Using Graphing Software in Education

Research Task

Thank you again for participating in the study, here's a couple of important notes about the provided software (Helix):

- click and drag to pan the plot and scroll to zoom,
- the button in the bottom-left of Helix creates a new input,
- you have to be clear with multiplication, `6x` is not accepted, `6 * x` is.

Also, in this document the coloured boxes correspond to tasks in Helix:

Info: This is an information box, telling you about what Helix can do.

Question: This is a question box, asking you to use Helix to solve a question.

Answer: This is an answer box, which will follow question boxes!

Most PDF readers should have the ability to rotate the document so you don't have to strain your neck!

Lines

For straight lines, the gradient is a fairly simple concept. Considering the function $f(x) = 6x + 2$, we can calculate the gradient like so:

$$\frac{f(x+1) - f(x)}{1} = (6x + 6 + 2) - (6x + 2) = 6,$$

or just by reading the coeffecient of x. We can even consider lines in 3D, like with the parametric equation $p(t) = (3t + 2, t - 3, 2t)$, we can see with a similar method that the gradient of $p(t)$ is represented by the vector $(3, 1, 2)$.

Info: We can do this in Helix by first defining the function in a new input: `f(x) = 6 * x + 2`, and in another input typing: `f(x + 1) - f(x)`. Hovering over the blue ‘`i`’ icon will show the result (note that this will also be plotted as `y = 6`, you can use the ‘eye’ button to hide this).

Question: We can also draw parametrics, like `(2 * t + 2, t - 3)`, in the software. What would the parametric plot corresponding to the function $p(t) = (3t + 2, t - 3, 2t)$?

Answer: We can simply write it like so: `(3 * t + 2, t - 3, 2 * t)`, being careful to add multiplication where necessary.

We can also consider the area captured between the line, the x-axis, and two values of x, by using the formula for the area of a trapezium:

$$\int_3^5 f(x)dx = \frac{(5-3) \cdot (f(3) + f(5))}{2} = 20 + 32 = 52.$$

Info: We can use our custom functions in functions we write, so we can define a trapezium-based area function like so:

```
tr(a, b) = (f(a) + f(b)) * ((b - a) / 2)
```

and use `tr(3, 5)` to get our answer, 52, or even use `tr(0, x)` to get the indefinite integral (at zero)!

Question: Can we write a function like `tr` that instead finds the derivative of a linear function? Think about what we did in the first info box.

Answer: Yes! We can use `gr() = f(x + 1) - f(x)`.

Curves

Things become more difficult when we introduce curves since their gradient is not constant. So, if we consider the function $g(x) = x^3 + 2x^2 + 2$, we can use:

$$\lim_{h \to 0} \left(\frac{g(x+h) - g(x)}{h} \right),$$

or just take h to be suitably small if we just want an estimate.

Info: We can find the derivative of $g(x)$ by using the in-built `der` function like so: `der(g(x))`. Also, we can expand the gradient function from the previous question to take some values for x and h:

```
gr(x, h) = (g(x + h) - g(x)) / h.
```

Then, we can try `gr(x, 0.01)`, which you should be able to see is very close to `der(g(x))`.

We may be interested in points on the curve where the derivative is zero, and the nature of these points. We can find where the derivative is equal to zero, and then evaluate the second derivative at those points. Looking at $g(x)$:

$$\frac{\mathrm{d}}{\mathrm{d}x} g(x) = 3x^2 + 4x = 0 \text{ when } x = 0 \text{ or } \frac{-4}{3},$$
$$\frac{\mathrm{d}^2}{\mathrm{d}x^2} g(x) = 6x + 4.$$

The derivative is the rate of change of the function, the second derivative is the rate of change of the gradient. Since the second derivative at $\frac{-4}{3}$ and 0 is -4 and 4 (respectively) we know that these points are a local maximum and minimum (respectively) since:

- at $x = \frac{-4}{3}$ the gradient is zero and decreasing,
- at $x = 0$ the gradient is zero and increasing.

Info: We can simply set `der(g(x)) = 0` and read the values by hovering over the blue '`i`' icon. Then, we can use `der(g(x), x, x)` to take the derivative with respect to x twice, and replace `x` with $\frac{-4}{3}$ and 0 (or write a function for the second derivative and use that instead).

Surfaces

We would like consider functions in two parameters, which can be interpreted as 3D surfaces. Taking $h(x, y) = x^2 + xy$, we have a choice for what we take the derivative with respect to, x and y:

$$\frac{\partial}{\partial x} h(x, y) = 2x + y,$$
$$\frac{\partial}{\partial y} h(x, y) = x,$$

where we assume the variable we aren't taking our derivative with respect to is constant (called a partial derivative). The value of partial derivative with respect to x, describes how the gradient changes for a given value of y in the direction of the vector $\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$ (and similarly for the partial derivative with respect to y).

Info: For the following tasks, you may want to:

- adjust limits on the plot using the 'Set Limits' button, a good range for z is -50 to 200, you can also use the home button to reset the view,
- adjust the plot quality using the input box at the bottom left of the 3D plotting window (higher is better but slower) - a good value is 1.

Info: Partial derivatives are handled the same as regular derivatives in Helix, for: `h(x, y) = x ^ 2 + x * y`, we can do: `der(h(x, y), x)` or similarly `der(h(x, y), y)`.

Using this, we can find the gradient at any point on the surface in any direction. If we consider the direction $\left(\begin{smallmatrix}2\\-2\end{smallmatrix}\right)$, we can calculate:

$$2\left(\frac{\partial}{\partial x} h(x, y)\right) - 2\left(\frac{\partial}{\partial y} h(x, y)\right) = 4x + 2y - 2x = 2(x + y).$$

So, the gradient in this direction at $\left(\begin{smallmatrix}2\\0\end{smallmatrix}\right)$ is 4.

Question: Calculate the gradient at $\left(\begin{smallmatrix}3\\6\end{smallmatrix}\right)$ in the direction $\left(\begin{smallmatrix}2\\2\end{smallmatrix}\right)$ on the surface $h(x, y) = x^2 + xy$.

Answer: We can calculate the gradient function: `der(h(x, y), x) * 2 + der(h(x, y), y) * 2`, giving us $6x + 2y$. Evaluating this at $\left(\begin{smallmatrix}3\\6\end{smallmatrix}\right)$ gives us 30.