Simplicial surfaces in GAP

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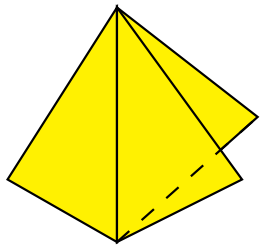
Basic data

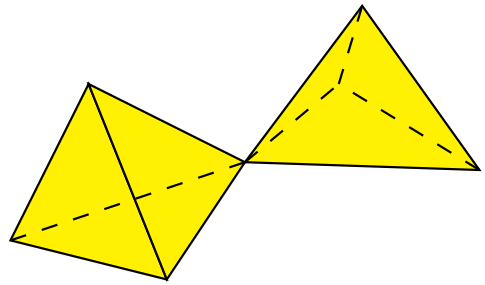
- Package name: `SimplicialSurfaces`
  - Not yet generally available
- Authors: Alice Niemeyer, Markus Baumeister
- based on current research at Lehrstuhl B including Plesken, Strzelczyk and others
- Internally used packages:
  - `AttributeScheduler` by Gutsche
  - `Digraphs` by De Beule, Mitchell, Pfeiffer, Wilson et al.
  - `GAPDoc` by Lübeck
  - `AutoDoc` by Gutsche

Motivation

Goal: Investigate paper folding

- rigid folding in $\mathbb{R}^3$
- consider surfaces built from triangles (simplicial surfaces)
  - not closed under folding
  - allow more general structures:





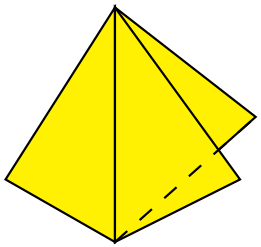
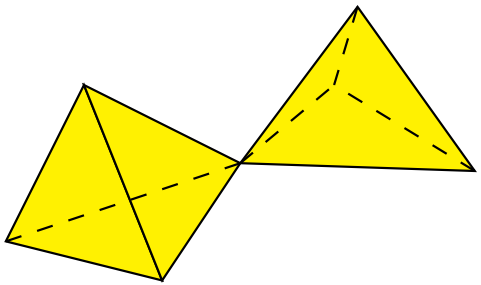
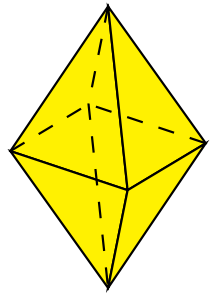
- embeddings are difficult to compute
  - some embeddings of an asymmetric icosahedron are not feasible to compute

$\rightsquigarrow$ focus on intrinsic properties

$\rightsquigarrow$ incidence geometry

Implementation in GAP

- can describe incidence geometry
- can manage hierarchy of structures
- works well with group–theoretic descriptions
- allows flexible access to the incidence geometry

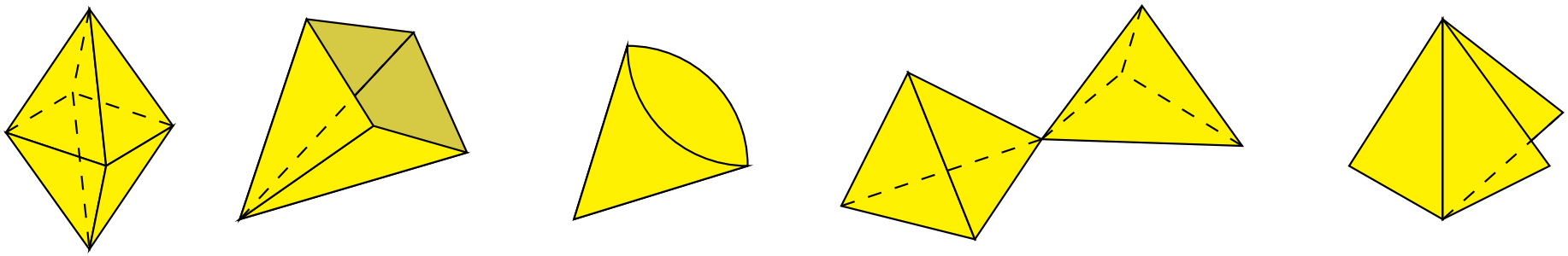




- difference to `FinInG`–package by De Beule, Neunhöffer et al.
  - only two dimensions but it can work with colourings and foldings

1. General simplicial surfaces
2. Edge colouring and group properties
3. Abstract folding

Triangular complexes

We want to describe different structures:

