ASTR 401 Midterm

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1 Lane-Emden Equation

The Lane-Emden equation (Equation (1) listed below) is very useful when it comes to the modeling of stellar structures.

$$\frac{1}{\xi^2}\frac{d}{d\xi}(\xi^2\frac{d\theta}{d\xi}) = -\theta^n \tag{1}$$

This equation can be used to solve the dimensionless function $\theta(\xi)$ in terms of ξ for a specific polytropic index n. This can lead directly to the profiling of density in respect to radius $\rho_c(r)$.

For example, when using equations (5.13) $\rho = \rho_c\theta^n$ and (5.16) $r = \alpha\xi$ in "The Theory of Stellar Structure" by Diana Prialnik, one can find the total mass of a polytropic star to be:

$$M = \int_0^R 4\pi r^2 dr \Rightarrow 4\pi\alpha^3\rho_c\int_0^{\xi_1}\xi^2\theta^n d\xi \Rightarrow -4\pi\alpha^3\rho_c\xi_1^2\Big(\frac{d\theta}{d\xi}\Big)_{\xi_1} \tag{2}$$

Where α is defined as:

$$\alpha^2 = \left[\frac{(n+1)K}{4\pi G\rho_C^{\frac{n-1}{n}}}\right] \tag{3}$$

We can also obtain a linear relationship between central density and average density $\bar{\rho}$ when combining the equations (5.16) (where r approaches R and ξ approaches ξ_1) and the solution of equation (2) to give a result of:

$$\rho_c = D_n\bar{\rho} = D_n\frac{M}{\frac{4pi}{3}R^3} \tag{4}$$

Where D_n can be derived from the solution of the Lane-Emden equation itself and is dependent on the value n:

$$D_n = -\Big[\frac{3}{\xi_1}\Big(\frac{d\theta}{d\xi}\Big)_{\xi_1}\Big]^{-1} \tag{5}$$

Furthermore, we can use equations (2), (3), and (5.18) to obtain a relationship between stellar mass and radius.

$$\Big(\frac{GM}{M_n}\Big)^{n-1}\Big(\frac{R}{r_n}\Big)^{3-n} = \frac{[(n+1)K]^n}{4\pi G} \tag{6}$$

Where constants $M_n = -\xi_1^2(d\theta/d\xi)_{\xi_1}$ and $R_n = \xi_1$ vary with the polytropic index n (which is solved next).

This relationship shows how the Lane-Emden equation is used when discussing stellar structures. It is useful to know the limitations of the Lane-Emden equation as well. For example, the equation only describes hydrostatic equilibrium and mass conservation, it does not contain any information about either energy transportation or energy generation within the star.

Below are some examples of the Lane-Emden equation where we ran n = 0, 1, 2, 3, 4, 5.

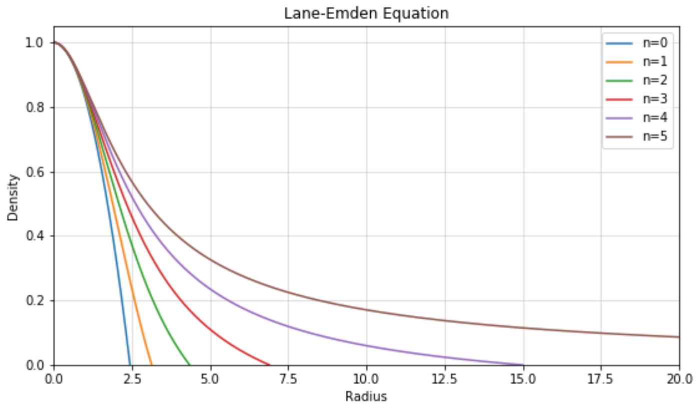

Figure 1: The plot of the Lane-Emden Equation with multiple n's.

n	Radius
0	2.448
1	3.141
3	4.353
4	6.9
5	Never Crosses

Figure 2: A table showing the ξ_1 intercept values.

