Simplicial surfaces in GAP

Markus Baumeister
(j/w Alice Niemeyer)

Lehrstuhl B für Mathematik
RWTH Aachen University

30.08.2017

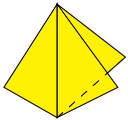
Basic data

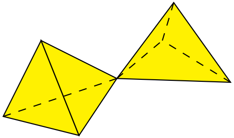
- Package name: `SimplicialSurfaces`
  - Not yet generally available
- Authors: Alice Niemeyer, Markus Baumeister
- based on current research at Lehrstuhl B including Plesken, Strzelczyk and others
- Internally used packages:
  - `AttributeScheduler` by Gutsche
  - `Digraphs` by De Beule, Mitchell, Pfeiffer, Wilson et al.
  - `GAPDoc` by Lübeck
  - `AutoDoc` by Gutsche

Motivation

Goal: Investigate paper folding

- rigid folding in $\mathbb{R}^3$
- consider surfaces built from triangles (simplicial surfaces)
    - not closed under folding
    - allow more general structures:





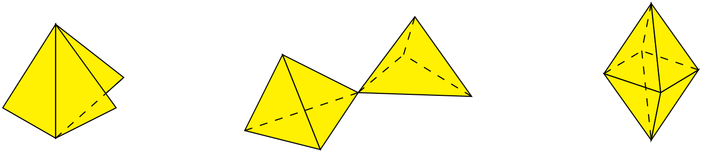
- embeddings are difficult to compute
    - some embeddings of an asymmetric icosahedron are not feasible to compute

$\rightsquigarrow$ focus on intrinsic properties

$\rightsquigarrow$ incidence geometry

Implementation in GAP

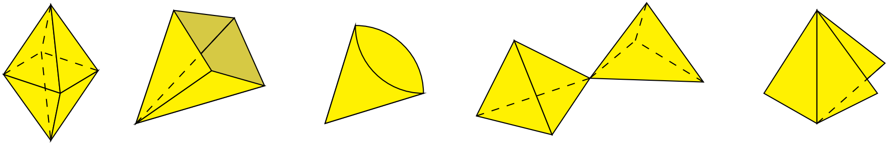
- can describe incidence geometry
- can manage hierarchy of structures
- works well with group–theoretic descriptions
- allows flexible access to the incidence geometry



- difference to `FinInG`–package by De Beule, Neunhöffer et al.
  - only two dimensions but it can work with colourings and foldings

Triangular complexes

We want to describe different structures:

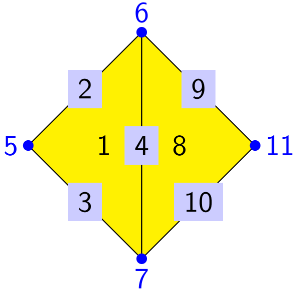


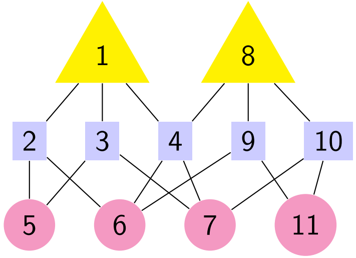
$\rightsquigarrow$ **triangular complexes**

- sets of vertices, edges and faces
- incidence relation between them
- every face is a triangle
- every vertex lies in an edge and every edge lies in a face

Isomorphism testing

Incidence structure can be interpreted as a coloured graph:





$\rightsquigarrow$ reduce to graph isomorphism problem

$\rightsquigarrow$ can be solved quite easily by `Nauty` (McKay, Piperno)

Interfaced by NautyTracesInterface (by Gutsche, Niemeyer, Schweitzer)

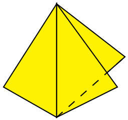
- direct C–interface without writing files
- also returns automorphism group

General properties

Some properties can be computed for all triangular complexes:

- Connectivity
- Euler–Characteristic

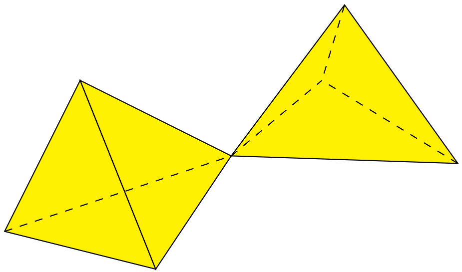
Orientability is **not** one of them. Counterexample:



$\Rightarrow$ every edge lies in at most two faces (for well–definedness)
$\rightsquigarrow$ **ramified simplicial surfaces**

Why ramified?

Typical example of ramified simplicial surface:



$\Rightarrow$ It is not a surface – there is a *ramification* at the central vertex
A **simplicial surface** does not have these ramifications.

Classification

Plesken/Strzelczyk classified all closed simplicial surfaces up to 20 triangles.

- only interesting for those without a 3–cycle of edges
- e. g. there are 87 non–isomorphic surfaces with 20 triangles
- e. g. there is only one surface with 10 triangles:

Progress report

Already implemented:

- surface hierarchy
- elementary properties (e. g. connectivity, orientability)
- isomorphism testing
- classification data base of small surfaces

Not yet implemented:

- automorphism group
- advanced properties (any wishes?)

