

Simplicial surfaces in GAP

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- 1 General polygonal complexes by incidence geometry
- 2 Edge colouring and group properties
- 3 Abstract folding

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Motivation

Goal: simplicial surfaces (and generalisations) in GAP



⇝ examples of **polygonal complexes**

No embedding

We do not work with embeddings (mostly)

- is very hard to compute
- if often unknown for an abstractly constructed surface
- is different from *intrinsic structure*

⇒ lengths and angles are not important

↪ incidence structure is intrinsic

Incidence structure of a polygonal complex

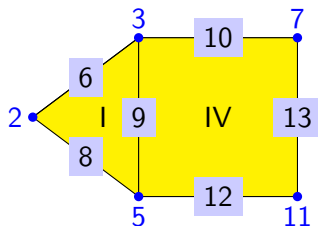
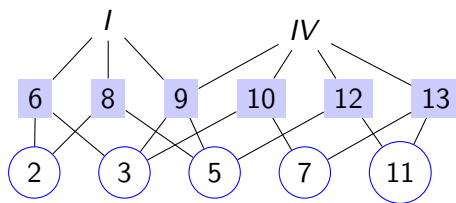
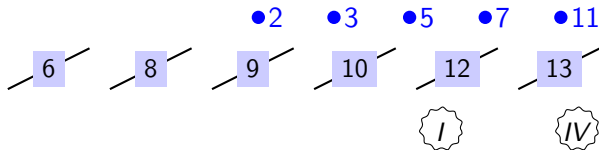
A **polygonal complex** consists of

- set of vertices \mathcal{V}

- set of edges \mathcal{E}

- set of faces \mathcal{F}

- transitive relation $\subseteq (\mathcal{V} \times \mathcal{E}) \uplus (\mathcal{V} \times \mathcal{F}) \uplus (\mathcal{E} \times \mathcal{F})$

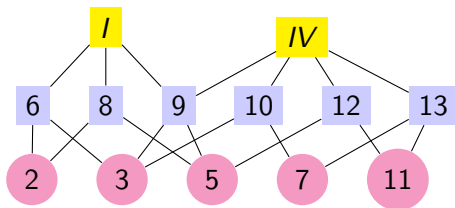


① Every face is a polygon

② Every vertex lies in an edge and every edge lies in a face

Isomorphism testing

Incidence geometry allows “easy” isomorphism testing. Incidence structure can be interpreted as a coloured graph:



↪ reduce to graph isomorphism problem

Solved by NautyTracesInterface (by Gutsche, Niemeyer, Schweitzer)

General properties

Some properties can be computed for all polygonal complexes:

- Connectivity
- Euler–Characteristic

Orientability is **not** one of them. Counterexample:

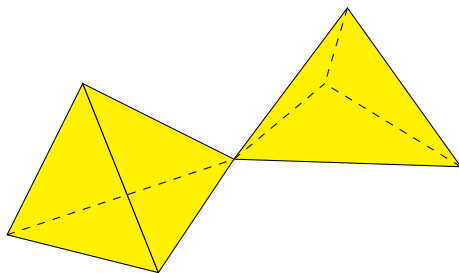


⇒ every edge lies in at most two faces (for well-definedness)

⇝ **ramified polygonal surfaces**

Why ramified?

Typical example of ramified polygonal surface:



⇒ It is not a surface – there is a *ramification* at the central vertex
A **polygonal surface** does not have these ramifications.

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Embedding question

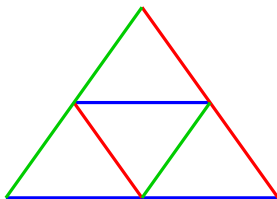
Given: A polygonal complex

- Can it be embedded?
- In how many ways?

Simplifications:

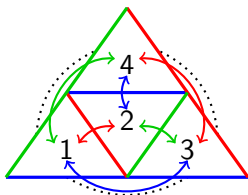
- 1 Only polygonal surfaces (surface that is build from polygons)
- 2 All polygons are triangles (**simplicial surfaces**)
- 3 All triangles are isometric

⇒ Edge-colouring encodes different lengths



Colouring as permutation

Consider tetrahedron with edge colouring

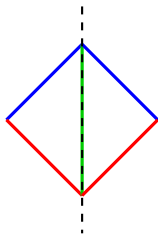


simplicial surface \Rightarrow at most two faces at each edge

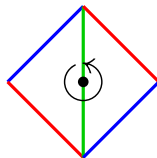
- \rightsquigarrow every edge defines transposition of incident faces
- \rightsquigarrow every colour class defines permutation of the faces
 - $(1,2)(3,4)$, $(1,3)(2,4)$, $(1,4)(2,3)$
- \rightsquigarrow group theoretic considerations
 - ▶ The connected components of the surface correspond to the orbits of $\langle \sigma_a, \sigma_b, \sigma_c \rangle$ on the faces

How do faces fit together?

