

# Simplicial surfaces in GAP

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- 1 General polygonal complexes by incidence geometry
- 2 Edge colouring and group properties
- 3 Abstract folding

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# Motivation

Goal: simplicial surfaces (and generalisations) in GAP



⇝ examples of **polygonal complexes**

# No embedding

We do not work with embeddings (mostly)

- is very hard to compute
- if often unknown for an abstractly constructed surface
- is different from *intrinsic structure*

⇒ lengths and angles are not important

↪ incidence structure is intrinsic

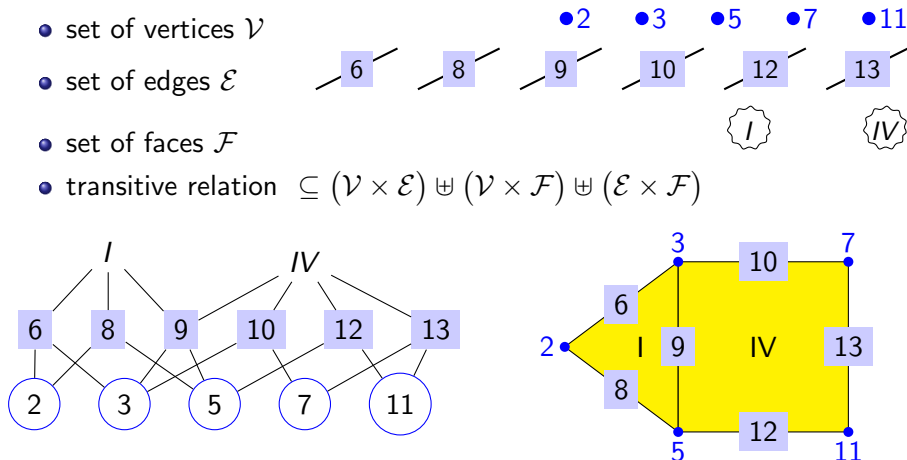
# Incidence structure of polygonal complex

- set of vertices  $\mathcal{V}$

- set of edges  $\mathcal{E}$

- set of faces  $\mathcal{F}$

- transitive relation  $\subseteq (\mathcal{V} \times \mathcal{E}) \uplus (\mathcal{V} \times \mathcal{F}) \uplus (\mathcal{E} \times \mathcal{F})$



- 1 Every edge has exactly two vertices

- 2 Every face is a polygon

- 3 Every vertex lies in an edge and every edge lies in a face

# Polygonal complexes

A **polygonal complex** is a two-dimensional incidence structure of vertices, edges and faces, such that:

① Every edge has exactly two vertices.



② Every face is a polygon.



③ Every vertex lies in an edge

④ Every edge lies in a face

# Isomorphism testing

Incidence geometry allows "easy" isomorphism testing. Incidence structure can be interpreted as a coloured graph:



↪ reduce to graph isomorphism problem

Solved by NautyTracesInterface (by Gutsche, Niemeyer, Schweitzer)



# General properties

Some properties can be computed for all polygonal complexes:

- Connectivity
- Euler–Characteristic

*Orientability* is **not** one of them. Counterexample:

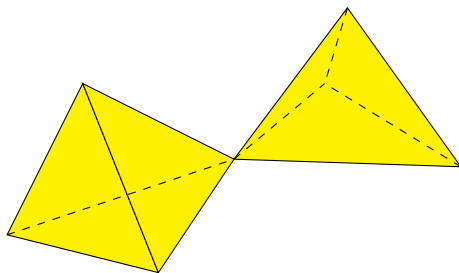


⇒ every edge lies in at most two faces (for well-definedness)

⇔ **ramified polygonal surfaces**

# Why ramified?

Typical example of ramified polygonal surface:



⇒ It is not a surface – there is a *ramification* at the central vertex  
A **polygonal surface** does not have these ramifications.

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# Embedding question

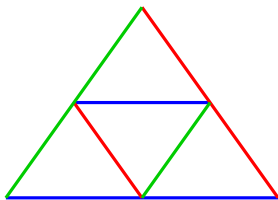
Given: A polygonal complex

- Can it be embedded?
- In how many ways?

