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Homework 1

Problem 1

(a) Since $\frac{\partial}{\partial \mathbf{x}_i}\|\mathbf{x}\|_2 = \frac{\partial}{\partial \mathbf{x}_i}\sqrt{\mathbf{x}_1^2+\mathbf{x}_2^2} = \frac{\mathbf{x}_i}{\sqrt{\mathbf{x}_1^2+\mathbf{x}_2^2}} = \frac{\mathbf{x}_i}{\|\mathbf{x}\|_2}$, we have $\frac{\mathrm{d}}{\mathrm{d}\mathbf{x}}\|\mathbf{x}\|_2 = \frac{\mathbf{x}}{\|\mathbf{x}\|_2}$. Thus

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}h}s[\gamma+h\mathbf{v}]\Big|_{h=0} &= \frac{\mathrm{d}}{\mathrm{d}h}\int_0^1 \|\gamma'(t)+h\mathbf{v}'(t)\|_2\mathrm{d}t\Big|_{h=0} = \int_0^1 \frac{\partial}{\partial h}\|\gamma'(t)+h\mathbf{v}'(t)\|_2\Big|_{h=0}\mathrm{d}t \\ &= \int_0^1 \frac{\gamma'(t)+h\mathbf{v}'(t)}{\|\gamma'(t)+h\mathbf{v}'(t)\|_2}\cdot\mathbf{v}'(t)\Big|_{h=0}\mathrm{d}t = \int_0^1 \frac{\gamma'(t)}{\|\gamma'(t)\|_2}\cdot\mathbf{v}'(t)\mathrm{d}t.\end{aligned}$$

(b) Recall that the directional derivation of a multivariate scalar function is defined by

$$\nabla_{\mathbf{v}}f(\mathbf{x}) = \frac{\mathrm{d}}{\mathrm{d}h}f(\mathbf{x}+h\mathbf{v})\Big|_{h=0} = \nabla f(\mathbf{x})\cdot\mathbf{v}.$$

If we view $\gamma(t)$ and $\mathbf{v}(t)$ as 'vectors' in the (infinite dimensional) linear space of smooth functions $\mathcal{C}^\infty([0,1])$ and the arc length s as a function(al) from $\mathcal{C}^\infty([0,1])$ to $\mathbb{R}_{\geqslant 0}$, then $\frac{\mathrm{d}}{\mathrm{d}h}s[\gamma+h\mathbf{v}]\big|_{h=0}$ just seems to be in a similar form as directional derivatives.

(c) For simplicity, we denote the parameterization by arc length $\gamma(\bar{s}) := \gamma\left(s^{-1}(\bar{s})\right)$ and $\gamma'(\bar{s}) := \frac{\mathrm{d}}{\mathrm{d}\bar{s}}\gamma\left(s^{-1}(\bar{s})\right)$, and similarly, $\mathbf{v}(\bar{s}) := \mathbf{v}\left(s^{-1}(\bar{s})\right)$, $\mathbf{v}'(\bar{s}) := \frac{\mathrm{d}}{\mathrm{d}\bar{s}}\mathbf{v}\left(s^{-1}(\bar{s})\right)$. By (a) (note that the expression in (a) holds for any parameterization of γ, specifically, it holds for the arc length parameterization), and using the fact that $\|\gamma'(\bar{s})\|_2 = 1$, we have

$$\begin{aligned}&\int_0^{s(1)}\mathbf{v}(\bar{s})\cdot\mathbf{w}(\bar{s})\mathrm{d}\bar{s} = \frac{\mathrm{d}}{\mathrm{d}h}s[\gamma+h\mathbf{v}]\Big|_{h=0} = \int_0^{s(1)}\frac{\gamma'(\bar{s})}{\|\gamma'(\bar{s})\|_2}\cdot\mathbf{v}'(\bar{s})\mathrm{d}\bar{s} \\ =&\int_0^{s(1)}\gamma'(\bar{s})\cdot\mathbf{v}'(\bar{s})\mathrm{d}\bar{s} = \gamma'(\bar{s})\cdot\mathbf{v}(\bar{s})\Big|_{\bar{s}=0}^{\bar{s}=s(1)} - \int_0^{s(1)}\gamma''(\bar{s})\cdot\mathbf{v}(\bar{s})\mathrm{d}\bar{s} = -\int_0^{s(1)}\gamma''(\bar{s})\cdot\mathbf{v}(\bar{s})\mathrm{d}\bar{s}.\end{aligned}$$

Since this equality holds for any $\mathbf{v}$, we have $\mathbf{w}(\bar{s}) = -\gamma''(\bar{s}) = -\kappa(\bar{s})\mathbf{n}(\bar{s})$, where κ is the curvature and $\mathbf{n}$ is the normal vector.

