

OpenGL Projection Matrix

Related Topics: [OpenGL Transformation](#), [OpenGL Matrix](#)

- [Overview](#)
- [Perspective Projection](#)
- [Orthographic Projection](#)

Updates: The MathML version is available [here](#).

Overview

A computer monitor is a 2D surface. A 3D scene rendered by OpenGL must be projected onto the computer screen as a 2D image. `GL_PROJECTION matrix` is used for this projection [transformation](#). First, it transforms all vertex data from the eye coordinates to the clip coordinates. Then, these clip coordinates are also transformed to the normalized device coordinates (NDC) by dividing with w component of the clip coordinates.

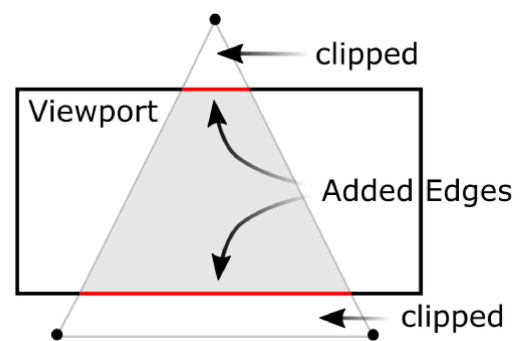
Therefore, we have to keep in mind that both clipping (frustum culling) and NDC transformations are integrated into

GL_PROJECTION matrix. The following sections describe how to build the projection matrix from 6 parameters; *left*, *right*, *bottom*, *top*, *near* and *far* boundary values.

Note that the frustum culling (clipping) is performed in the clip coordinates, just before dividing by w_c . The clip coordinates, x_c , y_c and z_c are tested by comparing with w_c . If any clip coordinate is less than $-w_c$, or greater than w_c , then the vertex will be discarded.

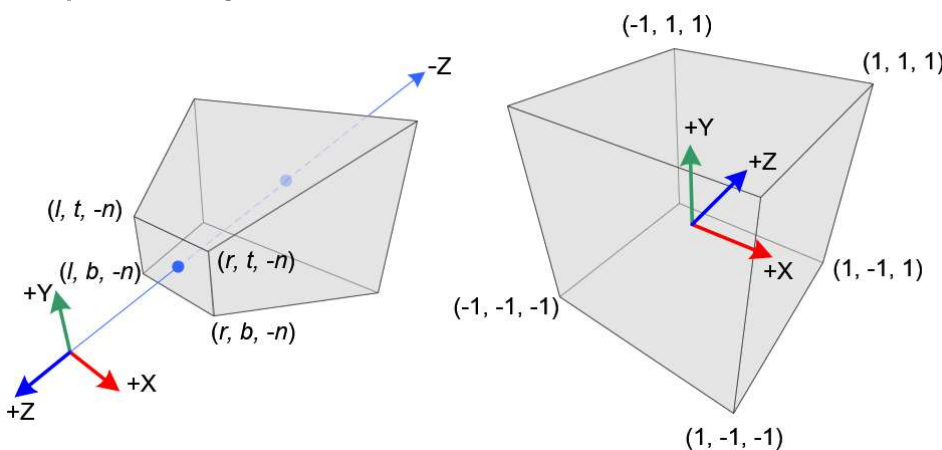
$$-w_c < x_c, y_c, z_c < w_c$$

Then, OpenGL will reconstruct the edges of the polygon where clipping occurs.



A triangle clipped by frustum

Perspective Projection



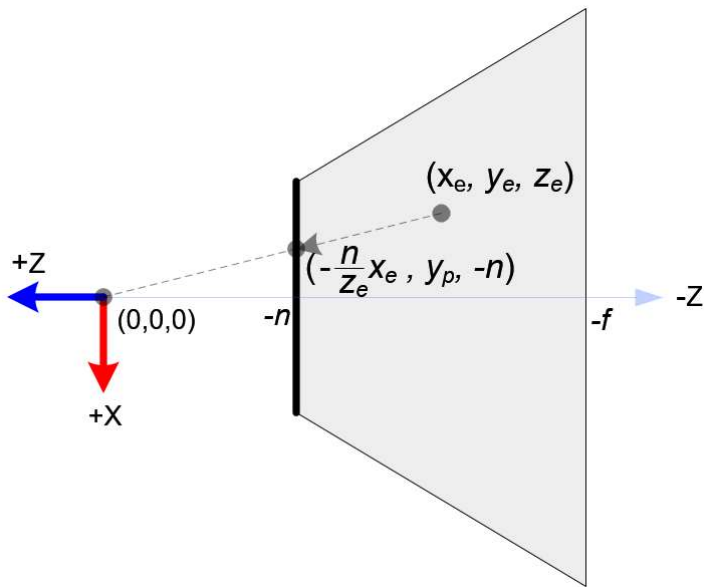
Perspective Frustum and Normalized Device Coordinates (NDC)