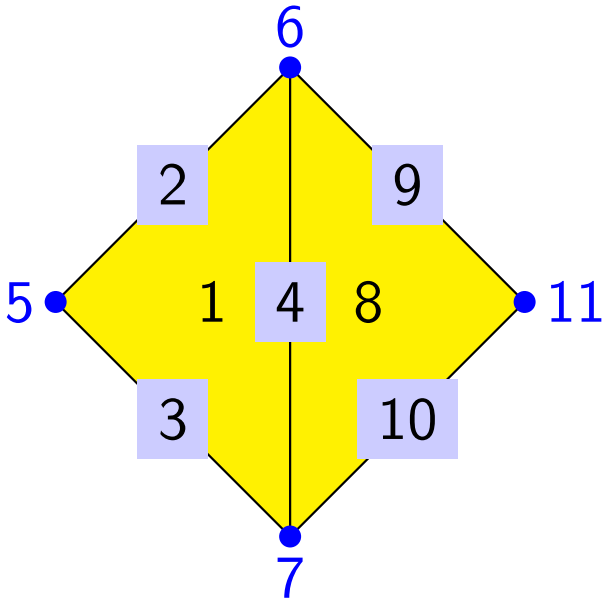


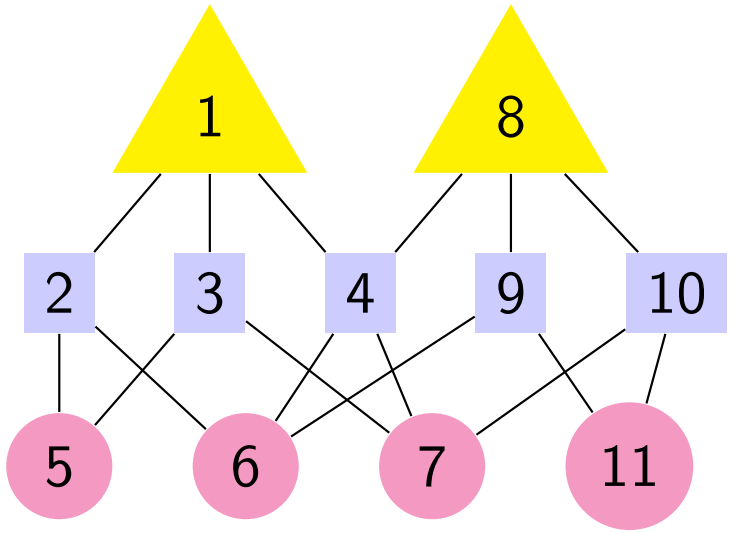
$\rightsquigarrow$ **triangular complexes**

- sets of vertices, edges and faces
- incidence relation between them
- every face is a triangle
- every vertex lies in an edge and every edge lies in a face

Isomorphism testing

Incidence structure can be interpreted as a coloured graph:





$\rightsquigarrow$ reduce to graph isomorphism problem

$\rightsquigarrow$ can be solved quite easily by `Nauty` (McKay, Piperno)

Interfaced by NautyTracesInterface (by Gutsche, Niemeyer, Schweitzer)

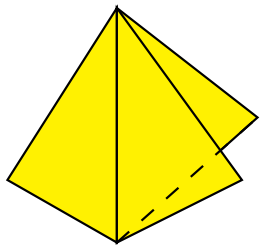
- direct C–interface without writing files
- also returns automorphism group

General properties

Some properties can be computed for all triangular complexes:

- Connectivity
- Euler–Characteristic

Orientability is **not** one of them. Counterexample:

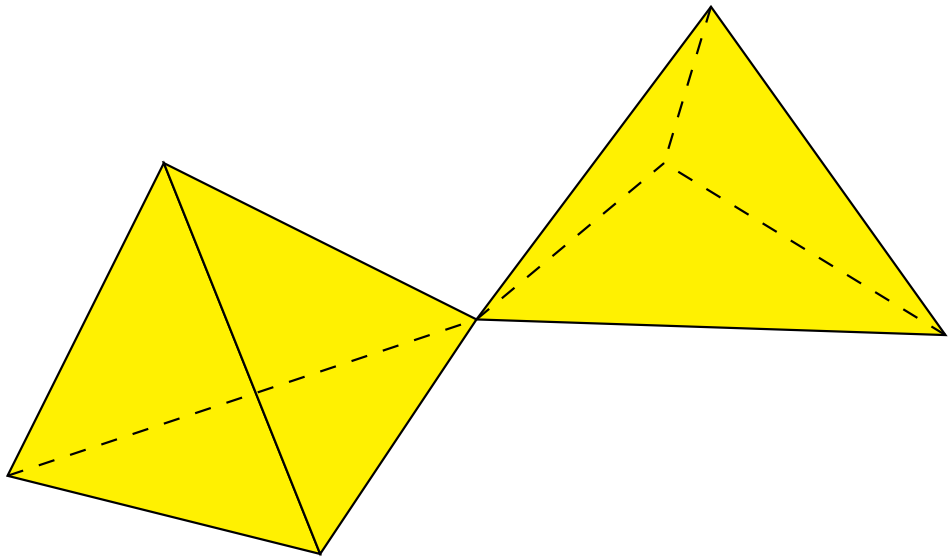


$\Rightarrow$ every edge lies in at most two faces (for well–definedness)

$\rightsquigarrow$ **ramified simplicial surfaces**

Why ramified?

Typical example of ramified simplicial surface:



$\Rightarrow$ It is not a surface – there is a *ramification* at the central vertex
A **simplicial surface** does not have these ramifications.

Classification

Plesken/Strzelczyk classified all closed simplicial surfaces up to 20 triangles.