As seen above on the graph, all values of n for ξ intercept zero except n = 5. We also plot the Lane-Emden Equation for a very large n below (where n = 500):

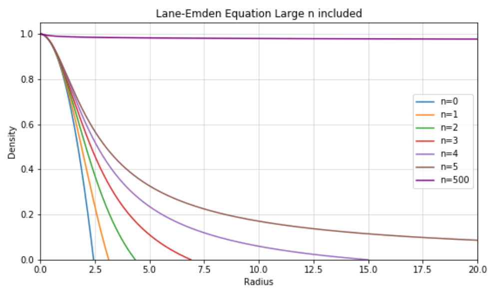

Figure 3: n=500 plotted next to other n values.

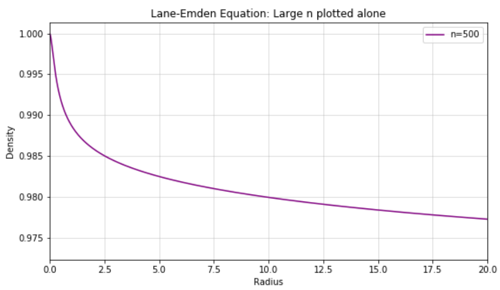

Figure 4: n=500 plotted alone to see the shape.

When observing specific n values, we can say there are only three analytic solutions to the Lane-Emden equation, specifically n = 0, 1, 5. At n = 0, the value (as given in Figure (2)) of $\xi_1 = \sqrt{6}$. At n = 1, the solution of the Lane-Emden equation is $\xi_1 = \pi$. And finally at n = 5, the solution is unique, even though $\xi_1 \Rightarrow \infty$ the mass is in fact finite. This does not carry over for values of $n > 5$ as shown in Figures (3) and (4) as plotting large n in the Lane-Emden equation proves it obvious that ξ will never intercept zero. Because of these values, we can confidentially say that the physical limits of n are constrained to $0 \leq n \leq 5$.

When calculating these values for n and ξ_1 I used python so that the calculations became easier, and it allowed me to plot the data a lot more effectively.

2 Hyades Star Cluster

The Hyades star cluster is found at ra = 66.75°, dec = $15.87°$, parallax(mas) = 21.052. To download this data of from Gaia, one must list these parameters: (ra $[63.75^{\circ} - 69.75^{\circ}]$, dec $[12.87°, 18.87°]$, parallax in (mas) [19.052-23.052]) this left me with a sample of 152 stars. The data downloaded with these stars should include ra, dec, parallax, phot g mean mag, bp-rp, and luminosity (to check our results of the derivation later). Remove stars with a magnitude above 16, as Gaia's scientific performance sheet states that any magnitudes above this value are subjected to errors due to how they collect data, and stars with a non-existent luminosity. This left us with a total sample of 81 stars. When plotting the Color Magnitude Diagram, we receive:

