LOGISTIC FAMILY AND COMPLEX DYNAMICS

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Abstract. We present a brief exposition of the dynamics of the logistic family, a central object in dynamical systems with wide-ranging applications. This is followed by a detailed study of the Julia set and the Mandelbrot set, with attention to both local and global structures of Julia sets, including those of polynomial maps. We also examine the technique of quasiconformal surgery introduced by Shishikura [23].

Contents

1. Introduction

The *logistic family* $f_\lambda(x) = \lambda x(1-x)$, with $x \in [0,1]$ and $0 < \lambda \leq 4$, is not only a central object in the field of dynamical systems, but also plays a fundamental role in modeling phenomena such as financial systems, population growth, the spread of diseases, and many other real-world applications [7]. This expository minor thesis provides a brief overview of the theory and conjectures surrounding the logistic family, along with a description of the structure of the Julia set and the Mandelbrot set, followed by a rudimentary introduction to the technique of quasiconformal surgery.

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Let $F : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a rational function, where $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ is the Riemann sphere. The study of *dynamics* of F is to understand the behaviors of iterations

$$F^n(x) := \underbrace{(F \circ F \circ \cdots \circ F)}_{n}(x).$$

The sequence $\{F^n(x)\}_{n\geq 0}$ is called the *orbit* of x. Points can be classified by the behavior of their orbits. A *periodic point* x satisfies $F^n(x) = x$ for some $n > 0$, and the smallest such n is called the *prime period*. The orbit of a periodic point is called a *cycle*. A *fixed point* of F is a point with period 1. An *eventually periodic point* is an x such that $F^n(x)$ is periodic for some $n > 0$. A periodic point x of prime period n is *attracting* if $|(F^n)'(x)| < 1$, *repelling* if $|(F^n)'(x)| > 1$, and *neutral* or *indifferent* if $|(F^n)'(x)| = 1$. Asymptotically, if the set A of limit points of an orbit $\{F^n(x)\}_{n\geq 0}$ is finite, then A must be the cycle of a periodic point. We say x is *attracted* to A, and A is an *attracting cycle*. On the other hand, an orbit could be unbounded, or bounded but not attracted to any cycles.

In the first two sections, we give an overview of the dynamics of the logistic family f_λ. Instead of the logistic family f_λ, we study the *quadratic family* $Q_c(x) = x^2 + c$, which is conjugate to f_λ for $c = \lambda/2 - \lambda^2/4$ and emerges to be more convenient in the subsequent sections.

In section 2, we study asymptotic behaviors of the orbit of the critical point $x = 0$ of the system Q_c for $c \in [-2, 1/4]$. We observe a curious phenomenon called *bifurcation*, where the length of attracting cycle doubles as c crosses a critical value. Moreover, the systems Q_c where the orbit of 0 is attracted to an attracting cycle are called *hyperbolic*, and they flourish on the interval $c \in [-2, 1/4]$.

Theorem 1.1 (Conjecture HD2$\mathbb{R}$ [16], Lyubich [13], Graczyk-Świątek [9])**.** *The values of c for which Q_c is hyperbolic are dense in $[-2, 1/4]$.*

In section 3, we study the system Q_c using complex tools. Concretely, we view $Q_c(z)$ as a function in a complex variable, with a complex parameter $c \in \mathbb{C}$. For each $c \in \mathbb{C}$, the *filled Julia set* K_c is the set of $z \in \mathbb{C}$ whose orbit under Q_c is bounded, and the *Julia set* J_c is the boundary of K_c. Finally, the celebrated *Mandelbrot set* $\mathcal{M}$ is the set of $c \in \mathbb{C}$ for which K_c is connected [7]. The topology of $\mathcal{M}$ encompasses a very rich structure, and the study of $\mathcal{M}$ has been especially fruitful [8, 10, 18, 24]. Some of the most central open problems in the field revolve around the Mandelbrot set. For example,

Conjecture 1.2 (MLC)**.** *$\mathcal{M}$ is locally connected.*

In section 4, we generalize the definition of the Julia set to any rational function $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$, and study in detail the structure of Julia set and its complement, the Fatou set. In particular, cycles of length n can be classified into several types according to its *multiplier* $\lambda = (f^n)'(z)$. We say a periodic orbit is *superattracting*,

attracting, *repelling*, and *indifferent* if $\lambda = 0$, $0 < |\lambda| < 1$, $|\lambda| > 1$, and $|\lambda| = 1$, respectively. Moreover, an indifferent cycle is *rationally indifferent* (resp. *irrationally indifferent*) if, when writing $\lambda = \exp(2\pi i\xi)$, the angle ξ is rational (resp. irrational).

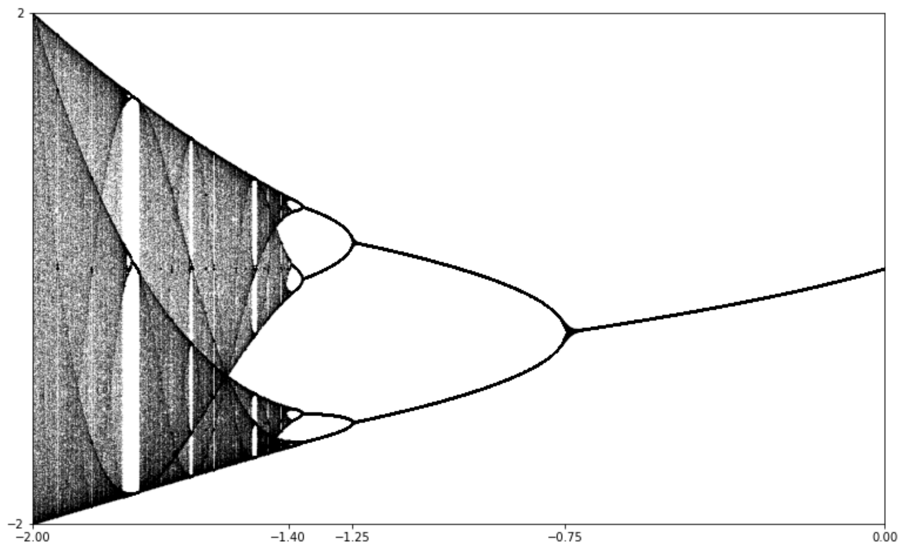
The local behavior of f near attracting and repelling cycles resembles complex multiplication $w \mapsto \lambda w$, whereas the local behavior of f near superattracting cycles resembles exponentiation $w \mapsto w^n$. The dynamics of f near rationally indifferent cycles are slightly more complicated, but they can still be well-described via the Leau-Fatou flower theorem. The dynamics of f near irrationally indifferent cycles are much wilder, and depends crucially on the number-theoretic properties of the irrational number ξ. In particular, an irrationally indifferent cycle is *Siegel* if a local linearization of f into $w \mapsto \lambda w$ is possible, and *Cremer* if otherwise.

