SimplicialSurfaces

Computation with simplicial surfaces and folding processes.

0.1

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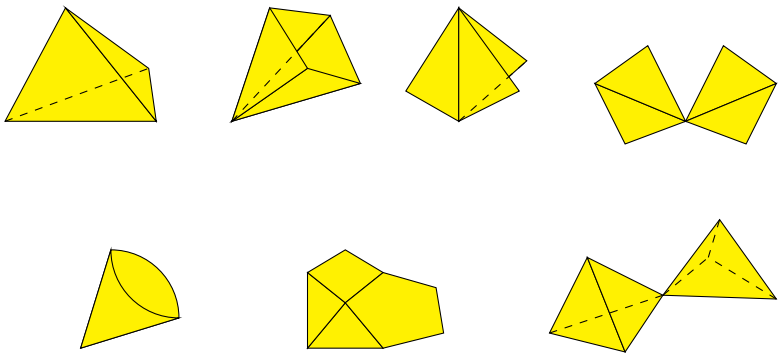
Chapter 1

Getting started

1.1 What can it do?

The `SimplicialSurface`-package contains this basic functionality:

1. It allows computations with simplicial surfaces (and generalisations of them, compare section 2), for example:



   Instead of working with an embedding of these structures, we see them as abstract surfaces and represent them by their incidence geometry (for more details see section 2.1).

2. Working with edge colourings of simplicial surfaces (in general and for the purpose of an embedding).

3. Treatment of folding and unfolding for these objects.

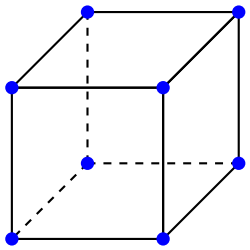
To use this package, you have to load it into GAP via

Example

```
gap> LoadPackage("SimplicialSurfaces");
true
```

1.2 Playing with simplicial surfaces

Since the platonic solids are pre-defined we use them to show a few capabilities of this package. We will use the cube as an example.



Example

```
gap> surface := Cube();;
```

We can compute elementary properties of the surface

Example

```
gap> NrOfVertices(surface);
8
gap> NrOfEdges(surface);
12
gap> NrOfFaces(surface);
6
gap> EulerCharacteristic(surface);
2
```

and we can show that the surface is homeomorphic to a sphere by verifying that it is closed, connected and orientable.

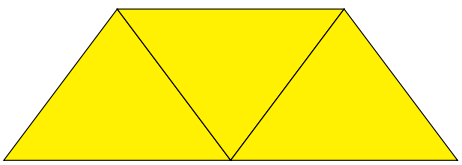
Example

```
gap> IsClosedSurface(surface);
true
gap> IsConnected(surface);
true
gap> IsOrientable(surface);
true
```

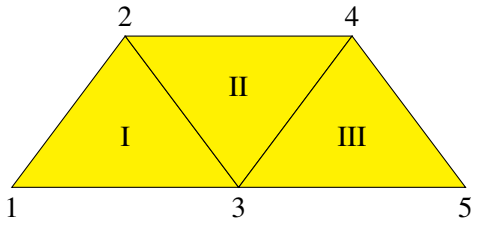
We can also compute more complicated properties like the automorphism group and check for isomorphisms between surfaces.

1.3 Constructing some surfaces

In the most cases one is not interested in the properties of platonic solids (usually one already knows a lot about them). Therefore we need a way to tell the package about the surfaces we are interested in. As a test case we consider a surface of three triangles that are connected by edges, like this:



Disregarding lengths and angles, we can describe this surface quite easily by labelling its faces and vertices. In our case each triangle is determined by its three vertices.



For example the face I consists of the vertices [1, 2, 3] and the face III has the vertices [3, 4, 5]. We can encode this information as a list with three entries (one for each face). The list entry at position p is a list of all vertices that are incident to the face with number p (their order is not important). In our example it looks like this:

Example

```
gap> verticesOfFaces := [ [1,2,3], [2,3,4], [3,5,4] ];
[ [ 1, 2, 3 ], [ 2, 3, 4 ], [ 3, 4, 5 ] ]
```

From this information we can construct a simplicial surface

Example

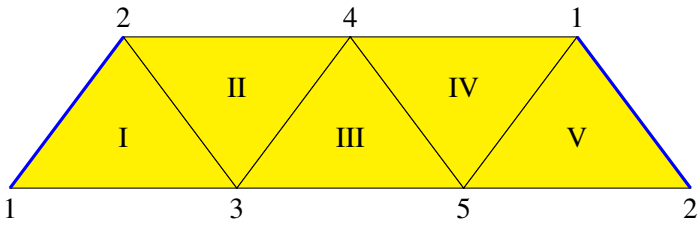
```
gap> surf := SimplicialSurfaceByVerticesInFaces( 5, 3, verticesOfFaces );;
```

that retains this information (note that the list [3, 5, 4] was internally changed into a set):

Example

```
gap> VerticesOfFaces(surf);
[ [ 1, 2, 3 ], [ 2, 3, 4 ], [ 3, 4, 5 ] ]
```

A slightly more complicated example is a Möbius-strip.



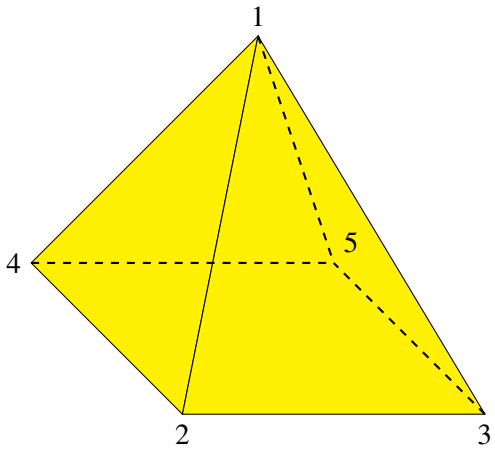
In this case the left-most and right-most edges are identified.

Example

```
gap> moebius := SimplicialSurfaceByVerticesInFaces(5,5,
>                          [[1,2,3],[2,3,4],[3,4,5],[4,5,1],[5,2,1]]);;
gap> IsOrientable(moebius);
false
```

1.3.1 Non-triangular faces

If we want to construct surfaces with non-triangular faces, the method `SimplicialSurfaceByVerticesInFaces` is a bit more subtle. We consider the example of a pyramid with a square base.



To encode this surface we have to enumerate the faces. But if we try to input the surface

Example

```
gap> pyr := SimplicialSurfaceByVerticesInFaces( 5, 5,
>                   [[2,3,4,5], [1,2,3], [1,3,5], [1,5,4],[1,2,4]] );;
```

we notice something strange: The resulting surface has the wrong number of edges (10 instead of 8) and is not closed.

Example

```
gap> NrOfEdges(pyr);
10
gap> IsClosedSurface(pyr);
false
```

The problem is: We did not tell our method what the edges should be. For triangular faces this is not an issue because there is an edge between any pair of vertices. But for the square face in our example the vertices do *not* determine its edges.

The method `SimplicialSurfaceByVerticesInFaces` will believe that two adjacent vertices in the given list are also connected by an edge of the face. Above, we gave the list [2, 3, 4, 5] for the square. If we compare it with our picture we can see that the vertices 2 and 4 are connected in the picture but not adjacent in our list. Likewise the vertices 2 and 5 are adjacent in the list (we imagine that the list wraps around) but don't have an edge between them in the picture.

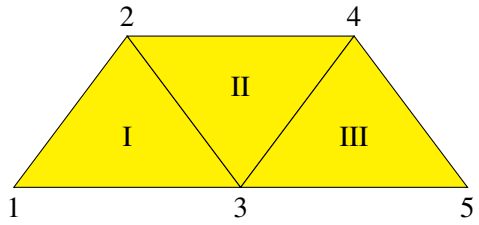
Instead we have to give the vertices in a proper cyclic ordering:

Example

```
gap> pyr := SimplicialSurfaceByVerticesInFaces( 5, 5,
>                   [[2,3,5,4], [1,2,3], [1,3,5], [1,5,4],[1,2,4]] );;
gap> NrOfEdges(pyr);
8
gap> IsClosedSurface(pyr);
true
```

1.4 Playing with vertices and faces.

After having learned how to construct a simplicial surface by the method `SimplicialSurfaceByVerticesInFaces`, we can use the labelling of vertices and faces to get more detailed information about the surface. We will use the example from section 1.3:



Example

```
gap> surf := SimplicialSurfaceByVerticesInFaces( 5, 3, [[1,2,3],[2,3,4],[3,4,5]] );;
```

It is easy to reclaim the complete incidence structure that went into the construction.

Example

```
gap> Vertices(surf);
[ 1, 2, 3, 4, 5 ]
gap> Faces(surf);
[ 1, 2, 3 ]
gap> VerticesOfFaces(surf);
[ [ 1, 2, 3 ], [ 2, 3, 4 ], [ 3, 4, 5 ] ]
```

By using the incidence-structure we can distinguish vertices that lie in a different number of faces.

Example

```
gap> UnsortedDegrees(surf);
[ 1, 2, 3, 2, 1]
```

The first entry of this list counts the number of faces that are incident to the vertex 1 (in general the i-th entry counts those for the vertex i).

In this case it is apparent that the third vertex is incident to three different faces and unique with that property. To distinguish vertices 1 and 5 (that are incident to one face each), we need to know which faces they are incident to.

Example

```
gap> FacesOfVertices(surf);
[ [ 1 ], [ 1, 2 ], [ 1, 2, 3 ], [ 2, 3 ], [ 3 ] ]
```

So the first vertex is incident to the face 1 and the fifth vertex is incident to face 3. We can also see that the second vertex is incident to the faces 1 and 2.

An additional advantage of the incidential information is that we can determine a concrete global orientation if the surface is orientable. By using `GlobalOrientation` we will get a list of permutations such that the i-th entry is a permutation of the vertices in face i.

Example

```
gap> GlobalOrientation(surf);
[ (1,3,2), (2,3,4), (3,5,4) ]
```

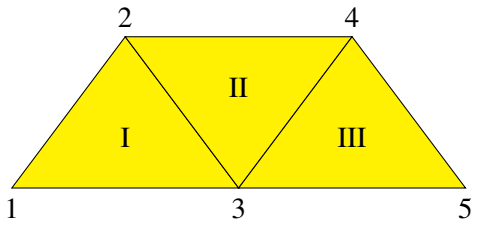
If we would rather work with lists instead of permutations (instead of the permutation (3,5,4) we would like the list [3, 5, 4]) we can use

Example

```
gap> GlobalOrientationByVerticesAsList(surf);
[ [ 1, 3, 2 ], [ 2, 3, 4 ], [ 3, 5, 4 ] ]
```

1.5 Adding edge numbering

Up until now we did not care about specific edges because they were not important for the construction of our examples so far. But let us assume that we want to know which edges are incident to exactly two faces in our example from before.



Example

```
gap> surf := SimplicialSurfaceByVerticesInFaces( 5, 3, [[1,2,3],[2,3,4],[3,4,5]] );;
```

We can see the edges by calling

Example

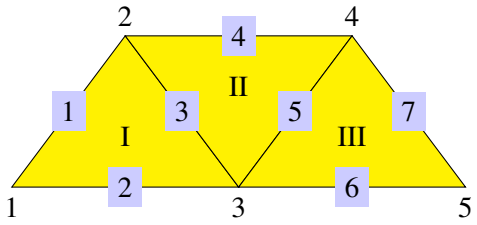
```
gap> Edges(surf);
[ 1, 2, 3, 4, 5, 6, 7 ]
```

but this does not tell us where they are. For that we may use

Example

```
gap> VerticesOfEdges(surf);
[ [ 1, 2 ], [ 1, 3 ], [ 2, 3 ], [ 2, 4 ], [ 3, 4 ], [ 3, 5 ], [ 4, 5 ] ]
```

As before, the first entry of this list contains the vertices of the first edge. In our example the edge with number 1 is incident to the vertices 1 and 2. We can update our picture:



If we now want to know which edges are incident to exactly two faces, we only have to check

Example

```
gap> FacesOfEdges(surf);
[ [ 1 ], [ 1 ], [ 1, 2 ], [ 2 ], [ 2, 3 ], [ 3 ], [ 3 ] ]
```

The positions with lists of two elements are the interesting edges. We can compute them easily by

Example

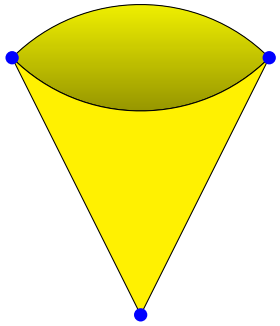
```
gap> Filtered( Edges(surf), e -> Size(FacesOfEdges(surf)[e]) = 2 );
[ 3, 5 ]
```

1.6 Constructors with vertex, edge and face data

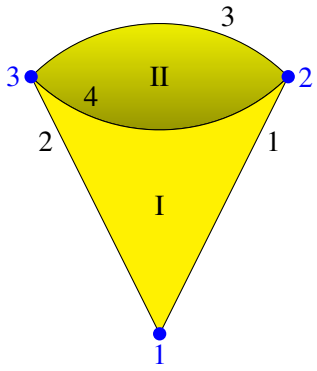
There are some cases in which we don't want to use `SimplicialSurfaceByVerticesInFaces` but a more versatile method, for example

1. We already have an edge labelling and want to keep it.
2. The faces of our surface are not determined by their vertices.

