Set Theory Notes

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These notes are not necessarily correct, consistent, representative of the course as it stands today, or rigorous. Any result of the above is not the author's fault.

These notes are in progress.

0 Notation

We commonly deal with the following concepts in Set Theory which I will abbreviate as follows for brevity:

Term	Notation
$\{0, 1, 2, \ldots\}$	$\mathbb{N}$

Contents

1 The Fundamentals

1.1 Axiom of Extensionality

For two sets a and b, we have that $a = b$ if and only if for all x we have that:

$$x \in a \Longleftrightarrow x \in b.$$

For two classes A and B, we have that $A = B$ if and only if for all x we have that:

$$x \in a \Longleftrightarrow x \in b.$$

1.2 Axiom of Pair Sets

For any sets x and y, there is a set $z = \{x, y\}$. This is the (unordered) pair set of x and y.

1.3 Axiom of the Powerset

For each set x, there exists a set which is the collection of the subsets of x, the powerset $\mathcal{P}(x)$.

For some set x, we have the powerset defined as follows $\mathcal{P}(x) = \{z : z \subseteq x\}$.

1.4 Axiom of the Empty Set

There exists a set with no members, the empty set $\varnothing$.

We have the empty set defined as follows $\varnothing = \{x : x \neq x\}$.

1.5 Axiom of Subsets

For some set x, we have that $\{y \in x : \Phi(y)\}$ is a set for some well-defined property of sets Φ.

1.6 Axiom of Unions

We have the basic union of two sets x_1 and x_2:

$$x_1 \cup x_2 = \{y : y \in x_1 \text{ or } y \in x_2\},$$

but for cases where we want to unify the members of the sets in a set X, we define:

$$\bigcup X = \{y : \exists x \in X, y \in x\}.$$

This axiom states that for a set X, $\bigcup X$ is a set.

1.7 Classes

We have that classes are collection of objects, these could also be sets. Classes that are not sets are called proper classes.

1.8 The Set ω

We have the set of natural numbers, $\mathbb{N} = \{0, 1, 2, \ldots\}$, and from this, we define ω:

$$\omega = \{0, 1, 2, \ldots\},$$

where for some n in ω,

$$n = \{0, 1, 2, \ldots, n-1\},$$

with 0_ω being the empty set. We can go beyond this definition, defining:

$$\begin{aligned}
\omega + 1 &= \{0, 1, 2, \ldots, \omega\}, \\
\omega + 2 &= \{0, 1, 2, \ldots, \omega, \omega + 1\}, \\
&\ldots \\
\omega + n &= \{0, 1, 2, \ldots, \omega, \omega + 1, \ldots \omega + n - 1\}.
\end{aligned}$$

1.9 Russell's Theorem

We have that $R = \{x : x \notin x\}$ is not a set.

Proof. Suppose we have a set z such that $z = R$, is z in R? If we suppose z is in R, we have that z is not in z by the definition of R (as $z = R$) but z is R so z is not in R, a contradiction. Thus, we have that there is no set z equal to R, so R is not a set but a proper class. □

1.10 The Universe of Sets

We define the universe of sets as $V = \{x : x = x\}$. We have that V is a proper class.

Proof. If we suppose V is a set, we apply the axiom of subsets with $\Phi(x) = x \notin x$ and reach a contradiction via Russell's theorem. $\square$

2 Relations

We will first state the significant properties relations can have. Taking a relation R on X with x, y, z arbitrary in X:

Name	Property
Reflexive	xRx
Irreflexive	$\neg(xRx)$
Symmetric	$xRy \Rightarrow yRx$
Antisymmetric	$[xRy \text{ and } yRx] \Rightarrow [x = y]$
Connected	$[x = y]$ or $[xRy]$ or $[yRx]$
Transitive	$[xRy \text{ and } yRz] \Rightarrow [xRz]$

For example, equivalence relations must satisfy reflexivity, symmetry, and transitivity.

2.1 Partial Orderings

We say that a relation $\prec$ on a set X is a (strict) partial ordering if it is irreflexive and transitive.

