# Development of triangulated surfaces

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# 1 Introduction

The aim of this work is to lay the foundations of certain parts of the theory of EUCLIDean simplical surfaces. More specifically we deal with simplicial surfaces composed of triangles which are all congruent to one control triangle. This may sound very special, but it turned out to be a rich area. The investigation was initiated by an interdisciplinary project on paper folding suggested by Martin Trautz and his working group in the architecture department of the RWTH Aachen.
There are three parts of this work. At the start we have the combinatorial group-theoretic part dealing with edge-coloured simplicial surfaces as combinatorial objects, the colours reflecting the lengths of the edges of the control triangle. Secondly there is the part dealing with simplicial surfaces as EUCLIDean 2-dimensional (compact) manifolds with finitely many singularities. Though this is by far the shortest part, the general theory here has a long history, the oldest result in the area goes back to DESCART's Theorem relating the EULER characteristic of the surface to the sum of the angle deficits at the singularisties, which was later generalized to become the GAUSS- BONNET-Theorem. As far as our investigation is concerned, a full description of these objects can be given from the combinatorics and the side lengths of the control triangle. The final part then concerns the embeddability of the abstract EUCLIDean simplicial surfaces into EUCLIDean three space, which is a major challenge to algorithmic and constructive methods in real algebraic geometry. Instead of deciding embeddability of individual surfaces, we aim at the simultaneous treatment of all surfaces with the same underlying combinatorial simplicial complex in terms of the side lenths of the control triangle. Relevant questions are: For which side length do embeddings exist, when are they unique, can one describe the singularities of the moduli space of embeddings and how do they relate to the way the embedded surfaces degenerate upon appoaching these singularities, how are volume of the enclosed 3-dimensional figures and the area of the surface are related etc.
Since the investigation on the one hand draws from many areas inside mathematics such as combinatorics, finite group theory, discrete geometry, algebraic topology, real algebraic geometry, and last not least certain algorithmic aspects of these, and on the other hand, since major parts of it should be readable also by non-mathematicians, we deal with the problems from scratch, i.e. start with basic definitions, give plenty of examples, try to illustrate the concepts by figures whenever they appear to be helpful. We have produced computer programs to compute examples of our objects and classify certain classes of examples. These programs should be considered to be part of the investigation and the reader is advised to use them to get a good feeling for the simplicial surfaces and their behaviour under combinatorial or metrical modifications.

A final word on related areas might be helpful. There is first of all the theory of convex polytopes in three space. We do not use it in a serious way, since our restriction about the congruence of all face triangles does not seem to have come up there except in rather special cases. The same applies to differece geometry. Simplicial complexes are of course a basic part of algebraic topology. So some elementary algebraic topology of surfaces such as EULER characteristic etc. lurks in the background, but again topology does not pay any attention to the congruence to the simplices. Finite groups play some role in what we are doing, because certain CAYLEY- and more generally SCHREIER-graphs will be used to constuct simplicial surfaces. This is maybe an essential point, where our objects or rather their construction deviate from the surfaces in difference geometry, where the surface constructions are often modelled on nets constructed in two dimensional lattices whereas finite groups give us an interesting tool to define surfaces globally. In this respect it should be pointed out that our intuition was inspired by Andreas Dress' theory of tilings, though this might not be visible from our write up. In fact our idea is to construct surfaces via their tilings by congruent triangles.

Going into more detail, we give a list of applications on what can be done with a finite simplicial complex. Let us first ignore the embeddings in EUCLIDean 3-space, and only look at realizations by congruent triangles as abstract EUCLIDean surface with singularities (in the vertices). These questions can already be satisfactorily answered on the combinatorial level. For instance one can decide whether or not there is a realization by general triangles by considering wild (edge) colourings for the combinatorial simplicial surface, where the colours stand for the three side lengths of the control triangle. For instance in the case of spherical simplicial surfaces, this is always possible as a consequence of the Four Colour Theorem. More ambitiously one might attach to each coloured side of the control triangle exactly one of the attributes $m$ or $r$, indicating that the construction of the surface from the control triangle works by identifying the given side with given colour to the identical neigbouring triangle either by respecting the end points in case of $m$ or interchanging them in case of $r$. In the first case the front and back sides of the triangle gets interchanged, in the latter case not. $m$ stands for mirror, and $r$ for rotation. Note, two $r$-neighboured triangles develop into the plane as a parallelogram, and two $m$ -neighboured triangles as a kite. In this situation of a highly decorated control triangle the whole simplicial surface might agree with this structure or not. This is measured by a holonomy group which is a subgroup of the combinatoiral symmetry group of the control triangle. If the holonomy group is trivial, arbitrary side lengths for the control triangle can be chosen; if its order is divisible by 3, the control triangle must be equiangular, if it is of order 2 it must be isoscelic. In any case, the triangle and also a pattern drwawn on the

triangle must be invariant under the holonomy group. In this way not only the combinatorial possibilities for the EUCLIDean surface are computable, but also the accompanying decoration patterns. So for instance for the (combinatorial) icosahedron one has to choose an equiangular control triangle, no matter how many sides are mirror type or of reflection type. Only in the case where all sides are of reflection type one can choose the pattern on the triangle to be onlyrotation invariant, whereas in the other cases it must be invariant under reflections as well. If one instists on having all faces congruent to one given general triangle, there is no possibility of an $mr$-structured control triangle, but only the weaker possibility of wild edge colouring as described earlier. Also there is a combinatorial covering theory going with the stronger set up, which often yields new interesting simplcial surfaces from given ones, carrying an $m, r$-structure, even if the original surface does not. We mention in passing that the group theretical background for this theory seems to be new to some extend and gives rise to interesting investigations obout finite groups generated by three involutions. In the opposite direction such groups give rise to various constructions of combinatorial simplicial surfaces.
The final challenge is the isometrical embedding of our abstract surfaces as piecewise linear surfaces in EUCLIDean 3-space. Especialy in the case of trivial holonomy groups the question of which control triangle are possible is a big challenge, cf. [paper]. An often computationally simpler question is to look only at those embeddings, for which certain combinatorial symmetries turn into symmetries induced by EUCLIDean motions. In any case both methods, which lend themselves for these kinds of problems, the real algebraic one using formal methods and the numerical one have their serious drawbacks. Whereas the first one is too time consuming the second one has its diffeculties because of the abundance of singularities and big derivaties. It therefore is necessary to deal with the problem geometrically by turning combinatorial constructions into geometric constructions when dealing with bigger problems. Coming back to the icosahedron above, the wild colourings allow a general congruence class for the face triangles and it is an intreaguing problem, to see how the regular icosahedron deforms under varying side lengths, when it is still convex, etc.

# 2 Combinatorial Simplicial Surfaces

Im unendlichen Fall Eckengrad beschrnkt statt endlich?

## 2.1 Definition and Notation

**Definition 2.1.** *A* **simplicial surface** $(X, <)$ *is a countable set* $X$ *together with a transitive relation* $<$ *on* $X$ *with* $X = X_0 \uplus X_1 \uplus X_2$ *with non empty* $X_i$ *such that the relation* $<$ *is a subset of the union* $X_0 \times X_1 \cup X_1 \times X_2 \cup X_0 \times X_2$ *satisfying conditions 1.) to 4.) below. The elements of* $X_i$ *are callled* $i$*-* **simplices***, or more specifically those of* $X_0$ **vertices***, the ones of* $X_1$ **edges** *and the ones of* $X_2$ **faces** *or* **triangles***, and* $<$ *is called* **adjacency***.*
*1.) For each edge* $e \in X_1$ *there are exactly two vertices* $V \in X_0$ *with* $V < e$*.*
*2.) For each face* $F \in X_2$ *there are exactly three edges* $e \in X_1$ *with* $e < F$ *and three vertices* $V \in X_0$ *with* $V < F$*. Moreover any of these three vertices is adjacent to exactly two of these three edges.*
*3.) For any edge* $e \in X_1$ *there are either exactly two faces* $F_1, F_2 \in X_2$ *with* $e < F_i$ *for* $i = 1, 2$ *or exactly one face* $F \in X_2$ *with* $e < F$*. In the first case* $F_1$ *and* $F_2$ *are called* **(**$e$**-)neighbours** *and* $e$ *an* **inner edge***. In the later case* $e$ *is called a* **boundary edge***.*
*4.) For any vertex* $V \in X_0$ *there are only finitely many faces* $F_i \in X_2$ *with* $V < F_i$*. These* $F_i$ *can be arranged in a sequence* $(F_1, \ldots, F_n)$ *such that* $F_{i+1}$ *and* $F_i$ *share a common edge adjacent to the vertex* $V$ *for* $i = 1, \ldots, n-1$*. In case the same holds true for* $F_n$ *and* $F_1$*,* $V$ *is called an* **inner vertex** *and the sequence can be viewed as a cycle, otherwise a* **boundary vertex***. The number* $n$ *of faces in the cycle is called the* **degree** *of the vertex* $V$*.*

Note, for a given vertex one usually has quite a few sequences as defined in 4.) all of which are considered to be equivalent, because they can be reversed and in the case of inner vertices cyclically permuted, and Condition 4.) ensures that all sequences for a fixed vertex are equivalent. Also a vertex is not uniquely determined by its sequences, cf. Example 2.7. Condition 4.) might look complicated at first sight, but in case it is violated it can easily be recovered by introducing new vertices so that the vertices are in bijection with the equivalence classes of sequences. It is useful to introduce the following notation:

**Definition 2.2.** *Let* $X$ *be a simplicial surface. For any* $i, j \in \{1, 2, 3\}$ *with* $i \neq j$ *and* $x \in X_i$ *define the set* $X_j(x)$ *as follows:*
$X_j(x) := \{y \in X_j | x < y\}$ *if* $i < j$ *and* $X_j(x) := \{y \in X_j | y < x\}$ *if* $j < i$*.*
*For a subset* $S \subseteq X_i$ *let* $X_j(S) := \cup_{x \in S} X_j(x)$*.*

**Remark 2.3.** *The conditions for a partially ordered set $(X, <)$ as in 2.1 to be a simplicial surface can be written as:*
*1.)* $|X_0(e)| = 2$ *for each* $e \in X_1$.
*2.)* $|X_1(F)| = 3$ *for each* $F \in X_2$.
*3.)* $1 \leq |X_2(e)| \leq 2$ *for each* $e \in X_1$.
*4.)* $|X_2(V)| < \infty$ *for each* $V \in X_0$ *and* $X_2(V)$ *can be ordered* $X_2(V) = \{F_1, \dots, F_n\}$ *such that* $|X_1(V) \cap X_1(F_i) \cap X_1(F_{i+1})| \geq 1$.

If not stated otherwise, simplicial surfaces are assumed to be finite, i.e. $|X| < \infty$.

Note, there are exactly three (different) vertices adjacent to one face. However, it is possible, that two faces share three vertices.

**Definition 2.4.** *Let* $(X, <)$ *and* $(Y, \prec)$ *be simplicial surfaces.*
*1.) An* **isomorphism** *from* $(X, <)$ *to* $(Y, \prec)$ *is a bijection* $\alpha : X \to Y$ *with* $A < B$ *in* $X$ *if and only if* $\alpha(A) \prec \alpha(B)$ *in* $Y$.
*2.) A* **covering** *of* $Y$ *by* $X$ *is a surjective map* $\alpha : X \to Y$ *with* $A < B$ *in* $X$ *implies* $\alpha(A) \prec \alpha(B)$ *in* $Y$. *Note a covering restricts to surjective mappings* $X_i \to Y_i$ *for* $i = 0, 1, 2$.

An isomorphism respects the EULER-**characteristic** $\chi(X) := |X_0| - |X_1| + |X_2|$ of a simplicial surface $X$ as well as the **orientability**, i.e. the possibility to arrange $X_0(F)$ for each $F \in X_2$ in three-cycles such that both possible orders for the $X_0(e)$ are induced for each inner edge $e \in X_1$ from the two neighbouring faces.

**Remark 2.5.** *A simplicial surface is called* **closed**, *if all its edges are inner. A closed simplicial surface necessarily has an even number* $f$ *of faces, namely*

$$f = \frac{2e}{3}$$

*with* $f := |X_2|, e := |X_1|$. *Also the number* $e$ *of edges is divisible by* 3 *in this case. In particular, with* $v := |X_0|$ *one has an* $n \in \mathbb{N}$ *with*

$$f = 2n, e = 3n, v = n + \chi(X).$$

To be able to give a concise description of finite simplicial surfaces, we introduce some notation, which at the same time is useful for computer algorithms. (It is the standard notation used in the GAP-package SimplicialS). It covers the most general case, i.e. if two different edges have the same vertices. Note, however, the symbol heavily depends on the numbering of the vertices, edges, and faces. Beyond that it is unique.

**Definition 2.6.** *Let $(X,<)$ be a finite simplicial surface. whose vertices $V_1,\ldots,V_n$, edges $e_1,\ldots,e_k$, and faces $F_1,\ldots,F_m$ are linearly ordered according to their numbering. The* **symbol** *assigned to $(X,<)$ is given by*

$$\mu((X,<)) := (n,k,m;(X_0(e_1),\ldots,X_0(e_k)),(X_1(F_1),\ldots,X_1(F_m))).$$

*Usually, in this symbol, we will replace $V_i$ by $i$, $e_j$ by $j$, and $F_l$ by $l$ and call the resulting symbol the* **ordinal symbol** *$\omega((X,<))$ of $(X,<)$.*

Here are some easy examples of orientable simplicial surfaces.

4 picture, alles nummerieren

**Example 2.7.** *1.) There is exactly one (up to isomorphism) simplicial surface with one face. It will be called* **one-face (surface)** *$\Delta$. Here is the ordinal symbol $\omega(\Delta)$:*

$$(3,3,1;(\{1,2\},\{1,3\},\{2,3\}),(\{1,2,3\})).$$

*2.) There is one closed simplicial surface with two faces. It has three inner edges and three inner vertices: $X_0 := \{V_1,V_2,V_3\}, X_1 = \{e_1,e_2,e_3\}, X_2 := \{F_1,F_2\}$ with $X_1(F_1) = X_1(F_2) = X_1$ and $X_0(e_1) := \{V_1,V_2\}, X_0(e_2) := \{V_1,V_3\}, X_0(e_3) := \{V_2,V_3\}$ so that the its ordinal symbol is*

$$(3,3,2;(\{1,2\},\{1,3\},\{2,3\}),(\{1,2,3\},\{1,2,3\})).$$

*We shall call it the* JANUS **head**. *There are two surfaces related to the Janus head, which are no longer closed:*
*a.) the* **open bag** *(by doubling the edge $e_3$) with $X_0 := \{V_1,V_2,V_3\}, X_1 = \{e_1,e_2,e_3,e_4\}, X_2 := \{F_1,F_2\}$ with $X_1(F_1) = X_1 - \{e_4\}, X_1(F_2) = X_1 - \{e_3\}$, and $X_0(e_1) := \{V_1,V_2\}, X_0(e_2) := \{V_1,V_3\}, X_0(e_3) := \{V_2,V_3\}, X_0(e_4) := \{V_2,V_3\}$. Hence the ordinal symbol is*

$$(3,4,2;(\{1,2\},\{1,3\},\{2,3\},\{2,3\}),(\{1,2,3\},\{1,2,4\})).$$

*b.) the* **butterfly** *with two faces, one inner and four boundary edges, and no inner and four boundary vertices.*
*All three simplicial surfaces in 2.) cover the one-face. The ordinal symbol can be chosen to be*

$$(4,5,2;(\{1,2\},\{1,3\},\{2,3\},\{2,4\},\{3,4\}),(\{1,2,3\},\{3,4,5\})).$$

Note, inner edges are the ones whose number occurs exactly twice in the last part of the ordinal symbol, and boundary edges just once. Of course we are mainly interested in connected simplicial surfaces:

**Definition 2.8.** *1.) A* **face path** *from the face $F \in X_2$ to the face $T \in X_2$ on the simplicial surface $(X, <)$ is a sequence of faces $(F_1 := F, F_2, \ldots, F_k := T)$ in $X_2$ for some $k$ such that $F_i$ and $F_{i+1}$ are neighbours for $i = 1, \ldots, k-1$.*
*2.) The simplicial surface $(X, <)$ is called* **connected***, if for any two faces $F, T \in X_2$ there is a face path between them.*

Check definition whether sufficient for colored surfaces?

## 2.2 Simplicial surfaces with 2-waists

Ausfhren als Gegenbeispiel e fr Vertextreue, Honweis auf Beweglichkeit, Klassen von Bumen und Ketten von Januskpfen und Tetraeder Kaleidozykel, , Allgemeiner Hinweis: Surgery

xxxxxxxxxxxxxxxx The Janus head is the simplest closed simplicial surface. In this section we analyse a sinplest way to cut two closed simplicial surfaces into two smaller simplicial surfaces and also the converse, how to glue two closed surfaces togoether. As for the cutting the situation is rather special, the glueing, however, is always possible.

**Definition 2.9.** *Let $(X, <)$ be a vertex-faithful closed simplicial surface. A* **2-waist** *is a closed path of two different edges. The set of all 2-waists of $X$ is denoted by $W_2(X)$. Usually we shall identify a 2-waist with its set of edges.*
*The* **degree** *of a vertex $P$ is the number $|X_2(P)|$ of faces adjacent to the vertex $P$. The set of all vertices of degree $k$ of $X$ is denoted by $V_k(X)$.*

**Remark 2.10.** *Let $X$ be a closed simplicial surface and let $P \in X_0$ be a vertex of degree 2.*
*1.) $P$ defines a 2-waist which consists of the edges of the two adjacent 2-simplices belonging to exactly one of them. In other words there is a map $V_2(X) \to W_2(X)$.*
*2.) One can remove $P$ and the faces adjacent to $P$ from $X$ and replace the two edges of the two faces in $X_2(P)$ not belonging to $X_1(P)$ by one edge with the same end points and adjacent to the two faces which do not lie in $X_2(p)$. (Ie. one glues the two edges together.). The remaining surface is called the* **$P$-truncated simplicial surface** *$X^{-P}$. Note, in case $X$ is not a Janus head, $X^{-P}$ is again a well defined closed simplicial surface of the same* Euler *characteristic as $X$. More precisely the number of vertices,*

*edges, resp. faces goes down by 1, 3, resp. 2.*
*3.) $X$ can be reconstructed from $X^{-P}$ as the connected sum of $X^{-P}$ with the Janus head along a pair of edges, cf. Definition 2.11 below.*

Here is a more general construction of closed simplicial surfaces with a 2-waist.

**Definition 2.11.** *Let $(X,<)$ and $(Y,<)$ be (not necessarily different) closed simplicial surfaces with one edge $e \in X_1$ and one edge $e' \in Y_1$ with $X_2(e)$ and $Y_2(e')$ disjoint. Let $P, Q$ be the vertices of $e$ and $P', Q'$ the vertices of $e'$. The $(P, e, Q) - (P', e', Q')$-**connected sum** or briefly the **connected sum along** $e.e'$ denoted by $X\#Y$ or more precisely $X_e\#_{e'}Y$ is defined as follows:*

$$\begin{aligned}(X\#Y)_0 &:= (X_0 - \{P, Q\}) \cup (Y_0 - \{P', Q'\})) \cup \{\tilde{P}, \tilde{Q}\} \\ (X\#Y)_1 &:= (X_1 - \{e\}) \cup (Y_0 - \{e'\}) \cup \{e_1, e_2.e_1', e_2'\} \\ (X\#Y)_2 &:= X_2 \cup Y_2\end{aligned}$$

*and the following partial order $<$.*

Note, after choosing $e, e'$ there are exactly 4 possibilities for $X_e\#_{e'}Y$ depending on which

**Proposition 2.12.** *Let $(X,<)$ be a closed connected simplicial surface, and $W$ a 2-waist of $X$ consisting of the edges $e_1, e_2. . Thenaclosedsurface X_W$ can be constructed as follows:*

### 2.3 Vertex-faithful simplicial surfaces

An important class of simplicial surfaces are those whose faces and edges are uniquely determined by adjacent vertices.

**Definition 2.13.** *A simplicial surfaces $(X,<)$ is **vertex-faithful**, if*

$$X_1 \cup X_2 \to \mathrm{Pot}(X_0) : S \mapsto X_0(S)$$

*is an injective map.*

So in Example 2.7 2.) the Janus head and the open bag are not vertex-faithful, whereas the butterfly is.

**Remark 2.14.** *1.) A vertex-faithful simplicial surface $(X,<)$ can be represented as set*

$$\{A| \ A \subseteq X_0(F), F \in X_2, A \neq \emptyset\}$$

*partially ordered by inclusion.*
*2.) Conversely, given a finite set $P$ and a non empty subset $\Sigma$ of $\mathrm{Pot}_3(P)$ with $P = \cup_{F \in \Sigma} F$, the pair $(P, \Sigma)$ represents a vertex-faithful surface $(X, <)$, if it satisfies the following condition:*
*For any element $V$ of $P$ contained in some $F \in \Sigma$ the set of all $F_i \in \Sigma$ with $V \in F_i$ can be arranged in a cyclic order $(F_1, F_2, \ldots, F_n)$ such that $|F_i \cap F_{i+1}| = 2$ for $i = 1, \ldots, n-1$ and $|F_i \cap F_j| = 2$ implies $|i - j| = 1$ or $\{i, j\} = \{1, n\}$.*
*In this case $X_i := \{A \subseteq F | F \in \Sigma, |A| = i + 1\}$ for $i = 0, 1, 2$ and $<$ is set-theoretic inclusion. Any vertex-faithful simplicial surface isomorphic to $(X, <)$ is called* **represented by** $\Sigma$ *and denoted by $X[\Sigma]$.*

cycle hier und in erster Definition kristisch beaeugen

**Example 2.15.** *1.) The tetrahedron can be represented by $\Sigma := \mathrm{Pot}_3(\{1, 2, 3, 4\})$. It does not cover the one-face $\Delta$.*
*2.) The double-tetrahedron can be represented by*

$$\Sigma := \underbrace{(\mathrm{Pot}_3(\{1, 2, 3, 4\}) \cup \mathrm{Pot}_3(\{2, 3, 4, 5\})) - \{\{2, 3, 4\}\}}_{\mathrm{Pot}_3(\{1,2,3,4\}) \triangle \mathrm{Pot}_3(\{2,3,4,5\}))}.$$

*i.e. by the symmetric difference of two tetrahedra intersecting in one face. Again it does not cover the one-face $\Delta$.*
*3.) The octahedron can be represented by*

$$\Sigma := \bigcup_{i=1}^{3} \{\{i, i + 1, 5\}, \{i, i + 1, 6\}\} \cup \{\{1, 4, 5\}, \{1, 4, 6\}\}$$

*or equivalently by the symmetric difference of two double-tetrahedra intersecting in two neighbouring faces, namely one from each tetrahedron. The octahedron covers the one-face, which is related to the possibility to fold a regular paper octahedron flat to triangle.*

### 2.4 Vertex faithful surfaces with 3-waists

The aim of this section is to create some insight into the combinatorial structure of simplicial surfaces. We restrict ourselves mainly to the most important class of such surfaces, namely the one of EULER characteristic two, i.e. of genus zero, or those whose topological realization is homeomorphic to a sphere. On the combinatorial side we shall assume that the surfaces are vertex-faithful. The following definition is fundamental for our point of view.

