Analysis 1 (TB2) Notes

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*An important note, these notes are absolutely **NOT** guaranteed to be correct, representative of the course, or rigorous. Any result of this is not the author's fault.*

1 Continuity

1.1 Continuous Functions

From Analysis 1A, we have that a function $f : A \to \mathbb{R}$ is continuous on A if:

$$\forall x \in A, \forall \epsilon > 0, \exists\, \delta > 0 \text{ such that } \forall y \in A, (|y - x| < \delta) \Rightarrow (|f(y) - f(x)| < \epsilon).$$

It's important to note that x is chosen given before we choose a δ. Thus, our choice for δ can depend on x as well as ϵ.

Uniform continuity requires that δ is independent of x.

A note, a function being continuous at a value (or set of values for that matter), it equivalent to saying that there exists a limit for the function at that value and that limit is the value of the function applied to that value.

1.2 Uniformly Continuous Functions

Uniform continuity is similar to continuity as we knew it in Analysis 1A. For a function $f : A \to \mathbb{R}$, f is uniformly continuous on A if:

$$\forall \epsilon > 0, \exists\, \delta > 0 \text{ such that } \forall x, y \in A, (|y - x| < \delta) \Rightarrow (|f(y) - f(x)| < \epsilon).$$

We can see that uniform continuity **implies** continuity but **not** vice versa.

A note, for uniform continuity, we are saying that given a value ϵ, we can always pick a distance (δ) such that if two values are within that distance of each other, the distance between the values after the function is applied to them will be less than ϵ. This is essentially testing for divergence to infinity at a value ($\frac{1}{x}$ is continuous but not uniformly continuous on $\mathbb{R}_{>0}$).

2 Convergence

We have the notion of convergence for sequences of real numbers from Analysis 1A, convergence in this section is similar but specifically for functions.

2.1 Pointwise Convergence

A sequence of functions $(f_n)_{n\in\mathbb{N}}$ from $A \to \mathbb{R}$ converges **pointwise** to the function f on A if:

$$\lim_{n\to\infty}(f_n(x)) = f(x). \qquad (\forall x \in A)$$

f is called the **pointwise limit** of $(f_n)_{n\in\mathbb{N}}$.

A note, for $f_n : [0,1] \to [0,1]; x \to x^n$, $f : [0,1] \to [0,1]; x \to \delta_1(x)$, f_n converges pointwise to f.

2.2 Uniform Convergence

A sequence of functions $(f_n)_{n\in\mathbb{N}}$ from $A \to \mathbb{R}$ converges **uniformly** to the function f on A if:

$$\forall \epsilon > 0, \exists N \in \mathbb{N} \text{ such that } \forall x \in A, \forall n \in \mathbb{N}, (n \geq N) \Rightarrow (|f(x) - f_n(x)| < \epsilon).$$

For the same functions outlined in the note under pointwise convergence, we have that f_n does not converge uniformly to f. Let $\epsilon \in (0,1)$, $x \in [0,1)$ and suppose f_n is uniformly convergent to f,

$$\begin{aligned}
&|f_n(x) - f(x)| = |x^n| < \epsilon \\
&\Rightarrow 0 \leq x^n < \epsilon < 1 \\
&\Rightarrow 0 \leq x < \epsilon^{\frac{1}{n}} < 1 \\
&\Rightarrow \epsilon = 1 \text{ as } x \in [0,1).
\end{aligned}$$

This is a contradiction by the definition of ϵ. Thus, we have the result.

2.3 Weierstrass' Theorem

For $a, b \in \mathbb{R}$ with $a < b$, if a sequence of continuous functions on $[a,b]$, $(f_n)_{n\in\mathbb{N}}$ converges uniformly to f on $[a,b]$, f is continuous on $[a,b]$.

Basically, uniform convergence preserves continuity (it also preserves regulation).

