

# Linear Algebra 2 Notes

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*An important note, these notes are absolutely **NOT** guaranteed to be correct, representative of the course, or rigorous. Any result of this is not the author's fault.*

# 1 Groups, Rings, and Fields

## 1.1 Definition of a Group

A group is a set  $G$  combined with a group operation  $\circ : G \times G \rightarrow G$  such that:

- For all  $g, h, j$  in  $G$ ,  $g(hj) = (gh)j$  (associativity)
- There exists  $e$  in  $G$  such that  $eg = ge = g$  for all  $g$  in  $G$
- For all  $g$  in  $G$ , there exists  $g^{-1}$  in  $G$  such that  $gg^{-1} = g^{-1}g = e$  where  $e$  is the identity of  $G$ .

## 1.2 Definition of a Homomorphism

A homomorphism between two groups  $G, H$  is a function  $f : G \rightarrow H$  such that  $f(gh) = f(g)f(h)$  for all  $g, h$  in  $G$ .

## 1.3 Properties of Homomorphisms

We can derive some properties of homomorphisms, for  $G, H$  groups, and  $f : G \rightarrow H$  a homomorphism:

- The image of the identity in  $G$  is the identity in  $H$
- The kernel of  $f$  is a subgroup of  $G$
- The image of  $f$  is a subgroup of  $H$
- Bijective homomorphisms are isomorphisms.

## 1.4 Definition of a Ring

A ring with unity is a set  $R$  along with an addition map  $+$ , and a multiplication map  $\circ$  where  $+, \circ : R \times R \rightarrow R$  such that:

- $(R, +)$  is an abelian group (of which the identity is called zero)
- The multiplication operation is associative
- The multiplication operation has a two-sided identity not equal to the zero identity (called one)
- For all  $a, b, c$  in  $R$ ,  $a(b + c) = ab + ac$  and  $(a + b)c = ac + bc$ .

A ring is commutative if the multiplication operation is commutative.

## 1.5 Definition of a Subring

For the ring  $R = (R', +, \circ)$  and  $S$  a set,  $S$  is a subring of  $R$  if  $S \subseteq R'$  and  $(S, +, \circ)$  is a ring.

## 1.6 Definition of a Ring Homomorphism

For rings with unity  $R$  and  $S$ ,  $f : R \rightarrow S$  is a ring homomorphism if for all  $a, b$  in  $R$ :

$$\begin{aligned}f(a + b) &= f(a) + f(b) \\f(ab) &= f(a)f(b) \\f(1_R) &= 1_S\end{aligned}$$

*Essentially, this says that  $f$  is a homomorphism for the groups formed by  $R$  and  $S$  under addition and multiplication.*

## 1.7 Definition of a Field

A field  $\mathbb{F}$  is a ring with unity with the following properties:

- $(\mathbb{F} \setminus \{0\}, \circ)$  is an abelian group.

## 1.8 Definition of the Field Characteristic

For a field  $\mathbb{F}$ , the field characteristic  $\text{char}(\mathbb{F})$  is the smallest positive integer  $n$  such that:

$$\sum_{i=1}^n 1 = 1 + 1 + \dots + 1 = 0,$$

or zero if no such value  $n$  exists.

## 1.9 Definition of the Algebraic Closure of Fields

A field  $\mathbb{F}$  is called algebraically closed if all non-constant polynomials with coefficients in  $\mathbb{F}$  also has a root in  $\mathbb{F}$ .

## 2 Vector Spaces

### 2.1 Definition of a Vector Space

A vector space over a field  $\mathbb{F}$  is a set  $V$  with an addition operation  $+: V \times V \rightarrow V$  and a scalar multiplication operations  $\circ: \mathbb{F} \times V \rightarrow V$  such that for all  $a, b$  in  $\mathbb{F}$  and  $v, w$  in  $V$ :

- $(V, +)$  is an abelian group
- $1 \circ v = v$  where 1 is the multiplicative identity of  $\mathbb{F}$
- $(ab) \circ v = a \circ (b \circ v)$
- $(a + b) \circ v = a \circ v + b \circ v$
- $a \circ (v + w) = a \circ v + a \circ w$ .