An example of the second situation is given by two triangles that share exactly two edges. They can be visualized as an "open bag".



Since both faces share the same vertices we can't use SimplicialSurfaceByVerticesInFaces here. Instead we need to label vertices, edges and faces individually:



Now we have to tell GAP which vertices are incident to which edges

Example

```
gap> verticesOfEdges := [[1,2],[1,3],[2,3],[2,3]];;
```

and which edges are incident to which face

Example

```
gap> edgesOfFaces := [[1,2,4],[1,2,3]];;
```

which allows us to use the constructor `SimplicialSurfaceByDownwardIncidence` (for an explanation of this name, see chapter ...)

Example

```
gap> bag := SimplicialSurfaceByDownwardIncidence(3, 4, 2, verticesOfEdges, edgesOfFaces);;
```

It would be nice if we were able to easily determine which edges/faces are not determined by their vertices alone. For that purpose we can use the following commands:

Example

```
gap> EdgeAnomalies(bag);
[ [ 1 ], [ 2 ], [ 3, 4 ] ]
gap> FaceAnomalies(bag);
[ [ 1, 2 ] ]
```

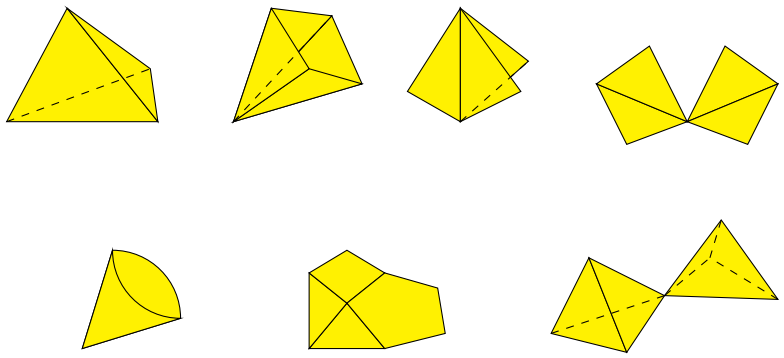
The list of edge anomalies is a partition of the edges such that two edges are in the same equivalence class if and only if they have the same vertices. So we see here that the edges 3 and 4 have the same vertices. We can do the same for the faces and see that the two faces share all their vertices.

Chapter 2

Polygonal Structures

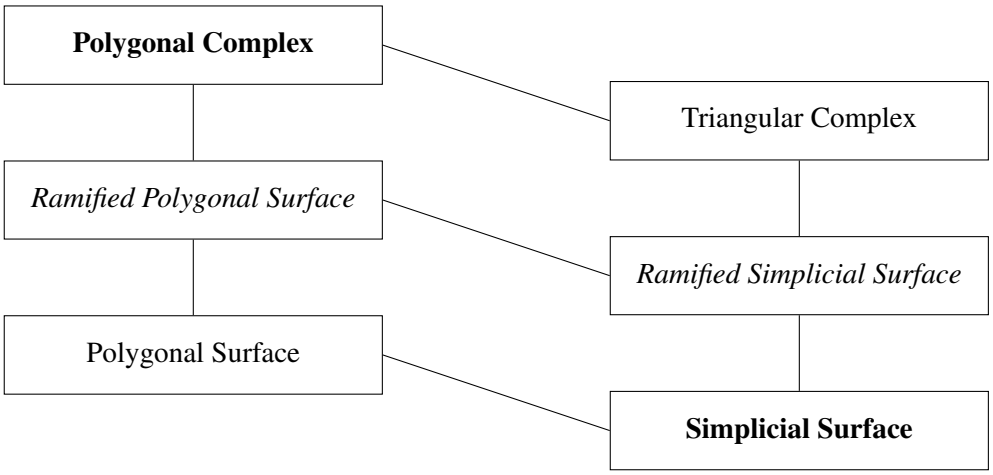
Although we are mostly interested in surfaces, the process of folding naturally leads us to more general structures. Therefore, the most general structure that we consider in this package is a polygonal complex (a generalisation of simplicial surfaces and complexes).

Polygonal Complexes are structures that are build from polygons (for the exact definition, see section 2.1), like



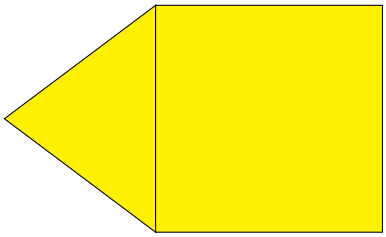
Since we are especially interested in surfaces we need a way to recognize them. For technical reasons we split this process into two separate steps, with the ramified polygonal surfaces as the intermediate structure between polygonal complexes and polygonal surfaces.

In many cases the definitions and algorithms simplify if we only consider triangles instead of general polygons. For this reason we also have equivalents of these structures that are only build from triangles. We can summarize the relationships graphically (higher means more general):

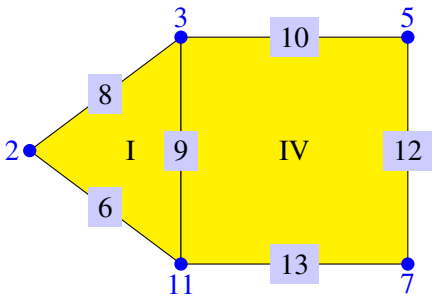


2.1 Polygonal and Triangular Complexes

A polygonal complex is defined by a two-dimensional incidence geometry that fulfills some regularity conditions (the complete definition will be given at the end of this section in definition 2.1.1). We will exemplify this by the following surface.



In a two-dimensional incidence geometry we have a set V of vertices, a set E of edges and a set F of faces. If we label our picture like this,



we have the sets

$V = \{2,3,5,7,11\}$
$E = \{6,8,9,10,12,13\}$
$F = \{I,IV\}$.

These sets can't encode the surface on their own. We also need the relations between vertices, edges and faces. For that we define a transitive relation in the union of $V \times E$, $V \times F$ and $E \times F$, where

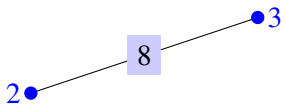
(a,b) is in the relation if and only if a is incident to b. Since the relation is transitive, it is sufficient to specify the tuples in $V \times E$ and $E \times F$. In our example the relation would be generated by

$\{(2,6),(2,8),(3,8),(3,9),(3,10),(5,10),(5,12),(7,12),(7,13),(11,6),(11,9),(11,13)\}$

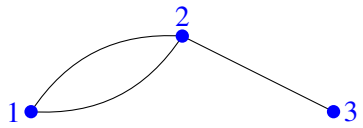
and

$\{(6,I),(8,I),(9,I),(9,IV),(10,IV),(12,IV),(13,IV)\}$.

Instead of a general incidence relation we only want to consider those relations that have similar properties to our example. For example, every edge should consist of exactly two vertices (more formally, for every edge there should be exactly two vertices that are incident to it).

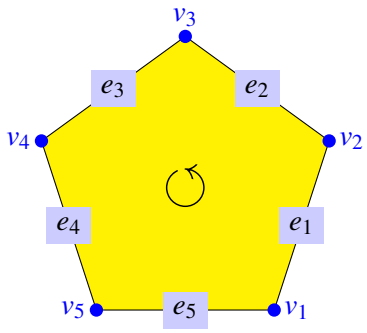


If we try to generalize this condition to the faces we encounter some difficulties. For example, if we only enforce that each face has exactly three edges and three vertices, it would be possible that two of those edges share all of their vertices. Therefore even the case of triangles is quite complicated.



We will solve this problem by imposing a cyclic ordering on the vertices and edges that are incident to each face, such that those are compatible with the incidence structure. Since this definition is equally valid for all possible faces (not just triangles), we formally define:

- For every face the number of incident vertices is equal to the number of incident edges. Additionally there is an enumeration of the vertices $(v_1, v_2, ..., v_k)$ and the edges $(e_1, e_2, ..., e_k)$ such that the vertices v_i and v_{i+1} are incident to the edge e_i (where we read indices $\bmod k$).



With these conditions we have guaranteed that our edges and faces behave as in our previous example. But they still allow the possibility of an incidence relation where we have vertices but no faces. Since we want to exclude singular cases like this from our considerations, we require that every vertex is incident to at least one edge and that every edge is incident to at least one face.

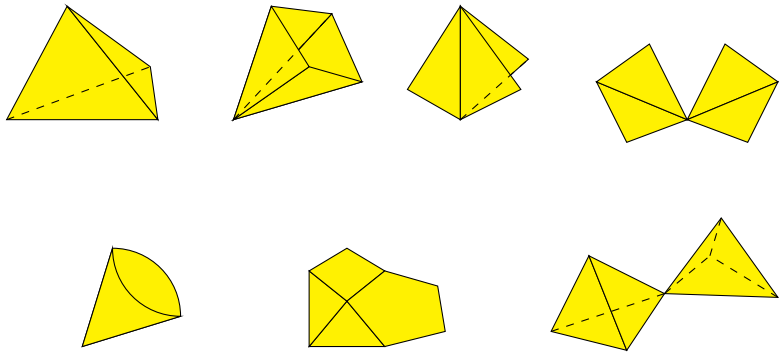
2.1.1 Definition (polygonal complex and triangular complex)

A *polygonal complex* is a two-dimensional incidence geometry consisting of vertices, edges and faces such that the following conditions hold (for a more detailed explanation see section 2.1):

1. For every edge there should be exactly two vertices that are incident to it.
2. For every face the number of incident vertices is equal to the number of incident edges. Additionally there is an enumeration of the vertices $(v_1, v_2, ..., v_k)$ and the edges $(e_1, e_2, ..., e_k)$ such that the vertices v_i and v_{i+1} are incident to the edge e_i (where we read indices $\bmod k$).
3. Every vertex is incident to an edge and every edge is incident to a face.

A *triangular complex* is a polygonal complex where every face has exactly three incident vertices.

It is now easy to verify that the examples in the picture



are all polygonal complexes.

2.1.2 IsPolygonalComplex

▷ `IsPolygonalComplex(`*`object`*`)` (category)

Returns: true or false

Checks whether *`object`* is a polygonal complex. A polygonal complex can be informally described as a structure that is constructed from polygons.

More formally (for a more extensive explanation see section 2.1) a polygonal complex is a two-dimensional incidence geometry of vertices, edges and faces such that the following conditions hold:

1. Every edge has exactly two incident vertices.
2. For every face the incident edges and vertices form a polygon. (See definition 2.1.1.)
3. Every vertex is incident to an edge and every edge is incident to a face.

2.1.3 IsTriangularComplex

▷ `IsTriangularComplex(`*`object`*`)` (property)

Returns: true or false

Checks whether *`object`* is a triangular complex (a polygonal complex with triangular faces). A triangular complex can be informally described as a structure that is constructed from triangles.

More formally (for a more extensive explanation see section 2.1) a triangular complex is a two-dimensional incidence geometry of vertices, edges and faces such that the following conditions hold:

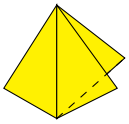
1. Every edge has exactly two incident vertices.
2. For every face the incident edges and vertices form a triangle. (See definition 2.1.1.)
3. Every vertex is incident to an edge and every edge is incident to a face.

2.2 Ramified polygonal and simplicial surfaces

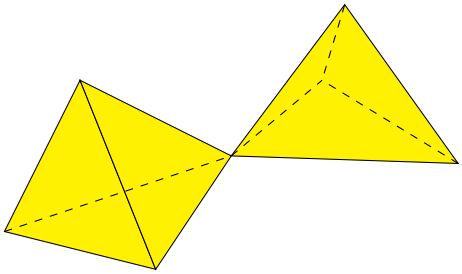
Ramified polygonal (simplicial) surfaces are an intermediate concept between polygonal (triangular) complexes (2.1) and polygonal (simplicial) surfaces (2.3).

A *ramified polygonal surface* is a polygonal complex where every edge is incident to at most two faces. If every face is a triangle, we call it a *ramified simplicial surface*.

The stereotypical polygonal complex that is *not* a ramified polygonal surface consists of three triangles that share a common edge.



A stereotypical example of a ramified polygonal surface would be two tetrahedrons that have one vertex in common.



Although these structures are not true surfaces, in many applications they behave like those (for example for questions of orientability).

2.2.1 IsRamifiedPolygonalSurface

▷ `IsRamifiedPolygonalSurface(`*`object`*`)` (property)

Returns: true or false

Checks whether *`object`* is a ramified polygonal surface (a polygonal complex where every edge is incident to at most two faces).