Furthermore, we briefly study the Julia set of polynomials, which have the characteristic property that ∞ is a superattracting fixed point, and $f^{-1}(\infty) = \{\infty\}$. We show that the Julia set is the boundary of the *filled Julia set*, defined to be the set of points not attracted to ∞. The main references for section 4 are [17, 6].

In section 5, we describe a procedure called quasiconformal surgery. This is a method to perturb the rational function f so that attracting fixed points become superattracting, and indifferent ones attracting. However, we cannot simply perturb f analytically, because of the identity theorem. The remedy is to consider conjugations by a specific type of homeomorphism, called *quasiconformal map*. Then one can conjugate f by a quasiconformal map φ that is analytic outside of an invariant domain, so that the resulting map $f_0 = \varphi^{-1} \circ f \circ \varphi$ is still rational, but has different types of fixed points [6]. Furthermore, Shishikura showed that this conjugation $f_\varepsilon = \varphi_\varepsilon^{-1} \circ g_\varepsilon \circ \varphi_\varepsilon$ (where g_ε is a modified version of f) depends analytically on a small parameter ε, and can be arranged so that f_ε has the same fixed points and cycles as f, but indifferent ones become attracting [23].

2. Bifurcation and Hyperbolicity

The asymptotic behavior of the quadratic family Q_c can be visualized by a *bifurcation diagram*. This is an image obtained by plotting the points $(c, Q_c^n(0))$ for each $-2 \le c \le 1/4$ and a certain number of iterations, say $n \le 1000$, and then discarding the first few iterations, say $n \le 100$, to only keep the asymptotic behaviors. A typical bifurcation diagram resembles the following [15].



It can be observed from the bifurcation diagram, as well as from direct computations, that

- When $c > 1/4$, the orbit of 0 tends to infinity, since $Q_c(x) - x \geq c - 1/4 > 0$.
- At $c = 1/4$, we encounter a *saddle-node bifurcation*, where a pair of fixed points emerges.
- When $-3/4 < c < 1/4$, the orbit of 0 is attracted to a fixed point $\alpha = (1 - \sqrt{1-4c})/2$. This fixed point is attracting when $-3/4 < c < 1/4$ and repelling when $c < -3/4$. The other fixed point $\beta = (1 + \sqrt{1-4c})/2$ is always repelling when $c < 1/4$.
- At $c = -4/3$, we encounter a *period-doubling bifurcation*, where the length of the attracting cycle doubles, and α flips to be repelling.
- When $-5/4 < c < -3/4$, the orbit of 0 is attracted to a 2-cycle $q_{\pm} = (-1 \pm \sqrt{-4c-3})/2$. This 2-cycle is attracting when $-5/4 < c < -3/4$, and repelling when $c < -5/4$.
- At $c = -5/4$, there is another period-doubling bifurcation, and the 2-cycle $q_{\pm}$ flips to be repelling.
- ...

Furthermore, it seems that as c passes -1.40 the orbit is no longer attracted to any cycles for most of c, but occasionally there is a "window" of attracting cycles appearing and disappearing. This corresponds to hyperbolic Q_c.

Definition 2.1. A rational map $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is *hyperbolic* if all critical points of f tend to attracting cycles under iteration.

Despite of their seemingly sporadic occurrences in the bifurcation diagram, hyperbolic maps are conjectured to be dense in various spaces of maps [16].

Conjecture 2.2 (HD)**.** *Hyperbolic maps are open and dense among all rational maps.*

Conjecture 2.3 (HD2)**.** *Hyperbolic maps are dense among quadratic polynomials.*

A theorem independently by Lyubich and Graczyk-Świątek asserts hyperbolic density in the real case.

Theorem 2.4 (Conjecture HD2$\mathbb{R}$ [16], Lyubich [13], Graczyk-Świątek [9])**.** *The values of c for which Q_c is hyperbolic are dense in $[-2, 1/4]$.*

On the other hand, a theorem of Jakobson [11] provides negative evidence to the hyperbolic density conjectures.

Theorem 2.5. *There is a set $\Lambda \subset [-2, 1/4]$ of positive Lebesgue measure such that Q_c is* not *hyperbolic for $c \in \Lambda$.*

It is also curious to note that a period-3 window emerges from chaos at $c \approx -1.75$, succeeded by period-doubling bifurcations. A celebrated theorem commonly known as "period three implies chaos", by Li-Yorke in their eponymous paper [12], describes the prominence of period-3 points.

Theorem 2.6 (Li-Yorke)**.** *If a continuous function $F : \mathbb{R} \to \mathbb{R}$ has a periodic point of prime period 3, then F also has periodic points of all other prime periods.*

Theorem 2.6 has a generalization by Sharkovsky. First order $\mathbb{N}$ in the following *Sharkovsky ordering.*

$$
\begin{aligned}
&3, 5, 7, 9, \dots \\
&2 \cdot 3, 2 \cdot 5, 2 \cdot 7, 2 \cdot 9, \dots \\
&4 \cdot 3, 4 \cdot 5, 4 \cdot 7, 4 \cdot 9, \dots \\
&\vdots \\
&2^k \cdot 3, 2^k \cdot 5, 2^k \cdot 7, 2^k \cdot 9, \dots \\
&\vdots \\
&\dots 2^n, \dots, 8, 4, 2, 1.
\end{aligned}
$$

Theorem 2.7 ([22])**.** *Suppose n_1 precedes n_2 in the Sharkovsky ordering. If a continuous function $F : \mathbb{R} \to \mathbb{R}$ has a periodic point of prime period n_1, then F also has a periodic point of prime period n_2.*

Both theorem 2.6 and theorem 2.7 can be proved in a rather elementary way, which is slightly surprising given their late appearance in the literature.

3. The Julia Set and the Mandelbrot Set

It is probably (not) surprising that the real families f_λ and Q_c are best studied via complex tools. We can view $Q_c(z) = z^2 + c$ as a system in one complex variable, and $c \in \mathbb{C}$ a complex parameter. The ubiquity of Q_c is justified by the fact that any complex quadratic system is conjugate to Q_c through a change of variable.

Definition 3.1. The *filled Julia set* K_c is the set of $z \in \mathbb{C}$ whose orbit under Q_c is bounded. The *Julia set* J_c is the boundary of K_c.