$+Z$ axis in NDC. Since `glFrustum()` accepts only positive values of *near* and *far* distances, we need to negate them during the construction of `GL_PROJECTION` matrix.

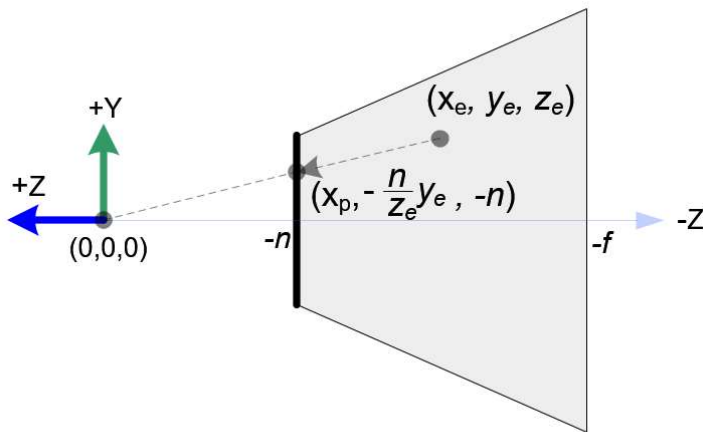
In OpenGL, a 3D point in eye space is projected onto the *near* plane (projection plane). The following diagrams show how a point (x_e, y_e, z_e) in eye space is projected to (x_p, y_p, z_p) on the *near* plane.

In perspective projection, a 3D point in a truncated pyramid frustum (eye coordinates) is mapped to a cube (NDC); the range of x -coordinate from $[l, r]$ to $[-1, 1]$, the y -coordinate from $[b, t]$ to $[-1, 1]$ and the z -coordinate from $[-n, -f]$ to $[-1, 1]$.

Note that the eye coordinates are defined in the right-handed coordinate system, but NDC uses the left-handed coordinate system. That is, the camera at the origin is looking along $-Z$ axis in eye space, but it is looking along



Top View of Frustum



Side View of Frustum

From the top view of the frustum, the x-coordinate of eye space, x_e is mapped to x_p , which is calculated by using the ratio of similar triangles;

$$\frac{x_p}{x_e} = \frac{-n}{z_e}$$

$$x_p = \frac{-n \cdot x_e}{z_e} = \frac{n \cdot x_e}{-z_e}$$

From the side view of the frustum, y_p is also calculated in a similar way;

$$\frac{y_p}{y_e} = \frac{-n}{z_e}$$

$$y_p = \frac{-n \cdot y_e}{z_e} = \frac{n \cdot y_e}{-z_e}$$

Note that both x_p and y_p depend on z_e ; they are inversely proportional to $-z_e$. In other words, they are both divided by $-z_e$. It is a very first clue to construct GL_PROJECTION matrix. After the eye coordinates are transformed by multiplying GL_PROJECTION matrix, the clip coordinates are still a [homogeneous coordinates](#). It finally becomes the normalized device coordinates (NDC) by divided by the w-component of the clip coordinates. (See more details on [OpenGL](#))

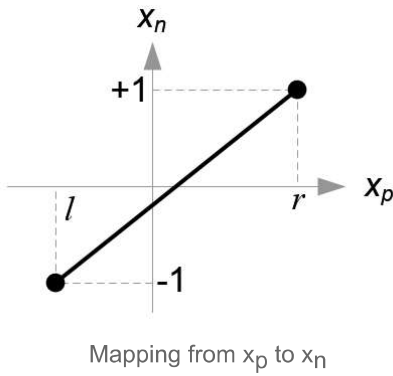
Transformation.)