More formally (for a more extensive explanation see sections 2.1 and 2.2) a ramified polygonal surface is a two-dimensional incidence geometry of vertices, edges and faces such that the following conditions hold:

1. Every edge has exactly two incident vertices.
2. For every face the incident edges and vertices form a polygon. (See definition 2.1.1.)
3. Every vertex is incident to an edge and every edge is incident to a face.
4. Every edge is incident to at most two faces.

2.2.2 IsRamifiedSimplicialSurface

▷ `IsRamifiedSimplicialSurface(`*`object`*`)` (property)

Returns: true or false

Checks whether `object` is a ramified simplicial surface (a polygonal complex with triangular faces where every edge is incident to at most two faces).

More formally (for a more extensive explanation see sections 2.1 and 2.2) a ramified simplicial surface is a two-dimensional incidence geometry of vertices, edges and faces such that the following conditions hold:

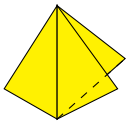
1. Every edge has exactly two incident vertices.
2. For every face the incident edges and vertices form a triangle. (See definition 2.1.1.)
3. Every vertex is incident to an edge and every edge is incident to a face.
4. Every edge is incident to at most two faces.

2.3 Polygonal and simplicial surfaces

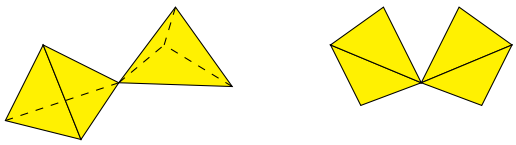
Polygonal (simplicial) surfaces are polygonal (triangular) complexes that behave like surfaces (two-dimensional manifolds with boundary). For example, they arise from the discretisation of a continuous surface.

There are two properties that distinguish polygonal/simplicial surfaces from polygonal/triangular complexes:

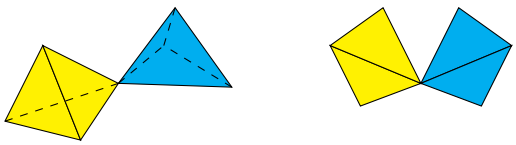
1. Every edge behaves "like a surface", i.e. it is incident to at most two faces. This excludes cases like three triangles that share one edge.



2. Every vertex behaves "like a surface". We will formalize this below but the aim of this condition is to exclude cases like those:



If only the first condition is fulfilled, we talk about *ramified polygonal/simplicial surfaces* (2.2). We now proceed to formalize the second condition. Intuitively we want to distinguish between the different coloured faces in these examples:

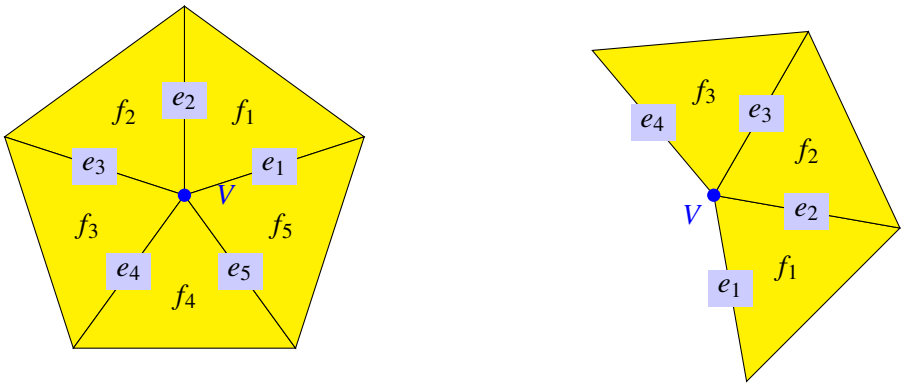


They can be distinguished by the fact that faces of the same colour are connected by edges that are incident to the given vertex. More formally:

2.3.1 Definition (Edge-Face-Path around a vertex)

An *edge-face-path around the vertex V* is a tuple $(e_1, f_1, e_2, f_2, \ldots, e_n, f_n, e_{n+1})$ with f_i pairwise distinct faces and e_i pairwise distinct edges (with the possible exception of $e_1 = e_{n+1}$ in the case $n > 1$), such that the vertex V is incident to all of them.

If $e_1 = e_{n+1}$ the path is called *closed* and we also denote it by $(e_1, f_1, e_2, f_2, \ldots, e_n, f_n)$.



With this definition in mind we can explicate the second property:

- For every vertex V all incident edges and faces can be arranged in an edge-face-path around V (compare definition 2.3.1).

For a general ramified polygonal surface we can't expect to get exactly one edge-face-path. In our examples above all incident edges and faces could be represented by two edge-face-paths. In general we can always find a set of edge-face-paths of a vertex such that all edges and faces incident to that vertex lie in exactly one of those paths. This is called the *edge-face-path partition* of the vertex.

2.3.2 EdgeFacePathPartitionsOfVertices

▷ `EdgeFacePathPartitionsOfVertices(ramSurf)` (attribute)

Returns: A list of edge-face-path partitions (who are lists themselves)

▷ `EdgeFacePathPartitionOfVertex(ramSurf, vertex)` (operation)

▷ `EdgeFacePathPartitionOfVertexNC(ramSurf, vertex)` (operation)

Returns: The edge-face-path partition of a vertex (as list)

For each vertex V the edge-face-path partition of this vertex is a list of edge-face-paths of this vertex (see definition 2.3.1) such that all edges and faces that are incident to V lie in exactly one of those edge-face-paths.

Each edge-face-path is given as a list $[e_1, f_1, e_2, f_2, \ldots, e_n, f_n]$ (for a closed path) or $[e_1, f_1, e_2, \ldots, e_n, f_n, e_{n+1}]$ (for an open path).

The attribute `EdgeFacePathPartitionsOfVertices` returns a list such that the entry V holds the edge-face-path partition of the vertex V. All other entries of the list are unbounded.

2.3.3 IsPolygonalSurface

▷ `IsPolygonalSurface(object)` (property)

Returns: true or false

Checks whether *object* is a polygonal surface. Informally a polygonal surface is a surface (with boundary) that is build only from polygons.

More formally (for more details and examples see sections 2.1, 2.2 and 2.3) a polygonal surface is a two-dimensional incidence geometry of vertices, edges and faces such that the following conditions hold:

1. Every edge has exactly two incident vertices.
2. For every face the incident edges and vertices form a polygon. (See definition 2.1.1.)
3. Every vertex is incident to an edge and every edge is incident to a face.
4. Every edge is incident to at most two faces.
5. For every vertex V all incident edges and faces can be arranged in an edge-face-path around V (compare definition 2.3.1).

2.3.4 IsSimplicialSurface

▷ `IsSimplicialSurface(`*`object`*`)` (property)

Returns: true or false

Checks whether *`object`* is a simplicial surface. Informally a simplicial surface is a surface (with boundary) that is build only from triangles.

More formally (for more details and examples see sections 2.1, 2.2 and 2.3) a simplicial surface is a two-dimensional incidence geometry of vertices, edges and faces such that the following conditions hold:

1. Every edge has exactly two incident vertices.
2. For every face the incident edges and vertices form a triangle. (See definition 2.1.1.)
3. Every vertex is incident to an edge and every edge is incident to a face.
4. Every edge is incident to at most two faces.
5. For every vertex V all incident edges and faces can be arranged in an edge-face-path around V (compare definition 2.3.1).

Chapter 3

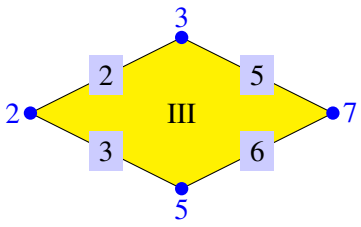
Access to the incidence geometry

In section 2.1 we introduced incidence geometries to describe polygonal complexes. This chapter describes several different ways to access these incidence structures.

In most cases the methods of section 3.2 (that return incident elements as a set) are sufficient. The subsequent sections contain more specific access methods.

3.1 Labels of vertices, edges and faces

In polygonal complexes (the most general supported incidence structure, compare chapter 2) all vertices, edges and faces are labelled by positive integers. These labels do not have to be disjoint as shown in the following example:



We can access the set of all those labels by `Vertices`, `Edges` and `Faces`. If only the number of vertices is relevant, we can use `NrOfVertices` instead (likewise for edges and faces).

3.1.1 Vertices (for IsPolygonalComplex)

▷ `Vertices(`*complex*`)` (operation)
▷ `VerticesAttributeOfPolygonalComplex(`*complex*`)` (attribute)

Returns: A set of positive integers

Return the set of vertices.

We have separated the operation `Vertices` from the corresponding attribute because there is a naming clash with the package grape.

3.1.2 Edges (for IsPolygonalComplex)

▷ `Edges(`*complex*`)` (attribute)

Returns: A set of positive integers

Return the set of edges.

3.1.3 Faces (for IsPolygonalComplex)

▷ `Faces(`*complex*`)` (attribute)

Returns: A set of positive integers

Return the set of faces.

3.1.4 NrOfVertices (for IsPolygonalComplex)

▷ `NrOfVertices(`*complex*`)` (attribute)

Returns: a non-negative integer

Return the number of vertices.

3.1.5 NrOfEdges (for IsPolygonalComplex)

▷ `NrOfEdges(`*complex*`)` (attribute)

Returns: a non-negative integer

Return the number of edges.

3.1.6 NrOfFaces (for IsPolygonalComplex)

▷ `NrOfFaces(`*complex*`)` (attribute)

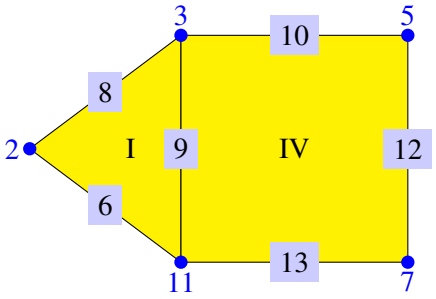
Returns: a non-negative integer

Return the number of faces.

3.2 Incidence between vertices, edges and faces

With the labels of vertices, edges and faces (which we can access by the methods of section 3.1) we can describe the incidence structure by lists of sets. All those methods have the form *Of*, e.g. `VerticesOfFaces` and `EdgesOfVertices`.

We will illustrate the general pattern of these methods by showcasing these two methods. For that we will use the following polygonal complex:



Example

```
gap> complex := PolygonalComplexByDownwardIncidence( 5, 6, 2,
>     [ , , , , , [2,11], , [2,3], [3,11], [3,5], , [5,7], [7,11] ],
>     [[6,8,9], , , [9,10,12,13]]);;
gap> Vertices(complex);
[ 2, 3, 5, 7, 11 ]
```

```
gap> Edges(complex);
[ 6, 8, 9, 10, 12, 13 ]
gap> Faces(complex);
[ 1, 4 ]
```

The method VerticesOfFaces tells us, which vertices are incident to which faces.

Example

```
gap> VerticesOfFaces(complex);
[ [ 2, 3, 11 ], , , [ 3, 5, 7, 11 ] ];
```

The first entry of this list contains a set of all vertices that are incident to face I. The second and third entries are not bounded since there are no faces II and III. Finally, the fourth entry contains all vertices that are incident to face IV.

So we have a list that contains sets of vertices and is indexed by the face labels.

The method EdgesOfVertices works in the same way: It returns a list that contains sets of edges and is indexed by the vertex labels.

Example

```
gap> EdgesOfVertices(complex);
[ , [ 6, 8 ], [ 8, 9, 10 ], , [ 10, 12 ], , [ 12, 13 ], , , , [ 6, 9, 13 ] ]
```

For example, if we consider the third entry of this list, we find the set [8, 9, 10]. Those are all edges that are incident to the vertex 3.

In the same way all other *Of*-methods are defined.

3.2.1 EdgesOfVertices (for IsPolygonalComplex)

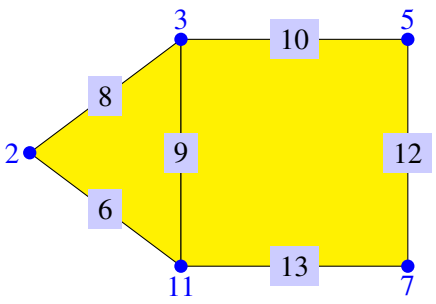
▷ EdgesOfVertices(*complex*) (attribute)
▷ EdgesOfVertex(*complex*, *vertex*) (operation)
▷ EdgesOfVertexNC(*complex*, *vertex*) (operation)

Returns: a list of sets of positive integers / a set of positive integers

The method EdgesOfVertex(*complex*, *vertex*) returns the set of all edges that are incident to *vertex*. The NC-version does not check whether the given *vertex* is a vertex of *complex*.