### 2.2 Definition of a Subspace

For  $V$  a vector space over the field  $\mathbb{F}$  and  $W$  a set,  $W$  is a subspace of  $V$  if it is a subset of  $V$  and is a vector space with respect to the addition and scalar multiplication defined by  $V$ .

It is sufficient to verify that for any  $a$  in  $\mathbb{F}$  and  $v, w$  in  $W$  we have that  $a(v + w)$  is in  $W$ .

### 2.3 Definition of a Linear Combination

For a set  $V$  with addition operation  $+$ , a field  $\mathbb{F}$  and  $n$  in  $\mathbb{N}$ , a linear combination of  $v_1, \dots, v_n$  in  $V$  is:

$$\sum_{i=1}^n a_i v_i,$$

for  $a_1, \dots, a_n$  in  $\mathbb{F}$ .

### 2.4 Definition of the Span

For a set  $V$  with addition operation  $+$  and a field  $\mathbb{F}$ , the span of  $W \subseteq V$  is the set of all the linear combinations of the values in  $W$ . Denoted by  $\text{span}(W)$ .

## 2.5 Definition of Linear Independence

For a vector space  $V$  and  $W \subseteq V$ , we say  $W$  is linearly dependent if there exists a non-trivial linear combination of all the vectors in  $W$  equal to zero (and linearly independent otherwise).

## 2.6 Properties of Linear Independence

For a vector space  $V$  with  $W \subseteq V$ :

- $0 \in W \Rightarrow W$  is linearly dependent
- $W$  linearly independent  $\Rightarrow$  any  $X \subseteq W$  is linearly independent
- If there's a linearly dependent subset of  $W$ , then  $W$  is linearly dependent.

## 2.7 Definition of a Basis

For a vector space  $V$  with  $W \subseteq V$ , if  $W$  is linearly independent and  $\text{span}(W) = V$ , we say that  $W$  is a basis of  $V$ .

Saying  $W$  is a basis is equivalent to saying that each vector in  $V$  can be **uniquely** written as a linear combination of vectors in  $W$ .

Additionally, for finite vector spaces, we have that all bases have the same amount of elements.

## 2.8 Definition of Dimension

For non-infinite bases, we say that the value of the basis is the dimension of the vector space it is a member of. Vector spaces with such bases are called finite-dimensional and all other vector spaces are infinite-dimensional.

By convention, for a vector space  $V$ ,  $\dim(\{0_V\}) = 0$ .

## 2.9 Isomorphisms from Dimension

For  $V, W$  finite-dimensional vector spaces over  $\mathbb{F}$  with  $\dim(V) = \dim(W)$ , then  $V \cong W$ .

If we set  $n = \dim(V)$ , we have that  $V \cong \mathbb{F}^n$ .

*Such an isomorphism can be found by mapping a vector in terms of some chosen basis vectors ( $v = a_1v_1 + a_2v_2 + \cdots + a_nv_n$ ) to the coefficients  $(a_1, a_2, \dots, a_n)$ .*

## 3 Linear Maps

### 3.1 Definition of a Linear Map

Let  $V, W$  be vector spaces over a field  $\mathbb{F}$ , we have that  $f : V \rightarrow W$  is a linear map if for all  $a, b$  in  $\mathbb{F}$  and  $u, v$  in  $V$ :

$$f(au + bv) = af(u) + bf(v).$$

A bijective linear map is called an isomorphism. If  $f : V \rightarrow W$  is an isomorphism, we say that  $V$  and  $W$  are isomorphic, denoted by  $V \cong W$ .

### 3.2 The Kernel of Linear Maps

Let  $V, W$  be vector spaces over a field  $\mathbb{F}$ , and  $f : V \rightarrow W$  be a linear map. We define the kernel of  $f$  as:

$$\text{Ker}(f) = \{v \in V : f(v) = 0_{\mathbb{F}}\}.$$

Saying  $\text{Ker}(f)$  is  $\{0_{\mathbb{F}}\}$  is equivalent to saying  $f$  is injective.

### 3.3 The Image of Linear Maps

Let  $V, W$  be vector spaces over a field  $\mathbb{F}$ , and  $f : V \rightarrow W$  be a linear map. We define the image of  $f$  as:

$$\text{Im}(f) = \{w \in W : \exists v \in V \text{ with } f(v) = w\}.$$

Saying  $\text{Im}(f)$  is  $W$  is equivalent to saying  $f$  is surjective.