Similarly, we say that a relation $\preceq$ on a set X is a non-strict partial ordering if it is reflexive, antisymmetric, and transitive.

A partial ordering $(X, \prec)$ is wellfounded if for any non-empty subset Y of X, Y has a least element under $\prec$.

2.2 Bounding

For a partially ordered set $(X, \prec)$:

- x_0 in X is the minimum of X if for all x in X, $x_0 \preceq x$,
- x' in X is minimal in X if for all x in X, $\neg(x \prec x')$,
- x_1 in X is the maximum of X if for all x in X, $x \preceq x_1$,
- x' in X is maximal in X if for all x in X, $\neg(x' \prec x)$.

Taking a non-empty subset Y of X, we consider the subordering $(Y, \prec)$ and for some α in X we say:

- α is a lower bound for Y if for all y in Y, $\alpha \prec y$,
- α is the infimum of Y if it's a lower bound and for all lower bounds λ of Y, $\alpha \preceq \lambda$,
- α is an upper bound for Y if for all y in Y, $y \prec \alpha$,
- α is the supremum of Y if it's an upper bound and for all upper bounds τ of Y, $\tau \preceq \alpha$.

2.3 Order Preserving Maps

We say that $f : (X, \prec_1) \to (Y, \prec_2)$ is an order preserving map if for each x_1, x_2 in X:

$$x_1 \prec_1 x_2 \Longrightarrow f(x_1) \prec_2 f(x_2).$$

Two orderings are (order) isomorphic if there is a bijective order preserving map between them.

2.4 Representation Theorem for Partially Ordered Sets

For a partially ordered set $(X, \prec)$, there is a set $Y \subseteq \mathcal{P}(X)$ which is such that $(X, \preceq)$ is order isomorphic to $(Y, \subseteq)$.

Proof. For some x in X, we set $X^x = \{x' \in X : x' \preceq x\}$, the set of elements preceding or equal to x. For x, y in X, $x \neq y$ implies that $X^x \neq X^y$ as these sets contain x and y (resp.) so $x \mapsto X^x$ is injective. This map is surjective trivially (mapping from X to $\{X^x : x \in X\}$). We have that:

$$x \preceq y \Longleftrightarrow X^x \subseteq X^y,$$

by our definition. Thus, $x \mapsto X^x$ is an order isomorphism. □

2.5 Total Orderings

A relation $\prec$ on a set X is a (strict) total ordering if it is a connected strict partial ordering.

Similarly, we say that a relation $\preceq$ on a set X is a non-strict total ordering if it is a connected non-strict partial ordering.

2.6 Well-orderings

A relation $\prec$ on a set X is a well-ordering if it is a strict total ordering and for any non-empty subset Y of X, Y has a least element under $\prec$. We denote this with $(X, \prec) \in WO$.

2.7 Ordered Pairs

For x, y sets, the ordered pair of x and y is the set:

$$\langle x, y\rangle = \{\{x\}, \{x, y\}\}.$$

2.7.1 Uniqueness of Ordered Pairs

For x, y, u, v sets, we have that:

$$\langle x, y\rangle = \langle u, v\rangle \iff (x = u) \text{ and } (y = v).$$

Proof. Suppose the former, if $x = y$ then $\langle x, y\rangle = \{\{x\}, \{x, x\}\} = \{\{x\}\}$. Thus, $\langle u, v\rangle = \{\{u\}\}$ as it is equal to $\langle x, y\rangle$ which has one element, hence $u = v$. By the Axiom of Extensionality, we have that $x = u$ and so $y = x = u = v$.

If $x \neq y$, then $\langle x, y\rangle$ and $\langle u, v\rangle$ both have the same two elements by our assumption (so $u \neq v$). We cannot have $\{x\} = \{u, v\}$ so $\{x\} = \{u\}$ which means $x = u$ by the Axiom of Extensionality. Thus, $\{u, v\} = \{x, y\} = \{u, y\}$ so $y = v$.

