# Approximation of functions with finite elements

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**PRELIMINARY VERSION**

## Contents

## ıist of Exercises





The finite element method is a powerful tool for solving differential eq
The method can easily deal with complex geometries and higher-order app
tions of the solution. Figure 1 shows a two-dimensional domain with a no
geometry. The idea is to divide the domain into triangles (elements) a
a polynomial approximations to the unknown functions on each triang
method glues these piecewise approximations together to find a global s
Linear and quadratic polynomials over the triangles are particularly po

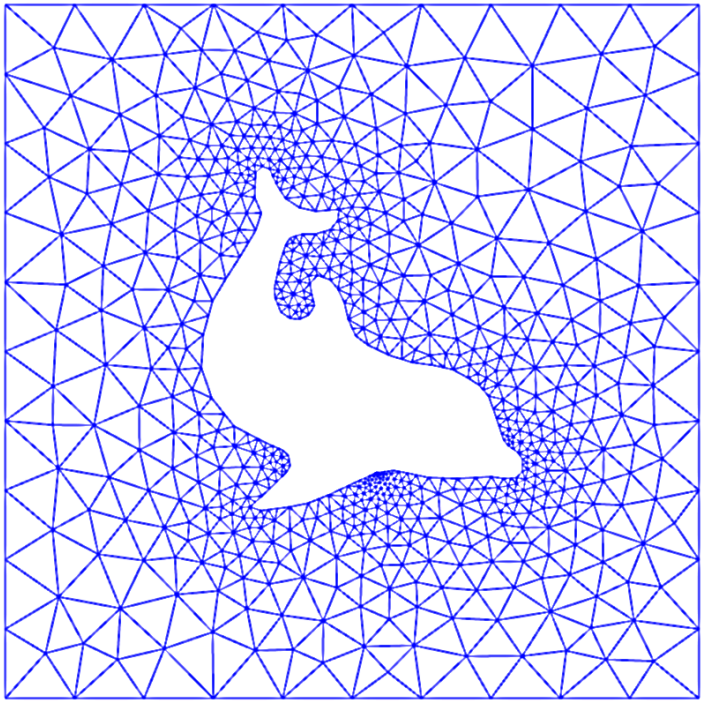


Figure 1: Domain for flow around a dolphin.

Many successful numerical methods for differential equations, includ
finite element method, aim at approximating the unknown function by

$$u(x) = \sum_{i=0}^{N} c_i \psi_i(x),$$

where $\psi_i(x)$ are prescribed functions and $c_0, \ldots, c_N$ are unknown coeffic
be determined. Solution methods for differential equations utilizing (
have a *principle* for constructing $N+1$ equations to determine $c_0, \ldots, c_I$
there is a *machinery* regarding the actual constructions of the equat
$c_0, \ldots, c_N$, in a particular problem. Finally, there is a *solve* phase for co
the solution $c_0, \ldots, c_N$ of the $N+1$ equations.

Especially in the finite element method, the machinery for constructing the iscrete equations to be implemented on a computer is quite comprehensive, with ıany mathematical and implementational details entering the scene at the same me. From an ease-of-learning perspective it can therefore be wise to introduce ıe computational machinery for a trivial equation: $u = f$. Solving this equation ith $f$ given and $u$ on the form (1) means that we seek an approximation to $f$. This approximation problem has the advantage of introducing most f the finite element toolbox, but with postponing demanding topics related ɔ differential equations (e.g., integration by parts, boundary conditions, and ɔordinate mappings). This is the reason why we shall first become familiar ith finite element *approximation* before addressing finite element methods for ifferential equations.

First, we refresh some linear algebra concepts about approximating vectors ı vector spaces. Second, we extend these concepts to approximating functions ı function spaces, using the same principles and the same notation. We present xamples on approximating functions by global basis functions with support ıroughout the entire domain. Third, we introduce the finite element type of ɔcal basis functions and explain the computational algorithms for working with ıch functions. Three types of approximation principles are covered: 1) the least quares method, 2) the $L_2$ projection or Galerkin method, and 3) interpolation r collocation.

# Approximation of vectors

Ve shall start with introducing two fundamental methods for determining the ɔefficients $c_i$ in (1) and illustrate the methods on approximation of vectors, ecause vectors in vector spaces give a more intuitive understanding than starting irectly with approximation of functions in function spaces. The extension om vectors to functions will be trivial as soon as the fundamental ideas are nderstood.

The first method of approximation is called the *least squares method* and ɔnsists in finding $c_i$ such that the difference $u - f$, measured in some norm, is ıinimized. That is, we aim at finding the best approximation $u$ to $f$ (in some orm). The second method is not as intuitive: we find $u$ such that the error $- f$ is orthogonal to the space where we seek $u$. This is known as *projection*, r we may also call it a *Galerkin method*. When approximating vectors and unctions, the two methods are equivalent, but this is no longer the case when pplying the principles to differential equations.

## .1 Approximation of planar vectors

uppose we have given a vector $\boldsymbol{f} = (3, 5)$ in the $xy$ plane and that we want to pproximate this vector by a vector aligned in the direction of the vector $(a, b)$. igure 2 depicts the situation.

We introduce the vector space $V$ spanned by the vector $\boldsymbol{\psi}_0 = (a, b)$:

$$V = \text{span}\,\{\boldsymbol{\psi}_0\}\,. \tag{2}$$



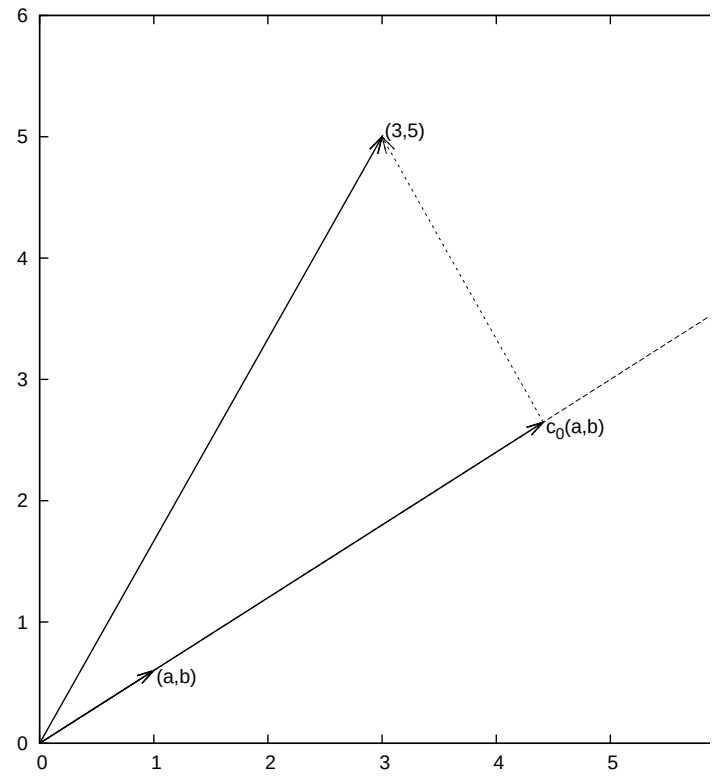



Figure 2: Approximation of a two-dimensional vector by a one-dime vector.

We say that $\boldsymbol{\psi}_0$ is a basis vector in the space $V$. Our aim is to f vector $\boldsymbol{u} = c_0\boldsymbol{\psi}_0 \in V$ which best approximates the given vector $\boldsymbol{f} = ($ reasonable criterion for a best approximation could be to minimize the l the difference between the approximate $\boldsymbol{u}$ and the given $\boldsymbol{f}$. The differ error $\boldsymbol{e} = \boldsymbol{f} - \boldsymbol{u}$, has its length given by the *norm*

$$||\boldsymbol{e}|| = (\boldsymbol{e}, \boldsymbol{e})^{\frac{1}{2}},$$

where $(\boldsymbol{e}, \boldsymbol{e})$ is the *inner product* of $\boldsymbol{e}$ and itself. The inner product, als *scalar product* or *dot product*, of two vectors $\boldsymbol{u} = (u_0, u_1)$ and $\boldsymbol{v} = (v$ defined as

$$(\boldsymbol{u}, \boldsymbol{v}) = u_0v_0 + u_1v_1\,.$$

**Remark 1.** We should point out that we use the notation $(\cdot, \cdot)$ for two things: $(a, b)$ for scalar quantities $a$ and $b$ means the vector starting in th

nd ending in the point $(a, b)$, while $(\boldsymbol{u}, \boldsymbol{v})$ with vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ means the inner roduct of these vectors. Since vectors are here written in boldface font there ıould be no confusion. We may add that the norm associated with this inner roduct is the usual Eucledian length of a vector.

**ɛmark 2.** It might be wise to refresh some basic linear algebra by consulting textbook. Exercises 1 and 2 suggest specific tasks to regain familiarity with ındamental operations on inner product vector spaces.