2.4 Supremum Norm

2.4.1 Definition of the Supremum Norm

For $a, b \in \mathbb{R}$ with $a < b$, let $f : [a, b] \to \mathbb{R}$ be a bounded function. The supremum norm of f on $[a, b]$ is denoted by $\|f\|_{[a,b]}$ and is defined by:

$$\|f\|_{[a,b]} := \sup \{\|f(x)| : x \in [a, b]\}.$$

The supremum norm is simply just the furthest distance from zero reached by a function over a closed interval. By definition, it is a real number and $\exists\, x \in [a, b]$ *such that* $f(x)$ *is the supremum norm.*

2.4.2 Properties of the Supremum Norm

There are a few key properties of the supremum norm, let a and b be as above and let $\lambda \in \mathbb{R}$, $f, g : [a, b] \to \mathbb{R}$ be bounded functions:

- $\|f\|_{[a,b]} > 0$
- $\|f\|_{[a,b]} = 0 \Leftrightarrow f = 0$ on $[a, b]$
- $\|\lambda f\|_{[a,b]} = |\lambda| \|f\|_{[a,b]}$
- $\|f + g\|_{[a,b]} = \|f\|_{[a,b]} + \|g\|_{[a,b]}$.

2.5 Cauchy Sequences of Functions

For $a, b \in \mathbb{R}$ with $a < b$, denote the set of continuous functions on $[a, b]$ by $C([a, b])$. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of functions in $C([a, b])$. We say $(f_n)_{n\in\mathbb{N}}$ is Cauchy if:

$$\forall \epsilon > 0, \exists\, N \in \mathbb{N} \text{ such that } \forall m, n \in \mathbb{N}, (m, n \geq N) \Rightarrow (\|f_n - f_m\|_{[a,b]} < \epsilon).$$

This obviously bears an extreme resemblance to the Cauchy sequences of Analysis 1A. Just replacing the sequences of reals with sequences of functions and the modulus with the supremum norm.

For each continuous function, there exists a Cauchy sequence such that the sequence converges uniformly to said function.

3 Integration

3.1 Step Functions

For $a, b \in \mathbb{R}$ with $a < b$, a partition of the interval $[a, b]$ is a set P of the form:

$$P = \{x_0, x_1, \ldots, x_n\} \text{ (for some } n \in \mathbb{N})$$
$$\text{where } a = x_0 < x_1 < \ldots < x_n = b.$$

We say a function $\psi : [a, b] \to \mathbb{R}$ is a step function if there exists a partition $P = \{x_0, \ldots, x_n\}$ and a set of constants in $\mathbb{R}$ ($\{c_0, c_1, \ldots, c_n\}$) such that:

$$\psi(x) = c_i \, (\forall x \in (x_{i-1}, x_i)).$$

In this case, P and ψ are adapted to each other.

$S[a, b]$ is the set of step functions over $[a, b]$.

3.2 Integration of Step Functions

3.2.1 Definition of integration on step functions

The integral of the step function is simple:

$$\int_a^b \psi(x)\, dx := \sum_{i=1}^{n} c_i(x_i - x_{i-1}).$$

As long as the partition is adapted to ψ, the integral doesn't change.

3.2.2 Properties of integration on step functions

Here are some properties of the integration of step functions, let ϕ, ψ be step functions over $[a, b]$, $y \in \mathbb{R}$ with $a < y < b$, $\alpha, \beta \in \mathbb{R}$:

- **Linearity**: $\int_a^b \alpha\psi(x) + \beta\phi(x)\, dx = \alpha \int_a^b \psi(x)\, dx + \beta \int_a^b \phi(x)\, dx$
- **Monotonicity**: $(\psi(x) \leq \phi(x)(\forall x \in [a, b])) \Rightarrow (\int_a^b \psi(x)\, dx \leq \int_a^b \phi(x)\, dx)$
- **Continuity**: $|\int_a^b \psi(x)\, dx| \leq (b - a)\|\psi(x)\|_{[a,b]}$
- **Additivity**: $\int_a^b \psi(x)\, dx = \int_a^y \psi(x)\, dx + \int_y^b \psi(x)\, dx$