Consider a face of the surface and a neighbouring face
The neighbour can be coloured in two ways:



mirror (m)



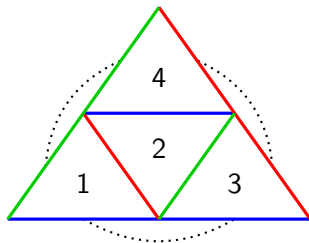
rotation (r)

This gives an **mr-assignment** for the edges.
Permutations and mr-assignment uniquely determine the surface.

Constructing surfaces from groups

A general mr-assignment leads to complicated surfaces.
Simplification: edges of same colour have the same type

Example



has an rrr-structure

The easiest structure is an mmm-structure.

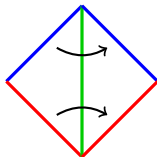
Covering

We want to characterize surfaces where all edges are mirrors.

Lemma

A simplicial surface has an mmm-structure iff it covers a single triangle, i. e. there is an incidence-preserving map to the simplicial surface consisting of exactly one face.

Consider



- Covering pulls back a colouring of the triangle.
- Colouring defines a map to the triangle.

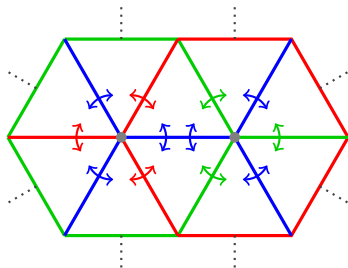
Construction from permutations

Start with three involutions σ_a , σ_b , σ_c (like generators of a finite group)

Lemma

There exists a coloured surface with the given involutions where all edges are mirror edges.

- The faces are the points moved by the involutions
- The edges are the cycles of the involutions
- The vertices are the orbits of $\langle \sigma_a, \sigma_b \rangle$ on the faces (for all pairs)

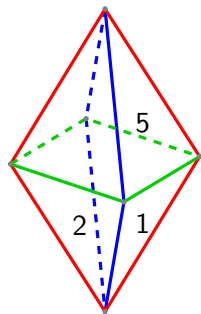
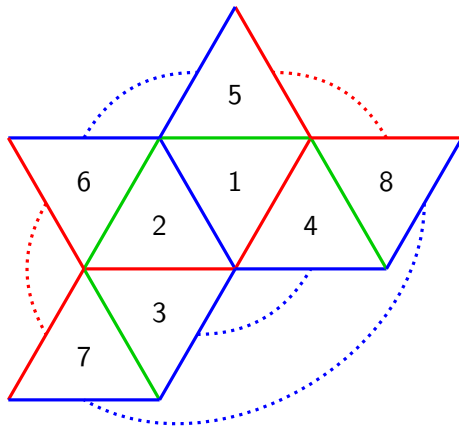


Construction example

$$\sigma_a = (1, 2)(3, 4)(5, 6)(7, 8)$$

$$\sigma_b = (1, 4)(2, 3)(5, 8)(6, 7)$$

$$\sigma_c = (1, 5)(2, 6)(3, 7)(4, 8)$$



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What kind of folding?

There are many different kinds of folding (e. g. Origami) Here:

- Folding of surface in \mathbb{R}^3
- Possible folding edges are fixed
- Folding should be rigid (no curvature)

Goal: Classify possible folding patterns (given a net)

Embeddings

Ideally, we would like to have embeddings.

But we want to define folding independently from an embedding, since:

- They are very hard to compute (even for small examples)
- We can only show foldability for specific small examples
 - ▶ Usually using regularity (like crystallographic symmetry)
 - ▶ No general method
- It is very hard to define iterated folding in an embedding

What is folding without embedding?

Central idea:

- Don't model folding process
- Describe result of folding process

⇒ Only consider changes in the topology (like identification of faces)

⇒ Incidence geometry (polygonal complex/surface)

- Ignores angles and lengths (mostly)
- Keeps folding restrictions (tetrahedron is rigid)
- Allows an abstract definition of folding