Embedding questions

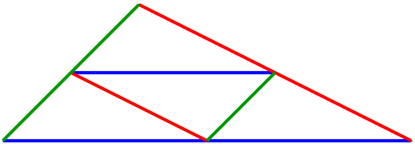
Given: A triangular complex

- Can it be embedded?
- In how many ways?

Simplifications:

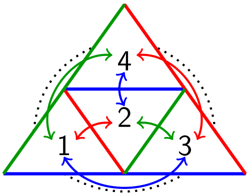
1. Only simplicial surfaces (that are built from polygons)
2. All triangles are isometric

$\rightsquigarrow$ Edge–colouring encodes different lengths



Colouring as permutation

Consider tetrahedron with edge colouring

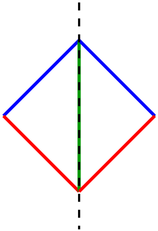


simplicial surface $\Rightarrow$ at most two faces at each edge

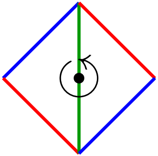
- $\rightsquigarrow$ every edge defines transposition of incident faces
- $\rightsquigarrow$ every colour class defines permutation of the faces
  - (1,2)(3,4) , (1,3)(2,4) , (1,4)(2,3)
- $\rightsquigarrow$ group theoretic considerations
  - The connected components of the surface correspond to the orbits of $\langle \sigma_a, \sigma_b, \sigma_c \rangle$ on the faces (fast computation for permutation groups)

How do faces fit together?

Consider a face of the surface and a neighbouring face
The neighbour can be coloured in two ways:



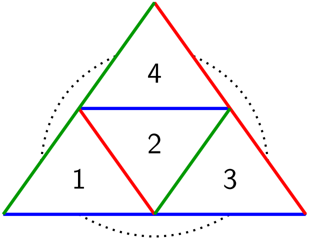
mirror (m)



rotation (r)

This gives an **mr–assignment** for the edges.
Permutations and mr–assignment uniquely determine the surface.

A general mr–assignment leads to complicated surfaces.
Simplification: edges of same colour have the same type
Example



has only r–edges.

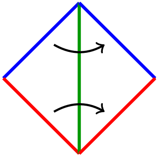
The mirror–case

If all edges are mirrors, the situation is simple.

Lemma

A simplicial surface has only mirror–edges iff it covers a single triangle, i. e. there is a surjective incidence–preserving map to the simplicial surface consisting of exactly one face.

Consider



$\Rightarrow$ Unique map that preserves incidence

- Covering pulls back a mirror–colouring of the triangle.
- Mirror–colouring defines a map to the triangle.

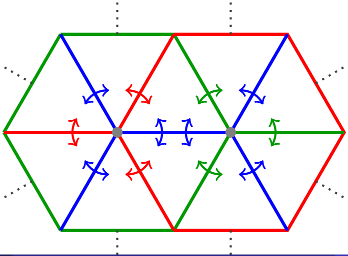
Construction from permutations

Start with three involutions σ_a, σ_b, σ_c in permutation representation (like generators of a finite group)

Lemma

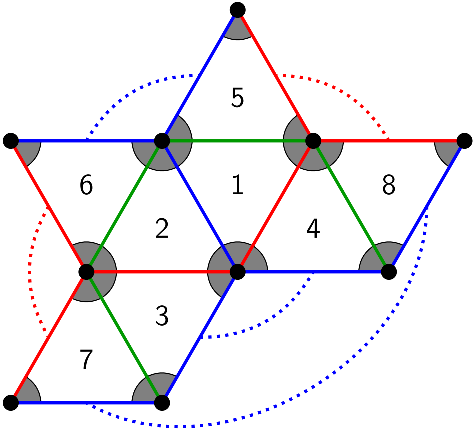
There exists a coloured surface with the given involutions where all edges are mirror edges.

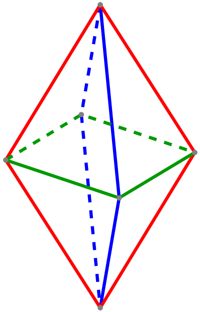
- The faces are the points moved by the involutions
- The edges are the cycles of the involutions
- The vertices are the orbits of $\langle \sigma_a, \sigma_b \rangle$ on the faces (for all pairs)



Construction example

$\sigma_a = (1,2)(3,4)(5,6)(7,8)$
$\sigma_b = (1,4)(2,3)(5,8)(6,7)$
$\sigma_c = (1,5)(2,6)(3,7)(4,8)$





Progress report

Implemented:

- computing all colourings of a given simplicial surface
- constructing all surfaces with given involutions
  1. up to (coloured) isomorphism
  2. with given mr–assignment
- drawing of simplicial surfaces
- constructing various coloured coverings

Still missing:

- Research TODO?

What kind of folding?