**'he least squares method.** We now want to find $c_0$ such that it minimizes $\boldsymbol{e}||$. The algebra is simplified if we minimize the square of the norm, $||\boldsymbol{e}||^2 = :, \boldsymbol{e})$, instead of the norm itself. Define the function

$$E(c_0) = (\boldsymbol{e}, \boldsymbol{e}) = (\boldsymbol{f} - c_0\boldsymbol{\psi}_0, \boldsymbol{f} - c_0\boldsymbol{\psi}_0)\,. \tag{4}$$

/e can rewrite the expressions of the right-hand side in a more convenient form ›r further work:

$$E(c_0) = (\boldsymbol{f}, \boldsymbol{f}) - 2c_0(\boldsymbol{f}, \boldsymbol{\psi}_0) + c_0^2(\boldsymbol{\psi}_0, \boldsymbol{\psi}_0)\,. \tag{5}$$

he rewrite results from using the following fundamental rules for inner product ɔaces:

$$(\alpha\boldsymbol{u}, \boldsymbol{v}) = \alpha(\boldsymbol{u}, \boldsymbol{v}), \quad \alpha \in \mathbb{R}, \tag{6}$$

$$(\boldsymbol{u} + \boldsymbol{v}, \boldsymbol{w}) = (\boldsymbol{u}, \boldsymbol{w}) + (\boldsymbol{v}, \boldsymbol{w}), \tag{7}$$

$$(\boldsymbol{u}, \boldsymbol{v}) = (\boldsymbol{v}, \boldsymbol{u})\,. \tag{8}$$

Minimizing $E(c_0)$ implies finding $c_0$ such that

$$\frac{\partial E}{\partial c_0} = 0\,.$$

ıifferentiating (5) with respect to $c_0$ gives

$$\frac{\partial E}{\partial c_0} = -2(\boldsymbol{f}, \boldsymbol{\psi}_0) + 2c_0(\boldsymbol{\psi}_0, \boldsymbol{\psi}_0)\,. \tag{9}$$

etting the above expression equal to zero and solving for $c_0$ gives

$$c_0 = \frac{(\boldsymbol{f}, \boldsymbol{\psi}_0)}{(\boldsymbol{\psi}_0, \boldsymbol{\psi}_0)}, \tag{10}$$

hich in the present case with $\boldsymbol{\psi}_0 = (a, b)$ results in

$$c_0 = \frac{3a + 5b}{a^2 + b^2}\,. \tag{11}$$

For later, it is worth mentioning that setting the key equation (9) to zero ɑn be rewritten as



$$(\boldsymbol{f} - c0\boldsymbol{\psi}_0, \boldsymbol{\psi}_0) = 0,$$

or

$$(\boldsymbol{e}, \boldsymbol{\psi}_0) = 0\,.$$

**The projection method.** We shall now show that minimizing $||\boldsymbol{e}||^2$ that $\boldsymbol{e}$ is orthogonal to *any* vector $\boldsymbol{v}$ in the space $V$. This result is visual clear from Figure 2 (think of other vectors along the line $(a, b)$: all of th lead to a larger distance between the approximation and $\boldsymbol{f}$). To see th mathematically, we express any $\boldsymbol{v} \in V$ as $\boldsymbol{v} = s\boldsymbol{\psi}_0$ for any scalar para recall that two vectors are orthogonal when their inner product vanish calculate the inner product

$$\begin{aligned}
(\boldsymbol{e}, s\boldsymbol{\psi}_0) &= (\boldsymbol{f} - c_0\boldsymbol{\psi}_0, s\boldsymbol{\psi}_0) \\
&= (\boldsymbol{f}, s\boldsymbol{\psi}_0) - (c_0\boldsymbol{\psi}_0, s\boldsymbol{\psi}_0) \\
&= s(\boldsymbol{f}, \boldsymbol{\psi}_0) - sc_0(\boldsymbol{\psi}_0, \boldsymbol{\psi}_0) \\
&= s(\boldsymbol{f}, \boldsymbol{\psi}_0) - s\frac{(\boldsymbol{f}, \boldsymbol{\psi}_0)}{(\boldsymbol{\psi}_0, \boldsymbol{\psi}_0)}(\boldsymbol{\psi}_0, \boldsymbol{\psi}_0) \\
&= s\left((\boldsymbol{f}, \boldsymbol{\psi}_0) - (\boldsymbol{f}, \boldsymbol{\psi}_0)\right) \\
&= 0\,.
\end{aligned}$$

Therefore, instead of minimizing the square of the norm, we could dema $\boldsymbol{e}$ is orthogonal to any vector in $V$. This method is known as *projection*, it is the same as projecting the vector onto the subspace. (The appro also be referred to as a Galerkin method as explained at the end of Sect

Mathematically the projection method is stated by the equation

$$(\boldsymbol{e}, \boldsymbol{v}) = 0, \quad \forall \boldsymbol{v} \in V\,.$$

An arbitrary $\boldsymbol{v} \in V$ can be expressed as $s\boldsymbol{\psi}_0$, $s \in \mathbb{R}$, and therefore (13)

$$(\boldsymbol{e}, s\boldsymbol{\psi}_0) = s(\boldsymbol{e}, \boldsymbol{\psi}_0) = 0,$$

which means that the error must be orthogonal to the basis vector in tl $V$:

$$(\boldsymbol{e}, \boldsymbol{\psi}_0) = 0 \quad \text{or} \quad (\boldsymbol{f} - c_0\boldsymbol{\psi}_0, \boldsymbol{\psi}_0) = 0\,.$$

The latter equation gives (10) and it also arose from least squares comp in (12).

## 1.2 Approximation of general vectors

Let us generalize the vector approximation from the previous section to in spaces with arbitrary dimension. Given some vector $\boldsymbol{f}$, we want to best approximation to this vector in the space

$$V = \text{span}\,\{\boldsymbol{\psi}_0, \ldots, \boldsymbol{\psi}_N\}\,.$$

We assume that the *basis vectors* $\boldsymbol{\psi}_0, \ldots, \boldsymbol{\psi}_N$ are linearly independent so that one of them are redundant and the space has dimension $N+1$. Any vector $\boldsymbol{u} \in V$ can be written as a linear combination of the basis vectors,

$$\boldsymbol{u} = \sum_{j=0}^{N} c_j \boldsymbol{\psi}_j,$$

here $c_j \in \mathbb{R}$ are scalar coefficients to be determined.

**The least squares method.** Now we want to find $c_0, \ldots, c_N$, such that $\boldsymbol{u}$ is the best approximation to $\boldsymbol{f}$ in the sense that the distance (error) $\boldsymbol{e} = \boldsymbol{f} - \boldsymbol{u}$ is minimized. Again, we define the squared distance as a function of the free parameters $c_0, \ldots, c_N$,

$$\begin{aligned} E(c_0, \ldots, c_N) &= (\boldsymbol{e}, \boldsymbol{e}) = (\boldsymbol{f} - \sum_j c_j \boldsymbol{\psi}_j, \boldsymbol{f} - \sum_j c_j \boldsymbol{\psi}_j) \\ &= (\boldsymbol{f}, \boldsymbol{f}) - 2\sum_{j=0}^{N} c_j (\boldsymbol{f}, \boldsymbol{\psi}_j) + \sum_{p=0}^{N}\sum_{q=0}^{N} c_p c_q (\boldsymbol{\psi}_p, \boldsymbol{\psi}_q)\,. \end{aligned} \tag{14}$$

Minimizing this $E$ with respect to the independent variables $c_0, \ldots, c_N$ is obtained by requiring

$$\frac{\partial E}{\partial c_i} = 0, \quad i = 0, \ldots, N\,.$$

The second term in (14) is differentiated as follows:

$$\frac{\partial}{\partial c_i} \sum_{j=0}^{N} c_j (\boldsymbol{f}, \boldsymbol{\psi}_j) = (\boldsymbol{f}, \boldsymbol{\psi}_i), \tag{15}$$

since the expression to be differentiated is a sum and only one term, $c_i(\boldsymbol{f}, \boldsymbol{\psi}_i)$, contains $c_i$ and this term is linear in $c_i$. To understand this differentiation in detail, write out the sum specifically for, e.g, $N = 3$ and $i = 1$.

The last term in (14) is more tedious to differentiate. We start with

$$\frac{\partial}{\partial c_i} c_p c_q = \begin{cases} 0, & \text{if } p \neq i \text{ and } q \neq i, \\ c_q, & \text{if } p = i \text{ and } q \neq i, \\ c_p, & \text{if } p \neq i \text{ and } q = i, \\ 2c_i, & \text{if } p = q = i, \end{cases} \tag{16}$$

then



$$\frac{\partial}{\partial c_i} \sum_{p=0}^{N}\sum_{q=0}^{N} c_p c_q (\boldsymbol{\psi}_p, \boldsymbol{\psi}_q) = \sum_{p=0, p\neq i}^{N} c_p (\boldsymbol{\psi}_p, \boldsymbol{\psi}_i) + \sum_{q=0, q\neq i}^{N} c_q (\boldsymbol{\psi}_q, \boldsymbol{\psi}_i) + 2c_i($$

The last term can be included in the other two sums, resulting in

$$\frac{\partial}{\partial c_i} \sum_{p=0}^{N}\sum_{q=0}^{N} c_p c_q (\boldsymbol{\psi}_p, \boldsymbol{\psi}_q) = 2\sum_{j=0}^{N} c_i (\boldsymbol{\psi}_j, \boldsymbol{\psi}_i)\,.$$

It then follows that setting

$$\frac{\partial E}{\partial c_i} = 0, \quad i = 0, \ldots, N,$$

leads to a linear system for $c_0, \ldots, c_N$:

$$\sum_{j=0}^{N} A_{i,j} c_j = b_i, \quad i = 0, \ldots, N,$$

where

$$\begin{aligned} A_{i,j} &= (\boldsymbol{\psi}_i, \boldsymbol{\psi}_j), \\ b_i &= (\boldsymbol{\psi}_i, \boldsymbol{f})\,. \end{aligned}$$

We have changed the order of the two vectors in the inner product acco (1.1):

$$A_{i,j} = (\boldsymbol{\psi}_j, \boldsymbol{\psi}_i) = (\boldsymbol{\psi}_i, \boldsymbol{\psi}_j),$$

simply because the sequence $i$-$j$ looks more aesthetic.

