

# Analysis 1 (TB2) Notes

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*An important note, these notes are absolutely **NOT** guaranteed to be correct, representative of the course, or rigorous. Any result of this is not the author's fault.*

# 1 Continuity

## 1.1 Continuous Functions

From Analysis 1A, we have that a function  $f : A \rightarrow \mathbb{R}$  is continuous on  $A$  if:

$$\forall x \in A, \forall \epsilon > 0, \exists \delta > 0 \text{ such that } \forall y \in A, (|y - x| < \delta) \Rightarrow (|f(y) - f(x)| < \epsilon).$$

It's important to note that  $x$  is chosen given before we choose a  $\delta$ . Thus, our choice for  $\delta$  can depend on  $x$  as well as  $\epsilon$ .

**Uniform** continuity requires that  $\delta$  is independent of  $x$ .

*A note, a function being continuous at a value (or set of values for that matter), it equivalent to saying that there exists a limit for the function at that value and that limit is the value of the function applied to that value.*

## 1.2 Uniformly Continuous Functions

Uniform continuity is similar to continuity as we knew it in Analysis 1A. For a function  $f : A \rightarrow \mathbb{R}$ ,  $f$  is uniformly continuous on  $A$  if:

$$\forall \epsilon > 0, \exists \delta > 0 \text{ such that } \forall x, y \in A, (|y - x| < \delta) \Rightarrow (|f(y) - f(x)| < \epsilon).$$

We can see that uniform continuity **implies** continuity but **not** vice versa.

*A note, for uniform continuity, we are saying that given a value  $\epsilon$ , we can always pick a distance ( $\delta$ ) such that if two values are within that distance of each other, the distance between the values after the function is applied to them will be less than  $\epsilon$ . This is essentially testing for divergence to infinity at a value ( $\frac{1}{x}$  is continuous but not uniformly continuous on  $\mathbb{R}_{>0}$ ).*

## 2 Convergence

We have the notion of convergence for sequences of real numbers from Analysis 1A, convergence in this section is similar but specifically for functions.

### 2.1 Pointwise Convergence

A sequence of functions  $(f_n)_{n \in \mathbb{N}}$  from  $A \rightarrow \mathbb{R}$  converges **pointwise** to the function  $f$  on  $A$  if:

$$\forall x \in A, \lim_{n \rightarrow \infty} (f_n(x)) = f(x).$$

$f$  is called the **pointwise limit** of  $(f_n)_{n \in \mathbb{N}}$ .

*A note, for  $f_n : [0, 1] \rightarrow [0, 1]; x \rightarrow x^n$ ,  $f : [0, 1] \rightarrow [0, 1]; x \rightarrow \delta_1(x)$ ,  $f_n$  converges pointwise to  $f$ .*

### 2.2 Uniform Convergence

A sequence of functions  $(f_n)_{n \in \mathbb{N}}$  from  $A \rightarrow \mathbb{R}$  converges **uniformly** to the function  $f$  on  $A$  if:

$$\forall \epsilon > 0, \exists N \in \mathbb{N} \text{ such that } \forall x \in A, \forall n \in \mathbb{N}, (n \geq N) \Rightarrow (|f(x) - f_n(x)| < \epsilon).$$

*For the same functions outlined in the note under pointwise convergence, we have that  $f_n$  does not converge uniformly to  $f$ . Let  $\epsilon \in (0, 1)$ ,  $x \in [0, 1)$  and suppose  $f_n$  is uniformly convergent to  $f$ ,*

$$\begin{aligned} |f_n(x) - f(x)| &= |x^n| < \epsilon \\ \Rightarrow 0 &\leq x^n < \epsilon < 1 \\ \Rightarrow 0 &\leq x < \epsilon^{\frac{1}{n}} < 1 \\ \Rightarrow \epsilon &= 1 \text{ as } x \in [0, 1). \end{aligned}$$

*This is a contradiction by the definition of  $\epsilon$ . Thus, we have the result.*

### 2.3 Weierstrass' Theorem

For  $a, b \in \mathbb{R}$  with  $a < b$ , if a sequence of continuous functions on  $[a, b]$ ,  $(f_n)_{n \in \mathbb{N}}$  converges uniformly to  $f$  on  $[a, b]$ ,  $f$  is continuous on  $[a, b]$ .

