

# Study Guide: Truncation Error Analysis

Hans Petter Langtangen<sup>1,2</sup>

Center for Biomedical Computing, Simula Research Laboratory<sup>1</sup>

Department of Informatics, University of Oslo<sup>2</sup>

Sep 18, 2013

# Motivation for studying truncation errors

- Definition: The *truncation error* is the discrepancy that arises from performing a finite number of steps to approximate a process with infinitely many steps.
- Widely used: truncation of infinite series, finite precision arithmetic, finite differences, and differential equations.
- Why? The truncation error is an error measure that is easy to compute.

# Abstract problem setting

Consider an abstract differential equation

$$\mathcal{L}(u) = 0.$$

Example:  $\mathcal{L}(u) = u'(t) + a(t)u(t) - b(t)$ .

The corresponding discrete equation:

$$\mathcal{L}_{\Delta}(u) = 0.$$

Let now

- $u$  be the numerical solution of the discrete equations, computed at mesh points:  $u^n$ ,  $n = 0, \dots, N_t$
- $u_e$  the exact solution of the differential equation

$$\mathcal{L}(u_e) = 0,$$

$$\mathcal{L}_{\Delta}(u) = 0.$$

$u$  is computed at mesh points

# Various error measures

- Dream: the true error  $e = u_e - u$ , but usually impossible
- Must find other error measures that are easier to calculate
  - Derive formulas for  $u$  in (very) special, simplified cases
  - Compute empirical convergence rates for special choices of  $u_e$  (usually non-physical  $u_e$ )
- To what extent does  $u_e$  fulfill  $\mathcal{L}_\Delta(u_e) = 0$ ?
- It does not fit, but we can measure the error  $\mathcal{L}_\Delta(u_e) = R$
- $R$  is the truncation error and it is easy to compute in general, without considering special cases

# Truncation errors in finite difference formulas

## Example: The backward difference for $u'(t)$

Backward difference approximation to  $u'$ :

$$[D_t^- u]^n = \frac{u^n - u^{n-1}}{\Delta t} \approx u'(t_n). \quad (1)$$

Define the truncation error of this approximation as

$$R^n = [D_t^- u]^n - u'(t_n). \quad (2)$$

The common way of calculating  $R^n$  is to

- 1 expand  $u(t)$  in a Taylor series around the point where the derivative is evaluated, here  $t_n$ ,
- 2 insert this Taylor series in (2), and
- 3 collect terms that cancel and simplify the expression.

# Taylor series

General Taylor series expansion from calculus:

$$f(x+h) = \sum_{i=0}^{\infty} \frac{1}{i!} \frac{d^i f}{dx^i}(x) h^i.$$

Here: expand  $u^{n-1}$  around  $t_n$ :

$$\begin{aligned} u(t_{n-1}) = u(t - \Delta t) &= \sum_{i=0}^{\infty} \frac{1}{i!} \frac{d^i u}{dt^i}(t_n) (-\Delta t)^i \\ &= u(t_n) - u'(t_n) \Delta t + \frac{1}{2} u''(t_n) \Delta t^2 + \mathcal{O}(\Delta t^3), \end{aligned}$$

- $\mathcal{O}(\Delta t^3)$ : power-series in  $\Delta t$  where the lowest power is  $\Delta t^3$
- Small  $\Delta t$ :  $\Delta t \gg \Delta t^3 \gg \Delta t^4$

## Taylor series inserted in the backward difference approximation

$$\begin{aligned}[D_t^- u]^n - u'(t_n) &= \frac{u(t_n) - u(t_{n-1})}{\Delta t} - u'(t_n) \\&= \frac{u(t_n) - (u(t_n) - u'(t_n)\Delta t + \frac{1}{2}u''(t_n)\Delta t^2 + \mathcal{O}(\Delta t^3))}{\Delta t} \\&\quad - u'(t_n) \\&= -\frac{1}{2}u''(t_n)\Delta t + \mathcal{O}(\Delta t^2)\end{aligned}$$

Result:

$$R^n = -\frac{1}{2}u''(t_n)\Delta t + \mathcal{O}(\Delta t^2). \quad (3)$$

The difference approximation is of *first order* in  $\Delta t$ . It is exact for linear  $u_e$ .