**The Galerkin or projection method.** In analogy with the "one-dim example in Section 1.1, it holds also here in the general case that minimi distance (error) $\boldsymbol{e}$ is equivalent to demanding that $\boldsymbol{e}$ is orthogonal to all

$$(\boldsymbol{e}, \boldsymbol{v}) = 0, \quad \forall \boldsymbol{v} \in V\,.$$

Since any $\boldsymbol{v} \in V$ can be written as $\boldsymbol{v} = \sum_{i=0}^{N} c_i \boldsymbol{\psi}_i$, the statement equivalent to saying that

$$(\boldsymbol{e}, \sum_{i=0}^{N} c_i \boldsymbol{\psi}_i) = 0,$$

for any choice of coefficients $c_0, \ldots, c_N$. The latter equation can be rew

$$\sum_{i=0}^{N} c_i (\boldsymbol{e}, \boldsymbol{\psi}_i) = 0\,.$$

this is to hold for arbitrary values of $c_0, \ldots, c_N$ we must require that each erm in the sum vanishes,

$$(\boldsymbol{e}, \boldsymbol{\psi}_i) = 0, \quad i = 0, \ldots, N\,. \tag{22}$$

hese $N+1$ equations result in the same linear system as (18):

$$(\boldsymbol{f} - \sum_{j=0}^{N} c_j \boldsymbol{\psi}_j, \boldsymbol{\psi}_i) = (\boldsymbol{f}, \boldsymbol{\psi}_i) - \sum_{j\in I} (\boldsymbol{\psi}_i, \boldsymbol{\psi}_j) c_j = 0,$$

nd hence

$$\sum_{j=0}^{N} (\boldsymbol{\psi}_i, \boldsymbol{\psi}_j) c_j = (\boldsymbol{f}, \boldsymbol{\psi}_i), \quad i = 0, \ldots, N\,.$$

o, instead of differentiating the $E(c_0, \ldots, c_N)$ function, we could simply use 21) as the principle for determining $c_0, \ldots, c_N$, resulting in the $N+1$ equations 22).

The names *least squares method* or *least squares approximation* are natural nce the calculations consists of minimizing $||\boldsymbol{e}||^2$, and $||\boldsymbol{e}||^2$ is a sum of squares f differences between the components in $\boldsymbol{f}$ and $\boldsymbol{u}$. We find $\boldsymbol{u}$ such that this m of squares is minimized.

The principle (21), or the equivalent form (22), is known as *projection*. lmost the same mathematical idea was used by the Russian mathematician oris Galerkin[1] to solve differential equations, resulting in what is widely known s *Galerkin's method*.

# Approximation of functions

et $V$ be a function space spanned by a set of *basis functions* $\psi_0, \ldots, \psi_N$,

$$V = \text{span}\,\{\psi_0, \ldots, \psi_N\},$$

ich that any function $u \in V$ can be written as a linear combination of the basis nctions:

$$u = \sum_{j\in I} c_j \psi_j\,. \tag{23}$$

he index set $I$ is defined as $I = \{0, \ldots, N\}$ and is used both for compact otation and for flexibility in the numbering of elements in sequences.

For now, in this introduction, we shall look at functions of a single variable : $u = u(x)$, $\psi_i = \psi_i(x)$, $i \in I$. Later, we will almost trivially extend the athematical details to functions of two- or three-dimensional physical spaces. he approximation (23) is typically used to discretize a problem in space. Other ethods, most notably finite differences, are common for time discretization, lthough the form (23) can be used in time as well.

[1] http://en.wikipedia.org/wiki/Boris_Galerkin



## 2.1 The least squares method

Given a function $f(x)$, how can we determine its best approximation $u($ A natural starting point is to apply the same reasoning as we did for in Section 1.2. That is, we minimize the distance between $u$ and $f$. H this requires a norm for measuring distances, and a norm is most conv defined through an inner product. Viewing a function as a vector of i many point values, one for each value of $x$, the inner product could int be defined as the usual summation of pairwise components, with sum replaced by integration:

$$(f, g) = \int f(x) g(x)\, \mathrm{d}x\,.$$

To fix the integration domain, we let $f(x)$ and $\psi_i(x)$ be defined for a $\Omega \subset \mathbb{R}$. The inner product of two functions $f(x)$ and $g(x)$ is then

$$(f, g) = \int_\Omega f(x) g(x)\, \mathrm{d}x\,.$$

The distance between $f$ and any function $u \in V$ is simply $f - u$, squared norm of this distance is

$$E = (f(x) - \sum_{j\in I} c_j \psi_j(x), f(x) - \sum_{j\in I} c_j \psi_j(x))\,.$$

Note the analogy with (14): the given function $f$ plays the role of th vector $\boldsymbol{f}$, and the basis function $\psi_i$ plays the role of the basis vector $\boldsymbol{\psi}_i$. rewrite (25), through similar steps as used for the result (14), leading t

$$E(c_i, \ldots, c_N) = (f, f) - 2\sum_{j\in I} c_j (f, \psi_i) + \sum_{p\in I}\sum_{q\in I} c_p c_q (\psi_p, \psi_q)\,.$$

Minimizing this function of $N+1$ scalar variables $\{c_i\}_{i\in I}$, requires differe with respect to $c_i$, for all $i \in I$. The resulting equations are very similar we had in the vector case, and we hence end up with a linear system of t (18), with basically the same expressions:

$$A_{i,j} = (\psi_i, \psi_j),$$
$$b_i = (f, \psi_i)\,.$$

## 2.2 The projection (or Galerkin) method

As in Section 1.2, the minimization of $(e, e)$ is equivalent to

$$(e, v) = 0, \quad \forall v \in V\,.$$

This is known as a projection of a function $f$ onto the subspace $V$. We r call it a Galerkin method for approximating functions. Using the same re as in (21)-(22), it follows that (29) is equivalent to

$$(e, \psi_i) = 0, \quad i \in I\,. \tag{30}$$

ıserting $e = f - u$ in this equation and ordering terms, as in the multi-imensional vector case, we end up with a linear system with a coefficient matrix ?7) and right-hand side vector (28).

Whether we work with vectors in the plane, general vectors, or functions ı function spaces, the least squares principle and the projection or Galerkin ıethod are equivalent.

## .3 Example: linear approximation

et us apply the theory in the previous section to a simple problem: given a arabola $f(x) = 10(x-1)^2 - 1$ for $x \in \Omega = [1, 2]$, find the best approximation $(x)$ in the space of all linear functions:

$$V = \text{span}\,\{1, x\}\,.$$

/ith our notation, $\psi_0(x) = 1$, $\psi_1(x) = x$, and $N = 1$. We seek

$$u = c_0\psi_0(x) + c_1\psi_1(x) = c_0 + c_1 x,$$

here $c_0$ and $c_1$ are found by solving a $2 \times 2$ the linear system. The coefficient ıatrix has elements

$$A_{0,0} = (\psi_0, \psi_0) = \int_1^2 1 \cdot 1 \,\mathrm{d}x = 1, \tag{31}$$

$$A_{0,1} = (\psi_0, \psi_1) = \int_1^2 1 \cdot x \,\mathrm{d}x = 3/2, \tag{32}$$

$$A_{1,0} = A_{0,1} = 3/2, \tag{33}$$

$$A_{1,1} = (\psi_1, \psi_1) = \int_1^2 x \cdot x \,\mathrm{d}x = 7/3\,. \tag{34}$$

he corresponding right-hand side is

$$b_1 = (f, \psi_0) = \int_1^2 (10(x-1)^2 - 1) \cdot 1 \,\mathrm{d}x = 7/3, \tag{35}$$

$$b_2 = (f, \psi_1) = \int_1^2 (10(x-1)^2 - 1) \cdot x \,\mathrm{d}x = 13/3\,. \tag{36}$$

olving the linear system results in

$$c_0 = -38/3, \quad c_1 = 10, \tag{37}$$

nd consequently

$$u(x) = 10x - \frac{38}{3}\,. \tag{38}$$



Figure 3 displays the parabola and its best approximation in the spa linear functions.