### 3.4 The Inverse of Linear Maps

For a bijective linear map  $f$ , the inverse of  $f$  is also linear.

### 3.5 Properties of the Set of Linear Maps

For  $V, W$  vector spaces over a field  $\mathbb{F}$ , we define  $\mathcal{L}(V, W)$  to be the set of all linear maps from  $V$  to  $W$ .

### 3.6 The Rank-Nullity Theorem

For  $V, W$  finite-dimensional vector spaces and  $f : V \rightarrow W$  a linear map, we have that:

$$\dim(V) = \dim(\text{Ker}(f)) + \dim(\text{Im}(f)).$$

Thus, for a linear map  $f : V \rightarrow V$ , if  $f$  is injective or surjective then it's an isomorphism.

## 4 Matrices

### 4.1 Definition of a Matrix

For  $m, n$  in  $\mathbb{Z}_{>0}$  and  $\mathbb{F}$  a field. An  $m \times n$  matrix with entries in  $\mathbb{F}$  is a map  $M : [m] \times [n] \rightarrow \mathbb{F}$ , more commonly written as  $M = (a_{ij})$  representing the rectangular array of values held by  $M$ .

The set of all  $m \times n$  matrices over  $\mathbb{F}$  is denoted by  $M_{m \times n}(\mathbb{F})$ .

### 4.2 Types of Matrix

For  $m, n$  in  $\mathbb{Z}_{>0}$  and  $\mathbb{F}$  a field, let  $M$  be in  $M_{m \times n}(\mathbb{F})$ . We have the following types of matrix:

- **Square:** where  $m = n$
- **Upper Triangular:** if  $a_{ij} = 0$  for  $i > j$
- **Lower Triangular:** if  $a_{ij} = 0$  for  $i < j$
- **Diagonal:** if  $a_{ij} = 0$  for  $i \neq j$
- **Symmetric:** if  $a_{ij} = a_{ji}$
- **Anti-symmetric:** if  $a_{ij} = -a_{ji}$ .

### 4.3 Properties of the Space of Matrices

For  $m, n$  in  $\mathbb{Z}_{>0}$  and  $\mathbb{F}$  a field, we have that  $M_{m \times n}(\mathbb{F})$  is a vector space over  $\mathbb{F}$  where matrices are added and multiplied by scalars component-wise. So, for  $M_1 = (a_{ij}), M_2 = (b_{ij})$  in  $M_{m \times n}$  and  $c$  in  $\mathbb{F}$  we have:

$$\begin{aligned} cM_1 &= (ca_{ij}) \\ M_1 + M_2 &= (a_{ij} + b_{ij}). \end{aligned}$$

Additionally, the zero vector is  $M_0 = (0)$  and the vector space has a basis consisting of  $M_{ij}$  where all entries are zero except the  $(i, j)^{\text{th}}$  entry. This leads to the conclusion that the dimension is  $mn$  and thus that  $M_{m \times n} \cong \mathbb{F}^{mn}$ .

### 4.4 Matrix Multiplication

For  $a, b, c$  in  $\mathbb{Z}_{>0}$  and a field  $\mathbb{F}$ , we can define the multiplication of the two matrices  $X = (x_{ij})$  in  $M_{a \times b}$  and  $Y = (y_{ij})$  in  $M_{b \times c}$  as follows:

$$XY = \left( \sum_{k=1}^b x_{ik} y_{kj} \right).$$

This operation is not commutative in general but is associative.

For  $A, B$  in  $M_n$ , we have that  $AB$  is also in  $M_n$ . This, along with matrix addition, makes  $M_n$  a ring with unity with multiplicative identity  $I_n = (\delta_{ij})$ . However, there exists  $A, B$  in  $M_n$  such that  $AB = 0$  so,  $M_n$  is not a field.