3.3 Regulated Functions

3.3.1 Definition of left and right limits

Let $A \subseteq \mathbb{R}$ and $f : A \to \mathbb{R}$. For some $\epsilon > 0$, $a \in A$, and $\alpha \in \mathbb{R}$:

1. f has a **right limit** of α at a if:
   $\exists\, \delta > 0$ such that $(0 < x - a < \delta) \Rightarrow (|f(x) - \alpha| < \epsilon)$

2. f has a **left limit** of α at a if:
   $\exists\, \delta > 0$ such that $(0 < a - x < \delta) \Rightarrow (|f(x) - \alpha| < \epsilon)$.

We can denote right limits by: $\lim_{x \downarrow a} f(x) = \alpha$. Similarly for left limits: $\lim_{x \uparrow a} f(x) = \alpha$.

There is a sequential definition too, for any sequence $(x_n)_{n\in\mathbb{N}}$ that satisfies $x_n > a$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} x_n = a$, if f has a right limit, $\lim_{n\to\infty} f(x_n) = \alpha$. There is a similar definition for left limits.

3.3.2 Definition of a regulated function

A function $f : [a, b] \to \mathbb{R}$ is regulated if:

- f has a left limit on all values in $(a, b]$
- f has a right limit on all values in $[a, b)$.

All continuous functions are regulated. All increasing and decreasing functions are regulated.

3.3.3 Properties of regulated functions

Let $R([a, b])$ be the set of functions regulated over $[a, b]$. We have that $R([a, b])$ is closed under:

- Scalar multiplication (over $\mathbb{R}$)
- Addition
- Multiplication
- Division (if the divisor is greater than zero over $[a, b]$)
- Composition
- The modulus.

Uniform convergence preserves regulation. Also, all step functions are regulated.

For f a regulated function over $[a, b]$, we have that:

$$\forall \epsilon > 0, \exists\, \psi \in S([a, b]) \text{ such that } \|\psi - f\| < \epsilon.$$

Basically, for any regulated function we can always choose an arbitrarily accurate approximation that is a step function.

3.4 Integeration of Regulated Functions

3.4.1 Definition of integration on regulated functions

For a function $f \in R([a, b])$, say we have two sequences of step functions, $(\psi_n)_{n\in\mathbb{N}}$ and $(\phi_n)_{n\in\mathbb{N}}$:

- $(\psi_n)_{n\in\mathbb{N}}$ is uniformly convergent to $f \Rightarrow (\int_a^b \psi_n(x)\, dx)_{n\in\mathbb{N}}$ is convergent
- $(\psi_n)_{n\in\mathbb{N}}$ and $(\phi_n)_{n\in\mathbb{N}}$ are uniformly convergent to $f \Rightarrow$ $\lim_{n\to\infty}(\int_a^b \psi_n(x)\, dx) = \lim_{n\to\infty}(\int_a^b \phi_n(x)\, dx)$.

Basically, we have that no matter what step function we choose to approximate our function, the value of the integral will tend to the same value.

We define the integral of a regulated function $f \in R([a, b])$ by choosing a sequence of step functions $(\psi_n)_{n\in\mathbb{N}}$ such that they converge uniformly to f:

$$\int_a^b f(x)\, dx := \lim_{n\to\infty} \int_a^b \psi_n(x)\, dx.$$

3.4.2 Properties of integration on regulated functions

The **linearity**, **continuity**, and **additivity** properties hold similarly to the properties of step functions. The **monotonicity** property holds also but the stated definition varies slightly:

- **Monotonicity**: For $f \in R([a, b])$ with $f(x) \geq 0$ for $x \in [a, b]$, we have that $\int_a^b f(x)\, dx \geq 0$.