- only interesting for those without a 3–cycle of edges
- e. g. there are 87 non–isomorphic surfaces with 20 triangles
- e. g. there is only one surface with 10 triangles:

Progress report of triangulated complexes

Already implemented:

- surface hierarchy
- elementary properties (e. g. connectivity, orientability)
- isomorphism testing
- classification data base of small surfaces

Not yet implemented:

- automorphism group
- advanced properties (any wishes?)

1 General simplicial surfaces

2 Edge colouring and group properties

3 Abstract folding

Embedding questions

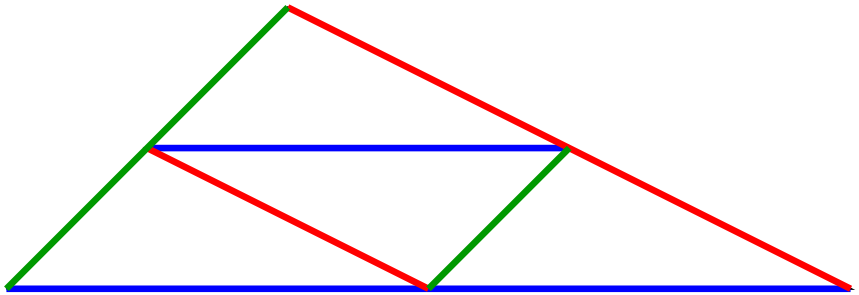
Given: A triangular complex

- Can it be embedded?
- In how many ways?

Simplifications:

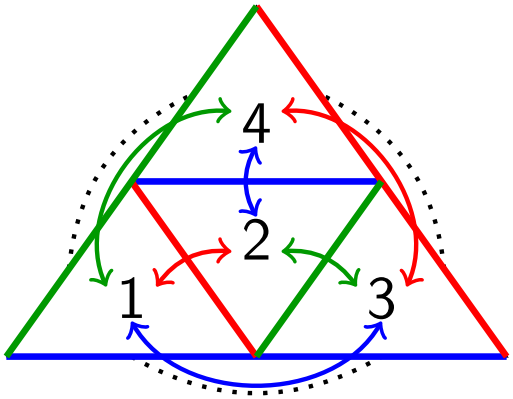
1. Only simplicial surfaces (that are built from polygons)
2. All triangles are isometric

$\rightsquigarrow$ Edge–colouring encodes different lengths



Colouring as permutation

Consider tetrahedron with edge colouring

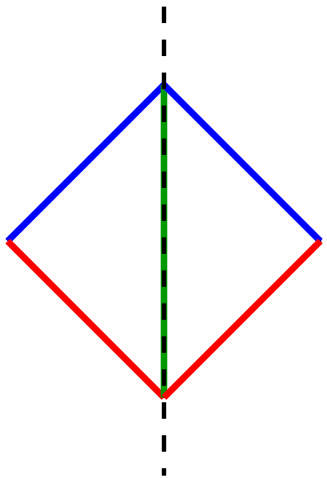


simplicial surface $\Rightarrow$ at most two faces at each edge

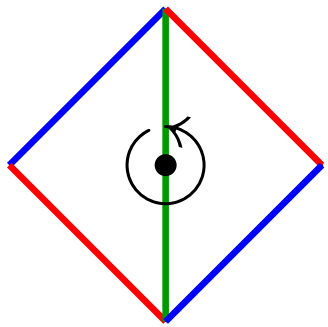
- $\rightsquigarrow$ every edge defines transposition of incident faces
- $\rightsquigarrow$ every colour class defines permutation of the faces
  - (1,2)(3,4) , (1,3)(2,4) , (1,4)(2,3)
- $\rightsquigarrow$ group theoretic considerations
  - The connected components of the surface correspond to the orbits of $\langle \sigma_a, \sigma_b, \sigma_c \rangle$ on the faces (fast computation for permutation groups)

How do faces fit together?

Consider a face of the surface and a neighbouring face
The neighbour can be coloured in two ways:



mirror (m)

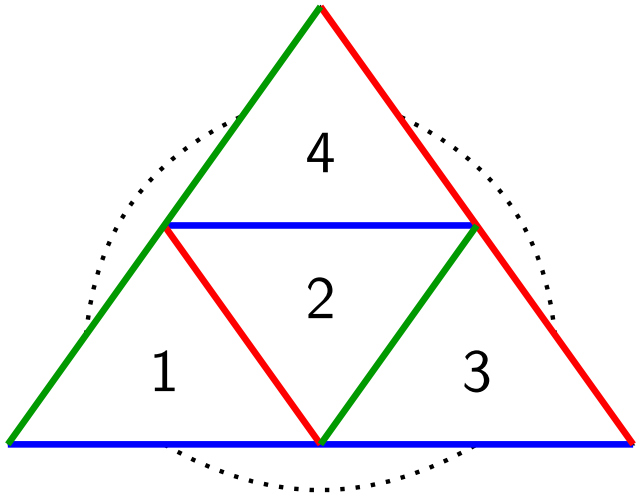


rotation (r)

This gives an **mr–assignment** for the edges.
Permutations and mr–assignment uniquely determine the surface.

