Optional Assignment 𝟙

Statement

Given the IVP $u' = f(u, t) = \frac{u^2+u}{t}$ over the interval $t \in [1,5]$ with initial value $u(1) = -2$, we are asked to appoximate...

We know that the solution to this ODE is $u(t) = \frac{2t}{1-2t}$

We are tasked to implement the following methods to approximate u at $t = 5$:

1. Forward Euler
2. Backward Euler
3. Explicit Trapezoidal
4. Explicit Midpoint
5. Heun's RK3
6. General RK4

Resolution

First, we need to define the IVP and import the necessary libraries to implement the RK methods and to plot the solution.

```python
import math, matplotlib.pyplot as plt

# Definition of the IVP problem and initial values
f = lambda u,t: (u**2+u)/t
f_prime = lambda u,t: (2*u+1)/t
u = lambda t: (2*t)/(1-2*t)
delta = lambda k: 1/(5*2**k)
a, b, k, u0 = 1, 5, delta(6), -2
```

Knowing the data we have on hand, we can design the RK methods accordingly so that it returns an array of approximations of u values of length $N = \lceil\frac{b-a}{k}\rceil + 1$.

```python
def forward_euler(f: callable, a: float, b: float, k: float, u0: float) -> list[float]:
  """Forward Euler Method"""
  N = math.ceil((b-a)/k) + 1
  k = (b-a)/N

  u = [u0]
  ui, ti = u0, a
  for _ in range(N-1):
    ui, ti = ui + k*f(ui, ti), ti+k
    u.append(ui)

  return u

def backward_euler(f: callable, f_prime: callable, a: float, b: float, k: float, u0: float, tolerance=0.0001, max_iterations=10) -> list[float]:
  """Backward Euler Method"""
  N = math.ceil((b-a)/k) + 1
  k = (b-a)/N

  u = [u0]
  ui, ti = u0, a
  for _ in range(N-1):
    ui, ti = newton_method(lambda x: x-ui-k*f(x,ti+k), lambda x: 1-k*f_prime(x,ti+k), ui, tolerance, max_iterations), ti+k
    u.append(ui)
  return u

def explicit_trapezoidal(f: callable, a: float, b: float, k: float, u0: float) -> list[float]:
  """Explicit Trapezoidal Runge-Kutta 2 Method"""
  N = math.ceil((b-a)/k) + 1
  k = (b-a)/N

  u = [u0]
  ui, ti = u0, a
  for _ in range(N-1):
    aux_u = ui + k*f(ui, ti)
    ui, ti = ui + (k/2)*(f(ui, ti) + f(aux_u, ti+k)), ti+k
    u.append(ui)
```

```python
    return u

def explicit_midpoint(f: callable, a: float, b: float, k: float, u0:
        float) -> list[float]:
    """Explicit Midpoint Runge-Kutta 2 Method"""
    N = math.ceil((b-a)/k) + 1
    k = (b-a)/N

    u = [u0]
    ui, ti = u0, a
    for _ in range(N-1):
        aux_u = ui + (k/2)*f(ui, ti)
        ui, ti = ui + k*f(aux_u, ti + k/2), ti+k
        u.append(ui)
    return u

def heun_rk3(f: callable, a: float, b: float, k: float, u0: float) ->
        list[float]:
    """Heun's Runge-Kutta 3 Method"""
    N = math.ceil((b-a)/k) + 1
    k = (b-a)/N

    u = [u0]
    ui, ti = u0, a
    for _ in range(N-1):
        f1, f2 = f(ui, ti), f(ui + k*f(ui, ti), ti+k)
        ui, ti = ui + (k/2)*(f1+f2), ti+k
        u.append(ui)
    return u

def rk4(f: callable, a: float, b: float, k: float, u0: float) ->
        list[float]:
    """Standard Runge-Kutta 4 Method"""
    N = math.ceil((b-a)/k) + 1
    k = (b-a)/N

    u = [u0]
    ui, ti = u0, a
    for _ in range(N-1):
        y1 = ui
        y2 = ui + (k/2)*f(y1, ti)
```

```python
        y3 = ui + (k/2)*f(y2, ti+k/2)
        y4 = ui + k*f(y3, ti+k/2)
        f1, f2, f3, f4 = f(y1, ti), f(y2, ti + k/2), f(y3, ti+k/2), f(y4, ti+k)
        ui, ti = ui + (k/6)*(f1+2*f2+2*f3+f4), ti+k
        u.append(ui)
    return u
```