### 4.5 Matrices of Linear Maps

For  $V, W$  vector spaces over a field  $\mathbb{F}$ , for some  $m, n$  in  $\mathbb{Z}_{>0}$  we have  $A = \{v_1, \dots, v_n\}$ ,  $B = \{w_1, \dots, w_n\}$  bases for  $V$  and  $W$  respectively. Given  $f$  in  $\mathcal{L}(V, W)$ , the matrix associated to  $f$  (with respect to the bases  $A$  and  $B$ ) is the  $m \times n$  matrix:

$$M_{BA}(f) = (a_{ij}),$$

where we define  $a_{ij}$  by:

$$f(v_j) = \sum_{i=1}^m a_{ij} w_i,$$

for each  $j$  in  $[n]$ .



## 4.6 Matrices of Composed Linear Maps

For  $U, V, W$  vector spaces over a field  $\mathbb{F}$ , for some  $l, m, n$  in  $\mathbb{Z}_{>0}$  we have  $A = \{u_1, \dots, u_n\}$ ,  $B = \{v_1, \dots, v_m\}$ ,  $C = \{w_1, \dots, w_l\}$  bases for  $U, V, W$  respectively. Given  $g, f$  in  $\mathcal{L}(V, W)$ , we have:

$$M_{CA}(g \circ f) = M_{CB}(g)M_{BA}(f).$$

## 4.7 Transition Matrices

For a finite-dimensional vector space  $V$ , with an identity  $I$  and bases  $A, A'$ , we call  $M_{A'A}(I) = C_{A'A}$  a transition matrix.

We have that  $C_{A'A}$  is invertible and  $C_{A'A}^{-1} = C_{AA'}$ .

*Essentially, the transition matrix transforms between bases.*

## 4.8 Matrix Transitions

For a finite-dimensional vector space  $V$ , with  $f : V \rightarrow V$  a linear operator, and bases  $A, B$ :

$$\begin{aligned} M_{BB}(f) &= C_{AB}^{-1} M_{AA}(f) C_{AB} \\ &= C_{BA} M_{AA}(f) C_{AB}. \end{aligned}$$

## 4.9 Similar Matrices

For matrices  $A', A$ , we say that  $A'$  and  $A$  are similar if there exists an invertible matrix  $C$  such that:

$$A' = C^{-1}AC.$$

This is denoted by  $A' \sim A$ . Similarity forms an equivalence relation on the space of square matrices.

If we have  $A \sim A'$  and  $A$  represents some linear operator  $f$  for some basis  $B$ , then we have that for some basis  $B'$ ,  $f$  has matrix  $A'$ .

## 5 Eigenvectors and Eigenvalues

### 5.1 Definition of an Eigenvectors and Eigenvalues

For a vector space  $V$  over  $\mathbb{F}$  with  $f : V \rightarrow V$  a linear operator, a non-zero vector  $v$  in  $V$  is an eigenvector if  $f(v) = \lambda v$  for some  $\lambda$  in  $\mathbb{F}$  which is called the eigenvalue corresponding to  $v$ .

### 5.2 Definition of an Eigenspace

For a vector space  $V$  over  $\mathbb{F}$  with  $f : V \rightarrow V$  a linear operator and some eigenvalue  $\lambda$ , we define the eigenspace of  $\lambda$  as the set of eigenvectors with eigenvalue  $\lambda$ .

This is denoted by  $E(\lambda)$  and  $E(\lambda) \cup \{0_V\}$  forms a subspace of  $V$ . The dimension of  $E(\lambda)$  is the geometric multiplicity of  $\lambda$ .

## 6 Direct Sums and Projections

### 6.1 Definition of a Direct Sum

For  $V, W$  vector spaces, we define the direct product of  $V$  and  $W$  as:

$$V \oplus W = \{(v, w) : v \in V, w \in W\},$$

with addition and scalar multiplication defined coordinate-wise and zero vector  $(0_V, 0_W)$ .

### 6.2 The Equivalence of Direct Sums

For  $V, W \subseteq U$ , we have that the following are equivalent:

- $U = V \oplus W$
- Each element in  $U$  can be written uniquely as the sum of elements in  $V$  and  $W$
- The map  $f : V \oplus W \rightarrow U; (v, w) \mapsto v + w$  is isomorphism.

### 6.3 The Addition Map for Direct Sums

For  $V, W$  subspaces of a vector space  $U$ , and  $f : V \oplus W \rightarrow U$  defined by:

$$f((v, w)) = v + w,$$

we have that:

- $f$  is linear
- $f$  is injective if and only if  $V \cap W = \{0\}$
- $f$  is surjective if and only if  $V \cup W$  spans  $U$ .