*Basically, uniform convergence preserves continuity (it also preserves regulation).*

## 2.4 Supremum Norm

For  $a, b \in \mathbb{R}$  with  $a < b$ , let  $f : [a, b] \rightarrow \mathbb{R}$  be a bounded function. The supremum norm of  $f$  on  $[a, b]$  is denoted by  $\|f\|_{[a,b]}$  and is defined by:

$$\|f\|_{[a,b]} := \sup \{ |f(x)| : x \in [a, b] \}.$$

*The supremum norm is simply just the furthest distance from zero reached by a function over a closed interval. By definition, it is a real number and  $\exists x \in [a, b]$  such that  $f(x)$  is the supremum norm.*

There are a few key properties of the supremum norm, let  $a$  and  $b$  be as above and let  $\lambda \in \mathbb{R}$ ,  $f, g : [a, b] \rightarrow \mathbb{R}$  be bounded functions:

- $\|f\|_{[a,b]} > 0$
- $\|f\|_{[a,b]} = 0 \Leftrightarrow f = 0$  on  $[a, b]$
- $\|\lambda f\|_{[a,b]} = |\lambda| \|f\|_{[a,b]}$
- $\|f + g\|_{[a,b]} = \|f\|_{[a,b]} + \|g\|_{[a,b]}.$

## 2.5 Cauchy Sequences of Functions

For  $a, b \in \mathbb{R}$  with  $a < b$ , denote the set of continuous functions on  $[a, b]$  by  $C([a, b])$ . Let  $(f_n)_{n \in \mathbb{N}}$  be a sequence of functions in  $C([a, b])$ . We say  $(f_n)_{n \in \mathbb{N}}$  is Cauchy if:

$$\forall \epsilon > 0, \exists N \in \mathbb{N} \text{ such that } \forall m, n \in \mathbb{N}, (m, n \geq N) \Rightarrow (\|f_n - f_m\|_{[a,b]} < \epsilon).$$

*This obviously bears an extreme resemblance to the Cauchy sequences of Analysis 1A. Just replacing the sequences of reals with sequences of functions and the modulus with the supremum norm.*

For each continuous function, there exists a Cauchy sequence such that the sequence converges uniformly to said function.

## 3 Integration

### 3.1 Step Functions

For  $a, b \in \mathbb{R}$  with  $a < b$ , a partition of the interval  $[a, b]$  is a set  $P$  of the form:

$$P = \{x_0, x_1, \dots, x_n\} \text{ (for some } n \in \mathbb{N}\text{)} \\ \text{where } a = x_0 < x_1 < \dots < x_n = b.$$

We say a function  $\psi : [a, b] \rightarrow \mathbb{R}$  is a step function if there exists a partition  $P = \{x_0, \dots, x_n\}$  and a set of constants in  $\mathbb{R}$  ( $\{c_0, c_1, \dots, c_n\}$ ) such that:

$$\psi(x) = c_i \text{ } (\forall x \in (x_{i-1}, x_i)).$$

In this case,  $P$  and  $\psi$  are adapted to each other.

$S[a, b]$  is the set of step functions over  $[a, b]$ .

### 3.2 Integration of Step Functions

The integral of the step function is simple:

$$\int_a^b \psi(x) dx = \sum_{i=1}^n c_i (x_i - x_{i-1}).$$

As long as the partition is adapted to  $\psi$ , the integral doesn't change.

Here are some properties of the integration of step functions, let  $\phi, \psi$  be step functions over  $[a, b]$ ,  $y \in \mathbb{R}$  with  $a < y < b$ ,  $\alpha, \beta \in \mathbb{R}$ :

- **Linearity:**  $\int_a^b \alpha \psi(x) + \beta \phi(x) dx = \alpha \int_a^b \psi(x) dx + \beta \int_a^b \phi(x) dx$
- **Monotonicity:**  $(\psi(x) \leq \phi(x) (\forall x \in [a, b])) \Rightarrow (\int_a^b \psi(x) dx \leq \int_a^b \phi(x) dx)$
- **Continuity:**  $|\int_a^b \psi(x) dx| \leq (b - a) \|\psi(x)\|_{[a, b]}$
- **Additivity:**  $\int_a^b \psi(x) dx = \int_a^y \psi(x) dx + \int_y^b \psi(x) dx$

### 3.3 Regulated Functions

Let  $A \subseteq \mathbb{R}$  and  $f : A \rightarrow \mathbb{R}$ . For some  $\epsilon > 0$ ,  $a \in A$ , and  $\alpha \in \mathbb{R}$ :

1.  $f$  has a **right limit** of  $\alpha$  at  $a$  if:  
 $\exists \delta > 0$  such that  $(0 < x - a < \delta) \Rightarrow (|f(x) - \alpha| < \epsilon)$
2.  $f$  has a **left limit** of  $\alpha$  at  $a$  if:  
 $\exists \delta > 0$  such that  $(0 < a - x < \delta) \Rightarrow (|f(x) - \alpha| < \epsilon)$ .

