Johnson Noise and the Boltzmann Constant

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1 Introduction

The purpose of this laboratory is to study *Johnson Noise* and to observe the Boltzmann constant k. You will also get to use a low-noise pre-amplifier, a nice bandwidth limiter, and a very good AC voltmeter to directly observe the Johnson noise on a series of metal-film resistors. This should give you an appreciation for the effect of Johnson noise in circuits with high-gain. You will also use a computer that is interfaced to a programmable function generator and the DMM for some of the data taking.

The calibration phase of this experiment uses a Python program written for this purpose. If you are familiar with Python programming, you are welcome to modify the program, but please discuss this with the instructor first.

References:

- Melissinos and Napolitano, Section 3.6 (p. 122 forward).
- *Elementary Statistical Physics,* C. Kittel, Chapter 29.
- J. B. Johnson, Phys. Rev. **32**, 97 (1928).
- H. Nyquist, Phys. Rev. **32**, 110 (1928).

1.1 Theory

Boltzmann's constant k relates microscopic degrees of freedom to temperature. It is probably most familiar as the constant in the ideal gas law, which relates the pressure P, volume V, number of particles N, and temperature T:

$$pV = NkT$$

Because the electrons (the charge carriers) in a metal can be thought of as a "gas" it should come as no surprise that their random motion can bring in Boltzmann's constant. Consider an ordinary metal resistor as a conductor made out of electrons. When we apply no electromotive force to the resistor, there is no potential difference across it *on the average*. But instantaneously, the random thermal motion of the charge carriers produces some instantaneous current, and so voltage. Although this voltage averages to zero, *its square does not* — there is a finite RMS voltage across the conductor that depends on temperature. That is, there is electrical **noise** associated with a resistor that cannot be removed except by cooling.

This effect was first studied experimentally by Johnson, and analyzed theoretically by Nyquist, in 1928. The power spectrum of this noise is given by the Nyquist formula

$$dP = 4kTdf \tag{1}$$

where dP is the amount of power in a frequency interval of width df, k is Boltzmann's constant, and T is the temperature. The factor of 4 in this formula is very interesting. I will not present a derivation of the Nyquist formula here, but the reference to Kittel's book above is excellent and readable. The derivation by Melissinos and Napolitano is sloppy and unconvincing, but provides a beautiful physical picture. The derivation by Nyquist is just physics salad — good luck with that.

An interesting and important feature of the Nyquist formula is that it is "flat." That is, the power spectral density dP/df is independent of the frequency. For this reason, Johnson noise is often called "white noise." It is clear that this must be a classical approximation. The total power cannot be infinite, so at some high frequency this formula cannot be correct. Equation (1) assumes classical electromagnetism. When the photon energy hf becomes comparable to kT Eq. (1) breaks down, and dP/df approaches 0. For room temperature, this frequency $f = kT/h$ is well into the THz range, so we will not be concerned with those effects for the low frequencies used in this lab.

1.2 Observation

If we consider a resistor as discussed above, the power is voltage squared divided by resistance. Then the root-mean-square (rms) voltage across the resistor is

$$d\langle V^2\rangle = 4kTRdf$$

Of course, to measure this we have to integrate df over some bandwidth which we observe. To make life easier, we will *amplify* this voltage before measuring it. If we have an amplifier with frequency dependent gain $g(f)$, then the voltage we measure will be:

$$\langle V^2\rangle_{\text{measured}} = 4kTR \int g^2(f)df \tag{2}$$

The integral at the end of Eq. (2) technically extends from zero to infinity. In order to make evaluation of the integral practical, we use a band-pass filter so that $g(f)$ is zero outside a relatively narrow range of frequencies, making the integral straightforward.

Let

$$B \equiv \int g^2(f)df \tag{3}$$

so

$$\langle V^2\rangle_{\text{measured}} = 4kTRB \tag{4}$$

1.3 Approaches

There are essentially three approaches to experimentally studying this equation:

- Hold B fixed and measure V as a function of R
- Hold R fixed and measure V as a function of B
- Hold R and B fixed and measure V as a function of T

We would like to study all three of these approaches. Unfortunately we do not have a good control over T at present. If time permits you can perform the last approach using room temperature and liquid nitrogen temperature.