Before, we continue, we are defining a relaible function that returns the exact value of u so that we can compute the errors. Moreover, backward euler needs of the Newton's Method so that it can find the values of $U_{n+1}(U_n, t_n)$. Therefore:

```python
def exact_solution(u: callable, a: float, b: float, k: float) -> float:
    """Computes the exact solution of the IVP"""
    N = math.ceil((b-a)/k) + 1
    k = (b-a)/N

    y = []
    for i in range(N):
        y.append(u(a+i*k))
    return y

def newton_method(f: callable, f_prime: callable, u0: float, tolerance: float, max_iterations: int) -> float:
    """Newton's Method to find the root of a function"""
    u1 = u0
    for _ in range(max_iterations):
        fu = f(u1)
        fpu = f_prime(u1)
        # Avoid division by zero
        if abs(fpu) < tolerance:
            raise ValueError("Derivative is too small; method fails.")
        u_new = u1 - fu / fpu
        # Check for convergence
        if abs(u_new - u1) < tolerance:
            return u_new
        u1 = u_new
    raise ValueError("Method did not converge within the maximum iterations.")
```

Now, we plot the absolute error of each method for each value of $\Delta t(k)$ s.t. $k \in$

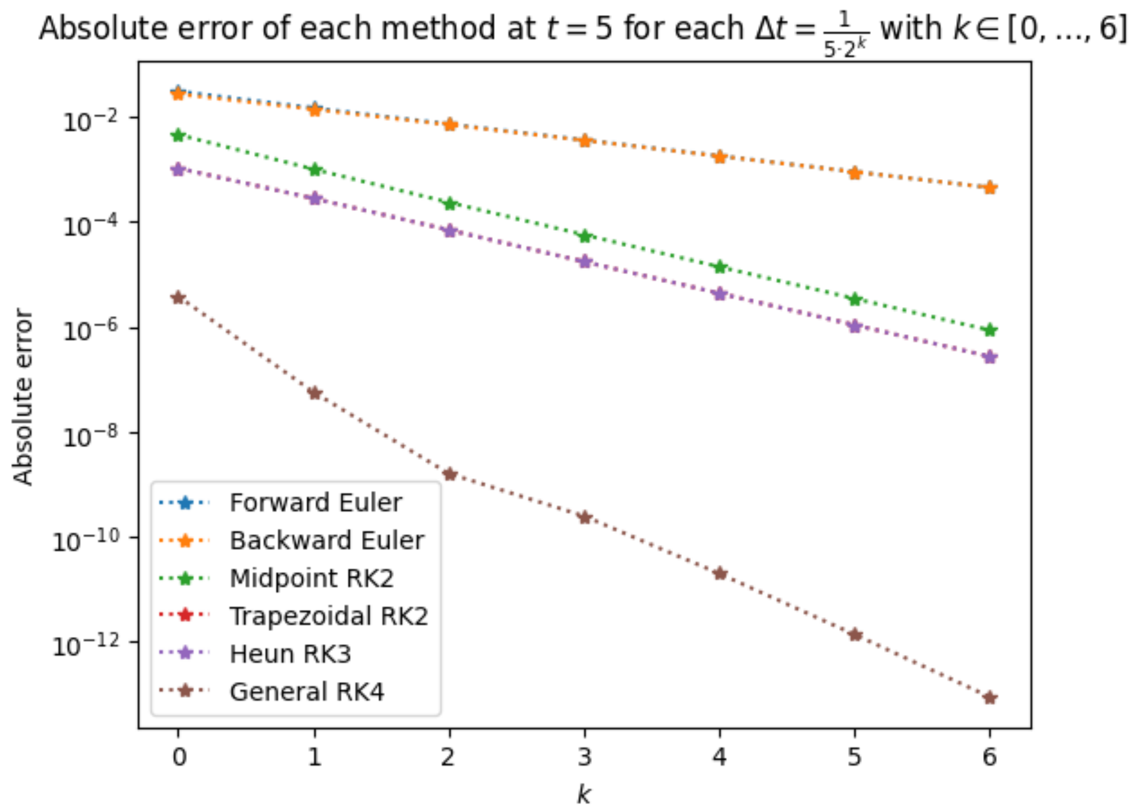
$\{0, \ldots, 6\}$

```
matrix = [[] for _ in range(6)]
for i in range(7):
    k = delta(i)