$$\begin{pmatrix} x_{clip} \\ y_{clip} \\ z_{clip} \\ w_{clip} \end{pmatrix} = M_{projection} \cdot \begin{pmatrix} x_{eye} \\ y_{eye} \\ z_{eye} \\ w_{eye} \end{pmatrix}, \quad \begin{pmatrix} x_{ndc} \\ y_{ndc} \\ z_{ndc} \end{pmatrix} = \begin{pmatrix} x_{clip}/w_{clip} \\ y_{clip}/w_{clip} \\ z_{clip}/w_{clip} \end{pmatrix}$$

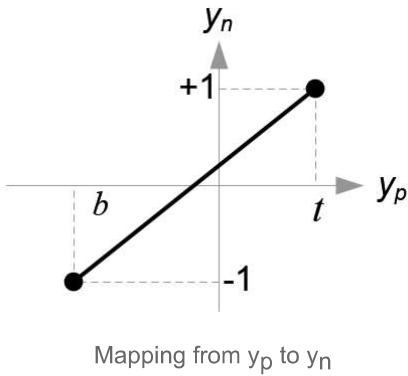
Therefore, we can set the w-component of the clip coordinates as $-z_e$. And, the 4th of GL_PROJECTION matrix becomes (0, 0, -1, 0).

$$\begin{pmatrix} x_c \\ y_c \\ z_c \\ w_c \end{pmatrix} = \begin{pmatrix} . & . & . & . \\ . & . & . & . \\ . & . & . & . \\ 0 & 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} x_e \\ y_e \\ z_e \\ w_e \end{pmatrix}, \quad \therefore w_c = -z_e$$

Next, we map x_p and y_p to x_n and y_n of NDC with linear relationship; $[l, r] \Rightarrow [-1, 1]$ and $[b, t] \Rightarrow [-1, 1]$.



$$\begin{aligned} x_n &= \frac{1 - (-1)}{r - l} \cdot x_p + \beta \\ 1 &= \frac{2r}{r - l} + \beta \quad (\text{substitute } (r, 1) \text{ for } (x_p, x_n)) \\ \beta &= 1 - \frac{2r}{r - l} = \frac{r - l}{r - l} - \frac{2r}{r - l} \\ &= \frac{r - l - 2r}{r - l} = \frac{-r - l}{r - l} = -\frac{r + l}{r - l} \\ \therefore x_n &= \frac{2x_p}{r - l} - \frac{r + l}{r - l} \end{aligned}$$



$$\begin{aligned} y_n &= \frac{1 - (-1)}{t - b} \cdot y_p + \beta \\ 1 &= \frac{2t}{t - b} + \beta \quad (\text{substitute } (t, 1) \text{ for } (y_p, y_n)) \\ \beta &= 1 - \frac{2t}{t - b} = \frac{t - b}{t - b} - \frac{2t}{t - b} \\ &= \frac{t - b - 2t}{t - b} = \frac{-t - b}{t - b} = -\frac{t + b}{t - b} \\ \therefore y_n &= \frac{2y_p}{t - b} - \frac{t + b}{t - b} \end{aligned}$$

Then, we substitute x_p and y_p into the above equations.

$$\begin{aligned}
x_n &= \frac{2x_p}{r-l} - \frac{r+l}{r-l} \quad (x_p = \frac{nx_e}{-z_e}) \\
&= \frac{2 \cdot \frac{n \cdot x_e}{-z_e}}{r-l} - \frac{r+l}{r-l} \\
&= \frac{2n \cdot x_e}{(r-l)(-z_e)} - \frac{r+l}{r-l} \\
&= \frac{\frac{2n}{r-l} \cdot x_e}{-z_e} - \frac{r+l}{r-l} \\
&= \frac{\frac{2n}{r-l} \cdot x_e}{-z_e} + \frac{\frac{r+l}{r-l} \cdot z_e}{-z_e} \\
&= \left(\underbrace{\frac{2n}{r-l} \cdot x_e + \frac{r+l}{r-l} \cdot z_e}_{x_c} \right) / -z_e
\end{aligned}$$

$$\begin{aligned}
y_n &= \frac{2y_p}{t-b} - \frac{t+b}{t-b} \quad (y_p = \frac{ny_e}{-z_e}) \\
&= \frac{2 \cdot \frac{n \cdot y_e}{-z_e}}{t-b} - \frac{t+b}{t-b} \\
&= \frac{2n \cdot y_e}{(t-b)(-z_e)} - \frac{t+b}{t-b} \\
&= \frac{\frac{2n}{t-b} \cdot y_e}{-z_e} - \frac{t+b}{t-b} \\
&= \frac{\frac{2n}{t-b} \cdot y_e}{-z_e} + \frac{\frac{t+b}{t-b} \cdot z_e}{-z_e} \\
&= \left(\underbrace{\frac{2n}{t-b} \cdot y_e + \frac{t+b}{t-b} \cdot z_e}_{y_c} \right) / -z_e
\end{aligned}$$

