INF5620 Lecture: Analysis of finite difference schemes for diffusion processes

Hans Petter Langtangen[1,2]

Center for Biomedical Computing, Simula Research Laboratory[1]

Department of Informatics, University of Oslo[2]

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Properties of the solution

The PDE

$$u_t = \alpha u_{xx} \tag{1}$$

admits solutions

$$u(x,t) = Qe^{-\alpha k^2 t}\sin\left(kx\right) \tag{2}$$

Observations from this solution:

- The initial shape $I(x) = Q\sin kx$ undergoes a damping $\exp\left(-\alpha k^2 t\right)$
- The damping is very strong for short waves (large k)
- The damping is weak for long waves (small k)
- Consequence: u is smoothened with time

Example

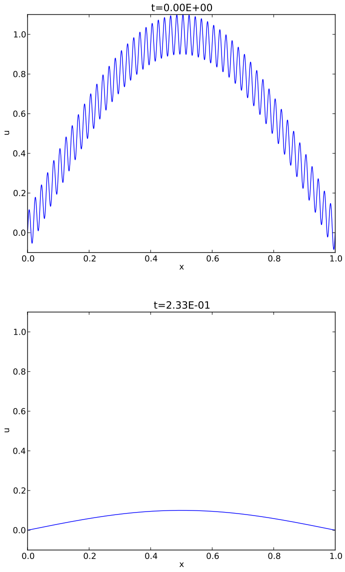
Test problem:

$$\begin{aligned} u_t &= u_{xx}, & x \in (0,1),\ \ t \in (0,T] \\ u(0,t) &= u(1,t) = 0, & t \in (0,T] \\ u(x,0) &= \sin(\pi x) + 0.1 \sin(100\pi x) \end{aligned}$$

Exact solution:

$$u(x,t) = e^{-\pi^2 t} \sin(\pi x) + 0.1 e^{-\pi^2 10^4 t} \sin(100\pi x) \tag{3}$$

Visualization of the damping in the diffusion equation



Fourier representation

Represent $I(x)$ as a Fourier series

$$I(x) \approx \sum_{k \in K} b_k e^{ikx} \tag{4}$$

The corresponding sum for u is

$$u(x,t) \approx \sum_{k \in K} b_k e^{-\alpha k^2 t} e^{ikx} \,. \tag{5}$$

Such solutions are also accepted by the numerical schemes, but with an amplification factor different from $exp(-\alpha k^2 t)$:

$$u_q^n = A^n e^{ikq\Delta x} = A^n e^{ikx} \tag{6}$$

Analysis of the finite difference schemes

Stability:

- $|A| < 1$: decaying numerical solutions (as we want)
- $A < 0$: *oscillating* numerical solutions (as we do not want)

Accurary:

- Compare numerical and exact amplification factor

$$A \quad \text{vs} \quad A_{\mathrm{e}} = \exp\left(-\alpha k^2 \Delta t\right)$$

Analysis of the Forward Euler scheme

$$[D_t^+ u = \alpha D_x D_x u]_q^n$$

Inserting

$$u_q^n = A^n e^{ikq\Delta x}$$

leads to

$$A = 1 - 4C\sin^2\left(\frac{k\Delta x}{2}\right), \quad C = \frac{\alpha\Delta t}{\Delta x^2} \tag{7}$$

The complete numerical solution is

$$u_q^n = \left(1 - 4C\sin^2 p\right)^n e^{ikq\Delta x}, \quad p = k\Delta x/2 \tag{8}$$

Results for stability

$A \leq 1$, but $A < -1$ is a possibility:

$$4C \sin^2 p \leq 2$$

The worst case is when $\sin^2 p = 1$, so a sufficient criterion for stability is

$$C \leq \frac{1}{2} \tag{9}$$

or:

$$\Delta t \leq \frac{\Delta x^2}{2\alpha} \tag{10}$$

Implications of the stability result.

Less favorable criterion than for $u_{tt} = c^2 u_{xx}$: halving Δx implies time step $\frac{1}{4}\Delta t$ (not just $\frac{1}{2}\Delta t$ as in a wave equation). Need very small time steps for fine spatial meshes!

Analysis of the Backward Euler scheme

$$[D_t^- u = \alpha D_x D_x u]_q^n$$

$$u_q^n = A^n e^{ikq\Delta x}$$

$$A = \left(1 + 4C\sin^2 p\right)^{-1} \tag{11}$$

$$u_q^n = \left(1 + 4C\sin^2 p\right)^{-n} e^{ikq\Delta x} \tag{12}$$

Stability

We see from (11) that $|A| < 1$ for all $\Delta t > 0$ and that $A > 0$ (no oscillations).