# The forward difference for $u'(t)$

Now consider a forward difference:

$$u'(t_n) \approx [D_t^+ u]^n = \frac{u^{n+1} - u^n}{\Delta t}.$$

Define the truncation error:

$$R^n = [D_t^+ u]^n - u'(t_n).$$

Expand  $u^{n+1}$  in a Taylor series around  $t_n$ ,

$$u(t_{n+1}) = u(t_n) + u'(t_n)\Delta t + \frac{1}{2}u''(t_n)\Delta t^2 + \mathcal{O}(\Delta t^3).$$

We get

$$R = \frac{1}{2}u''(t_n)\Delta t + \mathcal{O}(\Delta t^2).$$

# The central difference for $u'(t)$ (1)

For the central difference approximation,

$$u'(t_n) \approx [D_t u]^n, \quad [D_t u]^n = \frac{u^{n+\frac{1}{2}} - u^{n-\frac{1}{2}}}{\Delta t},$$

the truncation error is

$$R^n = [D_t u]^n - u'(t_n).$$

Expand  $u(t_{n+\frac{1}{2}})$  and  $u(t_{n-1/2})$  in Taylor series around the point  $t_n$  where the derivative is evaluated:

$$\begin{aligned} u(t_{n+\frac{1}{2}}) &= u(t_n) + u'(t_n) \frac{1}{2} \Delta t + \frac{1}{2} u''(t_n) \left(\frac{1}{2} \Delta t\right)^2 + \\ &\quad \frac{1}{6} u'''(t_n) \left(\frac{1}{2} \Delta t\right)^3 + \frac{1}{24} u''''(t_n) \left(\frac{1}{2} \Delta t\right)^4 + \mathcal{O}(\Delta t^5) \\ u(t_{n-1/2}) &= u(t_n) - u'(t_n) \frac{1}{2} \Delta t + \frac{1}{2} u''(t_n) \left(\frac{1}{2} \Delta t\right)^2 - \\ &\quad \frac{1}{6} u'''(t_n) \left(\frac{1}{2} \Delta t\right)^3 + \frac{1}{24} u''''(t_n) \left(\frac{1}{2} \Delta t\right)^4 + \mathcal{O}(\Delta t^5). \end{aligned}$$

## The central difference for $u'(t)$ (1)

$$u(t_{n+\frac{1}{2}}) - u(t_{n-\frac{1}{2}}) = u'(t_n)\Delta t + \frac{1}{24}u'''(t_n)\Delta t^3 + \mathcal{O}(\Delta t^5).$$

By collecting terms in  $[D_t u]^n - u(t_n)$  we find  $R^n$  to be

$$R^n = \frac{1}{24}u'''(t_n)\Delta t^2 + \mathcal{O}(\Delta t^4), \quad (4)$$

Note:

- Second-order accuracy since the leading term is  $\Delta t^2$
- Only even powers of  $\Delta t$

# Leading-order error terms in finite differences (1)

$$\begin{aligned}[D_t u]^n &= \frac{u^{n+\frac{1}{2}} - u^{n-\frac{1}{2}}}{\Delta t} = u'(t_n) + R^n, \\ R^n &= \frac{1}{24} u'''(t_n) \Delta t^2 + \mathcal{O}(\Delta t^4)\end{aligned}\tag{5}$$

$$\begin{aligned}[D_{2t} u]^n &= \frac{u^{n+1} - u^{n-1}}{2\Delta t} = u'(t_n) + R^n, \\ R^n &= \frac{1}{6} u'''(t_n) \Delta t^2 + \mathcal{O}(\Delta t^4)\end{aligned}\tag{6}$$

$$\begin{aligned}[D_t^- u]^n &= \frac{u^n - u^{n-1}}{\Delta t} = u'(t_n) + R^n, \\ R^n &= \frac{1}{2} u''(t_n) \Delta t + \mathcal{O}(\Delta t^2)\end{aligned}\tag{7}$$

$$\begin{aligned}[D_t^+ u]^n &= \frac{u^{n+1} - u^n}{\Delta t} = u'(t_n) + R^n, \\ R^n &= \frac{1}{2} u''(t_n) \Delta t + \mathcal{O}(\Delta t^2)\end{aligned}\tag{8}$$

## Leading-order error terms in finite differences (2)

$$\begin{aligned} [\bar{D}_t u]^{n+\theta} &= \frac{u^{n+1} - u^n}{\Delta t} = u'(t_{n+\theta}) + R^{n+\theta}, \\ R^{n+\theta} &= \frac{1}{2}(1 - 2\theta)u''(t_{n+\theta})\Delta t - \frac{1}{6}((1 - \theta)^3 - \theta^3)u'''(t_{n+\theta})\Delta t^2 + \mathcal{O}(\Delta t^3) \end{aligned} \quad (9)$$

$$\begin{aligned} [D_t^2 u]^n &= \frac{3u^n - 4u^{n-1} + u^{n-2}}{2\Delta t} = u'(t_n) + R^n, \\ R^n &= -\frac{1}{3}u'''(t_n)\Delta t^2 + \mathcal{O}(\Delta t^3) \end{aligned} \quad (10)$$

$$\begin{aligned} [D_t D_t u]^n &= \frac{u^{n+1} - 2u^n + u^{n-1}}{\Delta t^2} = u''(t_n) + R^n, \\ R^n &= \frac{1}{12}u''''(t_n)\Delta t^2 + \mathcal{O}(\Delta t^4) \end{aligned} \quad (11)$$

