Frank Cline
Math 307
HW 5

# 4.1

10. Show that the following are loop invariants for the loop:

    while $1 \leq m$ do
    $m := 2m$
    $n := 3n$

    (a) $n^2 \geq m^3$
    If $n^2 \geq m^3$ before entering the loop, then after $i$ iterations,
    $n^2 = (3^i n)^2 = 3^{2i}n^2$ and $m^3 = (2^i m)^3 = 2^{3i}m^3$. We know $n^2 \geq m^3$, so we need to show $3^{2i} \geq 2^{3i}$. By taking the $i$-th root on both sides we get $3^2 \geq 2^3$ which turns out to $9 \geq 8$ which is true, thus $n^2 \geq m^3$ is a loop invariant.

    (b) $2m^6 < n^4$
    If $2m^6 < n^4$ before entering the loop, then after $i$ iterations,
    $m = 2^i m$ and $n = 3^i n$. So $2m^6 < n^4$ becomes $2(2^i m)^6 < (3^i n)^4$ which equals $2^{6i}(2m^6) < (n^4)3^{4i}$. We know $2m^6 < n^4$, so we need to show $2^{6i} < 3^{4i}$. By taking the $i$-th root on both sides we get $2^6 < 3^4$ which turns out to $64 < 81$ which is true, thus $2m^6 < n^4$ is a loop invariant.

12. Consider the loop

    while $k \geq 1$ do
    $k := 2k$

    (a) Is $k^2 \equiv 1 \pmod 3$ a loop invariant? Explain.
    After $i$ iterations, $k = 2^i k$, so $k^2 = 4^i k^2$. We know $k^2 \equiv 1 \pmod 3$, so we need to show that $4^i \equiv 1 \pmod 3$. We also know that $4 \equiv 1 \pmod 3$, so $4^i \equiv 1^i \pmod 3$ which equals $4^i \equiv 1 \pmod 3$. Thus $k^2 \equiv 1 \pmod 3$ is a loop invariant.

    (b) Is $k^2 \equiv 1 \pmod 4$ a loop invariant? Explain.
    After $i$ iterations, $k = 2^i k$, so $k^2 = 4^i k^2$. We know $k^2 \equiv 1 \pmod 4$, so we need to show that $4^i \equiv 1 \pmod 4$. However, $4^i \equiv 0 \pmod 4$ because $4^i$ is divisible by 4. Thus $k^2 \equiv 1 \pmod 4$ is not a loop invariant.

22. (a) Is $5^k < k!$ an invariant of the following loop?

while $4 \leq k$ do
$k := k + 1$

No because, when $k = 4$, $5^k < k! = 5^4 < 4! = 625 < 24$ which is false.

(b) Can you conclude that $5^k < k!$ for all $k \geq 4$?
No because, when $k = 4$, $5^k < k! = 5^4 < 4! = 625 < 24$ which is false.

# 4.2

6. Prove $4 + 10 + 16 + ... + (6n - 2) = n(3n + 1)$ for all $n \in \mathbb{Z}^+$.

- (B) $n = 1$, $(6 \cdot 1 - 2) = 1(3 \cdot 1 + 1)$ so $4 = 4$ which is true.
- (I) For all $n \in \mathbb{Z}^+$ then $4 + 10 + 16 + ... + (6(n-1) - 2) = (n-1)(3(n-1) + 1)$
- Inductive Step:

$$\begin{aligned} n(3n+1) &= 4 + 10 + 16 + ... + (6(n-1) - 2) + (6n - 2) \\ &= (n-1)(3(n-1)+1) + (6n-2) && \text{(By Inductive Hypothesis)} \\ &= 3n^2 - 2n - 3n + 2 + 6n - 2 \\ &= 3n^2 + n \\ &= n(3n+1) \end{aligned}$$

- Thus $4 + 10 + 16 + ... + (6n - 2) = n(3n + 1)$ for all $n \in \mathbb{Z}^+$

8. Prove: $\frac{1}{1 \cdot 5} + \frac{1}{5 \cdot 9} + \frac{1}{9 \cdot 13} + ... + \frac{1}{(4n-3)(4n+1)} = \frac{n}{4n+1}$ for $n \in \mathbb{Z}^+$.

