Introduction to Group Theory Notes

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*An important note, these notes are absolutely **NOT** guaranteed to be correct, representative of the course, or rigorous. Any result of this is not the author's fault.*

1 The Basics of Groups

1.1 Binary operations

A binary operation on a set G is a function:

$$* : G \times G \to G.$$

It's just a function that takes two values and gives a single output. Examples are addition, multiplication, and composition.

Such an operation is called **commutative** if:

$$x * y = y * x. \qquad (\forall x, y \in G)$$

1.2 Definition of a Group

A group is a set G paired with a binary operation $*$ such that they satisfy the following:

- **Associativity**: For $x, y, z \in G$, $(x * y) * z = x * (y * z)$
- **Identity**: $\exists\, e \in G$ such that $\forall g \in G$, $e * g = g * e = g$
- **Inverses**: $\forall g \in G$, $\exists\, g^{-1} \in G$ such that $g * g^{-1} = g^{-1} * g = e$.

A group is called commutative or Abelian if all its elements commute with the given operation.

1.3 Consequences of the Definition

1.3.1 Left and right cancellation

We can left and right cancel with inverses:

$$(ax = bx) \Rightarrow (a = b) \qquad (\forall a, b, x \in G)$$
$$(xa = xb) \Rightarrow (a = b). \qquad (\forall a, b, x \in G)$$

However, $ax = xb$ does not imply $a = b$ unless the group is Abelian.

1.3.2 Uniqueness of the identity and inverses

We have uniqueness of certain elements:

- The identity of a group is unique
- The inverse of an element is unique.

1.3.3 Inverse properties

For a group G with elements x, y:

- $(x^{-1})^{-1} = x$
- $(xy)^{-1} = y^{-1}x^{-1}$.

1.3.4 Exponent properties

For a group G with an element x and m, n in $\mathbb{Z}$:

- $x^{-n} = (x^{-1})^n$
- $(x^n)(x^m) = x^{n+m}$.

However, $(xy)^n$ may not equal $x^n y^n$ unless G is Abelian.

2 Dihedral Groups

2.1 Definition of a Dihedral Group

The dihedral group D_{2n} is the group of symmetries of an n-sided polygon. This group has order $2n$ as is defined as:

$$\begin{aligned} D_{2n} &= \langle a \rangle \cap b\langle a \rangle \\ &= e, a, a^2, \dots, a^{n-1}, b, ba, ba^2, \dots, ba^{n-1}. \end{aligned}$$

Where a is a rotation of $\frac{2\pi}{n}$ radians around the centre of the polygon and b is a reflection in the line through vertex 1 and the centre of the polygon.

2.2 Properties of a Dihedral Group

For the dihedral group D_{2n}:

- $a^n = e$
- $b^2 = e$
- $a^n b = ba^{-n}$

3 Subgroups

3.1 Definition of a Subgroup

A subgroup is a subset H of a group G such that H is also a group under the binary operation defined by G ($H \leq G$). If we have a subset H of a group G, we can show it is a subgroup by showing the following properties hold for H:

- **Closure**: For $x, y \in H$, $xy \in H$
- **Identity**: $\exists\, e \in H$ such that for $x \in H$, $e * x = x * e = x$
- **Inverses**: For $x \in H$, $\exists\, x^{-1} \in H$ such that $x * x^{-1} = x^{-1} * x = e$.

A consequence of this definition is that the intersection of subgroups is a subgroup.

4 The Order of Elements

4.1 The Definition of Order for Elements

For x an element in some group G, we have that the order of x is defined by:

$$\operatorname{ord}(x) = \begin{cases} n \text{ such that } x^n = e & \text{if such } n \text{ exists} \\ \infty & \text{otherwise.} \end{cases}$$

The order is the ***least*** *possible integer such that $x^n = e$. To show the order of x is n, you need to show $x^n = e$ and $x^k \neq e$ for all $k \in \{1, 2, \ldots, n-1\}$.*

4.2 Properties of the Order of Elements

Let G be a group with element x:

- $\operatorname{ord}(x) = \infty \Rightarrow$ all x^i are distinct ($i \in \mathbb{Z}$)
- $|G| < \infty \Rightarrow \operatorname{ord}(x) < \infty$
- If $\operatorname{ord}(x) = n \in \mathbb{N}$, for $i \in \mathbb{N}$, $\operatorname{ord}(x^i) = \frac{n}{\gcd(n,i)}$.

5 Cyclic Groups

5.1 Definition of a Cyclic Group

For a group G, the cyclic group generated by x in G is defined by:

$$\langle x \rangle = \{x^i : i \in \mathbb{N}\}.$$

5.2 Properties of Cyclic Groups

For a group G with element x:

- $\langle x \rangle$ is a subgroup of G
- $|\langle x \rangle| = \operatorname{ord}(x)$
- Cyclic groups are Abelian
- Subgroups of cyclic groups are cyclic
- G is cyclic $\Leftrightarrow \exists\, x \in G$ such that $\operatorname{ord}(x) = |G|$.