# Leading-order error terms in mean values (1)

Weighted arithmetic mean:

$$\begin{aligned} [\bar{u}^{t,\theta}]^{n+\theta} &= \theta u^{n+1} + (1-\theta)u^n = u(t_{n+\theta}) + R^{n+\theta}, \\ R^{n+\theta} &= \frac{1}{2}u''(t_{n+\theta})\Delta t^2\theta(1-\theta) + \mathcal{O}(\Delta t^3). \end{aligned} \quad (12)$$

Standard arithmetic mean:

$$\begin{aligned} [\bar{u}^t]^n &= \frac{1}{2}(u^{n-\frac{1}{2}} + u^{n+\frac{1}{2}}) = u(t_n) + R^n, \\ R^n &= \frac{1}{8}u''(t_n)\Delta t^2 + \frac{1}{384}u'''(t_n)\Delta t^4 + \mathcal{O}(\Delta t^6). \end{aligned} \quad (13)$$

## Leading-order error terms in mean values (2)

Geometric mean:

$$u^{n-\frac{1}{2}} u^{n+\frac{1}{2}} = (u^n)^2 + R^n,$$
$$R^n = -\frac{1}{4} u'(t_n)^2 \Delta t^2 + \frac{1}{4} u(t_n) u''(t_n) \Delta t^2 + \mathcal{O}(\Delta t^4). \quad (14)$$

Harmonic mean:

$$u^n = \frac{2}{\frac{1}{u^{n-\frac{1}{2}}} + \frac{1}{u^{n+\frac{1}{2}}}} + R^{n+\frac{1}{2}},$$
$$R^n = -\frac{u'(t_n)^2}{4u(t_n)} \Delta t^2 + \frac{1}{8} u''(t_n) \Delta t^2. \quad (15)$$

# Software for computing truncation errors

- Can use sympy to automate calculations with Taylor series.
- Tool: course module `truncation_errors`

```
>>> from truncation_errors import TaylorSeries
>>> from sympy import *
>>> u, dt = symbols('u dt')
>>> u_Taylor = TaylorSeries(u, 4)
>>> u_Taylor(dt)
D1u*dt + D2u*dt**2/2 + D3u*dt**3/6 + D4u*dt**4/24 + u
>>> FE = (u_Taylor(dt) - u)/dt
>>> FE
(D1u*dt + D2u*dt**2/2 + D3u*dt**3/6 + D4u*dt**4/24)/dt
>>> simplify(FE)
D1u + D2u*dt/2 + D3u*dt**2/6 + D4u*dt**3/24
```

Notation:  $D1u$  for  $u'$ ,  $D2u$  for  $u''$ , etc.

See `trunc/truncation_errors.py`.



# Symbolic computing with difference operators

A class `DiffOp` represents many common difference operators:

```
>>> from truncation_errors import DiffOp
>>> from sympy import *
>>> u = Symbol('u')
>>> diffop = DiffOp(u, independent_variable='t')
>>> diffop['geometric_mean']
-D1u**2*dt**2/4 - D1u*D3u*dt**4/48 + D2u**2*dt**4/64 + ...
>>> diffop['Dtm']
D1u + D2u*dt/2 + D3u*dt**2/6 + D4u*dt**3/24
>>> diffop.operator_names()
['geometric_mean', 'harmonic_mean', 'Dtm', 'D2t', 'DtDt',
 'weighted_arithmetic_mean', 'Dtp', 'Dt']
```

Names in `diffop`: `Dtp` for  $D_t^+$ , `Dtm` for  $D_t^-$ , `Dt` for  $D_t$ , `D2t` for  $D_{2t}$ , `DtDt` for  $D_t D_t$ .

# Truncation errors in exponential decay ODE

$$u'(t) = -au(t)$$

# Truncation error of the Forward Euler scheme

The Forward Euler scheme:

$$[D_t^+ u = -au]^n. \quad (16)$$

Definition of the truncation error  $R^n$ :

$$[D_t^+ u_e + au_e = R]^n. \quad (17)$$

From (8):

$$[D_t^+ u_e]^n = u_e'(t_n) + \frac{1}{2}u_e''(t_n)\Delta t + \mathcal{O}(\Delta t^2).$$

Inserted in (17):

$$u_e'(t_n) + \frac{1}{2}u_e''(t_n)\Delta t + \mathcal{O}(\Delta t^2) + au_e(t_n) = R^n.$$