Note that we make both terms of each equation divisible by $-z_e$ for perspective division ($x_c/w_c, y_c/w_c$). And we set w_c to $-z_e$ earlier, and the terms inside parentheses become x_c and y_c of the clip coordiantes.

From these equations, we can find the 1st and 2nd rows of GL_PROJECTION matrix.

$$\begin{pmatrix} x_c \\ y_c \\ z_c \\ w_c \end{pmatrix} = \begin{pmatrix} \frac{2n}{r-l} & 0 & \frac{r+l}{r-l} & 0 \\ 0 & \frac{2n}{t-b} & \frac{t+b}{t-b} & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} x_e \\ y_e \\ z_e \\ w_e \end{pmatrix}$$

Now, we only have the 3rd row of GL_PROJECTION matrix to solve. Finding z_n is a little different from others because z_e in eye space is always projected to $-n$ on the near plane. But we need unique z value for the clipping and depth test. Plus, we should be able to unproject (inverse transform) it. Since we know z does not depend on x or y value, we borrow w -component to find the relationship between z_n and z_e . Therefore, we can specify the 3rd row of GL_PROJECTION matrix like this.

$$\begin{pmatrix} x_c \\ y_c \\ z_c \\ w_c \end{pmatrix} = \begin{pmatrix} \frac{2n}{r-l} & 0 & \frac{r+l}{r-l} & 0 \\ 0 & \frac{2n}{t-b} & \frac{t+b}{t-b} & 0 \\ 0 & 0 & A & B \\ 0 & 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} x_e \\ y_e \\ z_e \\ w_e \end{pmatrix}, \quad z_n = z_c/w_c = \frac{Az_e + Bw_e}{-z_e}$$

In eye space, w_e equals to 1. Therefore, the equation becomes;

$$z_n = \frac{Az_e + B}{-z_e}$$

To find the coefficients, A and B , we use the (z_e, z_n) relation; $(-n, -1)$ and $(-f, 1)$, and put them into the above equation.

$$\begin{cases} \frac{-An + B}{n} = -1 \\ \frac{-Af + B}{f} = 1 \end{cases} \rightarrow \begin{cases} -An + B = -n & (1) \\ -Af + B = f & (2) \end{cases}$$

To solve the equations for A and B , rewrite eq.(1) for B ;

$$B = An - n \quad (1')$$

Substitute eq.(1') to B in eq.(2), then solve for A ;

$$-Af + (An - n) = f \quad (2)$$

$$-(f - n)A = f + n$$

$$A = -\frac{f + n}{f - n}$$

Put A into eq.(1) to find B;

$$\left(\frac{f+n}{f-n}\right)n + B = -n \quad (1)$$

$$\begin{aligned} B &= -n - \left(\frac{f+n}{f-n}\right)n = -\left(1 + \frac{f+n}{f-n}\right)n = -\left(\frac{f-n+f+n}{f-n}\right)n \\ &= -\frac{2fn}{f-n} \end{aligned}$$

We found A and B. Therefore, the relation between z_e and z_n becomes;

$$z_n = \frac{-\frac{f+n}{f-n}z_e - \frac{2fn}{f-n}}{-z_e} \quad (3)$$

Finally, we found all entries of GL_PROJECTION matrix. The complete projection [matrix](#) is;

$$\begin{pmatrix} \frac{2n}{r-l} & 0 & \frac{r+l}{r-l} & 0 \\ 0 & \frac{2n}{t-b} & \frac{t+b}{t-b} & 0 \\ 0 & 0 & \frac{-(f+n)}{f-n} & \frac{-2fn}{f-n} \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