Problem 2

(a) We may define $s(\mathbf{x}) = \sum_{j=1}^{n-1}\|\mathbf{x}_{j+1}-\mathbf{x}_j\|_2$.

(b) According to 2(a) we have

$$\nabla_{\mathbf{x}_i}s(\mathbf{x}) = \nabla_{\mathbf{x}_i}\|\mathbf{x}_{i+1}-\mathbf{x}_i\|_2 + \nabla_{\mathbf{x}_i}\|\mathbf{x}_i-\mathbf{x}_{i-1}\|_2 = -\frac{\mathbf{x}_{i+1}-\mathbf{x}_i}{\|\mathbf{x}_{i+1}-\mathbf{x}_i\|_2} + \frac{\mathbf{x}_i-\mathbf{x}_{i-1}}{\|\mathbf{x}_i-\mathbf{x}_{i-1}\|_2}.$$

$\nabla_{\mathbf{x}_i}s(\mathbf{x})$ can be viewed as the summation of two unit vectors of direction $\mathbf{x}_i-\mathbf{x}_{i+1}$ and $\mathbf{x}_i-\mathbf{x}_{i-1}$, and thus has norm $2\sin\frac{\theta}{2}$.

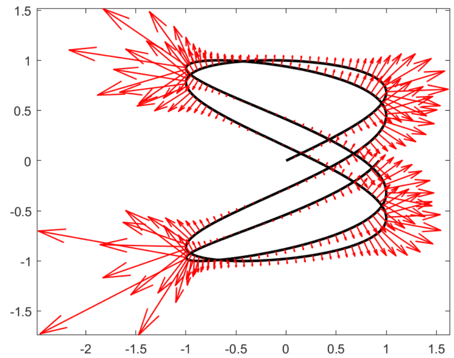


Figure 1: Derivative of arc length at each vertex (vectors in red), with $n = 300$ points sampled on the curve. The length of each vector is enlarged by ten times in order to make the plot clearer.

(c) Visualizing the derivative computed in 2(b), we get Figure 1.

(d) By 1(c), the discrete curvature $\kappa(\mathbf{x}_i)$ shall satisfy

$$\begin{aligned}\nabla s(\mathbf{x}) \cdot \mathbf{v} = \sum_{i=1}^{n} \nabla_{\mathbf{x}_i} s(\mathbf{x}) \cdot \mathbf{v}_i &= \frac{\mathrm{d}}{\mathrm{d}h} s(\mathbf{x} + h\mathbf{v})\Big|_{h=0} \\ &= -\sum_{i=1}^{n} \kappa(\mathbf{x}_i) \left(\frac{1}{2}\|\mathbf{x}_{i+1} - \mathbf{x}_i\| + \frac{1}{2}\|\mathbf{x}_i - \mathbf{x}_{i-1}\| \right) \mathbf{n}(\mathbf{x}_i) \cdot \mathbf{v}_i\end{aligned}$$

for any $\mathbf{v} \in \mathbb{R}^{2n}$, where $\mathbf{v}_i \in \mathbb{R}^2$ is the i-th component of $\mathbf{v}$ corresponding to $\mathbf{x}_i$. Here the last equality is by assuming that the curvature is locally constant in the neighbourhood of each $\mathbf{x}_i$, namely constant on half of the two edges connected to $\mathbf{x}_i$.

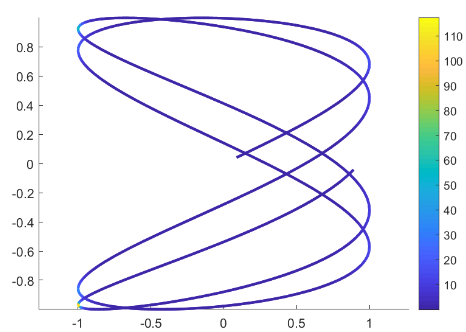


Figure 2: Discrete (unsigned) per-vertex curvature of a 2D discrete curve, with $n = 300$ points sampled on the curve.