Note:  $u_e'(t_n) + au_e^n = 0$  since  $u_e$  solves the ODE. Then

$$R^n = \frac{1}{2}u_e''(t_n)\Delta t + \mathcal{O}(\Delta t^2). \quad (18)$$

# Truncation error of the Crank-Nicolson scheme

Crank-Nicolson:

$$[D_t u = -a u]^{n+\frac{1}{2}}, \quad (19)$$

Truncation error:

$$[D_t u_e + a \overline{u_e}^t = R]^{n+\frac{1}{2}}. \quad (20)$$

From (5) and (13):

$$\begin{aligned} [D_t u_e]^{n+\frac{1}{2}} &= u'(t_{n+\frac{1}{2}}) + \frac{1}{24} u_e'''(t_{n+\frac{1}{2}}) \Delta t^2 + \mathcal{O}(\Delta t^4), \\ [a \overline{u_e}^t]^{n+\frac{1}{2}} &= u(t_{n+\frac{1}{2}}) + \frac{1}{8} u''(t_n) \Delta t^2 + \mathcal{O}(\Delta t^4) \end{aligned}$$

Inserted in the scheme we get

$$R^{n+\frac{1}{2}} = \left( \frac{1}{24} u_e'''(t_{n+\frac{1}{2}}) + \frac{1}{8} u''(t_n) \right) \Delta t^2 + \mathcal{O}(\Delta t^4) \quad (21)$$

$R^n = \mathcal{O}(\Delta t^2)$  (second-order scheme)

# Test the understanding!

Analyze the the truncation error of the Backward Euler scheme and show that it is  $\mathcal{O}(\Delta t)$  (first order scheme).

# Truncation error of the $\theta$ -rule

The  $\theta$ -rule:

$$[\bar{D}_t u = -a\bar{u}^{t,\theta}]^{n+\theta}.$$

Truncation error:

$$[\bar{D}_t u_e + a\bar{u}_e^{t,\theta} = R]^{n+\theta}.$$

Use (9) and (12) along with  $u'_e(t_{n+\theta}) + au_e(t_{n+\theta}) = 0$  to show

$$\begin{aligned} R^{n+\theta} = & \left(\frac{1}{2} - \theta\right) u''_e(t_{n+\theta}) \Delta t + \frac{1}{2} \theta (1 - \theta) u''_e(t_{n+\theta}) \Delta t^2 + \\ & \frac{1}{2} (\theta^2 - \theta + 3) u'''_e(t_{n+\theta}) \Delta t^2 + \mathcal{O}(\Delta t^3) \end{aligned} \quad (22)$$

Note: 2nd-order scheme if and only if  $\theta = 1/2$ .

# Using symbolic software

Can use sympy and the tools in `truncation_errors.py`:

```
def decay():
    u, a = sm.symbols('u a')
    diffop = DiffOp(u, independent_variable='t',
                    num_terms_Taylor_series=3)
    D1u = diffop.D(1)      # symbol for du/dt
    ODE = D1u + a*u        # define ODE

    # Define schemes
    FE = diffop['Dtp'] + a*u
    CN = diffop['Dt' ] + a*u
    BE = diffop['Dtm'] + a*u
    # Residuals (truncation errors)
    R = {'FE': FE-ODE, 'BE': BE-ODE, 'CN': CN-ODE}
    return R
```

The returned dictionary becomes

```
decay: {
  'BE': D2u*dt/2 + D3u*dt**2/6,
  'FE': -D2u*dt/2 + D3u*dt**2/6,
  'CN': D3u*dt**2/24,
}
```

$\theta$ -rule: see `truncation_errors.py` (long expression, very advantageous to automate the math!)

# Empirical verification of the truncation error (1)

Ideas:

- Compute  $R^n$  numerically
- Run a sequence of meshes
- Estimate the convergence rate of  $R^n$

For the Forward Euler scheme:

$$R^n = [D_t^+ u_e + a u_e]^n. \quad (23)$$

Insert correct  $u_e(t) = Ie^{-at}$  (or use method of manufactured solution in more general cases).



## Empirical verification of the truncation error (2)

- Assume  $R^n = C\Delta t^r$
- $C$  and  $r$  will vary with  $n$  - must estimate  $r$  for each mesh point
- Use a sequence of meshes with  $N_t = 2^{-k}N_0$  intervals,  
 $k = 1, 2, \dots$
- Transform  $R^n$  data to the coarsest mesh and estimate  $r$  for each coarse mesh point

See the text for more details and an implementation.