6 Groups from Modular Arithmetic

6.1 Congruence Classes

A congruence class $[a]$ of the set $\mathbb{Z}/n\mathbb{Z}$ is a set of integers congruent to $a \pmod n$. We define the following operations:

- **Addition**: $[a] + [b] = [a + b]$
- **Multiplication**: $[a][b] = [ab]$.

For example:

$$\mathbb{Z}/7\mathbb{Z} = \bigcup_{i=0}^{6} [i],$$

with distinct elements $0, 1, 2, 3, 4, 5, 6$.

6.2 The Set of Congruence Classes under Addition

We have that the set $\mathbb{Z}/n\mathbb{Z}$ with the operation of addition $(\mathbb{Z}/n\mathbb{Z}, +)$ is a cyclic group generated by 1.

This means it's also an Abelian group.

6.3 The Set of Congruence Classes under Multiplication

The trouble with multiplication is that certain congruence classes never have inverses and as a result, the set under multiplication can never be a group. We have that an element $[a]$ of $(\mathbb{Z}/n\mathbb{Z}, \times)$ has an inverse if:

$$\gcd(a, n) = 1.$$

We define the set U_n as follows:

$$U_n = \{a : a \in \mathbb{Z} \text{ with } \gcd(a, n) = 1\}.$$

Thus, we have $(U_n, \times)$ is an Abelian group.

6.4 The Set of Congruence Classes under the Direct Product

For m, n positive integers with $\gcd(m, n) = 1$, we have:

$$U_m \times U_n \cong U_{mn}.$$

7 Isomorphisms

7.1 Definition of an Isomorphisms

For $(G, *)$, $(H, \circ)$ groups, an isomorphism $\phi : G \to H$ is a bijective function such that:

$$\phi(x * y) = \phi(x) \circ \phi(y). \qquad (\forall x, y \in G)$$

7.2 Properties of Isomorphisms

For the groups G, H, K and an isomorphism $\phi : G \to H$:

- ϕ^{-1} is an isomorphism
- G and H are isomorphic ($G \cong H$)
- If there exists an isomorphism $\psi : H \to K$ then $G \cong K$ (transitive)
- $\phi(e_G) = e_H$
- $\phi(x^{-1}) = \phi(x)^{-1}$
- $\phi(x^i) = \phi(x)^i$ ($i \in \mathbb{Z}$)
- $\mathrm{ord}_G(x) = \mathrm{ord}_H(\phi(x))$
- $|G| = |H|$
- G is Abelian $\Leftrightarrow$ H is Abelian
- G is cyclic $\Leftrightarrow$ H is cyclic

8 Direct Products

8.1 Definition of the Direct Product

For G, H groups, $G \times H$ is the Cartesian product of G and H with the binary operation:

$$(x, y)(a, b) = (x * a, y * b). \qquad (\forall x, a \in G, y, b \in H)$$

This is itself a group.

8.2 Properties of the Direct Product

For H, K groups, $G = H \times K$:

- G is finite $\Leftrightarrow$ H and K are finite (in this case $|G| = |H||K|$)
- G is Abelian $\Leftrightarrow$ H and K are Abelian
- G is cyclic $\Rightarrow$ H and K are cyclic.

8.3 The Direct Product and Cyclic Groups

8.3.1 Order of elements

For H, K groups, $G = H \times K$, (x, y) in G:

$$\mathrm{ord}(x, y) = \mathrm{lcm}(\mathrm{ord}_H(x), \mathrm{ord}_K(y)).$$

8.3.2 Condition for a cyclic direct product

For H, K groups, $G = H \times K$, G is cyclic if and only if $\gcd(|H|, |K|) = 1$.

8.3.3 The direct product of cyclic groups

We denote the cyclic group of order n as C_n. We have that for C_n, C_m cyclic groups:

$$C_n \times C_m \cong C_{mn} \Leftrightarrow \gcd(m, n) = 1.$$

9 Lagrange's Theorem

9.1 Definition of Lagrange's Theorem

For a finite group G with $H \leq G$ a subgroup. We have that $|H|$ divides $|G|$.

9.2 Cyclic Subgroups

For G a finite group with order n, for x in G, $\mathrm{ord}(x)$ divides n (this is because $\langle x \rangle \leq G$).

9.3 Cosets

9.3.1 Definition of a coset

For a group G with $H \leq G$ and x in G, the left coset xH is and right coset Hx are the sets:

$$xH = \{xh : h \in H\},\ Hx = \{hx : h \in H\}.$$

*While this is a sub**set** of G, it is not necessarily a sub**group**.*

9.3.2 A bijection from a subgroup to its left coset

For a group G with $H \leq G$, x in G, and left coset xH, there exist a bijection from H to xH. This implies that their order is the same.