The attribute EdgesOfVertices(*complex*) collects all of those sets in a list that is indexed by the vertex labels, i.e. EdgesOfVertices(*complex*)[*vertex*] = EdgesOfVertex(*complex*, *vertex*). All other positions of this list are not bounded.

As an example, consider the polygonal complex that was introduced at the start of section 3.2:



Example

```
gap> EdgesOfVertex(complex, 2);
[ 6, 8 ]
```

```
gap> EdgesOfVertex(complex, 11);
[ 6, 9, 13 ]
gap> EdgesOfVertices(complex);
[ , [ 6, 8 ], [ 8, 9, 10 ], , [ 10, 12 ], , [ 12, 13 ], , , [ 6, 9, 13 ] ]
```

3.2.2 FacesOfVertices (for IsPolygonalComplex)

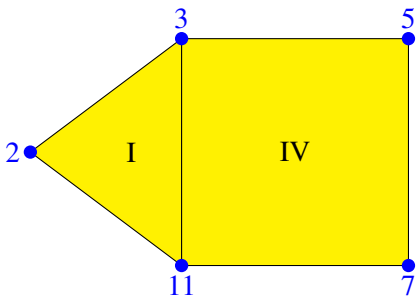
▷ `FacesOfVertices(`*complex*`)` (attribute)
▷ `FacesOfVertex(`*complex*`, `*vertex*`)` (operation)
▷ `FacesOfVertexNC(`*complex*`, `*vertex*`)` (operation)

Returns: a list of sets of positive integers / a set of positive integers

The method `FacesOfVertex(`*complex*`, `*vertex*`)` returns the set of all faces that are incident to *vertex*. The NC-version does not check whether the given *vertex* is a vertex of *complex*.

The attribute `FacesOfVertices(`*complex*`)` collects all of those sets in a list that is indexed by the vertex labels, i.e. `FacesOfVertices(`*complex*`)[`*vertex*`] = FacesOfVertex(`*complex*`, `*vertex*`)`. All other positions of this list are not bounded.

As an example, consider the polygonal complex that was introduced at the start of section 3.2:



Example

```
gap> FacesOfVertex(complex, 2);
[ 1 ]
gap> FacesOfVertex(complex, 11);
[ 1, 4 ]
gap> FacesOfVertices(complex);
[ , [ 1 ], [ 1, 4 ], , [ 4 ], , [ 4 ], , , [ 1, 4 ] ]
```

3.2.3 VerticesOfEdges (for IsPolygonalComplex)

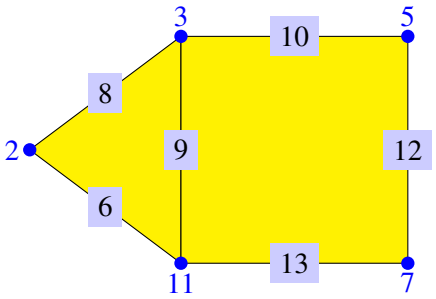
▷ `VerticesOfEdges(`*complex*`)` (attribute)
▷ `VerticesOfEdge(`*complex*`, `*edge*`)` (operation)
▷ `VerticesOfEdgeNC(`*complex*`, `*edge*`)` (operation)

Returns: a list of sets of positive integers / a set of positive integers

The method `VerticesOfEdge(`*complex*`, `*edge*`)` returns the set of all vertices that are incident to *edge*. The NC-version does not check whether the given *edge* is an edge of *complex*.

The attribute `VerticesOfEdges(`*complex*`)` collects all of those sets in a list that is indexed by the edge labels, i.e. `VerticesOfEdges(`*complex*`)[`*edge*`] = VerticesOfEdge(`*complex*`, `*edge*`)`. All other positions of this list are not bounded.

As an example, consider the polygonal complex that was introduced at the start of section 3.2:



Example

```
gap> VerticesOfEdge(complex, 8);
[ 2, 3 ]
gap> VerticesOfEdge(complex, 12);
[ 5, 7 ]
gap> VerticesOfEdges(complex);
[ , , , , , [ 2, 11 ], , [ 2, 3 ], [ 3, 11 ], [ 3, 5 ], , [ 5, 7 ], [ 7, 11 ] ]
```

3.2.4 FacesOfEdges (for IsPolygonalComplex)

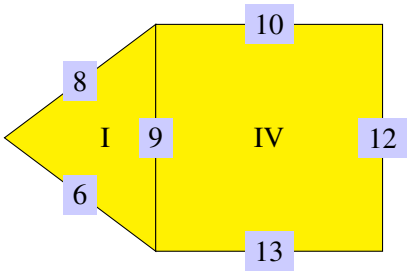
▷ `FacesOfEdges(`*complex*`)` (attribute)
▷ `FacesOfEdge(`*complex*`, `*edge*`)` (operation)
▷ `FacesOfEdgeNC(`*complex*`, `*edge*`)` (operation)

Returns: a list of sets of positive integers / a set of positive integers

The method `FacesOfEdge(`*complex*`, `*edge*`)` returns the set of all faces that are incident to *edge*. The NC-version does not check whether the given *edge* is an edge of *complex*.

The attribute `FacesOfEdges(`*complex*`)` collects all of those sets in a list that is indexed by the edge labels, i.e. `FacesOfEdges(`*complex*`)[`*edge*`] = FacesOfEdge(`*complex*`, `*edge*`)`. All other positions of this list are not bounded.

As an example, consider the polygonal complex that was introduced at the start of section 3.2:



Example

```
gap> FacesOfEdge(complex, 9);
[ 1, 4 ]
gap> FacesOfEdge(complex, 10);
[ 4 ]
gap> FacesOfEdges(complex);
[ , , , , , [ 1 ], , [ 1 ], [ 1, 4 ], [ 4 ], , [ 4 ], [ 4 ] ]
```

3.2.5 VerticesOfFaces (for IsPolygonalComplex)

▷ `VerticesOfFaces(`*complex*`)` (attribute)
▷ `VerticesOfFace(`*complex*`, `*face*`)` (operation)

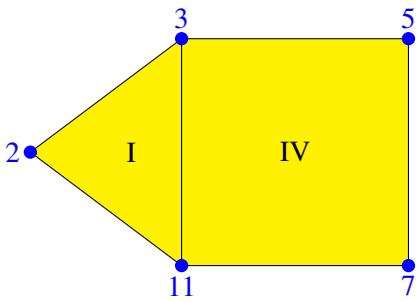
▷ `VerticesOfFaceNC(`*`complex, face`*`)` (operation)

Returns: a list of sets of positive integers / a set of positive integers

The method `VerticesOfFace(`*`complex`*`, `*`face`*`)` returns the set of all vertices that are incident to *face*. The NC-version does not check whether the given *face* is a face of *complex*.

The attribute `VerticesOfFaces(`*`complex`*`)` collects all of those sets in a list that is indexed by the face labels, i.e. `VerticesOfFaces(`*`complex`*`)[`*`face`*`] = VerticesOfFace(`*`complex`*`, `*`face`*`)`. All other positions of this list are not bounded.

As an example, consider the polygonal complex that was introduced at the start of section 3.2:



Example

```
gap> VerticesOfFace(complex, 1);
[ 2, 3, 11 ]
gap> VerticesOfFace(complex, 4);
[ 3, 5, 7, 11 ]
gap> VerticesOfFaces(complex);
[ [ 2, 3, 11 ], , , [ 3, 5, 7, 11 ] ]
```

3.2.6 EdgesOfFaces (for IsPolygonalComplex)

▷ `EdgesOfFaces(`*`complex`*`)` (attribute)

▷ `EdgesOfFace(`*`complex, face`*`)` (operation)

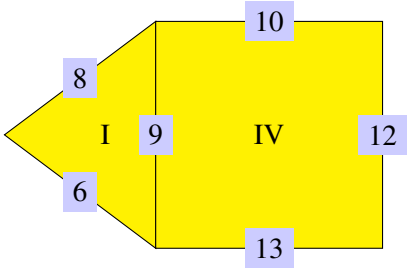
▷ `EdgesOfFaceNC(`*`complex, face`*`)` (operation)

Returns: a list of sets of positive integers / a set of positive integers

The method `EdgesOfFace(`*`complex`*`, `*`face`*`)` returns the set of all edges that are incident to *faces*. The NC-version does not check whether the given *face* is a face of *complex*.

The attribute `EdgesOfFaces(`*`complex`*`)` collects all of those sets in a list that is indexed by the face labels, i.e. `EdgesOfFaces(`*`complex`*`)[`*`face`*`] = EdgesOfFace(`*`complex`*`, `*`face`*`)`. All other positions of this list are not bounded.

As an example, consider the polygonal complex that was introduced at the start of section 3.2:



Example

```
gap> EdgesOfFace(complex, 1);
[ 6, 8, 9 ]
```

```
gap> EdgesOfFace(complex, 4);
[ 9, 10, 12, 13 ]
gap> EdgesOfFaces(complex);
[ [ 6, 8, 9 ], , , [ 9, 10, 12, 13 ] ]
```

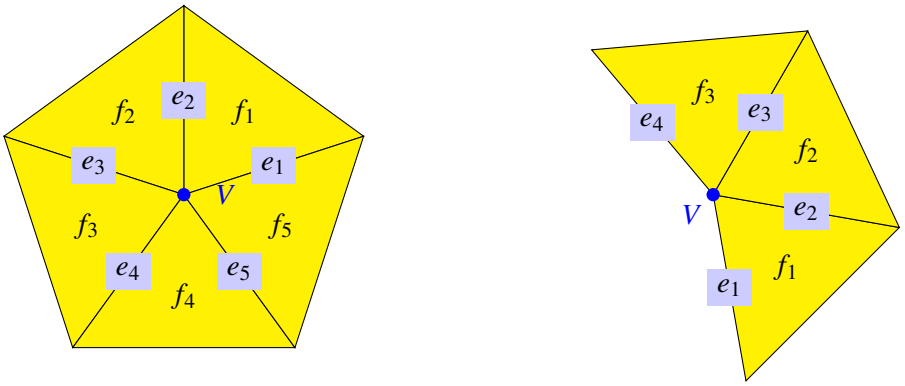
3.3 Face-induced order of incident vertices/edges

filler

3.4 Circular path of edges and faces around vertex

1. *Ordering around a vertex*

   For polygonal surfaces (that were introduced in section 2.3) there is a natural ordering of the edges and faces that are incident to a vertex. It is achieved by "travelling around the vertex" while staying on the surface.



   This is captured in the concept of an *edge-face-path*, which is defined in subsection 2.3.1.

Chapter 4

Simplicial Surfaces

A SimplicialSurface-object in GAP represents mathematical objects that are a generalization of simplicial surfaces. On the most basic level it consists of vertices, edges and faces, together with an incidence relation between them. This information is saved in the following form:

- the vertices are represented by a set of positive integers
- the edges are represented by a set of positive integers
- the faces are represented by a set of positive integers
- the incidence relation between vertices and edges (that is, which vertices lie in which edges) is represented by a list VerticesOfEdges. For each edge e the entry VerticesOfEdges[e] is a set of all vertices that are incident to the edge e.
- the incidence relation between edges and faces (that is, which edges lie in which faces) is represented by a list EdgesOfFaces. For each face f the entry EdgesOfFaces[f] is a set of all edges that are incident to the face f.

Every other incidence (like VerticesOfFaces or EdgesOfVertices) is represented in an analogous fashion. Furthermore we impose some restrictions onto these incidence relations:

- Every edge is incident to exactly two vertices
- Every face is incident to the same number of vertices and edges (at least three). Additionally these are cyclically oriented (to represent an n-gon)
- The incidence relations are transitive
- Every vertex and every edge is incident to at least one face

Note that it is allowed for an edge to be incident to more than two faces. In addition it is sometimes necessary to distinguish between the two sides of a single face. If this is irrelevant to your application, you can completely ignore this (it will be handled in the background). To use this distinction each side of a face f gets a name - the defaults are +f and -f (but you can give them custom names if you want to). To distinguish which side is which (not only by a name, but geometrically) we save a cyclic ordering of the vertices (or edges) of each face which we associate with the side +f. (In an embedding to $\mathbb{R}^3$ we could associate the cyclic ordering with the normal vector of the face that is defined by the right-hand-rule from physics.)

4.1 Constructors for Simplicial Surfaces

4.1.1 Janushead

▷ `Janushead()` (operation)

Returns: a simplicial surface

Return a simplicial surface that represents a janus head.

4.1.2 Tetrahedron

▷ `Tetrahedron()` (operation)

Returns: a simplicial surface

Return a simplicial surface that represents a tetrahedron.

4.1.3 Cube

▷ `Cube()` (operation)

Returns: a simplicial surface

Return a simplicial surface that represents a cube

4.1.4 Octahedron

▷ `Octahedron()` (operation)

Returns: a simplicial surface

Return a simplicial surface that represents a octahedron.

4.1.5 Dodecahedron

▷ `Dodecahedron()` (operation)

Returns: a simplicial surface

Return a simplicial surface that represents a dodecahedron.

