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Prime Numbers

Understanding How the Largest Primes Came to Be

Ibrahim Abdulhussein,
Robin Brusbo

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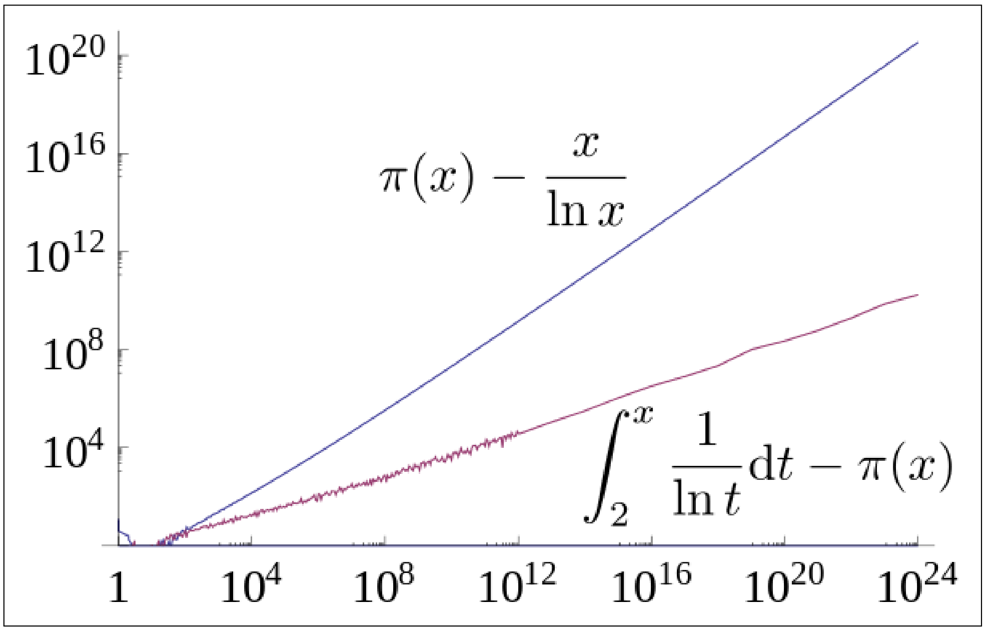

[1]

[1] The Prime Number Theorem [1]

Abstract

Contents

1 Introduction

1.1 History

Prime numbers have been studied since Ancient Greece, and have, perhaps, been known to us for even longer. [2]

1.2 Aim

The purpose of this essay is to investigate prime numbers. Investigating all the aspects of prime numbers would be a near impossible task; therefore, this essay will focus more on learning about the methods and algorithms of finding new primes. The study will also present ways to prove that a given number, n, is indeed prime. Primes are without a doubt beautiful mathematically, but what real life application do primes have?

The relevant questions for this study are the following:

1. What methods can be used to find new primes?
2. What are some algorithms for primality testing?
3. Are there "accurate" ways to measure the amount of primes under a given number, N?
4. Which algorithm is the most efficient and why?

2 Theory

2.1 What is a prime?

Prime numbers are defined as *positive integers* which only have the factors 1 and itself. Thus 4 is not a prime since $4 = 2 * 2$. On the other hand 5 is a prime since the only divisors of 5 is 1 and 5. An exception to this definition is 1, since it's the first natural number.

If a number, n, is not prime, it is refered to as a *composite number*.

2.2 The Fundamental Theory of Arithmetic

The Fundamental Theory of Arithmetic [3] states that all integers greater than 1 is either a prime, or can be expressed as a product of primes in a unique way. This means that all natural numbers, except for 1, has its own factorization containing only primes, unless it is a prime itself.

Important to know is that there is an infinite amount of primes. The proof is a quite easy by contradiction, but nonetheless beautiful:

> Assume that there is a finite amount of primes and make a list of them:
>
> $p_1, p_2, p_3, p_4, p_5, ...$
>
> Let the constant Q be the product of all the primes in the list and add 1:
>
> $Q = p_1 * p_2 * p_3 * ... + 1$
>
> According to the fundamental theorem of arithmetic, Q must be a prime since none of the primes in the list divide Q evenly because of the 1; therefore making the list incomplete and proving that you cannot make a finite list of all primes.