# Empirical verification of the truncation error in the Forward Euler scheme

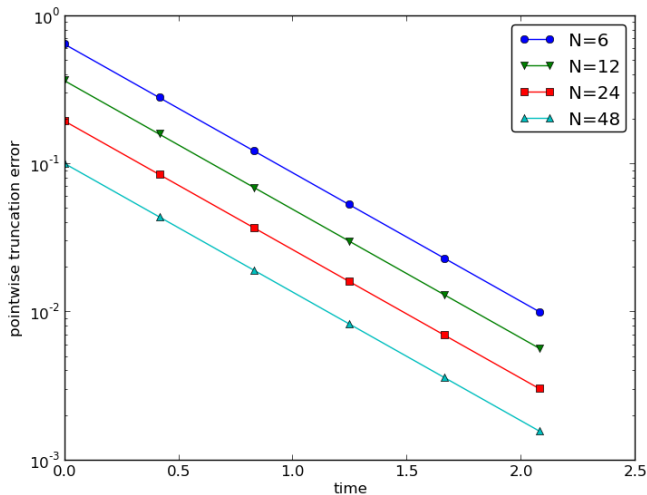


Figure: Estimated truncation error at mesh points for different meshes

# Empirical verification of the truncation error in the Forward Euler scheme

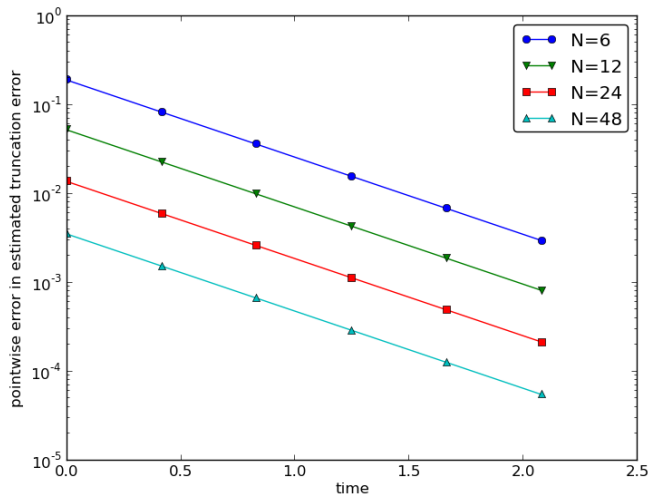


Figure: Difference between theoretical and estimated truncation error at

# Increasing the accuracy by adding correction terms

## Question.

Can we add terms in the differential equation that can help increase the order of the truncation error?

To be precise for the Forward Euler scheme, can we find  $C$  to make  $R \mathcal{O}(\Delta t^2)$ ?

$$[D_t^+ u_e + a u_e = C + R]^n. \quad (24)$$

$$\frac{1}{2} u_e''(t_n) \Delta t - \frac{1}{6} u_e'''(t_n) \Delta t^2 + \mathcal{O}(\Delta t^3) = C^n + R^n.$$

Choosing

$$C^n = \frac{1}{2} u_e''(t_n) \Delta t,$$

makes

$$R^n = \frac{1}{6} u_e'''(t_n) \Delta t^2 + \mathcal{O}(\Delta t^3).$$

## Lowering the order of the derivative in the correction term

- $C^n$  contains  $u''$
- Can discretize  $u''$  (requires  $u^{n+1}$ ,  $u^n$ , and  $u^{n-1}$ )
- Can also express  $u''$  in terms of  $u'$  or  $u$

$$u' = -au, \quad \Rightarrow \quad u'' = -au' = a^2 u.$$

Result for  $u'' = a^2 u$ : apply Forward Euler to a *perturbed ODE*,

$$u' = -\hat{a}u, \quad \hat{a} = a\left(1 - \frac{1}{2}a\Delta t\right), \quad (25)$$

to make a second-order scheme!

## With a correction term Forward Euler becomes Crank-Nicolson

Use the other alternative  $u'' = -au'$ :

$$u' = -au - \frac{1}{2}a\Delta t u' \quad \Rightarrow \quad \left(1 + \frac{1}{2}a\Delta t\right) u' = -au.$$

Apply Forward Euler:

$$\left(1 + \frac{1}{2}a\Delta t\right) \frac{u^{n+1} - u^n}{\Delta t} = -au^n,$$

which after some algebra can be written as

$$u^{n+1} = \frac{1 - \frac{1}{2}a\Delta t}{1 + \frac{1}{2}a\Delta t} u^n.$$

This is a Crank-Nicolson scheme (of second order)!

# Correction terms in the Crank-Nicolson scheme (1)

$$[D_t u = -a\bar{u}^t]^{n+\frac{1}{2}},$$

Definition of the truncation error  $R$  and correction terms  $C$ :

$$[D_t u_e + a\bar{u}_e^t = C + R]^{n+\frac{1}{2}}.$$

Must Taylor expand

- the derivative
- the arithmetic mean

$$C^{n+\frac{1}{2}} + R^{n+\frac{1}{2}} = \frac{1}{24} u_e'''(t_{n+\frac{1}{2}}) \Delta t^2 + \frac{a}{8} u_e''(t_{n+\frac{1}{2}}) \Delta t^2 + \mathcal{O}(\Delta t^4).$$

Let  $C^{n+\frac{1}{2}}$  cancel the  $\Delta t^2$  terms:

$$C^{n+\frac{1}{2}} = \frac{1}{24} u_e'''(t_{n+\frac{1}{2}}) \Delta t^2 + \frac{a}{8} u_e''(t_n) \Delta t^2.$$

## Correction terms in the Crank-Nicolson scheme (2)

- Must replace  $u'''$  and  $u''$  in correction term
- Using  $u' = -au$ :  $u'' = a^2u$  and  $u''' = -a^3u$

Result: solve the perturbed ODE by a Crank-Nicolson method,

$$u' = -\hat{a}u, \quad \hat{a} = a\left(1 - \frac{1}{12}a^2\Delta t^2\right).$$

and experience an error  $\mathcal{O}(\Delta t^4)$ .