**Definition 2.16.** *Let $(X, <)$ be a vertex-faithful closed simplicial surface. A* **3-waist** *is a closed path of three different edges not all of which are edges*

*of one face. The set of all 3-waists of $X$ is denoted by $W_3(X)$. Usually we shall identify a 3-waist with its set of edges.*
*The* **degree** *of a vertex $P$ is the number $|X_2(P)|$ of faces adjacent to the vertex $P$. The set of all vertices of degree $k$ of $X$ is denoted by $V_k(X)$.*

**Remark 2.17.** *Let $X$ be a closed vertex-faithful simplicial surface represented by $\Sigma \subseteq \mathrm{Pot}_3(X_0)$ for some set $X_0$ of vertices. Let $P \in X_0$ be a vertex of degree 3.*
*1.) $P$ defines a 3-waist which consists of the edges of the three adjacent 2-simplices belonging to exactly one of them. In other words there is a map $V_3(X) \to W_3(X)$.*
*2.) One can remove $P$ and the faces adjacent to $P$ from $X$ and replace them by a triangle to obtain the* **$P$-truncated simplicial surface** *$X^{-P}$ represented by*

$$\Sigma^{-P} := (\Sigma - \{F \in \Sigma | P \in F\}) \cup \{P_1, P_2, P_3\}$$

*where $P_1, P_2, P_3$ are the vertices other than $P$ of the three edges adjacent to $P$ in $X$. Note, in case $X$ is not a tetrahedron, $X^{-P}$ is again a well defined closed simplicial surface.*
*3.) $X$ can be reconstructed from $X^{-P}$ as the connected sum of $X^{-P}$ with the tetrahedron represented by $\mathrm{Pot}_3(\{P, P_1, P_2, P_3\})$ over the common face $\{P_1, P_2, P_3\}$, i.e. $\Sigma$ is the symmetric difference*

$$\Sigma = \Sigma^{-P} \triangle \mathrm{Pot}_3(\{P, P_1, P_2, P_3\}).$$

The following lemma is essential:

**Lemma 2.18.** *Let $X$ be a vertex-faithful closed surface with two neighbouring vertices of degree 3. Then $X$ is a tetrahedron.*

*Proof.* Let $P, Q$ be the two neighbouring vertices of degree 3. Then the edge joining $P$ and $Q$ is common edge of two triangles $F_1, F_2$. The other two edges adjacent to $P$ have as second vertex adjacent to them $P_i$ belonging to $F_i$ for $i = 1, 2$. The same holds for $Q$. Clearly $P, P_1, P_2$ must be vertices of a face $F_3$ and $Q, P_1, P_2$ must be vertices of a face $F_4$. In total we have a tetrahedron. □

**Corollary 2.19.** *Let $X$ be a vertex-faithful closed simplicial surface with a vertex $P \in X_0$ of degree 3. If $X$ is finite and not a tetrahedron, then $X^{-P}$ either has the same number of vertices of degree 3 as $X$ or exactly one less or is itself a tetrahedron.*

The attachment of a tetrahedron can also be seen as a subdivision of a face into three faces. This might be sometimes helpful, but the attachment-point-of-view is more in the line of the applications we have in mind.

This corollary in itself would justify to classify closed simplicial surfaces without vertices of degree 3 first, and start from there to attach tetrahedra to construct all closed simplical surfaces say up to some fixed number of faces. But one can do better by working with 3-waists rather than vertices of degree 3. The following might be helpful to see the advantage of the approach via 3-waists.

**Remark 2.20.** *Let $X, X'$ be two vertex-faithful closed simplicial surfaces represented by disjoint sets $\Sigma \subseteq \mathrm{Pot}_3(X_0)$ and $\Sigma' \subseteq \mathrm{Pot}_3(X_0')$. For any two faces $F$ of $X$, $F'$ of $X'$, and any bijection $\varphi : X_0(F) \to X_0'(F')$ the* **connected sum** *$X \#_\varphi X'$ denotes the closed simplicial surface represented by the symmetric difference $\Sigma \bigtriangleup_\varphi \Sigma'$, where $X_0(F) \subseteq \Sigma$ is identified via $\varphi$ with $X_0'(F') \subseteq \Sigma'$.*

$$|W_3(X \#_\varphi X')| = |W_3(X)| + |W_3(X')| + 1.$$

*Proof.* Clearly, the 3-waists of $X$ and $X'$ remain and the only new 3-waist of $X \#_\varphi X'$ comes from $X_1(F)$ identified with $X_1'(F')$, which no longer is the boundary of a triangle in $X \#_\varphi X'$. □

Note the special case, where $X'$ is a tetrahedron. The 3-waist created in this remark has the property that its removal makes the simplicial surface disconnected. To be precise we define:

**Lemma 2.21.** *Let $X$ be represented by $\Sigma \subseteq \mathrm{Pot}_3(X_0)$ and $W \subseteq X_1$ be a 3-waist. Then $X_2(X_0(W))$ splits into two classes defined as follows: $F, F' \in X_2(X_0(W))$ belong to the same class (i.e. lie on the same side of the 3-waist), if there is a face-path inside $X_2(X_0(W))$ joining the two in which two consecutive faces are e-neighbours with $e \in X_1(X_0(W)) - W$. For each $e' \in W$ the set $X_2(e')$ intersects each of the classes in exactly one element.*

*Proof.* Each of the six face cycles (ignoring orientation but marking the initial face) around any of the vertices of $W$ starting with a face adjacent to some $e' \in W$ can be split into two paths none of which crosses an edge in $W$. The desired classes are obtained by iterated union of sets of faces in any of these paths with nonempty intersection starting with the sets of faces belonging to the halfcycles just defined. The rest is clear. □

**Corollary 2.22.** *(Cutting through a 3-waist) Let $X, \Sigma, W$ be as in Lemma 2.21. Then a new closed simplicial surface $X^W$ can be formed by doubling the edges and vertices of $W$ and replacing the two new "holes" by two disjoint faces. For the* Euler*-characteristics one has:*

$$\chi(X^W) = \chi(X) + 2$$

*and for the number of 3-waists one has:*

$$|W_3(X^W)| = |W_3(X)| - 1.$$

*Proof.* Here are the formal details: For $i := 1, 2$ let $X_0(W)_i$ be two disjoint sets, each of which is also disjoint from $X_0$ with two bijections $X_0(W) \to X_0(W)_i : V \mapsto V_i$. Define $(X^W)_0 := (X_0 - X_0(W)) \uplus X_0(W)_1 \uplus X_0(W)_2$ and two maps

$$\varphi_i : X_0(W) \to (X^W)_0 : V \mapsto \begin{cases} V & V \notin X_0(W) \\ V_i & V \in X_0(W) \end{cases}$$

formal inkorrekt, schon iim vorigen Lemma sauber definieren

for $i = 1, 2$. Then $X^W$ is induced from $\Sigma^W \subseteq \mathrm{Pot}_3((X^W)_0)$ defined by

$$\Sigma^W := \varphi_1(X_0(W)) \cup \varphi_2(X_0(W)) \cup \{\varphi_1(X_0(W))\} \cup \{\varphi_2(X_0(W))\}.$$

The rest is clear. □

Now two situations may arise: The number of connected components of $X^W$ is the same as that of $X$ or it increases by one.

**Definition 2.23.** *Let $X$ be a vertex-faithful closed simplicial surface. A 3-waist $W \subseteq X_1$ is called* **separating**, *if the number of connected components of $X^W$ is bigger than that of $X$.*

Just like for any two 3-waists $W, W' \in W_3(X)$ one might have 0,1, or 2 vertices in the intersection $X_0(W) \cap X_0(W')$, the same holds for seperating 3-waists. Also like for any 3-waist $W$ cutting through $W$ reduces the number of 3-waists by one, similarly cutting through a seperating 3-waist $W$ reduces the number of seperating 3-waists by one. (Note this is no longer true, if $W$ is not seperating.) The following lemma is obvious.

**Lemma 2.24.** *Cutting through a separating 3-waist and taking connnected sums are inverse operations. More precisely:*
*1.) Suppose $X$ is a closed simplicial surface with seperating 3-waist $W$ such that $X^W = X^{(1)} \biguplus X^{(2)}$ with disjoint faces $F_i \in X_2^{(i)}$ corresponding to $W$. Then $X^{(1)} \#_\varphi X^{(2)}$ is isomorphic to $X$, where the bijection $\varphi : X_0^{(1)}(F_1) \to X_0^{(2)}(F_2)$ comes from the identification with $W$.*
*2.) Conversely the connected sum $X \#_\varphi X'$ as described in Remark 2.20 yields $(X \#_\varphi X')^W$ isomorphic to $X \biguplus X'$, where $W$ denotes the seperarting 3-waist of $X \#_\varphi X'$ created by the connected sum.*

Essential for the subsequent decomposition result is the following commutativity result, which is also obvious.

**Lemma 2.25.** *Let $W, W'$ be two different separating 3-waists of $X$. Denote the seperating 3-waist of $X^W$ corresponding to $W'$ also by $W'$ and the one of $X^{W'}$ corresponding to $W$ also by $W$. Then $(X^W)^{W'}$ and $(X^{W'})^W$ are isomorphic.*

Here is now the decomposition theorem for connected orientable simplicial surfaces.

**Theorem 2.26.** *Let $X$ be a vertex-faithful closed connected simplicial surface and denote the set of separating 3-waists of $X$ by $W_3^s(X)$.*
*1.) Cutting through the separating 3-waists of $X$ leads to $n := |W_3^s(X)|+1$ pairwise disjoint simplicial connected closed surfaces $X^{(1)}, \ldots, X^{(n)}$, each without separating 3-waists. The isomorphism types of the $X^{(i)}$ are uniquely determined by $X$ up to renumbering, in paticular they are independent of the sequential order in which the separating 3-waists are cut. The $X^{(i)}$ are called the* **building blocks** *of $X$.*
*2.) Furthermore $X$ defines a tree $T(X)$ whose vertices are the building blocks $X^{(i)}$ and whose edges are the separable 3-waists in $W_3^s(X)$. More precisely $X^{(i)}$ and $X^{(j)}$ for $j \neq i$ are connected by an edge in $T(X)$ if and only if there is a face of $X^{(i)}$ and one of $X^{(j)}$ corresponding to the cut though the same separating 3-waist of $X$. This tree (with or without the isomorphism types of the building blocks) also depends only on $X$.*
*3.) Finally in addition each $W \in W_3^s(X)$ together with each of the two faces $F_i$ in $X^{(i)}$ created by cutting through $W$ defines a bijection $\varphi_{W,i} : X_0(W) \to X_0(F_i)$ keeping track of the doubling of the points by the cut through $W$.*
*4.) There is a natural injective map from $X_2$ to $\biguplus_{i=1}^n X_2^{(i)}$ such that the faces in the complement of the image are the faces of the $X^{(i)}$ in 3.) created by a cut. In particular*

$$|X_2| = \sum_{i=1}^{n} |X_2^{(i)}| - 2|W_3^s(X)|.$$

*The number of faces of $X^{(i)}$ corresponding to a face of $X$ is*

$$|X_2^{(i)}| - |\{W \in W_3^s(X) |\ \text{the cut through } W \text{ creates a face of } X^{(i)}\}|,$$

*which is $|X_2^{(i)}| - 1$ if and only if $X^{(i)}$ is a leaf in $T(X)$.*
*5.) $X$ can be reconstructed from the data in 1), 2.), 3.) as an iterated connected sum.*

*Proof.* 1.) Follows easily from Lemma 2.25. The other statements are also clear from the perceeding lemmas and remarks, □

If we are only interested in surfaces of genus 0 (i. e. EULER characteristic 2), then the building blocks are also only of genus 0. The simplest possible building block is a tetrahedorn. To demonstrate the theorem we give an example.

**Example 2.27.** *Assume we want to construct all vertex-faithful closed simplicial surfaces $X$ of genus zero with exactly two vertices of degree 3 and whose building blocks are only tetrahedra. Then clearly the tree $T(X)$ is just a line of $n \geq 2$ vertices (i. e. tetrahedra). Here are the first few examples up to isomorphism (the number of faces is $2n+2$) in symmetric difference notation :*

| | |
|---|---|
| $n := 2$ | $\Sigma_2 := \mathrm{Pot}_3(\{1,2,3,4\}) \vartriangle \mathrm{Pot}_3(\{1,2,3,5\})$ |
| $n := 3$ | $\Sigma_3 := \Sigma_2 \vartriangle \mathrm{Pot}_3(\{1,2,5,6\})$ |
| $n := 4$ | $\Sigma_{4,1} := \Sigma_3 \vartriangle \mathrm{Pot}_3(\{1,2,6,7\})$ |
| | $\Sigma_{4,2} := \Sigma_3 \vartriangle \mathrm{Pot}_3(\{1,5,6,7\})$ |
| $n := 5$ | $\Sigma_{5,1} := \Sigma_{4,1} \vartriangle \mathrm{Pot}_3(\{1,6,7,8\})$ |
| | $\Sigma_{5,2} := \Sigma_{4,1} \vartriangle \mathrm{Pot}_3(\{1,2,7,8\})$ |
| | $\Sigma_{5,3} := \Sigma_{4,2} \vartriangle \mathrm{Pot}_3(\{5,6,7,8\})$ |
| | $\Sigma_{5,4} := \Sigma_{4,2} \vartriangle \mathrm{Pot}_3(\{1,6,7,8\})$ |
| | $\Sigma_{5,5} := \Sigma_{4,2} \vartriangle \mathrm{Pot}_3(\{1,5,7,8\})$ |

*There are nice "periodic" pattern which can be constructed this way:*

$$\Sigma_2 := \mathrm{Pot}_3(\{1,2,3,4\}) \vartriangle \mathrm{Pot}_3(\{2,3,4,5\}),$$

*and for $n \geq 2$*

$$\Sigma_{n+1} := \Sigma_n \vartriangle \mathrm{Pot}_3(\{n+1, n+2, n+3, n+4\})$$

*(If realized by regular tetrahedra in 3-space, one gets spiral staircases this way.)*

## 2.5 Surfaces without 3-waists

By the results of the last section, for the classification of general simplicial surfaces those without 3-waist form the main building blocks. In this section we report on the vertex-faithful simplicial surfaces of spherical types with up to 20 faces. Though we have derived these lists with our own methods, mainly using the crystallographic package CARAT for determining the automorphism groups and deciding isomorphism, we were pleased to learn that these lists are already in the literature. Here are some invariants which help to distinguish isomorphism classes of surfaces for hand calculations.

**Definition 2.28.** *Let $(X,<)$ be a finite simplicial surface.*
*1.) The* **vertex counter** *of $(X,<)$ is given by the formal product:*

$$\prod_{i\in\mathbb{N}} i^{a_i(X)} \text{ with } a_i(X) := |\{V \in X_0|\ |X_2(V)| = i\}|$$

*2.) The* **edge counter** *of $(X,<)$ is given by the formal product:*

$$\prod_{i\leq j}\{i,j\}^{a_{i,j}(X)} \text{ with}$$

$$a_{i,j}(X) := |\{e \in X_1|\ |X_2(V_1)| = i, |X_2(V_2)| = j \text{ where } \{V_1,V_2\} = X_0(e)\}|.$$

Of course, in the same spirit one might define the **face-counter** of $(X,<)$. But up to 14 faces a vertex-faithful simplical surface is characterized up to isomorphism by its vertex counter, and up to 18 faces by its edge counter. Of course, the edge counter determines the vertex counter:

**Remark 2.29.** *The vertex counter $\prod_{i\in\mathbb{N}} i^{a_i(X)}$ is determined by the edge counter $\prod_{i\leq j}\{i,j\}^{a_{i,j}(X)}$. More precisely*

$$a_i(X) = \frac{1}{i}(2a_{ii}(X) + \sum_{j\neq i} a_{i,j}(X)).$$

*For closed simplicial surfaces one has $|X_0| = \sum_i a_i(X)$, $|X_1| = \frac{1}{2}\sum_i ia_i(X)$, and $|X_2| = \frac{1}{3}\sum_i ia_i(X)$.*

Bemerkungen zum Beweis. Verwis auf Brinkamnn MacKay + alte Arbeut

**Example 2.30.** *In the following list all simplicial surfaces are vertex faithful of spherical type and are up to isomorphism uniquely determined by their respective edge-counter. We give a complete list of all simplicial surfaces without a 3-waist up to 16 faces.*

.

| $m$ | $n$ | *vertex counter* | *edge counter* | *aut. group* |
|---|---|---|---|---|
| 4 | 4 | $3^4$ | $\{3\}^6$ | $S_4$ |
| 8 | 6 | $4^6$ | $\{4\}^{12}$ | $C_2 \wr S_3$ |
| 10 | 7 | $4^5 5^2$ | $\{4\}^5\{4,5\}^{10}$ | $D_{20}$ |
| 12 | 8 | $4^6 6^2$ | $\{4\}^6\{4,6\}^{12}$ | $C_2 \times D_{12}$ |
| | | $4^4 5^4$ | $\{4\}^2\{5\}^4\{4,5\}^{12}$ | $D_8$ |
| 14 | 9 | $4^3 5^6$ | $\{5\}^9\{4,5\}^{12}$ | $D_{12}$ |
| | | $4^4 5^4 6^1$ | $\{4\}^2\{5\}^5\{4,5\}^8\{4,6\}^4\{5,6\}^2$ | $V_4$ |
| | | $4^5 5^2 6^2$ | $\{4\}^3\{4,5\}^6\{4,6\}^8\{5,6\}^4$ | $V_4$ |
| | | $4^7 7^2$ | $\{4\}^7\{4,7\}^{14}$ | $D_{28}$ |
| 16 | 10 | $4^8 8^2$ | $\{4\}^8\{4,8\}^{16}$ | $C_2 \times D_{16}$ |
| | | $4^6 5^2 7^2$ | $\{4\}^4\{4,5\}^6\{4,7\}^{10}\{5,7\}^4$ | $V_4$ |
| | | $4^5 5^3 6^1 7^1$ | $\{4\}^3\{5\}^2\{4,5\}^6\{4,6\}^3\{4,7\}^5\{5,6\}^3\{5,7\}^2$ | $C_2$ |
| | | $4^4 5^4 6^2$ | $\{4\}^2\{5\}^2\{4,5\}^8\{4,6\}^4\{5,6\}^8$ | $C_2^3$ |
| | | | $\{4\}\{5\}^3\{4,5\}^8\{4,6\}^6\{5,6\}^6$ | $C_2$ |
| | | | $\{4\}^2\{5\}^5\{6\}\{4,5\}^6\{4,6\}^6\{5,6\}^4$ | $C_2$ |
| | | $4^3 5^6 6^1$ | $\{5\}^9\{4,5\}^9\{4,6\}^3\{5,6\}^3$ | $S_3$ |
| | | $4^2 5^8$ | $\{5\}^{16}\{4,5\}^8$ | $D_{16}$ |
| | | $4^5 5^2 6^3$ | $\{4\}^2\{6\}^2\{4,5\}^6\{4,6\}^{10}\{5,6\}^4$ | $V_4$ |
| | | $4^6 6^4$ | $\{4\}^4\{6\}^4\{4,6\}^{16}$ | $D_8$ |

**Example 2.31.** *We list all simplicial surfaces are vertex-faithful of spherical type with eighteen faces without a 3-waist. Up to isomorphism they are still uniquely determined by their respective edge-counter.*

| *vertex counter* | *edge counter* | *aut. group* |
|---|---|---|
| $4^5 5^2 6^4$ | $\{4\}^3\{6\}^4\{4,5\}^4\{4,6\}^{10}\{5,6\}^6$ | $V_4$ |
| | $\{4\}^2\{6\}^3\{4,5\}^4\{4,6\}^{12}\{5,6\}^6$ | $C_2$ |
| $4^2 5^8 6^1$ | $\{5\}^{15}\{4,5\}^6\{4,6\}^2\{5,6\}^4$ | $V_4$ |
| $4^4 5^4 6^3$ | $\{4\}\{5\}^3\{6\}^2\{4,5\}^7\{4,6\}^7\{5,6\}^7$ | $C_1$ |
| | $\{5\}^2\{6\}\{4,5\}^8\{4,6\}^8\{5,6\}^8$ | $V_4$ |
| | $\{4\}\{5\}^5\{6\}^2\{4,5\}^5\{4,6\}^9\{5,6\}^5$ | $C_2$ |
| | $\{4\}^2\{5\}^3\{6\}^2\{4,5\}^6\{4,6\}^6\{5,6\}^8$ | $C_2$ |
| $4^6 6^5$ | $\{4\}^3\{6\}^6\{4,6\}^{18}$ | $D_{12}$ |
| $4^3 5^6 6^2$ | $\{5\}^7\{4,5\}^8\{4,6\}^4\{5,6\}^8$ | $C_2$ |
| | $\{4\}\{5\}^8\{4,5\}^6\{4,6\}^4\{5,6\}^8$ | $V_4$ |
| $4^7 6^2 7^2$ | $\{4\}^5\{4,6\}^8\{4,7\}^{10}\{6,7\}^4$ | $V_4$ |
| $4^6 5^1 6^3 7^1$ | $\{4\}^3\{6\}^2\{4,5\}^3\{4,6\}^{10}\{4,7\}^5\{5,6\}^2\{6,7\}^2$ | $C_2$ |
| $4^5 5^3 6^2 7^1$ | $\{4\}^3\{5\}^3\{6\}\{4,5\}^4\{4,6\}^6\{4,7\}^4\{5,6\}^3\{5,7\}^2\{6,7\}$ | $C_1$ |
| | $\{4\}^2\{5\}\{6\}\{4,5\}^7\{4,6\}^5\{4,7\}^4\{5,6\}^4\{5,7\}^2\{6,7\}$ | $C_1$ |
| | $\{4\}^2\{5\}\{6\}\{4,5\}^6\{4,6\}^6\{4,7\}^4\{5,6\}^4\{5,7\}^3$ | $C_2$ |
| $4^4 5^5 6^1 7^1$ | $\{4\}^2\{5\}^4\{4,5\}^8\{4,6\}\{4,7\}^3\{5,6\}^5\{5,7\}^4$ | $C_2$ |
| | $\{4\}\{5\}^5\{4,5\}^8\{4,6\}^2\{4,7\}^4\{5,6\}^4\{5,7\}^3$ | $C_2$ |
| | $\{4\}^2\{5\}^7\{4,5\}^6\{4,6\}^2\{4,7\}^4\{5,6\}^3\{5,7\}^2\{6,7\}$ | $C_2$ |
| $4^5 5^4 7^2$ | $\{4\}^2\{5\}^3\{4,5\}^8\{4,7\}^8\{5,7\}^6$ | $C_2$ |
| | $\{4\}^2\{5\}\{4,5\}^{10}\{4,7\}^6\{5,7\}^8$ | $V_4$ |
| $4^6 5^2 6^1 7^2$ | $\{4\}^3\{4,5\}^6\{4,6\}^4\{4,7\}^8\{5,7\}^4\{6,7\}^2$ | $V_4$ |
| $4^6 5^2 6^2 8^1$ | $\{4\}^4\{6\}\{4,5\}^4\{4,6\}^6\{4,8\}^6\{5,6\}^4\{5,8\}^2$ | $V_4$ |
| $4^6 5^3 7^1 8^1$ | $\{4\}^4\{5\}^2\{4,5\}^6\{4,7\}^4\{4,8\}^6\{5,7\}^3\{5,8\}^2$ | $C_2$ |
| $4^7 5^2 8^2$ | $\{4\}^5\{4,5\}^6\{4,8\}^{12}\{5,8\}^4$ | $V_4$ |
| $4^9 9^2$ | $\{4\}^9\{4,9\}^{18}$ | $D_{36}$ |

**Example 2.32.** *For simplicial surfaces made from 20 faces that do not contain any 3-waist there are excactly* 87 *isomorphism classes. The only isomorphism class without vertices of degree* 4 *is the icosahedron. Including the icosahedron there are* 23 *isomorphism classes with maximal vertex*

*degree $\leq 6$. More precisely we get the following table:*

| *maximal vertex degree* | *number of classes* |
|---|---|
| 4 | 0 |
| 5 | 1 |
| 6 | 21 |
| 7 | 43 |
| 8 | 18 |
| 9 | 3 |
| 10 | 1 |

*While the edge counter seperated the isomorphism classes up to 18 faces, there are 3 pairs of isomorphism classes with 20 faces, that each share the same edge counter.*

# 3 Groups and simplicial Surfaces

## 3.1 3-mirrored simplicial surfaces and group actions

In a general simplicial surfaces $(X, <)$ one can locally think of a neighboured face of some face $F \in X_2$ as coming about by a reflection fixing the common edge. Unfortunately is not always globally well defined. To be more explicit, one can start from the following lemma.