Some small notes on regulated functions, let $f \in R([a, b])$:

- $|\int_a^b f(x)\, dx| \leq \int_a^b |f(x)|\, dx$
- For $(f_n)_{n\in\mathbb{N}}$ uniformly convergent to f, $\lim_{n\to\infty} \int_a^b f_n(x)\, dx = \int_a^b f(x)\, dx$.

The first point is similar to the triangle inequality applied to summations. The second was covered similarly but strictly for step functions, not all regulated functions.

3.5 The Mean-Value Theorem of Integeration

For $f \in C([a, b])$, let $g \in R([a, b])$ and satisfy the following:

- $g(x) \geq 0$ for $x \in [a, b]$
- $\int_a^b g(x)\, dx > 0$

With these assumptions, we have that $\exists\, x \in (a, b)$ with:

$$f(x) \int_a^b g(t)\, dt = \int_a^b f(t)g(t)\, dt$$

Note that the function f is continuous. This is a stronger statement than just saying it's regulated. Also, consider $g = 1$:

$$\begin{aligned} f(x) \int_a^b g(t)\, dt &= f(x) \int_a^b 1\, dt \\ &= f(x)(b - a) \end{aligned} \tag{1}$$

$$\int_a^b f(t)g(t)\, dt = \int_a^b f(t)\, dt \tag{2}$$

$$\begin{aligned} &(1) \text{ and } (2) \;\Rightarrow \\ &\int_a^b f(t)\, dt = f(x)(b - a) \end{aligned}$$

4 Differentiation

4.1 Definition of Differentiation

For a function f defined on a δ-neighbourhood of some $a \in \mathbb{R}$, we have that f is differentiable at a if the following exists in $\mathbb{R}$:

$$\lim_{h \to 0} \left(\frac{f(a + h) - f(a)}{h} \right).$$

Differentiability at a value a implies continuity at a.

4.2 Properties of Differentiation

4.2.1 Closure of the set of differentiable functions

Let f, g be differentiable functions. The set of differentiable functions is closed under:

- **Addition**: $(f+g)' = f' + g'$
- **Multiplication**: $(fg)' = f'g + fg'$
- **Division**: $\frac{f}{g} = \frac{f'g - fg'}{g^2}$ (for g non-zero)
- **Composition**: $(f \circ g)' = g'(f' \circ g)$.

4.2.2 The implications of zero derivatives

For a differentiable function f:

- $f(x_0)$ is a maximum or minimum $\Rightarrow f'(x_0) = 0$
- $f'(x_0) = 0,\ f''(x_0) > 0 \Rightarrow f(x_0)$ is a minimum
- $f'(x_0) = 0,\ f''(x_0) < 0 \Rightarrow f(x_0)$ is a maximum
- $f'(x) = 0\,(\forall x \in [a,b]) \Rightarrow f$ is constant on $[a,b]$.

4.2.3 Rolle's Theorem and the Mean Value Theorem

For f continuous on $[a,b]$ and differentiable on (a,b),

$$(f(a) = f(b)) \Rightarrow (\exists\, x_0 \in (a,b) \text{ such that } f'(x_0) = 0) \qquad \text{(Rolle's Theorem)}$$

$$\exists\, x_0 \in (a,b) \text{ such that } f'(x_0) = \frac{f(b) - f(a)}{b - a}. \qquad \text{(Mean Value Theorem)}$$

Rolle's Theorem is a special case of the Mean Value Theorem. The Mean Value Theorem says that over an interval, the derivative is equal to the average derivative across the interval at some value.

4.2.4 Cauchy's Mean Value Theorem

For f, g continuous on $[a,b]$ and differentiable on (a,b),

$$\exists\, x_0 \in (a,b) \text{ such that } \left[(f(b) - f(a))g'(x_0) = (g(b) - g(a))f'(x_0)\right].$$

4.2.5 Other properties of the derivative

For f, g differentiable functions,

- $(f'(x) = g'(x)\,(\forall x \in [a,b])) \Rightarrow (f(x) = g(x) + c\,(c \in \mathbb{R}))$
- $f'(x) > 0\,(\forall x \in [a,b]) \Rightarrow f$ is strictly increasing (similarly for strictly decreasing).