We can denote right limits by:  $\lim_{x \downarrow a} f(x) = \alpha$ . Similarly for left limits:  $\lim_{x \uparrow a} f(x) = \alpha$ .

*There is a sequential definition too, for any sequence  $(x_n)_{n \in \mathbb{N}}$  that satisfies  $x_n > a$  for all  $n \in \mathbb{N}$  and  $\lim_{n \rightarrow \infty} x_n = a$ , if  $f$  has a right limit,  $\lim_{n \rightarrow \infty} f(x_n) = \alpha$ . There is a similar definition for left limits.*

A function  $f : [a, b] \rightarrow \mathbb{R}$  is regulated if:

- $f$  has a left limit on all values in  $(a, b]$
- $f$  has a right limit on all values in  $[a, b)$ .

*All continuous functions are regulated. All increasing and decreasing functions are regulated.*

Let  $R([a, b])$  be the set of functions regulated over  $[a, b]$ . We have that  $R([a, b])$  is closed under:

- Scalar multiplication (over  $\mathbb{R}$ )
- Addition
- Multiplication
- Division (if the divisor is greater than zero over  $[a, b]$ )
- Composition
- The modulus.

*Uniform convergence preserves regulation. Also, all step functions are regulated.*

For  $f$  a regulated function over  $[a, b]$ , we have that:

$$\forall \epsilon > 0, \exists \psi \in S([a, b]) \text{ such that } \|\psi - f\| < \epsilon.$$

*Basically, for any regulated function we can always choose an arbitrarily accurate approximation that is a step function.*

### 3.4 Integration of Regulated Functions

For a function  $f \in R([a, b])$ , say we have two sequences of step functions,  $(\psi_n)_{n \in \mathbb{N}}$  and  $(\phi_n)_{n \in \mathbb{N}}$ :

- $(\psi_n)_{n \in \mathbb{N}}$  is uniformly convergent to  $f \Rightarrow (\int_a^b \psi_n(x) dx)_{n \in \mathbb{N}}$  is convergent
- $(\psi_n)_{n \in \mathbb{N}}$  and  $(\phi_n)_{n \in \mathbb{N}}$  are uniformly convergent to  $f \Rightarrow \lim_{n \rightarrow \infty} (\int_a^b \psi_n(x) dx) = \lim_{n \rightarrow \infty} (\int_a^b \phi_n(x) dx)$ .

*Basically, we have that no matter what step function we choose to approximate our function, the value of the integral will tend to the same value.*

We define the integral of a regulated function  $f \in R([a, b])$  by choosing a sequence of step functions  $(\psi_n)_{n \in \mathbb{N}}$  such that they converge uniformly to  $f$ :

$$\int_a^b f(x) dx := \lim_{n \rightarrow \infty} \int_a^b \psi_n(x) dx.$$

The **linearity**, **continuity**, and **additivity** properties hold similarly to the properties of step functions. The **monotonicity** property holds also but the stated definition varies slightly:

- **Monotonicity:** For  $f \in R([a, b])$  with  $f(x) \geq 0$  for  $x \in [a, b]$ , we have that  $\int_a^b f(x) dx \geq 0$ .

Some small notes on regulated functions, let  $f \in R([a, b])$ :

- $|\int_a^b f(x) dx| \leq \int_a^b |f(x)| dx$
- For  $(f_n)_{n \in \mathbb{N}}$  uniformly convergent to  $f$ ,  $\lim_{n \rightarrow \infty} \int_a^b f_n(x) dx = \int_a^b f(x) dx$ .

*The first point is similar to the triangle inequality applied to summations. The second was covered similarly but strictly for step functions, not all regulated functions.*

### 3.5 The Mean-Value Theorem of Integration

For  $f \in C([a, b])$ , let  $g \in R([a, b])$  and satisfy the following:

- $g(x) \geq 0$  for  $x \in [a, b]$
- $\int_a^b g(x) dx > 0$

With these assumptions, we have that  $\exists x \in (a, b)$  with:

$$f(x) \int_a^b g(t) dt = \int_a^b f(t)g(t) dt$$

*Note that the function  $f$  is continuous. This is a stronger statement than just saying it's regulated. Also, consider  $g = 1$ :*

$$\begin{aligned} f(x) \int_a^b g(t) dt &= f(x) \int_a^b 1 dt \\ &= f(x)(b - a) \end{aligned} \tag{1}$$

$$\int_a^b f(t)g(t) dt = \int_a^b f(t) dt \tag{2}$$

(1) and (2)  $\Rightarrow$

$$\int_a^b f(t) dt = f(x)(b - a)$$