

## 4.1

10. Show that the following are loop invariants for the loop:

while  $1 \leq m$  do

$m := 2m$

$n := 3n$

- (a)  $n^2 \geq m^3$

If  $n^2 \geq m^3$  before entering the loop, then after  $i$  iterations,

$n^2 = (3^i n)^2 = 3^{2i} n^2$  and  $m^3 = (2^i m)^3 = 2^{3i} m^3$ . We know  $n^2 \geq m^3$ , so we need to show  $3^{2i} \geq 2^{3i}$ . By taking the  $i$ -th root on both sides we get  $3^2 \geq 2^3$  which turns out to  $9 \geq 8$  which is true, thus  $n^2 \geq m^3$  is a loop invariant.

- (b)  $2m^6 < n^4$

If  $2m^6 < n^4$  before entering the loop, then after  $i$  iterations,

$m = 2^i m$  and  $n = 3^i n$ . So  $2m^6 < n^4$  becomes  $2(2^i m)^6 < (3^i n)^4$  which equals  $2^{6i}(2m^6) < (n^4)3^{4i}$ . We know  $2m^6 < n^4$ , so we need to show  $2^{6i} < 3^{4i}$ . By taking the  $i$ -th root on both sides we get  $2^6 < 3^4$  which turns out to  $64 < 81$  which is true, thus  $2m^6 < n^4$  is a loop invariant.

12. Consider the loop

while  $k \geq 1$  do

$k := 2k$

- (a) Is  $k^2 \equiv 1 \pmod{3}$  a loop invariant? Explain.

After  $i$  iterations,  $k = 2^i k$ , so  $k^2 = 4^i k^2$ . We know  $k^2 \equiv 1 \pmod{3}$ , so we need to show that  $4^i \equiv 1 \pmod{3}$ . We also know that  $4 \equiv 1 \pmod{3}$ , so  $4^i \equiv 1^i \pmod{3}$  which equals  $4^i \equiv 1 \pmod{3}$ . Thus  $k^2 \equiv 1 \pmod{3}$  is a loop invariant.

- (b) Is  $k^2 \equiv 1 \pmod{4}$  a loop invariant? Explain.

After  $i$  iterations,  $k = 2^i k$ , so  $k^2 = 4^i k^2$ . We know  $k^2 \equiv 1 \pmod{4}$ , so we need to show that  $4^i \equiv 1 \pmod{4}$ . However,  $4^i \equiv 0 \pmod{4}$  because  $4^i$  is divisible by 4. Thus  $k^2 \equiv 1 \pmod{4}$  is not a loop invariant.

22. (a) Is  $5^k < k!$  an invariant of the following loop?

while  $4 \leq k$  do

$k := k + 1$

No because, when  $k = 4$ ,  $5^k < k! = 5^4 < 4! = 625 < 24$  which is false.

- (b) Can you conclude that  $5^k < k!$  for all  $k \geq 4$ ?

No because, when  $k = 4$ ,  $5^k < k! = 5^4 < 4! = 625 < 24$  which is false.

## 4.2

6. Prove  $4 + 10 + 16 + \dots + (6n - 2) = n(3n + 1)$  for all  $n \in \mathbb{Z}^+$ .

- (B)  $n = 1$ ,  $(6 \cdot 1 - 2) = 1(3 \cdot 1 + 1)$  so  $4 = 4$  which is true.
- (I) For all  $n \in \mathbb{Z}^+$  then  $4 + 10 + 16 + \dots + (6(n - 1) - 2) = (n - 1)(3(n - 1) + 1)$
- Inductive Step:

$$\begin{aligned}
 n(3n + 1) &= 4 + 10 + 16 + \dots + (6(n - 1) - 2) + (6n - 2) \\
 &= (n - 1)(3(n - 1) + 1) + (6n - 2) && \text{(By Inductive Hypothesis)} \\
 &= 3n^2 - 2n - 3n + 2 + 6n - 2 \\
 &= 3n^2 + n \\
 &= n(3n + 1)
 \end{aligned}$$

- Thus  $4 + 10 + 16 + \dots + (6n - 2) = n(3n + 1)$  for all  $n \in \mathbb{Z}^+$

8. Prove:  $\frac{1}{1 \cdot 5} + \frac{1}{5 \cdot 9} + \frac{1}{9 \cdot 13} + \dots + \frac{1}{(4n-3)(4n+1)} = \frac{n}{4n+1}$  for  $n \in \mathbb{Z}^+$ .