4.1.6 Icosahedron

▷ `Icosahedron()` (operation)

Returns: a simplicial surface

Return a simplicial surface that represents a icosahedron.

4.1.7 SimplicialSurfaceByUpwardIncidence (for IsSet, IsSet, IsSet, IsList, IsList)

▷ `SimplicialSurfaceByUpwardIncidence(`*`vertices, edges, faces, edgesOfVertices, facesOfEdges`*`)` (operation)

▷ `SimplicialSurfaceByUpwardIncidenceNC(`*`vertices, edges, faces, edgesOfVertices, facesOfEdges`*`)` (operation)

Returns: a simplicial surface

This constructor of a simplicial surface uses the following information:

- The set of vertices (alternatively a positive integer n, which will be interpreted as the set [1..n])
- The set of edges (alternatively a positive integer n, which will be interpreted as the set [1..n])

- The set of faces (alternatively a positive integer n, which will be interpreted as the set [1..n])
- The list edgesOfVertices. For each vertex v the entry edgesOfVertices[v] is a set of all edges that are incident to the vertex v.
- The list facesOfEdges. For each edge e the entry facesOfEdges[e] is a set of all faces that are incident to the edge e.

In this constructor there is no way of distinguishing the sides of each face, so this selection will be made randomly. The NC-version does not check whether the given input follows the following criteria:

- vertices, edges and faces have to be either positive integers or sets of positive integers.
- For each vertex v the entry edgesOfVertices[v] has to be a subset of the set of edges.
- Every edge has to be in one of the sets in the list edgesOfVertices.
- For each edge e the entry facesOfEdges[e] has to be a subset of the set of faces.
- Each face has to be in one of the sets in the list facesOfEdges.

4.1.8 SimplicialSurfaceByDownwardIncidence (for IsSet, IsSet, IsSet, IsList, IsList)

▷ `SimplicialSurfaceByDownwardIncidence(`*vertices, edges, faces, verticesOfEdges, edgesOfFaces*`)` (operation)
▷ `SimplicialSurfaceByDownwardIncidenceNC(`*vertices, edges, faces, verticesOfEdges, edgesOfFaces*`)` (operation)

Returns: a simplicial surface

This constructor of a simplicial surface uses the following information:

- The set of vertices (alternatively a positive integer n, which will be interpreted as the set [1..n])
- The set of edges (alternatively a positive integer n, which will be interpreted as the set [1..n])
- The set of faces (alternatively a positive integer n, which will be interpreted as the set [1..n])
- The list verticesOfEdges. For each edge e the entry verticesOfEdges[e] is a set of all vertices that are incident to the edge e.
- The list edgesOfFaces. For each face f the entry edgesOfFaces[f] is a set of all edges that are incident to the face f.

This constructor does not distinguish different sides of each face, so this selection will be made randomly. The NC-version does not check whether the given input follows the following criteria:

- vertices, edges and faces have to be either positive integers or sets of positive integers.
- For each edge e the entry verticesOfEdges[e] has to be a subset of the set of vertices.
- Every vertex has to be in one of the sets in the list verticesOfEdges.
- For each face f the entry edgesOfFaces[f] has to be a subset of the set of edges.
- Each edge has to be in one of the sets in the list edgesOfFaces.

4.1.9 SimplicialSurfaceByDownwardIncidenceWithOrientation (for IsSet, IsSet, IsSet, IsList, IsList)

▷ SimplicialSurfaceByDownwardIncidenceWithOrientation(*vertices, edges, faces, verticesOfEdges, edgesOfFaces[, namesOfFaces]*) (operation)
▷ SimplicialSurfaceByDownwardIncidenceWithOrientationNC(*vertices, edges, faces, verticesOfEdges, edgesOfFaces[, namesOfFaces]*) (operation)

Returns: a simplicial surface

This constructor of a simplicial surface uses the following information:

- The set of vertices (alternatively a positive integer n, which will be interpreted as the set [1..n])
- The set of edges (alternatively a positive integer n, which will be interpreted as the set [1..n])
- The set of faces (alternatively a positive integer n, which will be interpreted as the set [1..n])
- The list verticesOfEdges. For each edge e the entry verticesOfEdges[e] is a set of all vertices that are incident to the edge e.
- The list edgesOfFaces. For each face f the entry edgesOfFaces[f] is a list of all edges that are incident to the face f. This list has to have the property that two adjacent edges in this list (where we count the first and the last entry to be adjacent) have one vertex in common (this is always the case if the face is a triangle).
- The optional argument namesOfFaces. This is a list and for each face f it has an entry namesOfFaces[f] that is a list with exactly two elements. The first element will be the name of the primary side of the face, the second element will be the name of the secondary side of the face. If this argument is not given, the default naming scheme (+f for primary and -f for secondary) is used.

The ordering of the edges in the list edgesOfFaces defines which side of the the simplicial surface will consider as primary. The NC-version does not check whether the given input follows the following criteria:

- vertices, edges and faces have to be either positive integers or sets of positive integers.
- For each edge e the entry verticesOfEdges[e] has to be a subset of the set of vertices.
- Every vertex has to be in one of the sets in the list verticesOfEdges.
- For each face f the entry edgesOfFaces[f] has to be a subset of the set of edges.
- Each edge has to be in one of the sets in the list edgesOfFaces.
- The edge lists in the components of edgesOfFaces conform to the adjacency condition from before.
- For each face f the list namesOfFaces[f] has exactly two elements that are both integers.

4.1.10 SimplicialSurfaceByVerticesInFaces (for IsSet, IsSet, IsList)

▷ SimplicialSurfaceByVerticesInFaces(*vertices, faces, verticesOfFaces[, namesOfFaces]*) (operation)
▷ SimplicialSurfaceByVerticesInFacesNC(*vertices, faces, verticesOfFaces[, namesOfFaces]*) (operation)

Returns: a simplicial surface
This constructor of a simplicial surface uses the following information:

- The set of vertices (alternatively a positive integer n, which will be interpreted as the set [1..n])
- The set of faces (alternatively a positive integer n, which will be interpreted as the set [1..n])
- The list verticesOfFaces. For each face f the entry verticesOfFaces[f] is a set of all vertices that are incident to the face f.
- The optional argument namesOfFaces. This is a list and for each face f it has an entry namesOfFaces[f] that is a list with exactly two elements. The first element will be the name of the primary side of the face, the second element will be the name of the secondary side of the face. If this argument is not given, the default naming scheme (+f for primary and -f for secondary) is used.

We assume that two vertices that are adjacent in a list verticesOfFaces[f] (where the first and the last entry are adjacent) will have an edge between them. These edges will be constructed automatically. This constructor does not distinguish different sides of each face, so this selection will be made randomly. The NC-version does not check whether the given input follows the following criteria:

- vertices and faces have to be either positive integers or sets of positive integers.
- For each face f the entry verticesOfFaces[f] has to be a subset of the set of vertices.
- For each face f the list verticesOfFaces[f] has to contain at least three different vertices.
- For each face f the list namesOfFaces[f] has exactly two elements that are both integers.

4.2 Access to the incidence structure

4.2.1 Vertices (for IsSimplicialSurface)

▷ Vertices(*simpSurf*) (operation)
▷ VerticesAttributeOfSimplicialSurface(*simpSurf*) (attribute)

Returns: the set of vertices, a set of positive integers
Returns the vertices as a set. The vertices are positive integers.

4.2.2 Edges (for IsSimplicialSurface)

▷ Edges(*simpSurf*) (attribute)

Returns: the set of edges, a set of positive integers
Returns the edges as a set. The edges are positive integers.

4.2.3 Faces (for IsSimplicialSurface)

▷ `Faces(`*simpSurf*`)` (attribute)

Returns: the set of faces, a set of positive integers
Returns the faces as a set. The faces are positive integers.

4.2.4 NrOfVertices (for IsSimplicialSurface)

▷ `NrOfVertices(`*simpSurf*`)` (attribute)

Returns: a non-negative integer
Returns the number of vertices.

4.2.5 NrOfEdges (for IsSimplicialSurface)

▷ `NrOfEdges(`*simpSurf*`)` (attribute)

Returns: a non-negative integer
Returns the number of edges.

4.2.6 NrOfFaces (for IsSimplicialSurface)

▷ `NrOfFaces(`*simpSurf*`)` (attribute)

Returns: a non-negative integer
Returns the number of faces.

4.2.7 EdgesOfVertices (for IsSimplicialSurface)

▷ `EdgesOfVertices(`*simpSurf*`)` (attribute)

Returns: a list of sets of positive integers
Return a list edgesOfVertices such that

- if v is a vertex then edgesOfVertices[v] is the set of all edges that are incident to v.
- if v is not a vertex then edgesOfVertices[v] is not bound.

4.2.8 FacesOfVertices (for IsSimplicialSurface)

▷ `FacesOfVertices(`*simpSurf*`)` (attribute)

Returns: a list of sets of positive integers
Return a list facesOfVertices such that

- if v is a vertex then facesOfVertices[v] is the set of all faces that are incident to v.
- if v is not a vertex then facesOfVertices[v] is not bound.

4.2.9 VerticesOfEdges (for IsSimplicialSurface)

▷ `VerticesOfEdges(`*simpSurf*`)` (attribute)

Returns: a list of sets of positive integers
Return a list verticesOfEdges such that

- if e is an edge then verticesOfEdges[e] is the set of all vertices that are incident to e.

- if e is not an edge then verticesOfEdges[e] is not bound.

4.2.10 FacesOfEdges (for IsSimplicialSurface)

▷ `FacesOfEdges(`*simpSurf*`)` (attribute)

Returns: a list of sets of positive integers

Return a list facesOfEdges such that

- if e is an edge then facesOfEdges[e] is the set of all faces that are incident to e.
- if e is not an edge then facesOfEdges[e] is not bound.

4.2.11 VerticesOfFaces (for IsSimplicialSurface)

▷ `VerticesOfFaces(`*simpSurf*`)` (attribute)

Returns: a list of sets of positive integers

Return a list verticesOfFaces such that

- if f is a face then verticesOfFaces[f] is the set of all vertices that are incident to v.
- if f is not a face then verticesOfFaces[f] is not bound.

4.2.12 EdgesOfFaces (for IsSimplicialSurface)

▷ `EdgesOfFaces(`*simpSurf*`)` (attribute)

Returns: a list of sets of positive integers

Return a list edgesOfFaces such that

- if f is a face then edgesOfFaces[f] is the set of all edges that are incident to v.
- if f is not a face then edgesOfFaces[f] is not bound.

4.2.13 EdgeInFaceByVertices (for IsSimplicialSurface, IsPosInt, IsList)

▷ `EdgeInFaceByVertices(`*simpSurf, face, vertexList*`)` (operation)

Returns: the edge

Return the edge of the given face that is incident to the given vertices.

4.3 Basic properties of simplicial surfaces

4.3.1 EulerCharacteristic (for IsSimplicialSurface)

▷ `EulerCharacteristic(`*simpSurf*`)` (attribute)

Returns: an integer, the Euler characteristic.

Return the Euler characteristic of the given simplicial surface. The Euler characteristic is $|V| - |E| + |F|$, where $|V|$ is the number of vertices, $|E|$ is the number of edges and $|F|$ is the number of faces.

4.3.2 IsTriangleSurface (for IsSimplicialSurface)

▷ `IsTriangleSurface(`*simpSurf*`)` (property)

Returns: `true` or `false`

The property IsTriangleSurface is true if all faces of the simplicial surface are triangles (i.e. they consist of three edges).

4.3.3 IsPathConnected (for IsSimplicialSurface)

▷ `IsPathConnected(`*simpSurf*`)` (property)

Returns: `true` or `false`

If the simplicial surface is path connected (i.e. if every two faces are connected by a face-edge-path) this method returns true, otherwise false.

4.3.4 PathConnectedComponents (for IsSimplicialSurface)

▷ `PathConnectedComponents(`*simpSurf*`)` (attribute)
▷ `PathConnectedComponentOfFace(`*simpSurf, face*`)` (operation)
▷ `PathConnectedComponentOfFaceNC(`*simpSurf, face*`)` (operation)

Returns: a list of simplicial surfaces

Return a list of all path-connected components of the simplicial surface.

If a face is given additionally the path-connected component of this face is returned. The NC-version does not check if the given face actually lies in the simplicial surface.

4.3.5 IsConnected (for IsSimplicialSurface)

▷ `IsConnected(`*simpSurf*`)` (property)

Returns: `true` or `false`

Return if a simplicial surface is connected. If two faces share at least one vertex they are considered to be connected.

4.3.6 ConnectedComponentsAttributeOfSimplicialSurface (for IsSimplicialSurface)

▷ `ConnectedComponentsAttributeOfSimplicialSurface(`*simpSurf*`)` (attribute)
▷ `ConnectedComponents(`*arg*`)` (operation)
▷ `ConnectedComponentOfFace(`*simpSurf, face*`)` (operation)
▷ `ConnectedComponentOfFaceNC(`*simpSurf, face*`)` (operation)

Returns: a list of simplicial surfaces

Return a list of all connected components of the simplicial surface.

