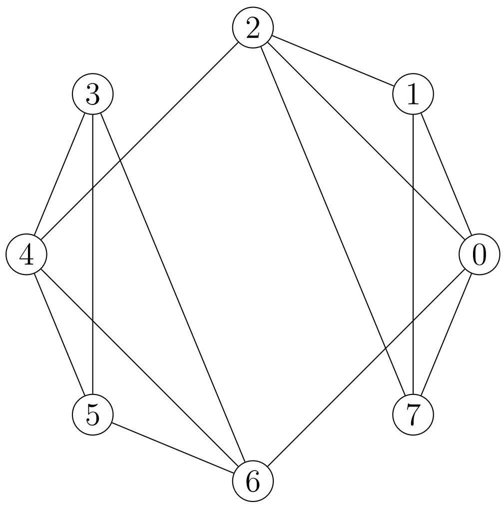
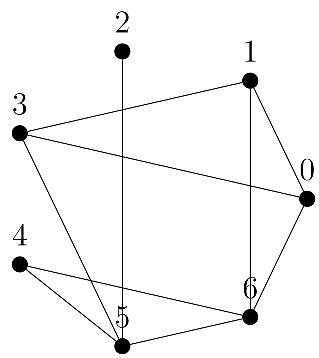
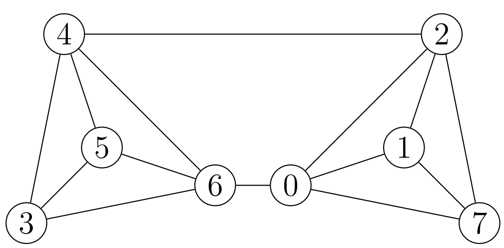
Frank Cline
Math 307
HW 9

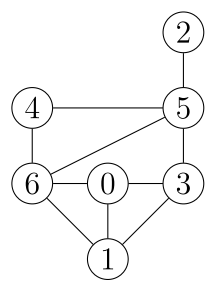
Problems 1-4

1. Find a plane drawing of the following graphs. Label your graph to demonstrate the isomorphism. For the left-hand graph, can you find a plane embedding with mirror symmetry?









2. (a) Suppose that G is a triangle-free, simple, connected plane graph with at least 3 vertices. Show that $E \leq 2V - 4$, by counting the edges around all the faces and using Euler's Theorem.

   Since G is a connected plane graph, we know $V - E + F = 2$. That means $F = 2 - V + E$.
   When $V = 1, E = 0$. $0 \not\leq 2(1) - 4$.
   When $V = 2, E = 2$. $2 \not\leq 2(2) - 4$
   When $V = 3, E \leq 2$. $2 \leq 2(3) - 4$

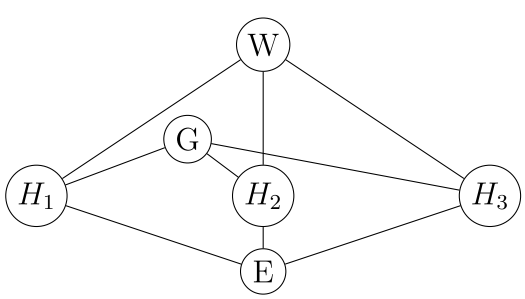
If $V \geq 3$, then

$$\begin{aligned}
2E &= 4(F_4) + 5(F_5) + 6(F_6) + ... + n(F_n) \\
&\geq 4(F_4) + 4(F_5) + 4(F_6) + ... + 4(F_n) \\
&= 4F \\
&= 4(2 - V + E) \qquad \text{(By Euler's Theorem)} \\
&= 8 - 4V + 4E \\
2E &\geq 8 - 4V + 4E \\
4V - 8 &\geq 2E \\
E &\leq 2V - 4
\end{aligned}$$

Thus, $E \leq 2V - 4$ when $V \geq 3$ and graph G is a triangle-free, simple, connected plane graph.

(b) Use the previous result to show that $K_{3,3}$ is not planar.
$K_{3,3}$ has the properties: $V = 6, E = 9$, and it is also a simple, connected graph with no triangle faces, so $E \leq 2V - 4$ which is $9 \leq 2(6) - 4$ and $9 \not\leq 8$.

(c) Suppose that there are three houses on a road. Each house needs to be connected to gas, water, and electricity, via an underground connection. Is there a way to make all nine connections without any of the lines crossing each other? (Having to dig on one plane is much cheaper than having to route some of the connections below the others!) If not, what is the fewest number of crossings possible?
The graph would be isomorphic to the $K_{3,3}$ graph because all each of the 3 houses connects to each of the 3 services, so using the previous question, it it not possible to make all nine connections without any of the lines crossing. The smallest number of crossings necessary is one, as illustrated below.



3. (a) Suppose that G is a connected simple plane graph, where every vertex is degree 3, that only has pentagons and hexagons as faces. Show that G must have exactly 12 pentagons.

Let F_5 and F_6 be the number of faces that are pentagons and hexagons. We know:

$$\begin{aligned}
2E &= 5(F_5) + 6(F_6) = \text{sum of deg(v) for } v \in (V(G)) = 3V \\
E &= \frac{5}{2}(F_5) + 3(F_6) \\
V &= \frac{5}{3}(F_5) + 2(F_6) \\
F &= F_5 + F_6 \\
2 &= V - E + F \\
&= \left(\frac{5}{3}(F_5) + 2(F_6)\right) - \left(\frac{5}{2}(F_5) + 3(F_6)\right) + (F_5 + F_6) = \frac{1}{6}(F_5) \\
2(6) &= 12 = F_5
\end{aligned}$$

Thus, we know there must be exactly 12 faces that are pentagons.

(b) In the above situation, what is the smallest graph G with this property? (Hint: what's the fewest number of hexagons possible in such a situation?)
The smallest graph would be a graph with no hexagons, so $F_6 = 0$.

$$\begin{aligned}
2E &= 3V = 5(F_5) + 6(F_6) \\
2E &= 3V = 5(12) = 60 \\
F &= F_5 + F_6 = 12 + 0 = 12 \\
E &= 60 \div 2 = 30 \\
V &= 60 \div 3 = 20
\end{aligned}$$

Thus the smallest graph G has 20 vertices, 30 edges, and 12 faces.

4. Go find out about a planarity testing algorithm and describe how it works.
The algorithm I found is the Hopcroft algorithm. The way it works is a tree is produced from a Depth First Search. It uses the edges in the DFS that form a cycle and then it uses the subgraphs that connect to the cycle. Each subgraph is then tested for planarity, and if it is plane, the plane drawing is added to the plane graph. This continues until the entire graph has been added to the plane drawing.