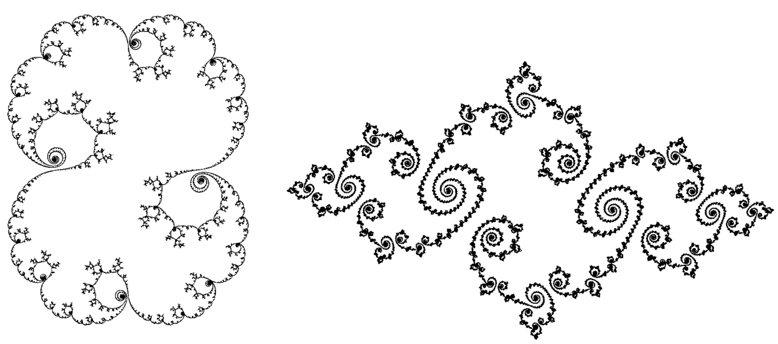
Definition 3.2. The *Mandelbrot set* $\mathcal{M}$ is the set of $c \in \mathbb{C}$ for which K_c is connected.

There is an alternate characterization of $\mathcal{M}$. It is the set of $c \in \mathbb{C}$ such that the orbit of 0 under Q_c is bounded. The following proposition is a consequence of propositions 4.20 and 4.21.

Proposition 3.3. *Either*

(1) *the orbit of 0 is unbounded and K_c consists of infinitely many connected components, or*
(2) *the orbit of 0 is bounded and K_c is connected.*

Here are two examples of Julia sets from [17]. The first example corresponds to the system $z \mapsto z^2 + (0.99 + 0.14i)z$ and is a simple closed curve. The second example corresponds to the system $z \mapsto z^2 + (-0.765 + 0.12i)$ and is totally disconnected.

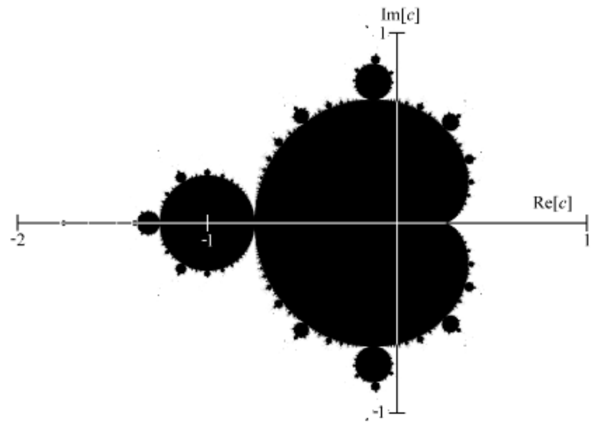


Since Q_c only has one critical point at $z = 0$, it follows that Q_c is hyperbolic if and only if either $|Q_c^n(0)| \to \infty$, or 0 is attracted to a cycle in $\mathbb{C}$. In light of conjecture 2.3, it is then natural to ask

Conjecture 3.4 (HD2' [16])**.** *If c lies in the interior of $\mathcal{M}$, then Q_c is hyperbolic.*

It is also known that if U is a component of int$(\mathcal{M})$ and Q_c is hyperbolic for one $c \in U$, then Q_c is hyperbolic for all $c \in U$. Furthermore, the boundary of $\mathcal{M}$ consists of Q_c with neutral attracting cycles, which are not hyperbolic [14]. Therefore, to resolve conjecture 2.3 and conjecture 3.4, it suffices to (not) find a $c \in$ int$(\mathcal{M})$ that is not hyperbolic.

An image of $\mathcal{M}$ is as follows [19]. Both the Mandelbrot set and the Julia set for many c are fractals, or self-similar shapes.



The main cardioid consists of c for which Q_c has an attracting fixed point. Basic calculus shows that it consists of parameters of the form $c(\mu) = (\mu/2)(1 - \mu/2)$ for $|\mu| \leq 1$. The largest bulb off the left of the main cardioid, which is a disk of radius $1/4$ centered at -1, consists of c for which Q_c has an attracting 2-cycle. More generally, for each $q \geq 2$ and a primitive q-th root of unity r_q, there is a bulb of attracting q-cycles attached to the main cardioid at $c(r_q)$.

The boundary of $\mathcal{M}$, which is known to be non-hyperbolic, has a Hausdorff dimension 2 [24], exhibiting its extremely fractal nature. The intersection of $\mathcal{M}$ and the real axis is the interval $[-2, 1/4]$, and the tangency points correspond to bifurcation points in the bifurcation diagram in section 2.

It is conjectured that

Conjecture 3.5 (MLC)**.** *$\mathcal{M}$ is locally connected.*

Furthermore, the work of Douady-Hubbard [8] shows that conjecture 3.5 implies conjecture 3.4 and conjecture 2.3.

4. The Structure of Julia Sets

We have defined the Julia set J_c of the quadratic system $Q_c(z) = z^2 + c$. In general, there are various definitions of the Julia set for a rational function $f = p/q : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$. We will assume $\deg f := \max\{\deg p, \deg q\} \geq 2$ throughout the rest of this minor thesis.

Definition 4.1. The *Fatou set* $P(f)$ is defined to be the set of $z \in \hat{\mathbb{C}}$ such that there is a neighborhood U about z such that the family $\{f^n\}$ restricted to U is a normal family. The *Julia set* $J(f)$ is the complement of $P(f)$ in $\hat{\mathbb{C}}$.

Another commonly known and equivalent definition, independently discovered by Julia and Fatou, is

Theorem 4.2. *The Julia set $J(f)$ is the closure of repelling periodic points of f.*

In particular, when f is a polynomial of degree 2 or more, define the *filled Julia set* $K(f)$ to be the set of z with bounded forward orbit.

Proposition 4.3. *When f is a polynomial of degree 2 or more, $J(f)$ is the topological boundary of $K(f)$.*

Note that proposition 4.3 agrees with previous definition of J_c. The main content of the rest of this section will be devoted to the proof of theorem 4.2 and proposition 4.3. We will first examine some straightforward properties of the Julia set.