We will begin by measuring B for a variety of band-pass settings. We can then select a particular band-pass setting, keeping B fixed, and measure V for a series of different resistors R. After that we can leave R fixed and measure V for the previously chosen values of B. Finally, we can think about liquid nitrogen.

In both cases (varrying R or varrying B) you measure the RMS voltage on a resistor R. The left-hand side of Eq. (4) is just the square of your measured RMS voltage. Thus, you can fit a straight line whose slope is $4kTB$ when you vary R or $4kTR$ when you vary B. Although Eq. (4) predicts that this line will have zero intercept, in practice it may not because of extra noise in the amplifer, filter, and voltmeter. You might expect the intercept to be the same whether you vary R or vary B. In either case, knowing the temperature in the lab (check the thermostat on the wall) will give you Boltzmann's constant k.

Evaluating B is straightforward, but you need to understand what it means. You will evaluate B as the numerical integral of $g^2(f)$. This means you need to pick a particular band-pass setting and measure $g(f)$. The noise voltage we want to measure is very small, which means we need a high-gain amplifier as well as a band-pass filter.

- The gain $g(f)$ is the ratio of the voltge input to the amplifer divide by the voltage output from the filter.

You have to understand that before continuing. Explain what $g(f)$ is to your lab partner to make sure you understand it.

2 Apparatus

Figure 1 shows a block diagram of the basic apparatus. You will also use a function generator and computer, which are not shown.

Now you should recognize what $g(f)$ is on Fig. 1. It is the voltage read by the RMS ac voltmeter divided by the voltage input to the differential inputs. Find these places on the apparatus. Understand it!

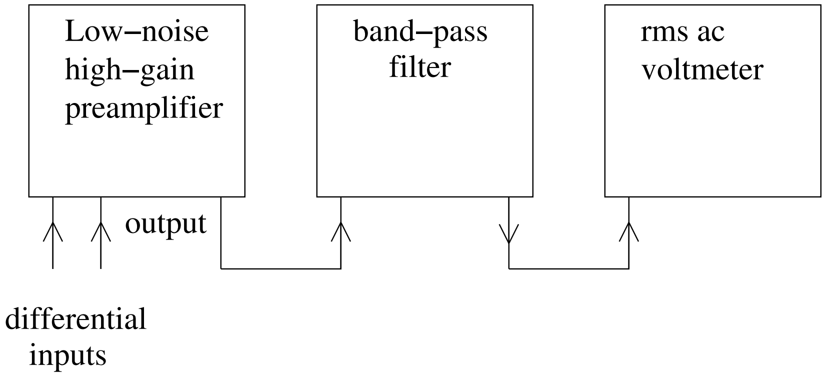

Figure 1: Simplified block diagram of the apparatus.

2.1 Low-noise preamp

The Stanford Research Systems SR-560 is a nice low-noise preamplifier. In order to measure the very small Johnson noise on resistors, we need high gain. In order to avoid other noise sources such as ground currents in the lab, we will operate the preamp in differential input mode. Set the input coupling for "DC", and source to "A-B". Set the gain to 1000. Configure the input with a 12 dB/octave high-pass filter at 1 Hz to block DC.

Connections to the input should be done carefully. Please note that there are two aluminum "Bud" boxes on the table, labeled **A** and **B**. On each of these boxes, one side has two female BNC connectors which connect to the preamp's two differential inputs. Connect the Bud-boxes to the preamp using two short male-male BNC cables. This is shown in Fig. 2.