If a face is given additionally the connected component of this face is returned. The NC-version does not check if the given face actually lies in the simplicial surface.

4.3.7 UnsortedDegrees (for IsSimplicialSurface)

▷ `UnsortedDegrees(`*simpSurf*`)` (attribute)

Returns: a list of positive integers

Return a list unsortedDegrees with the following property: For each vertex v the entry unsortedDegrees[v] contains the number of faces that are incident to that vertex (this is called the degree of the vertex).

4.3.8 SortedDegrees (for IsSimplicialSurface)

▷ `SortedDegrees(`*simpSurf*`)` (attribute)

Returns: a sorted list of positive integers

Return a sorted list sortedDegrees such that the degree of each vertex (which is defined as the number of incident faces) is contained in the list (counting repetitions).

4.3.9 VertexCounter (for IsSimplicialSurface)

▷ `VertexCounter(`*simpSurf*`)` (attribute)

Returns: a list of positive integers

Return the vertex counter of a simplicial surface. The vertex counter is defined as a list vertexSym, where vertexSym[i] counts the number of vertices that are incident to exactly i edges. If there are no vertices with i incident edges, this entry is unbounded.

4.3.10 EdgeCounter (for IsSimplicialSurface)

▷ `EdgeCounter(`*simpSurf*`)` (attribute)

Returns: a matrix of integers

Return the edge counter of a simplicial surface. The edge counter is a symmetric matrix M such that M[i,j] counts the number of edges such that the two vertices of the edge have edge-degrees i and j.

4.3.11 FaceAnomalyClasses (for IsSimplicialSurface)

▷ `FaceAnomalyClasses(`*simpSurf*`)` (attribute)

Returns: The face-anomaly-classes (as a list of sets)

Return the face-anomaly-classes of a simplicial surface.

Two faces are in the same face-anomaly-class if they contain the same vertices.

4.3.12 EdgeAnomalyClasses (for IsSimplicialSurface)

▷ `EdgeAnomalyClasses(`*simpSurf*`)` (attribute)

Returns: The edge-anomaly-classes (as a list of sets)

Return the edge-anomaly-classes of a simplicial surface (two edges are in the same edge-anomaly-class if they contain the same vertices).

4.3.13 IsAnomalyFree (for IsSimplicialSurface)

▷ `IsAnomalyFree(simpSurf)` (property)

Returns: true or false

Return whether the simplicial surface contains no anomalies (of faces or edges). This property is also known as vertex faithful. A simplicial surface is vertex faithful if all face and edge anomaly classes are trivial.

4.3.14 IncidenceGraph (for IsSimplicialSurface)

▷ `IncidenceGraph(simpSurf)` (attribute)

Returns: the coloured incidence graph

Return the coloured incidence graph of a simplicial surface.

- The vertex set of this graph consists of all vertices, edges and faces of the simplicial surface. All vertices, all edges and all faces are in individual colour classes.
- The edges are given by vertex-edge and edge-face pairs.

4.4 Functions for simplicial surfaces

4.4.1 SubsurfaceByFaces (for IsSimplicialSurface, IsSet)

▷ `SubsurfaceByFaces(simpSurf, faces)` (operation)

▷ `SubsurfaceByFacesNC(simpSurf, faces)` (operation)

Returns: a simplicial surface

Return the subsurface of a simplicial surface that is defined by the given set of faces.

The NC-version does not check if the given faces actually are faces of the simplicial surface.

4.4.2 SnippOffEars (for IsSimplicialSurface)

▷ `SnippOffEars(simpSurf)` (operation)

▷ `SnippOffEarsRecursive(simpSurf)` (operation)

Returns: a simplicial surface

Remove all ears of the simplicial surface and return the resulting surface. An ear is a face that has at most two vertices in common with all other faces.

The recursive-version applies this method recursively until the resulting simplicial surface has no more ears.

4.4.3 IsIsomorphic (for IsSimplicialSurface, IsSimplicialSurface)

▷ `IsIsomorphic(s1, s2)` (operation)

Returns: true or false

Check if two simplicial surfaces are isomorphic. This method only checks if they are isomorphic with respect to the incidence relation. It does not check if additional structure like a wild coloring is isomorphic (or even present).

4.4.4 AddVertexIntoEdge (for IsSimplicialSurface, IsPosInt)

▷ `AddVertexIntoEdge(`*simpSurf, edge*`)` (operation)

Returns: the modified simplicial surface

Add a vertex into an edge. This only works if there are exactly two faces adjacent to the edge.

4.5 Advanced properties of simplicial surfaces

Since the SimplicialSurface-objects in GAP can represent more general structures than just surfaces there is a property that checks whether a generic SimplicialSurface-objects represents an actual surface (the property IsActualSurface). To check whether a generic SimplicialSurface-object represents an actual surface we have to check the edges and the vertices.

- In an actual surface every edge is incident to at most two faces. This property is checked by IsEdgesLikeSurface.
- If each edge is incident to at most two faces, we can define face-edge-paths around each vertex. A face-edge-path around a vertex v is a list $(e_1, f_1, e_2, f_2, \ldots, e_n, f_n)$ or $(e_1, f_1, e_2, f_2, \ldots, e_n, f_n, e_{n+1})$ such that
  - all e_i are pairwise distinct edges incident to the vertex v
  - all f_i are pairwise distinct faces incident to the vertex v
  - if two elements are adjacent in a face-edge-path, they are incident in the simplicial surface
  - if the face-edge-path has even length (the first case), we require that e_1 and f_n are incident (this represents a closed path)
  - if the face-edge-path has odd length (the second case), we require that both e_1 and e_{n+1} are only incident to one face (this represents an open path) In general there may be many of those paths around a vertex (they partition the edges and faces incident to each vertex) but in an actual surface there is only one such path. The property IsVerticesLikeSurface checks this property.

4.5.1 IsVerticesLikeSurface (for IsSimplicialSurface)

▷ `IsVerticesLikeSurface(`*simpSurf*`)` (property)

Returns: `true` or `false`

Under the assumption that every edge is incident to at most two faces (which allows the definition of face-edge-paths) the property IsVerticesLikeSurface holds if there is only one face-edge-path (up to description) around each vertex.

4.5.2 IsEdgesLikeSurface (for IsSimplicialSurface)

▷ `IsEdgesLikeSurface(`*simpSurf*`)` (property)

Returns: `true` or `false`

The property IsEdgesLikeSurface holds if every edge of the simplicial surface is incident to at most two faces.

4.5.3 IsClosedSurface (for IsSimplicialSurface and IsEdgesLikeSurface)

▷ `IsClosedSurface(`*simpSurf*`)` (property)

Returns: `true` or `false`

The property IsActualSurface is true if both IsEdgesLikeSurface and IsVerticesLikeSurface are true. If we have a simplicial surface where every edge is incident to at most two faces, this method checks if the surface is closed. (A simplicial surface is closed if every edge is incident to exactly two faces.)

4.5.4 InnerEdges (for IsSimplicialSurface)

▷ `InnerEdges(`*simpSurf*`)` (attribute)

Returns: a set of edges

Return the set of all inner edges, that is edges with exactly two ajacent faces.

4.5.5 BoundaryEdges (for IsSimplicialSurface)

▷ `BoundaryEdges(`*simpSurf*`)` (attribute)

Returns: a set of edges

Return the set of all border edges, that is edges with only one adjacent face.

4.5.6 RamifiedEdges (for IsSimplicialSurface)

▷ `RamifiedEdges(`*simpSurf*`)` (attribute)

Returns: a set of edges

Return the set of all ramified edges, that is edges that have at least three adjacent faces.

4.5.7 FaceEdgePathsOfVertices (for IsSimplicialSurface and IsEdgesLikeSurface)

▷ `FaceEdgePathsOfVertices(`*simpSurf*`)` (attribute)
▷ `FaceEdgePathsOfVertex(`*simpSurf, vertex*`)` (operation)
▷ `FaceEdgePathsOfVertexNC(`*simpSurf, vertex*`)` (operation)

Returns: a list of lists of face-edge-paths

Return a list fep with the following conditions:

- for each vertex v the entry fep[v] contains a list of all face-edge-paths of this vertex. If there are several possible ways to describe a face-edge-path we choose the representative where the first face is minimal among all faces in the path. If this is not unique we choose the edge between the first and second face to be minimal among all remaining choices.
- every other entry of fep is not bounded.

If a vertex is given additionally, return only a list of face-edge-paths around this vertex. The NC-version does not check whether the given vertex actually is a vertex of the simplicial surface.

4.6 Local and global orientations

If we consider an embedding of a simplicial surface, each face gets mapped onto a polygon (usually a triangle). For each such polygon there are exactly two normal vectors that define the two "sides" of the polygon. Via the right-hand-rule from physics we can identify each normal vector with a cyclic permutation of the vertices in the polygon. For example, if a given face is incident to the vertices $\{1,2,3\}$, then the possible cyclic permutations are (1,2,3) and (1,3,2). If the polygon has more than three vertices then adjacent vertices in these permutations have to be connected by an edge.

The same construction can be used to derive a cyclic permutation of the incident edges. In addition it is sometimes useful to encode these permutations as lists, for example the permutation (1,2,3) corresponds to the list [1,2,3] (the first entry has to be the smallest one to make this choice of list unique).

A *local orientation* of a given face can now be described in four equivalent ways:

- ByVerticesAsPerm

  This is a cyclic permutation p of all vertices incident to the face such that v and p(v) are incident to an edge of the face (for each vertex v in the face).

- ByVerticesAsList

  This is a list of all vertices indicent to the face that encodes the permutation of *ByVerticesAsPerm*. The first entry is the smallest vertex among those incident to the face.

- ByEdgesAsPerm

  This is a cyclic permutation p of all edges incident to the face such that e and p(e) are incident to an edge of the face (for each edge e in the face).

- ByEdgesAsList

  This is a list of all edges indicent to the face that encodes the permutation of *ByEdgesAsPerm*. The first entry is the smallest edge among those incident to the face.

A standard local orientation is used to distinguish the different sides for each face. There is no necessary connection between local orientations of different faces. It is possible to give the sides different names. If no special measures are taken, the side that is distinguished by the standard local orientation is known as f, whereas the other side is known as -f.

If the simplicial surface is orientable (which is only well-defined if the property IsEdgesLikeSurface is true), we can assign a global orientation. More specifically there are 2^c different global orientations, where c denotes the number of path-connected components of the simplicial surface (two faces are path-connected if there exists a face-edge-path between them that is not necessarily incident to only one vertex).

To return a unique global orientation we pick the smallest face in each path-connected component and assign the standard local orientation to this face.

To describe the distinguished global orientation we have the same options as in the local case.

4.6.1 LocalOrientationByVerticesAsPerm (for IsSimplicialSurface)

▷ `LocalOrientationByVerticesAsPerm(`*`simpSurf`*`)` (attribute)
▷ `LocalOrientation(`*`simpSurf`*`)` (operation)

Returns: a list of permutations

Return a list localOr with the following properties:

- For each face f the entry localOr[f] consists of a cycle with all vertices that are incident to f. Adjacent vertices have an edge of the face in common.
- All other positions are not bounded.

This is the default interpretation for the method LocalOrientation.

4.6.2 LocalOrientationByVerticesAsList (for IsSimplicialSurface)

▷ `LocalOrientationByVerticesAsList(`*simpSurf*`)` (attribute)

Returns: a list of lists

Return a list localOr with the following properties:

- For each face f the entry localOr[f] consists of a list with all vertices that are incident to f. Adjacent vertices have an edge of the face in common.
- All other positions are not bounded.

4.6.3 LocalOrientationByEdgesAsPerm (for IsSimplicialSurface)

▷ `LocalOrientationByEdgesAsPerm(`*simpSurf*`)` (attribute)

Returns: a list of permutations

Return a list localOr with the following properties:

- For each face f the entry localOr[f] consists of a cycle with all edges that are incident to f. Adjacent edges have an vertex of the face in common.
- All other positions are not bounded.

4.6.4 LocalOrientationByEdgesAsList (for IsSimplicialSurface)

▷ `LocalOrientationByEdgesAsList(`*simpSurf*`)` (attribute)

Returns: a list of permutations

Return a list localOr with the following properties:

- For each face f the entry localOr[f] consists of a list with all edges that are incident to f. Adjacent edges have an vertex of the face in common.
- All other positions are not bounded.

4.6.5 NamesOfFaces (for IsSimplicialSurface)

▷ `NamesOfFaces(`*simpSurf*`)` (attribute)
▷ `NamesOfFace(`*simpSurf*`, `*face*`)` (operation)
▷ `NamesOfFaceNC(`*simpSurf*`, `*face*`)` (operation)

Returns: a list of lists of integers

Return alist names with the following properties:

- For each face f the entry names[f] is a list of two integers. The first integer is the name of the upper face-side, the second one is the name of the lower face-side (with respect to the local orientation).