Proposition 4.4. *For each $z \in \hat{\mathbb{C}}$, define the* grand orbit *of z to be*

$$GO(z, f) = \{z' \in \hat{\mathbb{C}} : f^m(z) = f^n(z') \text{ for some } m, n \geq 0\}.$$

Then $J(f)$ is completely invariant, *i.e., if $z \in J(f)$, then $GO(z, f) \subset J(f)$.*

Proof. It suffices to prove $z \in P(f)$ if and only if $f(z) \in P(f)$. Both directions are apparent by noting that both f and f^{-1} take open sets to open sets when f is holomorphic. □

Proposition 4.5. *$J(f^n) = J(f)$ for any $n > 0$.*

Proof. It is straightforward to verify the Fatou sets of f and f^n coincide. □

Proposition 4.6. *$J(f)$ is nonempty.*

Proof. If $J(f) = \varnothing$, then f^n would converge uniformly to a holomorphic limit g on all of $\hat{\mathbb{C}}$, but the degree of f^n diverges when $\deg f \geq 2$, a contradiction. □

Proposition 4.7. *If $z_0 \in J(f)$, then the set of all iterated preimages of z_0 is dense in $J(f)$. In particular, $GO(z_0, f)$ is dense in $J(f)$.*

Proof. Let U be any neighborhood about $z \in J(f)$. By Montel's theorem A.4, the sequence $\{f^n\}$ restricted to U misses an *exceptional set* E_z containing at most 2 points. Then $f^{-1}(E_z) \subset E_z$ by construction. One can then conjugate f so that $E_z = \{\infty\}$ or $E_z = \{0, \infty\}$, and in both cases E_z is contained in the Fatou set.

Now suppose $z_0 \in J(f)$, so $z_0 \notin E_z$. Then for any $z \in J(f)$ and any neighborhood U about z, there is $u \in U$ and $n > 0$ such that $f^n(u) = z_0$, or $u \in f^{-n}(z_0)$. It follows from proposition 4.4 that the set of all iterated preimages of z_0 is contained in $J(f)$, and density follows from previous arguments. □

Proposition 4.8. *$J(f)$ has no isolated points.*

Proof. Take any $z_0 \in J(f)$ and neighborhood U about z_0. Since $GO(z_0, f)$ is dense in $J(f)$ by proposition 4.7, there is $z_1 \in U \cap J(f)$ such that $f^n(z_1) = z_0$ for some $n > 0$. If z_0 is not periodic, then $z_1 \neq z_0$ and thus z_0 is not isolated. If z_0 is periodic, let n be its smallest period and consider the equation $f^n(z) = z_0$. If z_0 is the only solution, then z_0 would be superattracting for f^n and thus not in $J(f) = J(f^n)$, a contradiction. Thus let $z_1 \neq z_0$ be such that $f^n(z_1) = z_0$. By proposition 4.7 again there is a $z_2 \in U \cap J(f)$ such that $f^m(z_2) = z_1$. Furthermore, $z_2 \neq z_0$ since otherwise $z_0 = f^{n+m}(z_0)$, so m is a multiple of n by minimality of n. But this would imply $z_1 = f^m(z_0) = z_0$, a contradiction. □

4.1. **Local Structure.** For each periodic point z of f of period n, define the *multiplier* of z to be

$$\lambda = (f^n)'(z).$$

In particular, if $z = \infty$, then its multiplier is

$$\lambda = (A \circ f^n \circ A)'(0)$$

where $A(z) = 1/z$. We say z is *attracting* if $|\lambda| < 1$, *repelling* if $|\lambda| > 1$, and *indifferent* if $|\lambda| = 1$. An indifferent fixed point is *parabolic* or *rationally indifferent*, if λ is a root of unity but no iterations of f is the identity. It is *irrationally indifferent* if $|\lambda| = 1$ and λ is not a root of unity. In light of proposition 4.5, we begin by studying fixed points.

Proposition 4.9. *Any attracting fixed point is contained in the Fatou set. Any repelling or parabolic fixed point is contained in the Julia set.*

Proof. If z_0 is attracting, then there exists a disk D_r centered at z_0 such that $|f(z)| < c|z|$ for some $c < 1$ for all $z \in D_r$. Then the family $\{f^n\}$ restricted to D_r converges uniformly to the constant function z_0. If z_0 is repelling, then there is no holomorphic limit of $\{f^n\}$ since the derivatives $(f^n)'(z_0) = \lambda^n$ diverge. If z_0 is parabolic, suppose $z_0 = 0$ and $f^m(z)$ can be written as

$$f^m(z) = z + a_p z^p + \cdots$$

for some $m > 0$. Then $f^{mn}(z) = z + na_p z^p + \cdots$ and thus has no holomorphic limit, since the k-th derivatives diverge. □

Remark 4.10. The case of irrationally indifferent fixed points is more subtle and will be dealt with later in this section.

The local behavior of f near attracting and repelling fixed points can be well-understood.

Theorem 4.11 (Kœnigs Linearization Theorem [17])**.** *If the multiplier λ satisfies $|\lambda| \neq 0, 1$, then there exists a local holomorphic change of coordinate $w = \phi(z)$, with $\phi(z_0) = 0$, so that $\phi \circ f \circ \phi^{-1}$ is the linear map $w \mapsto \lambda w$ for all w in some neighborhood of the origin. Furthermore, ϕ is unique up to multiplication by a non-zero constant.*

Proof. Suppose $z_0 = 0$ and define $\varphi_n(z) = \lambda^{-n} f^n(z)$. Then $\varphi_n(f(z)) = \lambda\varphi_{n+1}(z)$. In the case $|\lambda| < 1$, we can show φ_n converges to a holomorphic limit φ uniformly, and thus the conjugation exists. When $|\lambda| > 1$, the inverse f^{-1} is locally defined, and z_0 is an attracting fixed point with multiplier λ^{-1}. We may then apply the above argument to f^{-1}. To show uniqueness, simply note that any holomorphic ψ satisfying $\psi(\lambda z) = \lambda\psi(z)$ must be multiplication by a constant when $\lambda \neq 0$ and λ is not a root of unity. □

Remark 4.12. This explains the abundance of spirals in the examples of Julia sets in section 3. Suppose $z_0 \in J(f)$ is a repelling fixed point (the set of which is dense in the Julia set by theorem 4.2) and its multiplier λ is not real. Then the preimages of any $w \in J(f)$ near z_0 are all in $J(f)$ by proposition 4.7. In the Kœnigs coordinates, the preimages are w/λ^n, which lie on a logarithmic spiral and converge to 0.

In the case $\lambda = 0$, known as a *superattracting fixed point*, there is a somewhat different local model.

Theorem 4.13 (Böttcher). *If f can be written as*

$$f(z) = z_0 + a_n(z - z_0)^n + a_{n+1}(z - z_0)^{n+1} \cdots \quad a_n \neq 0, n \geq 2$$

near a superattracting fixed point z_0, then there exists a local holomorphic change of coordinate $w = \phi(z)$, with $\phi(z_0) = 0$, so that $\phi \circ f \circ \phi^{-1}$ is the map $w \mapsto w^n$ for all w in some neighborhood of the origin. Furthermore, ϕ is unique up to multiplication by an $(n-1)$-th root of unity.