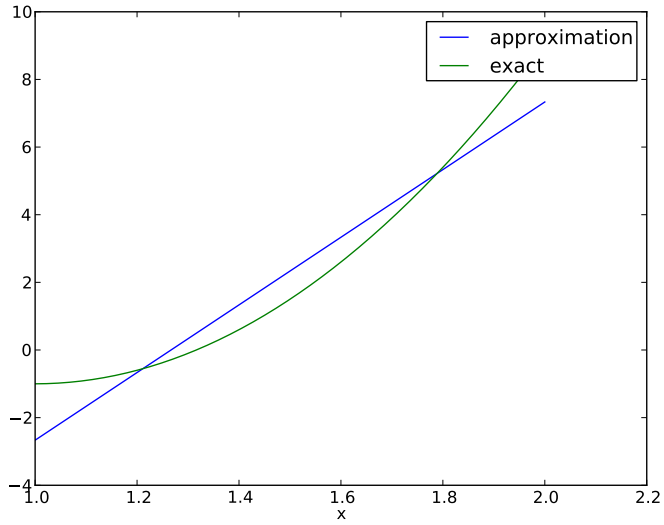



Figure 3: Best approximation of a parabola by a straight line.

## 2.4 Implementation of the least squares method

The linear system can be computed either symbolically or numerically (ı ical integration rule is needed in the latter case). Here is a function for s computation of the linear system, where $f(x)$ is given as a `sympy` expr involving the symbol `x`, `psi` is a list of expressions for $\{\psi_i\}_{i\in I}$, and `Om` 2-tuple/list holding the limits of the domain $\Omega$:

```
import sympy as sm

def least_squares(f, psi, Omega):
    N = len(psi) - 1
    A = sm.zeros((N+1, N+1))
    b = sm.zeros((N+1, 1))
    x = sm.Symbol('x')
    for i in range(N+1):
        for j in range(i, N+1):
            A[i,j] = sm.integrate(psi[i]*psi[j],
                                  (x, Omega[0], Omega[1]))
            A[j,i] = A[i,j]
        b[i,0] = sm.integrate(psi[i]*f, (x, Omega[0], Omega[1]))
    c = A.LUsolve(b)
    u = 0
    for i in range(len(psi)):
```

```
            u += c[i,0]*psi[i]
        return u
```

)bserve that we exploit the symmetry of the coefficient matrix: only the pper triangular part is computed. Symbolic integration in `sympy` is often me consuming, and (roughly) halving the work has noticeable effect on the aiting time for the function to finish execution.

Comparing the given $f(x)$ and the approximate $u(x)$ visually is done by ıe following function, which with the aid of `sympy`'s `lambdify` tool converts a `ympy` expression to a Python function for numerical computations:

```
lef comparison_plot(f, u, Omega, filename='tmp.pdf'):
    x = sm.Symbol('x')
    f = sm.lambdify([x], f, modules="numpy")
    u = sm.lambdify([x], u, modules="numpy")
    resolution = 401  # no of points in plot
    xcoor  = linspace(Omega[0], Omega[1], resolution)
    exact  = f(xcoor)
    approx = u(xcoor)
    plot(xcoor, approx)
    hold('on')
    plot(xcoor, exact)
    legend(['approximation', 'exact'])
    savefig(filename)
```

he `modules='numpy'` argument to `lambdify` is important if there are mathe- ıatical functions, such as `sin` or `exp` in the symbolic expressions in `f` or `u`, and ıese mathematical functions are to be used with vector arguments, like `xcoor` bove.

Both the `least_squares` and `comparison_plot` are found and coded in ıe file `approx1D.py`[2]. The forthcoming examples on their use appear in `x_approx1D.py`.

## .5 Perfect approximation

et us use the code above to recompute the problem from Section 2.3 where we ant to approximate a parabola. What happens if we add an element $x^2$ to the asis and test what the best approximation is if $V$ is the space of all parabolic ınctions? The answer is quickly found by running

```
>>> from approx1D import *
>>> x = sm.Symbol('x')
>>> f = 10*(x-1)**2-1
>>> u = least_squares(f=f, psi=[1, x, x**2], Omega=[1, 2])
>>> print u
10*x**2 - 20*x + 9
>>> print sm.expand(f)
10*x**2 - 20*x + 9
```

Now, what if we use $\psi_i(x) = x^i$ for $i = 0, 1, \ldots, N = 40$? The output from `east_squares` gives $c_i = 0$ for $i > 2$, which means that the method finds the erfect approximation.

[2] `http://tinyurl.com/jvzzcfn/fem/approx1D.py`



In fact, we have a general result that if $f \in V$, the least squa Galerkin/projection methods compute the exact solution $u = f$. The straightforward: if $f \in V$, $f$ can be expanded in terms of the basis fu $f = \sum_{j\in I} d_j\psi_j$, for some coefficients $\{d_i\}_{i\in I}$, and the right-hand side t entries

$$b_i = (f, \psi_i) = \sum_{j\in I} d_j(\psi_j, \psi_i) = \sum_{j\in I} d_j A_{i,j}\,.$$

The linear system $\sum_j A_{i,j}c_j = b_i$, $i \in I$, is then

$$\sum_{j\in I} c_j A_{i,j} = \sum_{j\in I} d_j A_{i,j}, \quad i \in I,$$

which implies that $c_i = d_i$ for $i \in I$.

## 2.6 Ill-conditioning

The computational example in Section 2.5 applies the `least_squares` f which invokes symbolic methods to calculate and solve the linear syste correct solution $c_0 = 9, c_1 = -20, c_2 = 10, c_i = 0$ for $i \geq 3$ is perfectly re

Suppose we convert the matrix and right-hand side to floating-poin and then solve the system using finite-precision arithmetics, which is w will (almost) always do in real life. This time we get astonishing resu to about $N = 7$ we get a solution that is reasonably close to the ex Increasing $N$ shows that seriously wrong coefficients are computed. I a table showing the solution of the linear system arising from approx a parabola by functions on the form $u(x) = c_0 + c_1x + c_2x^2 + \cdots +$ Analytically, we know that $c_j = 0$ for $j > 2$, but numerically we may ge for $j > 2$.

| exact | `sympy` | `numpy32` | `numpy64` |
|---|---|---|---|
| 9 | 9.62 | 5.57 | 8.98 |
| -20 | -23.39 | -7.65 | -19.93 |
| 10 | 17.74 | -4.50 | 9.96 |
| 0 | -9.19 | 4.13 | -0.26 |
| 0 | 5.25 | 2.99 | 0.72 |
| 0 | 0.18 | -1.21 | -0.93 |
| 0 | -2.48 | -0.41 | 0.73 |
| 0 | 1.81 | -0.013 | -0.36 |
| 0 | -0.66 | 0.08 | 0.11 |
| 0 | 0.12 | 0.04 | -0.02 |
| 0 | -0.001 | -0.02 | 0.002 |

The exact value of $c_j$, $j = 0, 1, \ldots, 10$, appears in the first column w other columns correspond to results obtained by three different metho

- Column 2: The matrix and vector are converted to the data st `sympy.mpmath.fp.matrix` and the `sympy.mpmath.fp.lu_solve` f is used to solve the system.

- Column 3: The matrix and vector are converted to `numpy` arrays with data type `numpy.float32` (single precision floating-point number) and solved by the `numpy.linalg.solve` function.

- Column 4: As column 3, but the data type is `numpy.float64` (double precision floating-point number).

/e see from the numbers in the table that double precision performs much better ıan single precision. Nevertheless, when plotting all these solutions the curves annot be visually distinguished (!). This means that the approximations look erfect, despite the partially very wrong values of the coefficients.

Increasing $N$ to 12 makes the numerical solver in `numpy` abort with the ıessage: "matrix is numerically singular". A matrix has to be non-singular to e invertible, which is a requirement when solving a linear system. Already when ıe matrix is close to singular, it is *ill-conditioned*, which here implies that the umerical solution algorithms are sensitive to round-off errors and may produce ery) inaccurate results.

The reason why the coefficient matrix is nearly singular and ill-conditioned that our basis functions $\psi_i(x) = x^i$ are nearly linearly dependent for large $i$. hat is, $x^i$ and $x^{i+1}$ are very close for $i$ not very small. This phenomenon is lustrated in Figure 4. There are 15 lines in this figure, but only half of them re visually distinguishable. Almost linearly dependent basis functions give rise ɔ an ill-conditioned and almost singular matrix. This fact can be illustrated by ɔmputing the determinant, which is indeed very close to zero (recall that a zero eterminant implies a singular and non-invertible matrix): $10^{-65}$ for $N = 10$ and $)^{-92}$ for $N = 12$. Already for $N = 28$ the numerical determinant computation eturns a plain zero.

On the other hand, the double precision `numpy` solver do run for $N = 100$, esulting in answers that are not significantly worse than those in the table bove, and large powers are associated with small coefficients (e.g., $c_j < 10^{-2}$ ɔr $10 \leq j \leq 20$ and $c < 10^{-5}$ for $j > 20$). Even for $N = 100$ the approximation ill lies on top of the exact curve in a plot (!).

