# Simplicial surfaces in GAP

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1. General polygonal complexes by incidence geometry
2. Edge colouring and group properties
3. Abstract folding

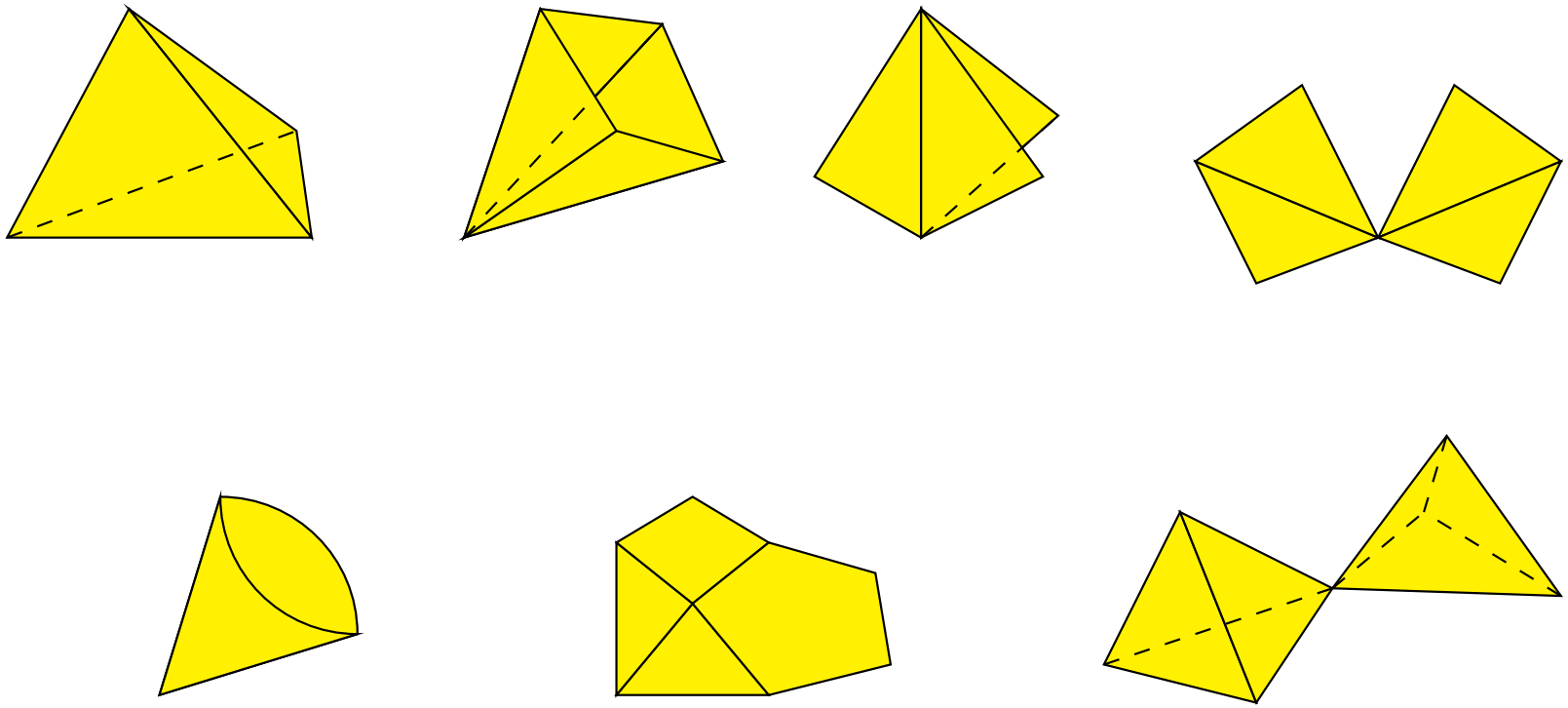
1 General polygonal complexes by incidence geometry

2 Edge colouring and group properties

3 Abstract folding

# Motivation

Goal: simplicial surfaces (and generalisations) in GAP



$\rightsquigarrow$ examples of **polygonal complexes**

# No embedding

We do not work with embeddings (mostly)