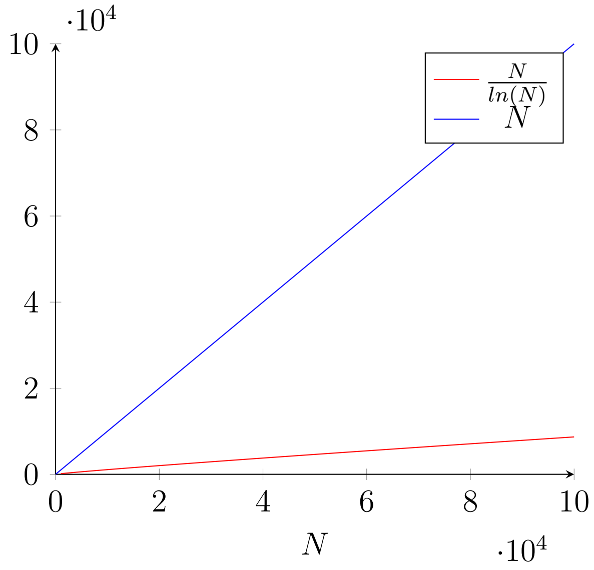

Figure 1: The graph of $\pi(N)$ and N from 0 to 10^5 letting us compare the relationship between the number N and the approximate amount of primes below it.

2.3 The Prime Number Theorem

The Prime Number Theorem [1] describes approximately how many primes there are less than or equal to a given number. The function $\pi(N) \sim \frac{N}{ln(N)}$ gives the expected amount of primes below a certain N. Graphing this function shows that primes become less common for greater N.

This proves that primes do not show up linearaly, meaning a computer that is twice as powerful will *not* produce twice as many primes. Instead, the most important and crucial part of generating and verifying primes is the optimization of the *algorithms*.

2.4 Time Complexity

Time complexity [4] is a concept within computer science, which describes the approximate time for a program to complete. The study will make heavy use of the Big O Notation [5], which notates how the run time increases as the input size increases. For example, $O(N)$ will grow linearly with the input size. Increasing the input size by a factor of 10, will also increase the run time by a factor of 10, as such $O(10N)$. On the other hand, $O(log(n))$ grows logarithmically, which is far more efficient for bigger input sizes, as $O(log(N))$ is strictly smaller than N for large enough values. It is important to note that when using logarithms in these instances, the base is irrelevant. The proof as to why the base is irrelevant will not be provided by this study.

This is very important when it comes to testing large numbers, because it makes it easier to determine whether the program will succeed, or run for a *very long time*[2]. Therefore it is important to write as efficient algorithms as possible, considering the fact that the largest known verified prime has $24,862,048$ digits [6].

2.4.1 Examples of Big O Notation

It is important to have an idea of how the Big O Notation is applied in code, therefore the following examples will show it thoroughly.

A program that runs in $O(N)$ would be a function that inputs an integer N and outputs every number up to N:

```
def linearTime(N):
    for number in range(N):
        print(number)
```

Since the program runs in $O(N)$ time, increasing the input by a factor of 10 would also increase the number of operations done by a factor of 10.

The second example is a program that runs in quadratic time, i.e. $O(N^2)$. Instead of printing every number up to N once, this program prints every number N times.

```
    def quadraticTime(N):
        for number in range(N):
            for number in range(N):
```

[2] Some programs will not finish until the sun explodes, which is quite impractical.

```
            print(number)
```

The third program describes $O(\sqrt{n})$. It is a simple program that sums all numbers up to $\sqrt{N}$.

```
import math

def sqrtTime(N):
    sum = 0
    for number in range(math.sqrt(N)):
        sum += number
    print(sum)
```