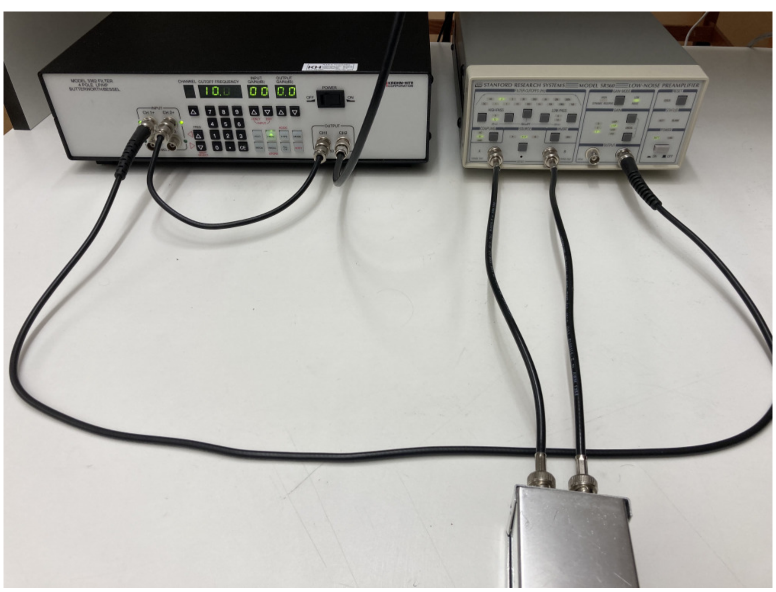

Figure 2: Connection of "Bud" box B to preamplifier.

2.2 Bandwidth filter

In order to obtain a precise value for the integral B in Eq. (3). We need to have the gain function $g(f)$ go to zero outside some well defined range. For this purpose, we will use a Krohn-Hite model 3362 4-pole filter. The model 3362 is actually a box with two independent filters. We will set the first filter to high-pass mode (blocking low frequencies) and the second filter to low-pass mode (blocking high frequencies). In this way, only frequencies between the low- and high-pass cut-offs are passed.

Setting the filter is a bit tricky. Before you turn the filter on, check that the cables are connected as follows:

- Connect the output of the preamp to the "CH 1+" filter input.
- Connect the CH1 output to the "CH 2+" filter input.
- Connect the CH2 output to the Fluke 8846A multimeter.

This is also shown in Fig. 2.

Now set up Channel 1:

- Look at the "Channel" display, and press the small button with a triangle under it until it reads "1".
- Press the "Mode" button until the "Cutoff Frequency" display reads "h.P." (high pass).
- Press the "Type" button until the same display reads "bu." (Butterworth).
- Press the "Freq" button once. Type "10" into the numeric display. Press the "kilo" button, and the then "Freq" button.

This sets Channel 1 to a high-pass filter with cut-off of 10 kHz.

Perform an analogous procedure to set Channel 2 for a low-pass filter with a cut-off of 30 kHz. You should now have a band-pass filter that passes signals between 10 and 30 kHz.

It is important to understand the previous procedure. You will be changing the setting on Channel 2 several times in this experiment.

2.3 Digital multimeter

We will make measurements with a Fluke 8846A digital multimeter (DMM). Connect the output of the Krohn-Hite 3362 filter to the voltage-input (not the sense input) of the fluke with a BNC-banana adapter (also known as a "Pomona connector").

This device makes it easy to measure noisy signals and get a useful measure of the uncertainty in their value. This is done with the yellow "Analyze" button. Pressing

"Analyze" and then "Stats" (blue button labeled F2) starts a series of measurements. The average and standard deviation of these measurements is continuously updated. You can program the DMM to analyze a set number of readings and stop. I suggest 20 as a good number to get a stable measurement without waiting too long.

3 Measuring the gain-squared bandwidth integral

First we need to measure the integral B from Eq. (3). To do this, we will put a known small signal into the preamplifier, and measure the output of the filter with the digital multimeter. We will do this for many different frequencies. The ratio of the filter output to the preamp input is the gain, $g(f)$. By plotting the square of this function, we can perform a numerical integral to get B.

