

# INF5620 Lecture: Analysis of finite difference schemes for diffusion processes

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# Properties of the solution

The PDE

$$u_t = \alpha u_{xx} \quad (1)$$

admits solutions

$$u(x, t) = Qe^{-\alpha k^2 t} \sin(kx) \quad (2)$$

Observations from this solution:

- The initial shape  $I(x) = Q \sin kx$  undergoes a damping  $\exp(-\alpha k^2 t)$
- The damping is very strong for short waves (large  $k$ )
- The damping is weak for long waves (small  $k$ )
- Consequence:  $u$  is smoothened with time

# Example

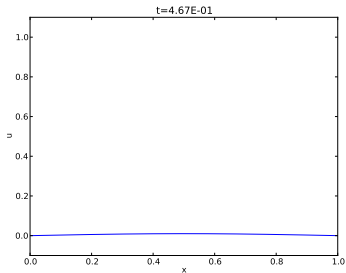
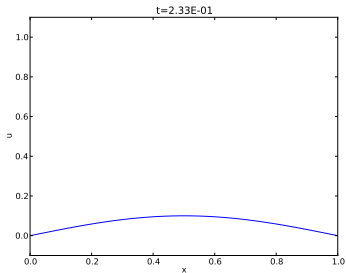
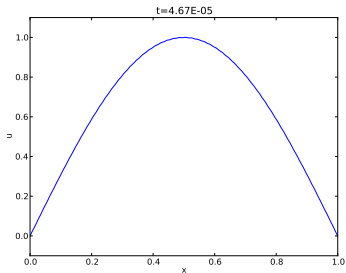
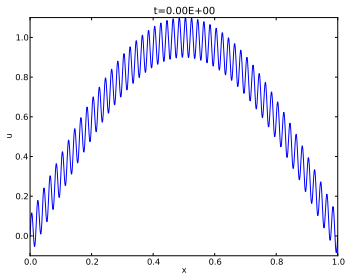
Test problem:

$$\begin{aligned}u_t &= u_{xx}, & x \in (0, 1), \quad t \in (0, T] \\u(0, t) &= u(1, t) = 0, & t \in (0, T] \\u(x, 0) &= \sin(\pi x) + 0.1 \sin(100\pi x)\end{aligned}$$

Exact solution:

$$u(x, t) = e^{-\pi^2 t} \sin(\pi x) + 0.1 e^{-\pi^2 10^4 t} \sin(100\pi x) \quad (3)$$

# Visualization of the damping in the diffusion equation



# Damping of a discontinuity; problem and model

## Problem.

Two pieces of a material, at different temperatures, are brought in contact at  $t = 0$ . Assume the end points of the pieces are kept at the initial temperature. How does the heat flow from the hot to the cold piece?

## Solution.

Assume a 1D model is sufficient (insulated rod):

$$\begin{aligned}u_t &= \alpha u_{xx} \\ u(x, 0) &= \begin{cases} U_L, & x < L/2 \\ U_R, & x \geq L/2 \end{cases} \\ u(0, t) &= U_L \\ u(L, t) &= U_R\end{aligned}$$

# Damping of a discontinuity; Backward Euler simulation

Movie

# Damping of a discontinuity; Forward Euler simulation

Movie

# Damping of a discontinuity; Crank-Nicolson simulation

Movie



# Fourier representation

Represent  $I(x)$  as a Fourier series

$$I(x) \approx \sum_{k \in K} b_k e^{ikx} \quad (4)$$

The corresponding sum for  $u$  is

$$u(x, t) \approx \sum_{k \in K} b_k e^{-\alpha k^2 t} e^{ikx}. \quad (5)$$

Such solutions are also accepted by the numerical schemes, but with an amplification factor different from  $\exp(-\alpha k^2 t)$ :

$$u_q^n = A^n e^{ikq\Delta x} = A^n e^{ikx} \quad (6)$$

# Analysis of the finite difference schemes

Stability:

- $|A| < 1$ : decaying numerical solutions (as we want)
- $A < 0$ : *oscillating* numerical solutions (as we do not want)

Accuracy:

- Compare numerical and exact amplification factor

$$A \quad \text{vs} \quad A_e = \exp(-\alpha k^2 \Delta t)$$

# Analysis of the Forward Euler scheme

$$[D_t^+ u = \alpha D_x D_x u]_q^n$$

Inserting

$$u_q^n = A^n e^{ikq\Delta x}$$

leads to

$$A = 1 - 4C \sin^2 \left( \frac{k\Delta x}{2} \right), \quad C = \frac{\alpha \Delta t}{\Delta x^2} \quad (7)$$

The complete numerical solution is

$$u_q^n = (1 - 4C \sin^2 p)^n e^{ikq\Delta x}, \quad p = k\Delta x/2 \quad (8)$$

## Results for stability

$A \leq 1$ , but  $A < -1$  is a possibility:

$$4C \sin^2 p \leq 2$$

The worst case is when  $\sin^2 p = 1$ , so a sufficient criterion for stability is

$$C \leq \frac{1}{2} \quad (9)$$

or:

$$\Delta t \leq \frac{\Delta x^2}{2\alpha} \quad (10)$$

### Implications of the stability result.

Less favorable criterion than for  $u_{tt} = c^2 u_{xx}$ : halving  $\Delta x$  implies time step  $\frac{1}{4}\Delta t$  (not just  $\frac{1}{2}\Delta t$  as in a wave equation). Need very small time steps for fine spatial meshes!