**Lemma 3.1.** *Let $F \in X_2$ be a face of the simplicial surface $(X, <)$.*
*1.) Let $X(F) := \{F\} \uplus X_1(F) \uplus X_0(F)$. The restriction of $<$ to $X(F) \times X(F)$ turns $X(F)$ into a vertex-faithful simplicial surface, namely the one isomorphic to the one-face $\Delta$, cf. 2.7.*
*2.) For any inner edge $e \in X_1(F)$ of $X$ there is a unique isomorphism $\sigma_{F,e} : X(F) \to X(F')$ fixing the two vertices of $e$, where $F'$ is the $e$-neighbour of $F$. The inverse isomorphism is $\sigma_{F',e}$.*
*3.) Let $P := (F_1, \ldots, F_n)$ be a face path on $X$. Let $e_i$ be an edge in $X_1(F_i) \cap X_1(F_{i+1})$ for $1 \leq i < n$. The composition of the $\sigma_{F_i,e_i}$ yields an isomorphism $\sigma_P : X(F_1) \to X(F_n)$ independent of the choice of the intermediate edges $e_i$.*

*Proof.* 1.), 2.) Obvious. 3.) Only the independence requires a proof. If there is more than one edge shared by $F_i$ and $F_{i+1}$ the analysis of the bag and the Janus head in 2.7 shows that the isomorphism are independent of the choice of the common edge. □

In general, a closed face path starting and ending in a face $F \in X_2$ gives rise to an isomorphism of $X(F)$ onto itself by composing the $\sigma_e$ of the edges $e$ in between two subsequent faces of the closed path. It might happen, that this composition is not the identity on $X(F)$. The following definition defines one natural characterising property of $(X,<)$ for this not to happen. From here on we often refer to a three element set $\{a,b,c\}$. The elements $a,b,c$ will be called **colours**. Later on it will become clear that these colours, which are ususally assigned to edges of triangles, are the combinatorial devices modelling the three lengths of the edges of a triangle, thus forcing all faces to be congruent. The first part of the subsequent definition will be taken up by itself in Section 3.4.

**Definition 3.2.** *Let $(X,<)$ be a simplicial surface.*
*1.) A map*

$$\lambda : X_1 \to \{a,b,c\}$$

*is called an* **edge colouring** *of $X$ or a* **wild colouring** *of $X$, if for every face $F \in X_2$, $\lambda$ restricts to a bijection*

$$X_1(F) \to \{a,b,c\}.$$

*In this case $(X,<,\lambda)$ or shorter $(X,\lambda)$ is called an* **edge coloured simplicial surface***.*
*2.) An edge coloured simplicial surface $(X,<,\lambda)$ is called* **3-mirrored***, if for each vertex $A \in X_0$ there exists a 2-element subset $\{i,j\} \subseteq \{a,b,c\}$ such that*

$$\lambda(X_1(A)) = \{i,j\}.$$

*In this situation $\lambda$ is called a* **3-mirror labelling***, an edge $e$ is called of* **type** *$i$ if $\lambda(e)=i$, a vertex $A$ as above is called of* **type** *$\{i,j\}$, the set of all edges of type $i$ is usually denoted by ${}^{i}X_1$ and the set of all vertices of type $\{i,j\}$ by ${}^{i,j}X_0 = {}^{j,i}X_0$.*

**Proposition 3.3.** *Let $(X,<)$ be a simplicial surface. It is 3-mirrored if and only if it covers the one-face $\Delta$.*

*Proof.* Label the three edges of $\Delta$ by $a,b,c$, i.e. write $\Delta_1 = \{a,b,c\}$. If the covering is $\alpha$, then its restriction $\lambda : X_1 \to \{a,b,c\}$ to $X_1$ defines a 3-mirror labelling. On the other hand any 3-mirror labelling $\lambda : X_1 \to \{a,b,c\}$ of $X$ clearly extends to a unique covering $\alpha : X \to \Delta$. $\square$

**Remark 3.4.** *1.) If $(X,<)$ is 3-mirrored, then for each inner vertex $V$ the number of adjacent edges is even, i. e. $2 \mid |X_1(V)| = |X_2(V)|$.*
*2.) The octahedron has a 3-mirror labelling.*
*3.) The converse of 1.) is not true.*
*4.) For a 3-mirrored connected simplicial surface $(X,<,\lambda)$ the labelling $\lambda$*

*is uniquely determined by its restriction to $X_1(F)$ for one $F \in X_2$.*
*5.) The simplicial surface $(X,<)$ has a 3-mirror labelling if and only if for any closed face path $P$ on $X$ the isomorphism $\sigma_P$ is the identity on $X(F)$ where $F$ is the first face of $P$.*

*Proof.* 1.), 2.), 4.) and 5.) are obvious. As for 3.) define for $r,s \in \mathbb{Z}_{\geq 2}$ the **Torus** $T(r,s)$ as simplicial surface as follows: First define an infinite 3-mirrored simplicial surface $S(\mathbb{R}^2)$, whose set of vertices is $S(\mathbb{R}^2)_0 := \mathbb{Z}^2$. Let $e_1 := (1,0), e_2 := (0,1), e_3 := (-1,-1)$. Then the set of edges consists of the segment of the straight line connecting $v \in \mathbb{Z}^2$ with one of $v + e_i$ for $i = a,b,c$. Finally an element of $S(\mathbb{R}^2)_2$ is the convex hull $F$ of the three vertices $v, v+e_i, v-e_j$ for $v \in \mathbb{Z}^2, 1 \leq i \neq j \leq 3$. Obviously the group $\mathbb{Z}^2$ acts by translation on this infinite simplicial surface and $T(r,s)$ is defined as the quotient surface $S(\mathbb{R}^2)/\{(x,y) | x \in r\mathbb{Z}, y \in s\mathbb{Z}\}$ in a self explanatory way.
One easily checks that $T(r,s)$ has six adjacent edges for each vertex. It is 3-mirrored if and only if both $r,s$ are divisible by 3. □

What makes the 3-mirrored simplicial surfaces special is that their neighbouring relation is the disjoint union of three relations, each one of which can be viewed as an involutive permutation of the set of faces, a feature which applies to all wild colourings. However, in the situation of 3-mirrored simplicial surfaces the vertices can easily be described in terms of the three involutions. At this point we mention that we also allow the degeneracy that an involution is the identity permutation. This arises sometimes in situations where the simplicial surface has a boundary.

**Proposition 3.5.** *Let $(X,<,\lambda)$ be 3-mirrored simplicial surface with $X_1 = {}^aX_1 \biguplus {}^bX_1 \biguplus {}^cX_1$ be the partitioning of the edges according to the fibres of the 3-mirror labelling $\lambda$.*
*1.) For each $i \in \{a,b,c\}$ there is a unique permutation $\sigma_i$ of $X_2$ such that for each inner edge in ${}^iX_1$ the two adjacent faces are interchanged and for each boundary edge in ${}^iX_1$ the adjacent face is fixed by $\sigma_i$.*
*2.) Let $G$ be the subgroup of the symmetric group on the set $X_2$ of faces generated by $\sigma_a, \sigma_b, \sigma_c$. The connected components of $X$ are in bijection to the orbits of $G$ on $X_2$. In case $G$ is transitive, the* Schreier*-Graph of $G$ with respect to $\sigma_a, \sigma_b, \sigma_c$ is the neighbour graph of $X$ with respect to the three neighbour relations induced by the ${}^iX_1$.*
*3.) $X$ can be reconstructed from $G$ and its generators $\sigma_a, \sigma_b, \sigma_c$ (viewed as permutations of $X_2$), as follows:*

- *a) ${}^iX_1$ is in natural bijection with the set $C_i$ of cycles $\{F, \sigma_i(F)\}$ of $\sigma_i$ for $i = a,b,c$ so that ${}^iX_1$ can be identified with $\{i\} \times C_i$ and* *b) for any 2-subset $\{i,j\}$ of $\{a,b,c\}$ the vertex set ${}^{i,j}X_0$ can be identified with the set of all pairs $(\{i,j\},B)$ where $B$ runs through the set of orbits of $\langle\sigma_i,\sigma_j\rangle$ on $X_2$.*

*c) The edge $(i,\{F,F'\})$ with $F\neq F'$ is an inner edge adjacent to the faces $F$ and $F'=\sigma_i(F)$ and the edge $(i,\{F\})$ with $\sigma_i(F)=F$ is a boundary edge only adjacent to the face $F$.*

*d) The vertex $(\{i,j\},B)$, which is usually represented by $(\{i,j\},F)$ for some face $F\in B$, is adjacent to the following edges only $(i,\{F,\sigma_i(F)\})$ and $(j,\{F,\sigma_j(F)\})$ with $F\in B$.*

*4.) The centralizer of $G$ in the symmetric group on $X_2$ is the automorphims group of the surface (or the* Schreier-*Graph) where the type of the neighbourhood is taken into account. Note, in case $G$ is transitive on $X_2$ this centralizer is isomorphic to $N_G(S)/S$, where $S$ is the stabilizer of some element of $X_2$ in $G$.*

itemize ohne Leerzeilen definireren

*Proof.* Straightforward. □

This result yields a large class of examples of 3-mirrored simplicial surfaces.

**Example 3.6.** *Let $G$ be a finite group generated by three elements $\sigma_a,\sigma_b,\sigma_c$ all of order 2. With these data we construct a 3-mirrored simplicial surface*

$$(X,<)=(X(G,\sigma_a,\sigma_b,\sigma_c),<)$$

*as follows. Define $X_2:=G$, ${}^iX_1:=\{(i,\{r,\sigma_ir\})|r\in G\}$ for $i=a,b,c$ and ${}^{i,j}X_0$ consisting of the pairs $(\{i,j\},g\langle\sigma_i,\sigma_j\rangle)$ of the 2-subset $\{i,j\}$ of $\{a,b,c\}$ and the left coset of any $g\in G$ with respect to $\langle\sigma_i,\sigma_j\rangle\leq G$. The partial order $<$ is canonical, in particular $r\in G$ is the neighbour of $\sigma_ir$ such that they share an edge of type $i$, namely $(i,\{r,\sigma_ir\})$. Hence there are $|G|$ faces, $\frac{3}{2}|G|$ edges, all of which are inner. If the order of $\sigma_i\sigma_j$ is $o_{ij}$ for $i\neq j$, then one has*

$$|G|(\frac{1}{2o_{ab}}+\frac{1}{2o_{ac}}+\frac{1}{2o_{bc}})$$

*vertices. The resulting surface is 3-mirrored, connected, and closed of* Euler-*characteristic*

$$\frac{|G|}{2}(\frac{1}{o_{ab}}+\frac{1}{o_{ac}}+\frac{1}{o_{bc}}-1).$$

*So, for instance the dihedral group $D_{2n}:=\langle s,t|s^2,t^2,(st)^n\rangle$ of order $2n$ with respect to the generators $\sigma_1:=s,\sigma_2:=s,\sigma_3:=t$ yields a simplicial surface* *of* EULER*-characteristic* 2*. It can easily be represented by folding paper in-various ways, thus demonstrating that the topological homeomorphism type is that of a sphere. Two* 3*-mirrored simplicial surfaces* $(X(G, \sigma_a, \sigma_b, \sigma_c), <)$ *and* $(X(H, \tau_a, \tau_b, \tau_c), <)$ *are isomorpic (as* 3*-mirrored simplicial surfaces, i. e. with an isomorphism respecting the colours of edges and vertices) if and only if the map* $\sigma_i \mapsto \tau_i, i = a, b, c$ *induces an isomorphism of the finite groups* $G$ *and* $H$*.*

The last example can be generalized by passing from the regular permutation representation of the group to any faithful transitive permutation representation of a finite group generated by three involutions.

**Example 3.7.** *Let* $M$ *be a finite set and* $\sigma_a, \sigma_b, \sigma_c$ *be three permutations of* $M$ *of orders (1 or) 2. To define a* 3*-mirrored simplicial surface*

$$(X(M, \sigma_a, \sigma_b, \sigma_c), <) := (X, <)$$

*with* $X_2 := M$*, let*

$${}^{i}X_1 := \{(i, \{m, \sigma_i(m)\}) | m \in M\}$$

*for* $i = a, b, c$ *and*

$${}^{i,j}X_0 := \{(\{i, j\}, B) | B \text{ an } \langle \sigma_i, \sigma_j \rangle\text{-orbit on } M\}$$

*for any 2-subset* $\{i, j\}$ *of* $\{a, b, c\}$*.*
$X$ *is connected iff* $\langle \sigma_a, \sigma_b, \sigma_c \rangle \leq S_M$ *is transitive on* $M$*.* $X$ *is closed iff none of the* $\sigma_i$ *has a fixed point in* $M$*. The* EULER*-characteristic of* $X$ *is*

$$|M| - \sum_{i \in \{a,b,c\}} |M/\langle \sigma_i \rangle| + \sum_{\{i,j\} \in \mathrm{Pot}_2(\{a,b,c\})} |M/\langle \sigma_i, \sigma_j \rangle|,$$

*where* $M/H$ *denotes the set of orbits of a group* $H \leq S_M$ *on* $M$*.*

*Verifiziere Axiome*

The obvious question arises, how the various connected 3-mirrored simplicial surfaces are interrelated if they come from the same finite group $G = \langle g_1, g_2, g_3 \rangle$ generated by three distinguished elements $g_i$ with $g_i^2 = 1$ or more precisely from $(P(g_1), P(g_2), P(g_3))$ for some (transitive) permutation representation $P$ of $G$. Clearly, if one such $P$ factors over some other, this results into what one might call a covering of the associated surfaces ramified in the vertices and edges.

Anheften von projektiven Ebenen und von Tori,
baryzentrische Unterteilung, 4-er Unterteilung

We summarize our discussion in the following theorem.

**Theorem 3.8.** *Let $\Sigma := C_2(a) * C_2(b) * C_2(c)$ be the free product of three copies of the cyclic group of order 2. There is a bijection between the isomorphism classes of finite $\Sigma$-sets and the isomorphism classes of the labelled 3-mirrored finite simplicial surfaces.*

*Proof.* Example 3.7 and Propositon 3.5 establish the correspondence, where $i \mapsto \sigma_i$ for $i \in \{a, b, c\}$. □

Clearly the theorem equally well holds for non finite simplicial complexes and infinite $\Sigma$-sets on which the product of any two of the $\sigma_i$ have finite order. There are two worthwhile supplements to this theorem describing invariants of the 3-mirrored simplicial surfaces in group theoretic terms.

Alles etwas problematisch, wenn man Raender hat. Genau pruefen.

**Remark 3.9.** *Let $(X, <, \lambda)$ be a connected 3-mirrored simplicial surface. The following three statements are equivalent.*
*1.) $(X, <)$ is orientable.*
*2.) There exists a map $c : X_2 \to \{-1, 1\}$ taking different values on any two neighboured faces.*
*3.) $\sigma_i \mapsto -1$ for $i = a, b, c$ defines an epimorphism of $\langle \sigma_1, \sigma_2, \sigma_3 \rangle \to \langle -1 \rangle \leq \mathbb{Q}^*$.*

Beweis

This remark can be used to draw 2-coloured pictures of orientable simplicial surfaces: Any two neighboured faces get different colours as suggested by the map $c$ above.

Bild von Zebraoktaeder, ein schnes und ein schematisches Bild

**Remark 3.10.** *Let $(X, <, \lambda)$ be a 3-mirrored simplicial surface. Then the automorphism group of $(X, <)$ as 3-mirrored surface, i. e. the stabilizer of $\lambda$ in the automorphism group of $(X, <)$ as simplicial surface, is $C_{S_{X_2}}(\langle \sigma_1, \sigma_2, \sigma_3 \rangle)$ in its action on $X_2$. In particular, if $\langle \sigma_1, \sigma_2, \sigma_3 \rangle$ acts regulary on $X_2$, then the automorphism group is isomorphic to $\langle \sigma_1, \sigma_2, \sigma_3 \rangle$.*

Later on, cf. Corollary 3.38, we shall see that a non orientable 3-mirrored simplicial surface has an orientable 2-fold 3-mirrored cover, e.g. the projective plane represented by 4 triangles is covered by an octahedron.

### 3.2 Simplicial surfaces covered by 3-mirrored simplicial surfaces

Having now a rather satisfying description of the 3-mirrored simplicial surfaces in terms of finite groups, the question arises, what can be done about the simplicial surfaces which are not necessarily 3-mirrored. Here is a construction yielding a 3-mirrored covering of any simplicial surface.

**Definition 3.11.** *Let $(X,<)$ be a simplicial surface. The simplicial surface $(\tilde{X},\tilde{<})$, called the* **sixfold** *mmm*-**cover** *of $X$ is given as follows:*
*1.) The set of faces of $(\tilde{X},\tilde{<})$ is*

$$\tilde{X}_2 := \{(F;e_a,e_b,e_c) | F \in X_2, \{e_a,e_b,e_c\} = X_1(F)\}.$$

*So each face of $X$ gives rise to exatly 6 faces of $\tilde{X}$. More precisely, the map*

$$\tilde{X}_2 \to X_2 : (F;e_a,e_b,e_c) \mapsto F$$

*is surjective with exactly 6 elements in each of its fibres.*
*2.) For $i = a,b,c$ define $\sigma_i : \tilde{X}_2 \to \tilde{X}_2$ by $\sigma_i((F;e_a,e_b,e_c)) := (F;e_a,e_b,e_c)$ if the edge $e_i$ of $F$ with respect to $X$ is a boundary edge and otherwise, cf. Lemma 3.1,*

$$\sigma_i((F;e_a,e_b,e_c)) := (\sigma_{F,e_i}(F);\sigma_{F,e_i}(e_a),\sigma_{F,e_i}(e_b),\sigma_{F,e_i}(e_c)).$$

*3.) $(\tilde{X},\tilde{<}) := (X(\tilde{X}_2,\sigma_a,\sigma_b,\sigma_c),<)$ as defined in 3.7.*

**Proposition 3.12.** *Let $(X,<)$ be a simplicial surface.*
*1.) The sixfold mmm-cover $(\tilde{X},\tilde{<})$ is a 3-mirrored simplicial surface.*
*2.) The map*

$$\tilde{X}_2 \to X_2 : (F;e_a,e_b,e_c) \mapsto F$$

*can be extended in a unique way to a covering map $\tilde{X} \to X$ by mapping the edge $(F;e_a,e_b,e_c)_i := \{(F;e_a,e_b,e_c),\sigma_i((F;e_a,e_b,e_c))\}$ of $(F;e_a,e_b,e_c)$ to $e_i$.*
*3.) In case $X$ is connected, $(\tilde{X},\tilde{<})$ is either connected, consists of two, three, or six pairwise isomorphic connected components. The last case occurs iff $(X,<)$ is 3-mirrored.*
*4.) $(\tilde{X},\tilde{<})$ is closed iff $(X,<)$ is.*

*Proof.* 1.) By Lemma 3.1 2.) the $\sigma_i^2 = \mathrm{Id}_{\tilde{X}_2}$ for all $i = a,b,c$ hence Example 3.7 applies.
2.) The assignment defines unique coverings of $X(F)$ by $X((F;e_a,e_b,e_c))$ for each $(F;e_a,e_b,e_c) \in \tilde{X}_2$ in a compatible way.
4.) Clear from definition.
3.) We give two different proofs. a) $G := \langle\sigma_a,\sigma_b,\sigma_c\rangle \leq S_{\tilde{X}_2}$ is normalized by the action of $S_{\{a,b,c\}}$

$$S_{\{a,b,c\}} \times \tilde{X}_2 \to \tilde{X}_2 : (\pi,(F;e_a,e_b,e_c)) \mapsto (F;e_{\pi^{-1}(a)},e_{\pi^{-1}(b)},e_{\pi^{-1}(c)})$$

Since $X$ is connected the semidirect product $G \rtimes S_{\{a,b,c\}}$ acts transitively on $\tilde{X}_2$. Hence the factor group $S_{\{a,b,c\}}$ acts on the $G$-orbits transitively. $G$-orbits in the same orbit under $S_{\{a,b,c\}}$ clearly give rise to isomorphic connected simplicial surfaces, where only the colours of the edges are permuted. Up to isomorphim there are four transitive $S_{\{a,b,c\}}$-actions, hence the claim follows. The 3-mirrored case is the case of regular $S_{\{a,b,c\}}$ action. The second proof will be given later.

Beweis via Pfade, Elemente von $\Sigma$ interpretieren

□

6-fold cover Tetraeder

The following example describes in detail the sixfold cover of the tetrahedron to help the reader to get used to the above notation.

**Example 3.13.** *The tetrahedron given on the left side of Figure 1 followed by its sixfold mmm-cover in the notation of Definition 3.11. The colours of the edges red, blue, refer to $\sigma_1, \sigma_2, sigma_3$ in this order, and the numbers $1, 2, 3$ refer to the positions in the symbol after the semicolon. So e.g., in the figure on the right the two neighboured faces sharing a green edge should have the same number in last position. Note, since the tetrahedron is vertex faithful, this also applies to its sixfold mmm-cover so that all information can be obtained out of the figure. For a detailed interpretation see Example 3.16 below.*

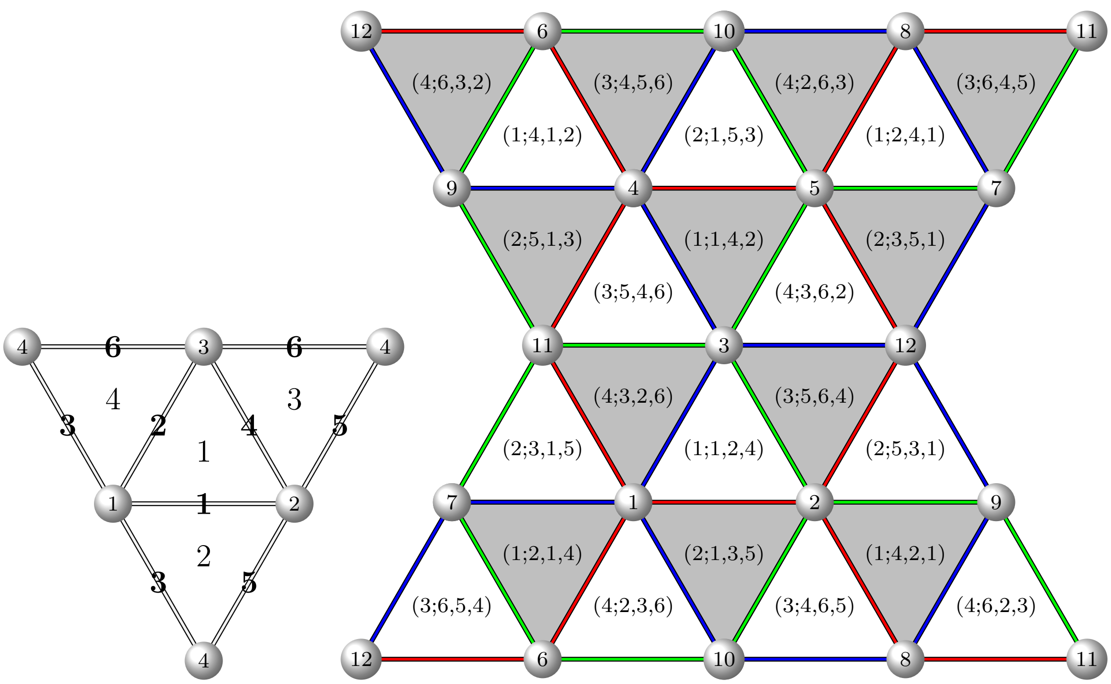


Figure 1: Tetrahdron and its sixfold $mmm$-cover

To work out some examples in detail one might take advantage of simplifications in the vertex-faithful case:

**Remark 3.14.** *Let $(X,<)$ be a vertex-faithful simplicial surface. Then $(\tilde{X},\tilde{<})$ is also vertex-faithful. In this case $\tilde{X}_2$ can be identified with*

$$\{(V_1,V_2,V_3)|\{V_1,V_2,V_3\} = X_0(F) \text{ for some } F \in X_2\}$$

The following examples of connected simplical surfaces $X$ cover all four possibilities for the number of connected components of $\tilde{X}$, except for 6 components, because that characterizes the property of $X$ to be 3-mirrored. But even in that case it might be interesting to investigate the connection between the image $\tilde{G}$ of the universal group $\Sigma$ from Theorem 3.8 in the symmetric group on $\tilde{X}_2$ and $G^6$, where $G$ is the image of $\Sigma$ in its action on $X_2$.