- is very hard to compute
- if often unknown for an abstractly constructed surface
- is different from *intrinsic structure*

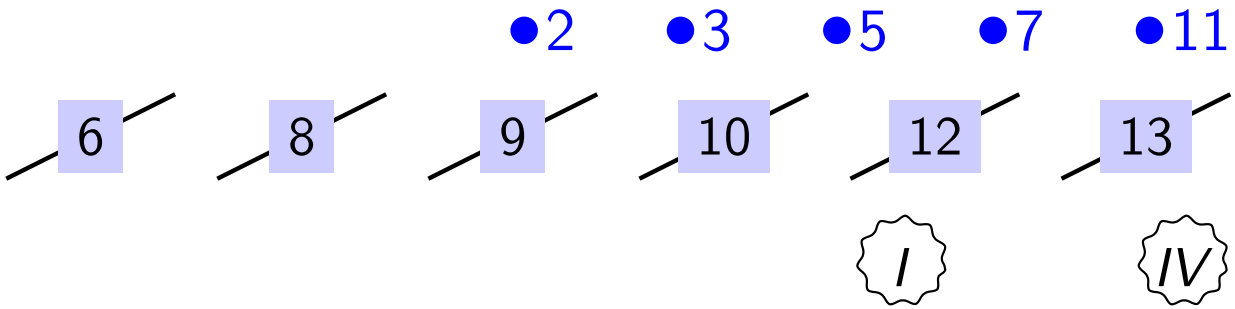
$\Rightarrow$ lengths and angles are not important

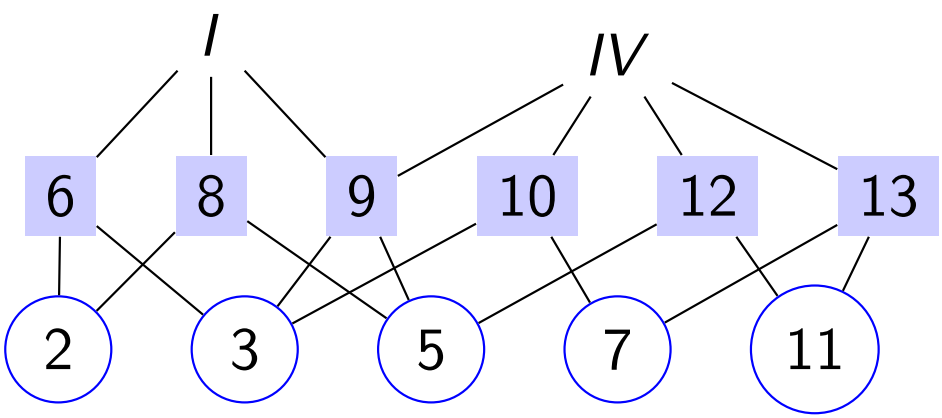
$\rightsquigarrow$ incidence structure is intrinsic

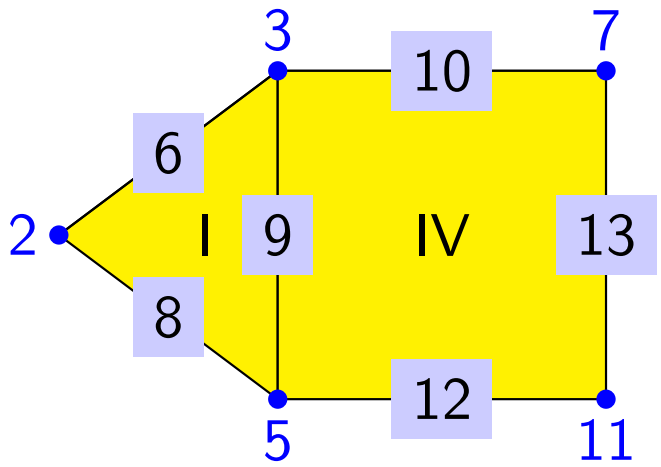
# Incidence structure of a polygonal complex

