tree and  $E' \subseteq E$ .

A spanning tree on  $G$  is minimal if there is no other spanning tree on  $G$  with a lower weight.

#### 9.6.1 Existence of spanning trees

We have that there is a spanning tree in a graph  $G$  if and only if  $G$  is connected.

### 9.6.2 Kruskal's algorithm

For a graph  $G = (V, E)$ , we have the following steps to the algorithm:

1. Generate a graph  $T = (V, \emptyset)$
2. Sort the edges by weight
3. For each edge  $(u, v)$  (in increasing order):
  - If  $u$  or  $v$  are not connected in  $T$ , add  $(u, v)$  to  $T$
  - Stop if there are  $|V| - 1$  edges in  $T$  or if we have run out of edges.

When this terminates, if the order of  $T$  is  $|V| - 1$  then  $T$  is a minimum spanning tree. Otherwise,  $T$  is an acyclic graph with  $n - k$  components.

## 10 Cliques and Independent Sets

### 10.1 Definition of a Triangle

We often call  $K_3$  (the complete graph on three vertices) a triangle. A graph  $G$  contains a triangle if a subgraph of  $G$  is isomorphic to  $K_3$ .

### 10.2 Mantel's Theorem

For  $G = (V, E)$  a graph of order  $n$  that contains no triangles, we have that:

$$|E| \leq \left\lfloor \frac{n^2}{4} \right\rfloor = \left\lfloor \left(\frac{n}{2}\right)^2 \right\rfloor.$$

We also have that:

$$\left[ |E| = \left\lfloor \frac{n^2}{4} \right\rfloor \right] \Rightarrow \left[ G \cong K_{k, n-k} \text{ where } k = \left\lfloor \frac{n}{2} \right\rfloor \right],$$

there always exists a graph where the equality above holds.

## 11 Planar Graphs

The motivator for understanding planar graphs is the problem of drawing graphs in the plane without intersecting edges.



## 11.1 Definition of an Arc

An arc is a subset of  $\mathbb{R}^2$  of the type  $\sigma : [0, 1] \rightarrow \mathbb{R}^2$  where  $\sigma$  is an injective, continuous map and  $\sigma(0), \sigma(1)$  are the endpoints of the arc. Injectivity here ensures the arc does not cross itself.

## 11.2 Definition of a Drawing

For a graph  $G = (V, E)$ , drawing it is equivalent to assigning:

- A point  $p$  in  $\mathbb{R}^2$  for each  $v$  in  $V$  (such that the map from vertices to points is injective)
- An arc  $\sigma$  for each  $e = (x, y)$  in  $E$  (such that  $\sigma$  intersects exactly two points, the points corresponding to  $x$  and  $y$ ).

## 11.3 Definition of a Planar Drawings and Graphs

A drawing with a set of arcs  $A$  is planar if for each  $\sigma_1, \sigma_2$  in  $A$ , we have that  $\sigma_1, \sigma_2$  either intersect at their endpoints or not at all.

A graph is planar if it admits at least one planar drawing. We have that  $K_5$  and  $K_{3,3}$  are not planar.

### 11.3.1 Non-Planar Subgraphs

For a graph  $G$  with  $G'$  a subgraph of  $G$  that is not planar, we have that  $G$  is not planar.

## 11.4 Definition of a Jordan Curve

An arc in the plane whose endpoints coincide is called a Jordan curve.

## 11.5 Jordan Curve Theorem

For any Jordan curve  $C$ ,  $C$  divides the plane into exactly two connected regions called the 'interior' and 'exterior'. The curve is the boundary of these regions.

## 11.6 Definition of a Face

For a planar graph  $G$ , a face of a drawing of  $G$  is a connected region bound by the drawing. The region going off to infinity is the outer face and the rest are inner faces.

## 11.7 Euler's Formula

For a connected graph  $G = (V, E)$  where  $F$  is the set of faces of a given drawing of  $G$ , we have that:

$$|V| - |E| + |F| = 2.$$

## 11.8 Edge Bound on Planar Graphs

For  $G = (V, E)$  a planar graph on at least three vertices:

$$|E| \leq 3(|V| - 2).$$