Constructing surfaces from groups

A general mr–assignment leads to complicated surfaces.
Simplification: edges of same colour have the same type
Example



has only r–edges.

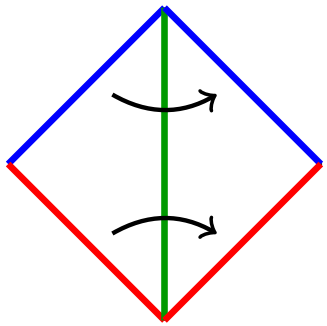
The mirror–case

If all edges are mirrors, the situation is simple.

Lemma

A simplicial surface has only mirror–edges iff it covers a single triangle, i. e. there is a surjective incidence–preserving map to the simplicial surface consisting of exactly one face.

Consider



$\Rightarrow$ Unique map that preserves incidence

- Covering pulls back a mirror–colouring of the triangle.
- Mirror–colouring defines a map to the triangle.

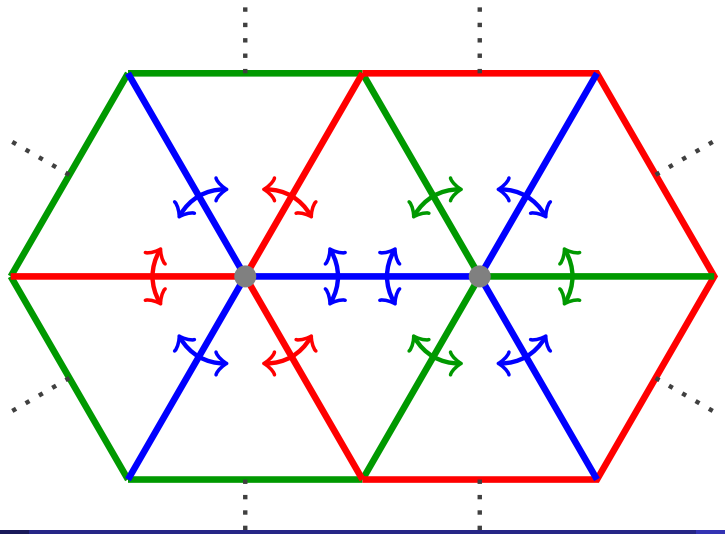
Construction from permutations

Start with three involutions σ_a, σ_b, σ_c in permutation representation (like generators of a finite group)

Lemma

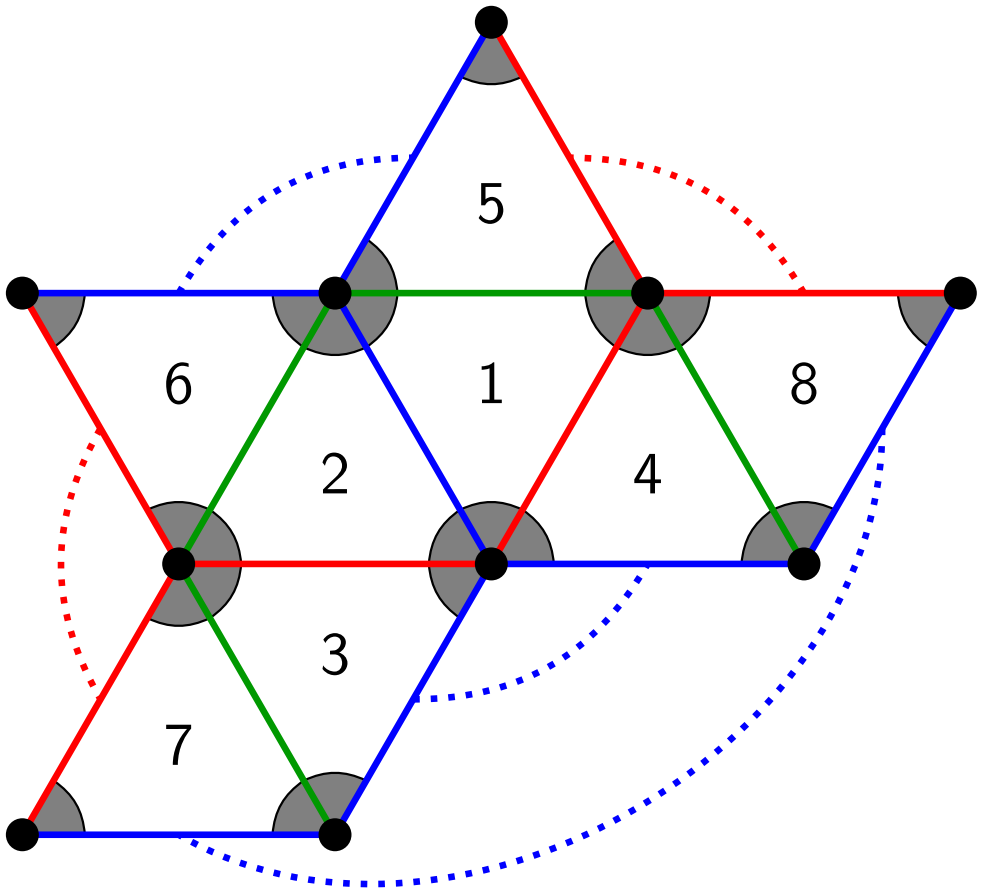
There exists a coloured surface with the given involutions where all edges are mirror edges.

