Frank Cline
Math 307
HW 4

3.4

1. Which of the following describe equivalence relations? For those that are not equivalence relations, specify which of $(R), (S), (T)$ fail and illustrate the failures with examples.

   (a) $L_1 || L_2$ for straight lines in the plane if L_1 and L_2 are the same or are parallel.
   Yes.

   (b) $L_1 \perp L_2$ for straight lines in the plane if L_1 and L_2 are perpendicular.
   No.
   Not (R) because a straight line can never be perpendicular to itself. $L_1 \not\perp L_1$.
   Not (T) because if $L_1 \perp L_2$ and $L_2 \perp L_3$, then $L_1 \not\perp L_3$. $L_1 || L_3$.

   (c) $p_1 \sim p_2$ for Americans if p_1 and p_2 live in the same state.
   No. There are some Americans that don't live in a state like in Washington D.C.

   (d) $p_1 \approx p_2$ for Americans if p_1 and p_2 live in the same state or in neighboring states.
   No. There are some Americans that don't live in a state like in Washington D.C.

   (e) $p_1 \approx p_2$ for people if p_1 and p_2 have a parent in common.
   No.
   Not (T) because if $p_1 \approx p_2$ and $p_2 \approx p_3$, it doesn't mean $p_1 \approx p_3$ because they could have different parents.

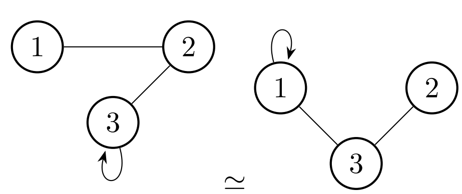
   (f) $p_1 \cong p_2$ for people if p_1 and p_2 have the same mother.
   Yes.

2. For each example of an equivalence relation in Exercise 1, describe the members of some equivalence class.

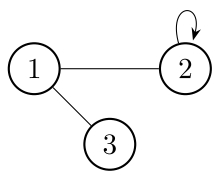
   (a) $L_1 || L_2$ for straight lines in the plane if L_1 and L_2 are the same or are parallel.
   If we describe the lines as vectors, then $[<1,1>] = \{<2,2>, <3,3>, <4,4>, ...\}$.

   (f) $p_1 \cong p_2$ for people if p_1 and p_2 have the same mother.
   The equivalence class of Michelle Obama is [Michelle] = {Sasha, Malia}.

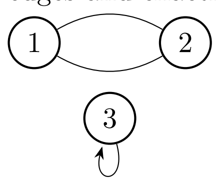
5. If G and H are both graphs with vertex set $\{1, 2, ..., n\}$, we say that G is isomorphic to H, and write $G \cong H$, in case there is a way to label the vertices of G so that it becomes H.



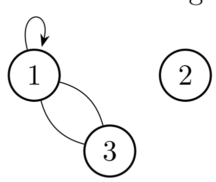
(a) Give a picture of another graph isomorphic to these two.



(b) Find a graph with vertex set $\{1, 2, 3\}$ that is not isomorphic to the graphs yet has three edges and exactly one is a loop.



(c) Find another example as in part(b) that isn't isomorphic to the answer of part(b) and the other two graphs.



(d) Show that $\simeq$ is an equivalence relation on the set of all graphs with the vertex set $\{1, 2, ..., n\}$.
$\simeq$ is reflexive, symmetric, and transitive. Thus the relation is an equivalence relation.

8. (a) For $m, n \in \mathbb{Z}$, define $m \sim n$ in case $m - n$ is odd. Is the relation reflexive? symmetric? transitive? Is it an equivalence relation?
(R) No because if $n = m$, then $m - n$ is 0, and zero is even, so $m \not\sim n$ thus the relation is not reflexive.
(S) Yes because $\forall m, n \in \mathbb{Z}$ if $m - n = odd$, then $n - m = -odd$ thus the relation is symmetric.
(T) No. Suppose $m = 6, n = 3$ and $\exists a \in \mathbb{Z} : a = 2$ then $m - n = 3$ and $n - a = 1$, so $m \sim n$ and $n \sim a$. However, $m - a = 4$ so $m \not\sim a$ thus the relation is not transitive.
It is not an equivalence relation because the relation is not reflexive or transitive.

(b) For a and b in $\mathbb{R}$, define $a \sim b$ in case $|a - b| \leq 1$. One could say that $a \sim b$ in case a and b are close enough or approximately equal. Answer the question in part (a).
(R) Yes because if $a = b$ then $a - b = 0 \leq 1$ so $a \sim b$ thus the relation is reflexive.
(S) Yes because $|a - b| = |b - a|$ so if $a \sim b$ then $b \sim a$ thus the relation is symmetric
(T) No. Suppose $a = 10, b = 9$, and $\exists c \in \mathbb{R} : c = 8$. $a - b = 1$ and $b - c = 1$ so $a \sim b$ and $b \sim c$. However, $a - c = 2$ thus the relation is not transitive because $a \sim b \wedge b \sim c$ but $a \not\sim c$.
It is not an equivalence relation because the relation is not transitive.

