Given $\frac{d}{dx}\left(\alpha(u)\frac{du}{dx}\right) = 0$ on $[0,1]$, $u(0)=0$, $u(1)=1$

Derive the discrete nonlinear equations using both the finite element and difference method.

Finite differences:

$$\frac{1}{\Delta x}\left(\alpha_{i+1/2}\frac{u_{i+1}-u_i}{\Delta x} - \alpha_{i-1/2}\frac{u_i-u_{i-1}}{\Delta x}\right) = 0\,, \quad \alpha_{i+1/2} = \alpha(u_{i+1/2})$$

Arithmetic mean for $\alpha_{i+1/2}$:

$$\alpha_{i+1/2} \approx \frac{1}{2}\left(\alpha(u_i) + \alpha(u_{i+1})\right)$$

Result:

$$F_i = \frac{1}{2\Delta x^2}\left(\left(\alpha(u_i)+\alpha(u_{i+1})\right)(u_{i+1}-u_i) - \left(\alpha(u_{i-1})+\alpha(u_i)\right)(u_i-u_{i-1})\right) = 0$$

Finite elements:

Need a variational formulation

$$\int_0^1 \frac{d}{dx}\left(\alpha\frac{du}{dx}\right) v\,dx = 0 \quad \forall\, v\in V,\ v(0)=v(1)=0 \text{ because of the Dirichlet cond. on } u$$

Integration by parts:

$$\int_0^1 \alpha(u)u'v'\,dx = 0 \quad \forall v\in V \quad \left(\text{note: } [vu']_0^1 = 0 \text{ since } v(0)=v(1)=0\right)$$

Linear system: $u = B(x) + \sum_{j=1}^{N-1} c_j\varphi_j$, $B(x) = x$ or $B(x) = \varphi_N(x)$

$$\sum_{j=1}^{N-1}\left(\int_0^1 \alpha(\underset{\sum_k c_k\varphi_k}{\underbrace{u}})\varphi_j'\varphi_i'\,dx\right)c_j = -\int_0^1 \alpha(\underset{\sum_k c_k\varphi_k}{\underbrace{u}})B'(x)\varphi_i'\,dx$$

The right-hand side only concerns Dirichlet values so we drop paying attention to it in the following.

Element matrix for element no. i:

$$\tilde{A}^{(i)}_{rs} = \int_{-1}^{1}\alpha(u)\frac{2}{h}\underset{\pm\frac{1}{2}}{\underbrace{\frac{d\tilde{\varphi}_r}{d\bar{X}}}}\cdot\frac{2}{h}\underset{\pm\frac{1}{2}}{\underbrace{\frac{d\tilde{\varphi}_s}{d\bar{X}}}}\frac{h}{2}\,d\bar{X}\,, \quad h = \text{element length}$$

$$= \frac{1}{2h}(-1)^r(-1)^s\int_{-1}^{1}\alpha(u)\,d\bar{X}$$

u can be expanded locally on the element by

$$u = \sum_r \tilde{\varphi}_r(\bar{X})\underset{\text{global coefficient}}{\underbrace{c_{q(i,r)}}}$$

In 1D with numbering of nodes and elements from left to right we have $q(i,r) = i+r$. Since $c_j = u$ at node no. j we introduce u_j for c_j since then we see the similarity to finite differences easily.

$u = \sum_r \tilde{\varphi}_r(\bar{X})u_{r+i}$ is the expansion on the reference element.

