# Linear Algebra 1 (TB2) Notes

*paraphrased by* Tyler Wright

*An important note, these notes are absolutely **NOT** guaranteed to be correct, representative of the course, or rigorous. Any result of this is not the author's fault.*

# 1 Vector Spaces, Fields, and Maps

## 1.1 Groups

A group is a *non-empty* set $(G)$ paired with a *binary group operation* $(*)$ denoted by $(G, *)$. The following properties hold for all groups (let $(G, *)$ be a group with elements $f, g, h$):

- **Associativity**: $f * (g * h) = (f * g) * h$
- **Identity**: $\exists e \in G : e * f = f * e = f$
- **Inverse**: $\exists x \in G : x * f = f * x = e$.

*A note, for a group $(G, *)$ with $g * h = h * g$ for all $g, h \in G$, this group is called **commutative** or **abelian**. However, it should be textitasised that this is **not** a necessary condition for a group.*

## 1.2 The Invertibility of Matrices

For a matrix $A \in M_{m,n}(\mathbb{F})$, the following are all **equivalent** statements:

- $A$ is **invertible**
- $\det A = 0$
- The **rows** of $A$ are **linearly independent**
- The **columns** of $A$ are **linearly independent**
- The **reduced row echelon form** of $A$ is the **identity**
- For all $\mathbf{b} \in \mathbb{F}^n$, $A\mathbf{x} = \mathbf{b}$ has a **unique solution**.

## 1.3 Fields

A field is a set $(F)$ defined under multiplication and division with the following properties:

- **Associativity** under multiplication and division
- **Commutativity** under multiplication and division
- $F$ contains an **identity** under multiplication and division
- All elements in $F$ contain an **inverse** under addition and multiplication (except 0 under multiplication)
- The defined multiplication is **distributive** across the defined addition.

## 1.4 Vector Spaces

A group $(V, +_V)$ ($+_V$ denotes addition defined with respect to the set $V$ as it can be ambigious in some cases) is a vector space over the field ($\mathbb{F}$) if the following holds (let $v, w \in V,\ \lambda, \mu \in \mathbb{F}$):

- $(V, +_V)$ is **abelian**
- $V$ is **closed under multiplication** with elements in $\mathbb{F}$
- $\lambda(v +_V w) = \lambda v + \lambda w$
- $(\lambda + \mu)v = \lambda v +_V \mu v$
- $(\lambda\mu)v = \lambda(\mu v)$
- $fv = v$ where $f$ is the **multiplicative identity** of $\mathbb{F}$.

## 1.5 Subspaces

Let $V$ be a vector space over $\mathbb{F}$, $U \subseteq V$ is a subspace if the following properties hold:

- $U$ is **non-empty**
- $U$ is **closed** under the **addition** defined by $V$
- $U$ is **closed** under the **multiplication** defined by $V$.

*Some notes on subspaces:*

- Subspaces are vector spaces
- The intersection of subspaces is a subspace
- The span of any non-empty subset of a given vector space is a subspace.

## 1.6 Linear Maps

For $V, W$ vector spaces over $\mathbb{F}$, the map $T : V \to W$ is called linear if the following properties hold (let $u, w \in V,\ \lambda \in \mathbb{F}$):

- $T(u + v) = T(u) + T(v)$
- $T(\lambda u) = \lambda T(u)$.

*A note, for a linear map ($T : V \to W$), if $V = W$, $T$ is sometimes referred to as a linear **operator**. Also, composed linear maps are also linear maps.*

## 1.7 The Kernel and Image

For a linear map ($T : V \to W$), the kernel is defined as follows:

$$\operatorname{Ker} T = \{v \in V : T(v) = 0\}.$$

The image is defined as follows:

$$\operatorname{Im} T = \{w \in W : \exists v \in V \text{ with } T(v) = w\}.$$

*Some notes on linear maps (let $T : V \to W$ be a linear map):*

- The kernel and image of $T$ are subspaces of $V$ and $W$ respectively
- For $U \subseteq V$, $T(U)$ is also a subspace (but of $W$ instead of $V$).

## 1.8 Bases and Dimension

### 1.8.1 Definition of linear independence

For $V$ a vector space, with $S \subseteq V$, let $s_1, s_2, ... \in S$,

- $S$ is linearly independent if $\sum_{n=1}^{|S|} \lambda_n s_n = 0 \Longleftrightarrow \lambda_i = 0 \ \forall i$
- S is linearly dependent if it's not linearly independent.

A result of linear dependence is that for a linear dependent set $S$, there exists $s \in S$ such that $\text{span}(S) = \text{span}(S \backslash \{s\})$.

*A note, if $S$ is linearly dependent, there's a vector in $S$ such that it can be written as the sum of other vectors in $S$.*

### 1.8.2 Definition of a basis

For a vector space $V$, we say $S \subseteq V$ is a basis of $V$ if:

- $S$ spans $V$
- $S$ is linearly independent.

### 1.8.3 Properties of bases

Let $V$ be a vector space:

- For $v \in V$, $B$ a basis for $V$, $v$ can be written uniquely as a linear combination of vectors in $B$
- $V$ is finitely dimensional if $|B| < \infty$
- If $V$ is finitely dimensional, there must exists a basis of $V$.

For $V$ a vector space with $S \subseteq V$ a linearly independent set. $S$ can be 'extended' to a basis of $V$. If $S$ spans $V$, it's already a basis. If not, we add a vector from $V \setminus \operatorname{span} S$. We can do this iteratively until we have a basis.

### 1.8.4 Definition of dimension

For a vector space $V$ with a basis $B$, the order of $B$ is the dimension of $V$, all bases of $V$ share the same order. This is denoted by $\dim V := |B|$.

### 1.8.5 Properties of dimension

Let $V$ be a finite dimensional vector space with $U, S \subseteq V$ where $U$ is a subspace:

- $S$ is linearly independent $\Rightarrow |S| < \dim V$
- $\operatorname{span} S = V \Rightarrow |S| \geq dimV$
- $(\operatorname{span} S = V) \wedge (|S| = \dim V) \Rightarrow S$ is a basis of $V$.
- $\dim U \leq \dim V$
- $\dim U = \dim V \Rightarrow U = V$

## 1.9 Direct Sums