For the above procedure to be accurate, we must choose a frequency range that is wide enough that the filter output has dropped to zero, and we must choose enough points to get an accurate numerical integral. A good choice is to choose a range of frequencies starting a factor of 10 lower than the high-pass filter setting (channel 1 on on the Krohn-Hite) and ending a factor of 10 higher than the low-pass filter setting (channel 2).

Before you start, consider that the preamplifier has a gain of 1000, and cannot sustain an output voltage much greater than a volt. This means that we will need to use a voltage divider to reduce the input from something easy to measure.

3.1 Setup

First, open up Bud-box A. You will see that the single input drops across two resistors, and that the two outputs are connected across the second resistor. This means that if you connect a sine wave to the input, the voltage across the two outputs will just be this input voltage multiplied by $R_2/(R_1 + R_2)$.

- You need to measure these two resistors now. Use the DMM, and the BNC-alligator clip lead on the table. Make sure nothing is connected to the Bud-box (such as the preamp) before making your measurements. (Do you see why?)

After you have measured the resistors, you will see that this box is designed to divide the input by approximately a factor of 1000. *Make sure your measurements support this.* Next, close up the box, and connect it to the preamplifier as shown in Fig. 2.

Look for the "Calibrate Filter" icon on the computer desktop and start this program. You will see a display similar to that shown in Fig. 3.

This program interfaces with the Stanford Research Systems DS-345 function generator and the Fluke DMM. It programs the function generator to generate a

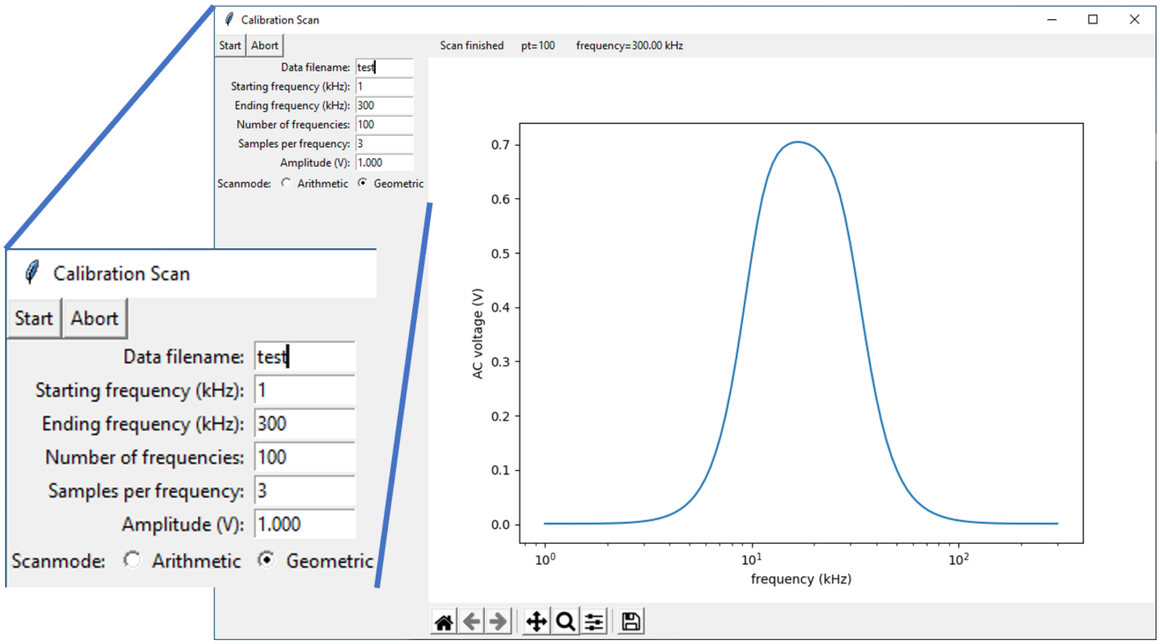

Figure 3: Python program for calibrating filter with inset showing important user-settable inputs.

sine wave with with a known amplitude and frequency, and then reads the ac RMS voltage from the Fluke DMM. It is designed to sweep the frequency from a low starting frequency to a high final frequency. Figure 3 shows the result of such a sweep of 100 different frequencies from a starting frequency of 1000 Hz to a final frequency of 300,000 Hz. At each frequency, the Fluke DMM was read three times, and the results averaged. The SRS function generator was programed with a 1 Volt Amplitude sine wave.