9.3.3 The intersection of cosets

For a group G with $H \leq G$, x, y in G:

$$xH \cap yH \neq \emptyset \Leftrightarrow xH = yH.$$

Cosets are distinct unless they are equal.

9.3.4 Index of a subgroup

For a group G with $H \leq G$ and x in G, the index of H in G $|G : H|$ is the number of left cosets of H in G. So, since all cosets of H are distinct, we have:

$$|G| = |H||G : H|.$$

9.4 Consequences of Lagrange's Theorem

9.4.1 Intersection of subgroups

For a group G with $H, K \leq G$, $\gcd(|H|, |K|) = 1$ implies $H \cap K = \{e\}$.

9.4.2 Prime order groups

For G a group with $|G| = p \in \mathbb{P}$ (prime):

- G is cyclic
- Every element of G except the identity has order p (and generates G)
- The only subgroups of G are G and $\{e\}$.

10 Fermat-Euler Theorem

10.1 Euler's ϕ Function

We define the Euler ϕ function over the naturals by:

$$\phi(n) = \left|\{a : a \in \mathbb{N}, \gcd(a, n) = 1\}\right|.$$

We have that $\phi(n)$ is the order of U_n (the group of congruence classes under multiplication). Also, for p in $\mathbb{P}$ (prime), $\phi p = p - 1$.

This is the number of values less than or equal to an integer that don't divide it.

10.2 Fermat-Euler Theorem

For a, n in $\mathbb{N}$ with $\gcd(a, n) = 1$, we have that:

$$a^{\phi(n)} \equiv 1 \pmod{n}.$$

So, for p in $\mathbb{P}$ (prime):

$$a^{p-1} \equiv 1 \pmod{p}.$$

11 Symmetric Groups

11.1 Definition of a Symmetric Group

For a set X, $S(X)$ is the group of all symmetries of X. For n in $\mathbb{N}$, S_n is the group of all symmetries of $\{1, \ldots, n\}$. We have that $|S_n| = n!$.

11.2 k-cycles in S_n

11.2.1 Definition of a k-cycle

For k, n in $\mathbb{N}$ with $k \leq n$. A k-cycle f in S_n is a permutation of the k distinct elements $\{i_1, i_2, \ldots, i_k\}$ in $\{1, \ldots, n\}$ of the form:

$$f(i_1) = i_2, f(i_2) = i_3, \ldots, f(i_k) = f(i_1)$$
$$f = (i_1, i_2, i_3, \ldots, i_k).$$

11.2.2 Properties of k-cycles

For f in S_n a k-cycle:

- f has order k
- $\mathrm{ord}(f) = 2 \Rightarrow f$ is a **transposition**.

11.3 Disjoint Cycles

11.3.1 Definition of a disjoint cycle

We call a set of cycles disjoint if no element of $\{1, \ldots, n\}$ is moved by more than one of the cycles.

11.3.2 Elements of S_n as a product of disjoint cycles

We have that for all f in S_n, f can be written as a product of disjoint cycles.

11.3.3 Order of elements of S_n

For f in S_n with $f = (f_1)(f_2) \cdots (f_k)$ a product of disjoint cycles:

$$\mathrm{ord}(f) = \mathrm{lcm}(\mathrm{ord}(f_1), \mathrm{ord}(f_2), \ldots, \mathrm{ord}(f_k)).$$

12 Transpositions

12.1 Elements of S_n as a Product of Transpositions

We have that for all f in S_n, f can be written as a product of transpositions.

12.2 Odd and Even Permutations

12.2.1 Definition of odd and even permutations

For each f in S_n, write f as the product of transpositions, let k be the number of transpositions needed:

- f is odd if k is odd
- f is even if k is even.

12.2.2 Composition of Permutations

For f, g in S_n, we have that:

- f, g both odd or both even $\Rightarrow fg$ even
- f, g odd and even (or vice-versa) $\Rightarrow fg$ odd.

12.2.3 k-cycles

For f in S_n a k-cycle:

- k odd $\Rightarrow f$ even
- k even $\Rightarrow f$ odd.

12.2.4 The alternating group

Let A_n be the set of all even permutations of S_n, we have:

- $|A_n| = \frac{|S_n|}{2}$ $(n \geq 1)$
- $A_n \leq S_n$.

13 Homomorphisms

13.1 Definition of a Homomorphism

For $(G, *)$, $(H, \circ)$ groups, a homomorphism $\phi : G \to H$ is a function such that:

$$\phi(x * y) = \phi(x) \circ \phi(y). \qquad (\forall x, y \in G)$$

This is an isomorphism without the requirement of bijectivity.

