Simplicial surfaces in GAP

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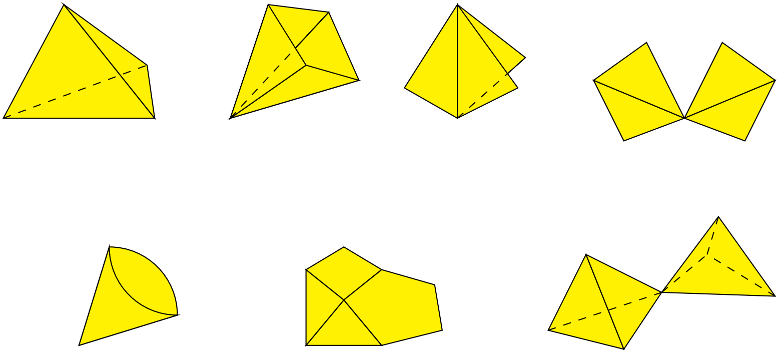
??.08.2017

Motivation

Goal: simplicial surfaces (and generalisations) in GAP



⇝ examples of **polygonal complexes**

No embedding

We do not work with embeddings (mostly)

- is very hard to compute
- if often unknown for an abstractly constructed surface
- is different from *intrinsic structure*

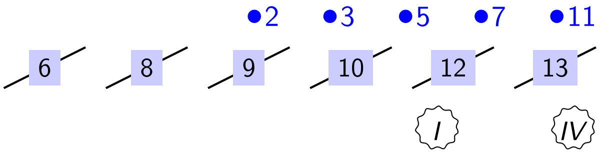
$\Rightarrow$ lengths and angles are not important

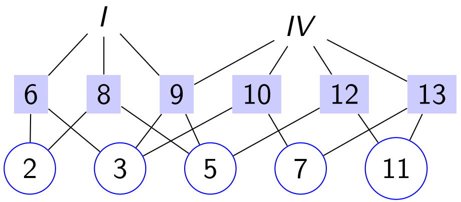
$\rightsquigarrow$ incidence structure is intrinsic

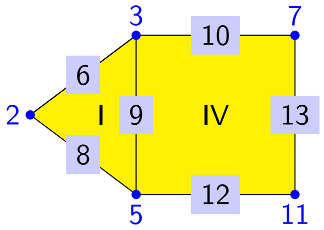
Incidence structure of a polygonal complex

A **polygonal complex** consists of

- set of vertices $\mathcal{V}$
- set of edges $\mathcal{E}$
- set of faces $\mathcal{F}$
- transitive relation $\subseteq \big(\mathcal{V} \times \mathcal{E}\big) \uplus \big(\mathcal{V} \times \mathcal{F}\big) \uplus \big(\mathcal{E} \times \mathcal{F}\big)$



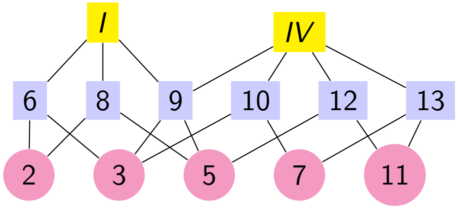




1. Every face is a polygon
2. Every vertex lies in an edge and every edge lies in a face

Isomorphism testing

Incidence geometry allows “easy” isomorphism testing. Incidence structure can be interpreted as a coloured graph:



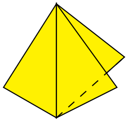
$\rightsquigarrow$ reduce to graph isomorphism problem
Solved by NautyTracesInterface (by Gutsche, Niemeyer, Schweitzer)

General properties

Some properties can be computed for all polygonal complexes:

- Connectivity
- Euler–Characteristic

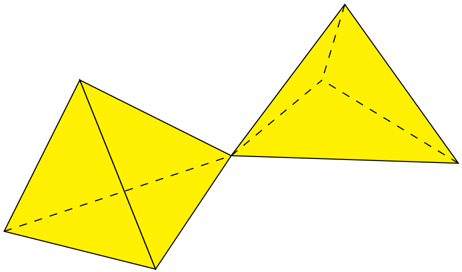
Orientability is **not** one of them. Counterexample:



$\Rightarrow$ every edge lies in at most two faces (for well–definedness)
$\rightsquigarrow$ **ramified polygonal surfaces**

Why ramified?

Typical example of ramified polygonal surface:



$\Rightarrow$ It is not a surface – there is a *ramification* at the central vertex
A **polygonal surface** does not have these ramifications.

Embedding question

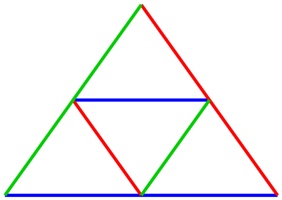
Given: A polygonal complex

- Can it be embedded?
- In how many ways?