Now an appropriate definition could be given by

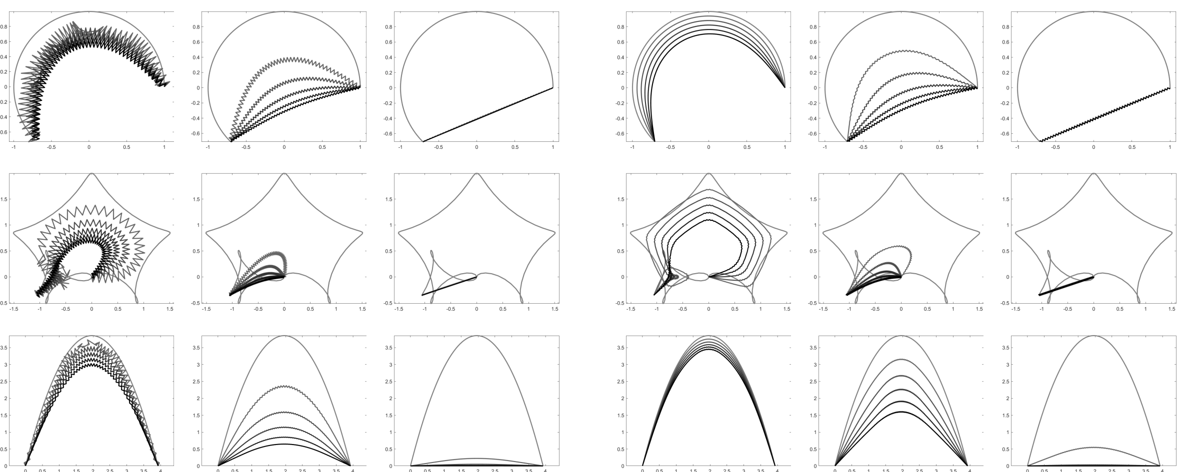
$$\nabla_{\mathbf{x}_i} s(\mathbf{x}) = \kappa(\mathbf{x}_i)\mathbf{n}(\mathbf{x}_i)\left(\frac{1}{2}\|\mathbf{x}_{i+1} - \mathbf{x}_i\| + \frac{1}{2}\|\mathbf{x}_i - \mathbf{x}_{i-1}\|\right),$$

and thus

$$\kappa(\mathbf{x}_i) = \frac{2\|\nabla_{\mathbf{x}_i} s(\mathbf{x})\|}{\|\mathbf{x}_{i+1} - \mathbf{x}_i\| + \|\mathbf{x}_i - \mathbf{x}_{i-1}\|} = \frac{4\sin\frac{\theta}{2}}{\|\mathbf{x}_{i+1} - \mathbf{x}_i\| + \|\mathbf{x}_i - \mathbf{x}_{i-1}\|},$$

where θ is the angle shown in the figure in 2(b). Visualizing this discrete curvature, we get Figure 2.

(e) If h is too large the iteration will not converge. In our implementation we tried $h = 1/\sqrt{k}$ (at the k-th step) and $h = 0.015$ respectively. The results are shown in Figure 3 and the supplemental video. From the figure we observe that with $h = 1/\sqrt{k}$ it converges faster, but less stable at the beginning - the curve becomes serrated at first but finally becomes smoother and converges to a straight line. For $h = 0.015$, the curve is smooth at the beginning, but it gradually becomes unstable and the convergence rate is relatively slow.



500 steps | 5000 steps | 50000 steps | 500 steps | 5000 steps | 5000 steps

(a) $h = 1/\sqrt{k}$ | (b) $h = 0.015$

Figure 3: Gradient descent on arc length, with $n = 300$ points sampled on the curve. Darker curves are generated later than then the lighter ones.

Problem 3

(a) Visualizing the curvature binormal at each vertex, we get Figure 4.