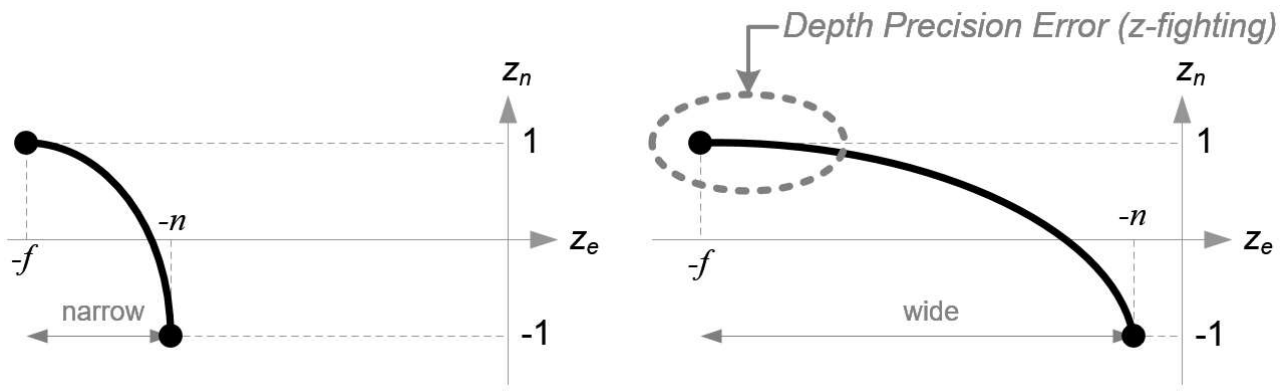
OpenGL Perspective Projection Matrix

This projection [matrix](#) is for a general frustum. If the viewing volume is symmetric, which is $r = -l$ and $t = -b$, then it can be simplified as;

$$\begin{cases} r+l=0 \\ r-l=2r \text{ (width)} \end{cases}, \quad \begin{cases} t+b=0 \\ t-b=2t \text{ (height)} \end{cases}$$

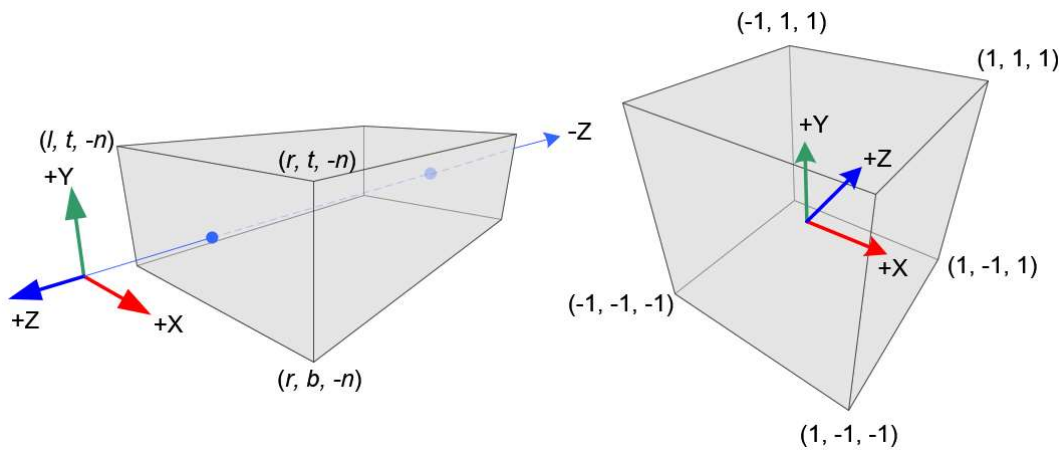
$$\begin{pmatrix} \frac{n}{r} & 0 & 0 & 0 \\ 0 & \frac{n}{t} & 0 & 0 \\ 0 & 0 & \frac{-(f+n)}{f-n} & \frac{-2fn}{f-n} \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

Before we move on, please take a look at the relation between z_e and z_n , eq.(3) once again. You notice it is a rational function and is non-linear relationship between z_e and z_n . It means there is very high precision at the *near* plane, but very little precision at the *far* plane. If the range $[-n, -f]$ is getting larger, it causes a depth precision problem (z-fighting); a small change of z_e around the *far* plane does not affect on z_n value. The distance between n and f should be short as possible to minimize the depth buffer precision problem.



