Set Theory Notes

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These notes are not necessarily correct, consistent, representative of the course as it stands today or, rigorous. Any result of the above is not the author's fault.

0 Notation

We commonly deal with the following concepts in Set Theory which I will abbreviate as follows for brevity:

Term	Notation
$\{0, 1, 2, \ldots\}$	$\mathbb{N}$

Contents

1 The Fundamentals

1.1 Axiom of Extensionality

For two sets a and b, we have that $a = b$ if and only if for all x we have that:

$$x \in a \Longleftrightarrow x \in b.$$

For two classes A and B, we have that $A = B$ if and only if for all x we have that:

$$x \in a \Longleftrightarrow x \in b.$$

1.2 Axiom of Pair Sets

For any sets x and y, there is a set $z = \{x, y\}$. This is the (unordered) pair set of x and y.

1.3 Axiom of the Powerset

For each set x, there exists a set which is the collection of the subsets of x, the powerset $\mathcal{P}(x)$.

For some set x, we have the powerset defined as follows $\mathcal{P}(x) = \{z \mid z \subseteq x\}$.

1.4 Axiom of the Empty Set

There exists a set with no members, the empty set $\varnothing$.

We have the empty set defined as follows $\varnothing = \{x \mid x \neq x\}$.

1.5 Axiom of Subsets

For some set x, we have that $\{y \in x \mid \Phi(y)\}$ is a set for some well-defined property of sets Φ.

1.6 Axiom of Unions

We have the basic union of two sets x_1 and x_2:

$$x_1 \cup x_2 = \{y \,|\, y \in x_1 \text{ or } y \in x_2\},$$

but for cases where we want to unify the members of the sets in a set X, we define:

$$\bigcup X = \{y \,|\, \exists\, x \in X, y \in x\}.$$

This axiom states that for a set X, $\bigcup X$ is a set.

1.7 Classes

We have that classes are collection of objects, these could also be sets. Classes that are not sets are called proper classes.

1.8 The Set ω

We have the set of natural numbers, $\mathbb{N} = \{0, 1, 2, \ldots\}$, and from this, we define ω:

$$\omega = \{0, 1, 2, \ldots\},$$

where for some n in ω,

$$n = \{0, 1, 2, \ldots, n-1\},$$

with 0_ω being the empty set. We can go beyond this definition, defining:

$$\begin{aligned}
\omega + 1 &= \{0, 1, 2, \ldots, \omega\}, \\
\omega + 2 &= \{0, 1, 2, \ldots, \omega, \omega + 1\}, \\
&\ldots \\
\omega + n &= \{0, 1, 2, \ldots, \omega, \omega + 1, \ldots \omega + n - 1\}.
\end{aligned}$$

1.9 Russell's Theorem

We have that $R = \{x \,|\, x \notin x\}$ is not a set.

Proof. Suppose we have a set z such that $z = R$, is z in R? If we suppose z is in R, we have that z is not in z by the definition of R (as $z = R$) but z is R so z is not in R, a contradiction. Thus, we have that there is no set z equal to R, so R is not a set but a proper class. $\square$

1.10 The Universe of Sets

We define the universe of sets as $V = \{x \mid x = x\}$. We have that V is a proper class.

Proof. If we suppose V is a set, we apply the axiom of subsets with $\Phi(x) = x \notin x$ and reach a contradiction via Russell's theorem. $\square$

2 Relations

We will first state the significant properties relations can have. Taking a relation R on X with x, y, z arbitrary in X:

Name	**Property**
Reflexive	xRx
Irreflexive	$\neg(xRx)$
Symmetric	$xRy \Rightarrow yRx$
Antisymmetric	$[xRy \text{ and } yRx] \Rightarrow [x = y]$
Connected	$[x = y]$ or $[xRy]$ or $[yRx]$
Transitive	$[xRy \text{ and } yRz] \Rightarrow [xRz]$

For example, equivalence relations must satisfy reflexivity, symmetry, and transitivity.

2.1 Partial Orderings

We say that a relation $\prec$ on a set X is a (strict) partial ordering if it is irreflexive and transitive.

Similarly, we say that a relation $\preceq$ on a set X is a non-strict partial ordering if it is reflexive, antisymmetric, and transitive.

A partial ordering $(X, \prec)$ is wellfounded if for any non-empty subset Y of X, Y has a least element under $\prec$.

2.2 Bounding

For a partially ordered set $(X, \prec)$:

- x_0 in X is the minimum of X if for all x in X, $x_0 \preceq x$,
- x' in X is minimal in X if for all x in X, $\neg(x \prec x')$,
- x_1 in X is the maximum of X if for all x in X, $x \preceq x_1$,
- x' in X is maximal in X if for all x in X, $\neg(x' \prec x)$.

Taking a non-empty subset Y of X, we consider the subordering $(Y, \prec)$ and for some α in X we say:

- α is a lower bound for Y if for all y in Y, $\alpha \prec y$,
- α is the infimum of Y if it's a lower bound and for all lower bounds λ of Y, $\alpha \preceq \lambda$,
- α is an upper bound for Y if for all y in Y, $y \prec \alpha$,
- α is the supremum of Y if it's an upper bound and for all upper bounds τ of Y, $\tau \preceq \alpha$.