All these parameters can be selected by the user. Refer to the left-hand side of the window; this is shown zoomed in in Fig. 3. It is very important to understand what each of these inputs does. In particular, you need to understand that "Number of frequencies" sets the number of frequency points in the sweep and "Samples per frequency" sets the number of time the DMM is read at each frequency. As you increase either of these quantities the time needed for a sweep will increase.

This program displays the results of its measurements as it scans. As you can see, Figure 3 shows the result of measuring across a band-pass filter with high-pass value of 10 kHz and low-pass value of 30 kHz.

- Make sure you understand why the "high-pass" value is lower than that "low-pass" value. You have to understand this!

The Python program can also allow the scan to be either an arithmetic or geometric progression of frequencies from the starting frequency to the stopping frequency. In the Figure, a geometric progression was chosen so that about the same number of points would be taken on the rising edge of the filter as on the falling edge. This is probably the easiest way to map out the gain function. If you want to use an arithmetic progression, you should understand why you want to. Feel free to discuss this with your instructor.

This program just displays the voltage being measured by the Fluke DMM. In order to convert this to gain g you need to understand what that means. Recall that the gain g is the voltage output from the filter divided by the voltage input to the preamp. This means you need to run the frequency scan *twice*: once with the the output of the function generator connected to the voltage divider input to the preamp, and once with the output of the function generator connected directly to the DMM. The gain is then the ratio of these two (corrected for the voltage divider).

- You have to understand this last paragraph! Don't even think about proceeding until you understand the previous paragraph.

3.2 Procedure

Let us start by measuring $g(f)$ for a series of different band-pass settings. This will allow us to calculate B for those different settings.

3.2.1 Recommendations

These are recommendations only. You could change this up. I recommend leaving the high-pass filter setting at 10 kHz and changing only the low-pass filter setting.

For the scan, I recommend using a starting frequency a factor of 10 lower than the high-pass filter setting. For a high-pass filter setting of 10 kHz as suggested above, that would mean 1 kHz. **Note:** The Python program accepts frequencies in kHz.

Similarly, I recommend a stopping frequency a factor of 10 greater than the low-pass filter setting. For a low-pass filter setting of 20 kHz this would mean ending the scan at 200 kHz.

Choosing a number of points and a number of samples per point depends on your patience. For good data I recommend setting the number of points to 100 and the number of samples per point to 3. These will be fast enough scans but give high-quality data.

3.2.2 Test Run

1. Enter "test.csv" into the "Filename" field.

2. Enter "1" for "Starting frequency" (because you have previously set the high-pass filter to 10 kHz).

3. Enter "300" for "Ending frequency" (because you have previously set the low-pass filter to 30 kHz).

4. Enter “25” for “Number of frequencies”.
5. Enter “1” for “Samples per frequency”.
6. Enter “1” for “Amplitude”.

Make sure the function generator is connected to the voltage divider box and that the output of the filter is connected to the DMM. Take a scan by clicking the “Start” button at the top left of the window.

Your result should look something like the the curve shown in Fig 3, although with many fewer frequency points.

Verify that your data are sensible! You should be seeing the transmission of a band-pass filter. The peak amplitude should be something near 0.7 Volts (since that is the RMS value of a 1 Volt amplitude sine wave).

- Note that a sine wave with a 1 V amplitude has a RMS of 0.71 V.

3.2.3 Measurements

You are now ready to start taking data. **This procedure takes a long time**, and you need to know what you are doing and work methodically.

First you will take a series of measurements like the one you just did, measuring the voltage out of the filter for a number of different filter settings using a different range of frequencies for each filter setting. However, you will increase the number of frequency points and also the number of samples at each frequency. Then you will measure the voltage going into the preamp for the same frequency ranges.

