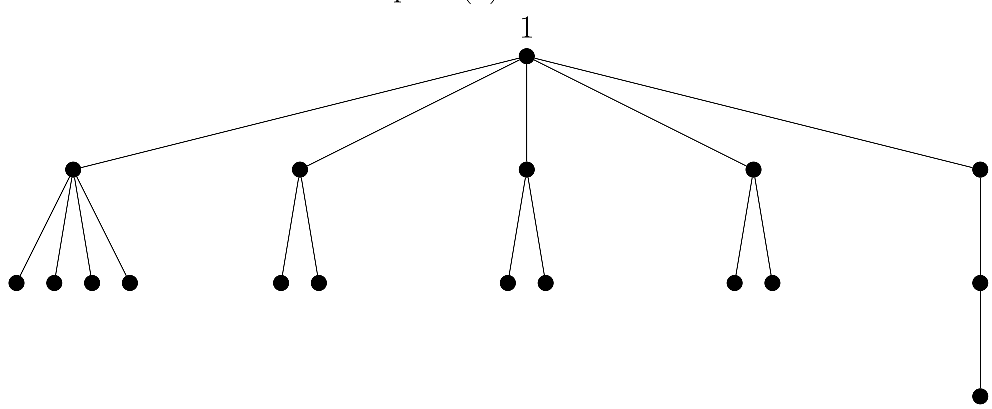
Frank Cline
Math 307
HW 7

6.3

8. Consider a tree with n vertices. It has exactly $n-1$ edges [Lemma 2], so the sum of its of the degrees of its vertices is $2n-2$.

   (a) A tree has two vertices of degree 5, three of degree 3, two of degree 2, and the rest of degree 1. How many vertices are in the graph?
   The tree has n vertices, so $n-7$ must be of degree 1.
   The sum of the degree of vertices is:
   $2|E(G)| = 2n-2 = 2(5)+3(3)+2(2)+(n-7)(1)$
   So $2n-2 = 10+9+4+n-7 \rightarrow 2n-2 = 16+n \rightarrow n = 18$.
   The tree has 18 vertices total.

   (b) Draw a tree as described in part (a).

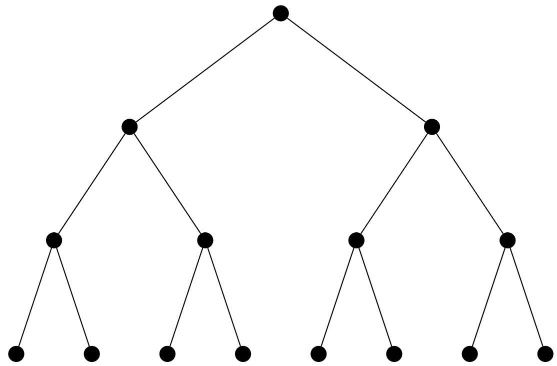


11. (a) Show that a forest with n vertices and m components has $n-m$ edges.
   Suppose the components have $n_1, n_2, n_3, ..., n_m$ vertices. The total vertices of the forest is then $n_1+n_2+n_3+...+n_m = n$. The ith component is a tree, so it has $n_i - 1$ edges by Theorem 4. The total amount of edges in the forest is then $(n_1-1)+(n_2-1)+(n_3-1)+...+(n_m-1) = n-m$.

6.4

6. (a) Draw each of the seven types of rooted trees of height 3 in which each node that is not a leaf has 2 children.

1



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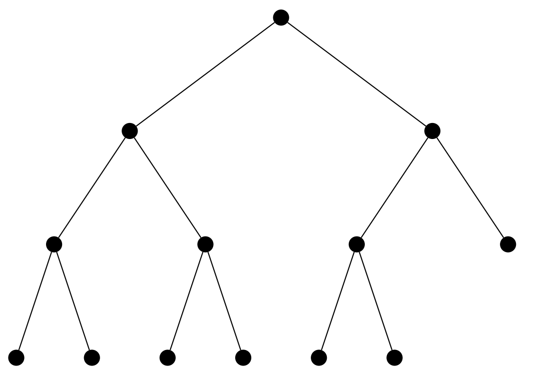

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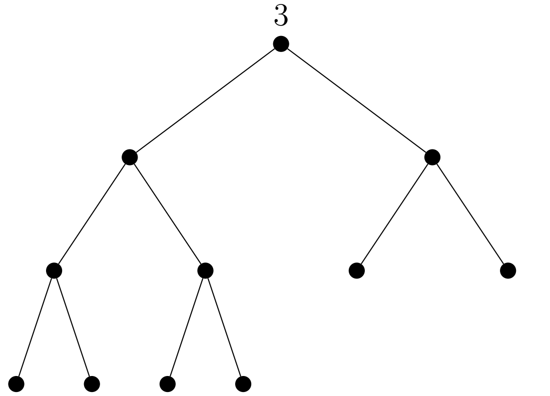