4.3 L'Hôpital's Rule

For f, g differentiable functions defined on a δ-neighbourhood of $a \in \mathbb{R}$, if:

- $g'(x) \neq 0\,(\forall x \in (a-\delta, a+\delta)\backslash\{a\})$
- $f(a) = g(a) = 0$
- $\lim_{x\to a} \frac{f'(x)}{g'(x)}$ exists in $\mathbb{R}$.

Then,

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}.$$

So, if we have two functions that equal zero at a value, this rule helps us find the derivative of the their quotient as long as the denominator isn't zero nearby.

5 Calculus

5.1 Differentiability on a Closed Interval

We say that a function $f : [a,b] \to \mathbb{R}$ is differentiable on $[a,b]$ if it's differentiable on (a,b) and the left and right derivatives at a and b exist in $\mathbb{R}$.

5.2 The Fundamental Theorem of Calculus

Let $f : [a,b] \to \mathbb{R}$ be a continuous function, we have the following is differentiable on $[a,b]$ and $F' = f$ on $[a,b]$:

$$F(x) := \int_a^x f(t)\,dt.$$

A result is that, for G differentiable on $[a,b]$ with $G' = f$:

$$\int_a^b f(x)\,dx = G(b) - G(a).$$

5.3 Integration by Parts

For $f, g : [a, b] \to \mathbb{R}$ continuously differentiable (differentiable with their derivatives being continuous):

$$\int_a^b f'(x)g(x)\,dx = [f(x)g(x)]_a^b - \int_a^b f(x)g'(x)\,dx.$$

5.4 Integration by Substitution

Let $f : [a, b] \to \mathbb{R}$ be a continuous function, suppose $\phi : [c, d] \to [a, b]$ is continuously differentiable (differentiable with their derivatives being continuous) and bijective. Then we have:

$$\int_a^b f(x)\,dx = \int_c^d f(\phi(t))\phi'(t)\,dt$$

5.5 Taylor's Theorem

5.5.1 Polynomial coefficients

If for $p : \mathbb{R} \to \mathbb{R}$ a polynomial function with degree n and coefficients $a_0, a_1, \ldots, a_n$ we have:

$$p(x) = \sum_{i=0}^{n} a_i x^i,$$

and we have that:

$$a_k = \frac{1}{k!} f^{(k)}(0). \qquad (k \in \{0, 1, \ldots, n\})$$

5.5.2 Taylor Polynomials

For $a \in \mathbb{R}$, suppose the function f is n times differentiable on a δ-neighbourhood of a, the Taylor polynomial for f degree n around a is defined by:

$$\begin{aligned} T_n(a, x) &:= f(a) + f'(a)(x - a) + \cdots + \frac{f^{(n)}(a)}{n!}(x - a)^n \qquad (x \in \mathbb{R}) \\ &:= \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x - a)^i. \end{aligned}$$

5.5.3 Taylor's Theorem

For $a \in \mathbb{R}$, suppose the function f is n times differentiable on a δ-neighbourhood of a. For some c between a and x:

$$f(x) = f(a) + f'(a)(x-a) + \cdots + \frac{f^{(n-1)}(a)}{(n-1)!}(x-a)^{n-1} + R_n$$

$$R_n = \frac{f^{(n)}(c)}{n!}(x-a)^n. \qquad \text{(Lagrange Form)}$$

6 Series

6.1 The Limit Superior and Limit Inferior

6.1.1 Definition of the limit superior and limit inferior

The limit inferior and superior gives bounds on the limit of a subsequence. This can be used in conjunction with the Bolzano-Weierstrass Theorem (bounded sequences have convergent subsequences). For a sequence $(a_n)_{n\in\mathbb{N}}$, we have the following definitions:

- **Limit superior**: $\limsup_{n\to\infty}(a_n) := \lim_{n\to\infty}(\sup\{a_k : k \geq n\})$
- **Limit inferior**: $\liminf_{n\to\infty}(a_n) := \lim_{n\to\infty}(\inf\{a_k : k \geq n\})$.