17. (a) Verify that the relation $\cong$ defined in Example 5b (the reachable Relation R on $V(G)$ by $(v, w) \in R$) is an equivalence relation on $V(G)$.
    $\cong$ is reflexive, symmetric, and transitive. Thus the relation is an equivalence relation.
    (b) Given a vertex v in $V(G)$, describe in words the equivalence class containing v.
    The equivalence class containing v is an equivalence class.

3.5

2. Find $nDIVm$ and $nMODm$ for the following values of n and m.

    (a) $n = 20, m = 3$
    $\lfloor 20/3 \rfloor = 6$
    $20 \mod 3 = 2$

    (b) $n = 20, m = 4$
    $\lfloor 20/4 \rfloor = 5$
    $20 \mod 4 = 0$

    (c) $n = -20, m = 3$
    $\lfloor -20/3 \rfloor = -7$
    $-20 \mod 3 = 1$

    (d) $n = -20, m = 4$
    $\lfloor -20/4 \rfloor = -5$
    $-20 \mod 4 = 0$

    (e) $n = 371, 246$, $m = 65$
    $\lfloor 371, 246/65 \rfloor = 5711$
    $371, 246 \mod 65 = 31$

    (f) $n = -371, 246$, $m = 65$
    $\lfloor -371, 246/65 \rfloor = -5712$
    $-371, 246 \mod 65 = 34$

4. (a) List all equvalence classes of $\mathbb{Z}$ for the equivalence relation congruence mod 4.
    $[0]_4$, $[1]_4$, $[2]_4$, $[3]_4$
    (b) How many different equivalence classes of $\mathbb{Z}$ are there with respect to congruence mod 73.
    There are 73 different equivalence classes with respect to congruence mod 73.

5. For each of the following integers m, find the unique integer r in $\{0, 1, 2, 3\}$ such that $m \equiv r \mod(4)$.

    (a) 17
    $r = 1$

(b) 7
   $r = 3$

(c) -7
   $r = 1$

(d) 2
   $r = 2$

(e) -88
   $r = 0$

8. (a) List the elements in the sets A_0, A_1, A_2 defined by
   $A_k = \{m \in \mathbb{Z} : -10 \leq m \leq 10 \text{ and } m \equiv k \bmod(3)\}$.
   $A_0 = \{-9, -6, -3, 0, 3, 6, 9\}$
   $A_1 = \{-8, -5, -2, 1, 4, 7, 10\}$
   $A_2 = \{-10, -7, -4, -1, 2, 5, 8\}$

   (b) What is A_3, A_4, A_5?
   $A_3, A_4, A_5 = \emptyset$

12. For $m, n \in \mathbb{N}$, define $m \sim n$ if $m^2 - n^2$ is a multiple of 3.

   (a) Show that $\sim$ is an equivalence relation on $\mathbb{N}$.
   (R) if $m = n$ then $m^2 - n^2 = 0$ and 0 is a multiple of 3 because $3 * 0 = 0$. Thus the relation is reflexive.
   (S) $m^2 - n^2 = -(n^2 - m^2)$ and if $m^2 - n^2$ is a multiple of 3, then $-(m^2 - n^2)$ is a multiple of 3 as well. Thus the relation is symmetric.
   (T) if $m^2 - n^2 = 3k$ and $n^2 - a^2 = 3j$ where $a \in \mathbb{N}$ and $k, j \in \mathbb{Z}$ then $n^2 = a^2 + 3j$, so $m^2 - (a^2 + 3j) = 3k$ which equals $m^2 - a^2 = 3(k + j)$ for some integers k and j. So $(m \sim n \wedge n \sim a) \rightarrow m \sim a$ thus the relation is transitive.
   Since the relation is reflexive, symmetric, and transitive, it's an eqivalence relation.

   (b) List four elements in the equivalence class [0].
   (5,4), (4,1), (3,0), (6,3)

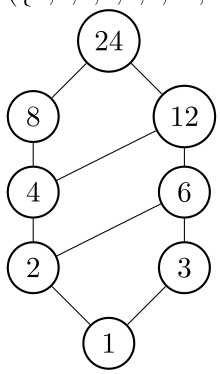
   (c) List four elements in the equivalence class [1].
   (1,3), (4,0), (5,0), (7,0)

   (d) Do you think there are any more equivalence classes?
   Yes, there is the equivalence class [2].