The conclusion is that visual inspection of the quality of the approximation ıay not uncover fundamental numerical problems with the computations. How-ver, numerical analysts have studied approximations and ill-conditioning for ecades, and it is well known that the basis $\{1, x, x^2, x^3, \ldots,\}$ is a bad basis. he best basis from a matrix conditioning point of view is to have orthogonal unctions such that $(\psi_i, \psi_j) = 0$ for $i \neq j$. There are many known sets of orthog-nal polynomials and other functions. The functions used in the finite element ıethods are almost orthogonal, and this property helps to avoid problems with ɔlving matrix systems. Almost orthogonal is helpful, but not enough when comes to partial differential equations, and ill-conditioning of the coefficient ıatrix is a theme when solving large-scale matrix systems arising from finite ement discretizations.



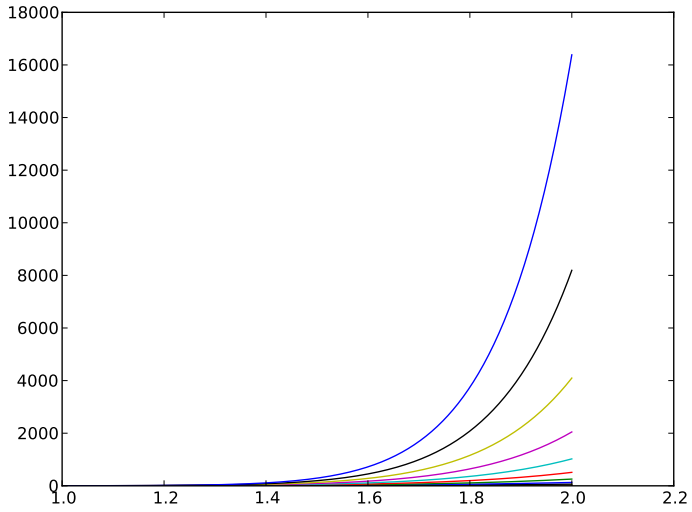



Figure 4: The 15 first basis functions $x^i$, $i = 0, \ldots, 14$.

## 2.7 Fourier series

A set of sine functions is widely used for approximating functions (the s also orthogonal as explained more in Section 2.6). Let us take

$$V = \text{span}\left\{\sin \pi x, \sin 2\pi x, \ldots, \sin(N+1)\pi x\right\}.$$

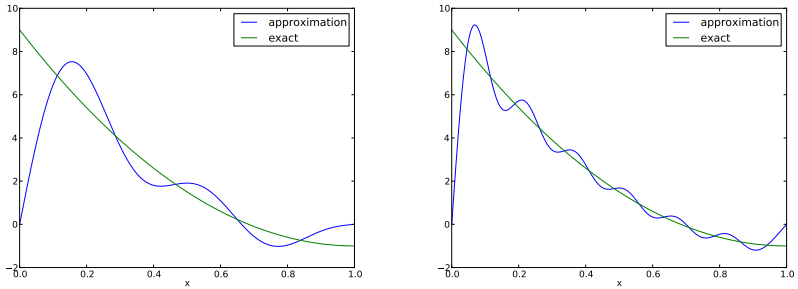
That is,

$$\psi_i(x) = \sin((i+1)\pi x), \quad i \in I\,.$$

An approximation to the $f(x)$ function from Section 2.3 can then be co by the `least_squares` function from Section 2.4:

```
N = 3
from sympy import sin, pi
x = sm.Symbol('x')
psi = [sin(pi*(i+1)*x) for i in range(N+1)]
f = 10*(x-1)**2 - 1
Omega = [0, 1]
u = least_squares(f, psi, Omega)
comparison_plot(f, u, Omega)
```

Figure 5 (left) shows the oscillatory approximation of $\sum_{j=0}^{N} c_j \sin((j$ when $N = 3$. Changing $N$ to 11 improves the approximation considera Figure 5 (right).




igure 5: Best approximation of a parabola by a sum of 3 (left) and 11 (right) ne functions.

There is an error $f(0) - u(0) = 9$ at $x = 0$ in Figure 5 regardless of how large $I$ is, because all $\psi_i(0) = 0$ and hence $u(0) = 0$. We may help the approximation o be correct at $x = 0$ by seeking

$$u(x) = f(0) + \sum_{j \in I} c_j \psi_j(x)\,. \tag{39}$$

owever, this adjustment introduces a new problem at $x = 1$ since we now get n error $f(1) - u(1) = f(1) - 0 = -1$ at this point. A more clever adjustment is o replace the $f(0)$ term by a term that is $f(0)$ at $x = 0$ and $f(1)$ at $x = 1$. A mple linear combination $f(0)(1 - x) + xf(1)$ does the job:

$$u(x) = f(0)(1 - x) + xf(1) + \sum_{j \in I} c_j \psi_j(x)\,. \tag{40}$$

his adjustment of $u$ alters the linear system slightly as we get an extra term $(f(0)(1 - x) + xf(1), \psi_i)$ on the right-hand side. Figure 6 shows the result of is technique for ensuring right boundary values: even 3 sines can now adjust ie $f(0)(1 - x) + xf(1)$ term such that $u$ approximates the parabola really well, t least visually.

## .8 Orthogonal basis functions

he choice of sine functions $\psi_i(x) = \sin((i + 1)\pi x)$ has a great computational dvantage: on $\Omega = [0, 1]$ these basis functions are *orthogonal*, implying that $_{i,j} = 0$ if $i \neq j$. This result is realized by trying

```
integrate(sin(j*pi*x)*sin(k*pi*x), x, 0, 1)
```

n WolframAlpha[3] (avoid `i` in the integrand as this symbol means the imaginary nit $\sqrt{-1}$). Also by asking WolframAlpha about $\int_0^1 \sin^2(j\pi x)\,\mathrm{d}x$, we find it to qual 1/2. With a diagonal matrix we can easily solve for the coefficients by and:

[3] `http://wolframalpha.com`



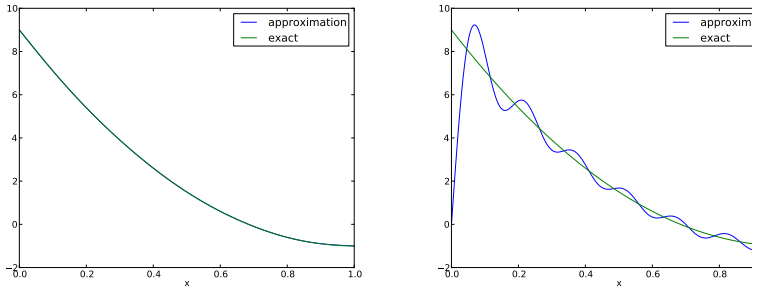



Figure 6: Best approximation of a parabola by a sum of 3 (left) and 1 sine functions with a boundary term.

$$c_i = 2\int_0^1 f(x)\sin((i + 1)\pi x)\,\mathrm{d}x, \quad i \in I,$$

which is nothing but the classical formula for the coefficients of the Fou series of $f(x)$ on $[0, 1]$. In fact, when $V$ contains the basic functions u Fourier series expansion, the approximation method derived in Section 2 in the classical Fourier series for $f(x)$ (see Exercise 6 for details).

With orthogonal basis functions we can make the `least_squares` f (much) more efficient since we know that the matrix is diagonal and c diagonal elements need to be computed:

```
def least_squares_orth(f, psi, Omega):
    N = len(psi) - 1
    A = [0]*(N+1)
    b = [0]*(N+1)
    x = sm.Symbol('x')
    for i in range(N+1):
        A[i] = sm.integrate(psi[i]**2, (x, Omega[0], Omega[1]))
        b[i] = sm.integrate(psi[i]*f,  (x, Omega[0], Omega[1]))
    c = [b[i]/A[i] for i in range(len(b))]
    u = 0
    for i in range(len(psi)):
        u += c[i]*psi[i]
    return u
```

This function is found in the file `approx1D.py`.

## 2.9 The interpolation (or collocation) method

The principle of minimizing the distance between $u$ and $f$ is an intuit of computing a best approximation $u \in V$ to $f$. However, there ar approaches as well. One is to demand that $u(x_i) = f(x_i)$ at some selecte $x_i$, $i \in I$:

$$u(x_i) = \sum_{j \in I} c_j \psi_j(x_i) = f(x_i), \quad i \in I\,.$$

his criterion also gives a linear system with $N+1$ unknown coefficients $\{c_i\}_{i\in I}$:

$$\sum_{j\in I} A_{i,j}c_j = b_i, \quad i \in I, \tag{43}$$

ith

$$A_{i,j} = \psi_j(x_i), \tag{44}$$
$$b_i = f(x_i)\,. \tag{45}$$

his time the coefficient matrix is not symmetric because $\psi_j(x_i) \neq \psi_i(x_j)$ in eneral. The method is often referred to as an *interpolation method* since some oint values of $f$ are given ($f(x_i)$) and we fit a continuous function $u$ that goes rough the $f(x_i)$ points. In this case the $x_i$ points are called *interpolation oints*. When the same approach is used to approximate differential equations, ne usually applies the name *collocation method* and $x_i$ are known as *collocation oints*.