A **polygonal complex** consists of

- set of vertices $\mathcal{V}$
- set of edges $\mathcal{E}$
- set of faces $\mathcal{F}$
- transitive relation $\subseteq (\mathcal{V} \times \mathcal{E}) \uplus (\mathcal{V} \times \mathcal{F}) \uplus (\mathcal{E} \times \mathcal{F})$



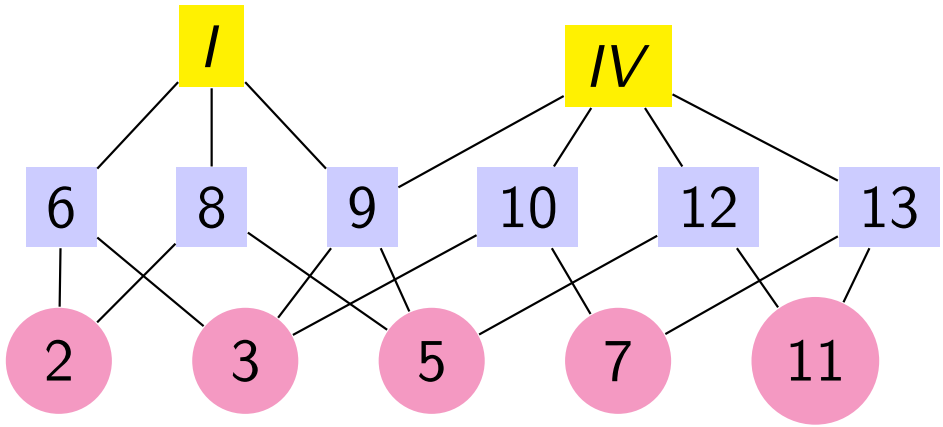




1. Every face is a polygon
2. Every vertex lies in an edge and every edge lies in a face

# Isomorphism testing

Incidence geometry allows "easy" isomorphism testing. Incidence structure can be interpreted as a coloured graph:



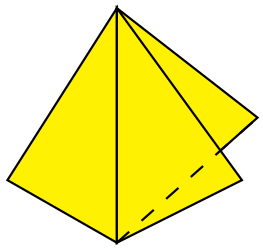

$\rightsquigarrow$ reduce to graph isomorphism problem
Solved by NautyTracesInterface (by Gutsche, Niemeyer, Schweitzer)

# General properties

Some properties can be computed for all polygonal complexes:

- Connectivity
- Euler–Characteristic

*Orientability* is **not** one of them. Counterexample:

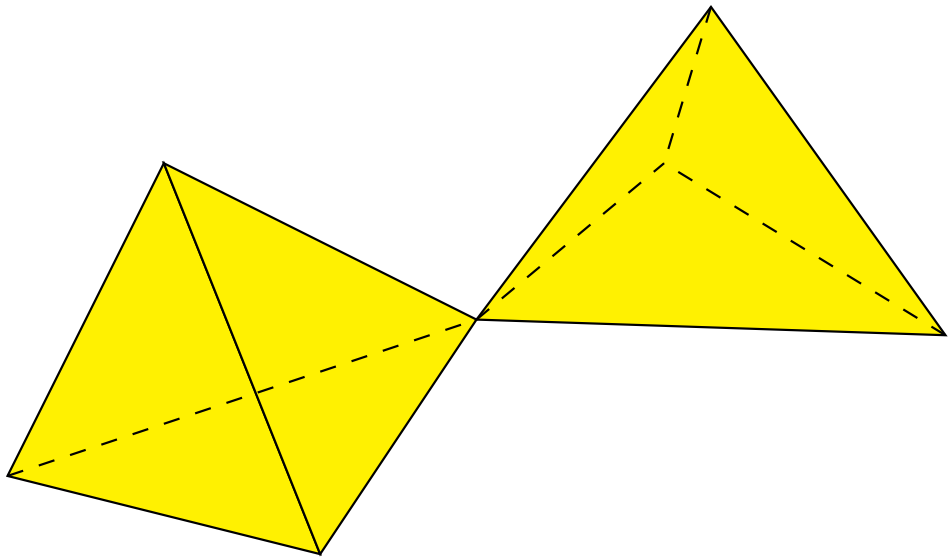


$\Rightarrow$ every edge lies in at most two faces (for well–definedness)