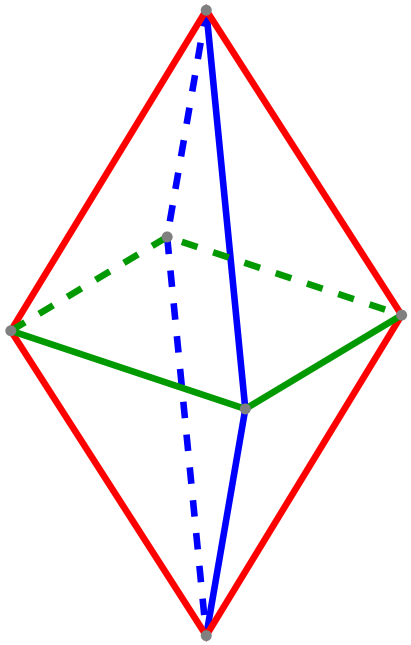
- The faces are the points moved by the involutions
- The edges are the cycles of the involutions
- The vertices are the orbits of $\langle \sigma_a, \sigma_b \rangle$ on the faces (for all pairs)



Construction example

$\sigma_a = (1,2)(3,4)(5,6)(7,8)$
$\sigma_b = (1,4)(2,3)(5,8)(6,7)$
$\sigma_c = (1,5)(2,6)(3,7)(4,8)$





Progress report of edge colouring

Implemented:

- computing all colourings of a given simplicial surface
- constructing all surfaces with given involutions
  1. up to (coloured) isomorphism
  2. with given mr–assignment
- drawing of simplicial surfaces
- constructing various coloured coverings

Still missing:

- Research TODO?

What kind of folding?

There are many different kinds of folding (e. g. Origami)
Here:

- Folding of surface in $\mathbb{R}^3$
- Fold only at given edges (no introduction of new folding edges)
- Folding should be rigid (no curvature)

Goal: Classify possible folding patterns (given a net)

Embeddings are very hard

- At every point in time the folding process has to be embedded
- We can only show foldability for specific small examples
  - Usually using regularity (like crystallographic symmetry)
  - No general method
- It is very hard to define iterated folding in an embedding

Folding without embedding

Central idea:

- Don't model folding process (needs embedding)
- Describe starting and final folding state
  - Only consider changes in the topology (like identification of faces)
  - allows abstraction from embedding

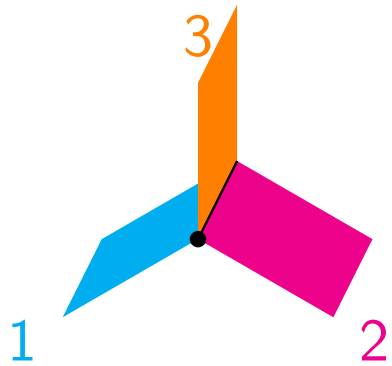
$\rightsquigarrow$ Incidence geometry (polygonal complex/surface)

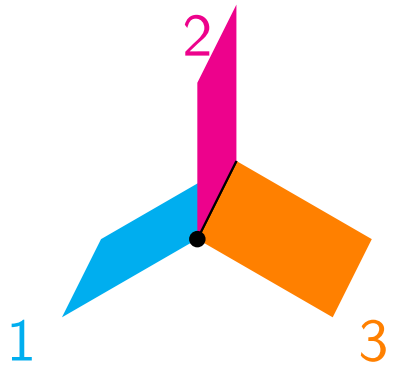
- Captures some folding restrictions (rigidity of tetrahedron)
- Still needs a lot of refinement

More than a triangular complex

- Concept should allow reversible folding
- We need an ordering of the faces:

- Adding a linear order on each face equivalence class is not enough:

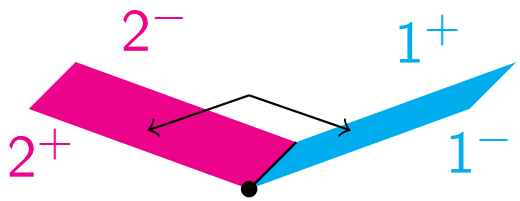


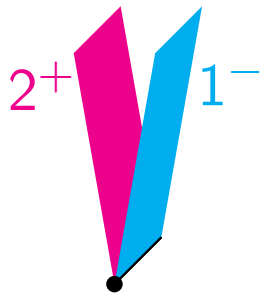


$\rightsquigarrow$ **folding complex**

How to describe folding?

Needs specification of two face sides:





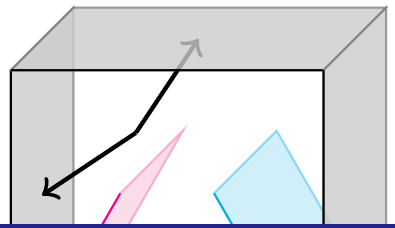
$\Rightarrow$ Describe folding by two face sides

$\rightsquigarrow$ **folding plan**

How does folding plan work?

