Understanding Root Spaces and Eigenspaces

by Tyler Wright

1 Preface

We will be considering f, id in $\mathcal{L}(V, V)$, where V is a vector space over the field K and id is the identity function on $\mathcal{L}(V, V)$.

2 The Root Space

The root space for some λ in K is $V(\lambda) \subseteq V$ where for all v in $V(\lambda)$:

$$(f - \lambda \mathrm{id})^r(v) = 0_V,$$

for some r in $\mathbb{Z}_{>0}$ called the height of v, denoted by $h(v)$. Note that the height of vectors in the root space may vary and:

- $V(\lambda) \neq \{0\}$ if and only if λ is an eigenvalue,
- $V(\lambda)$ is f-invariant,
- The intersection of two root spaces is not $\{0\}$ if and only if they are over the same value.

3 The Eigenspace

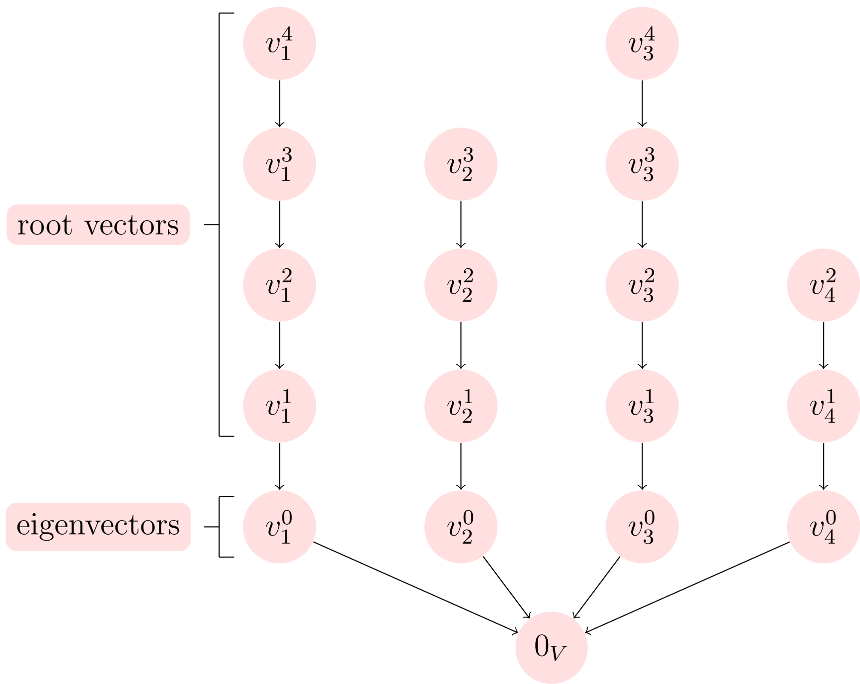
The eigenspace for some λ in K is $E(\lambda) \subseteq V(\lambda) \subseteq V$ where for all v in $E(\lambda)$:

$$(f - \lambda \mathrm{id})(v) = 0_V.$$

4 Mapping from the Root Space to the Eigenspace

For a given (non-zero) root space $V(\lambda)$, for each v in $V(\lambda)$, suppose we apply $(f-\lambda\mathrm{id})$ to it $h(v)-1$ times, take $w = (f-\lambda\mathrm{id})^{h(v)-1}(v)$. As $(f-\lambda\mathrm{id})^{h(v)}(v) = 0$ by definition, $(f-\lambda\mathrm{id})(w) = 0$, thus w is an eigenvalue.

We can visualise this with a graph (of stacks) where the directed edges represent applications of $(f-\lambda\mathrm{id})$:



Note that multiple eigenvectors can belong to the same eigenspace.

5 Eigenvalue Multiplicity

We have the height of a stack is the multiplicity of the eigenvalue of the stack in the **minimal polynomial**. Thus, for maps with $\dim(V)$ distinct eigenvalues, for each eigenvalue λ, $V(\lambda) = E(\lambda)$.

6 An Example

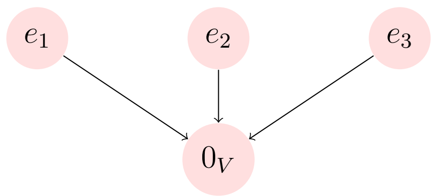
Take the following matrix:

$$A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

We clearly have:

$$\begin{aligned} p_A(x) &= (x-2)^3 \\ m_A(x) &= (x-2), \end{aligned}$$

demonstrating we have one eigenvalue, 2, with multiplicity 3 in the characteristic polynomial and multiplicity 1 in the minimal polynomial. We have a basis $\mathcal{B}_A$ for $E(2)$ where $\mathcal{B}_A = \{e_1, e_2, e_3\}$, with a stack representation:



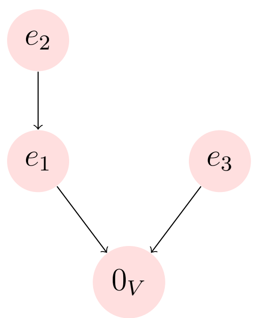
Now consider:

$$B = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \qquad C = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}.$$

It can be seen that B and C both have a single eigenvalue each, being 2. However, for example, e_2 is not in $E(2)$ of B or C. We will consider B, we have that:

$$\begin{aligned} p_B(x) &= (x-2)^3 \\ m_B(x) &= (x-2)^2, \end{aligned}$$

here the multiplicity of 2 in the minimal polynomial is 2 indicating that there is a stack of height 2. We have a basis $\mathcal{B}_B$ for $E(2)$ on B where $\mathcal{B}_B = \{e_1, e_3\}$ with a stack representation:



Now, for C,

$$p_C(x) = (x-2)^3$$
$$m_C(x) = (x-2)^3,$$

here the multiplicity of 2 in the minimal polynomial is 3 indicating that there is a stack of height 3. We have a basis $\mathcal{B}_C$ for $E(2)$ on C where $\mathcal{B}_C = \{e_1\}$ with a stack representation:

$$\begin{tikzcd} e_3 \arrow[d] \\ e_2 \arrow[d] \\ e_1 \arrow[d] \\ 0_V \end{tikzcd}$$