**Example 3.15.** *(Three connected components)*
*Take the double tetrahedron $\Sigma_2$ from Example 2.27. Then $\tilde{\Sigma_2}$ has exactly three connected components, each one of which is of* Euler *characteristic 2, i.e. a sphere on 12 faces. More precisely, each component is isomorphic to $X(D_{12},\sigma_a,\sigma_b,\sigma_c)$ in the notation of Example 3.6, where $\sigma_a,\sigma_b$ generate a dihedral group $D_6$ of order 6 and $\sigma_c$ generates the center $Z(D_{12})$. Curiously, the group describing $\tilde{\Sigma_2}$ is not the full direct product $D_{12}^3$, but of index 8 in it. Clearly, each component is a double 6-gon. The covering $\tilde{\Sigma_2} \to \Sigma_2$ restricts to a double cover $X(D_{12},\sigma_a,\sigma_b,\sigma_c) \to \Sigma_2$ of double tetrahedron $\Sigma_2$. Later we shall see that both of them can be embedded in* Euclidean *3-space.*

*cogwheel genauer anschauen, pass nicht ganz*

*We leave it to the reader to see that this setup can be generalized to a double $n$-gon for any odd $n \geq 3$ being covered by a double $2n$-gon with mmm-structure. Again the double $2n$-gon is of the form $X(D_{4n},\sigma_a,\sigma_b,\sigma_c)$ similarly as above.*

**Example 3.16.** *(One connected component)*
*Cf. 3.13 1.) Take the tetrahedron $T := X[\mathrm{Pot}_3(\{1,2,3,4\})]$, cf. Remark 2.14. It sixfold cover $\tilde{T}$ is connected, orientable of* Euler *characteristic 0, hence a 3-mirrored torus on 24 faces. In the following figure one can see the way it covers the tetrahedron. One can subdivide the torus into 6 connected parts, each of which can be considered as a tetrahedron cut open along the three edges meeting at one of the vertices. These are patched together along the arising cuts as the six triangles in $X(S_3,((1,2),(2,3),(1,3))$. Each of these six triangle is subdivided in 4 triangles according to the faces of the tetraheron. Another way of looking at it, is by rolling the tetrahedron across the surface by tilting over one of the edges touching the torus in such a way that the faces of the tetrahedron match the faces of the torus. We number the edges of the tetrahedron from 1 to 6 and write these numbers on the edges of the torus, whenever we hit them. Note the colours of the edges*

*of the torus come from the 3-mirrored structure of the torus. The faces whose edges have the same numbers form a fibre of the covering map from* $\tilde{T}_2 \to T_2$.

*draw pictures*

*The image group* $\tilde{G}$ *of* $\Sigma$ *as in Theorem 3.8 generated by the three involutions acting on* $\tilde{T}_2$ *is the symmetric group* $S_4$ *embedded via its regular action into* $S_{24}$. *Hence its centralizer in* $S_{24}$ *is also isomophic to* $S_4$. *It is the automorphism group of* $\tilde{T}$ *as 3-mirrored simplicial surface and acts regularly on the set* $\tilde{T}_2$ *of faces.*

*2.) Similarly the icosahedron* $I := I_{20}$ *can be treated. Again* $\Sigma$ *acts regularly on* $\tilde{I}_2$ *and induces a subgroup of* $S_{120}$ *isomorphic to* $C_2 \times A_5$, *whose centralizer in* $S_{120}$ *is therefore also isiomorphic to* $C_2 \times A_5$. *One should, however, not confuse the barycentric subsdivision of the icosahedron with its sixfold cover* $\tilde{I}_2$, *whereas the first is topolgically a sphere, the second is an orientable surface of genus 13.*
*3.) Similar considerations Similarly a surface of 56 equilateral triangles with 24 vertices with symmetry group* $\mathrm{PGL}(2,7)$ *such that every vertex is surrounded by 7 triangles can be considered. Again* $F$ *acts transitively. (Hint for the construction: Let* $\mathrm{PSL}(2,7)$ *act on the cosets of one of its 7-*SYLOW*-subgroups. There exists an orbit of length 56 on* $\mathrm{Pot}_3(\underline{24})$, *which gives the action of* $F$ *on* $\tilde{\Sigma}$.*) cf. also [5]*

*Einzelheiten*

bn The last two examples show that the universal cover is not always exciting as a 3-mirrored simplicial surface in its onwn right, but we shall see later its geometric meaning for the original surface.

**Example 3.17.** *(Two connected components)*
*The* MÖBIUS *strip* $X := X[\Sigma]$ *represented by*

$$\Sigma := \{\{1,2,3\},\{3,2,4\},\{3,5,4\},\{4,1,5\},\{1,2,5\}\}$$

*In this example* $\tilde{X}$ *has two connected components, each of which is again a* MÖBIUS *strip, this time on 15 faces. Choosing one of them gives a threefold covering of* $X$. *The group for* $\tilde{X}$ *does not act regularly on the set of 15 faces, but is of order 150. The regular representation then yields a torus.*

Weglassen oder anpassen

In other words, the edges can be labelled with $a, b, c$, in such a way that ...

**Example 3.18.**

$\Sigma := \{\{1,2,3\},\{1,2,4\},\{1,3,5\},\{1,4,5\},\{2,3,6\},\{2,4,6\},\{3,5,6\},\{4,5,6\}\}$

*describes a regular octahedron. It has the remarkable property, that $\tilde{\Sigma}$ splits into 6 orbits under the action of $F$. One such orbit is*

$$\{(1,2,3),(6,2,3),(1,5,3),(1,2,4),(6,5,3),(6,2,4),(1,5,4),(6,5,4)\}$$

*and the image $\overline{F}$ of $F$ describing the action on $\tilde{\Sigma}$ is elementary abelian of order 8 and acts reularly on each one of these six orbits. $E := \overline{F} \rtimes S_3 \cong C_2 \wr S_3$ acts regularly on $\tilde{\Sigma}$. The centralizer $C$ of the image $\overline{F}$ of $F$ via the permutation action on $\tilde{\Sigma}$ is obviously isomorphic to $C_2^3 \wr S_3$. From this one can recover the automorphism group of $\Sigma$, i. e. the stabilizer fo the action of $S_6$ on the set of six vertices as follows: each element of $C$ induces a bijection of one $F$-orbit $X$ onto some (possibly different) $F$-orbit. Each such bijection is induced by a unique element of $\sigma \in S_6$ satisfying*

$$(\sigma(a),\sigma(b),\sigma(c)) = \tau((a,b,d)) \textit{ for all } (a,b,c) \in X$$

*All these $\sigma$ taken together form the group of automorphims of $\Sigma$, which turns out to be the wreath product $C_2 \wr S_3$ on $\{1,2,\ldots,5\}$ with $\{\{i,j\} \subset \underline{6} | i+j=7\}$ as set of imprimitivity.*

Here are examples with a connected boundary and one innerpoint:

Sehr kritisch ansehen

**Example 3.19.** *1.)*

$$\Sigma := \{\{1,2,3\},\{1,2,5\},\{1,3,4\},\{1,4,5\}\}\}$$

*$F$ acts with six orbits, each of length 4 on $\tilde{\Sigma}$ and induces an elementary abelian group of order $2^3$. The centralizer in $S_{24}$ has the three orbits all of length 8 which are union of two $\overline{F}$-orbits. By inspecting the orbit of the tuple $((1,2,3),(1,4,3),(1,4,5),(1,2,5))$ of tuples under the centralizer, one finds the symmetry group of order 8, namely a $D_8$ induces by permutations of the vertices. (So the situation is similar to the octahedron example.)*

$$\begin{bmatrix} [1,2,3] & [1,4,3] & [1,2,5] & [1,4,5] \\ [1,4,3] & [1,2,3] & [1,4,5] & [1,2,5] \\ [1,2,5] & [1,4,5] & [1,2,3] & [1,4,3] \\ [1,3,2] & [1,5,2] & [1,3,4] & [1,5,4] \\ [1,4,5] & [1,2,5] & [1,4,3] & [1,2,3] \\ [1,5,2] & [1,3,2] & [1,5,4] & [1,3,4] \\ [1,3,4] & [1,5,4] & [1,3,2] & [1,5,2] \\ [1,5,4] & [1,3,4] & [1,5,2] & [1,3,2] \end{bmatrix}$$

*2.)*

$$\Sigma := \{\{1,2,3\},\{1,2,6\},\{1,3,4\},\{1,4,5\},\{1,5,6\}\}$$

*$F$ acts with three orbits, each of length $10$ on $\tilde{\Sigma}$ and induces a group of order $5^3 \cdot 2^2$, which is a subdirect product of three copies of dihedral groups $D_{10}$ of order 10. The centralizer in $S_{30}$ is a direct product of three copies of $D_{10}$ leaving the same three orbits as $\overline{F}$ itself. The action of $F$ on the first orbit is induced by a dihedral subgroup of $S_6$ of order 10.*
*3.)*

$$\Sigma := \{\{1,2,3\},\{1,2,7\},\{1,3,4\},\{1,4,5\},\{1,5,6\},\{1,6,7\}\}$$

*$F$ acts with six orbits, each of length $6$ on $\tilde{\Sigma}$ and induces a group of order $3^3 \cdot 2^2$. (The action on each orbit induces the regualr action of a $D_6$.) The centralizer in $S_{36}$ has the three orbits all of length 12 which are union of two $\overline{F}$-orbits. (So the situation is similar to the octahedronexample.) Anyhow, the symmetriy group can be obtained as above and is also a dihedral group of order 6 induced from a subgroup of the symmetrix group of the vertices. (The centralizer on each of its orbits is acting via a wreath produce $D_6 \wr S_2$ and is isomorphic to the direct product of three copies of $D_6 \wr S_2$. The bicentralizer of $\overline{F}$, i. e. the centralizer fo the centralizer just described, contains $\overline{F}$ as a subgroup of index 2.)*

It should be noted from the last example that the groups involved take no natice of the geometrical invariants as recored in the Gauss-Bonnet-theorem, but only of the combinatorial situation, as is to be expected.

### 3.3 Structures on surfaces

In section 3.1 we considered triangles with coloured edges such that the neighbouring triangle was obtained by reflection. There is however also the possibility of a rotation by an angle of $\pi$ around the center of the edge. To the former edges we assign the type $m$ for mirror, and the last ones the typ $r$ for rotation. We extend the Lemma 3.1 for $r$-edges.

**Lemma 3.20.** *Let $F \in X_2$ of the simplicial surface $(X, <)$.*
*1.) For any inner edge $e \in X_1(F)$ there is a unique isomorphism $\rho_{F,e} : X(F) \to X(F')$ interchanging the two vertices of $e$, where $F'$ is the $e$-neighbour of $F$. The inverse isomorphism is $\rho_{F',e}$.*
*2.) Let $P := (F_1, e_1, F_2, \ldots, e_{n-1}, F_n)$ be a sequence in $X$ such that $F_i \in X_2$ are faces with $F_i \neq F_{i+1}$ and $e_i \in X_1$ are common edges of $F_i$ and $F_{i+1}$, for short* **face-edge-path**. *(Note each $e_i$ must be an inner edge.) Assign*

*a type $\tau_i \in \{m,r\}$ to each edge $e_i$ above. Then the composition of the neighbouring isomorphisms which are $\sigma_{F_i,e_i}$ in case $\tau_i = m$ and $\rho_{F_i,e_i}$ in case $\tau_i = r$ respectively yields an isomorphism $\iota_{P,\tau} : F_1 \to F_n$.*

**Definition 3.21.** *The simplicial surface $(X,<)$ is called of type $(x,y,z)$, where $x,y,z \in \{m,r\}$ if there exists a map $\lambda : X_1 \to \{a,b,c\}$ satisfying the following two conditions:*
*1.) For every face $F \in X_2$ the map $\lambda$ restricts to a bijection $X_1(F) \mapsto \{a,b,c\}$.*
*2.) Let $\mu : \{a,b,c\} \to \{m,r\}$ be defined by the type assignment $\mu(a) = x, \mu(b) = y$ and $\mu(c) = z$. Then for two different faces $F,F' \in X_2$ with common edge $e$ the isomorphism $\iota_{((F,e,F'),\mu(\lambda(e))}$ respects the labelling $\lambda$ (and hence also the type $\mu \circ \lambda$).*
*In this situation $\lambda$ is called an $(x,y,z)$-**labelling** and $(X,<)$ carries an $(x,y,z)$-**structure**.*

The 3-mirrored type discussed in section 3 is the same as the type $(m,m,m)$. Of course type $(m,m,r)$ is the same as type $(r,m,m)$ up to renumbering. Here are some other examples.

**Example 3.22.** *1.) The tetrahedron represented by $\Sigma := Pot_3(\{a,b,c,4\})$ cf. Example, 2.15, can be given an $(r,r,r)$-structure such that the non-intersecting edges get the same value under the $(r,r,r)$-labelling $\lambda$. It is a good excercise to prove that a tetrahedron cannot carry an $(m,r,r)$ structure because the pairs of rotation edges do not match properly. Infinitely many examples of closed surfaces with $(r,r,r)$ structure ( of which the tetrahedron is the first) can be contructed by taking quotients of the stucture imposed of the $\mathbb{R}^2$ by a planar crystallographic group generated by two independent translation and one rotation by an angle of $\pi$, cf. proof Example 3.4 for an analogous construction.*
*2.) The octahedron represented as in Example 2.15 allows an $(m,m,r)$ structure besides the $(m,m,m)$ structure.*

Since the labelling is transported from each face to its neighbours one has the following:

**Remark 3.23.** *Given a connected simplicial surface $(X,<)$ and a $\mu = (x,y,z) \in \{m,r\}^3$. Let $(e_1,e_2,e_3) \in X_1^3$ enumerate the edges of some fixed face $F \in X_2$ in some order. Then there is at most one $\mu$-structure on $X$ with labelling $\lambda$ such that $\lambda(e_i) = i$.*

The first, the second, and the last statement of the next proposition are straight generalizations of the corresponding statements in Proposition 3.5, however the third statement is different and more subtle.

**Proposition 3.24.** *Let $(X,<)$ be a $(x,y,z)$-structured simplicial surface with $X_1 = X_1(a) \biguplus X_1(b) \biguplus X_1(c)$ be the partitioning of the edges according to the fibres of a $(x,y,z)$-labelling for $x,y,z \in \{m,r\}$.*
*1.) For each $i \in \{a,b,c\}$ there is a unique permutation $\sigma_i$ of $X_2$ such that for each inner edge in ${}^iX_1$ the two adjacent faces are interchanged and for each boundary edge in ${}^iX_1$ the adjacent face is fixed by $\sigma_i$.*
*2.) Let $G$ be the subgroup of the symmetric group on the set $X_2$ of faces generated by $\sigma_a, \sigma_b, \sigma_c$. The connected components of $X$ are in bijection to the orbits of $G$ on $X_2$. In case $G$ is transitive, the* Schreier*-graph of $G$ with respect to $\sigma_a, \sigma_b, \sigma_c$ is the neighbour graph of $X$ with respect to the three neighbour relations induced by the ${}^iX_1$.*
*3.) $X$ can be reconstructed as a simplicial surface with $(x,y,z)$-structure from $G$ and its generators $\sigma_a, \sigma_b, \sigma_c$ (viewed as permutations of $X_2$), in such a way that ${}^iX_1$ is the set of cycles of $\sigma_i$ for $i = a,b,c$. A vertex of $X$ is uniquely determined by an element $k$ of $X_2$ and a two-subset $\{i,j\}$ of $\{a,b,c\}$ referring to the two edges of $k$ intersecting in this vertex. $(k,\{i,j\})$ and $(k',\{i',j'\})$ represent the same vertex iff they are $(k,\{i,j\}) \sim (k',\{i',j'\})$ where $\sim$ is the equivalence relation on $X_2 \times \mathrm{Pot}_2(\{a,b,c\})$ given by the transitive closure of $(k,\{i,j\}) \sim (\sigma_i(k),\{i,\tilde{j}\})$ with $\tilde{j} = j$ in case $\mu(i) = m$ or $\sigma_i(k) = k$ and $\tilde{j}$ the unique element of $\{a,b,c\} - \{i,j\}$ in case $\mu(i) = r$ where $\mu_1 := a, \mu_2 := b$ and $\mu_3 := c$.*

*Ende des Obigen verkorkst*

*4.) The centralizer of $G$ in the symmetric group on $X_2$ is the automorphims group of the surface (or the* Schreier*-Graph) where the type of the neighbourhood is taken into account. Note, in case $G$ is transitive on $X_2$ this centralizer is isomorphic to $N_G(S)/S$, where $S$ is the stabilizer of some element of $X_2$ in $G$.*

*Proof.* Straightforward. □

The question arises whether a given triple $(\sigma_1,\sigma_2,\sigma_3) \in S_n^3$ of involutions and a triple $\mu := (x,y,z) \in \{m,r\}^3$ gives rise to a simplicial surface as in $(X,<)$ with $(x,y,z)$-structure with $|X_2| = n$ such that the $\sigma_i$ mark the pairs of faces sharing an edge of type $\mu_i$. This is the case if $\mu = (m,m,m)$, cf. Example 3.7. However for the other structure types obstructions arise:

**Lemma 3.25.** *Let $(\sigma_1,\sigma_2,\sigma_3) \in S_n^3$ be a triple of involutions and $\mu := (x,y,z) \in \{m,r\}^3$. Let $G := \langle \sigma_1,\sigma_2,\sigma_3 \rangle \leq S_n$ and let $\Sigma := \langle \hat{\sigma}_1, \hat{\sigma}_2, \hat{\sigma}_3 | \hat{\sigma}_i^2, i =$

$1, 2, 3\rangle$ *be the free product of three copies of the cyclic group of order 2.*
*1.)There is an action of* $\Sigma$ *on* $\underline{n} \times \mathrm{Pot}_2(\{a, b, c\})$ *defined by*

$$\hat{\sigma}_i(k, \{r, s\}) := (\sigma_i(k), \{r, s\}) \text{ in case } i \notin \{r, s\}$$
$$\hat{\sigma}_i(k, \{i, s\}) := (\sigma_i(k), \{i, s\}) \text{ in case } \mu_i = m$$
$$\hat{\sigma}_i(k, \{i, s\}) := (\sigma_i(k), \{i, r\}) \text{ with } i \neq r \neq s \text{ in case } \mu_i = r$$

*2.) Define*

$$X_2 := \underline{n}, X_1 := \{\{j, \sigma_i(j)\} |\ j \in \underline{n}, i \in \underline{3}\}, X_0 := (\underline{n} \times \mathrm{Pot}_2(\{a, b, c\}))/\sim$$

*with* $\sim$ *the equivalence relation given by the transitive closure of* $(k, \{i, j\}) \sim \hat{\sigma}_i((k), \{i, j\}) \sim \hat{\sigma}_j((k), \{i, j\})$. *Then* $(X, <)$ *with the canonically induced* $<$ *is a simplicial surface with* $(x, y, z)$*-structure via* $(\sigma_1, \sigma_2, \sigma_3)$ *iff for each vertex* $V \in (\underline{n} \times \mathrm{Pot}_2(\{a, b, c\}))/\sim$ *the map* $\nu : V \to \underline{n} : (k, \{i, j\}) \mapsto k$ *is injective.*

Beweis aufschreiben

Bedingungen fuer surgery

**Example 3.26.** *Let* $D_{12} := \langle \alpha, \beta | \alpha^2, \beta^2, (\alpha\beta)^6 \rangle \cong D_6 \times C_2$ *the dihedral group of order* 12 *and* $\rho$ *the regular representation of* $D_{12}$. *Furthermore let* $\zeta := (\alpha\beta)^3$ *the generator of the center. In the following three examples we have closed surfaces with* 12 *faces and* 18 *edges. The number of vertices can be obtained from the vertex counters. Most of the surfaces are not vertex-faithful.*
*1.) Let* $(\sigma_1, \sigma_2, \sigma_3) := (\rho(\alpha), \rho(\beta), \rho(\zeta))$. *Then by applying the construction in Lemma 3.25 there are the following possibilities for the structure types* $\mu$ *any of* $(m, m, m), (m, r, m), (r, m, m), (r, r, m), (r, r, r)$ *with corresponding vertex-counters* $4^6 12,\ 4^3 8^3,\ 4^3 8^3,\ 6^6,\ 9^4$.
*2.) Let* $(\sigma_1, \sigma_2, \sigma_3) := (\rho(\alpha), \rho(\beta\zeta), \rho(\zeta))$.*Then by applying the construction in Lemma 3.25 there are the following possibilities for the structure types* $\mu$ *any of* $(m, m, m), (m, m, r), (m, r, m), (r, m, m), (r, r, m)$ *with corresponding vertex-counters* $4^6 6^2,\ 6^2 12^2,\ 4^3 8^3,\ 4^3 8^3,\ 6^6$.

*The* $(r, r, m)$ *structured surface can be weakly embedded as a flat torus, with zero volume.*

*3.) Let* $(\sigma_1, \sigma_2, \sigma_3) := (\rho(\alpha), \rho(\beta), \rho(\alpha\beta\alpha\zeta))$.*Then by applying the construction in Lemma 3.25 there are the following possibilities for the structure types* $\mu$ *any of* $(m, m, m), (m, m, r), (m, r, m), (r, m, m), (r, r, r)$ *with corresponding vertex-counters* $6^2 12^2,\ 4^6 6^2,\ 8^3 12,\ 8^3 12,\ 6^6$.

