

Simplicial surfaces in GAP

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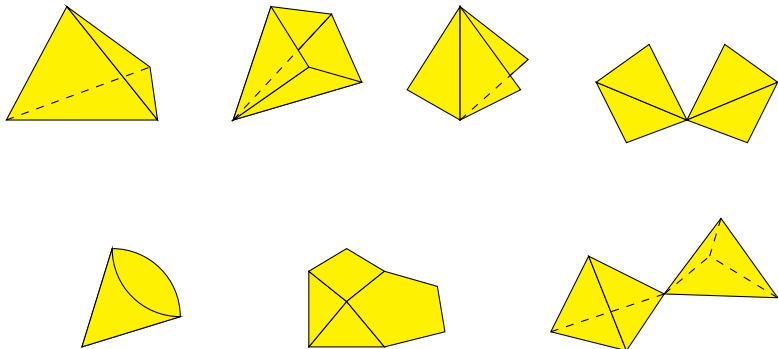
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- 1 General polygonal complexes by incidence geometry
- 2 Edge colouring and group properties
- 3 Abstract folding

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Motivation

Goal: simplicial surfaces (and generalisations) in GAP



⇝ examples of **polygonal complexes**

No embedding

We do not work with embeddings (mostly)

- is very hard to compute
- if often unknown for an abstractly constructed surface
- is different from *intrinsic structure*

⇒ lengths and angles are not important

↪ incidence structure is intrinsic

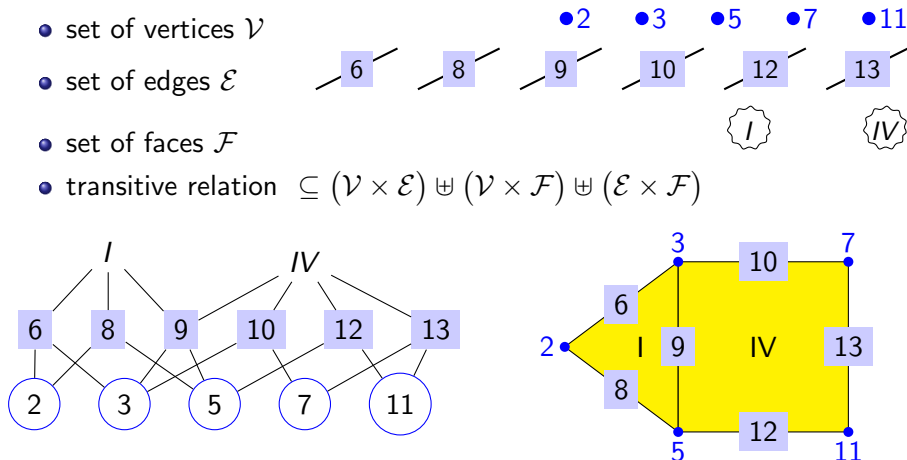
Incidence structure of polygonal complex

- set of vertices \mathcal{V}

- set of edges \mathcal{E}

- set of faces \mathcal{F}

- transitive relation $\subseteq (\mathcal{V} \times \mathcal{E}) \uplus (\mathcal{V} \times \mathcal{F}) \uplus (\mathcal{E} \times \mathcal{F})$



- 1 Every edge has exactly two vertices

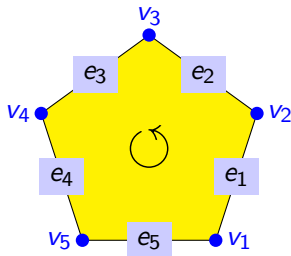
- 2 Every face is a polygon

- 3 Every vertex lies in an edge and every edge lies in a face

Polygonal complexes

A **polygonal complex** is a two-dimensional incidence structure of vertices, edges and faces, such that:

- 1 Every edge has exactly two vertices.
- 2 Every face is a polygon.



- 3 Every vertex lies in an edge
- 4 Every edge lies in a face

Isomorphism testing

Incidence geometry allows "easy" isomorphism testing. Incidence structure can be interpreted as a coloured graph:



↪ reduce to graph isomorphism problem

Solved by NautyTracesInterface (by Gutsche, Niemeyer, Schweitzer)

General properties

Some properties can be computed for all polygonal complexes:

- Connectivity
- Euler–Characteristic

Orientability is **not** one of them. Counterexample:

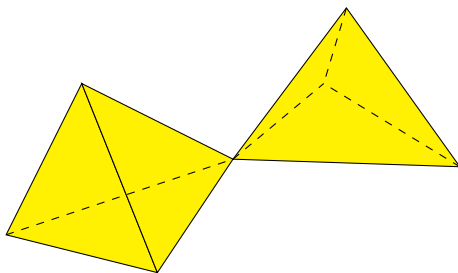


⇒ every edge lies in at most two faces (for well-definedness)

⇝ **ramified polygonal surfaces**

Why ramified?

