$\left(D_t D_t e^{i\omega\Delta t} \qquad u'' + \omega^2 u = 0 \quad \text{"vib"}\right)$

$$U_p^n = e^{i(kp\Delta x - \tilde{\omega} n \Delta t)}$$

$$D_t D_t U_p^n = e^{ikp\Delta x} \underbrace{D_t D_t e^{-i\tilde{\omega} n \Delta t}}_{-\frac{4}{\Delta t^2}\sin^2 \frac{\tilde{\omega}\Delta t}{2}} \qquad [D_t D_t u = c^2 D_x D_x u]_p^n$$

$$D_x D_x U_p^n = e^{-i\tilde{\omega}\Delta t} \underbrace{D_x D_x e^{ikx}}_{-\frac{4}{\Delta x^2}\sin^2 \frac{k\Delta x}{2}}$$

$\Rightarrow$

$$(*) \quad \sin^2 \frac{\tilde{\omega}\Delta t}{2} = C^2 \sin^2 \frac{k\Delta x}{2} \qquad \text{eq. for } \tilde{\omega}$$

solve for $\tilde{\omega}$:

$$\tilde{\omega} = \frac{2}{\Delta t} \arcsin\left(C_1 \sin \frac{k\Delta x}{2} \right) \qquad C_1 = c\frac{\Delta t}{\Delta x}$$

$$\tilde{\omega} = \text{function of } k, \underbrace{c, \Delta x, \Delta t}_{C_1}$$

numerical dispersion relations

$C_1 = 1$: $\tilde{\omega} = \omega$!!
the scheme is exact !!!
for any Δt, Δx !

$C_1 = 1.0015$: instability

$C_1 > 1$: Sin is max 1 $\Rightarrow$ $C_1 \sin\frac{k\Delta x}{2} > 1$ for $C_1 > 1$

$\arcsin(x > 1) \to$ complex value and complex $\tilde{\omega}$

$$e^{i(-\tilde{\omega}\Delta t)} = e^{i(-\tilde{\omega}_r \Delta t - i\tilde{\omega}_i \Delta t} \Rightarrow \underbrace{\left(e^{\tilde{\omega}_i \Delta t}\right)^n}_{\text{growth}}$$

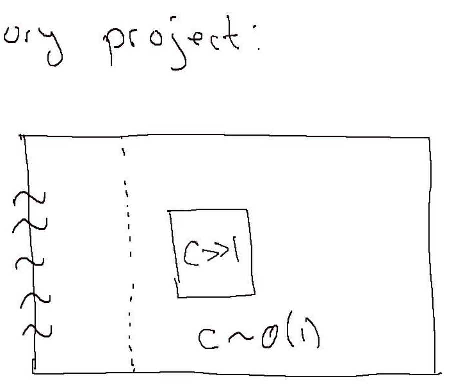
$\Rightarrow$ $C \leq 1$ is stability criterion.

$$\Delta t \leq \frac{\Delta x}{c}$$

Taylor series for accuracy:

$$\tilde{\omega} - \omega = \frac{2}{\Delta t}\arcsin(\dots) - \omega \sim \mathcal{O}(\Delta x^2) + \mathcal{O}(\Delta t^2)$$

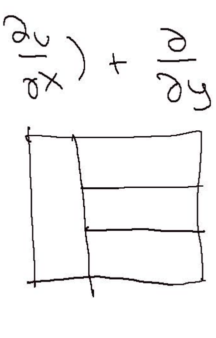
Compulsory project:



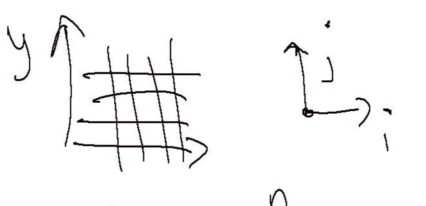
$\frac{\partial u}{\partial n} = 0$

$$u_{tt} = c^2 (u_{xx} + u_{yy})$$

$$u_{tt} = \frac{\partial}{\partial x}\left(c\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(c\frac{\partial u}{\partial y}\right)$$