- (B)  $n = 1$ ,  $\frac{1}{(4 \cdot 1 - 3)(4 \cdot 1 + 1)} = \frac{n}{4n+1}$  so  $\frac{1}{5} = \frac{1}{5}$  which is true.
- (I) For all  $\frac{1}{1 \cdot 5} + \frac{1}{5 \cdot 9} + \frac{1}{9 \cdot 13} + \dots + \frac{1}{(4(n-1)-3)(4(n-1)+1)} = \frac{n-1}{4(n-1)+1}$  for  $n \in \mathbb{Z}^+$

- Inductive Step:

$$\begin{aligned}
\frac{n}{4n+1} &= \frac{1}{1 \cdot 5} + \frac{1}{5 \cdot 9} + \frac{1}{9 \cdot 13} + \dots + \frac{n-1}{(4(n-1)-3)(4(n-1)+1)} + \frac{1}{(4n-3)(4n+1)} \\
&= \frac{n-1}{4(n-1)+1} + \frac{1}{(4n-3)(4n+1)} \\
&= \frac{n-1}{4n-3} + \frac{1}{(4n-3)(4n+1)} \\
&= \frac{n-1(4n+1)+1}{(4n-3)(4n+1)} \\
&= \frac{4n^2+n-4n}{(4n-3)(4n+1)} \\
&= \frac{(n)(4n-3)}{(4n-3)(4n+1)} \\
&= \frac{n}{4n+1}
\end{aligned}$$

- Thus  $\frac{1}{1 \cdot 5} + \frac{1}{5 \cdot 9} + \frac{1}{9 \cdot 13} + \dots + \frac{1}{(4n-3)(4n+1)} = \frac{n}{4n+1}$  for  $n \in \mathbb{Z}^+$ .

20. Prove  $1^3 + 2^3 + \dots + n^3 = (1 + 2 + \dots + n)^2$ .

- (B)  $n = 2$ ,  $1^3 + 2^3 = (1 + 2)^2$  so  $9 = 9$  which is true.
- (I)  $1^3 + 2^3 + \dots + (n-1)^3 = (1 + 2 + \dots + (n-1))^2$  so  $\sum_{i=1}^{n-1} i^3 = (\sum_{i=1}^{n-1} i)^2$
- Inductive Step:

$$1^3 + 2^3 + \dots + (n-1)^3 + n^3 = (1 + 2 + \dots + (n-1) + n)^2$$

$$\sum_{i=1}^{n-1} i^3 + n^3 = \left( \sum_{i=1}^{n-1} i + n \right)^2 \quad (\text{By inductive hypothesis})$$

$$\sum_{i=1}^{n-1} i^3 + n^3 = \left( \sum_{i=1}^{n-1} i \right)^2 + 2n \sum_{i=1}^{n-1} i + n^2$$

$$n^3 = \frac{2n(n-1)(n-1+1)}{2} + n^2 \quad \left( \text{By } \sum_{i=1}^{n-1} i = \frac{(n-1)(n-1+1)}{2} \right)$$

$$n^3 = n^3 - n^2 + n^2$$

$$n^3 = n^3$$

- Thus  $1^3 + 2^3 + \dots + n^3 = (1 + 2 + \dots + n)^2$ .

## 4.4

8. Let  $\Sigma = \{a, b\}$  and let  $s_n$  denote the number of words of length  $n$  that do not contain the string  $ab$ .

(a) Calculate  $s_0, s_1, s_2, s_3$ .

$$s_0 = 1$$

$$s_1 = 2$$

$$s_2 = 3$$

$$s_3 = 4$$

(b) Find a formula for  $s_n$  and prove it is correct.