Typical example of ramified polygonal surface:



⇒ It is not a surface – there is a *ramification* at the central vertex
A **polygonal surface** does not have these ramifications.

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Embedding question

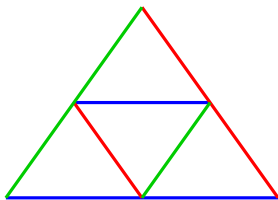
Given: A polygonal complex

- Can it be embedded?
- In how many ways?

Simplifications:

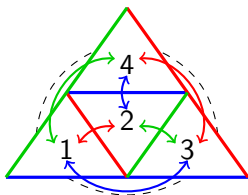
- 1 Only polygonal surfaces (surface that is build from polygons)
- 2 All polygons are triangles (**simplicial surfaces**)
- 3 All triangles are isometric

⇒ Edge-colouring encodes different lengths



Colouring as permutation

Consider tetrahedron with edge colouring



simplicial surface \Rightarrow at most two faces at each edge

\rightsquigarrow every edge defines transposition of incident faces

\rightsquigarrow every colour class defines permutation of the faces

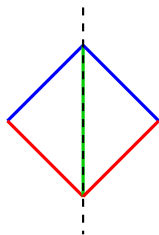
• $(1,2)(3,4)$, $(1,3)(2,4)$, $(1,4)(2,3)$

\rightsquigarrow group theoretic considerations

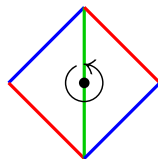
- ▶ The connected components of the surface correspond to the orbits of $\langle \sigma_a, \sigma_b, \sigma_c \rangle$ on the faces

How do faces fit together?

Consider a face of the surface and a neighbouring face
The neighbour can be coloured in two ways:



mirror (m)



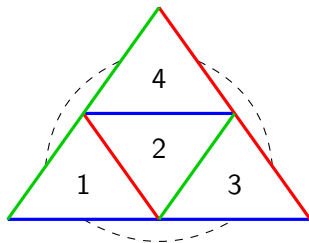
rotation (r)

This gives an **mr-assignment** for the edges.
Permutations and mr-assignment uniquely determine the surface.

Constructing surfaces from groups

A general mr-assignment leads to complicated surfaces.
Simplification: edges of same colour have the same type

Example



has an rrr-structure

The easiest structure is an mmm-structure.

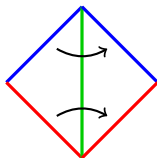
Covering

We want to characterize surfaces where all edges are mirrors.

Lemma

A simplicial surface has an mmm-structure iff it covers a single triangle, i. e. there is an incidence-preserving map to the simplicial surface consisting of exactly one face.

Consider



- Covering pulls back a colouring of the triangle.
- Colouring defines a map to the triangle.

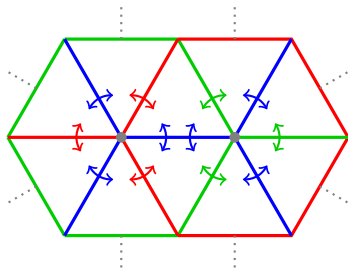
Construction from permutations

Start with three involutions σ_a , σ_b , σ_c (like generators of a finite group)

Lemma

There exists a coloured surface with the given involutions where all edges are mirror edges.

- The faces are the points moved by the involutions
- The edges are the cycles of the involutions
- The vertices are the orbits of $\langle \sigma_a, \sigma_b \rangle$ on the faces (for all pairs)

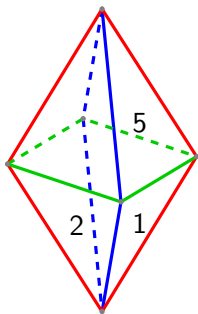
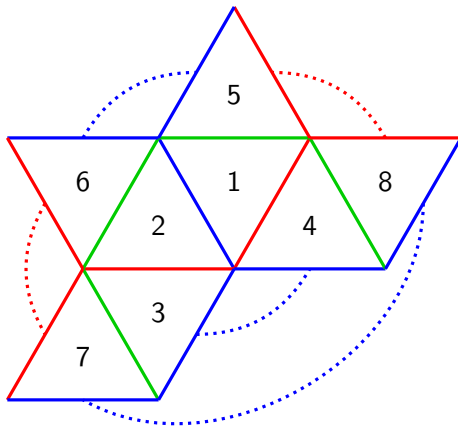


Construction example

$$\sigma_a = (1, 2)(3, 4)(5, 6)(7, 8)$$

$$\sigma_b = (1, 4)(2, 3)(5, 8)(6, 7)$$

$$\sigma_c = (1, 5)(2, 6)(3, 7)(4, 8)$$



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