Simplifications:

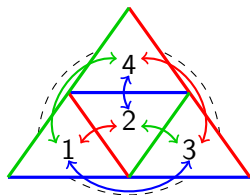
- 1 Only polygonal surfaces (surface that is build from polygons)
- 2 All polygons are triangles (**simplicial surfaces**)
- 3 All triangles are isometric

⇒ Edge-colouring encodes different lengths



# Colouring as permutation

Consider tetrahedron with edge colouring



*simplicial surface*  $\Rightarrow$  at most two faces at each edge

$\rightsquigarrow$  every edge defines transposition of incident faces

$\rightsquigarrow$  every colour class defines permutation of the faces

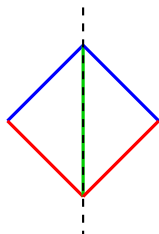
•  $(1,2)(3,4)$  ,  $(1,3)(2,4)$  ,  $(1,4)(2,3)$

$\rightsquigarrow$  group theoretic considerations

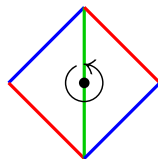
- ▶ The connected components of the surface correspond to the orbits of  $\langle \sigma_a, \sigma_b, \sigma_c \rangle$  on the faces

# How do faces fit together?

Consider a face of the surface and a neighbouring face  
The neighbour can be coloured in two ways:



mirror (m)



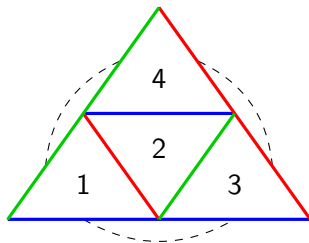
rotation (r)

This gives an **mr-assignment** for the edges.  
Permutations and mr-assignment uniquely determine the surface.

# Constructing surfaces from groups

A general mr-assignment leads to complicated surfaces.  
Simplification: edges of same colour have the same type

Example



has an rrr-structure

The easiest structure is an mmm-structure.

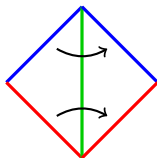
# Covering

We want to characterize surfaces where all edges are mirrors.

## Lemma

*A simplicial surface has an mmm-structure iff it covers a single triangle, i. e. there is an incidence-preserving map to the simplicial surface consisting of exactly one face.*

Consider



- Covering pulls back a colouring of the triangle.
- Colouring defines a map to the triangle.



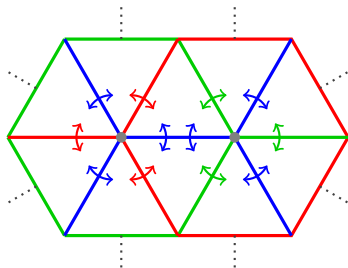
## Construction from permutations

Start with three involutions  $\sigma_a, \sigma_b, \sigma_c$  (like generators of a finite group)

## Lemma

*There exists a coloured surface with the given involutions where all edges are mirror edges.*

- The faces are the points moved by the involutions
- The edges are the cycles of the involutions
- The vertices are the orbits of  $\langle \sigma_a, \sigma_b \rangle$  on the faces (for all pairs)

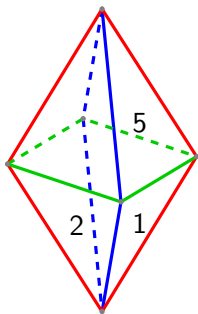
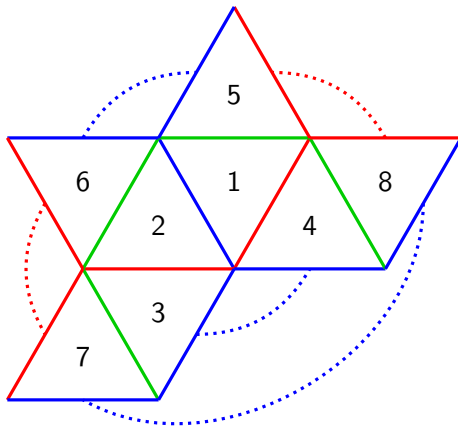


# Construction example

$$\sigma_a = (1, 2)(3, 4)(5, 6)(7, 8)$$

$$\sigma_b = (1, 4)(2, 3)(5, 8)(6, 7)$$

$$\sigma_c = (1, 5)(2, 6)(3, 7)(4, 8)$$



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