Frank Cline
Math 307
HW 12

5.3

2. Let $S = \{100, 101, 102, ..., 999\}$ so that $|S| = 900$.

   (a) How many numbers in S have at least one digit that is a 3 or a 7?
   First, find the amount of numbers that don't have a 3 or a 7 in them. That amount is equal to $\binom{7}{1}$ for the first digit because it can be numbers 1-9 excluding 3 and 7. The second two digits can be numbers 0-9 exculding 3 and 7, so that is $\binom{8}{1}$. Thus, the equation for numbers that don't have a single digit as 3 or 7 is $\binom{7}{1}\binom{8}{1}\binom{8}{1} = 448$. The amount of numbers that have at least one 3 or 7 is equal to the total amount of numbers minus the amount of numbers that don't have a 3 or a 7 in them which equals $900 - 448 = 454$.

   (b) How many numbers in S have at least one digit that is a 3 and at least one digit that is a 7?
   To find the amount of numbers that have at least one digit that is 3 and one digit that is 7, first choose 2 of the 3 digits to be 3 and 7 where order matters $P(3, 2)$. Then choose which Then pick the last number's digit which can be between 0-9 $\binom{10}{1}$. The total is equal to $P(3, 2) \times \binom{10}{1} = (6)(10) = 60$.

4. An investor has 7 \$1000 bills to distribute among 3 mutual friends.

   (a) In how many ways can she invest her money?
   $3^7 = 2187$.

   (b) In how many ways can she invest her money if each fund must get at least \$1000.
   $3^7 - (2^7\binom{3}{2} - 1^7\binom{2}{1}\binom{3}{2}) - 1^7\binom{3}{1} = 1806$.

9. Use the binomial theorem to expand the following:

   (c) $(3x + 1)^4$

$$
\begin{aligned}
(3x+1)^4 &= \sum_{r=0}^{4}\binom{4}{r}(3x)^r(1)^{4-r} \\
&= \binom{4}{0}(3x)^0(1)^{4-0} + \binom{4}{1}(3x)^1(1)^{4-1} + \binom{4}{2}(3x)^2(1)^{4-2} \\
&+ \binom{4}{3}(3x)^3(1)^{4-3} + \binom{4}{4}(3x)^4(1)^{4-4} \\
&= 1 + (4)(3x)^1 + (6)(3x)^2 + (6)(3x)^3 + (4)(3x)^4 \\
&= 1 + 12x + 54x^2 + 162x^3 + 324x^4
\end{aligned}
$$

(d) $(x+2)^5$

$$
\begin{aligned}
(x+2)^5 &= \sum_{r=0}^{5}\binom{5}{r}(x)^r(2)^{5-r} \\
&= \binom{5}{0}(x)^0(2)^{5-0} + \binom{5}{1}(x)^1(2)^{5-1} + \binom{5}{2}(x)^2(2)^{5-2} \\
&+ \binom{5}{3}(x)^3(2)^{5-3} + \binom{5}{4}(x)^4(2)^{5-4} + \binom{5}{5}(x)^5(2)^{5-5} \\
&= (2)^5 + 5x(2)^4 + 10x^2(2)^3 + 10x^3(2)^2 + 5x^4(2)^1 + x^5 \\
&= 32 + 80x + 80x^2 + 40x^3 + 10x^4 + x^5
\end{aligned}
$$

11. Prove that $2^n = \sum_{r=0}^{n}\binom{n}{r}$

(a) by setting $a = b = 1$ in the binomial theorem.

$$
\begin{aligned}
(2)^n &= (1+1)^n \\
(1+1)^n &= \sum_{r=0}^{n}\binom{n}{r}(1)^r(1)^{n-r} && \text{(by the binomial theorem)} \\
&= \sum_{r=0}^{n}\binom{n}{r}(1)(1) && \text{(because 1 to the power of anything is still 1)} \\
&= \sum_{r=0}^{n}\binom{n}{r}
\end{aligned}
$$

Thus $2^n = \sum_{r=0}^{n}\binom{n}{r}$.

12. Prove that $\sum_{r=0}^{n}\binom{n}{r}2^r = 3^n$ for all $n \in \mathbb{Z}^+$.

By setting a = 2 and b = 1 in the binomial theorem we get:

$$\begin{aligned}
(3)^n &= (2+1)^n \\
(2+1)^n &= \sum_{r=0}^{n} \binom{n}{r} (2)^r (1)^{n-r} && \text{(by the binomial theorem)} \\
&= \sum_{r=0}^{n} \binom{n}{r} (2)^r (1) && \text{(because 1 to the power of anything is still 1)} \\
&= \sum_{r=0}^{n} \binom{n}{r} (2)^r
\end{aligned}$$

Thus $3^n = \sum_{r=0}^{n} \binom{n}{r} 2^r$ for all $n \in \mathbb{Z}^+$.

13. (b) Prove $\sum_{k=m}^{n} \binom{k}{m} = \binom{n+1}{m+1}$ by induction on n for $n \geq m$.