6.1.2 Properties of the $\limsup$ and $\liminf$

For some sequence $(a_n)_{n\in\mathbb{N}}$, some direct consequences of the definition are:

- a_n bounded above $\Rightarrow \limsup_{n\to\infty}(a_n) \in \mathbb{R}$
- a_n bounded below $\Rightarrow \liminf_{n\to\infty}(a_n) \in \mathbb{R}$
- a_n not bounded above $\Rightarrow \limsup_{n\to\infty}(a_n) = \infty$
- a_n not bounded below $\Rightarrow \liminf_{n\to\infty}(a_n) = -\infty$
- If the $\limsup$ or $\liminf$ exists in $\mathbb{R}$, then there exists a subsequence such that the limit of the sequence is the $\limsup$ or $\liminf$ respectively
- A sequence is convergent if and only if it's $\limsup$ and $\liminf$ exist in $\mathbb{R}$ and are equal.

6.1.3 Alternate definition of the $\limsup$ and $\liminf$

We have that for some sequence $(a_n)_{n\in\mathbb{N}}$, let $a \in \mathbb{R}$, $\epsilon > 0$:

- $\limsup_{n\to\infty}(a_n) = a \Leftrightarrow$
  - $\exists\, N \in \mathbb{N}$ such that $a_n < a + \epsilon$ for $n \geq N$
  - $a_m > a - \epsilon$ for infinitely many $m \in \mathbb{N}$
- $\liminf_{n\to\infty}(a_n) = a \Leftrightarrow$
  - $\exists\, N \in \mathbb{N}$ such that $a_n > a - \epsilon$ for $n \geq N$
  - $a_m < a + \epsilon$ for infinitely many $m \in \mathbb{N}$.

6.2 Subsequential Limits

A subsequential limit of a sequence $(a_n)_{n\in\mathbb{N}}$ is a value $a \in \mathbb{R}$ such that there exists a subsequence $(a_{n_k})_{k\in\mathbb{N}}$ where:

$$\lim_{k\to\infty}(a_{n_k}) = a.$$

If we consider the set of all subsequential limits S, if the $\limsup$ or $\liminf$ of a_n exist in $\mathbb{R}$ we have:

- $\limsup_{n\to\infty}(a_n) = \max(S)$
- $\liminf_{n\to\infty}(a_n) = \min(S)$,

respectively.

6.3 Types of Convergence

6.3.1 Absolute Convergence

For a series $\sum_{n=1}^{\infty}(a_n)$, we have that it is absolutely convergent if $\sum_{n=1}^{\infty}(|a_n|)$ is convergent.

We have that all absolutely convergent series are convergent.

6.3.2 Conditional Convergence

For a series $\sum_{n=1}^{\infty}(a_n)$, we have that it is conditionally convergent if it's convergent but not absolutely convergent.

6.4 Limit Theorems for Sequences of Functions

6.4.1 Uniform convergence under integration

For $(f_n)_{n\in\mathbb{N}}$ a sequence of regulated functions on $[a, b]$, assume f_n converges uniformly on $[a, b]$ to some function f. Let:

$$\begin{aligned} F_n(x) &:= \int_a^x f_n(t)\, dt \\ F(x) &:= \int_a^x f(t)\, dt. \end{aligned}$$

We have that $(F_n)_{n\in\mathbb{N}}$ converges uniformly to F on $[a, b]$.

So, we have that uniform convergence is preserved under integration.