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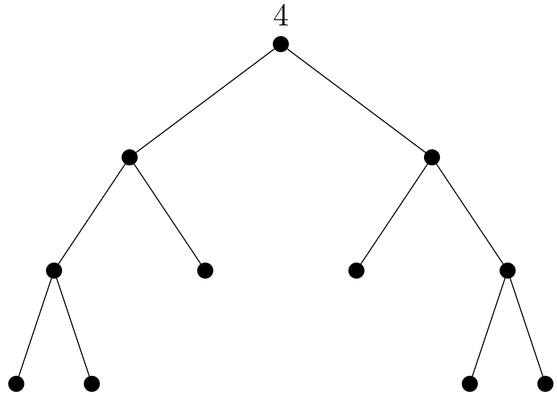

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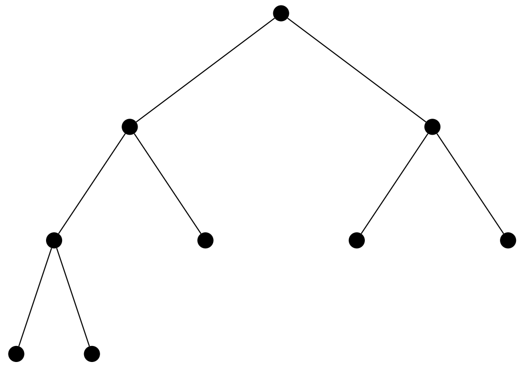

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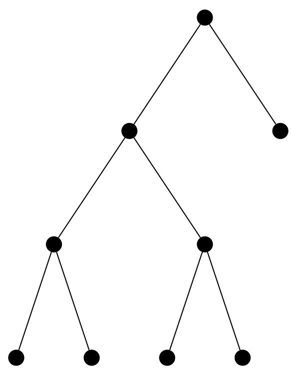

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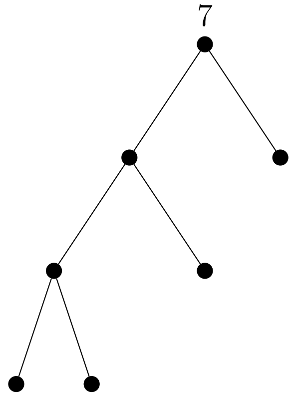
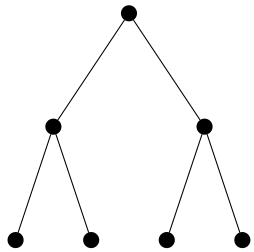


(b) How many different types of regular binary trees are there of height 3?

$\frac{2^{3+1}-1}{2-1} = \frac{15}{1} = 15$ so there are 15 different types of regular binary trees of height 3.

8. A 2-3 tree is a rooted tree such that each interior node, including the root if the height is 2 or more, has either two or three children and all paths from the root to the leaves habe the same length. There are seven different types of 2-3 trees of height 2. Draw one tree of each type.

1



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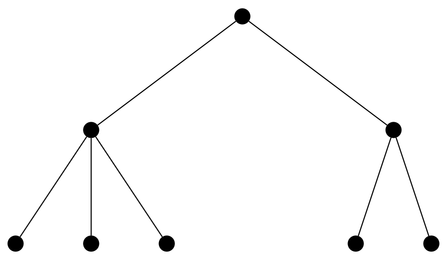

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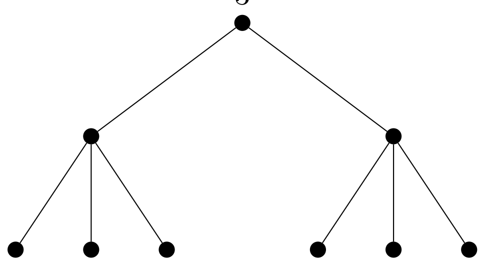

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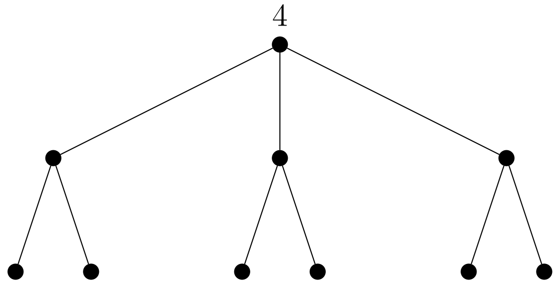

