Simplicial surfaces in GAP

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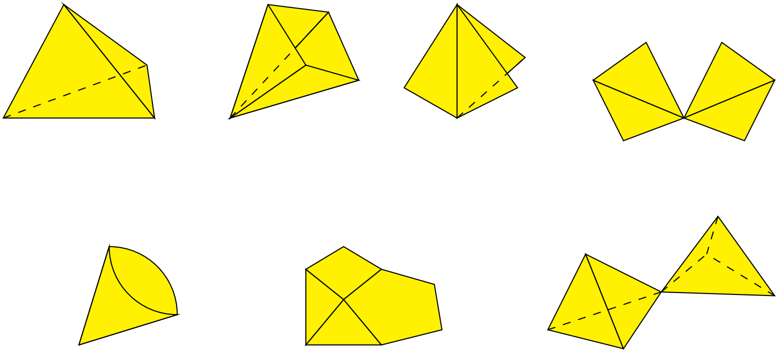
Motivation

Goal: simplicial surfaces (and generalisations) in GAP



$\rightsquigarrow$ examples of **polygonal complexes**

No embedding

We do not work with embeddings (mostly)

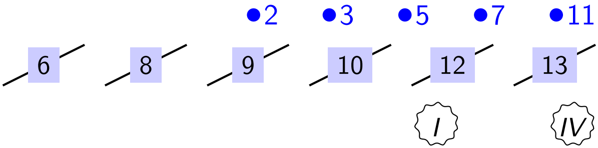
- is very hard to compute
- if often unknown for an abstractly constructed surface
- is different from *intrinsic structure*

$\Rightarrow$ lengths and angles are not important

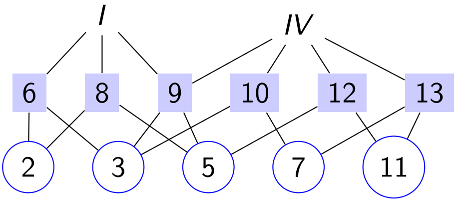
$\rightsquigarrow$ incidence structure is intrinsic

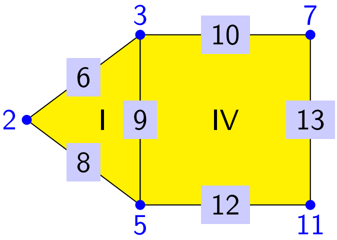
Incidence structure of polygonal complex

- set of vertices $\mathcal{V}$
- set of edges $\mathcal{E}$
- set of faces $\mathcal{F}$



- transitive relation $\subseteq \left(\mathcal{V} \times \mathcal{E}\right) \uplus \left(\mathcal{V} \times \mathcal{F}\right) \uplus \left(\mathcal{E} \times \mathcal{F}\right)$





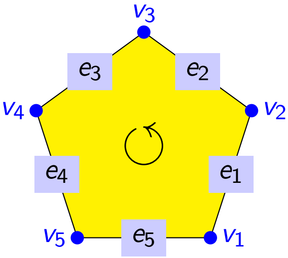
1. Every edge has exactly two vertices
2. Every face is a polygon
3. Every vertex lies in an edge and every edge lies in a face

Polygonal complexes

A **polygonal complex** is a two–dimensional incidence structure of vertices, edges and faces, such that:

1. Every edge has exactly two vertices.

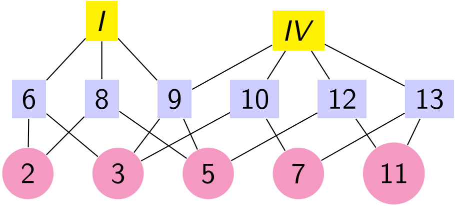
2. Every face is a polygon.



3. Every vertex lies in an edge
4. Every edge lies in a face

Isomorphism testing

Incidence geometry allows "easy" isomorphism testing. Incidence structure can be interpreted as a coloured graph:



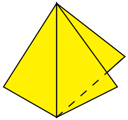
$\rightsquigarrow$ reduce to graph isomorphism problem
Solved by NautyTracesInterface (by Gutsche, Niemeyer, Schweitzer)

General properties

Some properties can be computed for all polygonal complexes:

- Connectivity
- Euler–Characteristic

Orientability is **not** one of them. Counterexample:

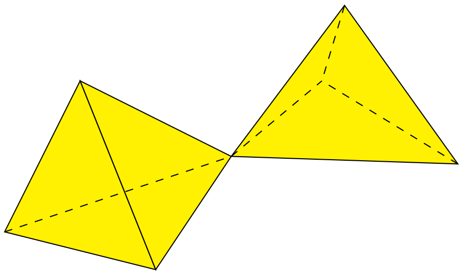


$\Rightarrow$ every edge lies in at most two faces (for well–definedness)

$\rightsquigarrow$ **ramified polygonal surfaces**

Why ramified?