    x_points = [a+k*i for i in range(math.ceil((b-a)/k)+1)]
    exact = exact_solution(u, a, b, k)[-1]
    y_feuler = forward_euler(f, a, b, k, u0)
    y_beuler = backward_euler(f, f_prime, a, b, k, u0,
        tolerance=0.0001, max_iterations=10)
    y_midpoint = explicit_midpoint(f, a, b, k, u0)
    y_trapezoid = explicit_trapezoidal(f, a, b, k, u0)
    y_heun = heun_rk3(f, a, b, k, u0)
    y_rk4 = rk4(f, a, b, k, u0)

    points = [
        y_feuler[-1],
        y_beuler[-1],
        y_midpoint[-1],
        y_trapezoid[-1],
        y_heun[-1],
        y_rk4[-1],
    ]
    for j in range(6):
        matrix[j].append(abs(points.pop(0)-exact))

labels = ['Forward Euler', 'Backward Euler', 'Midpoint RK2',
        'Trapezoidal RK2', 'Heun RK3', 'General RK4']
for i, label in enumerate(labels):
    plt.semilogy([j for j in range(7)], matrix[i], '*:', label=label)


plt.legend()
plt.title("Absolute error of each method at $t=5$ for each $\\Delta t
        = \\frac{1}{5·2^k}$ with $k\\in[0, \\ldots, 6]$")
plt.xlabel("$k$")
plt.ylabel("Absolute error")
plt.show()
```



It is clear, that RK4, the most computationally expensive converges the fastest to the solution with $\tau^n = O(k^4)$. Meanwhile, the other Runge-Kutta methods converges slower. Interestingly, the graphs of the RK Methods of the same order overlaps with each other.