- (B) $n = 1$, $\frac{1}{(4 \cdot 1 - 3)(4 \cdot 1 + 1)} = \frac{n}{4n+1}$ so $\frac{1}{5} = \frac{1}{5}$ which is true.
- (I) For all $\frac{1}{1 \cdot 5} + \frac{1}{5 \cdot 9} + \frac{1}{9 \cdot 13} + ... + \frac{1}{(4(n-1)-3)(4(n-1)+1)} = \frac{n-1}{4(n-1)+1}$ for $n \in \mathbb{Z}^+$

- Inductive Step:

$$\begin{aligned}
\frac{n}{4n+1} &= \frac{1}{1\cdot 5} + \frac{1}{5\cdot 9} + \frac{1}{9\cdot 13} + ... + \frac{n-1}{(4(n-1)-3)(4(n-1)+1)} + \frac{1}{(4n-3)(4n+1)} \\
&= \frac{n-1}{4(n-1)+1} + \frac{1}{(4n-3)(4n+1)} \\
&= \frac{n-1}{4n-3} + \frac{1}{(4n-3)(4n+1)} \\
&= \frac{n-1(4n+1)+1}{(4n-3)(4n+1)} \\
&= \frac{4n^2+n-4n}{(4n-3)(4n+1)} \\
&= \frac{(n)(4n-3)}{(4n-3)(4n+1)} \\
&= \frac{n}{4n+1}
\end{aligned}$$

- Thus $\frac{1}{1\cdot 5} + \frac{1}{5\cdot 9} + \frac{1}{9\cdot 13} + ... + \frac{1}{(4n-3)(4n+1)} = \frac{n}{4n+1}$ for $n \in \mathbb{Z}^+$.

20. Prove $1^3 + 2^3 + ... + n^3 = (1+2+...+n)^2$.

- (B) $n = 2$, $1^3 + 2^3 = (1+2)^2$ so $9 = 9$ which is true.
- (I) $1^3 + 2^3 + ... + (n-1)^3 = (1+2+...+(n-1))^2$ so $\sum_{i=1}^{n-1} i^3 = (\sum_{i=1}^{n-1} i)^2$
- Inductive Step:

$$\begin{aligned}
1^3 + 2^3 + ... + (n-1)^3 + n^3 &= (1+2+...+(n-1)+n)^2 \\
\sum_{i=1}^{n-1} i^3 + n^3 &= (\sum_{i=1}^{n-1} i + n)^2 && \text{(By inductive hypothesis)} \\
\sum_{i=1}^{n-1} i^3 + n^3 &= (\sum_{i=1}^{n-1} i)^2 + 2n\sum_{i=1}^{n-1} i + n^2 \\
n^3 &= \frac{2n(n-1)(n-1+1)}{2} + n^2 && \left(\text{By } \sum_{i=1}^{n-1} i = \frac{(n-1)(n-1+1)}{2}\right) \\
n^3 &= n^3 - n^2 + n^2 \\
n^3 &= n^3
\end{aligned}$$

- Thus $1^3 + 2^3 + ... + n^3 = (1+2+...+n)^2$.

## 4.4

8. Let $\Sigma = \{a, b\}$ and let $s_n$ denote the number of words of length $n$ that do not contain the string $ab$.

   (a) Calculate $s_0, s_1, s_2, s_3$.

   $s_0 = 1$

   $s_1 = 2$

   $s_2 = 3$

   $s_3 = 4$

   (b) Find a formula for $s_n$ and prove it is correct.

   (B) $s_0 = 1$

   (R) $s_n = s_{n-1} + 1$ for $n \in \mathbb{N}$

10. Consider the sequence defined by

    (B) SEQ(0) = 1, SEQ(1) = 0

    (R) SEQ($n$) =SEQ($n-2$) for $n \geq 2$.

    (a) List the first few terms of this sequence.

    SEQ(2) = 1, SEQ(3) = 0, SEQ(4) = 1, SEQ(5) = 0

    (b) What is the set of values of this sequence?

    {0,1}

## 4.6

2. For $n \in \mathbb{Z}^+$, prove

   (b) $\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + ... + \frac{1}{n(n+1)} = \frac{n}{n+1}$

   - (B) $n = 1$, $\frac{1}{1 \cdot 2} = \frac{1}{1+1}$ so $\frac{1}{2} = \frac{1}{2}$ which is true.
   - (I) $\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + ... + \frac{1}{(n-1)(n-1+1)} = \frac{n-1}{n-1+1}$

- Inductive Step:

$$\begin{aligned}
\frac{n}{n+1} &= \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + ... + \frac{1}{(n-1)(n-1+1)} + \frac{1}{n(n+1)} \\
&= \frac{n-1}{n-1+1} + \frac{1}{n(n+1)} && \text{(By inductive hypothesis)} \\
&= \frac{(n+1)(n-1)}{n(n+1)} + \frac{1}{n(n+1)} \\
&= \frac{(n+1)(n-1)+1}{n(n+1)} \\
&= \frac{n^2}{n(n+1)} \\
&= \frac{n}{n+1}
\end{aligned}$$

- Thus $\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + ... + \frac{1}{n(n+1)} = \frac{n}{n+1}$.