Suppose the latter, then the former holds trivially. $\square$

2.7.2 The Ordered k-tuple

We define the k-tuple inductively. The 2-tuple is already defined. We define the 3-tuple:

$$\langle x_1, x_2, x_3\rangle = \langle\langle x_1, x_2\rangle, x_3\rangle,$$

and for k in $\{3, 4, \ldots\}$:

$$\langle x_1, x_2, \ldots, x_k\rangle = \langle\langle x_1, x_2, \ldots, x_{k-1}\rangle, x_k\rangle.$$

2.7.3 The Product of Sets

For A, B sets, we define:

$$A \times B = \{\langle a, b\rangle : a \in A, b \in B\}.$$

Similarly to k-tuples, for $A_1, A_2, \ldots, A_k$ sets, we have $A_1 \times A_2$ defined, so we define:

$$A_1 \times A_2 \times \cdots \times A_k = (A_1 \times A_2 \times \cdots \times A_{k-1}) \times A_k,$$

defining the k-product for k in $\{2, 3, \ldots\}$. This is not associative.

2.8 Binary Relations

A binary relation R is a class of ordered pairs. We write $R^{-1} = \{\langle y, x\rangle : \langle x, y\rangle \in R\}$.

2.8.1 Domain and Range

For a relation R, we define:

$$\begin{aligned} \operatorname{dom}(R) &= \{x : \exists\, y \text{ where } \langle x, y\rangle \in R\}, \\ \operatorname{ran}(R) &= \{y : \exists\, x \text{ where } \langle x, y\rangle \in R\}, \\ \operatorname{Field}(R) &= \operatorname{dom}(R) \cup \operatorname{ran}(R). \end{aligned}$$

2.9 Functions

A relation F is a function if for all x in $\operatorname{dom}(F)$, there is a unique y in $\operatorname{ran}(F)$ with $\langle x, y\rangle$ in F.

If F is a function, it is injective if and only if for all x, x':

$$(\langle x, y\rangle \in F \text{ and } \langle x', y\rangle \in F) \Rightarrow (x = x').$$

2.9.1 Ranges and Restrictions

For $F : X \to Y$:

- $F''A = \{y \in Y : \exists\, x \in A \text{ such that } F(x) = y\}$ the range of F on A,
- $F \upharpoonright A = \{\langle x, y\rangle \in F : x \in A\}$ the restriction of F to A.

We can see that $F''A = \operatorname{ran}(F \upharpoonright A)$.

2.9.2 The Set of Functions

For X, Y sets, we have that ${}^{X}Y = \{F : F : X \to Y\}$.

2.9.3 Indexed Cartesian Products

For a set I with each i in I corresponding to a non-empty set A_i:

$$\prod_{i \in I} A_i = \{\text{functions } f : \text{dom}(f) = I \text{ and } f(i) \in A_i \text{ for all } i \in I\}.$$

3 Transitive Sets

A set x is transitive if and only if for all y in x, $y \subseteq x$. This can be abbreviated to $\cup x \subseteq x$.

3.1 The Successor Function

For a set x, $S(x) = x \cup \{x\}$ is the successor of x. $S(x) = x$ is equivalent to saying x is transitive.

3.2 Transitive Closure

For a set x, to find a superset of x which is transitive, the transitive closure TC of x, we recurse:

$$\bigcup^0 x = x,$$
$$\bigcup^{n+1} x = \bigcup \left(\bigcup^n x \right),$$

which we can write as:

$$TC(x) = \bigcup \left\{ \bigcup^n x : n \in \mathbb{N} \right\}.$$

The transitive closure is always transitive.

3.2.1 Properties of Transitive Closure

For a set x:

1. $x \subseteq TC(x)$,
2. If t is transitive and $x \subseteq t$ then $TC(x) \subseteq t$. $TC(x)$ is the smallest transitive set containing x,
3. By the above, $TC(x) = x$ if and only if x is transitive.

Proof. (1) This is true as $\bigcup^0 = x$.

(2) If $x \subseteq t$ then clearly $\bigcup^0 x \subseteq t$. We assume $\bigcup^k x \subseteq t$ and use the fact that:

$$\left[A \subseteq B \text{ with } B \text{ transitive} \right] \Rightarrow \bigcup A \subseteq B,$$

to deduce that $\bigcup^{k+1} x \subseteq t$. By induction we have that $TC(x) \subseteq t$ as required.

