

Data Structures and Algorithms Notes

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*An important note, these notes are absolutely **NOT** guaranteed to be correct, representative of the course, or rigorous. Any result of this is not the author's fault.*

1 Graph Theory

1.1 Definition of a Graph

A graph is a pair of sets $G = (V, E)$, where V is a set of vertices (or nodes) and E is a set of edges (or arcs).

1.2 Definition of an Edge

An edge of a graph $G = (V, E)$ is $e = \{u, v\}$ in E where u, v are vertices in V .

1.3 Definition of a Neighbourhood

For a graph $G = (V, E)$ with v in V , the neighbourhood of v is the set $V' \subseteq V$ of vertices connected to v by an edge in E .

The neighbourhood of v is denoted by $N(v)$.

The neighbourhood of a set of vertices is the union of the neighbourhoods of each vertex.

1.4 Definition of Degree

For a graph $G = (V, E)$ with v in V , the degree of v is the size of its neighbourhood.

The degree of v is denoted by $d(v)$.

1.5 Isomorphic Graphs

Graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are called isomorphic if there exists a bijection $f : V_1 \rightarrow V_2$ such that:

$$\{u, v\} \in E_1 \iff \{f(u), f(v)\} \in E_2.$$

This relationship is denoted by $G_1 \cong G_2$.

1.6 Definition of a Subgraph

A graph $G' = (V', E')$ is a subgraph of $G = (V, E)$ if $V' \subseteq V$ and $E' \subseteq E$.

1.7 Definition of an Induced Subgraph

An induced subgraph generated from $G = (V, E)$ by $V' \subseteq V$ is the graph $G' = (V', E')$ where:

$$E' = \{\{u, v\} \in E \text{ such that } u, v \in V'\}.$$

Essentially, you generate an induced subgraph from a subset of the vertices of a graph by selecting edges that join vertices in the subset.

1.8 Walks

1.8.1 Definition of a walk

A walk in a graph $G = (V, E)$ is a set of vertices in V connected by edges in E . The length of the walk is the number of edges traversed in the walk.

1.8.2 Definition of a path

A path is a walk where no vertices are repeated.

1.8.3 Definition of an Euler walk

An Euler walk is a walk such that every edge is traversed exactly once. Thus, for a graph $G = (V, E)$, the length is $|E|$.

1.8.4 Conditions for an Euler walk

For an Euler walk to be possible on a given graph, all vertices must have an even degree **or** exactly two vertices have odd degree.

If all vertices have even degree we have that the Euler walk is a cycle, if exactly two vertices have odd degree then we have that these vertices are the start and end points of our Euler walk.

1.9 Definition of a Connected Graph

A connected graph is a graph where for each pair of vertices, there is a path connecting them.

1.10 Definition of a Component

A component of a graph $G = (V, E)$ is a maximal connected induced subgraph of G . This means an induced subgraph of G that is connected but is not longer connected if a vertex is removed.

Connected graphs have a single component, the entire graph.

1.11 Digraphs

1.11.1 Definition of a digraph

A digraph (or directed graph) is a graph where each of the edges has a direction. This direction means the edge can only be traversed in a single direction.

1.11.2 Definition of a strongly connected digraph

A digraph $G = (V, E)$ is strongly connected if for each u, v in E , there exists a path from u to v **and** from v to u .

1.11.3 Definition of a weakly connected digraph

A digraph $G = (V, E)$ is weakly connected if for each u, v in E , there exists a path from u to v **or** from v to u .

1.11.4 Definition of components of digraphs

A strong component of a digraph is the maximal *strongly* connected induced subgraph.

A weak component of a digraph is the maximal *weakly* connected induced subgraph.

So, these are induced subgraphs that are strongly/weakly connected but are no longer strongly/weakly connected once a vertex is removed.

1.11.5 Definition of neighbourhoods in digraphs

The neighbourhood of a vertex in a digraph can be considered by looking at the edges *from* the vertex and the edges *to* the vertex.

The in-neighbourhood of a vertex v are the edges that enter v . The out-neighbourhood of a vertex v are the edges that exit v . These are denoted by $N^-(v)$ and $N^+(v)$ respectively.

1.11.6 Definition of degrees in digraphs

For a vertex v , the in-degree of the vertex $d^-(v)$ is the size of the in-neighbourhood and the out-degree of the vertex $d^+(v)$ is the size of the out-neighbourhood.

It can be seen that the degree of a given vertex is the sum of its in and out degree (in a digraph).

1.11.7 Conditions for an Euler walk in a digraph

For an Euler walk to be possible on a given digraph, we have two cases, either:

- the digraph is strongly connected and every vertex has equal in and out degrees, or
- one vertex has an in-degree one greater than its out-degree, another has an out-degree one greater than its in-degree, and all remaining vertices have equal in and out degrees.

In the first case we have that the Euler walk is a cycle, in the second we have that the special vertices are the start and end points of our Euler walk.

1.11.8 Cycles

1.11.9 Definition of a cycle

A cycle is a walk where the first and last vertices are the same and each vertex appears at most once (barring the first and last vertex).

1.11.10 Definition of a Hamiltonian cycle

A Hamiltonian cycle is a cycle where each vertex is visited.

1.11.11 Conditions for a Hamiltonian cycle

Whilst the conditions necessary for a Hamiltonian cycle in general are unknown, by Dirac's theorem, we know that for a graph with n vertices, if every vertex has degree $\frac{n}{2}$ or greater then a Hamiltonian cycle exists.

2 Types of Algorithms

2.1 Greedy Algorithms

These types of algorithms start with a trivial solution and iteratively optimise their solution based on the information available at the time. They do not retroactively change the solution based on new data, only add to it.