Sequence and recurrence

Flydexo

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1 Reasoning by recurrence

Theorem: $P(n)$ a property, we suppose that:

- $P(0)$ is true
- if $P(n)$ is true so $P(n+1)$ is true

So for every n, $P(n)$ is true

Theorem: Reasoning by recurrence from a certain rank $n_0 \in \mathbb{N}$, $n \geq n_0$, we suppose that:

- $P(n_0)$ is true
- if $n \geq n_0$, $P(n)$ is true so $P(n+1)$ is true

So for every $n \geq n_0$, $P(n)$ is true

Steps to demonstrate by recurrence with $n \geq n_0$

1. Initialization, verification of the truth of the property for $n = n_0$
2. Heredity, we suppose $P(n)$ is true, demonstrate that $P(n+1)$ is true
3. Conclusion, we conclude that $P(n)$ is true for all $n \geq n_0$

Property: Inequality of Bernoulli

$$a \in \mathbb{R}^*, n \in \mathbb{N}, (1+a)^n \geq 1 + na$$

2 Limit of a sequence

Definition: Sequence diverging to infinity (u_n) tends to $+\infty$ when n tends to $+\infty$, if for every real: $A > 0$ the interval $]A; +\infty[$ includes all the terms of the sequence from a certain rank. We say that (u_n) diverges

$$\lim_{n \to +\infty} u_n = +\infty$$

(u_n) tends to $-\infty$ when n tends to $+\infty$, if for every real $A > 0$ the interval $]-\infty; -A[$ includes all the terms of the sequence from a certain rank. (u_n) diverges.

$$\lim_{n \to +\infty} u_n = -\infty$$

Definition: Sequence converging a real number (u_n) tends to the real l when n tends to $+\infty$ if all the opened interval including l includes all the terms of the sequence from a certain rank. (u_n) converges

$$\lim_{n \to +\infty} u_n = l$$

Theorem: Unicity of the limit If a limit exists, it is unique.

3 Property of limits

Property: Limits of the reference sequences $\sqrt{n}, n$ and n^k where $k \in \mathbb{R}^*$ have

$$\lim_{n \to +\infty} = +\infty$$

$\frac{1}{\sqrt{n}}, \frac{1}{n}, \frac{1}{n^k}$ have

$$\lim_{n \to +\infty} = 0$$

Property: Sum and products of limits

(u_n) a pour limite	(v_n) a pour limite	(u_n+v_n) a pour limite	$(u_n \times v_n)$ a pour limite
ℓ	ℓ'	$\ell+\ell'$	$\ell \times \ell'$
ℓ	$+\infty$	$+\infty$	$+\infty$ si $\ell>0$ $-\infty$ si $\ell<0$ **indéterminée** si $\ell=0$
ℓ	$-\infty$	$-\infty$	$-\infty$ si $\ell>0$ $+\infty$ si $\ell<0$ **indéterminée** si $\ell=0$
$+\infty$	$+\infty$	$+\infty$	$+\infty$
$+\infty$	$-\infty$	**indéterminée**	$-\infty$
$-\infty$	$-\infty$	$-\infty$	$+\infty$
$-\infty$	$+\infty$	**indéterminée**	$-\infty$

Property: **Quotient of limits**

(u_n) a pour limite	(v_n) a pour limite	$\left(\frac{u_n}{v_n}\right)$ a pour limite
ℓ	$\ell' \neq 0$	$\frac{\ell}{\ell'}$
ℓ	$+\infty$ ou $-\infty$	0
$\ell \neq 0$	0^+	$+\infty$ si $\ell>0$ $-\infty$ si $\ell<0$
	0^-	$-\infty$ si $\ell>0$ $+\infty$ si $\ell<0$
$+\infty$	ℓ'	$+\infty$ si $\ell'>0$ ou $\ell'=0^+$ $-\infty$ si $\ell'<0$ ou $\ell'=0^-$
$-\infty$	ℓ'	$-\infty$ si $\ell'>0$ ou $\ell'=0^+$ $+\infty$ si $\ell'<0$ ou $\ell'=0^-$
$\pm\infty$	$\pm\infty$	**indéterminée**
0	0	**indéterminée**

4 Limit and comparison

<u>Theorem</u>: Theorem of comparison (u_n) and (v_n), $n \geq n_0$, $u_n \leq v_n$

- If $\lim_{n\to+\infty} u_n = +\infty$ then $\lim_{n\to+\infty} v_n = +\infty$
- If $\lim_{n\to+\infty} v_n = -\infty$ then $\lim_{n\to+\infty} u_n = -\infty$

<u>Theorem</u>: Theorem of gendarmes $(u_n), (v_n), (w_n), l$ a real.
We suppose that:

- n_0 exists, such as, $n \geq n_0, v_n \leq u_n \leq w_n$
- $\lim_{n\to+\infty} v_n = \lim_{n\to+\infty} w_n = l$

So the sequence (u_n) converges and $\lim_{n\to+\infty} u_n = l$

<u>Property</u>: Inequalities and limits Let (u_n) and (v_n) two sequences converging. We suppose n_0, such as $n \geq n_0$, $u_n \leq v_n$. So $\lim_{n\to+\infty} u_n \leq \lim_{n\to+\infty} v_n$

5 Geometrical sequences and monotone sequences

<u>Property</u>: Limit of a geometrical sequence Let q a real.

- If $q > 1$, so $\lim_{n\to+\infty} q^n = +\infty$
- If $q = 1$, so $\lim_{n\to+\infty} q^n = 1$
- If $-1 < q < 1$, so $\lim_{n\to+\infty} q^n = 0$
- If $q \leq -1$, so the sequence (q^n) has no limit

<u>Definition</u>: Increased, understated, bounded sequence Let (u_n) be a sequence defined from rank k.

- We say that (u_n) is increased if a real M exists such as for every integer, $n \geq k, u_n \leq M$
- We say that (u_n) is understated if a real m exists such as for every integer, $n \geq k, u_n \geq m$
- We say that (u_n) is bounded if (u_n) is understated and increased.

<u>Property</u>: Convergence of a monotone sequence

1. Every increasing increased sequence converges
2. Every increasing non-increased sequence diverges to $+\infty$
3. Every decreasing understated sequence converges
4. Every decreasing non-understated sequence diverges to $-\infty$