Proof. Without loss of generality, assume $z_0 = 0$ and $a_n = 1$ by a linear change of variable. Define

$$\varphi_n(z) := f^n(z)^{p^{-n}} = z(1 + O(z^n))^{p^{-n}}.$$

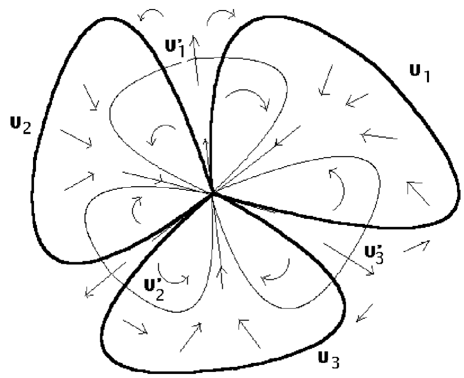
Then $\varphi_n(f(z)) = \varphi_{n+1}(z)^p$. One can show φ_n is well-defined and converges to a holomorphic limit φ uniformly in a neighborhood of the origin, and thus the conjugation exists. □

When $\lambda = 1$, the local behavior of f resembles a *Leau-Fatou flower.* First define the *multiplicity* of a parabolic fixed point to be the least $n > 1$ such that $f^{(n)}(z_0) \neq 0$. Assuming $z_0 = 0$ without loss of generality, one can write

$$f(z) = z + a(z - z_0)^n + \cdots$$

where $a \neq 0$.

Theorem 4.14 (Leau-Fatou). *If the origin is a fixed point of multiplicity $n + 1 \geq 2$, then there exist n disjoint attracting petals U_i and n disjoint repelling petals U_i' so that the union of these $2n$ petals, together with the origin itself, forms a neighborhood N_0 of the origin.*



A picture from [17] provides an illustration of the Leau-Fatou flower, with multiplicity $n+1=4$. The outward unit vector between attracting petals are called *repelling directions*. There are n evenly spaced repelling directions v characterized by the property that av^n is real and positive. Similarly, there are n evenly spaced *attracting directions* v between repelling petals, characterized by av^n being real and negative.

Any orbit that enters an attracting petal will stay in the attracting petal, and eventually converge to the origin, whereas any orbit that starts from a repelling petal will eventually leave the petal, but may return later, possibly infinitely many times.

Proof of theorem 4.14. Under the change of variable $w = c/z^n$ for $c = -1/(na)$, the sector between adjacent repelling directions will be mapped to the silted w-plane, with a neighborhood of 0 mapped to a neighborhood of infinity. The map f acts by

$$w \mapsto w' = w + 1 + o(1)$$

as $|w| \to \infty$. One can then construct a neighborhood of ∞ that is attracted to ∞. This corresponds to an attracting petal in the z-plane. □

If $\lambda \neq 1$ but $\lambda^q = 1$ for some $q > 1$, we may apply theorem 4.14 to f^q.

Proposition 4.15. *If the multiplier λ of a fixed point z_0 is a q-th root of unity, then the number n of attracting petals of f^q around z_0 must be a multiple of q.*

Proof. By computing the formal power series of $f \circ f^q = f^q \circ f$, one can show that f permutes the cyclic order of n attracting directions of f^q. This permutation has order q, so n must be a multiple of q. □

Finally, suppose $|\lambda| = 1$ but λ is not a root of unity. This case of is more subtle, and we will only outline the local structure. One can write $\lambda = \exp(2\pi i \xi)$ where $\xi \in (0, 1)$ is irrational. In this scenario, we hope to ask whether a local linearization like theorem 4.11 is possible, i.e., whether there exists φ such that

$f(\varphi(z)) = \varphi(\lambda z)$. There are two remarkable theorems by Cremer and Siegel, in different directions.

Theorem 4.16 (Cremer). *There is an uncountably dense subset of the unit circle such that if z_0 is a fixed point of any rational function with multiplier λ in the subset, then z_0 is the limit of an infinite sequence of periodic points. In other words, no linearization is possible.*

Theorem 4.17 (Siegel). *For almost every λ on the unit circle (outside of a Lebesgue-null set), any holomorphic function with a fixed point of multiplier λ is locally linearizable by a holomorphic change of coordinates.*

By definition, an irrationally indifferent fixed point is called a *Siegel point* if such linearization is possible, and a *Cremer point* if otherwise. It can be shown that Cremer points belong to the Julia set whereas Siegel points belong to the Fatou set. Moreover, generalizations of these properties, as well as more precise conditions under which linearization is possible, are closely related to number-theoretic properties of ξ, and have been pursued by Roth, Siegel, Bryuno, Perez-Marco, and Cremer, among others.

4.2. **Global Structure.** In this section we prove theorem 4.2. We first state the following result of Fatou.

Theorem 4.18. *Let $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a rational map of degree $d \geq 2$. Then f only has finitely many attracting or indifferent cycles.*

The original bound obtained by Fatou is $6d - 6$, which goes as follows. One can show that every attracting or rationally indifferent cycle attracts a critical point, so the number of them is bounded by the number of critical points, which is in turn bounded by $2d-2$. Furthermore, Fatou showed that the number of attracting cycles plus half of the number of indifferent cycles is bounded by $2d - 2$. One can then combine these bounds to obtain a grand bound of $6d - 6$ for attracting and indifferent cycles. We shall postpone the proof of theorem 4.18 to section 5, where we will discuss Shishikura's sharper bound of $2d-2$ using quasiconformal surgery. Here, we prove only one key lemma, which will also be used in section 5.

Lemma 4.19. *Each attracting cycle attracts at least one critical point.*

Proof. Let A be an attracting cycle of f. Put
$$U = \{z \in \hat{\mathbb{C}} : \lim_{n\to\infty} d(f^n(z), A) = 0\}$$
with the spherical metric. Then U is open, and $f^{-1}(U) = U$. Since the Julia set is nonempty and has no isolated points, it is uncountably infinite. In particular, U misses 3 points in $\hat{\mathbb{C}}$ since $U \cap J(f) = \varnothing$, and thus U is hyperbolic. If U contains no critical points, then $f : U \to U$ is a covering map, and thus a local isometry.

This contradicts the Schwarz-Pick theorem A.3, since $|f'(z_0)| < 1$ for some $z_0 \in A$ by the chain rule. Thus A must attract a critical point. □

Now we are ready to prove theorem 4.2, following Fatou.