13.2 Properties of Homomorphisms

For the groups G, H and a homomorphism $\phi : G \to H$:

- $\phi(e_G) = e_H$
- $\phi(x^{-1}) = \phi(x)^{-1}$
- $\phi(x^i) = \phi(x)^i$ $(i \in \mathbb{Z})$

13.3 Trivial Homomorphisms

For the groups G, H, the following are all homomorphisms:

- $\phi : G \to H$; $\phi(x) = e_H$
- $\phi : G \to G \times H$; $\phi(g) = (g, e_H)$
- $\phi : H \to G \times H$; $\phi(h) = (e_G, h)$
- $\phi : G \times H \to G$; $\phi(g, h) = g$
- $\phi : G \times H \to H$; $\phi(g, h) = h$.

13.4 The Kernel and Image

For the groups G, H and a homomorphism $\phi : G \to H$:

- $\mathrm{Ker}(\phi) = \{x : x \in G,\ \phi(x) = e_H\} \leq G$
- $\mathrm{Im}(\phi) = \{\phi(x) : x \in G\} \leq H$.

13.5 Injectivity

For the groups G, H and a homomorphism $\phi : G \to H$, ϕ is injective if and only if $\mathrm{Ker}(\phi) = \{e_G\}$.

14 Normal Subgroups

14.1 Definition of Normal Subgroups

A normal subgroup of group G is a subgroup $N \leq G$ such that $gNg^{-1} = N$ for all $g \in G$. This is denoted by $N \trianglelefteq G$.

We have, $gNg^{-1} = N \Rightarrow gN = Ng$. So, we can show a group is a normal subgroup by showing the left and right cosets are the same for a given g.

14.2 Abelian Groups

All subgroups of Abelian groups are normal.

14.3 The Kernel of Homomorphisms

For the groups G, H and a homomorphism $\phi : G \to H$, $\text{Ker}(\phi)$ is a normal subgroup of G.

15 Quotient Groups

15.1 Definition of Quotient Groups

For G a group with $N \trianglelefteq G$ a normal subgroup, the quotient group G/N is the set of cosets of N in G with the binary operation defined for x, y in G by:

$$(xN)(yN) = (xy)N.$$

15.2 The Quotient Homomorphism

For G a group with $N \trianglelefteq G$ a normal subgroup, we can define a homomorphism ϕ from G to the quotient group G/N:

$$\phi : G \to G/N$$
$$\phi(g) = gN.$$

It's easy to see that this is surjective also.

16 The Homomorphism Theorem

We have that for the groups G, H with a homomorphism $\phi : G \to H$, $\operatorname{Ker}(\phi) \trianglelefteq G$. So, it makes sense to construct the quotient group $G/\operatorname{Ker}(\phi)$. We have that this group is isomorphic to $\operatorname{Im}(\phi)$.

17 Group Actions

17.1 Definition of a Group Action

A group action of a group G on a set X is a function $(\cdot) : G \times X \to X$ where for all x in X, g, h in G:

- $e \cdot x = x$
- $g \cdot (h \cdot x) = (gh) \cdot x$.

17.2 The Trivial Group Action

For G a group, we have $(\cdot) : G \times G \to G$ the trivial group action defined for g, h in G by:

$$g \cdot h = gh.$$

17.3 Bijective Functions from Group Actions

For a group G acting on a set X, we have that for each g in G, f is bijective defined by:

$$f : X \to X$$
$$f(x) = gx.$$

17.4 The Orbit and Stabiliser

17.4.1 Definition of the orbit and the stabiliser

For G acting on X with x in X:

- The orbit of x ($G \cdot x$) is defined by:

$$G \cdot x = \{g \cdot x : g \in G\}.$$

- The stabiliser of x (G_x) is defined by:

$$G_x = \{g : g \in G, g \cdot x = x\}.$$

So, the orbit of an element is everything that it can be mapped to under the group action. The stabiliser of an element x is the set of elements that have no effect on x under the group action. To loosely put it, the 'identities' of x.

17.4.2 Disjoint property of orbits

For G acting on X with x, y in X, $G \cdot x$ and $G \cdot y$ are either disjoint or equal. So, we have that X is partitioned into orbits so that each element of x exists in exactly one orbit.

17.4.3 Subgroup property of stabilisers

For G acting on X with x in X, G_x is a subgroup of G.

17.4.4 The orbit-stabiliser theorem

For G acting on X with x in X:

$$|G : G_x| = |G \cdot X|,$$

and if G is finite:

$$|G| = |G \cdot X||G_x|$$

So, we have that the number of cosets of the stabiliser in G is equal to the amount of elements in the orbit. The second result follows from:

$$|G : G_x| = \frac{|G|}{|G_x|},$$

if G is finite as G_x is a subgroup.