6.4.2 Uniform convergence under differentiation

Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of continuously differentiable (differentiable with their derivatives being continuous) functions on $[a, b]$. Assume:

- $(f_n')_{n\in\mathbb{N}}$ converges uniformly to a function g
- $(f_n(a))_{n\in\mathbb{N}}$ converges in $\mathbb{R}$.

Then we have that $(f_n)_{n\in\mathbb{N}}$ converges uniformly to a function f such that $f' = g$ on $[a, b]$.

6.4.3 Uniform convergence of series

For $(g_n)_{n\in\ltimes}$ a sequence of real valued functions defined on $[a, b]$, define:

$$f := \sum_{k=1}^{\infty} g_k \tag{1}$$

$$f_n := \sum_{k=1}^{n} g_k. \tag{2}$$

We say f converges uniformly if f_n converges pointwise and uniformly to f on $[a, b]$.

6.5 Tests for Series Convergence

6.5.1 Root Test

For a series $\sum_{n=1}^{\infty}(a_n)$, where each a_k $(k \in \mathbb{N})$ is non-negative, set:

$$\lambda := \limsup_{n\to\infty}(a_n^{\frac{1}{n}}).$$

We have that:

- $\lambda < 1 \Rightarrow$ convergence
- $\lambda > 1 \Rightarrow$ divergence
- $\lambda = 1 \Rightarrow$ may be convergent or divergent

6.5.2 Alternating Series Test

For a series $\sum_{n=1}^{\infty}(a_n)$, where each a_k $(k \in \mathbb{N})$ is positive and a_n is decreasing with limit 0, we have that the following is convergent:

$$\sum_{n=1}^{\infty}(-1)^n a_n.$$

6.5.3 Weierstrass M-Test

For $(g_n)_{n\in\mathbb{N}}$ a sequence of continuous functions on $[a, b]$, assume that for each $k \in \mathbb{N}$ there exists M_k such that $\|g_k\| \leq M_k$ and $\sum_{k=1}^{\infty} M_k < \infty$. Then the following converges uniformly on $[a, b]$:

$$\sum_{k=1}^{\infty} g_k.$$

This is saying if we can find a sequence that is convergent as a series and bounds our sequence of functions then the series of our functions is uniformly convergent. Note that M_k can never be negative as the supremum norm of any function is non-negative.

6.5.4 Simpler Forms of the M-Test

For $(g_n)_{n\in\mathbb{N}}$ a sequence of continuously differentiable (differentiable with their derivatives being continuous) functions on $[a, b]$. For:

$$f := \sum_{n=1}^{\infty} g_n \Rightarrow f' := \sum_{n=1}^{\infty} g',$$

we have that the sequence of partial sums converges uniformly on $[a, b]$ to f and f is continuously differentiable on $[a, b]$ if:

$$\sum_{n=1}^{\infty} [\|g_n\| + \|g_n'\|] < \infty.$$

Additionally, for $(g_n)_{n\in\mathbb{N}}$ a sequence of regulated functions, we define:

$$f := \sum_{n=1}^{\infty} g_n.$$

So, if:

$$\sum_{n=1}^{\infty} \|g_n\| < \infty,$$

then f converges uniformly on $[a, b]$ and is regulated.

7 Power Series

7.1 Definition of Radius of Convergence

The radius of convergence R for a power series:

$$\mathcal{P} = \sum_{n=0}^{\infty} a_n x^n,$$

is defined as:

$$R := \begin{cases} \infty & \text{if } \mathcal{P} \text{ converges for all } x \in \mathbb{R} \\ r & \text{if } \mathcal{P} \text{ converges if and only if } |x| < r \in \mathbb{R} \\ 0 & \text{if } \mathcal{P} \text{ diverges for all } x \in \mathbb{R}\backslash\{0\}. \end{cases}$$

7.2 Tests for the Radius of Convergence

7.2.1 Cauchy-Hadamard Theorem

Let $(a_n)_{n\in\mathbb{N}\cup\{0\}}$ be a sequence of real numbers and define:

$$\alpha := \limsup_{n\to\infty}(a_n^{\frac{1}{n}})$$
$$R := \begin{cases} \infty & \text{for } \alpha = 0 \\ 0 & \text{for } \alpha = \infty \\ \frac{1}{\alpha} & \text{otherwise.} \end{cases}$$

We have that the series $\sum_{n=0}^{\infty} a_n x^n$ has radius of convergence R.