PRO-TIP: You are measuring small signals. Don’t move around during the measurments. Don’t jostle the cables. Try to keep signal sources like your phone away from the sensitive inputs.

- Leave the high-pass filter at 10 kHz and begin by setting the low-pass filter to 20 kHz.
- Set “Starting frequency” on the computer program to 1 kHz. (Understand why!)
- Set “Ending frequency” on the computer program to 200 kHz. (Understand why!)
- Set the “Number of frequencies” to 100.
- Set the “Samples per frequency” to 3.

- Enter a filename into the "Data filename" field that is sensible. Since you will be varying the low-pass filter setting, you could include that in the file name. If your file name does not end in ".csv" then that extension will be added to your file name.
- Initiate the scan by pressing the "Start" button at the top left of the program window.

Each time you take a scan, make sure that what you are seeing makes sense! Make sure the rising and falling edge are sensible, and that the voltage being read is sensible.

In addition to the scan just described, repeat the scan for low-pass filter settings of 25 kHz, 30 kHz, 35 kHz, 40 kHz, 45 kHz, and 50 kHz (seven scans).

- Each time make sure you change "Ending frequency" and the file name.

You now have measurements of the voltage output of the band-pass filter for seven different band-pass settings.

In order to get the gain, you must now measure the voltage that was input to the voltage divider during each of the previous scans. To get these, connect the output of the function generator directly to the input of the DMM. Now repeat the scans changing "Ending frequency" each time to the same value used in the previous scans (seven more scans).

Note that the voltage will not vary much! The variances will be expanded to fill the scale, but note the axis. The changes should be a small fraction of a volt.

You now have the data you need to calculate B for all seven different band-pass settings! When you do this, don't forget that B involves the integral of the gain *squared* and don't forget to include the voltage divider.

4 Measuring Johnson noise

4.1 Fixed B and vary R

Now that you have data for determining the integral B for several different band-pass settings, you will pick a particular band-pass and then measure the Johnson noise on a series of resistors. This will allow you to plot $\langle V^2 \rangle$ versus R. According to Eq. (2) the slope will be $4kTB$.

I recommend using the low-pass filter setting of 30 kHz. You can use any of the seven you measured, but this seems to work well. Make this setting on the filter now.

Next, you need to measure the resistance of the resistors you will use. In a box you will find 8 high quality metal-film resistors. Use the Fluke multimeter to measure the resistance of each one. Try to get ridiculously precise values. Also, be careful to handle the resistors as little as possible, and try to hold them by the wires. Remember the slope you are going to measure depends on the temperature in the lab. You don't want to heat up the resistors with your hands.

Next connect Bud-box **B** to the preamplifier. Inside you will see two alligator clips. Gently clamp one of the resistors with the alligator clips. Make sure the connection is good, and make sure the leads from the resistors do not touch the box. This is shown in Fig. 4.

Figure 4: Connection of resistors in "Bud"-box B.

For each resistor, measure the output of the filter with the Fluke multimeter. Remember you are looking for a small value, on the order of a milli-volt. Make sure you use the "Analyze" feature of the multimeter. We recommend letting it do about 20 averages. Don't forget to record the standard deviation – this is your uncertainty.

Do the 10 kΩ resistor last and leave it in the bud box when you are done.

Here is an example of the kind of data you should get:

R (kΩ)	V_{out} (μV)	ΔV_{out} (μV)
0.99890	694.6	3.7
2.00383	896.2	4.5
3.00156	1061.7	5.0
4.00195	1202.8	5.4
4.98222	1330.4	6.5
6.03090	1453.8	6.5
6.98997	1559.7	8.2
8.02847	1659.7	8.6
9.05905	1758.0	7.9
9.98767	1841.6	9.3

Again, this is *just an example.* The resistor values you will find are different from the ones above, and the voltage measured depends on the gain and the bandwidth. But please do note that the uncertainty (standard deviation) gently rises with the voltage. Remember: *this is the uncertainty in* V_{out} , you must calculate the uncertainty in $\langle V^2 \rangle$ from this.