# Analysis of the Backward Euler scheme

$$[D_t^- u = \alpha D_x D_x u]_q^n$$

$$u_q^n = A^n e^{ikq\Delta x}$$

$$A = (1 + 4C \sin^2 p)^{-1} \quad (11)$$

$$u_q^n = (1 + 4C \sin^2 p)^{-n} e^{ikq\Delta x} \quad (12)$$

We see from (11) that  $|A| < 1$  for all  $\Delta t > 0$  and that  $A > 0$  (no oscillations).

# Analysis of the Crank-Nicolson scheme

The scheme

$$[D_t u = \alpha D_x D_x \bar{u}^x]_q^{n+\frac{1}{2}}$$

leads to

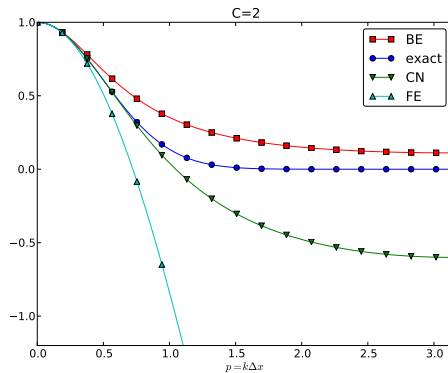
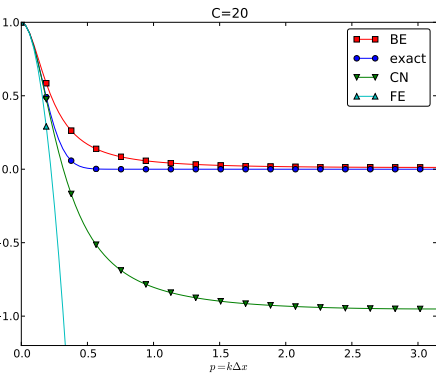
$$A = \frac{1 - 2C \sin^2 p}{1 + 2C \sin^2 p} \quad (13)$$

$$u_q^n = \left( \frac{1 - 2C \sin^2 p}{1 + 2C \sin^2 p} \right)^n e^{ikp\Delta x} \quad (14)$$

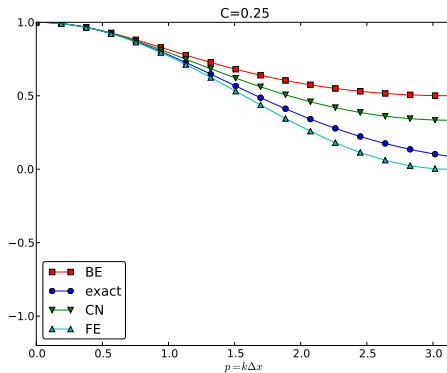
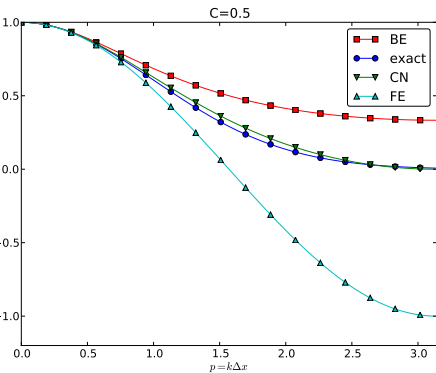
The criteria  $A > -1$  and  $A < 1$  are fulfilled for any  $\Delta t > 0$ .



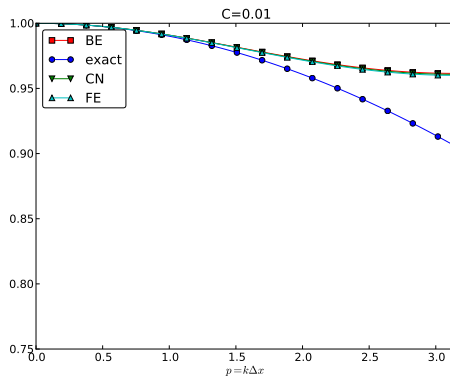
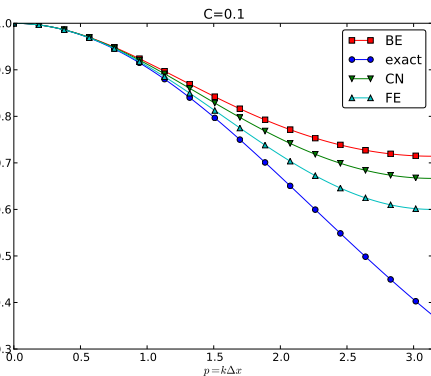
# Summary of accuracy of amplification factors; large time steps



# Summary of accuracy of amplification factors; time steps around the Forward Euler stability limit



# Summary of accuracy of amplification factors; small time steps



- Crank-Nicolson gives oscillations and not much damping of short waves for increasing  $C$ .
- These waves will manifest themselves as high frequency oscillatory noise in the solution.
- All schemes fail to dampen short waves enough

The problems of correct damping for  $u_t = u_{xx}$  is partially manifested in the similar time discretization schemes for  $u'(t) = -\alpha u(t)$ .