## Extension to variable coefficients

$$u'(t) = -a(t)u(t) + b(t)$$

Forward Euler:

$$[D_t^+ u = -au + b]^n. \quad (26)$$

The truncation error is found from

$$[D_t^+ u_e + au_e - b = R]^n. \quad (27)$$

Using (8):

$$u'_e(t_n) - \frac{1}{2}u''_e(t_n)\Delta t + \mathcal{O}(\Delta t^2) + a(t_n)u_e(t_n) - b(t_n) = R^n.$$

Because of the ODE,  $u'_e(t_n) + a(t_n)u_e(t_n) - b(t_n) = 0$ , and

$$R^n = -\frac{1}{2}u''_e(t_n)\Delta t + \mathcal{O}(\Delta t^2). \quad (28)$$

No problems with variable coefficients!

# Exact solutions of the finite difference equations

How does the truncation error depend on  $u_e$  in finite differences?

- One-sided differences:  $u_e'' \Delta t$  (lowest order)
- Centered differences:  $u_e''' \Delta t^2$  (lowest order)
- Only harmonic and geometric mean involve  $u_e'$  or  $u_e$

Consequence:

- $u_e(t) = ct + d$  will very often give exact solution of the discrete equations ( $R = 0$ )!
- Ideal for verification
- Centered schemes allow quadratic  $u_e$

Problem: harmonic and geometric mean (error depends on  $u_e'$  and  $u_e$ )

# Computing truncation errors in nonlinear problems (1)

$$u' = f(u, t) \quad (29)$$

Crank-Nicolson scheme:

$$[D_t u' = \bar{f}^t]^{n+\frac{1}{2}}. \quad (30)$$

Truncation error:

$$[D_t u'_e - \bar{f}^t = R]^{n+\frac{1}{2}}. \quad (31)$$

Using (13) for the arithmetic mean:

$$\begin{aligned} [\bar{f}^t]^{n+\frac{1}{2}} &= \frac{1}{2}(f(u_e^n, t_n) + f(u_e^{n+1}, t_{n+1})) \\ &= f(u_e^{n+\frac{1}{2}}, t_{n+\frac{1}{2}}) + \frac{1}{8}u_e''(t_{n+\frac{1}{2}})\Delta t^2 + \mathcal{O}(\Delta t^4). \end{aligned}$$

## Computing truncation errors in nonlinear problems (2)

With (5), (31) leads to  $R^{n+\frac{1}{2}}$  equal to

$$u_e'(t_{n+\frac{1}{2}}) + \frac{1}{24} u_e'''(t_{n+\frac{1}{2}}) \Delta t^2 - f(u_e^{n+\frac{1}{2}}, t_{n+\frac{1}{2}}) - \frac{1}{8} u_e''(t_{n+\frac{1}{2}}) \Delta t^2 + \mathcal{O}(\Delta t^4).$$

Since  $u_e'(t_{n+\frac{1}{2}}) - f(u_e^{n+\frac{1}{2}}, t_{n+\frac{1}{2}}) = 0$ , the truncation error becomes

$$R^{n+\frac{1}{2}} = \left( \frac{1}{24} u_e'''(t_{n+\frac{1}{2}}) - \frac{1}{8} u_e''(t_{n+\frac{1}{2}}) \right) \Delta t^2.$$

The computational techniques worked well even for this *nonlinear* ODE!

# Truncation errors in vibration ODEs

## Linear model without damping

$$u''(t) + \omega^2 u(t) = 0, \quad u(0) = I, \quad u'(0) = 0. \quad (32)$$

Centered difference approximation:

$$[D_t D_t u + \omega^2 u = 0]^n. \quad (33)$$

Truncation error:

$$[D_t D_t u_e + \omega^2 u_e = R]^n. \quad (34)$$

Use (11) to expand  $[D_t D_t u_e]^n$ :

$$[D_t D_t u_e]^n = u_e''(t_n) + \frac{1}{12} u_e''''(t_n) \Delta t^2,$$

Collect terms:  $u_e''(t) + \omega^2 u_e(t) = 0$ . Then,

$$R^n = \frac{1}{12} u_e''''(t_n) \Delta t^2 + \mathcal{O}(\Delta t^4). \quad (35)$$

# Truncation errors in the initial condition

- Initial conditions:  $u(0) = I$ ,  $u'(0) = V$
- Need discretization of  $u'(0)$
- Standard, centered difference:  $[D_{2t}u = V]^0$ ,  $R^0 = \mathcal{O}(\Delta t^2)$
- Simpler, forward difference:  $[D_t^+u = V]^0$ ,  $R^0 = \mathcal{O}(\Delta t)$
- Does the lower order of the forward scheme impact the order of the whole simulation?
- Answer: run experiments!