Simplifications:

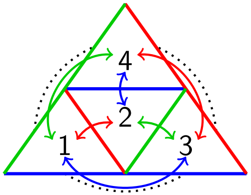
1. Only polygonal surfaces (surface that is build from polygons)
2. All polygons are triangles (**simplicial surfaces**)
3. All triangles are isometric

$\rightsquigarrow$ Edge–colouring encodes different lengths



Colouring as permutation

Consider tetrahedron with edge colouring

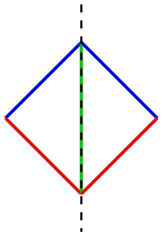


simplicial surface ⇒ at most two faces at each edge

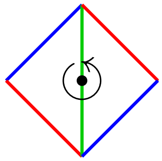
- ⇝ every edge defines transposition of incident faces
- ⇝ every colour class defines permutation of the faces
- (1,2)(3,4) , (1,3)(2,4) , (1,4)(2,3)
- ⇝ group theoretic considerations
  - The connected components of the surface correspond to the orbits of $\langle \sigma_a, \sigma_b, \sigma_c \rangle$ on the faces

How do faces fit together?

Consider a face of the surface and a neighbouring face
The neighbour can be coloured in two ways:



mirror (m)

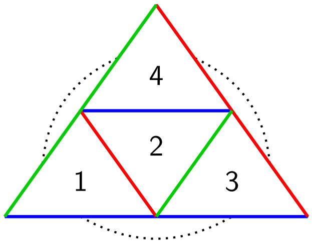


rotation (r)

This gives an **mr–assignment** for the edges.
Permutations and mr–assignment uniquely determine the surface.

Constructing surfaces from groups

A general mr–assignment leads to complicated surfaces.
Simplification: edges of same colour have the same type
Example



has an rrr–structure
The easiest structure is an mmm–structure.

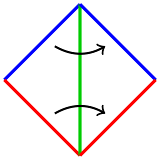
Covering

We want to characterize surfaces where all edges are mirrors.

Lemma

A simplicial surface has an mmm–structure iff it covers a single triangle, i. e. there is an incidence–preserving map to the simplicial surface consisting of exactly one face.

Consider



- Covering pulls back a colouring of the triangle.
- Colouring defines a map to the triangle.

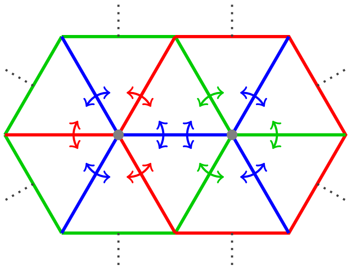
Construction from permutations

Start with three involutions σ_a, σ_b, σ_c (like generators of a finite group)

Lemma

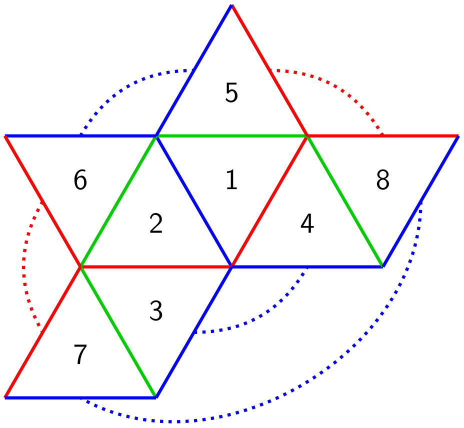
There exists a coloured surface with the given involutions where all edges are mirror edges.

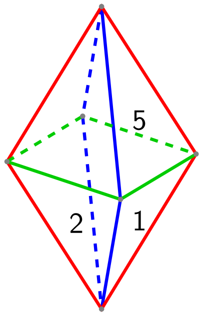
- The faces are the points moved by the involutions
- The edges are the cycles of the involutions
- The vertices are the orbits of $\langle \sigma_a, \sigma_b \rangle$ on the faces (for all pairs)



Construction example

$\sigma_a = (1,2)(3,4)(5,6)(7,8)$
$\sigma_b = (1,4)(2,3)(5,8)(6,7)$
$\sigma_c = (1,5)(2,6)(3,7)(4,8)$





What kind of folding?

There are many different kinds of folding (e. g. Origami) Here:

- Folding of surface in $\mathbb{R}^3$
- Possible folding edges are fixed
- Folding should be rigid (no curvature)

Goal: Classify possible folding patterns (given a net)

Why are embeddings hard?

Ideally, we would like to have embeddings.

But we want to define folding independently from an embedding, since:

- They are very hard to compute (even for small examples)
- We can only show foldability for specific small examples
  - Usually using regularity (like crystallographic symmetry)
  - No general method
- It is very hard to define iterated folding in an embedding

Is there an alternative?