- all other entries are unbounded.

If a face is given in addition, the corresponding entry of this list is returned. The NC-version does not throw an error if a non-face is given.

4.6.6 FaceByName (for IsSimplicialSurface, IsInt)

▷ `FaceByName(simpSurf, name)` (operation)

Returns: a positive integer

Return the face of the simplicial surface that has the given name as the name of one of its sides.

4.6.7 IsFaceNamesDefault (for IsSimplicialSurface)

▷ `IsFaceNamesDefault(simpSurf)` (property)

Returns: `true` or `false`

Return whether the naming scheme for the faces is the default one, meaning that the upper side of a face f is called f (a positive integer) and the lower side -f (a negative integer).

4.6.8 IsOrientable (for IsSimplicialSurface and IsEdgesLikeSurface)

▷ `IsOrientable(simpSurf)` (property)

Returns: `true` or `false`

If we have a simplicial surface where every edge is incident to at most two faces, this method checks if the surface is orientable. (A simplicial surface is orientable if we can assign a side for each face such that for every two adjacent sides either both or none are assigned.

4.6.9 GlobalOrientationByVerticesAsPerm (for IsSimplicialSurface and IsEdgesLikeSurface)

▷ `GlobalOrientationByVerticesAsPerm(simpSurf)` (attribute)

▷ `GlobalOrientation(simpSurf)` (operation)

Returns: a list of permutations or fail

Return the distinguished global orientation if the simplicial surface is orientable. Warning: The returned orientation depends on the chosen local orientation of the simplicial surface. This might not be equal even if the incidence structure is.

The orientation is returned as a list globalOr. For each face f the entry globalOr[f] contains a cycle of the vertices that are incident in the face f. The order of this cycle corresponds to the orientation of the face f with respect to the global orientability.

This method returns fail if the given surface is not orientable.

4.6.10 GlobalOrientationByVerticesAsList (for IsSimplicialSurface and IsEdgesLikeSurface)

▷ `GlobalOrientationByVerticesAsList(simpSurf)` (attribute)

Returns: a list of lists or fail

Return the distinguished global orientation if the simplicial surface is orientable. Warning: The returned orientation depends on the chosen local orientation of the simplicial surface. This might not be equal even if the incidence structure is.

The orientation is returned as a list globalOr. For each face f the entry globalOr[f] contains a list of the vertices that are incident in the face f. The order of this list corresponds to the orientation of the face f with respect to the global orientability.

This method returns fail if the given surface is not orientable.

4.6.11 GlobalOrientationByEdgesAsPerm (for IsSimplicialSurface and IsEdgesLikeSurface)

▷ `GlobalOrientationByEdgesAsPerm(`*simpSurf*`)` (attribute)

Returns: a list of permutations or fail

Return the distinguished global orientation if the simplicial surface is orientable. Warning: The returned orientation depends on the chosen local orientation of the simplicial surface. This might not be equal even if the incidence structure is.

The orientation is returned as a list globalOr. For each face f the entry globalOr[f] contains a cycle of the edges that are incident in the face f. The order of this cycle corresponds to the orientation of the face f with respect to the global orientability.

This method returns fail if the given surface is not orientable.

4.6.12 GlobalOrientationByEdgesAsList (for IsSimplicialSurface and IsEdgesLikeSurface)

▷ `GlobalOrientationByEdgesAsList(`*simpSurf*`)` (attribute)

Returns: a list of lists or fail

Return the distinguished global orientation if the simplicial surface is orientable. Warning: The returned orientation depends on the chosen local orientation of the simplicial surface. This might not be equal even if the incidence structure is.

The orientation is returned as a list globalOr. For each face f the entry globalOr[f] contains a list of the edges that are incident in the face f. The order of this list corresponds to the orientation of the face f with respect to the global orientability.

This method returns fail if the given surface is not orientable.

4.7 Technical functions (for development)

This section contains methods that concern the internal structure of simplicial surfaces. You only have to read this section if you want to understand the underlying implementation better or if you want to develop code that is derived from this. There are three unique features in the implementation of simplicial surfaces that especially concern the definition of specializes simplicial surfaces:

- The use of a method selection graph
- A general methods to help defining specialized classes
- A guide for easier initialization

Since the method selection graph is the most salient feature we will cover it first. It derives from a simple observation: If you know either VerticesOfEdges or EdgesOfVertices, you can calculate the other. If you additionally know either of EdgesOfFaces or FacesOfEdges, you can calculate all six of these attributes. This could have been implemented by a lot of specialized methods but the number of

these methods rised exponentially with the number of attributes that are connected. Instead we only implement methods for the "difficult" parts (where work has to be done) and delegate the "easy" parts (if we can calculate B from A and C from B, we can also calculate C from A) into a method selection graph. If an attribute should be part of the method selection graph (which it only should if you can calculate information inside the method selection graph from this attribute) you have to make two modifications:

- There has to be a method to calculate this attribute by the method selection graph, like
- For each "difficult" method there has to be a call that adds this possibility into the method selection graph, like

Secondly we guarantee unique methods for specialization. To consider a specific example, imagine we want to give certain simplicial surfaces an edge colouring. If one simplicial surface may have different edge colourings we can't implement this as an attribute of the simplicial surface. Instead we define a new type for this situation (as a subtype of SimplicialSurfaceType). The disadvantage of this procedure is that is becomes harder to take a simplicial surface as input and add an edge colouring (type changes are frowned upon in GAP). For this reason we offer a special method that does just that - it copies many attributes of the simplicial surface into an object of the new type.

4.7.1 ObjectifySimplicialSurface (for IsType,IsRecord,IsSimplicialSurface)

▷ `ObjectifySimplicialSurface(`*type, record, simpSurf*`)` (operation)

Returns: an object of type type

This function calls and afterwards copies all attributes and properties of the simplicial surface modelSurf that are declared in this section to the the new object. This method has to be overwritten for a specialization of this class. WARNING: The type can't be checked! Only types that are derived from IsSimplicialSurface can be used with impunity!

Finally we consider the constructors. Simplicial surfaces are defined by an incidence structure *and* a local orientation (with face names). However, only the incidence structure has to be given - the local orientation can be derived (in the sense that there are several possibilites and one of them will be picked). To facilitate this procedure we offer a method which can be called *after* the incidence structure is defined.

4.7.2 DeriveLocalOrientationAndFaceNamesFromIncidenceGeometry (for IsSimplicialSurface)

▷ `DeriveLocalOrientationAndFaceNamesFromIncidenceGeometry(`*simpSurf*`)` (operation)

▷ `DeriveLocalOrientationAndFaceNamesFromIncidenceGeometryNC(`*arg*`)` (operation)

Returns: nothing

This is a method which should only be used in code development. It should not be called by a normal user as it presupposes knowledge of the internal attribute storing system. A simplicial surface consists of two separate sets of attributes: One set of attributes to save the incidence geometry (Vertices, Edges, Faces, EdgesOfFaces, etc.), the other to save the local orientation of the faces (LocalOrientationOfVerticesAsPerm, NamesOfFaces, etc.). While the second set of attributes may be crucial for some applications (like folding), it is easy to ignore it in other applications. This is usually managed by a judicious constructor call that will handle the necessary overhead without burdening the user. If - for whatever reason - no constructor should be called (for example for a subcategory of

IsSimplicialSurface) this method can be used to initialize all necessary attributes of the second set. This method will throw an error if some of these attributes are already set. It will only check the attributes

- LocalOrientationByVerticesAsPerm
- LocalOrientationByVerticesAsList
- LocalOrientationByEdgesAsPerm
- LocalOrientationByEdgesAsList
- IsFaceNamesDefault
- NamesOfFaces

If other attributes interfere with these, they will not be checked! For this reason this method should only be called if one knows exactly which attributes are already set. The NC-version doesn't check whether attributes are set (it is therefore even more dangerous to use).

4.7.3 PrintStringAttributeOfSimplicialSurface (for IsSimplicialSurface)

▷ `PrintStringAttributeOfSimplicialSurface(`*simpSurf*`)` (attribute)

Returns: a string

Return the string that is printed by the PrintObj-method. This method is used since it may be expensive to compute the string.

4.7.4 AlternativeNames (for IsSimplicialSurface)

▷ `AlternativeNames(`*simpSurf*`)` (attribute)

Returns: a record

Return the record of alternative names. This attribute is usually set by some constructing operation and encodes information about the construction. The returned record has one component for each alternative naming scheme. Each of these components can have these three components: Vertices, Edges and Faces. Those are lists, which have an entry for every vertex/edge/face of the simplicial surface. Every other position is unbounded. If this attribute was not set explicitly, it will be an empty record.

4.7.5 CommonCover (for IsSimplicialSurface and IsEdgesLikeSurface and IsTriangleSurface,IsSimplicialSurface and IsEdgesLikeSurface and IsTriangleSurface,IsList, IsList)

▷ `CommonCover(`*surf1, surf2, mrType1, mrType2*`)` (operation)

Returns: the common cover

Compute the common cover of two simplicial surfaces. TODO

Chapter 5

Wild Simplicial Surfaces

5.1 Constructors for wild simplicial surfaces

5.1.1 WildSimplicialSurfaceByDownwardIncidenceAndEdgeColouring (for IsSet, IsSet, IsSet, IsList, IsList, IsList)

▷ `WildSimplicialSurfaceByDownwardIncidenceAndEdgeColouring(`*`vertices, edges, faces, verticesOfEdges, edgesOfFaces`*`)` (operation)

Returns: a wild simplicial surface

This is a constructor for a wild simplicial surface based on the following information: - vertices A set of vertices (a positive integer n may represent [1..n]) - edges A set of edges (a positive integer n may represent [1..n]) - faces A set of faces (a positive integer n may represent [1..n]) - verticesOfEdges A list of sets, where a set of two vertices is given for each edge - edgesOfFaces A list of sets, where a set of three edges is given for each face - coloursOfEdges A list that contains the numbers 1,2,3. For each edge the colour of this edge is given. There have to be an equal amount of edges for each colour and the edges of each face have to have different colours. The NC-version doesn't check whether the given information is consistent. coloursOfEdges

5.1.2 WildSimplicialSurfaceByDownwardIncidenceAndEdgeColouringNC (for IsSet, IsSet, IsSet, IsList, IsList, IsList)

▷ `WildSimplicialSurfaceByDownwardIncidenceAndEdgeColouringNC(`*`arg1, arg2, arg3, arg4, arg5, arg6`*`)` (operation)

5.1.3 WildSimplicialSurfaceByDownwardIncidenceAndGenerators (for IsSet, IsSet, IsSet, IsList, IsList, IsList)

▷ `WildSimplicialSurfaceByDownwardIncidenceAndGenerators(`*`arg1, arg2, arg3, arg4, arg5, arg6`*`)` (operation)

This is a constructor for a wild simplicial surface based on the following information: - vertices A set of vertices (a positive integer n may represent [1..n]) - edges A set of edges (a positive integer n may represent [1..n]) - faces A set of faces (a positive integer n may represent [1..n]) - verticesOfEdges A list of sets, where a set of two vertices is given for each edge - edgesOfFaces A list of sets, where

a set of three edges is given for each face - generators A list of three involutions whose cycles define the edge colouring. If this is not unique (that is, if there is are two edges that are incident to the same two faces, then an error is thrown). The NC-version doesn't check whether the given information is consistent.

5.1.4 WildSimplicialSurfaceByDownwardIncidenceAndGeneratorsNC (for IsSet, IsSet, IsSet, IsList, IsList, IsList)

▷ `WildSimplicialSurfaceByDownwardIncidenceAndGeneratorsNC(`*arg1, arg2, arg3, arg4, arg5, arg6*`)` (operation)

5.1.5 WildSimplicialSurfaceExtensionByEdgeColouring (for IsSimplicialSurface and IsActualSurface and IsTriangleSurface, IsList)

▷ `WildSimplicialSurfaceExtensionByEdgeColouring(`*arg1, arg2*`)` (operation)

This is a constructor for a wild simplicial surface based on the following information: - surface A simplicial surface which consists only of triangles and is an actual surface. - coloursOfEdges A list that contains the numbers 1,2,3. For each edge the colour of this edge is given. There have to be an equal amount of edges for each colour and the edges of each face have to have different colours. The NC-version doesn't check whether the given information is consistent.