Comparison of Depth Buffer Precisions

Orthographic Projection

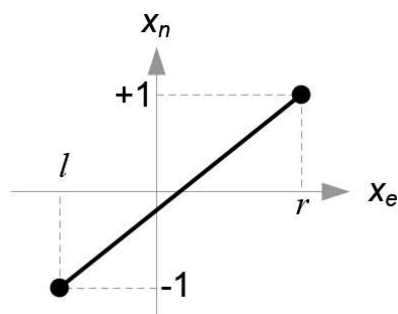


Orthographic Volume and Normalized Device Coordinates (NDC)

Constructing `GL_PROJECTION` [matrix](#) for orthographic projection is much simpler than perspective mode.

All x_e , y_e and z_e components in eye space are linearly mapped to NDC. We just need to scale a rectangular volume to a cube, then move it to the origin. Let's find out the elements of

`GL_PROJECTION` using linear relationship.



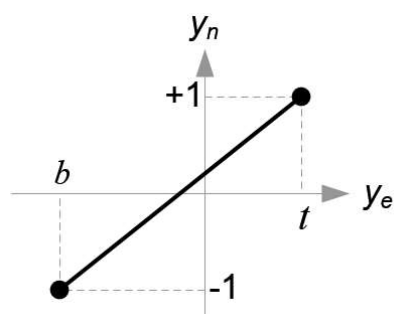
Mapping from x_e to x_n

$$x_n = \frac{1 - (-1)}{r - l} \cdot x_e + \beta$$

$$1 = \frac{2r}{r - l} + \beta \quad (\text{substitute } (r, 1) \text{ for } (x_e, x_n))$$

$$\beta = 1 - \frac{2r}{r - l} = -\frac{r + l}{r - l}$$

$$\therefore x_n = \frac{2}{r - l} \cdot x_e - \frac{r + l}{r - l}$$



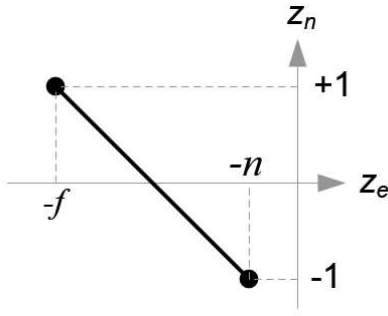
Mapping from y_e to y_n

$$y_n = \frac{1 - (-1)}{t - b} \cdot y_e + \beta$$

$$1 = \frac{2t}{t - b} + \beta \quad (\text{substitute } (t, 1) \text{ for } (y_e, y_n))$$

$$\beta = 1 - \frac{2t}{t - b} = -\frac{t + b}{t - b}$$

$$\therefore y_n = \frac{2}{t - b} \cdot y_e - \frac{t + b}{t - b}$$



Mapping from z_e to z_n

$$z_n = \frac{1 - (-1)}{-f - (-n)} \cdot z_e + \beta$$

$$1 = \frac{2f}{f - n} + \beta \quad (\text{substitute } (-f, 1) \text{ for } (z_e, z_n))$$

$$\beta = 1 - \frac{2f}{f - n} = -\frac{f + n}{f - n}$$

$$\therefore z_n = \frac{-2}{f - n} \cdot z_e - \frac{f + n}{f - n}$$

Since w-component is not necessary for orthographic projection, the 4th row of GL_PROJECTION matrix remains as (0, 0, 0, 1). Therefore, the complete GL_PROJECTION [matrix](#) for orthographic projection is;

$$\begin{pmatrix} \frac{2}{r-l} & 0 & 0 & -\frac{r+l}{r-l} \\ 0 & \frac{2}{t-b} & 0 & -\frac{t+b}{t-b} \\ 0 & 0 & \frac{-2}{f-n} & -\frac{f+n}{f-n} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

OpenGL Orthographic Projection Matrix

It can be further simplified if the viewing volume is symmetrical, $r = -l$ and $t = -b$.

$$\begin{cases} r + l = 0 \\ r - l = 2r \text{ (width)} \end{cases}, \begin{cases} t + b = 0 \\ t - b = 2t \text{ (height)} \end{cases}$$

$$\begin{pmatrix} \frac{1}{r} & 0 & 0 & 0 \\ 0 & \frac{1}{t} & 0 & 0 \\ 0 & 0 & \frac{-2}{f-n} & -\frac{f+n}{f-n} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$