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Project brief

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Title: Mixed Finite Element Analysis of Real and Complex Magnetodielectrical Waveguide Waves

Objective: To create an application which allows analysing dispersion characteristics of magnetodielectrical waveguide waves

Prerequisites:

Description: This project was completed as a part of collaboration program between DIT(Dublin Institute of Technology) and MSTU MIREA(Moscow State Technical University Institute of Radioengineering Electronics and Automation).

1. Project description
   1. Introduction

The purpose of this project is to implement an application which allows calculating dispersion characteristics and critical values of multilayer waveguide waves.

This application is implemented with combining of resources of Visual Studio 2010 and Matlab 7.0. These two systems are powerful enough to do calculations of this project and suitable to work with complicated structures.

The interface suitable for the user that is easy to use must be implemented in this project. The system should work in combination with Matlab 7.0+. The user should be able to use system without special preparation.

* 1. Project rationale

Waveguides are one of the means that are used to transmit the waves. According to critical wavelength, which is twice bigger than waveguide’s diameter, they are suitable for microwaves.

Power loss is small enough relatively to other types of transmission lines – that is their obvious advantage.

For more efficient usage of waveguide is a chance to improve its capacity. To do that waveguides are created with a variable index of refraction. To solve the problem of synthesis of multilayer waveguide it is necessary to calculate the dispersion curves with given refractive index. From mathematical point of view, the problem lies in the solution of Maxwell’s equations in a cylinder with variable index of refraction, which varies along the radios of cylinder.

During solving the problem can appear non-physical solutions and finite elements method helps to get rid of them.

* 1. Design approach
  2. Layout of the project

1. Finite elements analysis
   1. Common scheme

Finite elements analysis is based on idea of approximation continuous function with discrete model, which is based on a set of piecewise continuous functions defined on a finite number of subdomains called finite elements. On every element unknown function is approximated by test function (as general rule polynomial) and boundary conditions coincide with the boundary conditions of initial problem.

We consider usage of finite elements analysis on example of spectral problem in domain Ω:

A (1.1)



on domain boundary (1.2)



Then we reduce the original problem (1.1), (1.2) to a problem in the variational formulation.

To do this, we multiply on right on and take the scalar product of left and right parts of equation, finaly we have:



(A) (1.3)



on domain boundary (1.4)



So we have the problem equivalent to problem (1.1), (1.2). It is variational formulation of spectral problem and is solved in this formulation. Equality of differential equations and variational problems forms the basis of the choice of the computational scheme. Differential equations might be approximated with discrete system, using finite differences, and variational functional can be minimized on finite-dimensional space as in finite elements analysis.

We search the solution of the problem (1.3), (1.4) in form of expansion in the system of basic functions:



Here ci – coefficients of expansion, Ni(G) – basic functions.

Substitute (1.5) in (1.3) and set , then we have:



So, the problem transforms into system of algebraic equations, which looks as matrix:



where λ is eigenvalue and elements of matrices B and C are:

= (A) , = () (1.8)



Finally we have generalised problem of eigenvalues. Now we should define basic functions. In finite element analysis polynomials of different orders are used as basic hosts, they are called finite functions, which are not null in finite domains. In one-dimensional case are used equal segments, in two-dimensional – triangles and rectangles.

Further, we consider one-dimensional scalar problem on segment [a,b]. We set uniform grid {xi}: i = 0,1,..,M, x0 = a, xM = b, xi = ih, h – is grid spacing, M – number of finite elements. As finite elements are meant equal segments of which consists the original segment. Finite functions are defined be equation:

(1.9)



According to attachment x to segment [a,b] and in (1.9) t = (x-xi)/h, we have:

=

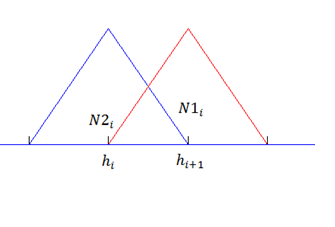


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In the aggregate these functions are designated as N, which includes both basic functions on one finite element.

Graphically it looks like:



Here are the first-order functions. In the same way are defined polynomials of second, third, etc. order. This will improve accuracy of the method, but at the same time matrices become less sparse and technical implementation of elements of higher order will be more difficult. With modern computers accuracy can be reached on the elements of the first order.

* 1. Problem statement

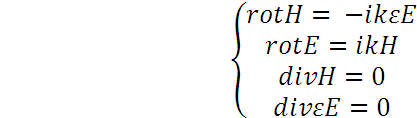
We consider cylindrical waveguide with circular cross-section Ω with unitary radius r=1. In one point on its axis we set cylindrical coordinating system, axis Oz goes along cylinder axis. Let the waveguide be filled with material with characteristics:



ε(r) is piecewise continuous in Ω, the waveguide walls are perfectly conducting.