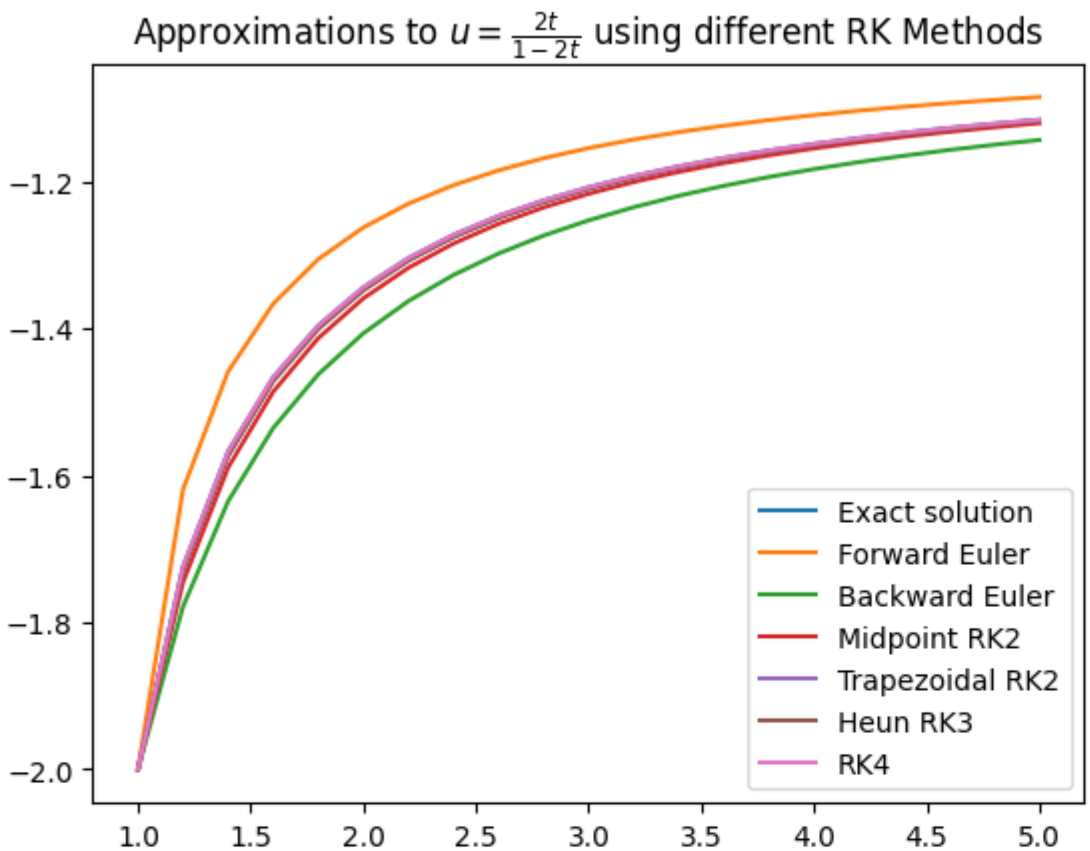
Other plots

```
k = delta(0)

# Compute the solutions using the different methods
x_points = [a+k*i for i in range(math.ceil((b-a)/k)+1)]


y_points = exact_solution(u, a, b, k)
y_feuler = forward_euler(f, a, b, k, u0)
y_beuler = backward_euler(f, f_prime, a, b, k, u0, tolerance=0.0001,
        max_iterations=10)
y_midpoint = explicit_midpoint(f, a, b, k, u0)
y_trapezoid = explicit_trapezoidal(f, a, b, k, u0)
y_heun = heun_rk3(f, a, b, k, u0)
y_rk4 = rk4(f, a, b, k, u0)
```

```
plt.plot(x_points, y_points, label='Exact solution')
plt.plot(x_points, y_feuler, label='Forward Euler')
plt.plot(x_points, y_beuler, label='Backward Euler')
plt.plot(x_points, y_midpoint, label='Midpoint RK2')
plt.plot(x_points, y_trapezoid, label='Trapezoidal RK2')
plt.plot(x_points, y_heun, label='Heun RK3')
plt.plot(x_points, y_rk4, label='RK4')
plt.title("Approximations to $u = \\frac{2t}{1-2t}$ using different RK
        Methods")
plt.legend()
plt.show()
```




```
abs_error = lambda u, v: [abs(u[i]-v[i]) for i in range(len(u))]

plt.semilogy(x_points, abs_error(y_points, y_feuler), label='Forward
        Euler')
plt.semilogy(x_points, abs_error(y_points, y_beuler), label='Backward
        Euler')
plt.semilogy(x_points, abs_error(y_points, y_midpoint),
        label='Midpoint RK2')
plt.semilogy(x_points, abs_error(y_points, y_trapezoid),
        label='Trapezoidal RK2')
plt.semilogy(x_points, abs_error(y_points, y_heun), label='Heun RK3')
```

```python
plt.semilogy(x_points, abs_error(y_points, y_rk4), label='RK4')
plt.title("Absolute error of each method")
plt.legend()
plt.show()
```