Given $f$ as a `sympy` symbolic expression `f`, $\{\psi_i\}_{i\in I}$ as a list `psi`, and a set of oints $\{x_i\}_{i\in I}$ as a list or array `points`, the following Python function sets up nd solves the matrix system for the coefficients $\{c_i\}_{i\in I}$:

```
def interpolation(f, psi, points):
    N = len(psi) - 1
    A = sm.zeros((N+1, N+1))
    b = sm.zeros((N+1, 1))
    x = sm.Symbol('x')
    # Turn psi and f into Python functions
    psi = [sm.lambdify([x], psi[i]) for i in range(N+1)]
    f = sm.lambdify([x], f)
    for i in range(N+1):
        for j in range(N+1):
            A[i,j] = psi[j](points[i])
        b[i,0] = f(points[i])
    c = A.LUsolve(b)
    u = 0
    for i in range(len(psi)):
        u += c[i,0]*psi[i](x)
    return u
```

he `interpolation` function is a part of the `approx1D` module.

We found it convenient in the above function to turn the expressions `f` nd `psi` into ordinary Python functions of `x`, which can be called with `float` alues in the list `points` when building the matrix and the right-hand side. he alternative is to use the `subs` method to substitute the `x` variable in an xpression by an element from the `points` list. The following session illustrates oth approaches in a simple setting:

```
>>> from sympy import *
>>> x = Symbol('x')
>>> e = x**2              # symbolic expression involving x
>>> p = 0.5               # a value of x
>>> v = e.subs(x, p)      # evaluate e for x=p
>>> v
0.250000000000000
>>> type(v)
sympy.core.numbers.Float
>>> e = lambdify([x], e)  # make Python function of e
>>> type(e)
>>> function
>>> v = e(p)              # evaluate e(x) for x=p
>>> v
0.25
>>> type(v)
float
```



A nice feature of the interpolation or collocation method is that it computing integrals. However, one has to decide on the location of the $x$ A simple, yet common choice, is to distribute them uniformly througho

**Example.** Let us illustrate the interpolation or collocation method by imating our parabola $f(x) = 10(x-1)^2 - 1$ by a linear function on $\Omega$ using two collocation points $x_0 = 1 + 1/3$ and $x_1 = 1 + 2/3$:

```
f = 10*(x-1)**2 - 1
psi = [1, x]
Omega = [1, 2]
points = [1 + sm.Rational(1,3), 1 + sm.Rational(2,3)]
u = interpolation(f, psi, points)
comparison_plot(f, u, Omega)
```

The resulting linear system becomes

$$\begin{pmatrix} 1 & 4/3 \\ 1 & 5/3 \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \end{pmatrix} = \begin{pmatrix} 1/9 \\ 31/9 \end{pmatrix}$$

with solution $c_0 = -119/9$ and $c_1 = 10$. Figure 7 (left) shows the r approximation $u = -119/9 + 10x$. We can easily test other interpolatio say $x_0 = 1$ and $x_1 = 2$. This changes the line quite significantly, see F (right).

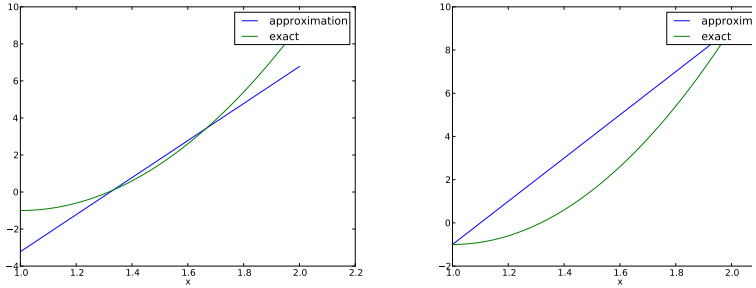



Figure 7: Approximation of a parabola by linear functions computed interpolation points: 4/3 and 5/3 (left) versus 1 and 2 (right).

## .10 Lagrange polynomials

ı Section 2.7 we explain the advantage with having a diagonal matrix: formulas ɔr the coefficients $\{c_i\}_{i\in I}$ can then be derived by hand. For an interpolation/- ɔllocation method a diagonal matrix implies that $\psi_j(x_i) = 0$ if $i \neq j$. One ɛt of basis functions $\psi_i(x)$ with this property is the *Lagrange interpolating ɔlynomials*, or just *Lagrange polynomials*. (Although the functions are named fter Lagrange, they were first discovered by Waring in 1779, rediscovered by uler in 1783, and published by Lagrange in 1795.) The Lagrange polynomials ave the form

$$\psi_i(x) = \prod_{j=0, j\neq i}^{N} \frac{x - x_j}{x_i - x_j} = \frac{x - x_0}{x_i - x_0} \cdots \frac{x - x_{i-1}}{x_i - x_{i-1}} \frac{x - x_{i+1}}{x_i - x_{i+1}} \cdots \frac{x - x_N}{x_i - x_N}, \quad (46)$$

ɔr $i \in I$. We see from (46) that all the $\psi_i$ functions are polynomials of degree $I$ which have the property

$$\psi_i(x_s) = \begin{cases} 1, & i = s, \\ 0, & i \neq s, \end{cases} \quad (47)$$

hen $x_s$ is an interpolation/collocation point. This property implies that $A_{i,j} = 0$ ɔr $i \neq j$ and $A_{i,j} = 1$ when $i = j$. The solution of the linear system is them mply

$$c_i = f(x_i), \quad i \in I, \quad (48)$$

nd

$$u(x) = \sum_{j\in I} f(x_i)\psi_i(x)\,. \quad (49)$$

The following function computes the Lagrange interpolating polynomial $\psi_i(x)$, iven the interpolation points $x_0, \ldots, x_N$ in the list or array `points`:

```
ɗef Lagrange_polynomial(x, i, points):
    p = 1
    for k in range(len(points)):
        if k != i:
            p *= (x - points[k])/(points[i] - points[k])
    return p
```

he next function computes a complete basis using equidistant points throughout :

```
ɗef Lagrange_polynomials_01(x, N):
    if isinstance(x, sm.Symbol):
        h = sm.Rational(1, N-1)
    else:
        h = 1.0/(N-1)
    points = [i*h for i in range(N)]
    psi = [Lagrange_polynomial(x, i, points) for i in range(N)]
    return psi, points
```



When `x` is an `sm.Symbol` object, we let the spacing between the interp points, `h`, be a `sympy` rational number for nice end results in the f for $\psi_i$. The other case, when `x` is a plain Python `float`, signifies nu computing, and then we let `h` be a floating-point number. Observe t `Lagrange_polynomial` function works equally well in the symbolic and nu case - just think of `x` being an `sm.Symbol` object or a Python `float`. interactive session illustrates the difference between symbolic and nu computing of the basis functions and points:

```
>>> import sympy as sm
>>> x = sm.Symbol('x')
>>> psi, points = Lagrange_polynomials_01(x, N=3)
>>> points
[0, 1/2, 1]
>>> psi
[(1 - x)*(1 - 2*x), 2*x*(2 - 2*x), -x*(1 - 2*x)]

>>> x = 0.5  # numerical computing
>>> psi, points = Lagrange_polynomials_01(x, N=3)
>>> points
[0.0, 0.5, 1.0]
>>> psi
[-0.0, 1.0, 0.0]
```

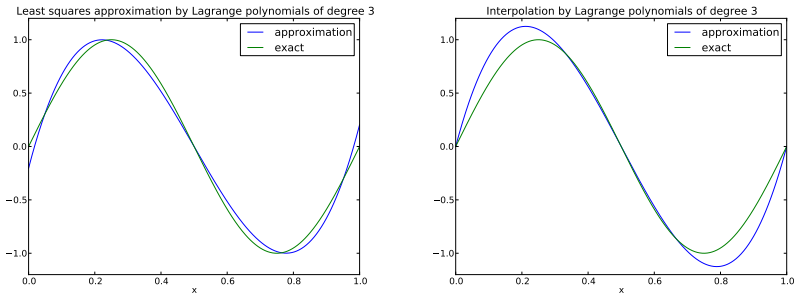
The Lagrange polynomials are very much used in finite element methods of their property (47).

**Successful example.** Trying out the Lagrange polynomial basis for mating $f(x) = \sin 2\pi x$ on $\Omega = [0, 1]$ with the least squares and the interp techniques can be done by

```
x = sm.Symbol('x')
f = sm.sin(2*sm.pi*x)
psi, points = Lagrange_polynomials_01(x, N)
Omega=[0, 1]
u = least_squares(f, psi, Omega)
comparison_plot(f, u, Omega)
u = interpolation(f, psi, points)
comparison_plot(f, u, Omega)
```

Figure 8 shows the results. There is little difference between the least squa the interpolation technique. Increasing $N$ gives visually better approxi

**Less successful example.** The next example concerns interpolating $|1 - 2x|$ on $\Omega = [0, 1]$ using Lagrange polynomials. Figure 9 shows a effect: the approximation starts to oscillate more and more as $N$ grow numerical artifact is not surprising when looking at the individual L polynomials. Figure 10 shows two such polynomials, $\psi_2(x)$ and $\psi_7(x)$, degree 11 and computed from uniformly spaced points $x_{x_i} = i/11$, $i = 0$ marked with circles. We clearly see the property of Lagrange polyn $\psi_2(x_i) = 0$ and $\psi_7(x_i) = 0$ for all $i$, except $\psi_2(x_2) = 1$ and $\psi_7(x_7) =$ most striking feature, however, is the significant oscillation near the bo The reason is easy to understand: since we force the functions to zero at




igure 8: Approximation via least squares (left) and interpolation (right) of a ne function by Lagrange interpolating polynomials of degree 3.

oints, a polynomial of high degree is forced to oscillate between the points. The henomenon is named *Runge's phenomenon* and you can read a more detailed xplanation on Wikipedia[4].