7.2.2 Ratio Test

Let $(a_n)_{n\in\mathbb{N}\cup\{0\}}$ be a sequence of real numbers such that for:

$$\beta := \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|,$$

if we have that β exists in $\mathbb{R}$, $\sum_{n=0}^{\infty} a_n x^n$ has radius of convergence R defined by:

$$R := \begin{cases} \infty & \text{for } \beta = 0 \\ 0 & \text{for } \beta = \infty \\ \frac{1}{\beta} & \text{otherwise.} \end{cases}$$

7.3 Consequences of the Radius of Convergence

7.3.1 Preservation of the Radius of Convergence

For a power series $\sum_{n=0}^{\infty} a_n x^n$ with radius of convergence R, the following have the same radius of convergence:

$$\sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$\sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}.$$

The radius of convergence is preserved by integration and differentiation.

Furthermore, if we have the power series $\sum_{n=0}^{\infty} a_n x^n$ with radius of convergence R, then we can define:

$$f : (-R, R) \to \mathbb{R}; f(x) = \sum_{n=0}^{\infty} a_n x^n.$$

We have that f is continuously differentiable on $(-R, R)$ with:

$$f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$\int_0^x f(t)\, dt = \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1},$$

defined on $(-R, R)$.

8 Elementary Functions

8.1 The Exponential and Logarithm

8.1.1 Definition of the Exponential

We define the exponential function $\exp : \mathbb{R} \to \mathbb{R}$ by:

$$\exp(x) := \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

8.1.2 Properties of the exponential

Let f be the exponential function:

- f is differentiable
- $f = f'$
- $f(0) = 1$
- $f(x)f(y) = f(x+y)$
- $f > 0$
- f is strictly increasing
- $f \to \infty$ as $x \to \infty$
- $f \to 0$ as $x \to -\infty$
- The range of f is $(0, \infty)$
- $\frac{f(x)}{x^k} \to \infty$ as $x \to \infty$ $\forall k \in \mathbb{N}$

8.1.3 Euler's number

We call Euler's number $e := \exp(1)$. We have that for $x \in \mathbb{R}$:

$$\exp(x) = e^x.$$

8.1.4 Definition of the natural logarithm

We define the natural logarithm as the inverse of the exponential.

8.1.5 Properties of the natural logarithm

Let f be the natural logarithm:

- f is differentiable
- f is increasing
- $f(xy) = f(x) + f(y)$
- $f(x/y) = f(x) - f(y)$
- $f(1/x) = -f(x)$
- $f(1) = 0$
- $f'(x) = 1/x$
- $f \to \infty$ as $x \to \infty$
- $f \to -\infty$ as $x \downarrow 0$
- $\frac{f(x)}{x^k} \to 0$ as $x \to \infty$ $\forall k \in \mathbb{N}$.

8.1.6 The natural logarithm as a power series

For $x \in (-1, 1)$:

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}.$$

8.1.7 Exponentials

For $a > 0$, $x \in \mathbb{R}$ we can write:

$$a^x := e^{xln(a)}.$$

8.2 Trigonometric Functions

8.2.1 Definition of sin amd cos

We can define the sin and cos functions as follows:

$$\sin(x) := x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$$
$$\cos(x) := 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \dots.$$

8.2.2 Differentiability of $\sin$ amd $\cos$

These are both differentiable with:

$$(\sin)' = \cos$$
$$(\cos)' = -\sin .$$

8.2.3 The Pythagorean Identity

We have that:

- $\sin^2(x) + \cos^2(x) = 1.$