2.3 Order Preserving Maps

We say that $f : (X, \prec_1) \to (Y, \prec_2)$ is an order preserving map if for each x_1, x_2 in X:

$$x_1 \prec_1 x_2 \Longrightarrow f(x_1) \prec_2 f(x_2).$$

Two orderings are (order) isomorphic if there is a bijective order preserving map between them.

2.4 Representation Theorem for Partially Ordered Sets

For a partially ordered set $(X, \prec)$, there is a set $Y \subseteq \mathcal{P}(X)$ which is such that $(X, \preceq)$ is order isomorphic to $(Y, \subseteq)$.

Proof. For some x in X, we set $X^x = \{x' \in X : x' \preceq x\}$, the set of elements preceding or equal to x. For x, y in X, $x \neq y$ implies that $X^x \neq X^y$ as these sets contain x and y (resp.) so $x \mapsto X^x$ is injective. This map is surjective trivially (mapping from X to $\{X^x : x \in X\}$). We have that:

$$x \preceq y \Longleftrightarrow X^x \subseteq X^y,$$

by our definition. Thus, $x \mapsto X^x$ is an order isomorphism. □

2.5 Total Orderings

A relation $\prec$ on a set X is a (strict) total ordering if it is a connected strict partial ordering.

Similarly, we say that a relation $\preceq$ on a set X is a non-strict total ordering if it is a connected non-strict partial ordering.

2.6 Well-orderings

A relation $\prec$ on a set X is a well-ordering if it is a strict total ordering and for any non-empty subset Y of X, Y has a least element under $\prec$. We denote this with $(X, \prec) \in WO$.

2.7 Ordered Pairs

For x, y sets, the ordered pair of x and y is the set:

$$\langle x, y\rangle = \{\{x\}, \{x, y\}\}.$$

2.7.1 Uniqueness of Ordered Pairs

For x, y, u, v sets, we have that:

$$\langle x, y\rangle = \langle u, v\rangle \iff (x = u) \text{ and } (y = v).$$

Proof. Suppose the former, if $x = y$ then $\langle x, y\rangle = \{\{x\}, \{x, x\}\} = \{\{x\}\}$. Thus, $\langle u, v\rangle = \{\{u\}\}$ as it is equal to $\langle x, y\rangle$ which has one element, hence $u = v$. By the Axiom of Extensionality, we have that $x = u$ and so $y = x = u = v$.

If $x \neq y$, then $\langle x, y\rangle$ and $\langle u, v\rangle$ both have the same two elements by our assumption (so $u \neq v$). We cannot have $\{x\} = \{u, v\}$ so $\{x\} = \{u\}$ which means $x = u$ by the Axiom of Extensionality. Thus, $\{u, v\} = \{x, y\} = \{u, y\}$ so $y = v$.

Suppose the latter, then the former holds trivially. $\square$

2.7.2 The Ordered k-tuple

We define the k-tuple inductively. The 2-tuple is already defined. We define the 3-tuple:

$$\langle x_1, x_2, x_3\rangle = \langle\langle x_1, x_2\rangle, x_3\rangle,$$

and for k in $\{3, 4, \ldots\}$:

$$\langle x_1, x_2, \ldots, x_k\rangle = \langle\langle x_1, x_2, \ldots, x_{k-1}\rangle, x_k\rangle.$$

2.7.3 The Product of Sets

For A, B sets, we define:

$$A \times B = \{\langle a, b\rangle : a \in A, b \in B\}.$$

Similarly to k-tuples, for $A_1, A_2, \ldots, A_k$ sets, we have $A_1 \times A_2$ defined, so we define:

$$A_1 \times A_2 \times \cdots \times A_k = (A_1 \times A_2 \times \cdots \times A_{k-1}) \times A_k,$$

defining the k-product for k in $\{2, 3, \ldots\}$. This is not associative.

2.8 Binary Relations

A binary relation R is a class of ordered pairs. We write $R^{-1} = \{\langle y, x\rangle : \langle x, y\rangle \in R\}$.

2.8.1 Domain and Range

For a relation R, we define:

$$\begin{aligned}
\mathrm{dom}(R) &= \{x : \exists\, y \text{ where } \langle x, y\rangle \in R\},\\
\mathrm{ran}(R) &= \{y : \exists\, x \text{ where } \langle x, y\rangle \in R\},\\
\mathrm{Field}(R) &= \mathrm{dom}(R) \cup \mathrm{ran}(R).
\end{aligned}$$

2.9 Functions

A relation F is a function if for all x in $\mathrm{dom}(F)$, there is a unique y in $\mathrm{ran}(F)$ with $\langle x, y\rangle$ in F.

If F is a function, it is injective if and only if for all x, x':

$$(\langle x, y\rangle \in F \text{ and } \langle x', y\rangle \in F) \Rightarrow (x = x').$$