4.2 Fixed R and vary B

You can now take the data where you leave R fixed and vary B. This step is important so that you can compare your value of the Boltzmann constant from the two ways to make sure your experiment is behaving sanely. You already have the data you need to calculate B for seven different band-pass settings. You just need to measure the voltage for a given resistor for each of those band-pass settings.

You should already have the 10 kΩ resistor in the bud box. So, just set the Krohn-Hite filter to the first of the settings you took data for in the calibration step, that is, set the pass band from 10 kHz to 20 kHz. Now, make sure the output of the filter is connected to the Fluke DMM and measure the AC voltage just like you did in the previous step.

Repeat this for low-pass filter settings of 25 kHz, 30 kHz, 35 kHz, 40 kHz, 55 kHz, and 50 kHz.

You can now plot $\langle V^2 \rangle$ versus B. The slope is just $4kTR$ where R is the value you measured for the 10 kΩ resistor.

4.3 Measuring temperature

The last thing you need to measure is the temperature in the lab. The thermostat on the East wall will suffice.

If there is time, it is fun to record a value of the noise voltage for two different temperatures. You instructor will help you prepare a cup of liquid nitrogen to see how that affects the noise voltage. This is optional, but it is a lot of fun.

This completes data acquition.

5 Analysis

Now you need to analyze your data. First you will determine B for each of the seven band-pass settings. Then you will plot $\langle V^2 \rangle$ versus R for the series of resistors you measured. The slope will be $4kTB$ where B is the value gotten for the band-pass setting used. From this you can obtain a value of k and its uncertainty. Next you will plot $\langle V^2 \rangle$ versus B for the series of band-pass settings. The slope will be $4kTR$ where R is the resistor used in that step. Then you can compare these two values of k.

5.1 Gain-squared-bandwidth integral

You should have seven data files with the voltage out of the filter and another seven with the voltage out of the function generator. These are "comma-separated-values"

(.csv) files and you can open them in Excell or other spreadsheet programs, or import them into various scripting languages.

Here is a procedure that works, but you can do this many different ways. For each band-pass setting, first open the file which was the scan taken with the function generator connected directly to the DMM. Highlight the two columns "Frequency" and "Mean" and copy these to the cut buffer. Now open the file for the same band-pass setting that had the function generator connected to the bud box. Paste those two columns in, *making sure that the frequency rows line up with the ones in this file*. If you divide the "Mean" column by the pasted one, and then divide that by the voltage divider ratio, you will get a column with the gain. Plot this column as a function of the frequency column, and you should get a curve that looks like a band-pass. It should be near zero on the edges and have a value of about 1000 in the pass band. If you don't have this, something is wrong and there is no point proceding until you get it right.

Remember that B is the integral of the gain *squared* so you need to square the gain and do a numerical integral of $g^2(f)$.

To make sure your value is sensible, the answer you get for B should be approximately equal to 1 million (gain $\approx$ 1000, squared) times the difference between the high-pass cutoff and low-pass cutoff frequencies of the filter. So, for 10 kHz and 30 kHz cutoff frequencies, you should get $B \approx 2\times10^{10}$ Hz. Hopefully, you can get quite a few significant figures.

5.2 Slope of the Johnson noise

You have measured the amplified Johnson noise as a function of resistance. You need to find the slope of $\langle V^2\rangle$ versus R, **and its uncertainty**.

The uncertainty is tricky! You **must** use a weighted least-squares fit, because your uncertainty will vary a lot from the low resistance to high. Why is this? You may think your uncertainty in the voltage was about the same for each different resistor. But wait! What you measured was $V_{\rm rms}$, or rather $(\langle V^2\rangle)^{1/2}$. Thus, you have to calculate your uncertainty in $\langle V^2\rangle$. Do you know how to do this?? (Hint: the error bars will increase with increasing voltage.)