**emedy for strong oscillations.** The oscillations can be reduced by a more ever choice of interpolation points, called the *Chebyshev nodes*:

$$x_i = \frac{1}{2}(a+b) + \frac{1}{2}(b-a)\cos\left(\frac{2i+1}{2(N+1)}pi\right), \quad i = 0\ldots, N, \tag{50}$$

n the interval $\Omega = [a, b]$. Here is a flexible version of the `Lagrange_polynomials_01` unction above, valid for any interval $\Omega = [a, b]$ and with the possibility to gener- te both uniformly distributed points and Chebyshev nodes:

```
lef Lagrange_polynomials(x, N, Omega, point_distribution='uniform'):
    if point_distribution == 'uniform':
        if isinstance(x, sm.Symbol):
            h = sm.Rational(Omega[1] - Omega[0], N)
        else:
            h = (Omega[1] - Omega[0])/float(N)
        points = [Omega[0] + i*h for i in range(N+1)]
    elif point_distribution == 'Chebyshev':
        points = Chebyshev_nodes(Omega[0], Omega[1], N)
    psi = [Lagrange_polynomial(x, i, points) for i in range(N+1)]
    return psi, points

lef Chebyshev_nodes(a, b, N):
    from math import cos, pi
    return [0.5*(a+b) + 0.5*(b-a)*cos(float(2*i+1)/(2*(N+1))*pi) \
            for i in range(N+1)]
```

ll the functions computing Lagrange polynomials listed above are found in the odule file `Lagrange.py`. Figure 11 shows the improvement of using Chebyshev odes (compared with Figure 9).

Another cure for undesired oscillation of higher-degree interpolating poly- omials is to use lower-degree Lagrange polynomials on many small patches of

[4] `http://en.wikipedia.org/wiki/Runge%27s_phenomenon`



the domain, which is the idea pursued in the finite element method. stance, linear Lagrange polynomials on $[0, 1/2]$ and $[1/2, 1]$ would yield a approximation to $f(x) = |1 - 2x|$ on $\Omega = [0, 1]$ since $f$ is piecewise linea

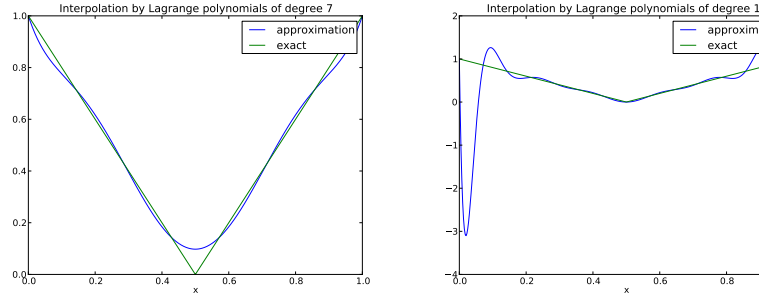



Figure 9: Interpolation of an absolute value function by Lagrange poly and uniformly distributed interpolation points: degree 7 (left) and 14 (

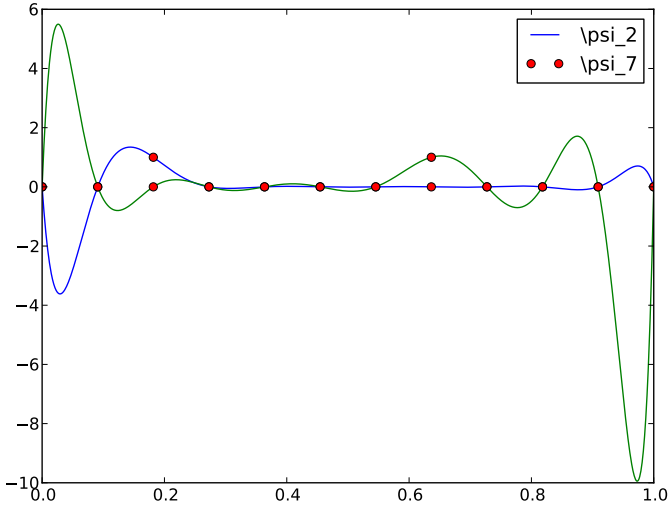
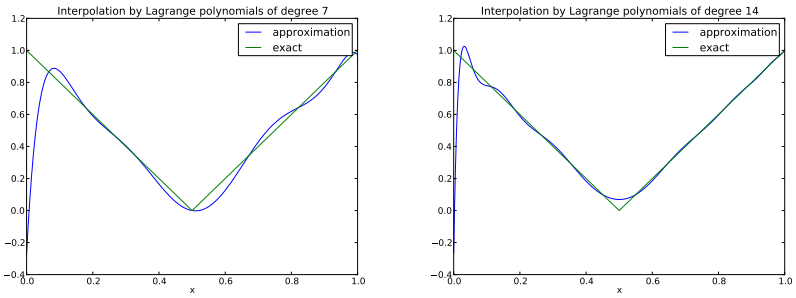



Figure 10: Illustration of the oscillatory behavior of two Lagrange poly based on 12 uniformly spaced points (marked by circles).

How does the least squares or projection methods work with L polynomials? Unfortunately, `sympy` has problems integrating the $f(x) =$ function times a polynomial. Other choices of $f(x)$ can also make the s integration fail. Therefore, we should extend the `least_squares` f




igure 11: Interpolation of an absolute value function by Lagrange polynomials nd Chebyshev nodes as interpolation points: degree 7 (left) and 14 (right).

ıch that it falls back on numerical integration if the symbolic integration is nsuccessful. In the latter case, the returned value from `sympy`'s `integrate` ınction is an object of type `Integral`. We can test on this type and utilize ıe `mpmath` module in `sympy` to perform numerical integration of high precision. ere is the code:

```
lef least_squares(f, psi, Omega):
    N = len(psi) - 1
    A = sm.zeros((N+1, N+1))
    b = sm.zeros((N+1, 1))
    x = sm.Symbol('x')
    for i in range(N+1):
        for j in range(i, N+1):
            integrand = psi[i]*psi[j]
            I = sm.integrate(integrand, (x, Omega[0], Omega[1]))
            if isinstance(I, sm.Integral):
                # Could not integrate symbolically, fallback
                # on numerical integration with mpmath.quad
                integrand = sm.lambdify([x], integrand)
                I = sm.mpmath.quad(integrand, [Omega[0], Omega[1]])
            A[i,j] = A[j,i] = I
        integrand = psi[i]*f
        I = sm.integrate(integrand, (x, Omega[0], Omega[1]))
        if isinstance(I, sm.Integral):
            integrand = sm.lambdify([x], integrand)
            I = sm.mpmath.quad(integrand, [Omega[0], Omega[1]])
        b[i,0] = I
    c = A.LUsolve(b)
    u = 0
    for i in range(len(psi)):
        u += c[i,0]*psi[i]
    return u
```

# Finite element basis functions

he specific basis functions exemplified in Section 2 are in general nonzero on the ıtire domain $\Omega$, see Figure 12 for an example where we plot $\psi_0(x) = \sin\frac{1}{2}\pi x$ nd $\psi_1(x) = \sin 2\pi x$ together with a possible sum $u(x) = 4\psi_0(x) - \frac{1}{2}\psi_1(x)$.



We shall now turn the attention to basis functions that have *compact* meaning that they are nonzero on only a small portion of $\Omega$. Morec shall restrict the functions to be *piecewise polynomials*. This means t domain is split into subdomains and the function is a polynomial on one subdomains, see Figure 13 for a sketch involving locally defined hat fu that make $u = \sum_j c_j\psi_j$ piecewise linear. At the boundaries between sub one normally forces continuity of the function only so that when connect polynomials from two subdomains, the derivative becomes discontinuous type of basis functions are fundamental in the finite element method.