(3) By (1), $x \subseteq TC(x)$. If x is transitive, we substitute it for t in (2) and get that $TC(x) \subseteq x$ as required. $\square$

4 Number Systems

4.1 Von Neumann Numerals

We have the von Neumann numerals defined as:

$$\begin{aligned}
0 &= \varnothing, \\
1 &= \{\varnothing\} = \{0\}, \\
2 &= \{\varnothing, \{\varnothing\}\} = \{1, 2\}, \\
&\cdots \\
n + 1 &= \{0, 1, \ldots, n\}.
\end{aligned}$$

4.2 Inductive Sets

A set X is called inductive if $\varnothing$ is in X and for all x in X, $S(x)$ is in X.

4.3 Axiom of Infinity

There exists an inductive set.

4.4 Natural Numbers

We say that x is a natural number if for all X:

$$X \text{ is an inductive set} \Rightarrow x \in X.$$

We define ω as the class of natural numbers, $\omega = \cap\{X : X \text{ is an inductive set}\}$. We have that ω is the smallest inductive set.

Proof. Let z be an inductive set (by the Axiom of Infinity it exists). By the Axiom of Subsets, we define a set N:

$$N = \{x \in z : \forall Y, Y \text{ is inductive} \Rightarrow x \in Y\},$$

the elements of z in every inductive set. But $N = \omega$, so ω is a set.

We know that $\varnothing$ is in every inductive set by definition, so $\varnothing$ is in ω as it is the intersection of all inductive sets. For any x in ω, we know that for any inductive set Y that x is in Y (by the definition of ω) and thus $S(x)$ is also in Y (by the definiton of an inductive set). Thus, $S(x)$ is also in ω as Y was chosen arbitrarily. Hence, ω is an inductive set and the smallest such set by its definition. $\square$

4.5 Principle of Mathematical Induction

We suppose Φ is a well-defined property of sets, then we have that:

$$\Big[\Phi(0) \text{ and } \forall\, x \in \omega \text{ we have that } \Phi(x) \Rightarrow \Phi(S(x))\Big] \Rightarrow \Big[\forall\, x \in \omega \text{ we have that } \Phi(x)\Big].$$

Proof. We assume the antecedent, it suffices to show that the collection of x in ω where $\Phi(x)$ holds is inductive (as we assume $\Phi(0)$ holds).

Let $Y = \{x \in \omega : \Phi(x)\}$. As we assumed $\Phi(0)$, we know that 0 is in Y. Then, by the second half of our assumption, we can see that Y is closed under the successor function. Thus, Y is inductive and as ω is the smallest inductive set, $\omega \subseteq Y$ as required. $\square$

4.6 Representation of Natural Numbers

We have that every natural number is either 0 or $S(x)$ for some natural number x.

Proof. Let $Z = \{y \in \omega : y = 0 \text{ or } \exists\, x \in \omega \text{ such that } S(x) = y\}$. It suffices to show that Z is inductive. Clearly, 0 is in Z. Suppose we have some u in Z, then u is in ω. As ω is inductive, $S(u)$ is also in ω so $S(u)$ is in Z. Thus, Z is inductive as required. $\square$

4.7 Transitivity of ω

We have that ω is transitive.

Proof. Let $X = \{n \in \omega : n \subseteq \omega\}$. If $X = \omega$ then by definition ω is transitive. It suffices to show that X is inductive. We know that $\varnothing$ is in X as 0 is in ω. Taking n in X, then clearly $\{n\} \subseteq \omega$ as n is in ω. Furthermore, $n \subseteq \omega$ as n is in X. Thus, $n \cup \{n\} \subseteq \omega$ so $S(n) \in X$ which means X is inductive as required. $\square$

4.8 Ordering on the Naturals

For m, n in ω, we define:

$$\begin{aligned} m < n &\iff m \in n, \\ m \leq n &\iff m = n \text{ or } m \in n. \end{aligned}$$

By definition, $n < S(n)$.