(B)  $s_0 = 1$

(R)  $s_n = s_{n-1} + 1$  for  $n \in \mathbb{N}$

10. Consider the sequence defined by

(B)  $\text{SEQ}(0) = 1, \text{SEQ}(1) = 0$

(R)  $\text{SEQ}(n) = \text{SEQ}(n-2)$  for  $n \geq 2$ .

(a) List the first few terms of this sequence.

$$\text{SEQ}(2) = 1, \text{SEQ}(3) = 0, \text{SEQ}(4) = 1, \text{SEQ}(5) = 0$$

(b) What is the set of values of this sequence?

$$\{0, 1\}$$

## 4.6

2. For  $n \in \mathbb{Z}^+$ , prove

(b)  $\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \dots + \frac{1}{n(n+1)} = \frac{n}{n+1}$

• (B)  $n = 1, \frac{1}{1 \cdot 2} = \frac{1}{1+1}$  so  $\frac{1}{2} = \frac{1}{2}$  which is true.

• (I)  $\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \dots + \frac{1}{(n-1)(n-1+1)} = \frac{n-1}{n-1+1}$

- Inductive Step:

$$\begin{aligned}
\frac{n}{n+1} &= \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \dots + \frac{1}{(n-1)(n-1+1)} + \frac{1}{n(n+1)} \\
&= \frac{n-1}{n-1+1} + \frac{1}{n(n+1)} && \text{(By inductive hypothesis)} \\
&= \frac{(n+1)(n-1)}{n(n+1)} + \frac{1}{n(n+1)} \\
&= \frac{(n+1)(n-1) + 1}{n(n+1)} \\
&= \frac{n^2}{n(n+1)} \\
&= \frac{n}{n+1}
\end{aligned}$$

- Thus  $\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \dots + \frac{1}{n(n+1)} = \frac{n}{n+1}$ .

6. Recursively define  $a_0 = 1, a_1 = 2$  and  $a_n = \frac{a_{n-1}^2}{a_{n-2}}$  for  $n \geq 2$ .

(a) Calculate the first few terms of the sequence.

$$a_2 = \frac{2^2}{1} = 4$$

$$a_3 = \frac{4^2}{2} = 8$$

$$a_4 = \frac{8^2}{4} = 16$$

(b) Using part (a), guess the general formula for  $a_n$ .

$$a_n = 2^n$$

(c) Prove the guess in part (b).

- (B)  $n = 1, a_1 = 2 = 2^1$
- (I)  $a_{n-1} = 2^{n-1}$  and  $a_{n-2} = 2^{n-2}$
- Inductive Step:

$$\begin{aligned}
a_n &= \frac{a_{n-1}^2}{a_{n-2}} \\
&= \frac{(2^{n-1})^2}{2^{n-2}} \\
&= 2^{2n-2-n+2} \\
&= 2^n
\end{aligned}$$

- Thus  $a_n = 2^n$ .

12. Recursively define:

$$a_0 = 1,$$

$$a_1 = 3,$$

$$a_2 = 5$$

$$a_n = 3a_{n-2} + 2a_{n-3} \text{ for } n \geq 3.$$

(a) Calculate  $a_n$  for  $n = 3, 4, 5, 6, 7$ .

$$a_3 = 11$$

$$a_4 = 21$$

$$a_5 = 43$$

$$a_6 = 85$$

$$a_7 = 171$$

(b) Prove that  $a_n > 2^n$  for  $n \geq 1$ .

- (B)  $n = 1$ ,  $a_1 > 2^1 = 3 > 2$  which is true.
- (I)  $a_{n-2} > 2^{n-2}$  and  $a_{n-3} > 2^{n-3}$
- Inductive Step:

$$a_n = e3 \qquad = 2$$

• .

(c) Prove that  $a_n < 2^{n+1}$  for  $n \geq 1$ .

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(d) Prove that  $a_n = 2_a n - 1 + -1^{n-1}$  for  $n \geq 1$ .

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