$\rightsquigarrow$ **ramified polygonal surfaces**

# Why ramified?

Typical example of ramified polygonal surface:



$\Rightarrow$ It is not a surface – there is a *ramification* at the central vertex
A **polygonal surface** does not have these ramifications.

1. General polygonal complexes by incidence geometry

2. Edge colouring and group properties

3. Abstract folding

# Embedding question

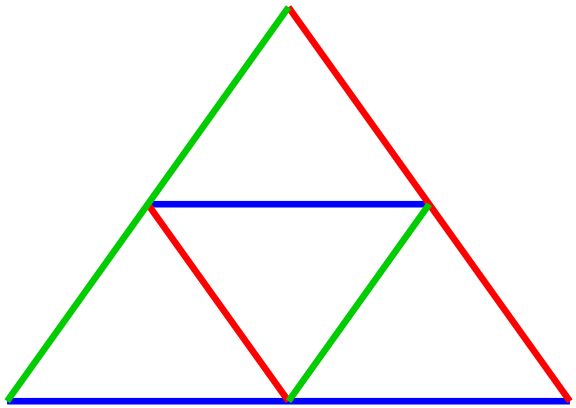
Given: A polygonal complex

- Can it be embedded?
- In how many ways?

Simplifications:

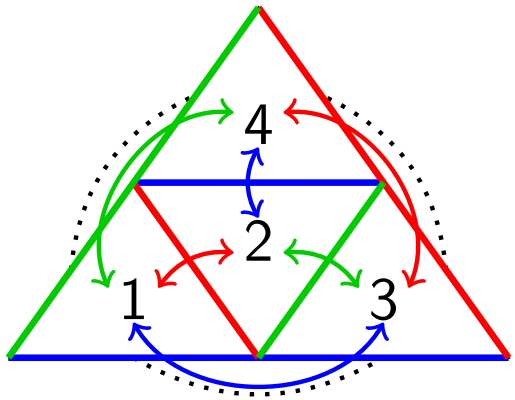
1. Only polygonal surfaces (surface that is build from polygons)
2. All polygons are triangles (**simplicial surfaces**)
3. All triangles are isometric

$\rightsquigarrow$ Edge–colouring encodes different lengths



# Colouring as permutation

Consider tetrahedron with edge colouring

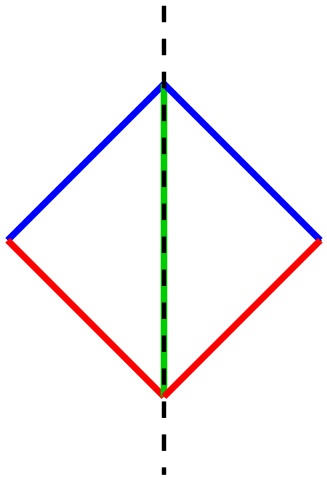



*simplicial surface* $\Rightarrow$ at most two faces at each edge

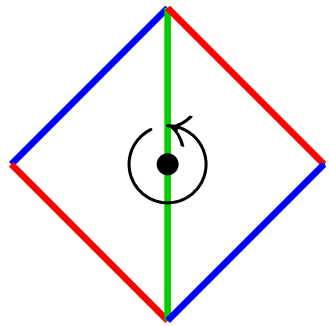
- $\rightsquigarrow$ every edge defines transposition of incident faces
- $\rightsquigarrow$ every colour class defines permutation of the faces
  - (1,2)(3,4) , (1,3)(2,4) , (1,4)(2,3)
- $\rightsquigarrow$ group theoretic considerations
  - The connected components of the surface correspond to the orbits of $\langle \sigma_a, \sigma_b, \sigma_c \rangle$ on the faces