Extensions to 2D



n: time counter

$$u(x_i, y_j, t_n) = U_{i,j}^n$$

Equation

$$\left[D_{tt} u = c^2 (D_{xx} u + D_{yy} u)\right]_{i,j}^n$$

$\Downarrow$

$$\frac{U_{i,j}^{n+1} - 2U_{i,j}^n + U_{i,j}^{n-1}}{\Delta t^2} = \quad \downarrow \quad \frac{U_{i,j+1}^n - 2U_{i,j}^n + U_{i,j-1}^n}{\Delta y^2}$$

$\downarrow$ Unknown: $U_{i,j}^{n+1}$

$U_{i,j}^{n+1} = \dots\dots\dots$ $\qquad i = 1, \dots, N_x - 1, \; j = 1, \dots, N_y - 1$

internal points

$U_{i,j}^{n+1} = 0$ at the boundary: $i = 0, \; j = 0, \dots, N_y$
$i = N_x, \; j =$ — " —
$j = 0, \; i = 1, \dots N_x$
$j = N_y, \; i =$ — " —

Challenge: $\frac{\partial u}{\partial n} = 0$ at the boundary

$$\frac{\partial}{\partial x}\left(c(x,y) \frac{\partial u}{\partial x} \right)$$

$$\left.\frac{\partial}{\partial x}\left(c(x) \frac{\partial u}{\partial x} \right)\right|_i \approx \frac{\left(c\frac{\partial u}{\partial x}\right)_{i+1/2} - \left(c\frac{\partial u}{\partial x}\right)_{i-1/2}}{\Delta x}$$

discretize first

$$\approx \frac{c_{i+1/2}\frac{u_{i+1} - u_i}{\Delta x} - c_{i-1/2}\frac{u_i - u_{i-1}}{\Delta x}}{\Delta x}$$

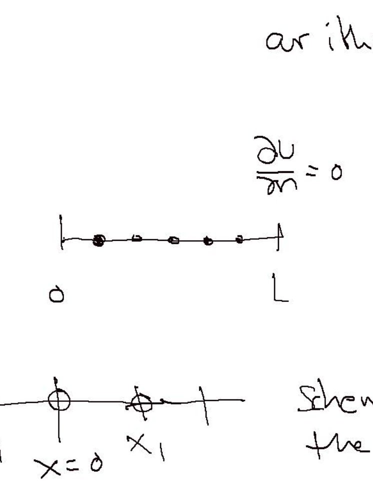
$$= \frac{1}{\Delta x^2}\left(c_{i+1/2}(u_{i+1} - u_i) - c_{i-1/2}(u_i - u_{i-1}) \right)$$

?

$$c(x_{i+\frac{1}{2}}) = c(x_i + \frac{\Delta x}{2})$$

Sometimes you don't have a formula $c(x)$, but c at the mesh points: $c_0, c_1, c_2, \dots$

$$c_{i+1/2} \approx \underbrace{\frac{1}{2}(c_i + c_{i+1})}_{\text{arithmetic mean}} \quad \text{or} \quad \underbrace{2\frac{1}{\frac{1}{c_i} + \frac{1}{c_{i+1}}}}_{\text{harmonic mean}}$$



Scheme at the boundary

$$\frac{\partial u}{\partial n} = n \cdot \nabla u$$

$\nabla \cdot (\lambda \nabla u)$

$\hookrightarrow \lambda \frac{\partial u}{\partial n} = \dots$ on $\partial\Omega$

$$\frac{\partial u}{\partial n} = 0 \quad [D_{2x} u]_i^n : \frac{U_{i+1}^n - U_{i-1}^n}{2\Delta x}$$

$$\hookrightarrow x = 0 \quad \frac{U_1^n - U_{-1}^n}{2\Delta x} = 0 \Rightarrow U_{-1}^n = U_1^n$$

Combine with the scheme!

$$U_i^{n+1} = -U_i^{n-1} + 2U_i^n + \left(c\frac{\Delta t}{\Delta x}\right)^2 \left(U_{i+1}^n - 2U_i^n + U_{i-1}^n\right)$$

$i = 0$

$= U_{i+1}^n$ for $i = 0$

$\Rightarrow$ Special formula for boundary points where $\frac{\partial u}{\partial n} = 0$