- (Base Case):
  The base case is when $n = m$. $\sum_{k=m}^{n} \binom{k}{m} = \binom{n}{m} = \binom{m}{m} = 1$. $\binom{n+1}{m+1} = \binom{m+1}{m+1} = 1$. $1 = 1$ so our base case holds.
- (Ind Hyp):
  $\sum_{k=m}^{n-1} \binom{k}{m} = \binom{n-1+1}{m+1}$ for some $n > m$.
- (Ind Step):

$$\begin{aligned}
\sum_{k=m}^{n} \binom{k}{m} &= \sum_{k=m}^{n-1} \binom{k}{m} + \binom{n}{m} && (1) \\
&= \binom{n-1+1}{m+1} + \binom{n}{m} && (2) \\
&= \binom{n}{m+1} + \binom{n}{m} \qquad \text{(By Ind Hyp)} && (3) \\
&= \binom{n+1}{m+1} && (4)
\end{aligned}$$

Thus by induction, $\sum_{k=m}^{n} \binom{k}{m} = \binom{n+1}{m+1}$

5.5

2. (a) A sack contains 50 marbles of 4 different colors. Explain why there are at least 13 marbles of the same color.

4 different colors $\rightarrow$ 4 boxes. 50 marbles $\rightarrow$ 50 balls. $(4)(12) = 48 < 50$, so there are more than 12 marbles (≥ 13) of the same color by the pigeonhole principle.

(b) If exactly 8 marbles are red, explain why there are at least 14 marbles of the same color.
If 8 marbles are red, that leaves 42 marbles of 3 different colors.
3 different colors $\rightarrow$ boxes. 42 marbles $\rightarrow$ balls. $(3)(13) = 39 < 42$, so there must be more than 13 marbles (≥ 14) that are the same color by the pigeonhole principle.

4. (a) Let B be a 12-element subset of $\{1,2,3,4,5,6\} \times \{1,2,3,4,5,6\}$. Show that B containts two different ordered pairs, the sums of whose entries are equal.
$(2,3) \in B$ and $(1,4) \in B$. $2+3=5$ and $1+4=5$, so their sums are equal.

(b) How many times can a pair of dice be tossed without obtaining the same sum twice?
There 11 different sums (2,3,4,5,6,7,8,9,10,11,12) $\rightarrow$ boxes. So a pair of dice can be tossed $\boxed{11}$ times before obtaining the same sum twice.

6. Let S be a 3-element set of integers. Show that S has two different nonempty subsets such that the sums of the numbers in each of the subsets are congruent mod 6.
Congruent mod 6 has 6 equivalence classes (0,1,2,3,4,5) $\rightarrow$ boxes. A 3-element set of integers has $|P(S)|$ subets. $|P(S)| = 2^{|S|} = 2^3 = 8 \rightarrow$ balls. There are 7 balls (8 minus the empty set) and only 6 boxes, so there must be at least one box with 2 or more balls. This means that there must be at least 2 different nonempty subsets of S such that their sums are congruent mod 6.

13. Let $n_1, n_2,$ and n_3 be distinct positive integers. Show that at least one of $n_1, n_2, n_3, n_1 + n_2, n_2 + n_3, n_1 + n_2 + n_3$ is divisible by 3.

Supplementary Exercises

11. A box contains tickets of 4 colors: red, blue, yellow, and green. Each ticket has a number from {0,1,...,9} written on it.

(a) What is the largest number of tickets the box can contain without having at least 21 tickets of the same color?
4 colors $\rightarrow$ boxes. $21 - 1 = 20 \rightarrow$ balls. $(4)(20) = 80$ so the box can contain $\boxed{80}$ tickets without there being a box with at least 21 tickets of the same color by the pigeonhole principle.

(b) What is the smallest number of tickets the box must contain to be sure that it contains at least two tickets of the same color with the same number on them?
4 colors of 10 different numbers is $(4)(10) = 40 \rightarrow$ boxes. $2 - 1 = 1 \rightarrow$ balls. $(40)(1) = 40$ so there must be at least $\boxed{41}$ tickets for there to be at least 2 tickets with the same color and number on them.

39. Let $D = \{1, 2, 3, 4, 5\}$ and let $L = \{a, b, c\}$.

(a) How many functions are there from the set D to the set L?
There are 5 elements in D each of which can map to one of 3 elements in L. Thus, the total number of functions is equal to $\binom{3}{1}\binom{3}{1}\binom{3}{1}\binom{3}{1}\binom{3}{1} = \boxed{243}$ functions.

(b) How many of these functions map D onto L?
For D to map ont L, all of the elements in L must be mapped to. To do so, first choose 3 of the 5 elements in D; these elements will map to the 3 elements in L. Then the other 2 elements in D can be mapped to any of the elements in L. Thus, the equation is $\binom{5}{3}\binom{3}{1}\binom{3}{1} = \boxed{90}$ of the functions are onto.