Folding of two faces can force folding of other faces:

- Can apply to arbitrary many faces
- The forced identification is not unique



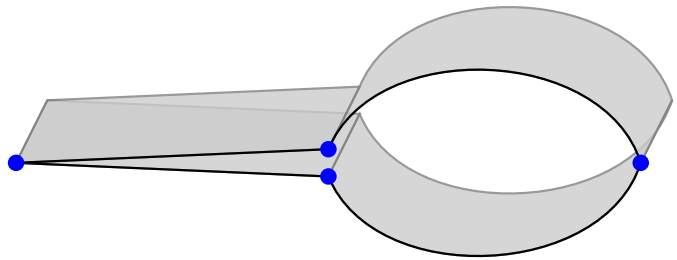
How does folding plan work?

Folding of two faces can force folding of other faces:

- Can apply to arbitrary many faces
- The forced identification is not unique

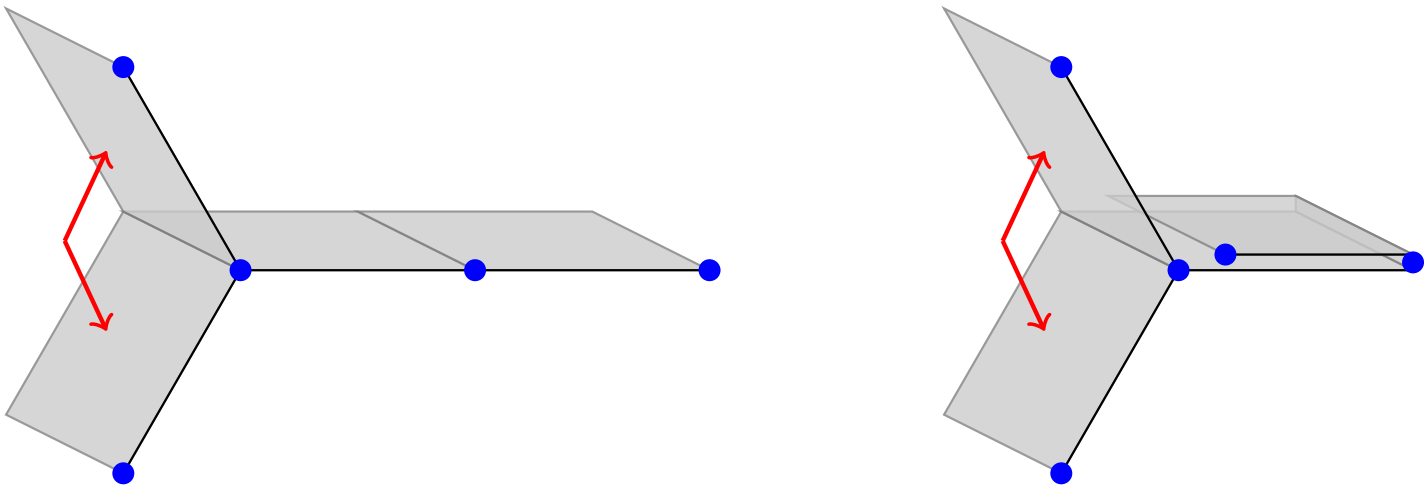
$\Rightarrow$ Identify only two faces at a time

- $\rightsquigarrow$ Relax the rigidity–constraint:
  - Allow non–rigid configurations as transitional states