*The* $(r, r, r)$ *structured can be embedded as a flat torus with volume 0 but only by iscoceles triangles, the last* $r$ *is independent*

**Example 3.27.** *The following example with mmm-structure yields a torus, which can be paper-embedded into 3-space: gap¿ A;B;C;*

$$\begin{aligned}A &:= (1,28)(2,27)(3,26)(4,25)(5,32)(6,31)(7,30)(8,29)(9,23)(10,24)(11,21)\\&\qquad(12,22)(13,19)(14,20)(15,17)(16,18),\\B &:= (1,6)(2,5)(3,8)(4,7)(9,14)(10,13)(11,16)(12,15)(17,22)(18,21)(19,24)\\&\qquad(20,23)(25,30)(26,29)(27,32)(28,31),\\C &:= (1,18)(2,20)(3,17)(4,19)(5,10)(6,12)(7,9)(8,11)(13,31)(14,29)(15,32)\\&\qquad(16,30)(21,27)(22,25)(23,28)(24,26)\end{aligned}$$

*It comes from*

$$a := (1,7)(2,8)(3,6)(4,5), b := (1,2)(3,4)(5,6)(7,8), c := (1,5)(2,3)(4,8)(6,7)$$

*See paper model.*

**Example 3.28.** *We discuss the various xyz-covers of the Janus-head.*
*1.) The universal mmm cover has 6 connected components. Hence the Janusz-head has an mmm-structure and the corresponding holonomy group is trivial. So one can draw any picture on one of the faces and map it via reflection onto the other face. No matter which edge is used as reflection line, one will have the same picture on the other face. Also the control-triangle can be chosen to be any shape.*
*2.) The universal rrr-cover has two connected components, so that the holonomy group is of order* 3*. This we Interprete as follows: One has to choose the faces as equiangular triangles and one can draw any picture fixed by the threefold rotation of the on one of the faces and copy it via a rotation of an angle* $\pi$*. No matter which edge has been chosen, one gets the same picture.*
*3.) The universal mmr-cover has 3 connected components, so that the holonomy group has order 2. This we interprete as follows: The two edges interchanged by generator of the holonomy group must be of the same length, i. e. we have an isosceles triangle with the two sides of equal length corresponing to reflections (m-sides) and the remaining side corresponds to a rotation. If we now have a pattern on the first face of the Janus-head, which is invariant under the reflction of the holonomy group, this pattern is mapped onto the same pattern of the second side, independent of the map, i.e. one of the reflections at the equal length sides or the rotation around the midpoint of the third side. Discuss the devolopment of the Janus-head on its connected mmr-cover.*
*4.) The universal mrr cover is connected. Hence the corresponding holonomy group is the full symmetric group* $S_3$*. So our triangle must be equilateral and one can draw only* $S_3$*-invariant picture on one of the faces and*

*map it via reflection onto the other face. No matter which one of the three transformations is used, one will have the same picture on the other face.*

*Abrollen auf connected universal cover diskutieren*

## 3.4 Wild colouring

Satz in Euclidian simplicial Surfaces: Ein Dreieckstyp mit 3 verschieden Kantenlngen fr eine EusiSu liefert genau eine wilde Kantenfrbung Isometrie Isomorphie

**Definition 3.29.** *Let $(X, <)$ be a simplicial surface. A* **wild colouring** *of $X$ is a map $\omega : X_1 \to \{a, b, c\}$ such that for each $F \in X_2$ the restriction of $\omega$ to $X_1(F)$ is bijective. $\omega(e)$ is called the* **colour** *of the edge $e \in X_1$. Associated with $\omega$ are three involutions $\sigma_a, \sigma_b, \sigma_c \in S_{X_2}$ of the faces of $X$, where for each edge $e \in X_1$ with $\omega(e) = i$ one has $\sigma_i(F_1) := F_2$ if $\{F_1, F_2\}$ is the set of all faces in $X_2$ having $e$ as edge, for $i \in \{a, b, c\}$. We call $(\sigma_a, \sigma_b, \sigma_c)$ the* **involution triple** *of the wild colouring $\omega$.*

**Remark 3.30.** *In the above definition $X_2$ together with the involution triple $(\sigma_a, \sigma_b, \sigma_c)$ determines $X_1$, the restriction $<_{X_1 \times X_2}$, and the wild colouring $\omega : X_1 \to \{a, b, c\}$. More precisely the (boundary and inner) edges in $X_1$ with colour $i \in \{a, b, c\}$ are in natural bijection to the (one and two)-cycles of $\sigma_i$. In particular all simplical surfaces with face set $X_2$ and involution triple $(\sigma_a, \sigma_b, \sigma_c)$ can and will be viewed to have the same set $X_1$ of coloured edges.*

We have seen already that structures such as $mmm$- or $mmr$-structures on $X$ give rise to a wild colouring. However, the general context in which wild colourings come up is more general and described in Chapter 5. The following lemma is clear.

**Lemma 3.31.** *Let $\omega$ be a wild colouring of the simplicial surface $(X, <)$ with involution triple $(\sigma_a, \sigma_b, \sigma_c)$. Then for each inner edge $e \in X_1$ we assign the type $m$ resp. $r$ if the isomorphism between the two adjacent faces of $e$ respecting the colours of the edges fixes the two vertices of $e$ respectively interchanges them. Let $i := \omega(e)$ be the colour of $e$. Then the 2-cycle of $\sigma_i$ corresponding to $e$ is also assigned type $m$ resp. $r$.*

**Definition 3.32.** *For an involution triple $(\sigma_a, \sigma_b, \sigma_c)$ of permutations, a map*

$$\alpha : \{(\tau, x) | x \in \{a, b, c\}, \tau \text{ a 2-cycle of } \sigma_x\} \to \{m, r\}$$

*is called an $mr$-***assigment** *of the involution triple $(\sigma_a, \sigma_b, \sigma_c)$. If the $mr$-assigment arises from a wild coloured simplicial surface $(X, <)$ with involution triple $(\sigma_a, \sigma_b, \sigma_c)$, then it is called* **admissible**.

It might well be, that two simplicial surfaces with wild colourings, which are not isomorphic, give rise to the same triple of involutions. However, for a fixed and admissible $mr$-assignment, the isomorphism type of the edge-coloured surface is determined.

**Remark 3.33.** *Let $(X,<)$ and $(X',<)$ be simplicial surfaces with wild colourings. Let $(\sigma_a,\sigma_b,\sigma_c)$ and $(\sigma'_a,\sigma'_b,\sigma'_c)$ be the associated triples of involutions on a set $X_2$ and $X'_2$ respectively. Then $X$ and $X'$ are isomorphic as simplicial surfaces with wild colourings, if and only if there is a bijection $\iota : X_2 \to X'_2$ such that*
*1.) $\iota\sigma_k\iota^{-1} = \sigma'_k$ for $k := a,b,c$ and*
*2.) for each 2-cycle $\zeta$ of $\sigma_k$ its type $m$ or $r$ is the same as for the 2-cycle $\iota\zeta\iota^{-1}$ of $\sigma'_k$ again for each $k := a,b,c$.*

Vielleicht Permutation der Farben zulassen

*Proof.* The bijection $\iota : X_2 \to X'_2$ is already given. Since 2-cycles correspond to inner edges and 1-cycles to boundary edges Condition 1.) yields a bijection between $X_1$ and $X'_1$ induced by $\iota$. Also the partial ordering is carried over so far. Concerning the vertices, note that a vertex $V \in X_0$ can be given the name $(F,\{i,j\})$ where $F \in X_2$ is a face with $V < F$ and $i,j \in \{a,b,c\}$ are the two colours of the two edges adjacent to $F$ intersecting in $V$. Though $(F,\{i,j\})$ uniquely determines $V$, the number of such names for $V$ is equal to the degree of $V$. More precisely, if $e \in X_1$ is the edge adjacent to $F$ of colour $i$ with $V \in X_0(e)$, then there are two different flags $(V,e,F)$ and $(V,e,F')$. In case $e$ is an $m$-edge $(F',\{i,j\})$ is also a name for $V$. In $e$ is $r$-edge $(F',\{i,k\})$ with $k \in \{a,b,c\} - \{i,j\}$. Walking around $V$ this gives all possible names for $V$. In particular, the assignment of types $m$ and $r$ to the edges completely determines each vertex. The claim of the remark is now an easy consequence. $\square$

The question arises about the properties of the colourings of the 2-cycles of an involutions triple to define a surface. The answer can be given by using the SCHREIER-graph and needs some preparation. Without loss of generality we restrict to the case where the three involutions generate a transitive permutation group, i.e. where the associated surface is connected. To be precise we fix the notation for the SCHREIER-graph.

**Definition 3.34.** *Let $M$ be a (finite) set and $(\sigma_a,\sigma_b,\sigma_c)$ a triple of involutions generating the transitive subgroup $G = G((\sigma_a,\sigma_b,\sigma_c)$ of the symmetric group $S_M$.*
*1.) The* SCHREIER**-graph** *$\Gamma := \Gamma(\sigma_a,\sigma_b,\sigma_c)$ has $M$ as its set of vertices and its set $E$ of edges is partitioned in $E_i := E_i(\Gamma) := \{\{m,\sigma_i(m)\}_i | m \in$

*$M\}$ for $i = a, b, c$, i.e. $E := E(\Gamma) := E_a \uplus E_b \uplus E_c$.*
*2.) A vertex $m$ of $\Gamma$ is called* **inner vertex**, *if it is not fixed by any of $\sigma_a, \sigma_b, \sigma_c$. There are two types of* **outer vertices**, *namely the ones fixed by exactly one $\sigma_i$, called* **integrated**, *and the ones fixed by exactly two of the $\sigma_i$, called* **exposed**.

To construct all simplicial surfaces with wild colouring out of $\Gamma$ we need the concept of vertex defining paths in $\Gamma$. Certain sets of them will be called "nets". A net immediately translates into a coloured simplicial surface because it determines all vertices of the corresponding surface.

vernuenftig einrcken

**Definition 3.35.** *Let $\Gamma = \Gamma(\sigma_a, \sigma_b, \sigma_c)$ be as above.*
*1.) A* **closed reduced path** *in $\Gamma$ is a sequence $s : \mathbb{Z} \to M \cup \{a, b, c\}$ of the form $\dots s_i, s_{i+1}, s_{i+2}, \dots$ wtih the following properties:*
*a) (periodicity) $s$ is periodic of minimal period at least 4,*
*b) (alternating) $s_i \in M$ implies $s_{i+1} \in \{a, b, c\}$ and $\sigma_{s_{i+1}}(s_i) = s_{i+2} \in M$ as well as $s_i \in \{a, b, c\}$ implies $s_{i-1} \in M$ and $\sigma_{s_i}(s_{i-1}) = s_{i+1} \in M$ for all $i \in \mathbb{Z}$.*
*c) (no repeated colours) If $s_i \in \{a, b, c\}$ then $s_{i+2} \neq s_i$,*
*d) (no repeated faces) Any two $s_i$ in a minimal period of $s$ which lie in $M$ are distinct.*
*Two such sequences are identified, if they are related by a shift or a reflection followed by a shift.*
*2.) A* **transversing reduced path** *in $\Gamma$ is a sequence $s : \{0, 1, \dots, 2n\} \to M \cup \{a, b, c\}$ wtih the following properties:*
*a) (start at boundary) $s_0 \in \{a, b, c\}$ and $s_1 \in M$ with $\sigma_{s_0}(s_1) = s_1$,*
*b) (alternating) For even $i$ with $0 < i < 2n$ we have $s_i \in \{a, b, c\}$ and $s_{i-1} \in M$ and $\sigma_{s_i}(s_{i-1}) = s_{i+1} \in M$.*
*c) (end in boundary) $s_{2n} \in \{a, b, c\}$ and $s_{2n-1} \in M$ with $\sigma_{s_{2n}}(s_{2n-1}) = s_{2n-1}$,*
*d) (no repeated colours) For even $i \in \{0, 1, \dots, 2n\}$ one has $s_{i+2} \neq s_i$,*
*e) (no repeated faces) For any two odd $i, j \in \{1, \dots, 2n-1\}$ one has $s_i \neq s_j$.*
*Each such path is identified with its reverse path.*
*3.) An* **vertex defining path** *is one of 1.) or 2.).*
*4.) A* **net** *on $\Gamma$ is a set $N$ of vertex defining paths such that any sequence $(i, m, j)$ with $i, j \in \{a, b, c\}, m \in M$ and $i \neq j$ occurs as a subsequence in exactly one of the paths of $N$. (Note reversing).*

Naturally the closed reduced paths will give rise to inner vertices since each minimal period is a closed walk around the vertex repeating no face, while the transversing reduced paths will yield boundary vertices.

**Theorem 3.36.** *Let $M$ be a (finite) set and $(\sigma_a, \sigma_b, \sigma_c)$ a triple of involutions generating the transitive subgroup $G = G(\sigma_a, \sigma_b, \sigma_c)$ of the symmetric group $S_M$. Let $\mathcal{S}(M, \sigma_a, \sigma_b, \sigma_c)$ be the set of all simplicial surfaces $(X, <)$ with $X_2 = M$, and $X_1$ with its wild colouring determined by the involution triple $(\sigma_a, \sigma_b, \sigma_c)$ as in Remark 3.30. Furthermore let $\mathcal{N}(M, \sigma_a, \sigma_b, \sigma_c)$ be the of all nets on $\Gamma(\sigma_a, \sigma_b, \sigma_c)$. Then there is a bijection*

$$\nu : \mathcal{S}(M, \sigma_a, \sigma_b, \sigma_c) \to \mathcal{N}(M, \sigma_a, \sigma_b, \sigma_c)$$

*Note, we already have a bijection between $\mathcal{S}(M, \sigma_a, \sigma_b, \sigma_c)$ and the set of admissible mr-assingments of $(\sigma_a, \sigma_b, \sigma_c)$.*

*Proof.* Let $(X, <)$ with $X_2 = M$ and wild colouring $\omega : X_1 \to \{a, b, c\}$ corresponding to $(\sigma_a, \sigma_b, \sigma_c)$. Each vertex $V \in X_0$ then defines a vertex defining path in the SCHREIER-graph $\Gamma := \Gamma(\sigma_a, \sigma_b, \sigma_c)$, more precisely each inner vertex defines a closed reduced path by taking a circular face-edge path around the vertex $V$ and each boundary vertex $V$ a transversing reduced path also by taking the face-edge path around the vertex $V$ from one boundary edge of $X$ adjacent to $V$ the other boundary edge of $X$ adjacent to $V$. All these vertex defining paths in $\Gamma$ form a net $\nu(X)$ in $\Gamma$, because a vertex $V$ in $X$ is uniquely described by $(F, \{i, j\})$ where $F$ is a face adjacent to $V$ and $i, j$ are the colours of the two edges of $F$ adjacent to $V$, which translates into the sequence $(i, F, j)$ (read forwards or backwards) occuring as subsequence in exactly one of the vertex defining paths in $\nu(X)$, namely of the path coming from the walk arount $V$. Hence $X \to \nu(X)$ defines a map $\nu : \mathcal{S}(M, \sigma_a, \sigma_b, \sigma_c) \to \mathcal{N}(M, \sigma_a, \sigma_b, \sigma_c)$. The injectivity of this map is immediate.
To construct the inverse map $\xi : \mathcal{N}(M, \sigma_a, \sigma_b, \sigma_c) \to \mathcal{S}(M, \sigma_a, \sigma_b, \sigma_c)$ let $N$ be a net on $\Gamma$. Define the wild coloured simplicial surface $X = (X, <)$ by $X_2 := M$ and $X_1$ and $\omega : X_1 \to \{a, b, c\}$ as above. Define $X_0 := \{(F, \{i, j\}) | F \in M, i, j \in \{a, b, c\}, i \neq j\}/ \equiv$ where $\equiv$ is the equivalence relation defined by $(F, \{i, j\}) \equiv (F', \{i', j'\})$ if and only if $(i, F, j)$ and $(i', F', j')$ occur as subsequences of the same vertex defining path of $N$. One easily checks the axioms for simplicial surfaces, to see that $X$ and therefore also $\zeta$ are well defined. Clearly $\nu$ and $\zeta$ are inverse to each other. ☐

So our future notation for simplicial surfaces with wild edge colouring is very concise: the involution together with an admissble $mr$-assingment indicated by decorating each $m$-cycle with a bar and each $r$-cycle by a dot. Here is an example:

**Example 3.37.**
*1.)* $(\sigma_a, \sigma_b, \sigma_c) := ((1, 3)(2, 4)(5, 6), (1, 3)(2, 4)(5, 6), (1, 2)(3, 5)(4, 6))$ *gen-*

*erate a symmetric group $S_3$ in its regular permutation representation. There are 4 admissible mr-assignments*

$$((\overline{1,2})(\overline{3,6})(\overline{4,5}),(\overline{1,2})(\overline{3,6})(\overline{4,5}),(\overline{1,4})(\overline{2,3})(\overline{5,6}))$$
$$((\overline{1,2})(\overline{3,6})(\overline{4,5}),(\overline{1,2})(\overline{3,6})(\overline{4,5}),(\overline{1,4})(2\dot{,}3)(5\dot{,}6))$$
$$((\overline{1,2})(\overline{3,6})(\overline{4,5}),(\overline{1,2})(\overline{3,6})(\overline{4,5}),(1\dot{,}4)(\overline{2,3})(5\dot{,}6))$$
$$((\overline{1,2})(\overline{3,6})(\overline{4,5}),(\overline{1,2})(\overline{3,6})(\overline{4,5}),(1\dot{,}4)(2\dot{,}3)(\overline{5,6}))$$

*Their surfaces are twofold covers of the two surfaces with boundary below. (For the first two examples faces $i, i+3$ correspond to face $i$ for $i = 1, 2, 3$.) Note, by Remark 3.33 the last three surfaces on 6 faces are isomorphic as wild coloured simplicial surfaces. Note also, the open bags correspond to identical 2-cycles in different generators, and faces with two boundary edges correspond to common fixed points of two generators.*

$$((\overline{1,2}),(\overline{1,2}),(\overline{2,3}))$$
$$((\overline{1,2}),(\overline{1,2}),(2\dot{,}3))$$

*2.) Passing to three different generators, there is essentially just one choice of involution triple in $S_3$. It has 2 different admissible mr-assingments:*

$$((\overline{1,2}),(\overline{1,3}),(\overline{2,3}))$$
$$((1\dot{,}2),(1\dot{,}3),(2\dot{,}3))$$

*The first surface is a* MÖBIUS*-strip, the second a triangular pyramide without base. We note, if embedded into $\mathbb{R}^3$, the first surface contains two open bags. (It can be composed of cloth so that it can be turned inside out in two different ways yielding each time a different set of open bags.)*

*Now we pass to the regular permutation representation of $S_3$, which yields 8 surfaces as follows:*

| *involution triple with admissible mr-assignment* | *vertex counter* |
|---|---|
| $((\overline{1,5})(\overline{2,4})(\overline{3,6}),(\overline{1,6})(\overline{2,5})(\overline{3,4}),(\overline{1,4})(\overline{2,6})(\overline{3,5}))$ | $6^3$ |
| $((1\dot{,}5)(2\dot{,}4)(3\dot{,}6),(1\dot{,}6)(2\dot{,}5)(3\dot{,}4),(1\dot{,}4)(2\dot{,}6)(3\dot{,}5))$ | $6^3$ |
| $((\overline{1,5})(\overline{2,4})(\overline{3,6}),(\overline{1,6})(\overline{2,5})(\overline{3,4}),(1\dot{,}4)(2\dot{,}6)(3\dot{,}5))$ | $4^3 \cdot 6$ |
| $((\overline{1,5})(\overline{2,4})(\overline{3,6}),(1\dot{,}6)(2\dot{,}5)(3\dot{,}4),(\overline{1,4})(\overline{2,6})(\overline{3,5}))$ | $4^3 \cdot 6$ |
| $((1\dot{,}5)(2\dot{,}4)(3\dot{,}6),(\overline{1,6})(\overline{2,5})(\overline{3,4}),(\overline{1,4})(\overline{2,6})(\overline{3,5}))$ | $4^3 \cdot 6$ |
| $((\overline{1,5})(2\dot{,}4)(3\dot{,}6),(1\dot{,}6)(2\dot{,}5)(\overline{3,4}),(1\dot{,}4)(\overline{2,6})(3\dot{,}5))$ | $4^3 \cdot 6$ |
| $((1\dot{,}5)(\overline{2,4})(3\dot{,}6),(\overline{1,6})(2\dot{,}5)(3\dot{,}4),(1\dot{,}4)(2\dot{,}6)(\overline{3,5}))$ | $4^3 \cdot 6$ |
| $((1\dot{,}5)(2\dot{,}4)(\overline{3,6}),(1\dot{,}6)(\overline{2,5})(3\dot{,}4),(\overline{1,4})(2\dot{,}6)(3\dot{,}5))$ | $4^3 \cdot 6$ |

*The first surface is a torus covering the* MÖBIUS*-strip above twice. (For the general construction passing form a non orientable surface to an orientable 2-fold cover cf. Corollary 3.38 below. (It can also be realized by cloth, cf. above.) The second surface is also a torus embeddable into $\mathbb{R}^3$. The second and last surfaces both cover the open pyramide above twice. The last six surfaces are all non orientable. All these covers are such that the preimages of face $i \in \{1, 2, 3\}$ consist of the faces with number $i, i+3$.*

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*Further the surfaces numbered 6.5.8 are isomorphic in the light of Remark 3.33. Also 3,4,5 become isomorphic upon permutation of the colours. All six surfaces have the double hexagon $4^6 \cdot 6^2$ as 2-fold cover.*

Note, by Example 3.7, with given involution triple there is always one wild colouring yielding a simplicial surface, namely the $mmm$-structure assigning an $m$ to each $2$-cycle.

We formalize the observation that each triangle in $X_2$ has two sides, say an upper and a lower side, by introducing $X_2^{\pm} := \{+, -\} \times X_2$. To be brief, we denote the elements of $X_2^{\pm}$ by $F$ and $-F$ instead of $(+, F), (-, F)$ for $F \in X_2$. Within $X_2(P)$ for any $P \in X_0$ one clearly can keep track of the upper and lower sides of the faces, once one upper face is declared. This is simply by observing that an edge of type $m$ interchanges upper and lower faces, whereas an edge of type $r$ keeps upper and lower faces. Any closed face-edge-path around $P$ with faces in $X_2(P)$ only will give the same assignment of $\pm$ to the sides of faces, because clearly the composition of the neighbouring isomorphisms, cf. Lemma 3.20, yields the identity of the starting face, as one easily checks by using the existence of the wild colouring $\omega$.

Involution: 1-Element zugelassen

**Corollary 3.38.** *In the situation of Lemma 3.31 one obtains a simplicial surface $X^{\pm}$ covering $X$. In detail: $X_2^{\pm}$ as above. The involution triple $(\sigma_a, \sigma_b, \sigma_c)$ defines an involution triple $(\sigma_a^{\pm}, \sigma_b^{\pm}, \sigma_c^{\pm})$ in $S_{X_2^{\pm}}$ by replacing each 2-cycle $(F, F')$ of type $m$ in any of the involutions for $X$ by $(F, -F')(-F, F')$, both of type $m$, and each 2-cycle $(F, F')$ of type $r$ by $(F, F')(-F, -F')$, both of type $r$. Any 1-cycle $(F)$ of $\sigma_i$ gives rise to two 1-cycles $(F)(-F)$ in $\sigma_i^{\pm}$ for $i \in \{a, b, c\}$. Then $X^{\pm}$ is defined to be the simplicial surface corresponding to the $(\sigma_a^{\pm}, \sigma_b^{\pm}, \sigma_c^{\pm})$ with the $m, r$-assignment just defined in the sense of Theorem 3.36. Then:*
*1.) $X_2^{\pm} \to X_2 : F \mapsto F, -F \mapsto F$ defines a (2-fold) covering of $X$ by $X^{\pm}$.*
*2.) In case $X$ is connected, the following three statements are equivalent:*

*a) $X$ is orientable,*
*b) $X^{\pm}$ is connected,*
*c) $(\sigma_a^{\pm}, \sigma_b^{\pm}, \sigma_c^{\pm})$ generate a transitive group on $X_2^{\pm}$.*
*3.) For the* EULER*-characteristics one has $\chi(X^{\pm}) = 2\chi(X)$.*
*4.) $X^{\pm}$ is orientable.*
*5.) $\langle \sigma_a^{\pm}, \sigma_b^{\pm}, \sigma_c^{\pm} \rangle \to \langle \sigma_a, \sigma_b, \sigma_c \rangle : \sigma_i^{\pm} \to \sigma_i$ defines an epimorphism of groups, which is an isomorphism if and only if $X$ is orientable.*

4 beweisen

Standard group theoretic constructions yield new involution triples of permutations from given ones. It is interesting to decide whether these constructions carry over to constructions of surfaces.