Proof of theorem 4.2. Since there are finitely many attracting and indifferent cycles of f, and all repelling cycles belong to $J(f)$, we may suppose to the contrary that there is an open disk U that meets $J(f)$ and U contains no fixed points of any f^m. We may also assume U contains no poles or critical values of f. For any $z_0 \in U \cap J(f)$, the equation $f(z) = z_0$ has at least two distinct solutions z_1 and z_2 with $f'(z_i) \neq 0$ since $d \geq 2$ and z_0 is not a critical value. By the inverse function theorem and shrinking U if necessary, there are two branches φ_1 and φ_2 of f^{-1} on U with $\varphi_i(z_0) = z_i$. Define

$$g_n(z) = \frac{f^n(z) - \varphi_1(z)}{f^n(z) - \varphi_2(z)} \cdot \frac{z - \varphi_2(z)}{z - \varphi_1(z)}.$$

Since U contains no periodic points, the family g_n omits $\{0, 1, \infty\}$ on U and thus is normal by Montel's theorem A.4. But this would imply the family f^n is also normal, contradicting $z \in J(f)$. This shows cycles are dense in $J(f)$. Since $J(f)$ contains no isolated points and there are only finitely many attracting or indifferent cycles, we can exclude them and conclude that repelling cycles are dense in $J(f)$. □

4.3. **Julia Set of Polynomials.** In this section we prove propositions 3.3 and 4.3. Recall that the *filled Julia set* $K(f)$ is the set of points whose orbit under a polynomial f is bounded. It is a closed set, so $\partial K(f) \subset K(f)$.

Proof of proposition 4.3. Both $K(f)$ and $\partial K(f)$ are completely invariant under f by construction. Moreover, if $z \in \partial K(f)$, then there are points arbitrarily close to z whose orbits diverge to infinity, so $\{f^n\}$ cannot be normal in any neighborhood and $z \in J(f)$. Now $\partial K(f)$ is a completely invariant subset of $J(f)$, so it is dense by proposition 4.7. Since $\partial K(f)$ is closed, there is $J(f) = \partial K(f)$. □

The following two propositions are due to Fatou and Julia independently.

Proposition 4.20. *The Julia set $J(f)$ is connected if and only if each critical point in $\mathbb{C}$ belongs to $K(f)$.*

Proposition 4.21. *If $f^n(q) \to \infty$ for each critical point q, then $J(f)$ is a Cantor set, i.e., a perfect totally disconnected set.*

Note that propositions 4.20 and 4.21 cover all cases in the quadratic case, so proposition 3.3 is obtained as a corollary. Fatou conjectured that proposition 4.21 is also a necessary condition for $J(f)$ to be a Cantor set. However, this was disproved by Brolin [5], who constructed a cubic polynomial f whose Julia set is a Cantor set and contains one critical point. In general, Branner-Hubbard [4] gives necessary and sufficient conditions on when the Julia set of a cubic polynomial is a Cantor set, and similar results in higher degrees have been proved by Qiu-Yin [21], see also [3, 20]. We shall only present the proof of proposition 4.20 here.

Proof of proposition 4.20. Let $A_\infty = \hat{\mathbb{C}} - K(f)$. Since ∞ is superattracting for f, it follows from Böttcher's local model that there is a local holomorphic diffeomorphism φ such that

$$\varphi(f(z)) = \varphi(z)^d$$

when $|z| > C$, where $d = \deg f$. The function $G(z) = \log|\varphi(z)|$ is called *Green's function* for f. Note that $G(z)$ solves *Green's functional equation*

$$G(f(z)) = d \cdot G(z).$$

We can extend G to $\mathbb{C}$ by setting $G(z) = 0$ for $z \in K(f)$ and $G(z) = G(f^n(z))/d^n$ for $z \in A_\infty$, where n is big enough so that $|f^n(z)| > C$.

Suppose first that $K(f)$ contains all critical points. For each $r > 0$, define

$$V(r) = \{z \in \mathbb{C} : G(z) > r\} \subset A_\infty.$$

Then f maps $V(r)$ in a d-to-one fashion onto $V(dr)$. We can then extend φ analytically from $V(r)$ to $V(r/d)$ by setting $\varphi(z) = \varphi(f(z))^{1/d}$. Note that

$$A_\infty = \bigcup_{r>0} V(r) \cup \{\infty\}$$

Setting $\varphi(\infty) = \infty$, we can extend φ to A_∞ so that φ maps A_∞ conformally onto $\hat{\mathbb{C}} - D$ where D is the closed disk. Then A_∞ is simply connected, so its boundary $J(f)$ is connected.

Otherwise, we would encounter a $r_0 > 0$ where we cannot extend φ past $V(r_0)$, and the level set $\partial V(r_0) = \{G(z) = r_0\}$ contains a critical point z_0 of f. As z approaches z_0 from different directions in $\partial V(r_0)$, $\varphi(z)$ approaches at least two different values, so $\partial V(r_0)$ contains at least two simple closed curves that meet at z_0. The interior of each curve must contain points in $J(f)$, since otherwise $G(z)$ would be harmonic and positive, and hence constant in the interior. This shows J is disconnected. □

5. Quasiconformal Surgery

In this section, we discuss the method of quasiconformal surgery, introduced by Shishikura in [23] to obtain a sharper bound of $2d-2$ for theorem 4.18. More specifically, Shishikura obtained the following.

Theorem 5.1. *Denote by $z_0, \ldots, z_N$ all non-repelling periodic points of f. There exist, for $0 < \varepsilon < \varepsilon_0$, rational functions f_ε of degree d and points $z_0^\varepsilon, \ldots, z_N^\varepsilon$ of $\hat{\mathbb{C}}$ such that*

(1) *When $\varepsilon \to 0$, there is $f_\varepsilon \to f$ uniformly and $z_i^\varepsilon \to z_i$ with respect to the spherical metric of $\hat{\mathbb{C}}$.*
(2) *If $f(z_i) = z_j$, then $f_\varepsilon(z_i^\varepsilon) = z_j^\varepsilon$. Each z_i^ε is an attractive periodic point of f_ε with the same period as z_i.*

Therefore, the number of attracting and indifferent cycles of f is less than or equal to the number of attracting cycles of f_ε.

Intuitively, condition (1) of theorem 5.1 means f_ε is a *perturbation*, and condition (2) means f_ε has the same periodic cycles as f, but the indifferent ones are perturbed into attracting ones. In light of lemma 4.19, theorem 5.1 gives a sharper bound for theorem 4.18 as a corollary.