Structure of multiple foldings

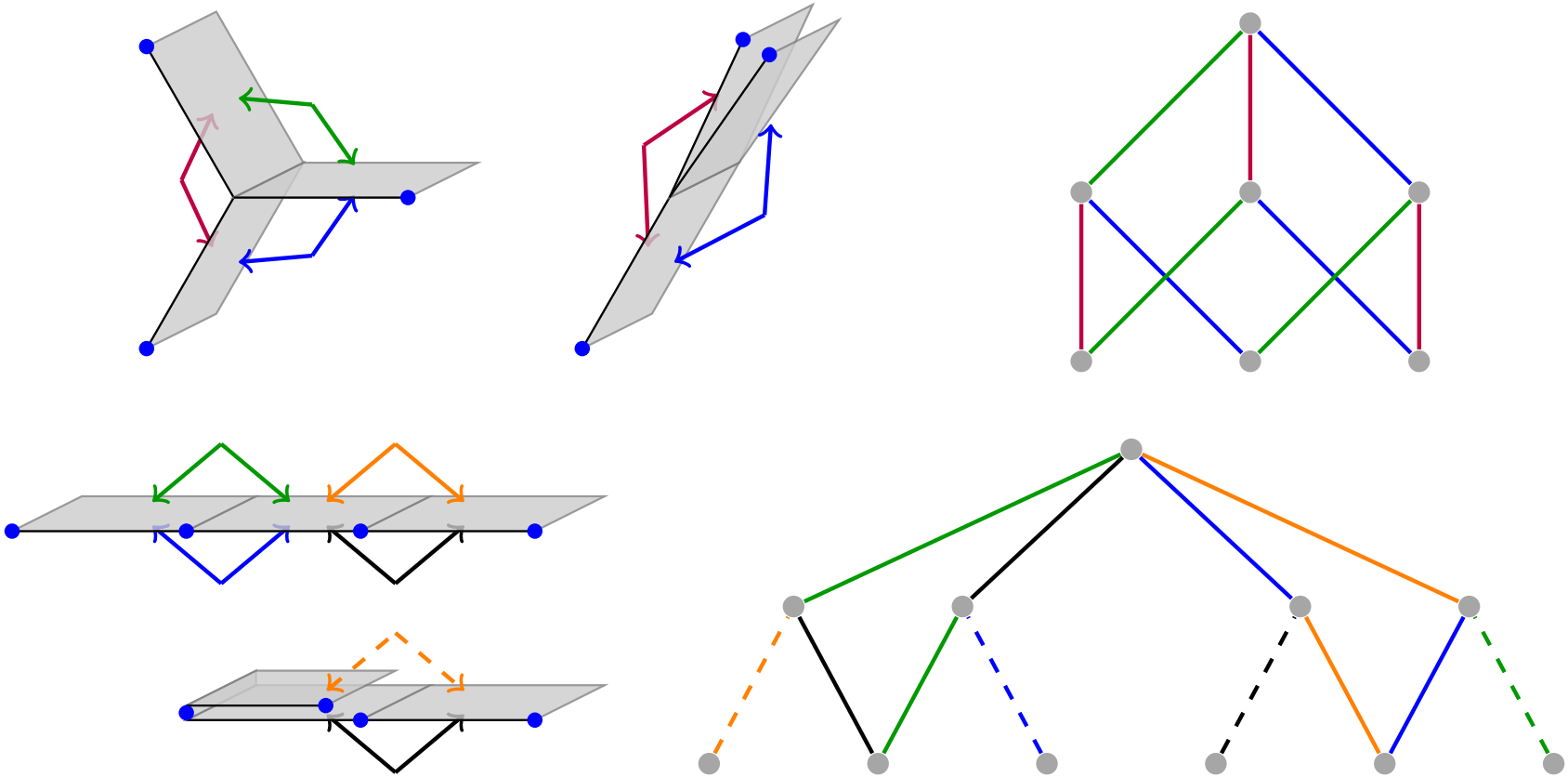
With folding plans we can perform the same folding in different folding complexes



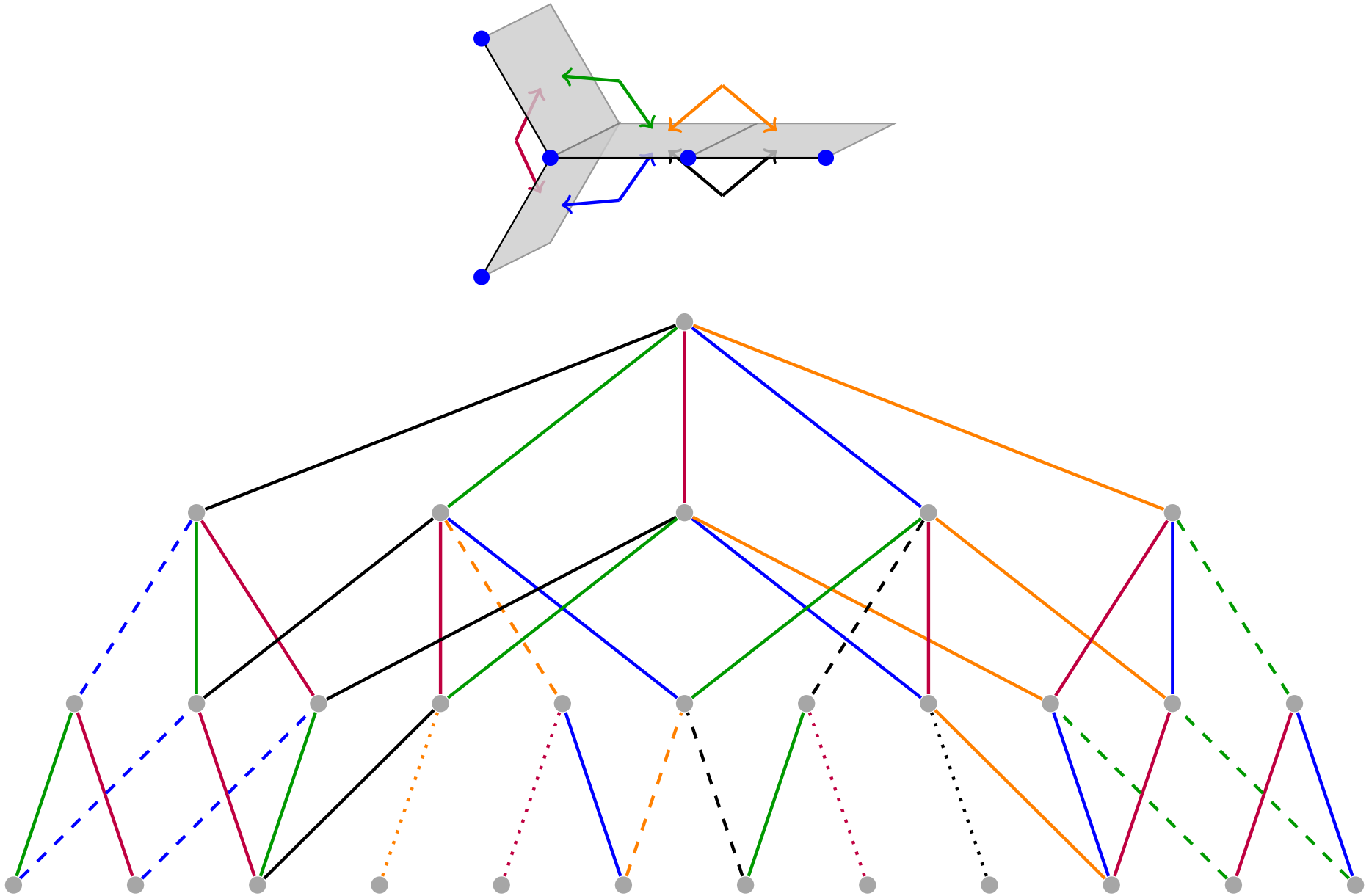
$\rightsquigarrow$ more structure on the set of possible foldings