Typical example of ramified polygonal surface:



$\Rightarrow$ It is not a surface – there is a *ramification* at the central vertex
A **polygonal surface** does not have these ramifications.

Embedding question

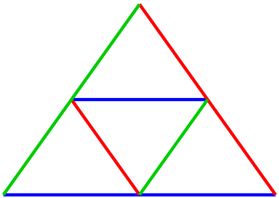
Given: A polygonal complex

- Can it be embedded?
- In how many ways?

Simplifications:

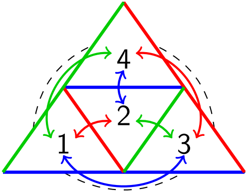
1. Only polygonal surfaces (surface that is build from polygons)
2. All polygons are triangles (**simplicial surfaces**)
3. All triangles are isometric

$\rightsquigarrow$ Edge–colouring encodes different lengths



Colouring as permutation

Consider tetrahedron with edge colouring



simplicial surface $\Rightarrow$ at most two faces at each edge

$\rightsquigarrow$ every edge defines transposition of incident faces

$\rightsquigarrow$ every colour class defines permutation of the faces

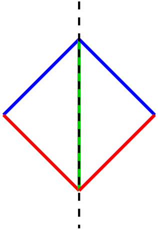
- (1,2)(3,4) , (1,3)(2,4) , (1,4)(2,3)

$\rightsquigarrow$ group theoretic considerations

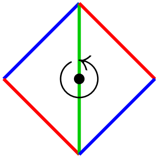
- The connected components of the surface correspond to the orbits of $\langle \sigma_a, \sigma_b, \sigma_c \rangle$ on the faces

How do faces fit together?

Consider a face of the surface and a neighbouring face
The neighbour can be coloured in two ways:



mirror (m)

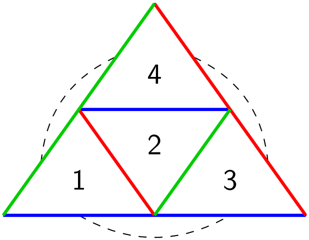


rotation (r)

This gives an **mr–assignment** for the edges.
Permutations and mr–assignment uniquely determine the surface.

Constructing surfaces from groups

A general mr–assignment leads to complicated surfaces.
Simplification: edges of same colour have the same type
Example



has an rrr–structure
The easiest structure is an mmm–structure.

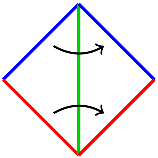
Covering

We want to characterize surfaces where all edges are mirrors.

Lemma

A simplicial surface has an mmm–structure iff it covers a single triangle, i. e. there is an incidence–preserving map to the simplicial surface consisting of exactly one face.

Consider



- Covering pulls back a colouring of the triangle.
- Colouring defines a map to the triangle.

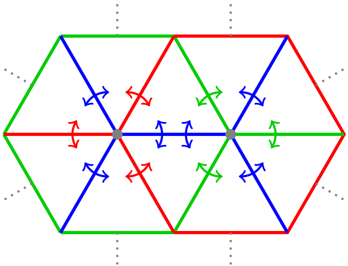
Construction from permutations

Start with three involutions σ_a, σ_b, σ_c (like generators of a finite group)

Lemma

There exists a coloured surface with the given involutions where all edges are mirror edges.

- The faces are the points moved by the involutions
- The edges are the cycles of the involutions
- The vertices are the orbits of $\langle \sigma_a, \sigma_b \rangle$ on the faces (for all pairs)



Construction example

$\sigma_a = (1,2)(3,4)(5,6)(7,8)$
$\sigma_b = (1,4)(2,3)(5,8)(6,7)$
$\sigma_c = (1,5)(2,6)(3,7)(4,8)$