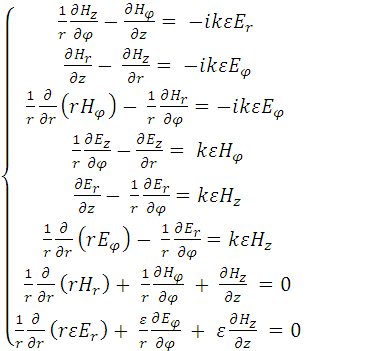
Electromagnetic field inside the waveguide is described by a system of eight Maxwell’s equations for 6 unknowns.

(2.1)



We expand the equation for rotor and rewrite them in cylindrical coordinate system:

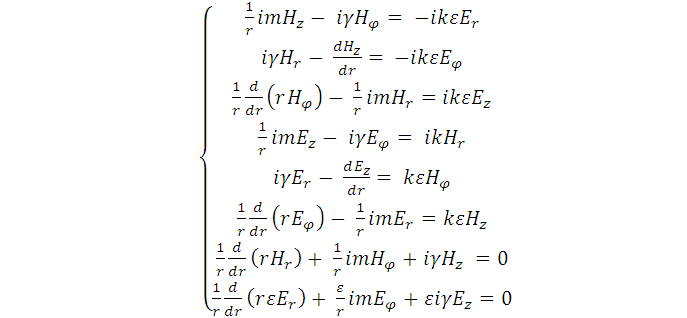
(2.2)



The solution will be in form of normal waves – functions that depend on r, z and φ:

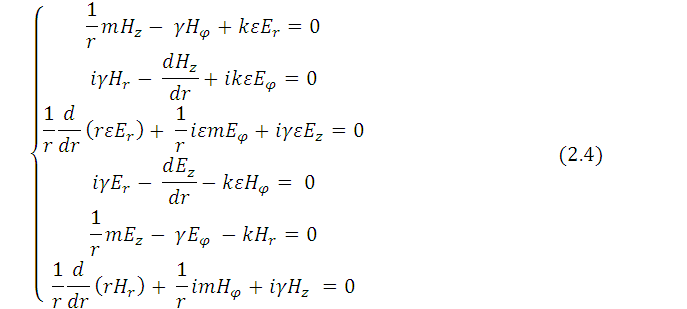
E, H = E(r)eiγz+imφ.

Substituting these normal waves in (2.2), reduce exponential factor – and we come to the problem of finding eigenvalues on segment [0; 1], where γ is eigenvalue:



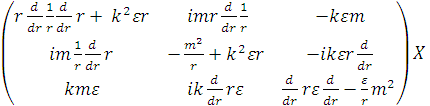
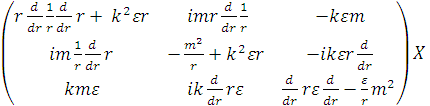
(2.3)

To solve the problem we choose 6 equations:



Then, we set X = (Hr, Hφ, Ez)T = (H┴, Ez)T, Y = (εEr, εEφ, Hz)T = (εE┴, Hz)T

If we substitute Y from first 3 equations (2.4) in last 3, we have the problem of finding eigenvalues:



(2.5)

X belongs to the set of vectors from C∞[0,1] and satisfies boundary conditions:

Hr(0) = 0 and Ez(0) = 0

Hr(1) = 0 and Ez(1) = 0,



and to Maxwell’s equation:



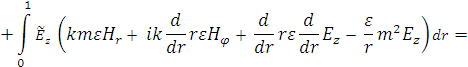
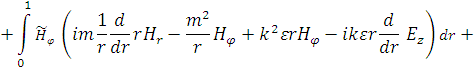
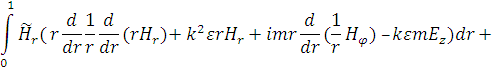
and (in case of distontinued ε) the conditions of conjugation are specified:

s = s = 0, s = s = 0

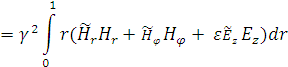


* 1. Variational functional of the problem

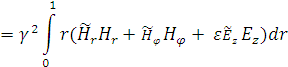
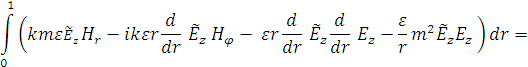
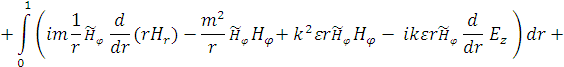
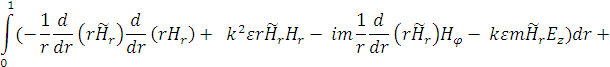
We write down the variational functional (weak formulation) for the original problem. To do this, we multiply (2.5) on the left by arbitrary vector = () and integrate over r from 0 to 1. Finally, we have:



