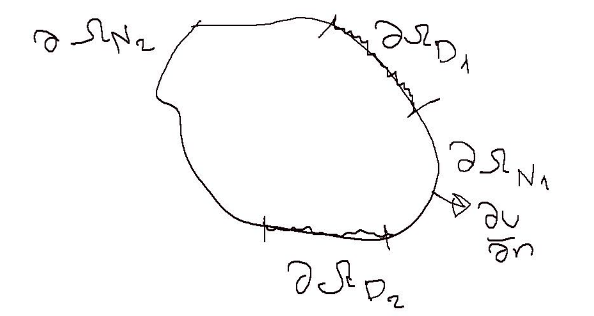
The diffusion equation

$$u_t = \alpha \nabla^2 u \qquad \text{1D: } u_t = \alpha u_{xx}$$



0 — L

$u(0) = C$ or $u'(0) = C$ $\qquad$ $u(L) = D$ or $u'(L) = D$

Rule: one condition at each point of the boundary
(true for PDEs w/ second-order derivatives in space)

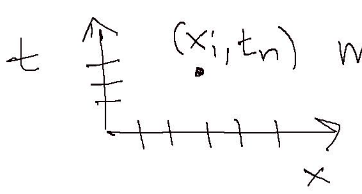
Initial condition: u known at $t=0$
(one init. cond. because of first-order derivative in time)

Model problem:

$$u_t = \alpha u_{xx}, \quad x \in (0,L), \quad t \in (0,T]$$

$$u(0) = 0$$
$$u'(L) = D$$
$$u(x,0) = I(x)$$

Finite differences:



$$u_t(x_i,t_n) = \alpha u_{xx}(x_i,t_n)$$
$$\downarrow \qquad\qquad \downarrow$$
$$[D_t^+ u]_i^n = \alpha [D_x D_x u]_i^n$$
$$\downarrow$$
$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = \alpha \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\Delta x^2}$$

only unknown

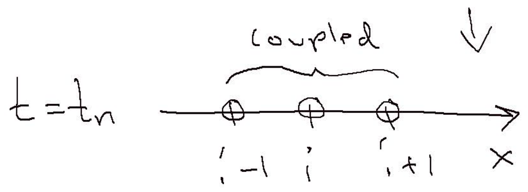
$$\Rightarrow u_i^{n+1} = u_i^n + \alpha \frac{\Delta t}{\Delta x^2}\left(u_{i+1}^n - 2u_i^n + u_{i-1}^n\right)$$

explicit formula for new values (explicit scheme)
Same alg. & programming as for the wave eq.

Alternative: Backward Euler approx. in time

$$u_t(x_i,t_n) \approx [D_t^- u]_i^n = \frac{u^n - u^{n-1}}{\Delta t}$$

$$\Rightarrow \underbrace{u_i^n + \alpha \frac{\Delta t}{\Delta x^2}\left(u_{i-1}^n - 2u_i^n + u_{i+1}^n\right)}_{\text{unknowns}} = \underbrace{u_i^{n-1}}_{\text{known}}$$



Cannot solve wrt u_i^n, because the formula involves other unknown quantities u_{i-1}^n, u_{i+1}^n

We have a linear system:

$A_{i,j}$: $\quad A_{i,i} = 1 + 2C, \quad C = \alpha \frac{\Delta t}{\Delta x^2}$
$\quad A_{i,i-1} = -C$
$\quad A_{i,i+1} = -C$

$$\begin{pmatrix} 1+2C & -C & & 0 \\ -C & 1+2C & \ddots & \\ 0 & & \ddots & \end{pmatrix} \begin{pmatrix} u_1^n \\ \vdots \\ u_N^n \end{pmatrix} = \begin{pmatrix} u_1^{n-1} \\ \vdots \\ u_N^{n-1} \end{pmatrix}$$

Tridiagonal system at every time level