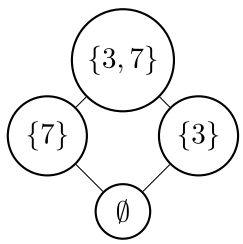
11.1

1. Draw Hasse diagrams for the following posets.

(a) $(\{1,2,3,4,6,8,12,24\}, |)$ where $m \mid n$ means m divides n.



(b) The set of subsets of $\{3,7\}$ with $\subseteq$ as a partial order.



15. Define the relations $<, \leq, \preceq$ on the plane $\mathbb{R} \times \mathbb{R}$ by

$(x, y) < (z, w)$ if $x^2 + y^2 < z^2 + w^2$,

$(x, y) \leq (z, w)$ if $(x, y) < (z, w)$ or $(x, y) = (z, w)$,

$(x, y) \preceq (z, w)$ if $x^2 + y^2 \leq z^2 + w^2$.

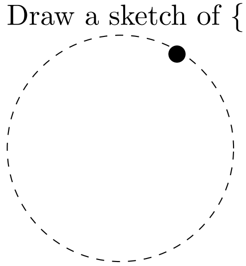
(a) Which of these relations are partial orders? Explain.

$\leq$ is a partial order.

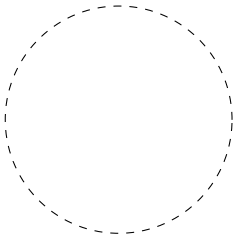
$<$ is not a partial order because $\forall n \in \mathbb{R}, n \not< n$ thus it is not reflexive.

$\preceq$ is not a partial order because if $(x, y) = (0, 1)$ and $(z, w) = (1, 0)$ then $0^2 + 1^2 = 1^2 + 0^2$ so $x^2 + y^2 = z^2 + w^2$ so (x, y) is related to (z, w) and (z, w) is related to (x, y) but $(x, y) \neq (z, w)$ thus $\preceq$ is not antisymmetric.

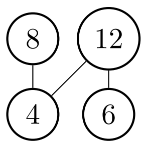
(c) Draw a sketch of $\{(x, y) : (x, y) \leq (3, 4)\}$.



(d) Draw a sketch of $\{(x, y) : (x, y) \preceq (3, 4)\}$.



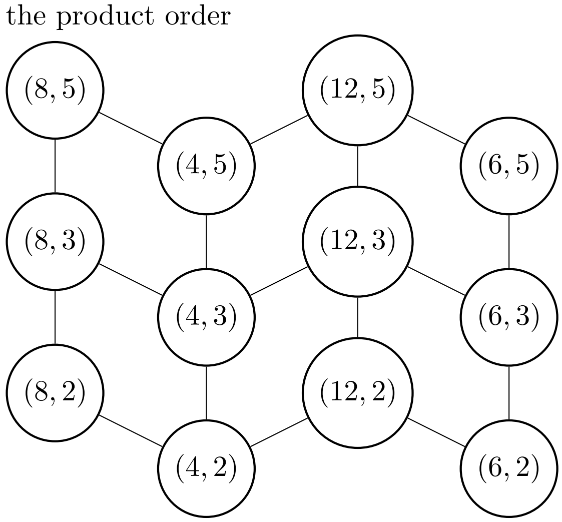
1. Draw a Hasse diagram for $S = \{4, 6, 8, 12\}$ with the partial order $|$.



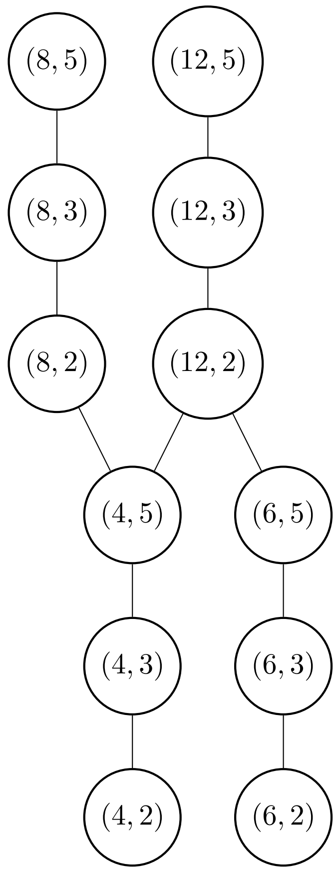
11.2

1. Draw a Hasse diagram for the given order on $S \times T$, where $S = \{4, 6, 8, 12\}$ with the partial order $|$, and $T = \{2, 3, 5\}$ with the partial order $\leq$.

   (a) the product order

   (b) the lexicographic order


(8, 5)
(12, 5)
(8, 3)
(12, 3)
(8, 2)
(12, 2)
(4, 5)
(6, 5)
(4, 3)
(6, 3)
(4, 2)
(6, 2)