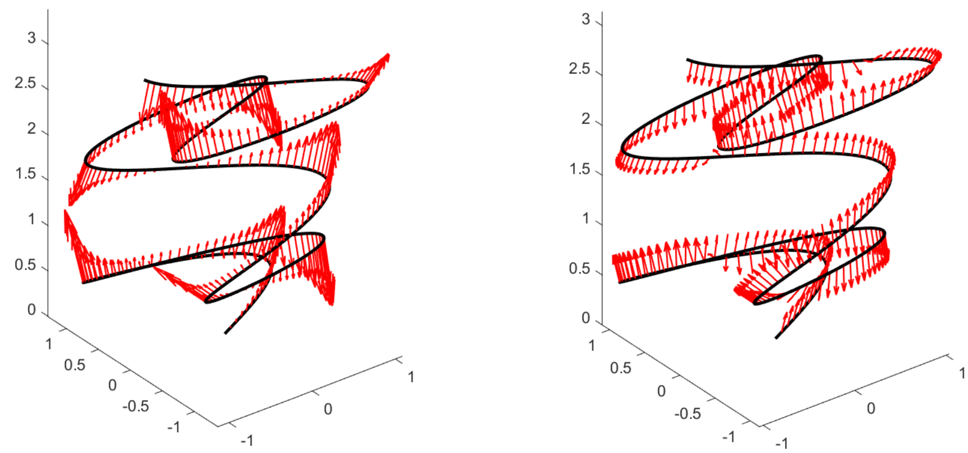
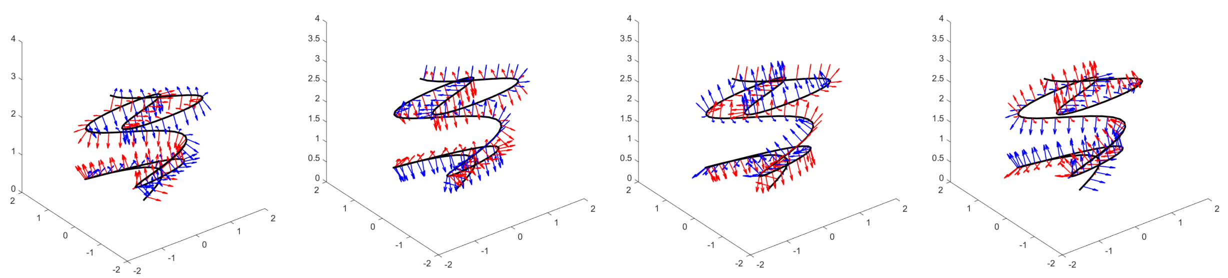


Figure 4: The discrete curvature binormal (vectors in red), with $n = 300$ points sampled on the curve. Left: The Darboux vector (curvature binormal) $(\kappa \mathbf{b})_i$ at each vertex i. The length of each vector is enlarged by three times in order to make the plot clearer. Right: Direction of the binormal vector.

(b) By modifying the value of `theta` code, we get the Bishop frame $(\mathbf{u}, \mathbf{v}, \mathbf{t})$ with different choices (Figure 5 and the supplemental video).



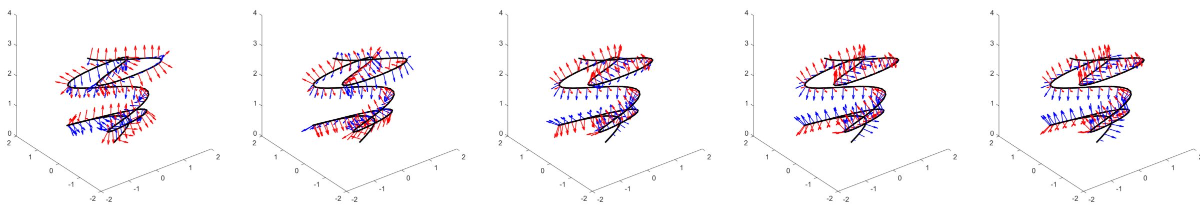
`theta`$= 0°$ `theta`$= 90°$ `theta`$= 180°$ `theta`$= 270°$

Figure 5: Bishop frames with different initial choices, with $n = 100$ points sampled on the curve.

(c) We apply gradient descent to the twist energy on the angle between material frame and Bishop frame

$$E_{twist} = \sum_{i=1}^{n} \beta \frac{(\theta_i - \theta_{i-1})^2}{|\boldsymbol{e}^{i-1}| + |\boldsymbol{e}^i|} = \sum_{i=1}^{n} \beta \frac{m_i^2}{\bar{l}_i}.$$

In our implementation, we set $\beta = 1$ and `stepSize`$= 0.038$ (which seems to be the maximal step size that could guarantee the convergence). We notice that the material frame gradually become untwisted and finally aligns with the Bishop frame (Figure 6, roughly compared with `theta`$= 270°$ in Figure 5, and the supplemental video).



1 step | 5000 steps | 10000 steps | 15000 steps | 20000 steps

Figure 6: Gradient descent on the angle between material frame and Bishop frame, with $n = 100$ points sampled on the curve.