# How do faces fit together?

Consider a face of the surface and a neighbouring face
The neighbour can be coloured in two ways:



mirror (m)

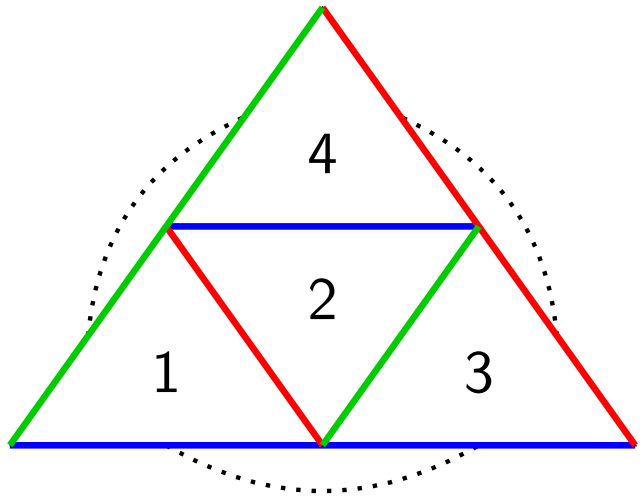


rotation (r)

This gives an **mr–assignment** for the edges.
Permutations and mr–assignment uniquely determine the surface.

# Constructing surfaces from groups

A general mr–assignment leads to complicated surfaces.
Simplification: edges of same colour have the same type
Example




has an rrr–structure
The easiest structure is an mmm–structure.

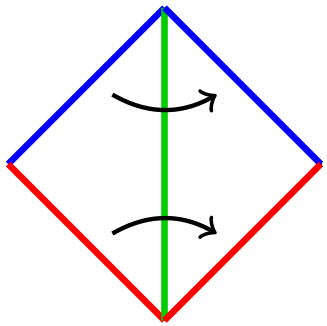
# Covering

We want to characterize surfaces where all edges are mirrors.

## Lemma

*A simplicial surface has an mmm–structure iff it covers a single triangle, i. e. there is an incidence–preserving map to the simplicial surface consisting of exactly one face.*

Consider



- Covering pulls back a colouring of the triangle.
- Colouring defines a map to the triangle.

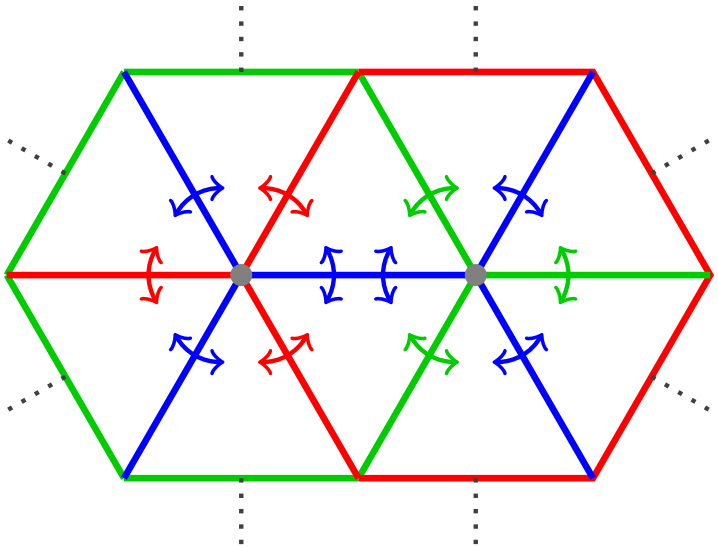
# Construction from permutations

Start with three involutions $\sigma_a$, $\sigma_b$, $\sigma_c$ (like generators of a finite group)

## Lemma

*There exists a coloured surface with the given involutions where all edges are mirror edges.*

- The faces are the points moved by the involutions
- The edges are the cycles of the involutions
- The vertices are the orbits of $\langle \sigma_a, \sigma_b \rangle$ on the faces (for all pairs)