6. Recursively define $a_0 = 1, a_1 = 2$ and $a_n = \frac{a_{n-1}^2}{a_{n-2}}$ for $n \geq 2$.

   (a) Calculate the first few terms of the sequence.
   $a_2 = \frac{2^2}{1} = 4$
   $a_3 = \frac{4^2}{2} = 8$
   $a_4 = \frac{8^2}{4} = 16$

   (b) Using part (a), guess the general formula for $a_n$.
   $a_n = 2^n$

   (c) Prove the guess in part (b).

   - (B) $n = 1$, $a_1 = 2 = 2^1$
   - (I) $a_{n-1} = 2^{n-1}$ and $a_{n-2} = 2^{n-2}$
   - Inductive Step:

$$\begin{aligned}
a_n &= \frac{a_{n-1}^2}{a_{n-2}} \\
&= \frac{(2^{n-1})^2}{2^{n-2}} \\
&= 2^{2n-2-n+2} \\
&= 2^n
\end{aligned}$$

   - Thus $a_n = 2^n$.

12. Recursively define:

$a_0 = 1,$

$a_1 = 3,$

$a_2 = 5$

$a_n = 3a_{n-2} + 2a_{n-3}$ for $n \geq 3$.

(a) Calculate $a_n$ for $n = 3, 4, 5, 6, 7$.

$a_3 = 11$

$a_4 = 21$

$a_5 = 43$

$a_6 = 85$

$a_7 = 171$

(b) Prove that $a_n > 2^n$ for $n \geq 1$.

- (B)

$$\begin{aligned} n &= 1, & a_1 > 2^1 = 3 > 2 = \text{True.} \\ n &= 2, & a_2 > 2^2 = 5 > 4 = \text{True.} \\ n &= 3, & a_3 > 2^3 = 11 > 8 = \text{True.} \end{aligned}$$

- (I) $a_{n-2} > 2^{n-2}$ and $a_{n-3} > 2^{n-3}$ for $n \geq 3$
- Inductive Step:

$$\begin{aligned} a_n &= 3a_{n-2} + 2a_{n-3} \\ &> 3(2^{n-2}) + 2(2^{n-3}) && \text{(By the inductive hypothesis)} \\ &= 3(2^{n-2}) + 2^{n-2} \\ &= 4\left(2^{n-2}\right) \\ &= 2^{n-2+2} \\ &= 2^n \end{aligned}$$

- Thus $a_n > 2^n$ for $n \geq 1$.

(c) Prove that $a_n < 2^{n+1}$ for $n \geq 1$.

- (B)

$$\begin{aligned} n &= 1, & a_1 &< 2^{1+1} = 3 < 4 = \text{True.} \\ n &= 2, & a_2 &< 2^{2+1} = 5 < 8 = \text{True.} \\ n &= 3, & a_3 &< 2^{3+1} = 11 < 16 = \text{True.} \end{aligned}$$

- (I) $a_{n-2} < 2^{n-2+1}$ and $a_{n-3} < 2^{n-3+1}$ for $n \geq 3$
- Inductive Step:

$$\begin{aligned} a_n &= 3a_{n-2} + 2a_{n-3} \\ &< 3(2^{n-2+1}) + 2(2^{n-3+1}) && \text{(By the inductive hypothesis)} \\ &= 3(2^{n-1}) + 2^{n-1} \\ &= 4\left(2^{n-1}\right) \\ &= 2^{n-1+2} \\ &= 2^{n+1} \end{aligned}$$

- Thus $a_n < 2^{n+1}$ for $n \geq 1$.

(d) Prove that $a_n = 2a_{n-1} + (-1)^{n-1}$ for $n \geq 1$.

- (B)

$$\begin{aligned} n &= 1, & a_1 &= 2a_0 + (-1)^0 = 2(1) + 1 = 3 \\ n &= 2, & a_2 &= 2a_1 + (-1)^1 = 2(3) + (-1) = 5 \\ n &= 3, & a_3 &= 2a_2 + (-1)^2 = 2(5) + (1) = 11 \end{aligned}$$

- (I) $a_{n-1} = 2a_{n-2} + (-1)^{n-2}$ and $a_{n-2} = 2a_{n-3} + (-1)^{n-3}$ for $n \geq 3$
- Inductive Step:

$$\begin{aligned} a_n &= 3a_{n-2} + 2a_{n-3} \\ &= 3a_{n-2} + (a_{n-2} - (-1)^{n-3}) && \text{(By inductive hypothesis)} \\ &= 4a_{n-2} - (-1)^{n-3} \\ &= 2(a_{n-1} - (-1)^{n-2}) - (-1)^{n-3} && \text{(By inductive hypothesis)} \\ &= 2a_{n-1} - 2(-1)^{n-2} + (-1)^{n-2} \\ &= 2a_{n-1} - (-1)^{n-2} \\ &= 2a_{n-1} + (-1)^{n-1} \end{aligned}$$

- Thus $a_n = 2a_{n-1} + (-1)^{n-1}$ for $n \geq 1$.