# Computing correction terms

- Can we add terms to the ODE such that the truncation error is improved?

$$[D_t D_t u_e + \omega^2 u_e = C + R]^n,$$

- Idea: choose  $C^n$  such that it absorbs the  $\Delta t^2$  term in  $R^n$ ,

$$C^n = \frac{1}{12} u_e''''(t_n) \Delta t^2.$$

- Downside: got a  $u''''$  term
- Remedy: use the ODE  $u'' = -\omega^2 u$  to see that  $u'''' = \omega^4 u$ .
- Just apply the standard scheme to a modified ODE:

$$[D_t D_t u + \omega^2 (1 - \frac{1}{12} \omega^2 \Delta t^2) u = 0]^n,$$

- Accuracy is  $\mathcal{O}(\Delta t^4)$ .



## Model with damping and nonlinearity

Linear damping  $\beta u'$ , nonlinear spring force  $s(u)$ , and excitation  $F$ :

$$mu'' + \beta u' + s(u) = F(t). \quad (36)$$

Central difference discretization:

$$[mD_t D_t u + \beta D_{2t} u + s(u) = F]^n. \quad (37)$$

Truncation error is defined by

$$[mD_t D_t u_e + \beta D_{2t} u_e + s(u_e) = F + R]^n. \quad (38)$$

# Carrying out the truncation error analysis

Using (11) and (6) we get

$$[mD_t D_t u_e + \beta D_{2t} u_e]^n = mu_e''(t_n) + \beta u_e'(t_n) + \left( \frac{m}{12} u_e''''(t_n) + \frac{\beta}{6} u_e'''(t_n) \right) \Delta t^2 + \mathcal{O}(\Delta t^4)$$

The terms

$$mu_e''(t_n) + \beta u_e'(t_n) + \omega^2 u_e(t_n) + s(u_e(t_n)) - F^n,$$

correspond to the ODE (= zero).

Result: accuracy of  $\mathcal{O}(\Delta t^2)$  since

$$R^n = \left( \frac{m}{12} u_e''''(t_n) + \frac{\beta}{6} u_e'''(t_n) \right) \Delta t^2 + \mathcal{O}(\Delta t^4), \quad (39)$$

Correction terms: complicated when the ODE has many terms...

## Extension to quadratic damping

$$mu'' + \beta |u'|u' + s(u) = F(t). \quad (40)$$

Centered scheme:  $|u'|u'$  gives rise to a nonlinearity.

Linearization trick: use a geometric mean,

$$[|u'|u']^n \approx |[u']^{n-\frac{1}{2}}|[u']^{n+\frac{1}{2}}.$$

Scheme:

$$[mD_tD_tu]^n + \beta |[D_tu]^{n-\frac{1}{2}}|[D_tu]^{n+\frac{1}{2}} + s(u^n) = F^n. \quad (41)$$

# The truncation error for quadratic damping (1)

Definition of  $R^n$ :

$$[mD_t D_t u_e]^n + \beta |[D_t u_e]^{n-\frac{1}{2}}| [D_t u_e]^{n+\frac{1}{2}} + s(u_e^n) - F^n = R^n. \quad (42)$$

Truncation error of the geometric mean, see (14),

$$\begin{aligned} |[D_t u_e]^{n-\frac{1}{2}}| [D_t u_e]^{n+\frac{1}{2}} &= [|D_t u_e| D_t u_e]^n - \frac{1}{4} u'(t_n)^2 \Delta t^2 + \\ &\quad \frac{1}{4} u(t_n) u''(t_n) \Delta t^2 + \mathcal{O}(\Delta t^4). \end{aligned}$$

Using (5) for the  $D_t u_e$  factors results in

$$\begin{aligned} [|D_t u_e| D_t u_e]^n &= |u'_e + \frac{1}{24} u_e'''(t_n) \Delta t^2 + \mathcal{O}(\Delta t^4)| \times \\ &\quad (u'_e + \frac{1}{24} u_e'''(t_n) \Delta t^2 + \mathcal{O}(\Delta t^4)) \end{aligned}$$

## The truncation error for quadratic damping (2)

For simplicity, remove the absolute value. The product becomes

$$[D_t u_e D_t u_e]^n = (u_e'(t_n))^2 + \frac{1}{12} u_e(t_n) u_e'''(t_n) \Delta t^2 + \mathcal{O}(\Delta t^4).$$