Folding graph

- Vertices are folding complexes (modelling folding states)
- Edges are folding plans connecting two folding complexes

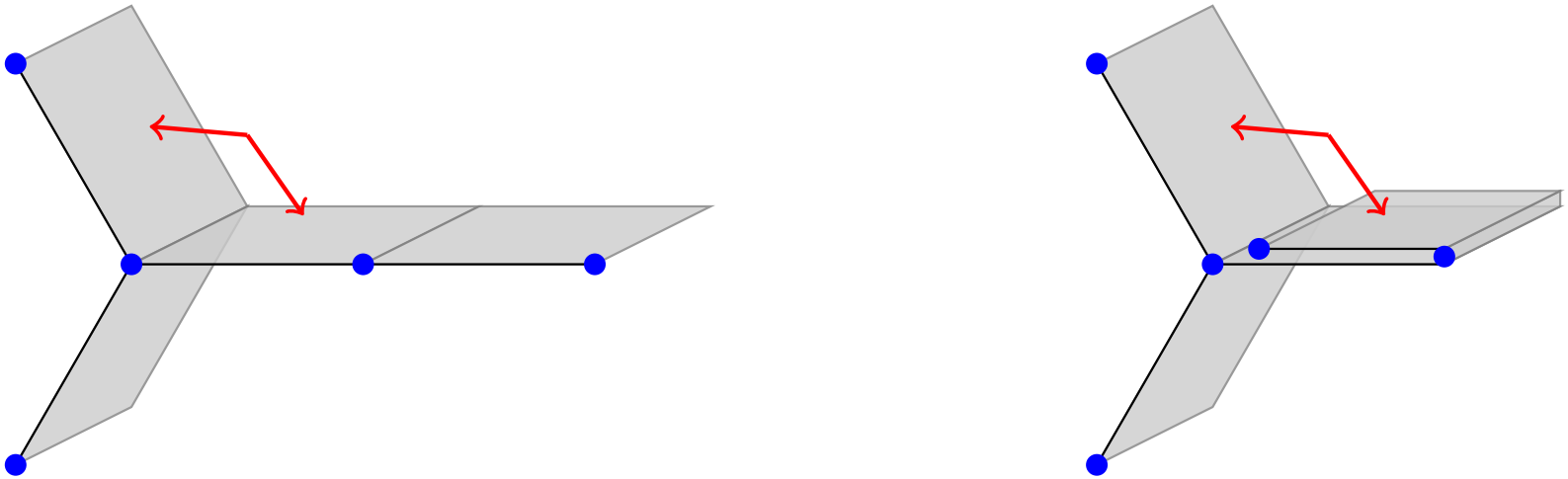


Larger graph



Drawback of folding plans

Some foldings that "should" be the same, aren't:



$\Rightarrow$ If you know the folding structure of a small complex, you can't easily find the folding structure of an extended complex

$\rightsquigarrow$ Folding plans are not optimal to model folding

Progress of abstract folding

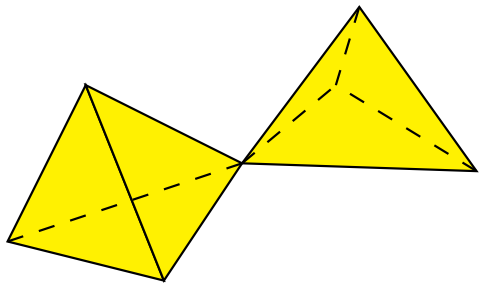
In development:

- folding complex
- folding plans
- folding graph

Missing:

- better folding description
- properties of folding graphs

Summary



Triangulated complexes

- mostly complete

Edge colouring

- current theory implemented
- a lot of theory missing

Abstract folding

- framework exists
- needs proper implementation