Analysis of the Crank-Nicolson scheme

The scheme

$$[D_t u = \alpha D_x D_x \overline{u}^x]_q^{n+\frac{1}{2}}$$

leads to

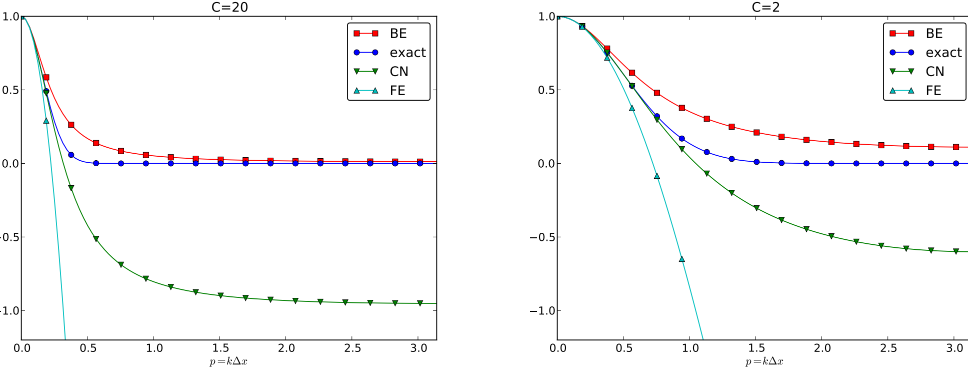
$$A = \frac{1 - 2C\sin^2 p}{1 + 2C\sin^2 p} \tag{13}$$

$$u_q^n = \left(\frac{1 - 2C\sin^2 p}{1 + 2C\sin^2 p}\right)^n e^{ikp\Delta x} \tag{14}$$

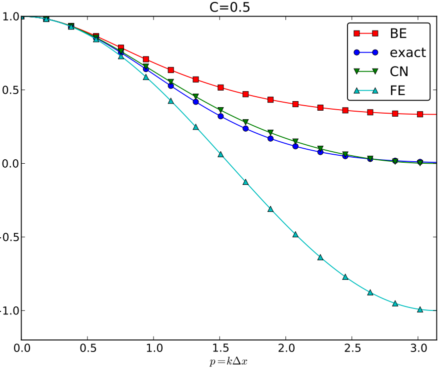
Stability

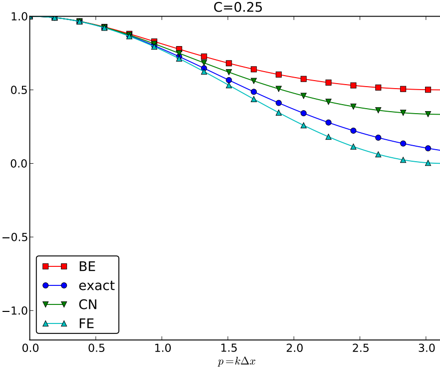
The criteria $A > -1$ and $A < 1$ are fulfilled for any $\Delta t > 0$.

Summary of accuracy of amplification factors; large time steps

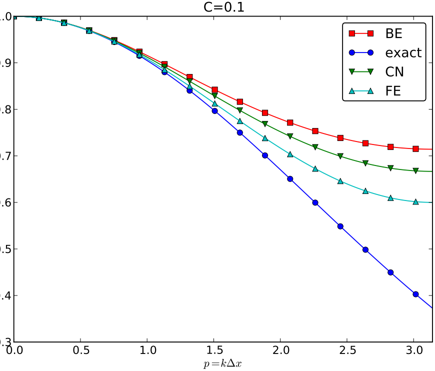


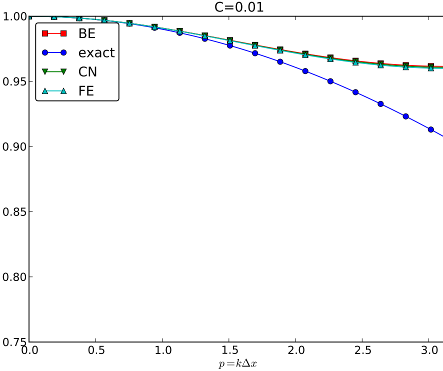
Summary of accuracy of amplification factors; time steps around the Forward Euler stability limit





Summary of accuracy of amplification factors; small time steps





Observations

- Crank-Nicolson gives oscillations and not much damping of short waves for increasing C.
- These waves will manifest themselves as high frequency oscillatory noise in the solution.
- All schemes fail to dampen short waves enough

The problems of correct damping for $u_t = u_{xx}$ is partially manifested in the similar time discretization schemes for $u'(t) = -\alpha u(t)$.