Corollary 5.2. *The number of attracting and indifferent cycles of f is bounded above by $2d - 2$.*

To obtain the desired perturbation, Shishikura first obtained individual perturbations for cycles that are attracting, repelling, Siegel, and Cremer. Then the perturbations are combined to obtain a single one that works for all non-repelling cycles. In this section, we describe a few key lemmas introduced by Shishikura that make quasiconformal surgery possible. However, we omit the specific constructions corresponding to attracting, repelling, Siegel, and Cremer cycles, respectively.

Along the way, it is difficult to glue holomorphic functions directly because of identity theorem. Instead, we consider whether conjugations by certain homeomorphisms can be glued together, and whether the resulting map is conjugate to a holomorphic one. The homeomorphisms used are *quasiconformal maps*.

Definition 5.3 ([1, 23])**.** Let Ω and Ω' be domains of $\mathbb{C}$. A homeomorphism $\varphi : \Omega \to \Omega'$ is a K-*quasiconformal map*, or K-*qc-map*, if

(1) φ is absolutely continuous on almost all lines parallel to the real axis, and almost all lines parallel to the imaginary axis.
(2) The map $\mu_\varphi(z)$ defined by the *Beltrami equation*

$$\frac{\partial \varphi}{\partial \bar{z}} = \mu_\varphi(z) \cdot \frac{\partial \varphi}{\partial z}$$

satisfies

$$k = \sup_z |\mu_\varphi(z)| = \frac{K-1}{K+1} < 1$$

almost everywhere (with respect to the Lebesgue measure). μ_φ is called the *complex dilation* of φ.

A *quasi-regular map* is the composition of a qc-map and a holomorphic function.

Remark 5.4. Note that the partial derivatives in condition (2) exist almost everywhere because of absolute continuity in condition (1). Also, a 1-qc-map is a conformal map.

The following lemma answers our questions in the previous paragraph. One can glue quasi-regular maps that are conjugate to holomorphic maps on their individual domains into a map that is globally conjugate to a rational map by qc-map.

Lemma 5.5 (Fundamental lemma for qc-surgery [23])**.** *Let $g : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a quasi-regular map. Suppose there are disjoint open sets $E_i \subset \hat{\mathbb{C}}$, qc-maps $\Phi : E_i \to E_i' \subset \hat{\mathbb{C}}$ $(1 \leq i \leq m)$ and an integer $N \geq 0$ satisfying*

(1) $g(E) \subset E$ where $E = E_1 \cup \cdots \cup E_m$

(2) $\Phi \circ g \circ \Phi_i^{-1}$ is holomorphic in $E_i' = \Phi_i(E_i)$, where $\Phi : E \to \hat{\mathbb{C}}$ is defined by $\Phi|_{E_i} = \Phi_i$

(3) $\partial g/\partial \bar{z} = 0$ almost everywhere on $\hat{\mathbb{C}} - g^{-N}(E)$.

Then there exists a qc-map φ on $\hat{\mathbb{C}}$ such that $\varphi \circ g \circ \varphi^{-1}$ is a rational function. Moreover, $\varphi \circ \Phi_i^{-1}$ is conformal in E_i' and $\partial\varphi/\partial\bar{z} = 0$ almost everywhere on $\hat{\mathbb{C}} - \bigcup_{n\geq 0} g^{-n}(E)$.

Proof. We first define a new measurable conformal structure σ on $\hat{\mathbb{C}}$. Let σ_0 be the standard conformal structure defined by $|dz|$. Set $\sigma = \Phi^*\sigma_0$ on E. Now $g^*\sigma = \sigma$ on E by assumption (2), so we may extend σ to $\bigcup_{n\geq 0} g^{-n}(E)$. Finally, set $\sigma = \sigma_0$ elsewhere. Then σ satisfies $g^*\sigma = \sigma$ by construction and condition (3).

Now suppose Φ is K_1-qc and g is K_2-quasi-regular, and $\sigma = |dz + \mu \cdot d\bar{z}|$ a.e. Then we can derive that

$$\sup_z |\mu(z)| \leq k = \frac{K-1}{K+1} < 1$$

where $K = K_1 K_2^N$. It follows from measurable Riemann mapping theorem A.5 that there exists a K-qc-map φ with dilation μ, so $\varphi^*\sigma_0 = \sigma$ a.e. Then $f = \varphi \circ g \circ \varphi^{-1}$ satisfies $\partial f/\partial\bar{z} = 0$ a.e. on $\hat{\mathbb{C}}$. It follows from elliptic theory of the Cauchy-Riemann operator that f is a holomorphic function away from finitely many poles, so f is rational. ☐

Furthermore, we can perform the conjugation in a family fashion to obtain a perturbation into maps that are conjugate to rational functions. We first set up a preliminary lemma.

Lemma 5.6. *Let $h(z)$ be a polynomial of degree k. Define $H_\varepsilon : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ for $\varepsilon \in \mathbb{C}$ by*

$$H_\varepsilon(z) = z + \varepsilon \cdot h(z) \cdot \rho(|\varepsilon|^{1/k} \cdot |z|)$$

and $H_\varepsilon(\infty) = \infty$. Here ρ is a smooth function on $\mathbb{R}$ such that $0 \leq \rho \leq 1$, $\rho = 1$ on $[0, 1]$ and $\rho = 0$ on $[2, \infty)$. Then for small ε, the map H_ε is a qc-map. Furthermore, $H_\varepsilon \to \mathrm{id}_{\hat{\mathbb{C}}}$ uniformly with respect to the spherical metric and $\sup|\mu_{H_\varepsilon}| \to 0$ as $\varepsilon \to 0$.

Proof. It follows from definition of ρ that H_ε is holomorphic when $|z| < |\varepsilon|^{-1/k}$ or $|z| > 2|\varepsilon|^{-1/k}$, so $\mu_{H_\varepsilon} = 0$ in these cases. When $|\varepsilon|^{-1/k} < |z| < 2|\varepsilon|^{-1/k}$, one may calculate that

$$\frac{1}{|\mu_{H_\varepsilon}|} \geq 1 + \frac{1 - C_2|\varepsilon|}{C_1|\varepsilon|}$$

where C_1 and C_2 are bounds that only depend on h and h' and hold when ε is small. Thus we see that when $|\varepsilon| < 1/C_2$, the map H_ε is a qc-map and $\sup |\mu_{H_\varepsilon}| \to 0$ as $\varepsilon \to 0$. Furthermore,

$$|H_\varepsilon(z) - z| = |\varepsilon \cdot h(z) \cdot \rho(|\varepsilon|^{1/k} \cdot |z|)| \leq C_1|\varepsilon|$$

and thus $H_\varepsilon \to \mathrm{id}_{\hat{\mathbb{C}}}$ uniformly as $\varepsilon \to 0$. □

Lemma 5.7. *Suppose a polynomial $h(z)$ and open sets $E_\varepsilon \subset \hat{\mathbb{C}}$ for small ε satisfy*

(1) $E_0 \subset E_\varepsilon$ and E_ε are uniformly bounded in $\hat{\mathbb{C}}$.
(2) $f(\infty) \in E_0$
(3) $f \circ (\mathrm{id} + \varepsilon \cdot h)(E_\varepsilon) \subset E_\varepsilon$.