(2.6)



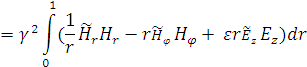
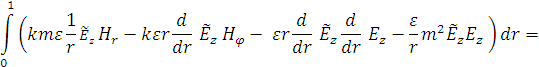
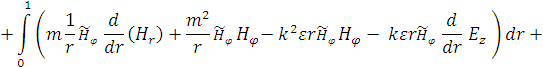
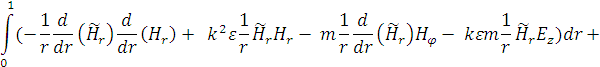
Using boundary conditions, we integrate (2.6) by parts:



We make the following changes to simplify the calculations in the future, as well as get rid of imaginary one:

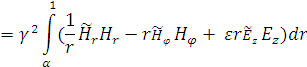
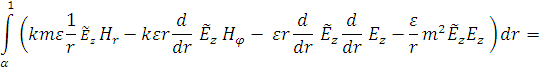
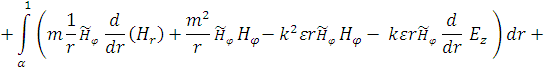
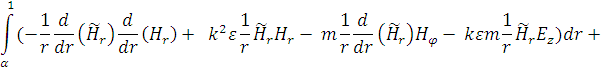


Finally, we have variational functional with complex numbers:



(2.7)

The problem (2.7) has peculiarity in 0. At integration appears indeterminacy – natural logarithm of zero. To avoid it, we put on cylinder axis thin conducting cylinder with radius α and will integrate from α:



(2.8)

Tend α to zero, then, by theorem of Samarsky, eigenvalues of the problem (2.8) will tend to eigenvalues of original problem (2.7).

The final problem is to find eigenvalues γ – the propagation constant – and to construct dispersion curves – dependence of propagation constant on the wavenumber k.

* 1. Mixed finite elements method

The problem has infinite core:

X = ()



where φ is arbitrary function.

While using standard finite elements method operators core approximates in the wrong way, that causes emergence of non-physical solutions, “spirits” of specter, which locate between genuine values – so we can’t tell the difference. Mixed finite elements analysis helps to avoid it. Method consists of approximation components of vector X with polynomials of different order. In our problem we will approximate Hr and Ez by continuous polynomials of first order and Hφ by discontinuous polynomials of zero order.

We set on segment [α, 1] uniform grid {xi}: i = 0,1,..,n, x0 = a, xn = b, xi = ih, h – is grid spacing, n – number of finite elements.

We assume permittivity to be constant on every finite element. Further, we expend in the basic functions every component of vectors X and

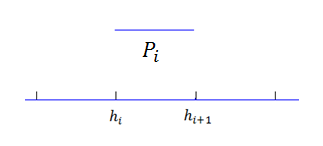
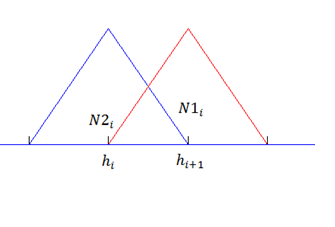


Now we have changes:

|  |  |  |
| --- | --- | --- |
|  |  |  |
|  |  |  |

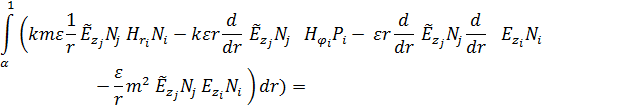
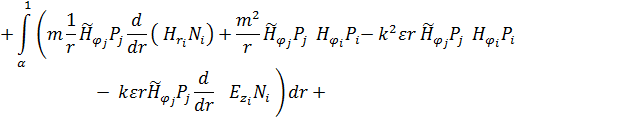
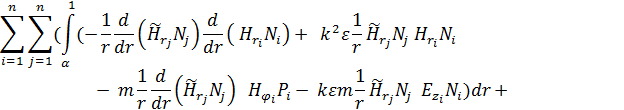
Where Ni = θN1i+σN2i on segment [hi, hi+1] (on finite element with number i).

Graphically it looks as:

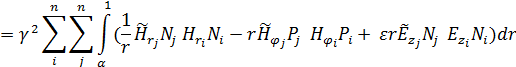


Finite element analysis allows approximate operators core, which tends into zero state, multiplicity of which is about one third of dimension of the matrix eigenvalue problem.

Applying changes, we rewrite variational functional in this way:



(2.9)



In this way have generalized problem of eigenvalues:

AX = γ2 BX

where A and B are matrices of expansion coefficients of X in the basic functions.

1. Application
2. Testing
3. Conclusion

References

1. Appendices