**Proposition 3.39.** *Let $\sigma := (\sigma_a, \sigma_b, \sigma_c)$ be an involution triple of permutations generating a transitive permutation group $G$ on $M := \{1, \ldots, n\}$. Let $S$ be the stabilizer of $1$ in $G$ and $U \lneqq S$ of index $k > 1$. Denote the permutation representation of $G$ on the cosets of $U$ in $G$ by $\pi : G \to S_{nk}$. The $G$-map from $\tilde{M} := \{1, \ldots, nk\}$ to $\{1, \ldots, n\}$ is denoted by $\rho$.*
*1.) $\tilde{\sigma} := (\pi(\sigma_a), \pi(\sigma_b), \pi(\sigma_c))$ is an involution triple of permutations of degree $nk$ generating a group isomorphic to $G$.*
*2.) For every admissible $mr$-assignment $\mu$ for $\sigma$, there is an admissible $mr$-assignment $\tilde{\mu}$ for $\tilde{\sigma}$ defined as follows: Let $x \in \{a, b, c\}$ and $(i, j)$ be a 2-cycle of $\tilde{\sigma_x}$. If $\rho(i) \neq \rho(j)$ then $\tilde{\mu}((i, j)) := \mu((\rho(i), \rho(j)))$ otherwise $\tilde{\mu}((i, j)) := m$.*

*Proof.* By Theorem 3.36 it suffices to discuss nets in SCHREIER-graphs. So we have to reformulate the concept of admissible $mr$-assigments of the involution triple in the laguage of walks in the SCHREIER-graph $\Gamma$ and to relate the vertex defining paths in $\Gamma(\sigma)$ with the ones in $\Gamma(\tilde{\sigma})$.
To start with, in the presence of the $mr$-assignment $\mu$ a vertex defining path in $\Gamma(\sigma)$ is uniquely defined by any subsequence $(i, F, j)$ with two different $i, j \in \{a, b, c\}$ and an $F \in M$, because $\mu(\sigma(j))$ determines how the sequence is extended to the right and $\mu(\sigma(i))$ how it is extended to the left. If $\tilde{\mu}$ is an admissible $mr$-assignment, then the same applies to $\Gamma(\tilde{\sigma})$. Since obviously $\tilde{\mu}$ - in case it is admissible- induces $\mu$, we only have to prove that $\tilde{\mu}$ is admissible. This will be done by constructing vertex defining paths in $\Gamma(\tilde{\sigma})$ via lifting from vertex-defining paths in $\Gamma(\sigma)$ and then observing that the same paths can be obtained via $\tilde{\mu}$.
Let $(i, \tilde{F}, j)$ be a sequence with $i \neq j$ in $\{a, b, c\}$ and $\tilde{F} \in \tilde{M}$. Then there is a unique vertex defining path $\Pi$ in $\Gamma(\sigma)$ passing through $(i, F, j)$ with $F := \rho(\tilde{F})$. In case $\Pi$ is a closed reduced path in $\Gamma(\sigma)$ then its lifing (following the colour pattern used in $\Pi$) yields a unique closed path $\tilde{\Pi}$ in $\Gamma(\tilde{\sigma})$ the length of which is a multiple of the length $\Pi$ because there are

no fixed points in the sequence in $\Gamma$ and therefore also not in the lifted sequence. Also the closed path in $\tilde{\Pi}$ projects onto $\Pi$ with constant number of elements in each fibre. Therefore it is a vertex defining path, i.e. no face is repeated since the projection onto $\Pi$ determines its neighbouring colours. Note, using the colour pattern of $\Pi$ is tantamount to the first part of the definition of $\tilde{\mu}$.

In case $\Pi$ is a transversing reduced path, then we lift the path starting from $(i, \tilde{F}, j)$ in both directions by using the colour pattern of $\Pi$ until we have to lift one of the end points of $\Pi$. The resulting path in $\Gamma(\tilde{\sigma})$ is called the leaf of the lift of $\Pi$ through $\tilde{F}$. Note the faces in each leaf above $\Pi$ are pairwise distinct, since $\Pi$ is vertex defining. Note, the same leaves would also be obtained just by using $mr$-assignment $\tilde{\mu}$ in the the first part of its definition. In case the right end $(i, \tilde{F}, j)$ of a leaf satifies $\tilde{\sigma}_j(\tilde{F}) = \tilde{F}$, we have also reached the right end of the lift $\tilde{\Pi}$ to be constructed. (Similarly for left ends $(i, \tilde{F}, j)$.) In case $\tilde{\sigma}_j(\tilde{F}) \neq \tilde{F}$, then $\tilde{\sigma}_j(\tilde{F})$ is a different face over $F := \rho(\tilde{F})$ and therefore a left end of a different leaf over $\Pi$, thus the new leaf is traversed in reversed order (i. e. from right to left). For left ends $(i, \tilde{F}, j)$ of a leaf the obvious changes apply. Note, now already, this construction corresponds the second part of the definition of $\tilde{\mu}$. That no face is repeated is now obvious since all the faces in a leaf are pairwise different and all leaves are by construction pairwise disjoint. If a leaf in the present case is part of a closed path, the length of the path is an even multiple of the length of $\Pi$. In any case all lifts of $\Pi$ are vertex-defining paths and could be obtained via $\tilde{\mu}$.

It remains to prove that all the lifts of all the vertex defining paths from the net defined by $\mu$ define a net on $\Gamma(\tilde{\sigma})$. But this is a partition property for the sequences $(i, F, j)$, which clearly is inherited by the inverse image construction above. $\square$

The last proposition has a RIEMANN-HURWITZ-type of corollary in case of closed surfaces $X$, i.e. if none of the involutions have fixed points. For a vertex $\tilde{P}$ of the covering simplicial surface $\tilde{X}$ lying above the vertex $P$ below, call the quotient

$$e(\tilde{P}) := \frac{|\tilde{X}_2(\tilde{P}|}{|X_2(P)|}$$

the **ramification index** of $\tilde{P}$. Clearly the ramification indices of all vertices in the fibre over some vertex $V$ of $X$ add up to $k$, which is the number of faces in the fibre above a face of $X$. A straight forward comparison of the number of faces, edges, and vertices involved gives the following result.

**Corollary 3.40.** *Assume in the situation of Proposition 3.39 that none of the $\sigma_x$ have fixed points. Then the* EULER*-characteristics of $\tilde{X}$ and $X$ are* *related as follows:*

$$\chi(\tilde{X}) = k\chi(X) - \sum_{\tilde{P} \in \tilde{X}_1} (e(\tilde{P}) - 1).$$

In a sense the regular permutation representation of a group is the most important one. Therefore we comment on this special case. Also it is clear that it is not really necessary that $G$ is a permutation group on $X_2$, it suffices to assume that $G$ acts on $X_2$, i.e. not necessarily faithfully. Here is a corollary of this slight extension:

**Corollary 3.41.** *Let the finite group $G$ generated by three involutions act trivially on the surface $X$ consisting of just one face so that the involution triple $(\sigma_a, \sigma_b, \sigma_c)$ is trivial. Then there exists a (unique) simplicial surface $X$ with $mmm$-structure, such the regular $G$-set can be taken as $X_2$ with $mmm$-structure.*

Of course from this regular case, one can easily proceed to the general case and therefore recover the characterization of simplicial surfaces with $mmm$-structures as coverings of a triangle.

regulaere Ueberlagerung, regulaer in Cor ueberfluessig?

Starting from a simplicial surface, one can try to find all wild edge colourings going with this surface, each one being represented by an involution triple, since the surface is already given. The $mr$-assignment of the edges is then determined by the wild colouring. The other possibility is that one starts from an involution triple (without a surface) and then computes all admissible $mr$-assignments.

Given a simplicial surface the question arises whether there exist involution triples which yields the surface. The following algorithm decides this question and enumerates all such involution triples. Note that the question is equivalent to finding all TAIT colourings of the trivalent graph dual to edge graph of the simplicial surface, where the vertices of the dual graph correspond to the faces of the simplicial surface and the edges of the dual graph are in natural bijection to the edges of the simplicial surface. In particular (see [2, p. 432]), the Four Colour Theorem guarantees the existence of an involution triple in case the simplicial surface is topologically a sphere.

**Algorithm 3.42.** *(InvolutionTripleFinder)*
**Input***: A multiset $C$ of* 1- *and* 2-*cycles in the symmetric group $S_M$ such that each element $f \in M$ occurs in exactly three elements of $C$.*

**Output**: *A list of involution triples $\sigma = (\sigma_a, \sigma_b, \sigma_c)$, such that the cycles of disjoint cycle decomposition of the $\sigma_i$ taken together form the multiset $C$.*
**Algorithm**: *The algorithm maintains three lists, each of which consists of pairwise disjoint cycles chosen from $C$, representing candidates for subsets of cycles of $\sigma_i$. It recursively repeats the obvious extension procedure to extend such a triple of lists by a single element c from the remaining elements of $C$ maintaining the property of being pairwise disjoint within each list.*

An efficient implementation of the above algorithm will depend on an efficient choice of the next cycle $c$ in the list.

**Example 3.43.**
*1.) Number the faces of the (regular) icosahedron from 1 to 20 and read off a list of edges by indentifying them with the set of their adjacent faces. Interprete the list of edges as a (multi)set of 2-cycles as described above., e.g.:*

$$(1,2),(1,14),(1,5),(2,3),(2,6),(3,4),(3,8),(4,10),(4,5),(5,12),(6,15),$$
$$(6,7),(7,8),(7,17),(8,9),(9,10),(9,18),(10,11),(11,12),(11,19),(12,13),$$
$$(13,14),(13,20),(14,15),(15,16),(16,17),(16,20),(17,18),(18,19),(19,20)$$

*The above algorithm will find exactly ten possible involution triples (neglecting the order). Here is one of them:*

$$(1,2)(3,4)(5,12)(6,7)(8,9)(10,11)(13,14)(15,16)(17,18)(19,20),$$
$$(1,5)(2,6)(3,8)(4,10)(7,17)(9,18)(11,19)(12,13)(14,15)(16,20),$$
$$(1,14)(2,3)(4,5)(6,15)(7,8)(9,10)(11,12)(13,20)(16,17)(18,19)$$

*Each of the ten triples generates a group of order $60 \cdot 2^{10}$. At first sight it is surprising no group isomorphic to the symmetry group $A_5$ of the icosahedron is among them. But our involutions do not describe symmetries but neigbouring relations.*
*2.) The involution triple generating the* Mathieu *group $M_{12}$ given in Example 3.46 below gives rise to a genuine multiset of 2-cycles, where two of them have multiplicity 2. The above algorithm tells us that this multiset gives rise to no other involution triple of degree 12.*

To make the previous discussions constructive in the practical sense we start with an algorithm finding all nets for a given involution triple. A key subroutine is the following trying to extend a partial vertex-defining path by one face and one edge.

**Algorithm 3.44.**
**Input**: *An involution triple* $\sigma := (\sigma_a, \sigma_b, \sigma_c)$ *generating a transitive subgroup of the symmetric group* $S_M$ *with* $|M| = n$*, an alternating sequence* $s := (i_1, F_1, i_2, F_2, i_3, F_3, \ldots, F_{k-1}, i_k)$ *with* $i_r \in \{a, b, c\}, F_r \in M$ *satisfying the conditions 1.b), 1.c), 1.d) of Definition 3.35, for short a partial vertex-defining path of length* $2k-1$.
**Output**: $s$ *in case* $s$ *is a vertex-defining path. Otherwise the set of all partial vertex-defining paths of length* $2k+1$ *containing* $s$ *as subsequence (which might be empty or consists of exactly two elements).*
**Algorithm**:
*Step 1: Define* $F_k := \sigma_{i_k}(F_{k-1})$*. If* $F_k = F_1$*, then return* $s$ *as a closed reduced path. If* $F_k = F_{k-1}$ *and* $\sigma_{i_1}(F_1) = F_1$*, then return* $s$ *as a transversing reduced path. If* $F_k \in \{F_2, \ldots, F_{k-2}\}$ *then return the empty set.*
*Step 2: If* $F_k = F_{k-1}$ *and* $\sigma_{i_1}(F_1) \neq F_1$ *then continue with Step 3 with the reversed* $s$.
*Step 3: Return* $\{(s, F_k, i_{k+1}) \mid i_{k+1} \in \{a, b, c\} - \{i_k\}\}$.

Clearly, the recursive application of Algorithm 3.44 starting with sequence $(i, F, j)$ naming a vertex will finally yield all vertex-defining paths containing $(i, F, j)$. To use this we need a further subroutine which extends a partial net by one further vertex-defining path.

**Algorithm 3.45.**
**Input**: *An involution triple* $\sigma := (\sigma_a, \sigma_b, \sigma_c)$ *generating a transitive subgroup of the symmetric group* $S_n$ *and list* $N$ *of pairwise disjoint vertex-defining paths for* $\sigma$*, for short a partial net on* $\sigma$ *and one* $(i, F, j)$ *with* $F \in M, i \neq j \in \{a, b, c\}$ *not occuring in any path of* $N$.
**Output**: *The (possibly empty) set of all partial nets for* $\sigma$ *extending* $N$ *by any vertex-defining path containing* $(i, F, j)$.
**Algorithm**: *Recursively apply Algorithm 3.44 starting with* $(i, F, j)$ *to find all vertex-defining paths containing* $(i, F, j)$*. Return the set of partial nets each extending* $N$ *by any one of these which does not share a common subsequence* $(i', F', j')$ *with a path in* $N$.

A recursive application of Algorithm 3.45 starting with an empty net and using some linear ordering the set of sequences $(i, F, j)$, to find the first vertex not already represented, one obtains a duplicate-free list of all nets for $\sigma$. For the clarity of the idea we have supressed the tricks for efficiency.

Here is an example, where we start from an involution triple and construct all admissible $mr$-assignments. It turns out that some surfaces obtained are surprising simple.

**Example 3.46.** *The* MATHIEU *Group $M_{12}$ is generated by the involution triple $(\sigma_a, \sigma_b, \sigma_c)$ given with one admissible mr-assignment:*

$$(\overline{1,2})(3\overset{\bullet}{,}4)(\overline{5,6})(\overline{7,11})(\overline{8,12})(9\overset{\bullet}{,}10),$$
$$(\overline{1,4})(2\overset{\bullet}{,}11)(\overline{3,8})(\overline{5,6})(7\overset{\bullet}{,}10)(\overline{9,12}),$$
$$(\overline{1,4})(\overline{2,10})(3\overset{\bullet}{,}11)(5\overset{\bullet}{,}8)(\overline{6,7})(\overline{9,12})$$

*There are 32 admissible mr-assignments alltogether, of which 8 have 8 vertices each and therefore are topologically spheres, and 8 with 6 vertices all nonorientable, and 16 with 7 vertices. The indicated mr-assignment yields a double tetrahedron (connected sum of two tetrahera along a 3-waist) with 3 ears, i.e. open bags glued to the double tetrahedron along 2-waists.*

*figure*

*If one cuts off the three ears, the resulting involution triple is*

$$((2\overset{\bullet}{,}3)(\overline{7,11})(8\overset{\bullet}{,}10),(2\overset{\bullet}{,}11)(\overline{3,8})(7\overset{\bullet}{,}10),(\overline{2,10})(3\overset{\bullet}{,}11)(7\overset{\bullet}{,}8))$$

*generating a symmetric group $S_4$ acting on the six 6 points (or rather remaining faces)* $2, 3, 7, 8, 10, 11$.

A particularly interesting case arises, when the group $G$ generated by the involution triple $\sigma$ acts regularly on the set $M$ (of faces). In this case the closed reduced paths are relators in the $\sigma_i$ for $G$. Usually they are not defining but define an infinite group together with an infinite simplicial surface which is wild-coloured and inherits an admissible $mr$-assignment from the assignment of the original surface. This infinite surface covers the original surface without producing new types of vertices. To be precise we define the notion of the type of a vertex.

**Definition 3.47.** *Let $M$ be a (finite) set and $(\sigma_a, \sigma_b, \sigma_c)$ a triple of involutions generating the transitive subgroup $G = G(\sigma_a, \sigma_b, \sigma_c)$ of the symmetric group $S_M$ acting regularly on $M$. Let $N$ be a net on the* CAYLEY*-graph $\Gamma = \Gamma(\sigma_a, \sigma_b, \sigma_c)$ with corresponding simplicial surface $X$.*
*1.) A* **vertex-relator** *is a word in $a, b, c$ coming from a minimal period of a closed reduced path $\Pi$ in $N$ corresponding to a vertex $V$ recording only the subsequence of generators in $\Pi$. Note each vertex of degree $d$ gives rise to at most $2d$ conjugate vertex-relators in the free group on $a, b, c$, each one of length $d$.*

*2.) The factor group of the free group $F(a, b, c)$ on $a, b, c$ modulo the normal subgoup generated by the vertex relators of $X$ and $a^2, b^2, c^2$ is called the* **vertex-relator group** *$\mathcal{V}(X)$ of $X$ of $N$. To be brief, we denote its three generators again by $a, b, c$. The natural epimorphism of $\mathcal{V}(X)$ onto $G$ mapping $i \in \{a, b, c\}$ onto $\sigma_i$ is denoted by $\nu_N$ or $\nu_X$.*

Analysing the proof of Theorem 3.36 shows that the finiteness assumption for the set $M$ of faces is not necessary. Therefore one can use the CAYLEY graph of $\mathcal{V}(X)$ and repeat the construction there to obtain a surface of possibly infinitely many faces corresponding to the $mr$-assignment by lifting the $mr$-assignment of $\sigma$ back to the cycles of the involutions induced by $a, b, c$ on $\tilde{M} := \mathcal{V}(X)$ by multiplication from the left. The resulting surface is denoted by $\tilde{X}$ and called the **universal vertex cover** of $X$, because we have the following result:

**Corollary 3.48.** *1.) The natural epimorphism $\nu_X : \mathcal{V}(X) \to G$ induces a covering $\zeta_X : \tilde{X} \to X$.*
*2.) For any vertex $\tilde{V}$ of $\tilde{X}$ the restriction of $\zeta$ to $\tilde{X}_2(\tilde{V})$ induces a bijection of $\tilde{X}_2(\tilde{V})$ onto $X_2(\zeta(\tilde{V}))$, i.e. the covering is unramified. In particular, the vertex-relators of $\tilde{X}$ are the same as those of $X$ and $\mathcal{V}(\tilde{X}) = \mathcal{V}(X)$.*

*Proof.* 1.) See discussion above. 2.) Generalize Proposition 3.39 appropriately in a straight forward manner. □

In case the edge colouring is not wild, i.e. defines a structure such as $mmm$, or $mmr$ etc. on the simplicial surface, more can be said about the vertex group. In the terminology of this section it means that the $mr$-assignment is constant on the cycles of each of the three involution $\sigma_x$ with $x \in \{a, b, c\}$.

Definition 3.21 berarbeiten

**Proposition 3.49.** *Let $(X, <)$ be a simplicial surface with xyz-structure, cf. 3.21, i.e. an admissible mr-assighment $\mu$ which yields the constant value $x, y$, resp. $z$ on the cycles of $\sigma_a, \sigma_b$, resp. $\sigma_c$. Assume that the group generated by $\sigma_a, \sigma_b, \sigma_c$ acts regularly of the set of faces. Then one has the following vertex relators, which are unique up to cyclic conjugacy:*
*1.) In case of mmm one has $(ab)^{k_1}$ (for all ab-vertices), $(ac)^{k_2}$ (for all ac-vertices), $(bc)^{k_3}$ (for all bc-vertices) for certain (minimally choosen) $k_i \in \mathbb{N}$.*
*2.) In case of mmr one has either $(ab)^k$ (for all ab-vertices) for some $k \in \mathbb{N}$ or $(acbc)^l$ (for all ac- and bc-vertices) for some $l \in \mathbb{N}$*
*3.) In case of mrr one has $(abcacb)^k$ (for all vertices) for some $k \in \mathbb{N}$ .*
*4.) In case of rrr one has $(abc)^k$ (for all vertices) for some $k \in \mathbb{N}$ .*

*Proof.* 2.) Same $l$ in both cases because of regularity) □

**Corollary 3.50.** *A closed simplicial surface $(X, <)$ does allow a structure, if one of its vertex degrees is not a multiple of 2 or 3.*

**Example 3.51.** *The group $C_2 \times D_8 := \langle z, x, y \mid z^2, x^2, y^2, (zx)^2, (zy)^2, (xy)^4 \rangle$ yields with the involution triple $(a, b, c) := (x, y, yz)$ in its regular permutation representation. Examples for the following four structures with their corresponding vertex groups with presentation $\langle a, b, c \mid a^2, b^2, c^2, w_? \rangle$ where missing relator(s) $w_?$ as follows:*

| *structure* | EULER-*characteristic* | *missing relator(s)* $w_?$ |
|---|---|---|
| $mmm$ | 0 | $(ab)^4, (bc)^2, (ac)^4$ |
| $mmr$ | $-4$ | $(ab)^4, (acbc)^4$ |
| $mrr$ | 0 | $abcacb$ |
| $rrr$ | 0 | $(abc)^2$ |

*All four surfaces on 16 faces are orientable, in particular three of them are tori. The vertex groups are infinite in all four cases, the second one even with infinitely many $\mathrm{PSL}(2, q)$-quotients, cf. [Jam????], [Fab], whereas the first and last obviously define planar crystallographic groups. So in all cases one has infinitely many simplicial surfaces with the respective types whose XXX-group is a factor group of one of the vertex groups above.*

*Wieviele Relatoren machen die Gruppen dann endlich*

*Gruppe liefert wild colourings deren Vertexrelationen vllig anders sind*

The next example demonstrates that structures on the regular simplicial surfaces do not inherit themselves to surfaces corresponding to quotient groups of the neigbouring group.

**Example 3.52.** *Let $C_2 \times D_8$ with involution triple as above in Example 3.51. As we saw, this group allows a regular $rrr$-structured simplicial surface. It has two factor groups isomorphic to the dihedral group $D_8$, namely by factoring out by $\langle z \rangle$ or by $\langle z(xy)^2 \rangle$. In the second case the given involution triple is mapped onto n involution triple which again supports an $rrr$-structure, whereas in the first case one gets a generating involution triple not supporting an $rrr$-structure. Note in the first case, the two last elements in the involution triple are identical.*

**Example 3.53.** *In this example we discuss the planar cristallogrphic groups which occur as vertex groups. Here is a list:*

| *structure* | *defining relators* | *name of group* |
|---|---|---|
| $mmm$ | $a^2, b^2, c^2, (ab)^4, (bc)^2, (ac)^4$ | $p4mm$ |
| | $a^2, b^2, c^2, (ab)^6, (bc)^2, (ac)^3$ | $p6mm$ |
| | $a^2, b^2, c^2, (ab)^3, (bc)^3, (ac)^3$ | $p3m1$ |
| $mmr$ | $a^2, b^2, c^2, (ab)^2, (acbc)^2$ | $c2mm$ |
| $mrr$ | $a^2, b^2, c^2, abcacb$ | $p2mg$ |
| $rrr$ | $a^2, b^2, c^2, (abc)^2$ | $p2$ |

The following question is of interest: Given a vertex group of some regular simplicial surface with some structure. Which are the factor groups of this vertex group which still allow the same type of structure for their regular simplicial surfaces. The answer seems to be related to the property that no vertex relator factors in two relators for the factor group. Details have to be investigated.

# 4 Interpretation of neighbouring group

Papierstriefen im Strukturfall

Abrollen vershcieder Flchen aufeinander (Strukturfall)

Universeller Fall, (relativ) universelle berlagerung

induzierte Permutationsgruppe=Nachbarschaftsgruppe im Strukturfall

Vergleich verschiedener Flchen

Was bertrgt sich auf den wilden Fall

Abrollen in der Ebene als Bewegungsspiel

## 4.1 Development in the presence of edge colourings

Among the edge-coloured simplicial surfaces $(X, <, \sigma)$ we distinguish between the **regular** ones, i.e. the neigbouring group $G := G_\sigma := \langle \sigma_a, \sigma_b, \sigma_c \rangle$ acts regularly on the set $X_2$ of faces of $X$ and the others, for short **non regular** ones. Here is a geometric distinction between the two.

**Remark 4.1.** *1.) $X$ is a regular edge-coloured simplicial surface, if and only if any reduced word $w := w(a, b, c) \in C_2(a) * C_2(b) * C_2(c)$ in the free product either yields $w_\sigma := w(\sigma_a, \sigma_b, \sigma_c)$ fixed point free on $X_2$ or equal to the identity.*
*2.) We view $w$ as a* **paper strip** *of edge-coloured triangles, which can be laid on $X$ starting with one face $i \in X_2$ and following the colours ending*

*with* $w_\sigma(i)$.
*3.) In case* $X$ *is regular the epimorphism*

$$\pi_X : C_2(a) * C_2(b) * C_2(c) \to G : w \mapsto w_\sigma$$

*factors over the vertex group of* $X$.
*4.) Reduced words in* $\mathrm{Ker}\pi_X$ *are called* $X$**-closed paper strips**. *Vertex relators are examples of these, but generally speaking not the only ones.*

This remark opens up the possibility to compare two different edge coloured simplicial surfaces:

**Definition 4.2.** *Let* $(X, \sigma)$ *and* $(Y, \tau)$ *be two edge-coloured regular simplicial surfaces.*
*1.) The orbits of* $\pi_X(\mathrm{Ker}\pi_Y)$ *on* $X_2$ *are called the* $Y$*-sets in* $X$ *and elements of* $X_2$ *in the same* $Y$*-set are called* $Y$**-equal**. *If* $\pi_X(\mathrm{Ker}\pi_Y)$ *acts transitively on* $X_2$*, then* $X$ *is called* $Y$**-alien**.
*2.)* $X$ *is called a* **specialization** *of* $Y$ *or* **more special** *than* $Y$ *or* **developable** *into* $Y$*, if* $\mathrm{Ker}\pi_Y \subseteq \mathrm{Ker}\pi_X$.