- Enter your $V_{\rm rms}$ versus R data into columns in a spreadsheet. (One row for each resistance.)
- Create two new columns for $\langle V^2\rangle$, **and its uncertainty**.
- Using the formulae for weighted least squares, find the slope of $\langle V^2\rangle$ versus R and the uncertainty in this slope. You will need to create several new columns in your spreadsheet for the sums involved in the least squares fit.
- Note that you cannot simply ask Excel to generate a trend line! It will not weight the points correctly.

You might want to plot the fitted line and its residuals to show how nice a job you did. Figure 5 shows an example. Note how the error bars increase with resistance. This is because the uncertainty in voltage was approximately constant, so the uncertainty in voltage-squared increases with voltage.

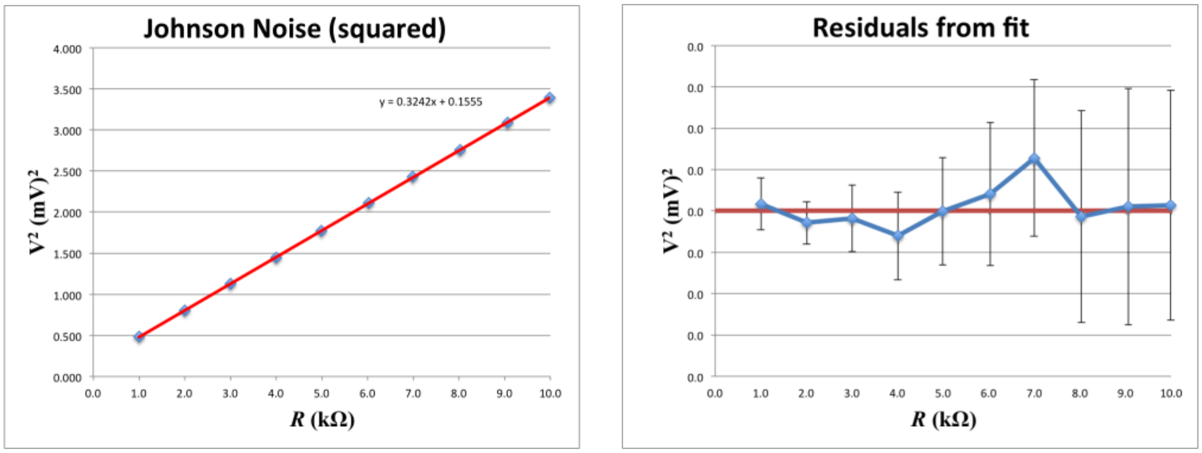

Figure 5: Fit of Johnson-noise data. Note the error bars.

Do the same thing for the measurements of $\langle V^2 \rangle$ as a function of B for fixed R.

6 Boltzmann's Constant

Finally, you are in a position to extract Boltzmann's constant. Eq. (2) says the slope of the line just measured is equal to $4kTB$. From your slope, your measured value of B and from the temperature in the lab, derive a value for Boltzmann's constant **and its uncertainty**. Make sure you include some uncertainty in the temperature in your estimate. Half a degree Fahrenheit might be a good estimate.

For the sample data shown in this write-up, the value obtained is:

$$k = 1.389(9) \times 10^{-23} \text{ J/K}$$

This is to be compared with the known value of 1.38066×10^{-23} J/K. For the sample data, we are within about half a standard deviation, with no systematic effect.

Can you do better?

How does your value of k for the two different procedures compare?

7 Discussion

- The gain-squared-bandwidth integral B is sort of proportional to the bandwidth. We used 10 kHz to 30 kHz, which is 20 kHz of bandwidth. Why not run the bandwidth way up more and get more signal? Why not set the upper cut-off of the filter to 100 kHz, or even 1 MHz?

- Along the same lines, why not increase the gain? Instead of 1000, why not use 10000?
- Since the Johnson-noise voltage (squared) is proportional to resistance (we did get a nice straight line) why not use bigger resistors? We only used 1 kΩ to 10 kΩ. Why not use 100 kΩ or even a MΩ?
- Can you think of any way to improve the measurement?