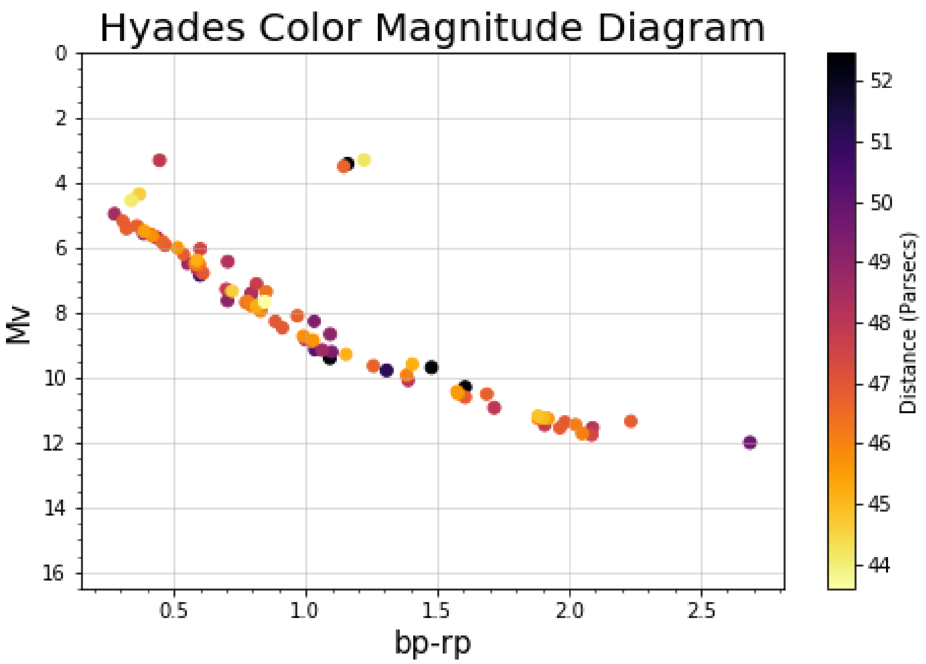

Figure 5

To find the brightest star in the cluster, one must apply these two equations:

$$M = m + 5 - 5log(D) \tag{7}$$

$$M = 4.77 - 2.5log_{10}\left(\frac{L}{L_o}\right) \tag{8}$$

Where m is the apparent magnitude of the star (our M_v values), D is the distance to the star in parsecs, M is the absolute magnitude of the desired star, L is the luminosity of the desired star, 4.77 is the absolute magnitude of the sun, and L_o is the luminosity of the sun ($3.862 x 10^{16}$ Watts).

After using equations (7) and (8), we have found the Brightest star in the Hyades star cluster to have a luminosity of roughly 76.36 L_o. Furthermore, we can also use the same equations to find the main sequence turn-off point of the star cluster. When measuring the last data point on the main sequence (seen in Figure (5) to have an apparent magnitude of 5.17, and a bp-rp of 0.307) we can find the luminosity of the turn-off point to be 14.64 L_o. Comparing this to the luminosity turnoff point of the same star that Gaia has measured: 13.97 L_o, we can confidentially say that the cutoff luminosity of the Hyades star cluster is around 14 L_o.

When comparing this measurement to the main-sequence turn-off point discussed in class (Fig. (6) seen below).

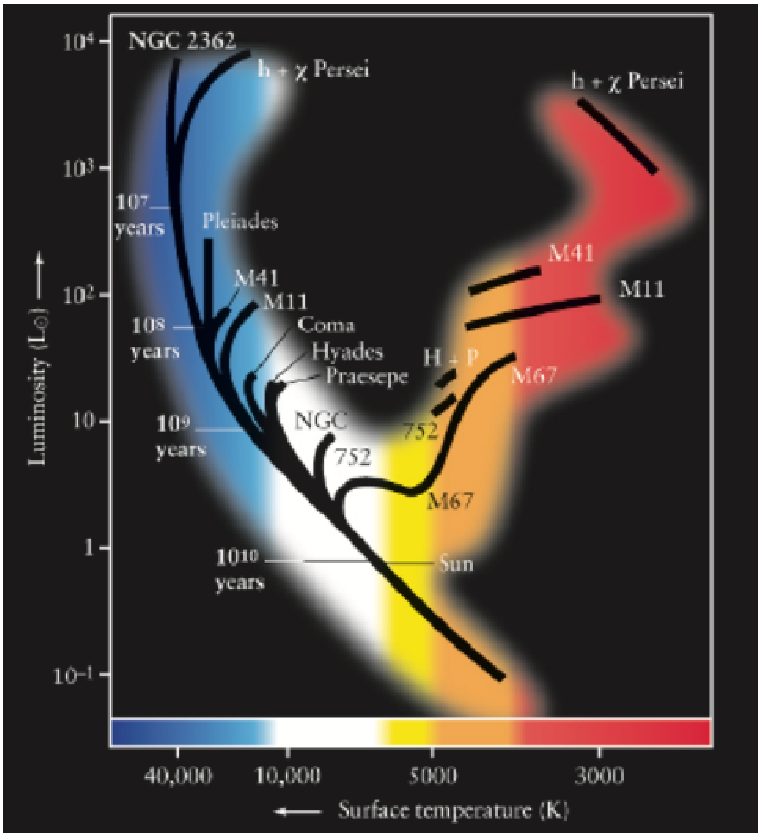

Figure 6: Main Sequence timescales of star clusters covered in lecture.

We can say that the luminosity we found of the Hyades Star cluster agrees with the luminosity that was given in lecture.