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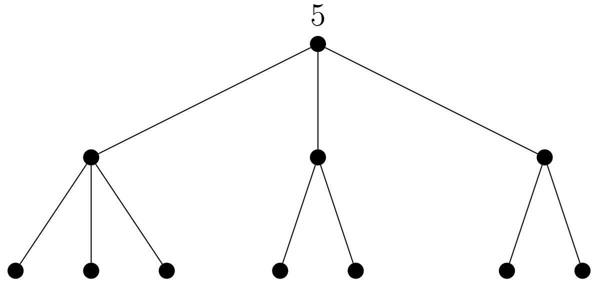

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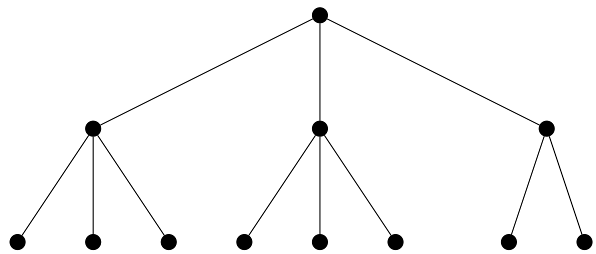

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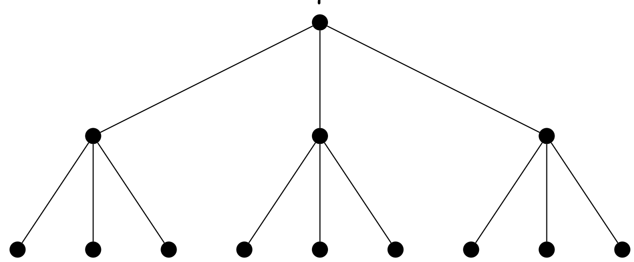


10. Consider a full binary tree T of height h.

    (a) How many leaves does T have?
    A full binary tree has 2^h leaves.

    (b) How many vertices does T have?
    A full m-ary tree has $\frac{m^{h+1}-1}{m-1}$ vertices, so
    a full binary tree has $\frac{2^{h+1}-1}{2-1} = 2^{h+1} - 1$ vertices.

12. Give some real-life examples of information storage that can be viewed as labeled trees.
    A labeled tree can be used to visualize a family tree, and a labeled tree can also be used to visualize the syntax of a language.

1. Additional Problem: Draw a tree with Prufer code (5,2,1,4,4,1,6,1,1), with no crossing

edges. Show some work.

$$(5,2,1,4,4,1,6,1,1) \to \qquad (\cancel{5},2,1,4,4,1,6,1,1) \to$$
$$\{1,2,3,4,5,6,7,8,9,10,11\} \to \qquad \{1,2,\cancel{3},4,5,6,7,8,9,10,11\} \to$$

$$(\cancel{5},\cancel{2},1,4,4,1,6,1,1) \to \qquad (\cancel{5},\cancel{2},\cancel{1},4,4,1,6,1,1) \to$$
$$\{1,2,\cancel{3},4,\cancel{5},6,7,8,9,10,11\} \to \qquad \{1,\cancel{2},\cancel{3},4,\cancel{5},6,7,8,9,10,11\} \to$$

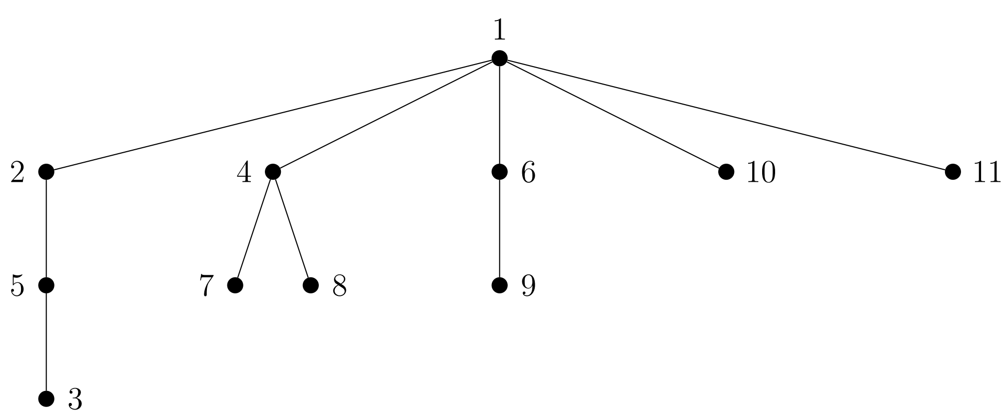
$$(\cancel{5},\cancel{2},\cancel{1},\cancel{4},4,1,6,1,1) \to \qquad (\cancel{5},\cancel{2},\cancel{1},\cancel{4},\cancel{4},1,6,1,1) \to$$
$$\{1,\cancel{2},\cancel{3},4,\cancel{5},6,\cancel{7},8,9,10,11\} \to \qquad \{1,\cancel{2},\cancel{3},4,\cancel{5},6,\cancel{7},\cancel{8},9,10,11\} \to$$