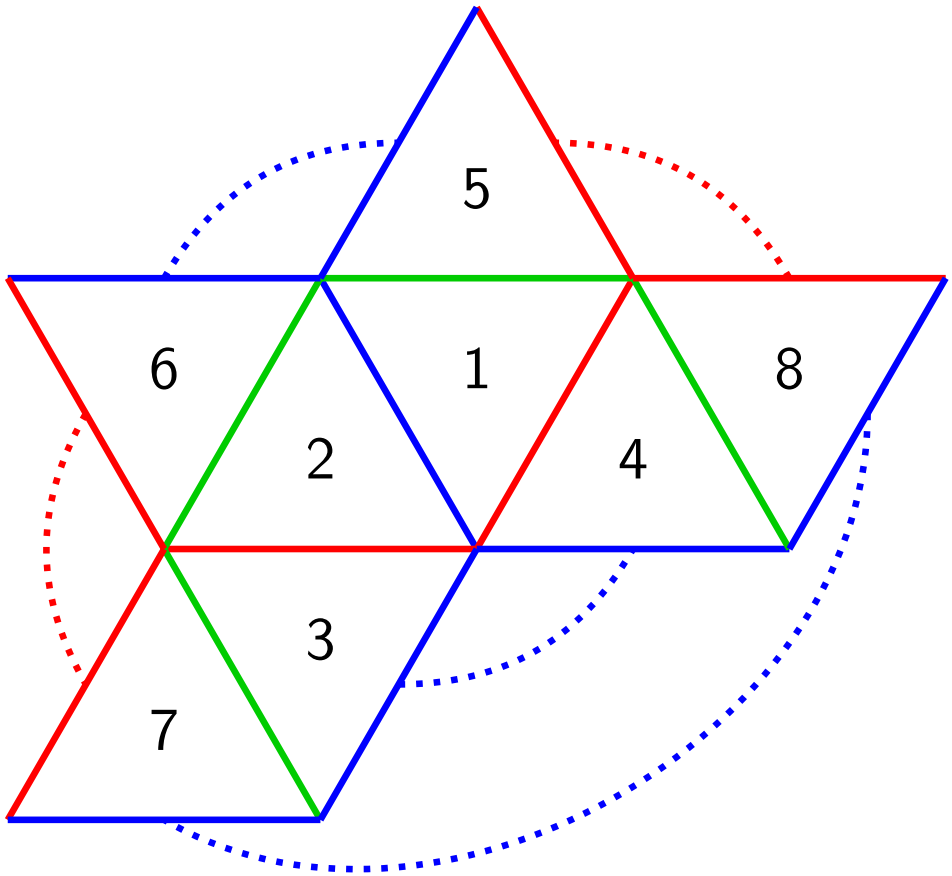


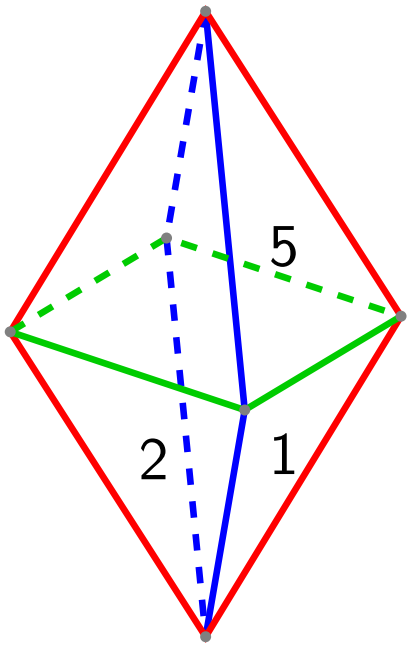
# Construction example

$\sigma_a = (1,2)(3,4)(5,6)(7,8)$

$\sigma_b = (1,4)(2,3)(5,8)(6,7)$

$\sigma_c = (1,5)(2,6)(3,7)(4,8)$



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# What kind of folding?