### 6.4 Projections

For  $V, W$  subspaces of  $U$  with  $U = V \oplus W$ , the projection **onto**  $V$  along  $W$  is the linear operator  $P_{V,W} : U \rightarrow U$  where:

$$P_{V,W}(u) = v,$$

where  $u = v + w$  for some unique  $v$  in  $V$  and  $w$  in  $W$ .

We have that for a linear operator  $P$ ,  $P$  is a projection if and only if  $P \circ P = P$ .

### 6.5 $f$ -invariance

For a vector space  $V$  with  $U \subseteq V$  a subspace and  $f : U \rightarrow U$  a linear operator, we have that  $U$  is  $f$ -invariant if for all  $u$  in  $U$  we have  $f(u)$  in  $U$ .

The eigenspaces of  $f$  are examples of  $f$ -invariant spaces.

### 6.6 Matrices of Linear Maps (using $f$ -invariance)

For  $U, W \subseteq V$  subspaces of the vector space  $V$  such that  $V = U \oplus W$ , let  $B_U, B_W$  be finite bases of  $U$  and  $W$  respectively. If we have a linear operator  $f : V \rightarrow V$  such that  $U$  and  $W$  are  $f$ -invariant, we have that the matrix with respect to the basis  $B = B_U \cup B_W$  of  $f$  has the following block form:

$$M_{BB}(f) = \begin{pmatrix} M_{B_U B_U}(f) & 0 \\ 0 & M_{B_W B_W}(f) \end{pmatrix}.$$

## 7 Quotient Spaces

### 7.1 Definition of a Quotient Space

For a vector space  $V$  with  $W \subseteq V$  a subspace. We define an equivalence relation on  $V$  by declaring:

$$v_1 \sim v_2 \text{ if } v_1 - v_2 \in W.$$

The set of equivalence classes is called the quotient of  $V$  by  $W$  and is denoted by  $V/W$ . For some  $v$  in  $V$ , we denote the class containing  $v$  by  $v + W$  (similarly to cosets in Introduction to Group Theory). So, we have:

$$V/W = \{v + W : v \in V\},$$

with addition and multiplication defined for  $v_1, v_2$  in  $V$  and  $a$  in the field:

$$\begin{aligned}(v_1 + W) + (v_2 + W) &= (v_1 + v_2) + W \\ a(v_1 + W) &= av_1 + W.\end{aligned}$$

### 7.2 Linear Map to the Quotient Space

For a vector space  $V$  with  $W \subseteq V$  a subspace, we can define  $\pi : V \rightarrow V/W$  for some  $v$  in  $V$  by  $\pi(v) = v + W$ . We have that  $\pi$  is linear and its kernel is  $W$ .

### 7.3 Isomorphisms formed by Linear Maps

For  $V, W$  vector spaces and  $f : V \rightarrow W$  a linear map, we have an isomorphism  $\text{Im}(f) \cong V/\text{Ker}(f)$ .

### 7.4 Existence of a Linear Operator on the Quotient Space

For a vector space  $V$  with  $W \subseteq V$  a subspace and a linear operator  $f : V \rightarrow V$ , there exists a well-defined operator  $\bar{f} : V/W \rightarrow V/W$  if and only if  $W$  is  $f$ -invariant. We call this the induced operator on  $V/W$ .

## 7.5 Matrices formed using Quotient Spaces

Consider a finite-dimension vector space  $V$  and  $f : V \rightarrow V$  a linear operator with  $W$  an  $f$ -invariant subspace of  $V$ . If we have  $B_W$  a basis for  $W$ , that we extend to a basis  $B$  of  $V$  and set  $A$ :

$$A = \{v + W : v \in B \setminus B_W\},$$

a basis of  $V/W$  and we can form a matrix in block form:

$$M_{BB}(f) = \begin{pmatrix} M_{B_W B_W}(f) & * \\ 0 & M_{AA}(\bar{f}) \end{pmatrix},$$

where  $\bar{f}$  is the induced operator on  $V/W$  and  $*$  marks the area of the matrix which we cannot determine.