$$(\cancel{5},\cancel{2},\cancel{1},\cancel{4},\cancel{4},\cancel{1},6,1,1) \to \qquad (\cancel{5},\cancel{2},\cancel{1},\cancel{4},\cancel{4},\cancel{1},\cancel{6},1,1) \to$$
$$\{1,\cancel{2},\cancel{3},\cancel{4},\cancel{5},6,\cancel{7},\cancel{8},9,10,11\} \to \qquad \{1,\cancel{2},\cancel{3},\cancel{4},\cancel{5},6,\cancel{7},\cancel{8},\cancel{9},10,11\} \to$$

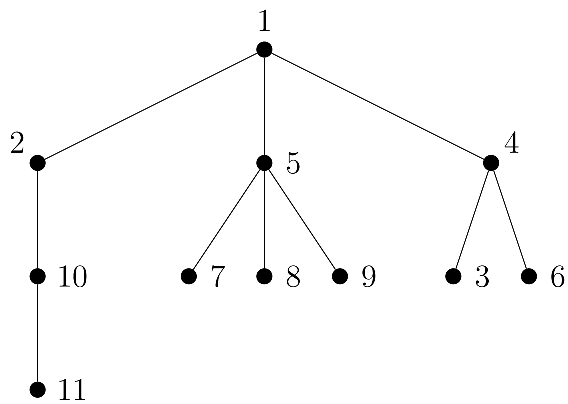
$$(\cancel{5},\cancel{2},\cancel{1},\cancel{4},\cancel{4},\cancel{1},\cancel{6},\cancel{1},1) \to \qquad (\cancel{5},\cancel{2},\cancel{1},\cancel{4},\cancel{4},\cancel{1},\cancel{6},\cancel{1},\cancel{1}) \to$$
$$\{1,\cancel{2},\cancel{3},\cancel{4},\cancel{5},\cancel{6},\cancel{7},\cancel{8},\cancel{9},10,11\} \to \qquad \{1,\cancel{2},\cancel{3},\cancel{4},\cancel{5},\cancel{6},\cancel{7},\cancel{8},\cancel{9},\cancel{10},11\} \to$$

$$(\cancel{5},\cancel{2},\cancel{1},\cancel{4},\cancel{4},\cancel{1},\cancel{6},\cancel{1},\cancel{1}) \to$$
$$\{\cancel{1},\cancel{2},\cancel{3},\cancel{4},\cancel{5},\cancel{6},\cancel{7},\cancel{8},\cancel{9},\cancel{10},\cancel{11}\} \to$$

$$\to \{(5,3),(2,5),(1,2),(4,7),(4,8),(1,4),(6,9),(1,6),(1,10),(1,11)\}$$



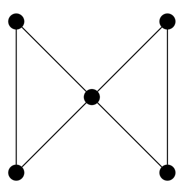
2. Additional Problem: Find the Prüfer code of the following tree. Show some work.



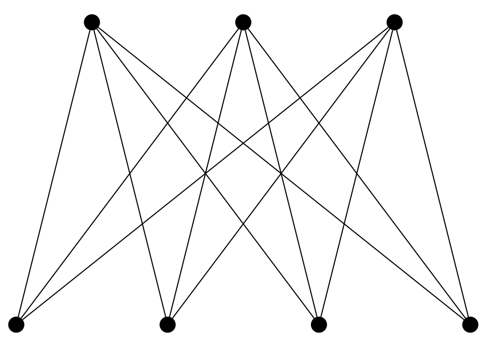
{Leaf (Neighbor),} $\rightarrow$ {3 (4), 6 (4), 4 (1), 7 (5), 8 (5), 9 (5), 5 (1), 1 (2), 2 (10)}
$\rightarrow (4, 4, 1, 5, 5, 5, 1, 2, 10)$

6.5

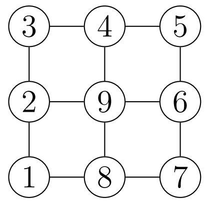
2. For each graph, give a Hamilton circuit or explain why none exists.