Central idea:

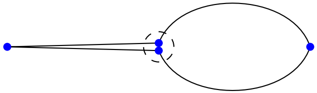
- Don't model folding process (needs embedding)
- Describe starting and final folding state
    - Only consider changes in the topology (like identification of faces)
    - allows abstraction from embedding

$\rightsquigarrow$ Incidence geometry (polygonal complex/surface)

- Captures some folding restrictions (rigidity of tetrahedron)
- Still needs a lot of refinement

Important properties of folding

- The class of surfaces is not closed under folding
- Folding can be undone by *unfolding*
- Identification of two faces might force identification of two other faces
  - Can apply to arbitrary many faces
  - The forced identification is not unique
  - $\Rightarrow$ Identify only two faces at a time

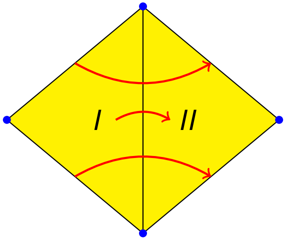


How to define abstract folding?

We need to define two structures:

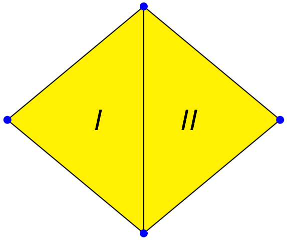
1. A folding state
    - Based on polygonal complexes
    - Describe “is folded together” by an equivalence relation
    - Describe order of faces in folding state
2. The folding steps
    - Only two faces at a time
    - Explain “unordered folding” (e. g. covering)
    - Modify to include face order relations

Unordered Folding (Covering)



Why do we need more than a polygonal complex?
Naive folding definition: surjective map that respects incidence
Problem: Can't be unfolded
$\Rightarrow$ Folding state should not forget original structure

Unordered Folding (Covering)



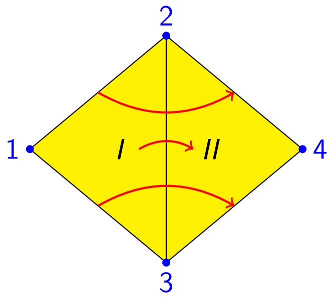
Represent folding by equivalence relation

- Separate relation on vertices, edges and faces
- Two elements are equivalent if they are folded together
- If two edges are equivalent, then their vertices have to be as well (likewise for faces)
- The vertices of an edge are not equivalent (likewise for faces)

$\Rightarrow$ Unordered folding is coarsening of equivalence relation

How does folding work?

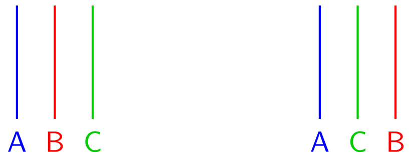
1. Choose two faces that are not folded together
2. Choose how to identify them (like $I \sim II$ and $1 \sim 4$)
3. Add those pairs to the equivalence relation



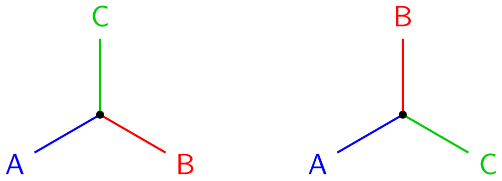
Restriction:
Two vertices in an edge can't be identified (slightly generalized)

Restrictions of unordered folding

We can't work with ordering of faces:



Adding a linear order on each face equivalence class is not enough:



$\rightsquigarrow$ define order of faces around edges (we will skip the details)

Folding complex

Definition

A **folding complex** *is a polygonal complex together with*

1. *An equivalence relation on vertices, edges and faces ("is folded together")*
2. *A linear ordering on each face equivalence class*
3. *A cyclical ordering of the faces around each edge equivalence class*

such that the orderings are compatible (in an appropriate sense).

To identify faces with each other, we have to combine those orderings.

- linear orderings get concatenated
- cyclical orderings are opened at one point and combined

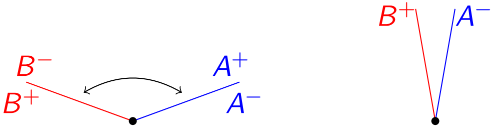
!! compatibility is not easily transfered, but can be calculated

Changed definition of folding

Folding with ordering:

1. Choose two faces that are not folded together
2. Choose how to identify them and extend the equivalence relation
3. Choose the sides of the faces that will meet and modify the orderings

$\rightsquigarrow$ Each face has two sides



$\Rightarrow$ Define folding by two face sides (**folding plan**)
$\rightsquigarrow$ Allows reversible (un)folding