There are many different kinds of folding (e. g. Origami) Here:

- Folding of surface in $\mathbb{R}^3$
- Possible folding edges are fixed
- Folding should be rigid (no curvature)

Goal: Classify possible folding patterns (given a net)

# Why are embeddings hard?

Ideally, we would like to have embeddings.

But we want to define folding independently from an embedding, since:

- They are very hard to compute (even for small examples)
- We can only show foldability for specific small examples
  - Usually using regularity (like crystallographic symmetry)
  - No general method
- It is very hard to define iterated folding in an embedding

# Is there an alternative?

Central idea:

- Don't model folding process (needs embedding)
- Describe starting and final folding state
  - Only consider changes in the topology (like identification of faces)
  - allows abstraction from embedding

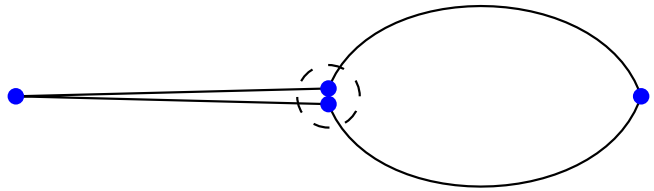
$\rightsquigarrow$ Incidence geometry (polygonal complex/surface)

- Captures some folding restrictions (rigidity of tetrahedron)
- Still needs a lot of refinement

# Important properties of folding

- The class of surfaces is not closed under folding
- Folding can be undone by *unfolding*
- Identification of two faces might force identification of two other faces
  - Can apply to arbitrary many faces
  - The forced identification is not unique

  $\Rightarrow$ Identify only two faces at a time

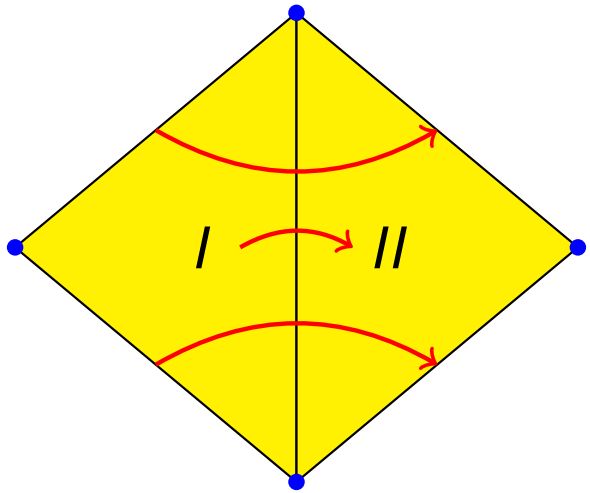


# How to define abstract folding?

We need to define two structures:

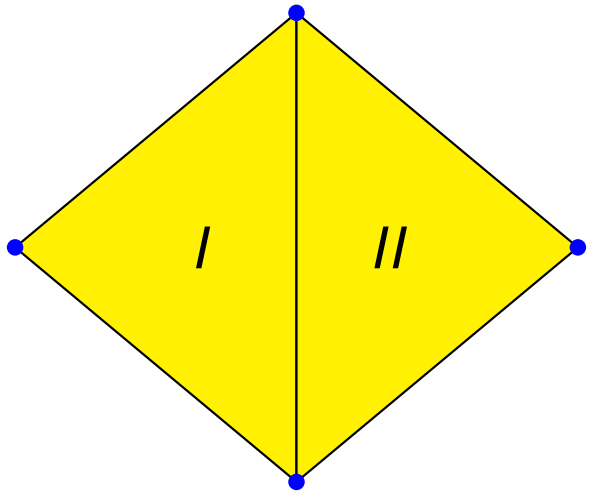
1. A folding state
   - Based on polygonal complexes
   - Describe "is folded together" by an equivalence relation
   - Describe order of faces in folding state
2. The folding steps
   - Only two faces at a time
   - Explain "unordered folding" (e. g. covering)
   - Modify to include face order relations

# Unordered Folding (Covering)




Why do we need more than a polygonal complex?
Naive folding definition: surjective map that respects incidence
Problem: Can't be unfolded
$\Rightarrow$ Folding state should not forget original structure