   (a) 

   A Hamilton circuit does not exist because the center vertex must always be passed twice in order to reach all the vertices.

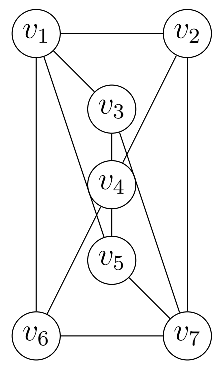
   (b) 

   Observe that the graph is bipartite because there exists 2 subsets V_1 and V_2 where every edge in the graph connects a vertex in V_1 to V_2. Suppose The vertices of V_1 are the top 3 and the vertices of V_2 are the bottom 4: $|V_1| = 3$ and $|V_2| = 4$. Notice that $|V_1| \neq |V_2|$ so there can't be a Hamilton circuit by Theorem 4.

(c) 

Observe that the graph is bipartite by splitting the the graph into subgraphs $V_1 = \{1, 3, 5, 7, 9\}$ and $V_2 = \{2, 4, 6, 8\}$. Notice that $|V_1| = 5$ and $|V_2| = 4$. $|V_1| \neq |V_2|$ so by Theorem 4 there can't be a Hamilton circuit.

4. Consider the graph shown.



(a) Is this a Hamiltonian graph?

Observe that the graph is bipartite by my answer to 4(c). Notice that $|V_1|$ from 4(c) equals 3 and $|V_2|$ from 4(c) equals 4. Since the graph is bipartite, and $|V_1| \neq |V_2|$ because $3 \neq 4$, there is no Hamilton circuit by Theorem 4. Thus, the this is not a Hamiltonian graph.

(b) Is this a complete graph?

This is not a complete graph because not all of the vertices are connected by a unique edge. For example, v_1 is not connected to v_7.

(c) Is this a bipartite graph?

Observe that by splitting the graph into two disjoint sets V_1 and V_2 where $V_1 = \{v_1, v_4, v_7\}$ and $V_2 = \{v_2, v_3, v_5, v_6\}$, we get two sets where every edge in the graph connects a vertex in V_1 to V_2. Thus, by definition, the graph is bipartite.

(d) Is this a complete bipartite graph?

Notice that each vertex in the set V_1 from 4(c) has an edge connecting to each of the vertices of V_2 from 4(c), and each vertex in V_2 has an edge connecting it to each of the vertices in V_1. Thus, the graph is a complete bipartite graph.

8. Give two examples of Gray codes of length 3 that are not equivalent to:

$$000, 001, 011, 111, 101, 100, 110, 010$$

(a) 111, 110, 100, 000, 010, 011, 001, 101

(b) 000, 100, 101, 111, 110, 010, 011, 001

4.6

8. Recursively define $a_0 = 1, a_1 = 2,$ and $a_n = \frac{a_{n-1}^2 - 1}{a_{n-2}}$ for $n \geq 2$.

   (a) Calculate the first few terms of the sequence.
   $a_2 = \frac{a_1^2 - 1}{a_0} = \frac{4-1}{1} = 3$
   $a_3 = \frac{a_2^2 - 1}{a_1} = \frac{9-1}{2} = 4$

   (b) Using part (a), guess a general formula for a_n.
   $a_n = n + 1$

   (c) Prove the guess in part (b).

   - (Statement)
     $a_n = \frac{a_{n-1}^2 - 1}{a_{n-2}} = n + 1$ when $n \geq 2$.
   - (Base)
     When $n = 2$, $a_2 = \frac{a_1^2 - 1}{a_0} = \frac{4-1}{1} = 3$ which equals $n + 1$ so our base case holds true.
   - (Induction Hypothesis)
     $a_{n-1} = (n - 1) + 1 = n$ for $n \geq 2$.
     $a_{n-2} = (n - 2) + 1 = n - 1$ for $n \geq 2$.
   - (Inductive Step)

$$\begin{aligned}
a_n &= \frac{a_{n-1}^2 - 1}{a_{n-2}} && (1) \\
&= \frac{n^2 - 1}{n - 1} \qquad \text{(By Induction Hypothesis)} && (2) \\
&= \frac{(n-1)(n+1)}{n-1} && (3) \\
&= n + 1 && (4)
\end{aligned}$$

Thus, by induction, $a_n = n + 1$ for $n \geq 2$.