We have that:

1. This ordering is transitive,
2. For all n in ω and for all m we have that $m < n$ if and only if $S(m) < S(n)$,
3. For all n in ω, $n \not< n$.

Proof. (1) This follows from the transitivity of set inclusion.

(2) We take $\Phi(k) = [(m < k) \Rightarrow (S(m) < S(k))]$. We see $\Phi(0)$ holds. Supposing $\Phi(k)$ holds for some k, let $m < S(k)$ then m is in $k \cup \{k\}$. If m is in k then by $\Phi(k)$ we have that $S(m) < S(k) < S(S(k))$. If $m = k$ then $S(m) = S(k) < S(S(k))$. Thus, by induction, we have our result.

Assume $S(m) < S(n)$, m is in $S(m) = m \cup \{m\}$ which is in $S(n) = n \cup \{n\}$. If $S(m) = n$, then m is in n so $m < n$. If $S(m)$ is in n then m is in n as n is transitive.

(3) We know that $0 \not< 0$ as $0 \notin 0$. If $k \notin k$ then $S(k) \notin S(k)$ by Part (ii). Thus, $X = \{k \in \omega : k \notin k\}$ is inductive which makes it equal to ω as required. □

4.9 Total Ordering on the Naturals

We have that $<$ is a (strict) total ordering on the naturals.

4.10 Well-ordering Theorem for ω

Let $X \subseteq \omega$, then either $X = \varnothing$ or there is some n_0 in X such that for any m in X either $n_0 = m$ or $n_0 < m$.

Proof. Suppose $X \subseteq \omega$ but has no least element. Let $Z = \{k \in \omega : \forall\, n < k, n \notin X\}$. We want to show Z is inductive, meaning $Z = \omega$ and so $X = \varnothing$.

Vacuously, 0 is in Z. Suppose we have k in Z, we let $n < S(k) = k \cup \{k\}$ and consider:

- If $n \in k$ then $n \notin X$ as $n < k \in Z$,
- If $n = k$ then $n \notin X$ because if n was in X then it would be the least element of X, a contradiction.

Thus, $S(k)$ is in Z so Z is inductive. □

4.11 Recursion Theorem on ω

Let A be any set with a in A and $f : A \to A$ any function. There exists a unique function $h : \omega \to A$ such that for any n in ω:

$$h(0) = a,$$
$$h(S(n)) = f(h(n)).$$

Proof. We will find h as a union of k-approximations where u is a k-approximation if it is a function with $\mathrm{dom}(u) = k$ and for:

- If $k > 0$ then $u(0) = a$,
- If $k > S(n)$ then $u(S(n)) = f(u(n))$.

From this, we see that $\{\langle 0, a\rangle\}$ is a 1-approximation in particular. Furthermore, if u is a k-approximation and $l \leq k$ then $u \restriction l$ is an l-approximation, finally if $u(k-1) = c$ for some c, then $u' = u \cup \{\langle k, f(c)\rangle\}$ is a $(k+1)$-approximation.

Agreement on Domain
If u is a k-approximation and v is a k'-approximation for some $k \leq k'$ then $v \restriction k = u$ (hence $u \subseteq v$).

Proof. We appeal to the contrary with $0 \leq m < k$ being the least natural such that $u(m) \neq v(m)$. We know that $m \neq 0$ as $u(0) = a = v(0)$. So, $m = S(m')$ for some m'. As m is chosen minimally, $u(m') = v(m')$. We can then see that $u(m) = f(u(m')) = f(v(m')) = v(m)$, a contradiction. □

Uniqueness
If h exists, it is unique.

Proof. Suppose h and h' are two different functions with domain ω satisfying the theorem. We take $0 \leq m < \omega$ to be the least natural such that $h(m) \neq h'(m)$ and apply the same reasoning to the above. □

Existence
Let u be in B if and only if there exists k in ω such that u is a k-approximation. For any u, v in B either $u \subset v$ or vice-versa by our previous results. We take $h = \bigcup B$.

