Lab00-Proof

CS214-Algorithm and Complexity, Xiaofeng Gao, Spring 2021.

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1. Prove that for any integer $n > 2$, there is a prime p satisfying $n < p < n!$. (Hint: consider a prime factor p of $n! - 1$ and prove by contradiction)

   Proof. Consider the number $n! - 1$.

   If $n! - 1$ is a prime, then the proposition is proved.

   Otherwise, if $n! - 1$ is a composite number, then $\exists$ a prime number p which is a prime factor of $n! - 1$.

   If $p > n$, then the proposition is proved.

   Otherwise, if $p \leqslant n$, then p is a prime factor of $n!$.

   Now that $p|n!$ and $p|n! - 1$, we get $p|1$, which is obviously impossible. □

2. Use the minimal counterexample principle to prove that for any integer $n \geq 7$, there exists integers $i_n \geq 0$ and $j_n \geq 0$, such that $n = i_n \times 2 + j_n \times 3$.

   Proof. If there are values of n for which there does not exists such i_n and j_n, then there must be a smallest such value, say $n = k$.

   Since $0 \times 2 + 0 \times 3 = 0$, we have $i_n \geqslant 1$ or $j_n \geqslant 1$.

   Since k is the smallest value that cannot be written in that form, then $k - 1$ can be written in that form, which means there exists integers $i_{k-1} \geqslant 0$ and $j_{k-1} \geqslant 0$, such that $k - 1 = i_{k-1} \times 2 + j_{k-1} \times 3$.

   However, we have

$$\begin{aligned} k &= k - 1 + 1 \\ &= i_{k-1} \times 2 + j_{k-1} \times 3 + 3 - 2 \\ &= (i_{k-1} - 1) \times 2 + (j_{k-1} + 1) \times 3 \end{aligned}$$

   and

$$\begin{aligned} k &= k - 1 + 1 \\ &= i_{k-1} \times 2 + j_{k-1} \times 3 + 2 \times 2 - 3 \\ &= (i_{k-1} + 2) \times 2 + (j_{k-1} - 1) \times 3 \end{aligned}$$

   Since at least one of i_{k-1} and j_{k-1} is not 0, we can make sure k can be written in that form as well. We have derived a contradiction, which allows us to conclude that our original assumption is false. □

3. Suppose the function f be defined on the natural numbers recursively as follows: $f(0) = 0$, $f(1) = 1$, and $f(n) = 5f(n-1) - 6f(n-2)$, for $n \geq 2$. Use the strong principle of mathematical induction to prove that for all $n \in N$, $f(n) = 3^n - 2^n$.

Proof. We proof the proposition is true for $n \geqslant 0$ by induction.

Basis step. When $n = 0$, $f(0) = 3^0 - 2^0 = 0$, and the proposition is obviously true.

Introduction Hypothesis. Assume when $0 \leqslant i \leqslant k$ the proposition is true, which means $f(i) = 3^i - 2^i$.

Proof of Induction Step. Now let us prove that when $n = k + 1$ the proposition is true.

$$\begin{aligned} f(k+1) &= 5f(k) - 6f(k-1) \\ &= 5 \times (3^k - 2^k) - 6 \times (3^{k-1} - 2^{k-1}) \\ &= (5 - 6 \div 3) \times 3^k - (5 - 6 \div 2) \times 2^k \\ &= 3 \times 3^k - 2 \times 2^k \\ &= 3^{k+1} - 2^{k+1} \end{aligned}$$

□

4. An n-team basketball tournament consists of some set of $n \geq 2$ teams. Team p beats team q iff q does not beat p, for all teams $p \neq q$. A sequence of distinct teams p_1, p_2,..., p_k, such that team p_i beats team p_{i+1} for $1 \leq i < k$ is called a ranking of these teams. If also team p_k beats team p_1, the ranking is called a *k-cycle*.

   Prove by mathematical induction that in every tournament, either there is a "champion" team that beats every other team, or there is a 3-cycle.

   Proof. Define $P(n)$ be the statement that"in every tournament, either there is a 'champion' team that beats every other team, or there is a 3-cycle". We try to prove that $P(n)$ is true for every $n \geqslant 2$ by induction.

   Basis step. $P(2)$ is true, since there is obviously a team beat another when there are only two teams. $P(3)$ is true as well, since when there are three teams, either a team beats the other two, or there is a 3-cycle.

   Introduction Hypothesis. Assume that $P(k)$ is true for $k \geqslant 2$.

   Proof of Induction Step. Let us prove $P(k + 1)$.

   We divide the $k + 1$ teams into two parts, with k teams in one part and only one team in the other. Since $P(k)$ is true, either there is a "champion" team that beats every other team in the first part, or there is a 3-cycle in the first part.

   If the there is a 3-cycle in the first part, then obviously the 3-cycle still exists in the $k + 1$ teams. Then $P(k + 1)$ is true.

   If there is a "champion" team that beats every other team in the first part, we can define the "champion" team as p_c in $p_1, p_2 \ldots, p_k$, and the team in the other part as p_{k+1}.

   If p_{k+1} beats every team in the first part, then p_{k+1} is the "champion" team in the $k+1$ teams, $P(k + 1)$ is true.

   If p_{k+1} does not beat all the teams in the first part, then there is a p_i beats p_{k+1}. If p_c beats p_{k+1}, then p_c is the "champion" team in the $k + 1$ teams, $P(k + 1)$ is true.

   If p_c does not beat p_{k+1},then p_i is not p_c. So p_i beats p_{k+1}, p_{k+1} beats p_c and p_c beats p_i: there is a 3-cycle, which means $P(k + 1)$ is true.

   □

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