Set $g_\varepsilon = f \circ H_\varepsilon$. Then for small $\varepsilon > 0$, there exist qc-maps φ_ε on $\hat{\mathbb{C}}$ such that $f_\varepsilon := \varphi_\varepsilon \circ g_\varepsilon \circ \varphi_\varepsilon^{-1}$ are rational functions and $\varphi_\varepsilon \to \mathrm{id}_{\hat{\mathbb{C}}}$ and $f_\varepsilon \to f$ uniformly when $\varepsilon \to 0$.

Proof. We hope to apply lemma 5.5 for each ε to the data g_ε, E_ε, $\Phi = \mathrm{id}_{E_\varepsilon}$, and $N = 1$. When ε is small enough, the map g_ε is quasi-regular by lemma 5.6. Put $V_\varepsilon = \{z \in \hat{\mathbb{C}} : |z| > |\varepsilon|^{-1/k}\}$. Because E_ε is uniformly bounded, $E_\varepsilon \cap V_\varepsilon = \varnothing$ for small ε. Then $H_\varepsilon(z) = z + \varepsilon h(z)$ for all $z \in E_\varepsilon$ and

$$g_\varepsilon(E_\varepsilon) = f \circ (\mathrm{id} + \varepsilon \cdot h)(E_\varepsilon) \subset E_\varepsilon.$$

Moreover, $\Phi \circ g_\varepsilon \circ \Phi^{-1} = g_\varepsilon = f \circ (\mathrm{id} + \varepsilon \cdot h)$ on E_ε, which is holomorphic. Finally, since $f(\infty) \in E_0$, when ε is small we know $g_\varepsilon(\bar{V}_\varepsilon) \subset E_0$, and thus $\partial g_\varepsilon / \partial \bar{z} = 0$ on $\hat{\mathbb{C}} - g^{-1}(E_\varepsilon) \subset \hat{\mathbb{C}} - \bar{V}_\varepsilon$. We can now apply lemma 5.5 to obtain φ_ε. The continuous dependence of φ_ε on ε follows from the parametrized measurable Riemann mapping theorem A.5 and the proof of lemma 5.5. □

To prove theorem 5.1, Shishikura constructs the data of h and E_ε individually for each non-repelling cycle, according to whether it is attracting, rationally indifferent, Siegel, or Cremer. Finally, in order to apply lemma 5.7 to f with a single h and E_ε, the following auxiliary lemma is used.

Lemma 5.8. *Let $\zeta_1, \ldots, \zeta_m$ be distinct points in $\mathbb{C}$, and $B_1, \ldots, B_m$ pairwise disjoint closed sets in $\mathbb{C}$ homeomorphic to the closed disk. For each j, let h_j be a holomorphic function in a neighborhood of B_j. Suppose $\zeta_i \in B_j$, $h_j(\zeta_j) = 0$, and $h_j'(\zeta_j) = -1$. Then, for any $\delta > 0$, there is a polynomial $h(z)$ such that*

$$h(\zeta_j) = 0, \quad h'(\zeta_j) = -1 \quad (1 \leq j \leq m)$$

and $|h - h_j| < \delta$ on $B_1 \cup \cdots \cup B_m$.

Proof. Let p be a polynomial satisfying

$$p(\zeta_j) = 0, \quad p'(\zeta_j) = -1 \quad (1 \le j \le m)$$

and put $q(z) = \prod_i (z - \zeta_i)^2$. Then $(h_j - p)/q$ is holomorphic in a neighborhood of B_j. By Runge's theorem A.6, there is a polynomial r such that

$$\left| \frac{h_j - p}{q} - r \right| < \frac{\delta}{\sup_{\zeta \in B_j} |q(\zeta)|} \quad \text{on } B_j.$$

Then $h(z) = p + qr$ is the desired polynomial. □

Appendix A. List of Theorems

In this appendix, we state several important theorems used in this thesis, without proof.

Theorem A.1 (Uniformization)**.** *Any simply connected Riemann surface is conformally isomorphic to either $\mathbb{C}$, the closed disk $D \subset \mathbb{C}$, or $\hat{\mathbb{C}}$.*

Theorem A.2 (Picard)**.** *Any holomorphic map from $\mathbb{C}$ to $\hat{\mathbb{C}}$ which omits three different values must be constant. More generally, if the Riemann surface S admits some non-constant holomorphic map to the thrice punctured sphere Σ_3, then S must be Hyperbolic.*

Theorem A.3 (Schwarz-Pick)**.** *If $f : S \to T$ is a holomorphic map between hyperbolic surfaces, then*

$$\rho_T(f(z), f(w)) \le \rho_S(z, w)$$

where ρ denotes the hyperbolic metric on corresponding surfaces. Furthermore, if equality holds for some $z \ne w$, then f is a local isometry. That is, f preserves the infinitesimal distance element ds, and hence preserves the distances between nearby points.

Theorem A.4 (Montel)**.** *Let S be any Riemann surface. If a collection $\mathcal{F}$ of holomorphic maps from S to $\hat{\mathbb{C}}$ misses at least three points, then $\mathcal{F}$ is normal.*

Theorem A.5 (Measurable Riemann Mapping Theorem [1, 2])**.** *For any measurable μ with* $\sup \mu < 1$, *there exists a unique solution f to the Beltrami equation*

$$\frac{\partial f}{\partial \bar{z}} = \mu(z) \cdot \frac{\partial f}{\partial z}$$

that is a qc-map on $\mathbb{C}$ and fixes the points 0, 1, and ∞. Furthermore, if μ depends holomorphically on a parameter $\lambda \in U$, the f also depends holomorphically on the parameter $\lambda \in U$.

Theorem A.6 (Runge)**.** *Suppose that K is a compact subset of $\mathbb{C}$, and that f is a function taking complex values which is holomorphic on some domain Ω containing K. Suppose that $\mathbb{C} - K$ is path-connected. Then f is uniformly approximable by polynomials.*

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