We have that h is a function:

Proof. We appeal to the contrary. If $\langle n, c\rangle$ and $\langle n, d\rangle$ are in h with $c \neq d$, then we have u, v in B with $u(n) = c$ and $v(n) = d$ but this is impossible by **Agreement on Domain**. □

We have that $\mathrm{dom}(h) = \omega$:

Proof. We appeal to the contrary and suppose $\varnothing \neq X = \{n \in \omega : n \notin \mathrm{dom}(h)\}$. By the definition of h this means that:

$$X = \{n \in \omega : \text{There's no } u\text{-approximation with } n \in \mathrm{dom}(u)\}.$$

We saw that there is a 1-approximation, so 0 is not the least element of X. We suppose $n_0 = S(m)$ is the least element of X. As m is not in X, there must be an n_0-approximation n with $n(m) = c$ for some c. But, we saw that we can extend k-approximations, so we can generate a $(n_0 + 1)$-approximation which is a contradiction. Thus, $X = \varnothing$. $\square$

Thus, we have that h exists and is a unique function as required. $\square$

5 Well-orderings and Ordinals

5.1 The Principle of Transfinite Induction

Let $\langle X, \prec \rangle$ be a well-ordering. We have that:

$$[\forall\, x \in X, (\forall\, y \prec x, \Phi(y)) \Rightarrow \Phi(x)] \Rightarrow \forall\, x \in X, \Phi(x).$$

Proof. We appeal to the contrary and assume the antecedent but suppose that $\varnothing \neq Z = \{x \in X : \neg\Phi(x)\}$. As $\langle Z, \prec \rangle$, there is $\prec$-least element z_0. But then for all $x \prec z_0$, $\Phi(x)$ holds. But, by the antecedent, this means $\Phi(z_0)$ holds, a contradiction. $\square$

5.2 Initial Segments

For a well-ordering $\langle X, \prec \rangle$, the $\prec$-initial segment of some element z in X is the set of predecessors of z, denoted by X_z. Note that X_z does not contain z.

5.3 Order Preserving Maps on Well-orderings

For a well-ordering $\langle X, \prec \rangle$ with $f : \langle X, \prec \rangle \to \langle X, \prec \rangle$ an order preserving map, we have that for all x in X, $x \prec f(x)$.

Proof. We appeal to the contrary, that for some x in X, we have $f(x) \prec x$. As $\langle X, \prec \rangle$ is a well-ordering, there's a $\prec$-least x_0 in X with the property that $f(x_0) \prec x_0$. But $f(f(x_0)) \prec f(x_0)$ as f is order preserving. Thus, a contradiction to the minimality of x_0. $\square$

5.3.1 Uniqueness of Order Isomorphisms

For well-orderings $\langle X, \prec_x \rangle$, $\langle Y, \prec_y \rangle$ with $f : \langle X, \prec_x \rangle \to \langle Y, \prec_y \rangle$ an order isomorphism. We have that f is unique.

Proof. Suppose we have two such isomorphisms f and g. We have that $(f^{-1} \circ g)$ is also an order isomorphism. Taking x arbitrary in X:

$$\begin{aligned} & x \preceq_x (f^{-1} \circ g)(x) \\ \Longrightarrow \quad & f(x) \preceq_y f(f^{-1} \circ g)(x) \\ \Longrightarrow \quad & f(x) \preceq_y g(x). \end{aligned}$$

By applying this argument again with the roles of f and g swapped, we can also see that $g(x) \preceq_y f(x)$. Thus, $f(x) = g(x)$.

In particular, if $\langle X, \prec_x \rangle = \langle Y, \prec_y \rangle$ then this isomorphism is the identity map. $\square$

5.4 Non-existence of Order Isomorphisms to Segments

A well-ordered set is not order isomorphic to any segment of itself.

Proof. We appeal to the contrary and suppose there is such an order isomorphism on a well-ordering $\langle X, \prec \rangle$ to $\langle X_z, \prec \rangle$ for some z in X. But, we have that $x \preceq f(x)$ for any x in X and $f(z) \prec z$ as $f(z)$ is in X_z. Thus, we have that $z \preceq f(z)$ and $z \succ f(z)$, a contradiction. $\square$