**Remark 4.3.** *1.)* $C_2 * C_2 * C_2$ *acts similarly on* $X_2$ *and* $Y_2$ *if and only if* $X$ *is developable into* $Y$ *and* $Y$ *into* $X$. *In particular Algorithm 3.45 finds up to equivalence all possibilies for* $X$ *with given* $Y$.
*2.)* $X$ *is* $Y$*-alien if and only if* $Y$ *is* $X$*-alien.*
*3.)* $X$ *is developable into* $Y$ *iff and only if one (and therefore all)* $Y$*-sets in* $X$ *consist of just one element, i.e.* $Y$*-equality is equality on* $X_2$. *Another characterization of this situation is, that there is a* $C_2(a) * C_2(b) * C_2(c)$*-equivariant surjective map* $Y_2 \to X_2$.

**Example 4.4.** *Assume* $X$ *and* $Y$ *are structured regular simplicial surfaces. If* $X$ *is develople into* $Y$ *then each vertex relation for* $X$ *powers to a vertex relation for* $Y$. *For instance take*

$$a \mapsto (1,2)(3,7)(4,8)(5,6), b \mapsto (1,3)(2,4)(5,7)(6,8), c \mapsto (1,5)(2,6)(3,4)(7,8)$$

*to obtain a dihedral group* $D_8$ *as epimorphic image of* $C_2(a) * C_2(b) * C_2(c)$. *By factoring out the center* $Z(D_8) = \langle (1,6)(2,5)(3,8)(4,7) \rangle$*, one gets* $V_4$ *as epimorphic image*

$$a \mapsto (1,2)(3,4), b \mapsto (1,3)(2,4), c \mapsto (1,2)(3,4).$$

*They yield simplicial surfaces* $Y, X$ *both with structure mrm, namely an octahedron and a double Janus-head. We get a 2-fold covering of* $X$ *by* $Y$. *The vertex relation ac for* $X$ *is no relation for* $Y$*, but gives rise to the relation* $(ac)^2$. *However, the vertex relation cbab for* $X$ *is also a vertex relation for* $Y$.

*Bilder*

As discussed in Section 3.4 the same group can give rise to many topological different simplicial surfaces. One might find it disturbing that they become closely aquainted by the above process.

Beispiel D8: Okaeder und Torus lassen sich 1-1 aufeinander abrollen

If one want to avoid this, one can take the $mr$-assignment into account. To keep matters reasonably simple we stick to structures and assume that $X$ and $Y$ carry structures of the same type, e.g. both $mmm$ or both $mrm$. We then can repeat the above definitions by adding **structurally** to any of the concept to obtain notions like e.g. **structurally developable** or **structurally $X$-equal**. Then the structual version of Remark 4.3 reads as follows:

**Proposition 4.5.** *Let $X, Y$ be two simplicial surfaces with structures of the same type.*
*1.) The following three statements are equivalent:*
*a) $C_2 * C_2 * C_2$ acts similarly on $X_2$ and $Y_2$:*
*b) $X$ is structurally developable into $Y$ and $Y$ into $X$;*
*c) $X$ and $Y$ are isomorphic as structured simplicial surfaces.*
*2.) $X$ is structurally developable into $Y$ iff and only if one (and therefore all) $Y$-sets in $X$ consist of just one element, i.e. $Y$-equality is equality on $X_2$.*

It should be noted that with this more restrictive notions the paper strip view of things becomes more convincing as there is physically speaking exactly one paper strip for each reduced word in $C_2 * C_2 * C_2$, because now one knows how to attach two triangles to each other, whereas before there were two possibilities for each inner edge.

Now we understand the fact that each allows for mmm.

Concerning the $X$-sets on $Y$ and the $Y$-sets on $X$ one has the following reciprocity result.

**Proposition 4.6.** *Let $X$ and $Y$ be regular simplicial surfaces with structures of the same type. Let $x$ be an $X$-set in $Y$ and $y$ a $Y$-set in $X$. Then*

$$\frac{|X_2|}{|y|} = \frac{|Y_2|}{|x|}.$$

*Proof.* Let $G_{XY}$ be the group acting on $X_2 \times Y_2$ generated by $\sigma_{XY} := ((\sigma_a, \tau_a), (\sigma_b, \tau_b), (\sigma_c, \tau_c))$ diagonally. Then $G_{XY}$ is a subdirect product of $G_X$ and $G_Y$. We choose one orbit of this action, which clearly yields a regular representation. According to Theorem 3.36 we have to define a net on the Cayley-graph $\Gamma(\sigma_{XY})$ for $G_{XY}$ yielding a structure (in form

of a wild colouring) of the same type as for $X$ and for $Y$. To define a vertex-defining path in $\Gamma(\sigma_{XY})$ let $(g,h)$ be a vertex in $\Gamma(\sigma_{XY})$, i. e. a group element of $G_{XY}$ and choose $i \neq j$ in $\{a,b,c\}$. If $(i,g,j)$ proceeds with $(j,\sigma_j g,k)$ in $\Gamma(\sigma)$ then $(i,h,j)$ proceeds with $(j,\tau_j h,k)$ in $\Gamma(\tau)$ with the same $k \in \{a,b,c\}$, namely $k=i$ in case $\sigma_j$ and $\tau_j$ are of type $m$ and $k \neq i,j$ in case $\sigma_j$ and $\tau_j$ are of type $r$. We therefore define $(i,(g,h),j)$ to be proceeded by $(j,(\sigma_j g,\tau_j h),k)$ again with the same $k$, to get the new vertex-defining path in $\Gamma(\sigma_{XY})$. With this definition we clearly get a net in $\Gamma(\sigma_{XY})$ which defines a strucure of the same type as for $X$ and $Y$.
Since everything is well defined now, the result follows from the subdirect product structure of $G_{XY}$ by looking at the orbits of the normal subgroup generated by the two kernels of the projections of $G_{XY}$ onto $G_X$ and $G_Y$.

Es sollte einen sehr einfachen Beweis geben, der nicht die neue Flche konstruiert. Die Existenz dieser Flche sollte extra herausgestellt werden.

□

Here is now an example of the tetrahedron with $rrr$-structure, how it developes on various other regular simplicial surfaces with $rrr$-structure.

**Example 4.7.** *1.) The tetrahedron $(T,<)$ with rrr-structure has the* Klein-*4-group $V_4$ as neighbouring group acting regularly on $T_2$. It so happens that $V_4$ is already the vertex group of $T$.*
*2.) If we take the infinite simplicial surface corresponding to the planar crystallographic group $p2 := \langle a,b,c | a^2, b^2, c^2, (abc)^2 \rangle$ as neigbouring group of type $rrr$ we get a well known tesselation of the plane by congruent triangles, which can be thought of as created by $T$ as a roller handstamp. If we number the faces of $T$ by $1,2,3,4$, then the triangles in the plane with the same number form a $T$-equality class for the triangles in the plane.*

*Bild, wo eine Klasse eingefaerbt ist*

*Conversely developing the plane into $T$ corresponds to an epimorphism $p2$ onto $V_4$ and results into a covering of $T$ by the plane.*
*3.) Taking a normal subgroup $N$ of $p2$ which is contained in the kernel of the epimorphism $p2 \to V_4$ of finite index $k$ gives us a regular $rrr$-simplicial torus $X$ on $4k$-faces, where the developing experiment with $T$ (cf. 2.) ) can be repeated in very much the same way so that the equation in Proposition 4.6 reads $4k/k = 4/1$. We challange the reader to find out, what happens in the case that $N$ is not contained in the kernel of $p2 \to V_4$.*
*4.) One might ask what happens, if we give up the regular group action on $X_2$. As an example we subdivide each face of the torus $T$ into 4 triangular faces and retaining the $rrr$-structure at the same time.*

*Bild Unterteilung rrr Dreieck*

*The result is a spherical simplicial surface on 16 faces with $rrr$-structure. It has (inherited from $T$) 4 vertices of degree* $3$ *and has 6 vertices of degree* $6$ *in addition. The neigbouring group $U$ is of order 32 so that the action is not regular. If one passes over to the regular action, one gets an example of 3.) with* $k = 8$ *which at the same time defines a 2-fold cover of the present 16-face sphere (ramified at the 4 vertices of degree 3). The 16-face sphere demonstrates in a nice way that one should stick with the notion of vertex groups to regular simplicial surfaces: The 4 degree-3-vertices give relations like abc whereas the 6 degree-6-vertices relations like* $(abc)^2$. *But the abc-relation turns the vertex group already in a* KLEIN*-4-group.*
*5.) Clearly there are simplicial surfaces with $rrr$-structure which are completely unrelated (alien) to the tetrahedron $T$, when it comes to developing, i.e. where the equation in Proposition 4.6 reads* $1 = 1$. *As an example we mention the alternating group $A_5$ with generating involution triple*
$((1,3)(2,4),(1,2)(3,4),(1,4)(2,5))$.
*The regular surface $Y$ for this is orientable of* EULER *characteristic* $-10$ *and yields vertex relations of the type* $(abc)^3$. *So $Y_2$ is the $T$-equalitiy class on $Y$ and $T_2$ the $Y$-equality class on $Y_2$.*

## 4.2 Obstructions for the existence of wild colourings

**Definition 4.8.** *Let $(X. <)$ be a simplicial surface and let $\Delta$ be the one-face with $\Delta_1 =: \{a, b, c\}$, i.e. the edges of $\Delta$ are identified with their three different colours.*
*1.) An $mr$-**assignment** of $X$ is a map*

$$\tau : X_1^\circ \to \{m, r\},$$

*where $X_1^\circ$ denotes the set of inner edges of $X$.*
*2.) For a face of $F \in X_2$ of $X$ the **holonomy group** $\mathrm{Hol}_{\tau,F}(X)$ is defined as the group of all automorphims of $X(F)$ consisting of all elements of the form $\iota_{P,\tau}$, where $P := (F_1, e_1, F_2, \ldots, e_{n-1}, F_n)$ is a closed **face-edge-path** with $F_1 = F_n = F$, cf Lemma 3.20 and $\iota_{P,\tau}$ is the composition of the neighbouring isomorphisms $\hat{\tau}_{F_i,e_i}$ which are given by $\hat{\tau}_{F_i,e_i} := \sigma_{F_i,e_i}$ in case $\tau(e_i) = m$ and $\hat{\tau}_{F_i,e_i} := \rho_{F_i,e_i}$ in case $\tau(e_i) = r$ respectively.*

The following definition is completely analogous to the one in the case of $mmm$-structures, cf 3.11, with a slightly different notation more suitable for this general case.

**Definition 4.9.** *Let $(X, <)$ be a simplicial surface, $\Delta$ be the one-face with $\Delta_1 =: \{a, b, c\}$, i.e. the edges of $\Delta$ are identified with their three different*

*colours and* $\tau : X_1^\circ \to \{m,r\}$ *an mr-assignment. The* **sixfold $\tau$-cover** $\tilde{X}(\tau)$ *of* $X$ *is defined in three steps:*
*1.) The set of faces of* $(\tilde{X}(\tau), \tilde{<})$ *is*

$$\tilde{X}(\tau)_2 := \{(F,\beta) | F \in X_2, \beta : \Delta_1 \to X_1(F) \text{ induced by an isomorphism } \Delta \to X(F)\}$$

*In particular each face of* $X$ *gives rise to exactly 6 faces of* $\tilde{X}(\tau)$*. More precisely, the map*

$$\tilde{X}(\tau)_2 \to X_2 : (F,\beta) \mapsto F$$

*is surjective with 6 elements in each of its fibres.*
*2.) On* $\tilde{X}(\tau)_2$ *an involution triple is defined as follows. For* $i \in \Delta_1 = \{a,b,c\}$ *define* $\sigma_i : \tilde{X}(\tau)_2 \to \tilde{X}(\tau)_2$ *by* $\sigma_i((F,\beta)) := (F,\beta)$ *if the edge* $e_i$ *of* $F$ *with respect to* $X$ *is a boundary edge and otherwise, cf. Lemma 3.1,*

$$\sigma_i((F,\beta)) := (\hat{\tau}_{F,e_i}(F), \hat{\tau}_{F,e_i} \circ \beta)$$

*with* $\hat{\tau}_{F,e_i}$ *as defined in Definition 4.8 2.)*
*3.) The set* $\tilde{X}(\tau)_1$ *of edges is defined the set of all* $(i,\zeta)$ *with* $i \in \Delta_1$ *and* $\zeta$ *a one-cycle of* $\sigma_i$ *for outer edges and a two-cycle of* $\sigma_i$ *for inner edges. The set of vertices* $\tilde{X}(\tau)_0$ *is defined below in the proof of Theorem 4.10*

**Theorem 4.10.** *Let* $(X. <)$ *be a simplicial surface with an mr-assignment* $\tau : X_1^\circ \to \{m,r\}$*.*
*1.) There exists a sixfold* $\tau$*-cover* $\tilde{X}(\tau)$ *with* $\tilde{X}(\tau)_2$ *und* $\tilde{X}(\tau)_1$ *as described in Definition 4.9 and a covering* $\kappa : \tilde{X}(\tau) \to X$ *mapping* $(F,\beta) \in \tilde{X}(\tau)_2$ *to* $F \in X_2$ *such that each fibre of a face of* $F \in X_2$ *consists of exactly* 6 *faces in* $\tilde{X}(\tau)$*.*
*2.)* $\tilde{X}(\tau)$ *has a wild colouring given by the* $(\sigma_a, \sigma_b, \sigma_c)$*, cf. Definition 4.9, inducing the mr-assigmment* $\tilde{\tau}$ *which is compatible with* $\tau$ *via the covering, i.e.* $\tau \circ \kappa_1^\circ = \tilde{\tau}$*, where* $\kappa_1^\circ$ *is the restriction of* $\kappa$ *to* $\tilde{X}(\tau)_1^\circ$*, is also compatible*
*3.)* $\tilde{X}(\tau)$ *is closed if and only if* $X$ *is closed.*
*4.) In case* $X$ *is connected,* $\tilde{X}(\tau)$ *decomposes into one, two, three, or six connected components, which are isomophic after suitable renaming the colours. The case of six components occurs if and only if* $X$ *has a wild colouring inducing* $\tau$*.*

*Proof.* 1.) We first have to define the vertices of $\tilde{X}(\tau)$. Since we have already defined the faces and the edges, we can define

$$\tilde{X}(\tau)_0 := (\tilde{X}(\tau)_2 \times \mathrm{Pot}_2(\{a,b,c\}))/\approx$$

for a certain equivalence relation $\approx$.
To describe $\approx$ let $V \in X_0$ first be an inner vertex of $X$. Then $X_2(V)$ can be cyclically arranged in a closed face-edge-path $P := (F_1, e_1, F_2, \ldots, e_{n-1}, F_n)$,

i.e. $F_n = F_1$ and otherwise $F_i \neq F_j$ for $i \neq j$, cf. Lemma 3.20. This path determines $V$ and is called an **inner vertex tour** around $V$. Now $P$ lifts to possibly several closed face-edge-paths in $\tilde{X}(\tau)$ without repetition, all of which will be used to define inner vertices of $\tilde{X}(\tau)$ (namely the ones above $V$) by turning them into inner vertex tours of $\tilde{X}(\tau)$. Let $(F_1, \beta_1)$ be a face of $\tilde{X}(\tau)$. Define $i_1 := \beta_1^{-1}(e_1) \in \{a, b, c\}$ and define $(F_2, \beta_2) := \sigma_{i_1}((F_1, \beta_1))$, where the $\sigma's$ are defined in Definition 4.9. Proceeding in this way we get $i_2$, $(F_3, \beta_3)$ etc. until $(F_n, \beta_n)$. Though $F_n = F_1$ it it not necessarily the case that $\beta_n = \beta_1$. Therefore one keeps going until one reaches $(F_1, \beta_1)$ for the first time (which will be after $kn$ steps for some $k \leq 6$, actually $k \mid 6$ as one can see later). Note that this actuallly constructs a closed face-edge-path, the edge between $(F_j, \beta_j)$ and $(F_{j+1}, \beta_{j+1})$ being the 2-cycle $((F_j, \beta_j), (F_{j+1}, \beta_{j+1}))$ of $\sigma_{i_j}$ for $j = 1, \ldots, kn$.
Next let $V \in X_0$ be a boundary vertex of $X$. Then $X_2(V)$ can be arranged according to a maximal face-edge-path $P := (F_1, e_1, F_2, \ldots, e_{n-1}, F_n)$ without repetitions. This path determines $V$ and is called a **boundary vertex tour** around $V$. Now $P$ lifts to six face-edge-paths in $\tilde{X}(\tau)$ without repetition, all of which will be used to define boundary vertices of $\tilde{X}(\tau)$ (namely the ones above $V$) by turning them into boundary vertex tours of $\tilde{X}(\tau)$. Let $(F_1, \beta_1)$ be a face of $\tilde{X}(\tau)$. Define $i_1 := \beta_1^{-1}(e_1) \in \{a, b, c\}$ and define $(F_2, \beta_2) := \sigma_{i_1}((F_1, \beta_1))$, as above. Proceeding in this way we get $i_2$, $(F_3, \beta_3)$ etc. until $(F_n, \beta_n)$. Note, $F_1$ and $F_n$ are boundary faces both adjacent to $V$ and boundary edges $e_0$ and $e_n$, the latter being 1-cycles of the two involutions $\sigma_{i_0}$ and $\sigma_{i_n}$ they define.
Now the equivalence relation $\approx$ on $\tilde{X}(\tau)_2 \times \mathrm{Pot}_2(\{a, b, c\}$ is obvious, namely $((F, \beta), \{e, f\}) \approx ((F', \beta'), \{e', f'\})$ if and only if $(e, (F, \beta), f)$ or $(f, (F, \beta), e)$ on the one side and $(e', (F', \beta'), f')$ or $(f', (F', \beta'), e')$ on the other hand occur in the same lifted vertex tour just constructed, where in the case of boundary vertex tours the boundary edges have to be added in the obvious way.
One easily verifies that $\tilde{X}(\tau)$ is a simplicial surface and $\kappa : \tilde{X}(\tau) \to X$ a sixfold cover (possibly ramified at the vertices).
2.) and 3.) Clear from the proof of 1.).
4.) Analogoous to the proof of Proposition 3.12 3.), where the special 3-mirrored case is treated. Another proof will be given below using holonomy groups.

erster Teil von Bewies 4.) ist falsch!

□

We now proceed to a better understanding of part 4.) of the last theorem. We shall show for a simplicial surface with $mr$-assignment that the holonomy group is the obstruction for the existence of a wild colouring inducing the $mr$-assignment .

**Theorem 4.11.** *Let $(X,<)$ be a connected simplicial surface with mr-assignment $\tau : X_1^\circ \to \{m,r\}$. Let $F \in X_2$ be a face.*
*1.) The holonomy group $\mathrm{Hol}_{\tau,F}(X)$ is isomorphic to a subgroup of the symmetric group $S_3$.*
*2.) Its isomorphism type is independent of the choice of the face $F$.*
*3.) Let $\zeta$ be the number of connected components of the sixfold cover $\tilde{X}(\tau)$. then*

$$\zeta \cdot |\mathrm{Hol}_{\tau,F}(X)| = 6.$$

*In particular there exists a wild colouring on $X$ inducing $\tau$ if and only if $|\mathrm{Hol}_{\tau,F}(X)| = 1$.*

*Proof.* The set $\mathrm{FE}(X)$ of face-edge-paths on $X$ is a groupoid with respect to concatenation $\circ$ of face-edge-paths, where the second path commences at the destination of the first. More precisely, let

$$\alpha : \mathrm{FE}(X) \to X_2 :(F_1, e_1, F_2, \dots, e_{n-1}, F_n) \mapsto F_1$$
$$\omega : \mathrm{FE}(X) \to X_2 :(F_1, e_1, F_2, \dots, e_{n-1}, F_n) \mapsto F_n$$

then composition of two face-edge-paths $P$ and $P'$ is defined only if $\omega(P) = \alpha(P')$. The identity element for each face $F$ is $(F)$ and one has the rule $(F, e, F') \circ (F', e, F) = (F)$ and obvious consequences.
In the presence of an $mr$-assignment $\tau$, every path $P \in \mathrm{FE}(X)$ defines an isomorphism

$$\iota_{P,\tau} : X(\alpha(P)) \to X(\omega(P))$$

as defined in 4.8, whose inverse is $\iota_{P^{-1},\tau}$, where $P^{-1}$ is defined to be $P$ read backwards. This construction defines a finite image of the groupoid $\mathrm{FE}(X)$, which is relevant for us and lies in the groupoid

$$\mathrm{FI}(X_2) := \biguplus_{F,F' \in X_2} \mathrm{Iso}(X(F), X(F'))$$

where $\mathrm{Iso}(F, F')$ consists of all isomorphism of $X(F)$ onto $X(F')$. Since both $X(F)$ and $X(F')$ are isomorphic to the one-face $\Delta$, each one of these $\mathrm{Iso}(X(F), X(F'))$ consists of exacly 6 elements, in particular the automorphism group of a face is isomorphic to $S_3$. The composition law of this groupoid is clear: composition of maps if possible. The groupoid interesting for us is a subgroupoid of $\mathrm{FI}(X_2)$ and an epimorphic image of $\mathrm{FE}(X)$, namely

$$\mathrm{FI}(X,\tau) := \biguplus_{F,F' \in X_2} \underbrace{\{\iota_{P,\tau} | P \in \mathrm{FE}(X), \alpha(P) = F, \omega(P) = F'\}}_{{}_F\mathrm{FI}(X,\tau)_{F'}}$$

For $F' = F$, note ${}_F\mathrm{FI}(X,\tau)_F$ is a group isomorphic to $\mathrm{Hol}_{\tau,F}(X)$. Composition of matching path induced isomorphisms form "non-degenerate" maps

$$ {}_F\mathrm{FI}(X,\tau)_{F'} \times {}_{F'}\mathrm{FI}(X,\tau)_F \to {}_F\mathrm{FI}(X,\tau)_F $$

proving that all ${}_F\mathrm{FI}(X,\tau)_{F'}$ have the same cardinality, which multiplied with the index $\mathrm{Aut}(X(F)) : \mathrm{Hol}_{\tau,F}(X)$ is $6 = |\mathrm{Aut}(X(F))|$. Now clearly

$$ |\tilde{X}(\tau)_2| = |\biguplus_{F\in X_2} \mathrm{Aut}(X(F))|. $$

Since we know already that it suffices to look at one face only (because $X$ is connected) and $\mathrm{Hol}_{\tau,F}(X) \leq \mathrm{Aut}(X(F))$, the result follows readily. $\square$

### 4.3 Viewing developments as new surfaces

Assume we have two simplicial surfaces $X, X'$ with $mr$-assignments $\tau : X_1 \to \{m,r\}$ and $\tau' : X_1' \to \{m,r\}$. Note this time our $mr$-assigments are defined on all of $X_1$ not only on the set of inner edges. Developing $X$ onto $X'$ intuitively means that a face $F \in X_2$ is placed upon a face $F' \in X_2'$ so that the edges match. Tilting one or both surfaces around one of the now common three edges, leads to a placement for the correspondig neighbouring faces. Continuing in this way, one gets many such placements, which of course are somehow connected. To make this precise we define a new simplical surface describing all the possible placements and their interelation and covering the two original surfaces $X, X'$.