With

$$m[D_t D_t u_e]^n = m u_e''(t_n) + \frac{m}{12} u_e''''(t_n) \Delta t^2 + \mathcal{O}(\Delta t^4),$$

and using  $mu'' + \beta(u')^2 + s(u) = F$ , we end up with

$$R^n = (\frac{m}{12} u_e''''(t_n) + \frac{\beta}{12} u_e(t_n) u_e'''(t_n)) \Delta t^2 + \mathcal{O}(\Delta t^4).$$

Second-order accuracy! Thanks to

- difference approximation with error  $\mathcal{O}(\Delta t^2)$
- geometric mean approximation with error  $\mathcal{O}(\Delta t^2)$

# The general model formulated as first-order ODEs

$$mu'' + \beta|u'|u' + s(u) = F(t). \quad (43)$$

Rewritten as first-order system:

$$u' = v, \quad (44)$$

$$v' = \frac{1}{m} (F(t) - \beta|v|v - s(u)). \quad (45)$$

To solution methods:

- Forward-backward scheme
- Centered scheme on a staggered mesh

# The forward-backward scheme

Forward step for  $u$ , backward step for  $v$ :

$$[D_t^+ u = v]^n, \quad (46)$$

$$[D_t^- v = \frac{1}{m}(F(t) - \beta|v|v - s(u))]^{n+1}. \quad (47)$$

- Note:

- step  $u$  forward with known  $v$  in (46)
- step  $v$  forward with known  $u$  in (47)
- Problem:  $|v|v$  gives nonlinearity  $|v^{n+1}|v^{n+1}$ .
- Remedy: linearized as  $|v^n|v^{n+1}$

$$[D_t^+ u = v]^n, \quad (48)$$

$$[D_t^- v]^{n+1} = \frac{1}{m}(F(t_{n+1}) - \beta|v^n|v^{n+1} - s(u^{n+1})). \quad (49)$$

# Truncation error analysis

- Aim (as always): turn difference operators into derivatives + truncation error terms
- One-sided forward/backward differences: error  $\mathcal{O}(\Delta t)$
- Linearization of  $|v^{n+1}|v^{n+1}$  to  $|v^n|v^{n+1}$ : error  $\mathcal{O}(\Delta t)$
- All errors are  $\mathcal{O}(\Delta t)$
- First-order scheme? No!
- "Symmetric" use of the  $\mathcal{O}(\Delta t)$  building blocks yields in fact a  $\mathcal{O}(\Delta t^2)$  scheme (!)
- Why? See next slide...



# A centered scheme on a staggered mesh

Staggered mesh:

- $u$  is computed at mesh points  $t_n$
- $v$  is computed at points  $t_{n+\frac{1}{2}}$

Centered differences in (44)-(44):

$$[D_t u = v]^{n-\frac{1}{2}}, \quad (50)$$

$$[D_t v = \frac{1}{m}(F(t) - \beta|v|v - s(u))]^n. \quad (51)$$

- Problem:  $|v^n|v^n$ , because  $v^n$  is not computed directly
- Remedy: Geometric mean,

$$|v^n|v^n \approx |v^{n-\frac{1}{2}}|v^{n+\frac{1}{2}}.$$

# Truncation error analysis (1)

Resulting scheme:

$$[D_t u]^{n-\frac{1}{2}} = v^{n-\frac{1}{2}}, \quad (52)$$

$$[D_t v]^n = \frac{1}{m}(F(t_n) - \beta |v^{n-\frac{1}{2}}| v^{n+\frac{1}{2}} - s(u^n)). \quad (53)$$

The truncation error in each equation is found from

$$[D_t u_e]^{n-\frac{1}{2}} = v_e(t_{n-\frac{1}{2}}) + R_u^{n-\frac{1}{2}},$$

$$[D_t v_e]^n = \frac{1}{m}(F(t_n) - \beta |v_e(t_{n-\frac{1}{2}})| v_e(t_{n+\frac{1}{2}}) - s(u^n)) + R_v^n.$$

Using (5) for derivatives and (14) for the geometric mean:

$$u'_e(t_{n-\frac{1}{2}}) + \frac{1}{24} u_e'''(t_{n-\frac{1}{2}}) \Delta t^2 + \mathcal{O}(\Delta t^4) = v_e(t_{n-\frac{1}{2}}) + R_u^{n-\frac{1}{2}},$$

and

$$v'_e(t_n) = \frac{1}{m}(F(t_n) - \beta |v_e(t_n)| v_e(t_n) + \mathcal{O}(\Delta t^2) - s(u^n)) + R_v^n.$$

Resulting truncation error is  $\mathcal{O}(\Delta t^2)$ :

$$R_u^{n-\frac{1}{2}} = \mathcal{O}(\Delta t^2), \quad R_v^n = \mathcal{O}(\Delta t^2).$$

### Observation.

Comparing The schemes (52)-(53) and (48)-(49) are equivalent. Therefore, the forward/backward scheme with ad hoc linearization is also  $\mathcal{O}(\Delta t^2)$ !