There are many different kinds of folding (e. g. Origami)
Here:

- Folding of surface in $\mathbb{R}^3$
- Fold only at given edges (no introduction of new folding edges)
- Folding should be rigid (no curvature)

Goal: Classify possible folding patterns (given a net)

Embeddings are very hard

- At every point in time the folding process has to be embedded
- We can only show foldability for specific small examples
  - Usually using regularity (like crystallographic symmetry)
  - No general method
- It is very hard to define iterated folding in an embedding

Folding without embedding

Central idea:

- Don't model folding process (needs embedding)
- Describe starting and final folding state
  - Only consider changes in the topology (like identification of faces)
  - allows abstraction from embedding

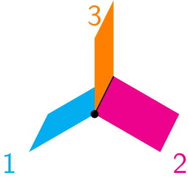
$\rightsquigarrow$ Incidence geometry (polygonal complex/surface)

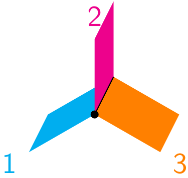
- Captures some folding restrictions (rigidity of tetrahedron)
- Still needs a lot of refinement

More than a triangular complex

- Concept should allow reversible folding
- We need an ordering of the faces:

- Adding a linear order on each face equivalence class is not enough:

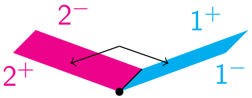


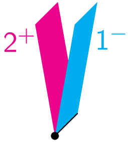


$\rightsquigarrow$ **folding complex**

How to describe folding?

Needs specification of two face sides:





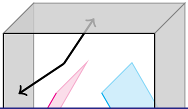
$\Rightarrow$ Describe folding by two face sides

$\rightsquigarrow$ **folding plan**

How does folding plan work?

Folding of two faces can force folding of other faces:

- Can apply to arbitrary many faces
- The forced identification is not unique



How does folding plan work?

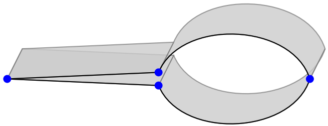
Folding of two faces can force folding of other faces:

- Can apply to arbitrary many faces
- The forced identification is not unique

$\Rightarrow$ Identify only two faces at a time

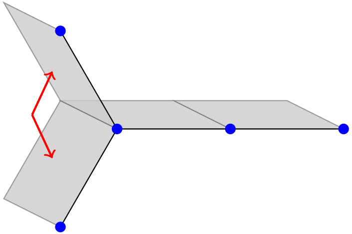
  $\rightsquigarrow$ Relax the rigidity–constraint:

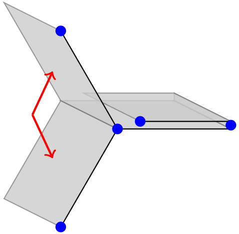
  - Allow non–rigid configurations as transitional states



Structure of multiple foldings

With folding plans we can perform the same folding in different folding complexes

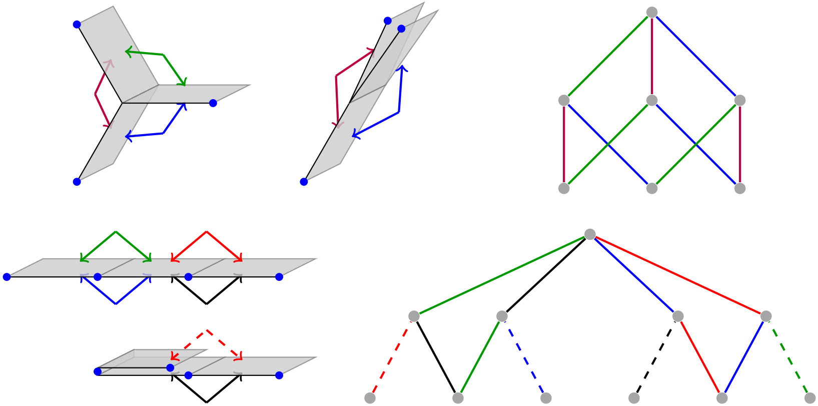




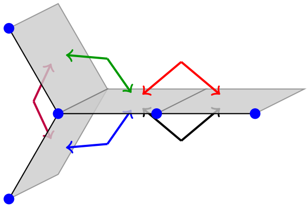
$\rightsquigarrow$ more structure on the set of possible foldings

Folding graph

- Vertices are folding complexes (modelling folding states)
- Edges are folding plans connecting two folding complexes

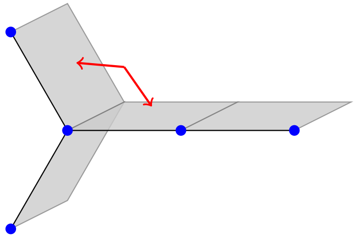


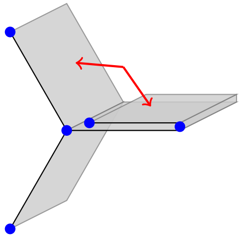
Larger graph



Drawback of folding plans

Some foldings that "should" be the same, aren't:





$\Rightarrow$ If you know the folding structure of a small complex, you can't easily find the folding structure of an extended complex

$\rightsquigarrow$ Folding plans are not optimal to model folding

Questions?