5.1.6 WildSimplicialSurfaceExtensionByEdgeColouringNC (for IsSimplicialSurface and IsActualSurface and IsTriangleSurface, IsList)

▷ `WildSimplicialSurfaceExtensionByEdgeColouringNC(`*arg1, arg2*`)` (operation)

5.1.7 WildSimplicialSurfaceExtensionByGenerators (for IsSimplicialSurface and IsActualSurface and IsTriangleSurface, IsList)

▷ `WildSimplicialSurfaceExtensionByGenerators(`*arg1, arg2*`)` (operation)

This is a constructor for a wild simplicial surface based on the following information: - surface A simplicial surface which consists only of triangles and is an actual surface. - generators A list of three involutions whose cycles define the edge colouring. If this is not unique (that is, if there is are two edges that are incident to the same two faces, then an error is thrown). The NC-version doesn't check whether the given information is consistent.

5.1.8 WildSimplicialSurfaceExtensionByGeneratorsNC (for IsSimplicialSurface and IsActualSurface and IsTriangleSurface, IsList)

▷ `WildSimplicialSurfaceExtensionByGeneratorsNC(`*arg1, arg2*`)` (operation)

5.1.9 WildSimplicialSurfaceByFaceEdgesPathsAndEdgeColouring (for IsSet, IsSet, IsSet, IsList, IsList)

▷ `WildSimplicialSurfaceByFaceEdgesPathsAndEdgeColouring(`*arg1, arg2, arg3, arg4, arg5*`)` (operation)

This is a constructor for a wild simplicial surface based on the following information: - vertices A set of vertices (a positive integer n may represent [1..n]) - edges A set of edges (a positive integer n may represent [1..n]) - faces A set of faces (a positive integer n may represent [1..n]) - faceEdgePaths A list of the face-edge-paths for each vertex. - coloursOfEdges A list that contains the numbers 1,2,3. For each edge the colour of this edge is given. There have to be an equal amount of edges for each colour and the edges of each face have to have different colours. The NC-version doesn't check whether the given information is consistent.

5.1.10 WildSimplicialSurfaceByFaceEdgesPathsAndEdgeColouringNC (for IsSet, IsSet, IsSet, IsList, IsList)

▷ `WildSimplicialSurfaceByFaceEdgesPathsAndEdgeColouringNC(`*arg1, arg2, arg3, arg4, arg5*`)` (operation)

5.1.11 WildSimplicialSurfaceByColouredFaceEdgePaths (for IsSet, IsSet, IsList)

▷ `WildSimplicialSurfaceByColouredFaceEdgePaths(`*arg1, arg2, arg3*`)` (operation)

This is a constructor for a wild simplicial surface based on the following information (the edges will be constructed internally): - vertices A set of vertices (a positive integer n may represent [1..n]) - faces A set of faces (a positive integer n may represent [1..n]) - colfaceEdgePaths A list of the coloured face-edge-paths for each vertex. The NC-version doesn't check whether the given information is consistent.

5.1.12 WildSimplicialSurfaceByColouredFaceEdgePathsNC (for IsSet, IsSet, IsList)

▷ `WildSimplicialSurfaceByColouredFaceEdgePathsNC(`*arg1, arg2, arg3*`)` (operation)

5.2 Attributes and properties of wild coloured simplicial surfaces

5.2.1 Generators (for IsWildSimplicialSurface)

▷ `Generators(a, wild, simplicial, surface)` (attribute)

Returns: a dense list of permutations

Returns the generators of the wild simplicial surface in a list.

5.2.2 GroupOfWildSimplicialSurface (for IsWildSimplicialSurface)

▷ `GroupOfWildSimplicialSurface(a, wild, simplicial, surface)` (attribute)

Returns: a group

Return the group that is generated by the generators of the wild simplicial surface.

5.2.3 VertexGroup (for IsWildSimplicialSurface)

▷ `VertexGroup(arg)` (attribute)

Returns: finitely presented group.

Given a wild coloured simplicial surface simpsurf, this function determines the vertex group of the simplicial surface. The vertex group of the simplicial surface simpsurf is defined to be F_3/R, where F_3 is the free group on three generators and R is the set of relations given by the vertex defining paths of the inner vertices.

5.2.4 ColoursOfEdges (for IsWildSimplicialSurface)

▷ `ColoursOfEdges(a, wild, simplicial, surface)` (attribute)

Returns: a list

Return the edge colors. We return a list which has unbounded entries for all edges (and only those). At the position 'edge' the colour of this edge is saved - that is, to which generator this edge belongs.

5.2.5 ColourOfEdge (for IsWildSimplicialSurface, IsPosInt)

▷ `ColourOfEdge(arg1, arg2)` (operation)

5.2.6 ColourOfEdgeNC (for IsWildSimplicialSurface, IsPosInt)

▷ `ColourOfEdgeNC(arg1, arg2)` (operation)

5.2.7 ColouredEdgesOfFaces (for IsWildSimplicialSurface)

▷ `ColouredEdgesOfFaces(a, wild, simplicial, surface)` (attribute)

Returns: a list

Return a list which is indexed by the faces. For each face we get a list of the incident edges such that the first entry corresponds to the edge of the first colour, the second entry to the edge of the second colour and the third entry to the edge of the third colour.

5.2.8 ColouredEdgeOfFace (for IsWildSimplicialSurface, IsPosInt, IsPosInt)

▷ `ColouredEdgeOfFace(arg1, arg2, arg3)` (operation)

5.2.9 ColouredEdgeOfFaceNC (for IsWildSimplicialSurface, IsPosInt, IsPosInt)

▷ `ColouredEdgeOfFaceNC(arg1, arg2, arg3)` (operation)

5.2.10 ColouredFaceEdgePathsOfVertices (for IsWildSimplicialSurface)

▷ ColouredFaceEdgePathsOfVertices(a, *wild, simplicial, surface*) (attribute)

Returns: a list

A coloured face edge path of an inner vertex v of a wild simplicial surface is a list $[c_1, f_1, c_2, f_2, ..., c_n, f_n]$, where f1,...,fn are the faces incident to v and c1,..,cn are colours such that the edges of face fi with ci and c(i+1) are those incident to v and c(n+1)=c1. A coloured face edge path of a boundary vertex v of a wild simplicial surface is a list [c1,f1,c2,f2,...,cn,fn, c(n+1)], where f1,...,fn are the faces incident to v and c1,..,c(n+1) are colours such that the edges of face fi with ci and c(i+1) are those incident to v and the edges of f1 and fn with colours c1 and c(n+1), respectively, are boundary edges.

5.2.11 ColouredFaceEdgePathOfVertex (for IsWildSimplicialSurface,IsPosInt)

▷ ColouredFaceEdgePathOfVertex(*arg1, arg2*) (operation)

5.2.12 ColouredFaceEdgePathOfVertexNC (for IsWildSimplicialSurface,IsPosInt)

▷ ColouredFaceEdgePathOfVertexNC(*arg1, arg2*) (operation)

5.2.13 EdgesOfColours (for IsWildSimplicialSurface)

▷ EdgesOfColours(a, *wild, simplicial, surface*) (attribute)

Returns: a list of sets of edges

Return a list of three sets, where the first set consists of all edges of the first colour, and so on.

5.2.14 EdgesOfColour (for IsWildSimplicialSurface, IsPosInt)

▷ EdgesOfColour(*arg1, arg2*) (operation)

5.2.15 EdgesOfColourNC (for IsWildSimplicialSurface, IsPosInt)

▷ EdgesOfColourNC(*arg1, arg2*) (operation)

5.2.16 MRTypeOfEdges (for IsWildSimplicialSurface)

▷ MRTypeOfEdges(*simpsurf*, a, *simplicial, surface, object, as, created*) (attribute)

Returns: a list of three lists, each of which contains the entries 0, 1 or 2.

Given a wild coloured simplicial surface simpsurf, this function determines the mr-type of each of the edges of simpsurf. The mr-type of an edge of simpsurf is either "m" (for mirror) or "r" (for rotation). It is defined as followed. Suppose the edge e is incident to the vertices v_1 and v_2 and to the two faces F and F'. Let x and y be the edges of incident incident to F and F' and to the same vertex v_1, say. Then e is of type m if both x and y have the same colour, and e is of type r if x and y are different. As we assume the surface to be wild coloured, this means that the colours of the other edges incident to e and both faces F and F' are then also determined. As the # $'$ edges of the simplicial surface are

pairs of points, the mr-type of the simplicial surface simpsurf can be encoded as a list of length 3. Each of the entries is in turn a list encoding the mr-type of all edges of a certain colour. Suppose that mrtype[1] is the list encoding the mr-type of the red edges. Then mrtype[1][i] = 0 if the mr-type of the red edge incident to the vertex i is unknown, mrtype[1][i] = 1 if the mr-type of the red edge incident to the vertex i is "m", and mrtype[1][i] = 2 if the mr-type of the red edge incident to the vertex i is "r". by WildSimplicialSurface

5.2.17 MRTypeOfEdgesAsNumbers (for IsWildSimplicialSurface)

▷ `MRTypeOfEdgesAsNumbers(arg)` (attribute)

5.2.18 IsSurfaceWithStructure (for IsWildSimplicialSurface)

▷ `IsSurfaceWithStructure(wildSurf)` (property)

Returns: true or false

If the mr-types of all edges with the same colour are identical, we call this a structure of the surface. This property checks whether this is the case.

5.2.19 MRTypeOfSurfaceWithStructure (for IsWildSimplicialSurface and IsSurfaceWithStructure)

▷ `MRTypeOfSurfaceWithStructure(wildSurf)` (attribute)

Returns: a list

Return the MR-type of a surface with structure in the form of a list [type of first colour, type of second colour, type of third colour].

5.2.20 MRTypeOfSurfaceWithStructureAsNumbers (for IsWildSimplicialSurface and IsSurfaceWithStructure)

▷ `MRTypeOfSurfaceWithStructureAsNumbers(arg)` (attribute)

5.3 Functions for wild coloured simplicial surfaces

5.3.1 EdgeByFacesAndColour (for IsWildSimplicialSurface, IsSet, IsPosInt)

▷ `EdgeByFacesAndColour(wildSimplicialSurface, setOfFaces, colour)` (operation)

Returns: an edge

Given a set of faces and a colour, return the edge such that the set of faces is the set of faces incident to the edge and that the edge has the given colour. The NC-version doesn't check if the given colour is valid.

5.3.2 EdgeByFacesAndColourNC (for IsWildSimplicialSurface, IsSet, IsPosInt)

▷ `EdgeByFacesAndColourNC(arg1, arg2, arg3)` (operation)

5.3.3 DoubleCover (for IsWildSimplicialSurface)

▷ `DoubleCover(wildSimplicialSurface)` (operation)

Returns: the double cover

Return the double cover of the wild simplicial surface, keeping the MR-type fixed.

5.3.4 SixFoldCover (for IsSimplicialSurface, IsList)

▷ `SixFoldCover(surface, mrtype, bool)` (operation)

Returns: a wild coloured simplicial surface if bool is true or not given, the mrtype specifies the behaviour of the σ_i, if bool is false, the mrtype specifies the behaviour of the edges.

The function SixFoldCover takes as input a generic description of a simplicial surface. The six fold cover of a simplicial surface is the following surface. If f is a face of the original face with edge numbers e_a, e_b and e_c, then the face is covered by the six faces of the form (f, e1, e2, e3), for which {e1, e2, e3} = {e_a, e_b, e_c}. See Proposition 3.XX in the paper. If the optional argument mrtype is given, it has to be a list of length 3 and each entry has to be 1, or 2. In this case the six fold cover will treat the position i for $i \in \{1,2,3\}$ of the three edges around a faces either as a reflection (mirror), if the entry in position i of mrtype is 1, or as a rotation, if the entry in position i is 2. That is, the cover surface is generated by three transpositions σ_i for $i = 1,2,3$. For $i = 1$, suppose f and f' are faces of the surface surf such that the edges of f are e_1, e_2 and e_3 and the edges of f' are e_1, e_a, e_b are the edges e_1, e_2 and e_a intersect in a common vertex and the edges e_1, e_3 and e_b intersect in a common vertex. For $i = 1$ and mrtype of position 1 being mirror (i.e. 1), then

$$\sigma_1(f, e_1, e_2, e_3) = (f', e_1, e_a, e_b),$$

whereas if the mrtype of position 1 is a reflection (i.e. 2), then

$$\sigma_1(f, e_1, e_2, e_3) = (f', e_1, e_b, e_a).$$

The definition of σ_2 and σ_3 are analogous, with e_2, respectively e_3 taking the role of the common edge e_1. If the optional argument mredges is given, and mredges is a list of length equal to the number of edges of the surface *surf* and an entry for an edge e is either 1 or 2. If the entry is 1 then the six fold cover will treat the edge as a reflection (mirror) and if the entry is 2 then the edge is treated as a rotation. The six fold cover is always a wild colourable simplicial surface.

5.3.5 ImageWildSimplicialSurface (for IsWildSimplicialSurface, IsPerm)

▷ `ImageWildSimplicialSurface(surface, perm)` (operation)

Returns: The new wild simplicial surface

Compute the action of a permutation on the faces of a wild simplicial surface. The permutation acts on the faces, while the names of vertices and edges may change.

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