**Definition 4.12.** *The* **common cover** $C := \mathrm{CC}(X,X',\tau,\tau')$ *of* $(X,\tau)$ *and* $(X',\tau')$ *is a simplicial surface with mr-assignment given by:*

$$ C_2 := \{\varphi : X(F) \to X'(F') | F \in X_2, F' \in X_2', \varphi \text{ isomorphism } \} $$

*For an isomorphism $\varphi : X(F) \to X'(F')$ viewed as a face of $C$, define the three edges of $\varphi$ as follows: Choose an edge $e \in X_1(F)$ and let $e' := \varphi(e)$ the correspondig edge of $F'$. Then*

$$ \psi := \hat{\tau}'_{F',e'} \circ \varphi \circ \hat{\tau}_{F,e}^{-1} : X(\hat{\tau}_{F,e}(F)) \to X'(\hat{\tau}'_{F',e'}(F')) $$

*is the $e$-neighbouring face of $\varphi$, and the indexed 2-cycle $(\varphi,\psi)_e$ is the edge between $\varphi$ and $\psi$ in case $\varphi \neq \psi$, and the indexed 1-cycle $(\varphi)_e$ in case $\varphi = \psi$. Note this edge is determined by $\varphi$ and $e$. Therefore the edge will also be denoted by $\varphi_e$ or equivalently $\psi_e$. Since $\psi$ is obtained from $\varphi$ just by knowing $F$ and $e$ only, we use the notation $F_e(\varphi)$ for $\psi$.*
*The three vertices of $\varphi$ are denoted by $\varphi_V$ where $V$ is a vertex of $F$. Note, if $e \in X_1(F)$ is an edge with $V$ adjacent to $e$, then $\varphi_V = F_e(\varphi)_V$, so that*

*the formal definition of a vertex of $C$ is that of an equivalence class on the set*

$$\{\varphi_V | V \in X_0, \varphi : X(F) \to X'(F') \text{ for some } F \in X_2, F' \in X'_2, V \leq F\}.$$

*The equivalence relation is the finest so that in the above notation $\varphi_V$ and $F_e(\varphi)_V$ become equivalent.*

Note, since $X(F), X'(F')$ are isomorphic to the one-face $\Delta$, one has $|C_2| = |X_2||X'_2|6$. Concerning the vertices, it is tempting to define a vertex of $C$ for a fixed $V \in X_0$ to be the set of all $\varphi_V$ for all $\varphi$ whose domain of definition is $X(F)$ with $V < F$. It might happen, however, that this attempt of a definition violates condition 4.) of Definition 2.1 for simplicial surfaces, namely that this set splits into more than one equivalence class in the sense of the the above definition, cf. also the explanations right behind Definition 2.1. Therefore one is forced to preceed as described above. With these remarks in mind it is also easy to write down a face-edge path characterizing a given vertex of $C$. The next example will demonstrate, why we insisted to have all of $X_1$ and $X'_1$ as domain of definition for the $mr$-assignment.

**Example 4.13.** *Let $X$ be any simplicial surface and denote by $\Delta$ the one-face surface. Assume that we have an $mr$-assignment $xyz : \Delta_1 = \{a, b, c\} \to \{m, r\}$ for $\Delta$. Then $\mathrm{CC}(\Delta, X, \tau, \tau')$ is a sixfold cover of $X$ with $xyz$-structure defined as follows:*

**Remark 4.14.** $\mathrm{CC}(X, X', \tau, \tau')$ *is isomorphic to* $\mathrm{CC}(X', X, \tau', \tau)$. *One automorphism is characterized by*

$$\mathrm{CC}(X, X', \tau, \tau')_2 \to \mathrm{CC}(X', X, \tau', \tau)_2 : \varphi \mapsto \varphi^{-1}.$$

# 5 EUCLIDean simplicial surfaces

## 5.1 Constructions

This section is concerned with the realization of combinatorial simplicial surfaces as EUCLIDean simplicial surfaces, i. e. as abstract metric 2-dimensional manifolds (possibly with boundary) where the metric is outside a finite set of points locally isometric to certain open subsets of the EUCLIDean plane.

**Definition 5.1.** *A* **EUCLIDean simplicial surface** *is a compact two-dimensional manifold $M$ with the following properties:*
*1.) There exists a finite subset $S$ of $M$ such that the manifold $M - S$ has an $\mathrm{E}_2(\mathbb{R})$-compatible atlas, where $\mathrm{E}_2(\mathbb{R})$ denotes the group of isometries of the EUCLIDean 2-space $\mathbb{R}^2$.*

*2.) There exists a (combinatorial) simplicial surface $(X, \leq)$ and a bijection $\varphi : X_0 \to S$.*
*a) Moreover each element of $e \in X_1$ corresponds to an open line segment $\hat{e}$ in $\mathbb{R}^2$ allowing an isometry $\varphi_e : \hat{e} \to M - S$ such that the closure of $\varphi_e(e)$ satisfies*

$$\overline{\varphi_e(\hat{e})} = \varphi_e(\hat{e}) \uplus \biguplus_{V \in X_0(e)} \varphi(V).$$

*b) For each $F \in X_2$ corresponds to an open triangle $\hat{F}$, i. e.. the interior of the convex Hull of three non collinear points in $\mathbb{R}^2$, allowing an injective isometry $\varphi_F : F \to M - S$ such that the closure satisfies*

$$\overline{\varphi_F(\hat{F})} = \varphi_F(F) \uplus \biguplus_{V \in X_0(F)} \varphi(V) \uplus \biguplus_{e \in X_1(F)} \varphi_e(\hat{e}).$$

*c)*

$$M = S \uplus \biguplus_{e \in X_1} \varphi_e(\hat{e}) \uplus \biguplus_{F \in X_2} \varphi_F(\hat{F})$$

*If all the $\hat{F}$ with $F \in X_2$ are congruent, $M$ is called a* **uni-faced** EUCLID**ean surface**. *The elements of $S$ are called* **vertices** *of $M$ and the elelemnts of $S$ whose sum of adjacent angles is not $2\Pi$ are called* **singularities** *of $M$.*

**Remark 5.2.** *A* EUCLID**ean simplicial surface** *carries a unique metric extending the* EUCLID*ean metrics on each of its triangles, because one has the concept of the length of a curve.*

With as slight modification one defines EUCLIDean simplicial surfaces with boundary. Also the notion of isometry of EUCLIDean simplicial surfaces is clear. For arbitrary (finite) simplicial surfaces one has the follwoing existence and uniqueness result.

**Theorem 5.3.** *Let $(X, <)$ be a finite simplicial surface and $a > 0$. Then there is up to isometry exactly one* EUCLID*ean simplicial surface build from equiangular triangles of edge length $a$. This gadget will be called the* **standard abstract model $\mathcal{S}_a(X) = \mathcal{S}(X)$ of $X$.**

*Proof.* We prove the result for simplicial surfaces with boundary be induction on the number of faces. Both, existence and uniqueness is clear, in case of $|X_2| = 1$. 1. Construction: Assume that $\mathcal{S}(X)$ exists, in case $|X_2| = n$. We prove that this implies the existence of $\mathcal{S}(X)$ for any simplicial surface $X$ with $|X_2| = n+1$. So assume $X$ is a simplicial surface with $|X_2| = n+1$.

Choose some fixed $F \in X_2$ and define a simplicial surface $X'$ by

$$\begin{aligned} X_2' &:= X_2 - \{F\} \\ X_1' &:= X_1 - \{e \in X_1(F) | e \text{ outer edge w. r. to } X\} \\ X_0' &:= X_0 - \{V \in X_0(F) | V < e \text{ for two edges of } F \text{ outer w. r. to } X\} \end{aligned}$$

and incidence by restriction from $X$. By induction $\mathcal{S}(X')$ exists. Also $\mathcal{S}(X(F))$ exists. Out of the two $\mathcal{S}(X)$ will be constructed as a connected sum as follows: In case $|X_1'| = |X_1| - 3$, one simply takes the disjoint union. In case $|X_1'| = |X_1| - 2$, one again take the disjoint union $\mathcal{S}(X') \uplus \mathcal{S}(X(F))$ and identifies as follows: From the point of view of $X$ there is a unique edge $e$ of $F$ which is an inner edge $e$ of $X$. Call the corresponding eges of $X'$ and $X(F)$ by the names $e'$ and $e_F$. Now $e'$ and $e_F$ get isometrically identified in such a way that the each matching pair of vertices becomes one vertex (as ithas been before).
In case $|X_1'| = |X_1| - 1$ one proceeds in the same way, where this time two pairs of edges get identified.
Finally the last case $|X_1'| = |X_1|$ is the same, with three edges getting identified.
In all four cases, the details of the construction are easily verified and can easily be performed with paper triangles. Note $X'$ is never closed. Note also since all edges have the same length, the identification is well defined.
As for the definition of the $\mathrm{E}_2(\mathbb{R})$-compatible atlas for $\mathcal{S}(X')$ with the vertices removed, the open cover can by obtained from the interor of rombi, which are constructed by identifying two trinagles along a common edge.
2. By the very definition, the isometry is patched together from isometries of the 2-simplices, i. e. triangles without any additional restrictions. Hence, the isometry is unique and well defined by the assignments of the vertices and of the 2-simplices. □

As an example, one might discuss the metric of $\mathcal{S}(X)$, where $X$ is a Janus-head. We now come to the case of simplicial surfaces with wild edge colourings.

**Theorem 5.4.** *Let $(X, <)$ a (finite) simplicial surface with a wild edge colouring $\omega : X_1^0 \to \{a, b, c\}$ . Then for any choice $a, b, c \in \mathbb{R}_{>0}$ with $a > b+c, b > a+c, c > a+b$ there is up to isometry exactly one* EUCLID*ean simplicial surface build from congruent triangles of edge lengths $a, b, c$. This gadget will be called the* **standard abstract model $\mathcal{S}_{a,b,c}(X)$ of $X$.**

*Proof.* One can either adjust the proof of the last theorem to the new situation, e. g. the rhombi become parallegrams in the case of $r$-edges and kites in the case of $m$-edges. The other possibility is to take $\mathcal{S}_a(X)$ as a

2-dimenisonal manifold and then change its metric so that the triangles get the desired shape. This shows that all the $\mathcal{S}_{a,b,c}(X)$ are not only homeomorphic, but even locally affinely related, a term that has to be properly defined. $\square$

This last theorem has a slightly more general variant, which starts from $mr$-assignments rather than wild colourings. Of course a realization of a Surface with an $mr$-assignment means that the isometry of two neighboured faces are by a mirror or a rotation according to the assignment.

**Theorem 5.5.** *Let $(X, <)$ be a finite connected simplicial surface with $mr$-assignment $\tau : X_1^0 \to \{m, r\}$. If the holonomy group $H$ of a face is nontrivial, the folowing holds:*
*1.) For $|H| = 2$, then one has realizations as above with isosceles triangles only (respecting the reflection of the holonomy).*
*2.) For $|H| \in \{3, 6\}$, realizations are only possible with equiangular triangles.*

The proof is left to the reader. It can be either given by the above principles or by using the above results for the coloured case and then investigating the covering map retricted to one connected component of the 6-fold cover.

### 5.2 Geometry of Euclidean simplicial surfaces

The Euclidean simplicial surfaces in itself are interesting geometric objects. We mention only two topics here, the geodesics and the Gauss-Bonnet-Theorem. Concerning Geodesics, it is clear, how the shortest path between two point in the same triangle looks like. The same is true if they lie in two neighboured triangles sharing only one edge, which is an $r$-edge. The situation gets more complicated soon. The general recipe is to develop the surface into the plane along the candidate for the geodesic, cf. [4]. One might have to consider several cases, but crossing an edge must be by a stretched angle, except posiibly for the vertices.

Project: Prove that for closed Euclidean simplicial surfaces always closed geodesics exist, How can one find a shortest one?

The second topic concerns a predecessor of the Gauss-Bonnet-Theorem, namely a Theorem due the Gauss-Bonnet-Theorem for simplicial surfaces in Euclidean space due to Descartes: It expresses the Euler-characteristic in terms of certain sums of angles around the vertices. We refer to [1] for details.

# 6 Examples

## 6.1 Icosahedron and related surfaces

Starting with the icosahedron as simplicial surface, one has various options to proceed according to the previous ideas. First of all one might try to find an involution triple, i.e. a wild colouring. Since the combinatorial automorphsim group of the icosahedron is isomorphic to $C_2 \times A_5$, the involution triples come in orbits under this group. Using Algorithm 3.42 one finds exactly one orbit consisting of 10 triples. Since we know the icosahedron, this gives immedeately an $mr$-assignment. In the next example we proceed in a slightly different way, to get more information.

**Example 6.1.** *The following involution triple with the given admissible mr-assignment, cf discussion before Example 3.37, yields a unique combinatorial icosahedron.*

$$\begin{aligned}
\sigma_a &:= (1\overset{\bullet}{,}2)(\overline{3,4})(5\overset{\bullet}{,}10)(\overline{6,11})(7\overset{\bullet}{,}12)(8\overset{\bullet}{,}13)(\overline{9,14})(15\overset{\bullet}{,}20)(\overline{16,17})(18\overset{\bullet}{,}19)\\
\sigma_b &:= (1\overset{\bullet}{,}5)(2\overset{\bullet}{,}7)(\overline{3,8})(4\overset{\bullet}{,}9)(\overline{6,15})(\overline{10,14})(11\overset{\bullet}{,}16)(\overline{12,17})(13\overset{\bullet}{,}18)(19\overset{\bullet}{,}20)\\
\sigma_c &:= (\overline{1,6})(\overline{2,3})(4\overset{\bullet}{,}5)(7\overset{\bullet}{,}11)(8\overset{\bullet}{,}12)(9\overset{\bullet}{,}13)(10\overset{\bullet}{,}15)(\overline{14,19})(16\overset{\bullet}{,}20)(\overline{17,18})
\end{aligned}$$

*Up to isomorphism (and permutation of colours) it is the only possible description of the icosahedron by an involution triple with any admissible mr-assingment. In fact fixing the involution triple, Algorithm 3.45 finds 30843 wild colourings of which 186 yield orientable surfaces, namely 125 with* EULER*-characteristic -4, and 60 with -2, and exactly one with* EULER*-characteristic 2, namely the icosahedron. The group $G := \langle \sigma_a, \sigma_b, \sigma_c \rangle$ is of order $2^{10} \cdot 3 \cdot 5$ and isomorphic to*

$$G \cong V_4 \times (A_5 \ltimes (2^{4'} \oplus 2^{4'}))$$

*in the notation of perfect groups in [3]. Clearly the orbits of the center $Z(G) \cong V_4$ on the faces are blocks for $G$. They are given by the orbit of*

$$B := \{3, 14, 6, 17\}$$

*under the action of $G$. Note, the cycles in the $\sigma_i$ moving a point of $B$ are exactly the ones whose mr-assignment is m. Clearly, the induced action of the $\sigma_i$ on the block system of $B$ define a new involution triple for which it is natural to assign rs to each cycle. One easily verifies that this assignment is admissible. The resulting simplicial surface on 5 faces is given by the involution triple*

$$\tau_a := (B\overset{\bullet}{,}C)(D\overset{\bullet}{,}E), \tau_b := (A\overset{\bullet}{,}D)(B\overset{\bullet}{,}E), \tau_c = (A\overset{\bullet}{,}B)(C\overset{\bullet}{,}D)$$

*where $B$ is as above, and $A, C, D, E$ complete the block system. More precisely $A$ can be chosen to consist of the green neigbours of the elements of*

*$B$, and $C$ and $E$ the red resp. blue neigbours, and finally $D$ the remaining faces.*
*The surface defined by triple $\tau$ (which generates an $A_5$) clearly has a boundary consiting of three edges. It is orientable, and has* EULER*-characteristic 2, i.e. it has only three vertices. (It looks like a threefold bag.) It is tempting to pass over to the transitive permutation representation of $A_5$ on twenty points to see whether it also allows an $rrr$-structure, which indeed it does. The resulting simplicial surface is spherical and has vertex counter $4^3 5^6 6^3$. We have not found an obvious relation to the structure on the icosahedron. Concerning the embeddings of the icosahedron with the above combinatorial data into* EUCLID*ean 3-space, there is worthwhile to assign two colours to the faces, say black to the ones in $B$ and white to the remaining 16, because one might construct the icosahedron from one triangle, which shows its front side in the 16 faces and its back side in the $B$-faces, cf. also Corollary 3.38. One can even go one step further and more colours either to mark the faces in the blocks $A, B, C, D, E$ or to use 4 colours to mark certain distinguished sets of representatives of the blocks. More interesting is the behaviour with respect to the side lengths of the triangles. Here are some examples.*

We next apply the construction of o

$$
\begin{bmatrix}
5 & 1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 & -5 \\
1 & 5 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -5 & -1 & -1 \\
1 & 1 & 5 & -1 & -1 & 1 & 1 & -1 & -5 & -1 & 1 & -1 \\
1 & 1 & -1 & 5 & 1 & -1 & -1 & 1 & 1 & -1 & -5 & -1 \\
1 & -1 & -1 & 1 & 5 & 1 & -5 & -1 & 1 & 1 & -1 & -1 \\
1 & -1 & 1 & -1 & 1 & 5 & -1 & -5 & -1 & 1 & 1 & -1 \\
-1 & 1 & 1 & -1 & -5 & -1 & 5 & 1 & -1 & -1 & 1 & 1 \\
-1 & 1 & -1 & 1 & -1 & -5 & 1 & 5 & 1 & -1 & -1 & 1 \\
-1 & -1 & -5 & 1 & 1 & -1 & -1 & 1 & 5 & 1 & -1 & 1 \\
-1 & -5 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 5 & 1 & 1 \\
-1 & -1 & 1 & -5 & -1 & 1 & 1 & -1 & -1 & 1 & 5 & 1 \\
-5 & -1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & 1 & 5
\end{bmatrix}
$$

$$
\begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 \\
1 & 1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & -1 & -1 \\
1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & -1 & 1 & -1 \\
1 & 1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & -1 \\
1 & -1 & -1 & 1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\
1 & -1 & 1 & -1 & 1 & 1 & -1 & -1 & -1 & 1 & 1 & -1 \\
-1 & 1 & 1 & -1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 \\
-1 & 1 & -1 & 1 & -1 & -1 & 1 & 1 & 1 & -1 & -1 & 1 \\
-1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & 1 & -1 & 1 \\
-1 & -1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & 1 & 1 \\
-1 & -1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & 1 \\
-1 & -1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & 1 & 1
\end{bmatrix}
$$

$$\begin{bmatrix} 85 & 10 & 10 & 10 & 10 & 10 & -35 & -35 & -35 & -35 & -35 & 40 \\ 10 & 115 & 25 & 25 & -20 & -20 & 25 & 25 & -20 & -110 & -20 & -35 \\ 10 & 25 & 115 & -20 & -20 & 25 & 25 & -20 & -110 & -20 & 25 & -35 \\ 10 & 25 & -20 & 115 & 25 & -20 & -20 & 25 & 25 & -20 & -110 & -35 \\ 10 & -20 & -20 & 25 & 115 & 25 & -110 & -20 & 25 & 25 & -20 & -35 \\ 10 & -20 & 25 & -20 & 25 & 115 & -20 & -110 & -20 & 25 & 25 & -35 \\ -35 & 25 & 25 & -20 & -110 & -20 & 115 & 25 & -20 & -20 & 25 & 10 \\ -35 & 25 & -20 & 25 & -20 & -110 & 25 & 115 & 25 & -20 & -20 & 10 \\ -35 & -20 & -110 & 25 & 25 & -20 & -20 & 25 & 115 & 25 & -20 & 10 \\ -35 & -110 & -20 & -20 & 25 & 25 & -20 & -20 & 25 & 115 & 25 & 10 \\ -35 & -20 & 25 & -110 & -20 & 25 & 25 & -20 & -20 & 25 & 115 & 10 \\ 40 & -35 & -35 & -35 & -35 & -35 & 10 & 10 & 10 & 10 & 10 & 85 \end{bmatrix}$$

$$\begin{bmatrix} 22 & 19 & 19 & 19 & 19 & 19 & -20 & -20 & -20 & -20 & -20 & -17 \\ 19 & 16 & 16 & 16 & -29 & -29 & 22 & 22 & -23 & -23 & -23 & 16 \\ 19 & 16 & 16 & -29 & -29 & 16 & 22 & -23 & -23 & -23 & 22 & 16 \\ 19 & 16 & -29 & 16 & 16 & -29 & -23 & 22 & 22 & -23 & -23 & 16 \\ 19 & -29 & -29 & 16 & 16 & 16 & -23 & -23 & 22 & 22 & -23 & 16 \\ 19 & -29 & 16 & -29 & 16 & 16 & -23 & -23 & -23 & 22 & 22 & 16 \\ -20 & 22 & 22 & -23 & -23 & -23 & 28 & 28 & -17 & -17 & 28 & -5 \\ -20 & 22 & -23 & 22 & -23 & -23 & 28 & 28 & 28 & -17 & -17 & -5 \\ -20 & -23 & -23 & 22 & 22 & -23 & -17 & 28 & 28 & 28 & -17 & -5 \\ -20 & -23 & -23 & -23 & 22 & 22 & -17 & -17 & 28 & 28 & 28 & -5 \\ -20 & -23 & 22 & -23 & -23 & 22 & 28 & -17 & -17 & 28 & 28 & -5 \\ -17 & 16 & 16 & 16 & 16 & 16 & -5 & -5 & -5 & -5 & -5 & -38 \end{bmatrix}$$

The mathematics of structures

## 7 Icosahedra of regular triangles in 3-space

The aim of this section is to discuss the immersed icosahedra with 12 different vertices and all faces ragular triangles of sidelegth 1. Here are generators of the combinatorial symmetry group (isomorphic to $A_5 \times C_2$):

$$\begin{aligned} a &:= (1,2)(3,4)(5,7)(6,8)(9,11)(10,12), \\ b &:= (1,10)(3,9)(2,12)(4,11)(5,6)(7,8), \\ c &:= (1,7)(2,3)(4,11)(5,12)(6,8)(9,10), \\ d &:= (1,12)(3,9)(2,10)(4,11)(5,7)(6,8). \end{aligned}$$

The actual EUCLIDean symmetry groups can be viewed as subgroups of this group. Here is a list of the relevant subgroups up to conjugacy:

| structure | representative | number of immersions | number of orbits |
|---|---|---|---|
| $C_1$ | | $2(30)$ | $30 = 30_1$ |
| $C_2$ | $\langle a \rangle$ | $5(16)$ | $16 = 2_1 + 14_2$ |
| $C_2$ | $\langle ad \rangle$ | $4(13) + 6(17)$ | $17 = 4_1 + 13_2$ |
| $V_4$ | $\langle a, b \rangle$ | $1(9)$ | $9 = 3_2 + 6_4$ |
| $V_4$ | $\langle ad, bd \rangle$ | $2(3) + 3(10)$ | $10 = 2_1 + 2_2 + 6_4$ |
| $D_6$ | $\langle ad, cbcd \rangle$ | $2(3)$ | $7 = 4_3 + 3_6$ |
| $D_{10}$ | $\langle ad, cd \rangle$ | $2(2) + 1(4)$ | $5 = 4_5 + 1_{10}$ |
| $C_2 \times D_6$ | $\langle d, a, cbc \rangle$ | $2(4)$ | $4 = 3_6 + 1_{12}$ |
| $C_2 \times D_{10}$ | $\langle d, ad, cd \rangle$ | $4$ | $3 = 3_{10}$ |
| $C_2 \times A_5$ | $\langle d, a, b, c \rangle$ | $2(1)$ | $1 = 1_{30}$ |

The numbers in the brackets in the third column indicate the number of different values for the squared lengths taken by the diagonals of combinatorial distance 2. Note, each such diagonal can be associated to the edge whose adjacent triangles have as third vertex one of the endpoints fo the diagonal, the first two vertices being the end points of the edge. Note, this number is bounded above by the number of orbits of the symmetry group on the set of edges, which is 30, 16($= 2_1 + 14_2$), 17($= 4_1 + 13_2$), 9($= 3_2 + 6_4$), 10($= 2_1 + 2_2 + 6_4$), 7($= 4_3 + 3_6$), 5($= 4_5 + 1_{10}$), 4($= 3_6 + 1_{12}$), 3($= 3_{10}$), 1($= 1_{30}$).

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