# Unordered Folding (Covering)



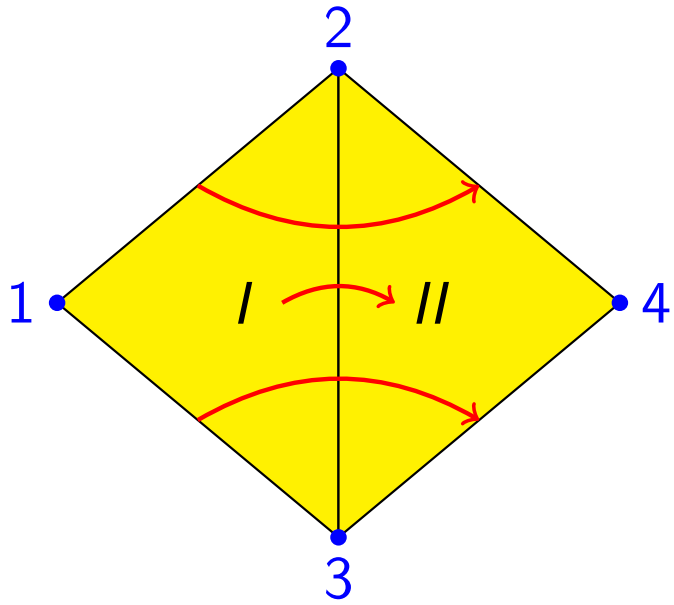

Represent folding by equivalence relation

- Separate relation on vertices, edges and faces
- Two elements are equivalent if they are folded together
- If two edges are equivalent, then their vertices have to be as well (likewise for faces)
- The vertices of an edge are not equivalent (likewise for faces)

$\Rightarrow$ Unordered folding is coarsening of equivalence relation

# How does folding work?

1. Choose two faces that are not folded together
2. Choose how to identify them (like $I \sim II$ and $1 \sim 4$)
3. Add those pairs to the equivalence relation

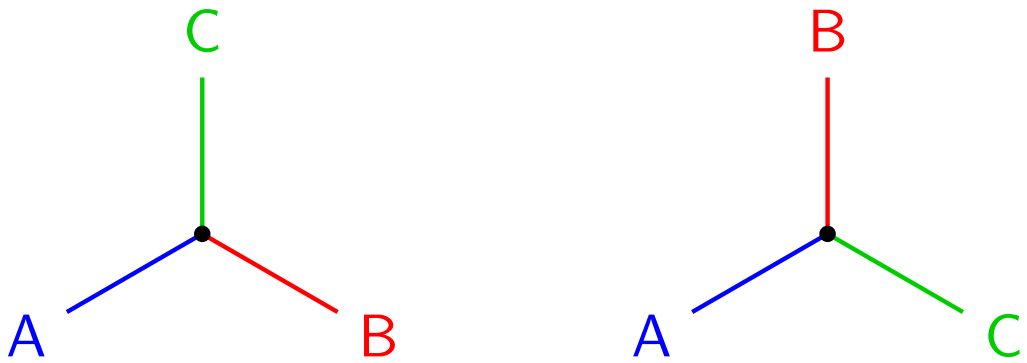



Restriction:
Two vertices in an edge can't be identified (slightly generalized)

# Restrictions of unordered folding

We can't work with ordering of faces:

Adding a linear order on each face equivalence class is not enough:




$\rightsquigarrow$ define order of faces around edges (we will skip the details)

# Folding complex

## Definition

*A* **folding complex** *is a polygonal complex together with*

1. *An equivalence relation on vertices, edges and faces ("is folded together")*
2. *A linear ordering on each face equivalence class*
3. *A cyclical ordering of the faces around each edge equivalence class*

*such that the orderings are compatible (in an appropriate sense).*

To identify faces with each other, we have to combine those orderings.

- linear orderings get concatenated
- cyclical orderings are opened at one point and combined

!! compatibility is not easily transfered, but can be calculated