Trapezoidal integration:

$$\int_{-1}^{1} g(\bar{X})\,dx \approx g(-1) + g(1)$$

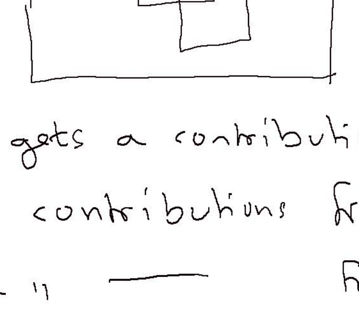
$$\int_{-1}^{1}\alpha(u)\,dx \approx \alpha\Big(\sum_r \underset{\neq 0 \text{ for } r=0}{\underbrace{\tilde{\varphi}_r(-1)}}u_{r+i}\Big) + \alpha\Big(\sum_r \underset{\neq 0 \text{ for } r=1}{\underbrace{\tilde{\varphi}_r(1)}}u_{r+i}\Big)$$

$$= \alpha(u_i) + \alpha(u_{i+1})$$

$$\Rightarrow \tilde{A}^{(i)}_{rs} = \frac{1}{h}\underset{\text{arithmetic mean!}}{\underbrace{\frac{1}{2}\left(\alpha(u_i)+\alpha(u_{i+1})\right)}}\underset{\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}}{\underbrace{(-1)^r(-1)^s}}$$

$$\tilde{A}^{(i)} = \frac{1}{2}\left(\alpha(u_i)+\alpha(u_{i+1})\right)\underset{\text{the usual element matrix arising from a } u'' \text{ term}}{\underbrace{\frac{1}{h}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}}}$$

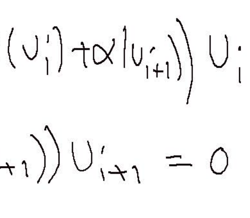
Assembly:

$\tilde{A}^{(i)} \rightarrow$  $\leftarrow$ row no. i represents equation no. i

Row no. i gets a contribution to $A_{i-1,i}$ from $\tilde{A}^{(i-1)}_{1,0}$

$A_{i,i}$ gets contributions from $\tilde{A}^{(i-1)}_{1,1}$ and $\tilde{A}^{(i)}_{0,0}$

$A_{i,i+1}$ — " — from $\tilde{A}^{(i)}_{0,1}$



$$A_{i,i-1}u_{i-1} + A_{i,i}u_i + A_{i,i+1}u_{i+1} = 0 \quad \text{eq. no. } i$$

$$-\frac{1}{h}\frac{1}{2}\left(\alpha(u_{i-1})+\alpha(u_i)\right)u_{i-1} + \frac{1}{h}\frac{1}{2}\left(\alpha(u_{i-1})+\alpha(u_i)\right)u_i + \frac{1}{h}\frac{1}{2}\left(\alpha(u_i)+\alpha(u_{i+1})\right)u_i$$
$$-\frac{1}{h}\frac{1}{2}\left(\alpha(u_i)+\alpha(u_{i+1})\right)u_{i+1} = 0$$

Can be reordered to

$$F_i = \frac{1}{2\Delta x^2}\left(\left(\alpha(u_i)+\alpha(u_{i+1})\right)(u_{i+1}-u_i) - \left(\alpha(u_{i-1})+\alpha(u_i)\right)(u_i-u_{i-1})\right) = 0$$

(divide by h, set $\Delta x = h$)

Conclusion: Same discrete equations as in the finite difference method.

Remark: Without the Trapezoidal integration we would not be able to write anything more specific but $\int_{-1}^{1}\alpha(u)\,d\bar{X}$.

Exercise 2: Use the group finite element method for the above problem. Show that we get the same equations.

Group finite elements: $\alpha(u) \approx \sum_j \varphi_j\alpha(c_j)$ or $\sum_j \varphi_j\alpha(u_j)$

Over a reference element:

$$\alpha(u) \approx \sum_r \tilde{\varphi}_r(\bar{X})\alpha(u_{i+r})$$

$$\int_{-1}^{1}\alpha(u)\,d\bar{X} \approx \alpha(u)\big|_{\bar{X}=-1} + \alpha(u)\big|_{\bar{X}=1}$$

$$= \sum_r \underset{\neq 0 \text{ for } r=0}{\underbrace{\tilde{\varphi}_r(-1)}}\alpha(u_{r+i}) + \sum_r \underset{\neq 0 \text{ for } r=1}{\underbrace{\hat{\varphi}_r(1)}}\alpha(u_{r+i})$$

$$= \alpha(u_i) + \alpha(u_{i+1}) \text{ — the rest is the same!}$$