The fourth and last example will be about $O(log(n))$. The algorithm that is used in this example is called *Binary Search* [7]. It’s an algorithm used for quickly guessing a number between, for example, 1 and 100. Usually two players are required to play this guessing game, but with a computer the user will give an input that the computer will try to guess. The output will be the amount of ”guesses” the algorithm perfomed.

```
def binarySearch():
    print("The number will be between _ and _\n")
    begin = int(input())
    end = int(input())
    value = int(input("What value will you input?\n"))
    guesses = 0

    while True:
        guesses += 1
        mid = (begin + end) / 2

        if mid == valueToGuess:
            break
        elif mid > valueToGuess:
            end = mid + 1
        elif mid < valueToGuess:
            begin = mid - 1

    print(guesses)
```

3 Materials and Methods

3.1 Resources

The most important resource to be used in this study will be Wikipedia and the countless articles written regarding this subject. The programming language used to implement and time the algorithms will be Python[3] [8].

As of today, the largest primes are found using supercomputers running these algorithms. Using a supercomputer is not plausible for this study, therefore the maxmimum limit for primes will be 10^{14}.

3.2 Methods

During the primality testing [9] there will be two test cases for the algorithms. The first one will determine if n is a prime or composite, and the second one will generate the largest prime within a time limit. At the start of the code a driver function will be defined to serve as the "starter" code for the program - deciding what test to perform. The driver function will call a primality testing function called **primeTest** which will return a boolean[4] value.

Two groups of algorithms will be tested: the "simple" methods and the probabilistic tests. The driver function will be called to start a timer and execute one of the following algorithms:

1. "Simple" methods
   - (a) Brute-force
   - (b) Smart Brute-force
   - (c) Sieve of Eratosthenes [10]
2. Probabilistic tests [9]
   - (a) Fermat Primality

[3] Although Python is one of the slowest languages, the simplicity of it makes it an invaluable tool for this study.

[4] A function that returns *True* or *False*

(b) Miller–Rabin

The function will be called upon three times to average an "accurate" runtime. The same integers will be used across all tests. These runtimes are noted and later compared in the discussion. The following Python code describes the setup:

```
import time
# from ALGORITHM_FILE import ALGORITHM

def driver():
    numbers = [...] # Values to be appended
    times = []
    for n in numbers:
        start = time.time()
        primeTest(n)
        runTime = time.time() - start
        times.append(runTime)
    print(sum(times) / len(times))

def primeTest(n):
    if Algorithm(n): # Algorithm function
        return True
    return False

if __name__ == "__main__":
    driver()
```

Important to note is that the algorithms will be stored in a seperate file. At the beginning of the main file the algorithms will be imported. To see further details and the *.py* files, refer to the essays GitHub page [11].

To generate the largest prime within 30 seconds, the **primeTest** function will remain the same, however the driver function will recieve some changes. Study the following code:

```
    def driver():
        n = 2
        primes = []
        timelimit = time.time() + 30  # Thirty seconds limit
        while (time.time() < timelimit):
            if primeTest(n):
                primes.append(n)
            n += 1
        print(max(primes))
```

3.3 Algorithms

3.3.1 Brute-force

3.3.2 Smart Brute-force

3.3.3 Sieve of Eratosthenes

3.3.4 Fermat Primality

3.3.5 Miller–Rabin

4 Results

4.1 Primality Testing

Brute-force	Smart brute-force	Sieve of Eratosthenes
0.x seconds	0.x seconds	0.x seconds
0.x seconds	0.x seconds	0.x seconds
0.x seconds	0.x seconds	0.x seconds

Table 1: The runtime for each algorithm three times

AVERAGE 1	AVERAGE 2	AVERAGE 3

Table 2: The summed averages of the above results

4.2 Generating Largest Prime

5 Discussion

6 Conclusion

7 Acknowledgments

Loke Gustafsson
The one and only Loke. Finding a word to describe you is extremely hard since there really are no fitting words in the English dictionary. Therefore, the most accurate word would be ”Lokeism”.

Björn Noren

Jonas Ingensson

Christiane Miller

8 References

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