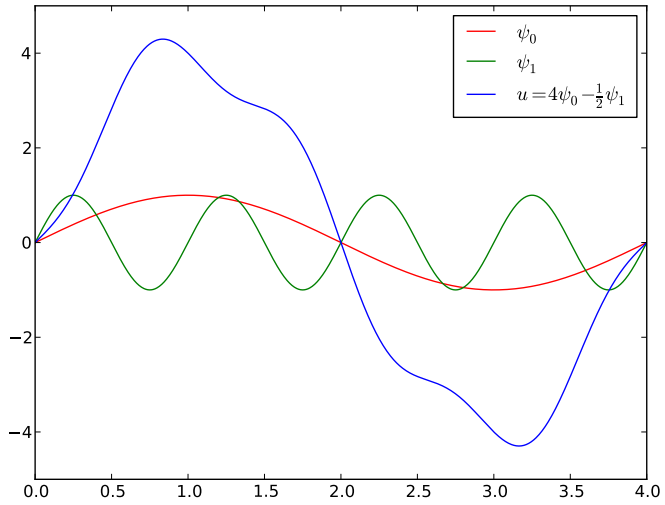



Figure 12: A function resulting from adding two sine basis functio

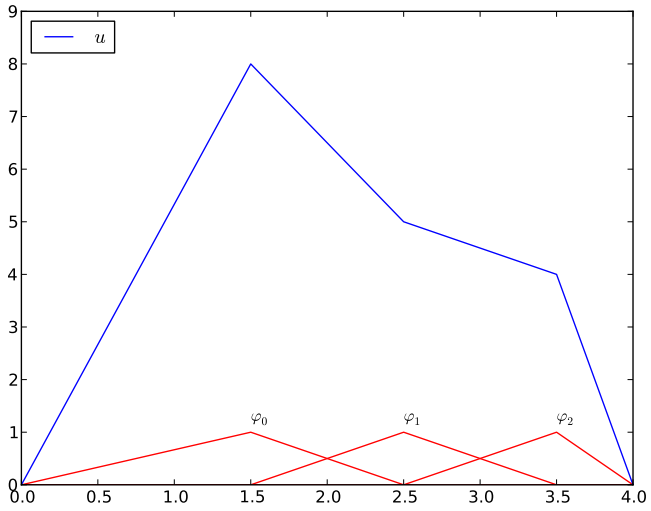
We first introduce the concepts of elements and nodes in a simplistic as often met in the literature. Later, we shall generalize the concep element, which is a necessary step to treat a wider class of approximation the family of finite element methods. The generalization is also compatil the concepts used in the FEniCS[5] finite element software.

## 3.1 Elements and nodes

Let us divide the interval $\Omega$ on which $f$ and $u$ are defined into non-over subintervals $\Omega^{(e)}$, $e = 0, \ldots, N_e$:

$$\Omega = \Omega^{(0)} \cup \cdots \cup \Omega^{(N_e)}.$$

[5] `http://fenicsproject.org`




igure 13: A function resulting from adding three local piecewise linear (hat) nctions.

/e shall for now refer to $\Omega^{(e)}$ as an *element*, having number $e$. On each element e introduce a set of points called *nodes*. For now we assume that the nodes re uniformly spaced throughout the element and that the boundary points f the elements are also nodes. The nodes are given numbers both within an ement and in the global domain. These are referred to as *local* and *global* node umbers, respectively. Figure 14 shows element boundaries with small vertical nes, nodes as small disks, element numbers in circles, and global node numbers nder the nodes.

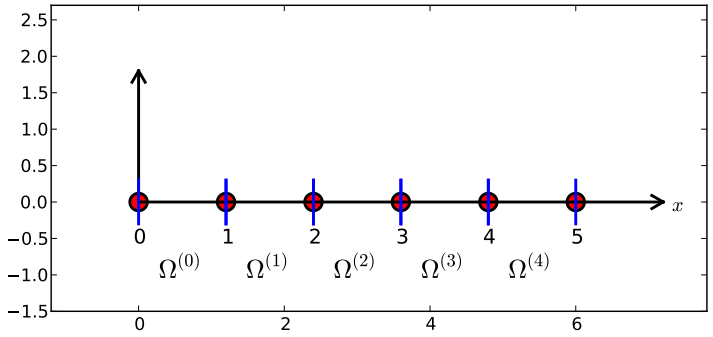



Figure 14: Finite element mesh with 5 elements and 6 nodes.



Nodes and elements uniquely define a *finite element mesh*, whicl discrete representation of the domain in the computations. A common case is that of a *uniformly partitioned mesh* where each element has tl length and the distance between nodes is constant.

**Example.** On $\Omega = [0, 1]$ we may introduce two elements, $\Omega^{(0)} = [0, ($ $\Omega^{(1)} = [0.4, 1]$. Furthermore, let us introduce three nodes per element, spaced within each element. Figure 15 shows the mesh. The three r element number 0 are $x_0 = 0$, $x_1 = 0.2$, and $x_2 = 0.4$. The local and glol numbers are here equal. In element number 1, we have the local nodes $x$ $x_1 = 0.7$, and $x_2 = 1$ and the corresponding global nodes $x_2 = 0.4$, $x$ and $x_4 = 1$. Note that the global node $x_2 = 0.4$ is shared by the two el

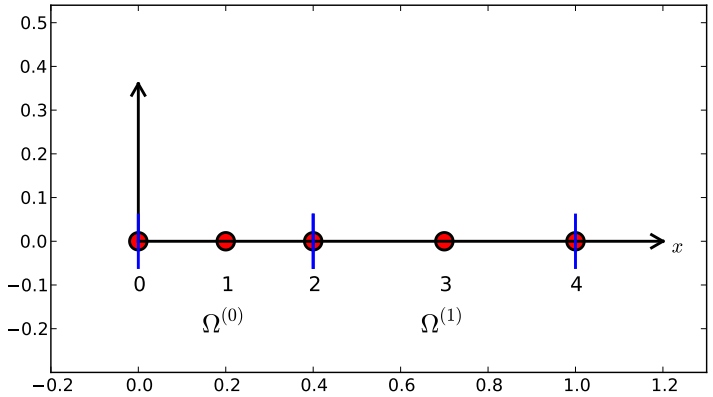



Figure 15: Finite element mesh with 2 elements and 5 nodes.

For the purpose of implementation, we introduce two lists or arrays for storing the coordinates of the nodes, with the global node numbers as and `elements` for holding the global node numbers in each element, v local node numbers as indices. The `nodes` and `elements` lists for the mesh above take the form

```
nodes = [0, 0.2, 0.4, 0.7, 1]
elements = [[0, 1, 2], [2, 3, 4]]
```

Looking up the coordinate of local node number 2 in element 1 is here `nodes[elements[1][2]]` (recall that nodes and elements start their nu at 0).

The numbering of elements and nodes does not need to be regular. F shows and example corresponding to

```
nodes = [1.5, 5.5, 4.2, 0.3, 2.2, 3.1]
elements = [[2, 1], [4, 5], [0, 4], [3, 0], [5, 2]]
```

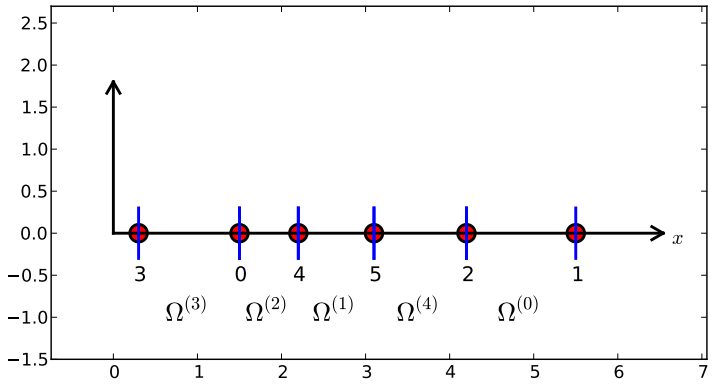



Figure 16: Example on irregular numbering of elements and nodes.

## .2 The basis functions

**onstruction principles.** Finite element basis functions are in this text rec-gnized by the notation $\varphi_i(x)$, where the index now in the beginning corresponds a global node number. In the current approximation problem we shall simply ake $\psi_i = \varphi_i$.

Let $i$ be the global node number corresponding to local node $r$ in element umber $e$. The finite element basis functions $\varphi_i$ are now defined as follows.

- If local node number $r$ is not on the boundary of the element, take $\varphi_i(x)$ to be the Lagrange polynomial that is 1 at the local node number $r$ and zero at all other nodes in the element. On all other elements, $\varphi_i = 0$.
- If local node number $r$ is on the boundary of the element, let $\varphi_i$ be made up of the Lagrange polynomial that is 1 at this node in element number $e$ and its neighboring element. On all other elements, $\varphi_i = 0$.

visual impression of three such basis functions are given in Figure 18.

**roperties of $\varphi_i$.** The construction of basis functions according to the princi-les above lead to two important properties of $\varphi_i(x)$. First,

$$\varphi_i(x_j) = \begin{cases} 1, & i = j, \\ 0, & i \neq j, \end{cases} \tag{52}$$

hen $x_j$ is a node in the mesh with global node number $j$. This result arises ecause the Lagrange polynomials are constructed to have exactly this property. he property also implies a convenient interpretation of $c_i$ as the value of $u$ at ode $i$. To show this, we expand $u$ in the usual way as $\sum_j c_j\psi_j$ and choose $_i = \varphi_i$:

$$u(x_i) = \sum_{j\in I} c_j\psi_j(x_i) = \sum_{j\in I} c_j\varphi_j(x_i) = c_i\varphi_i(x_i) = c_i\,.$$



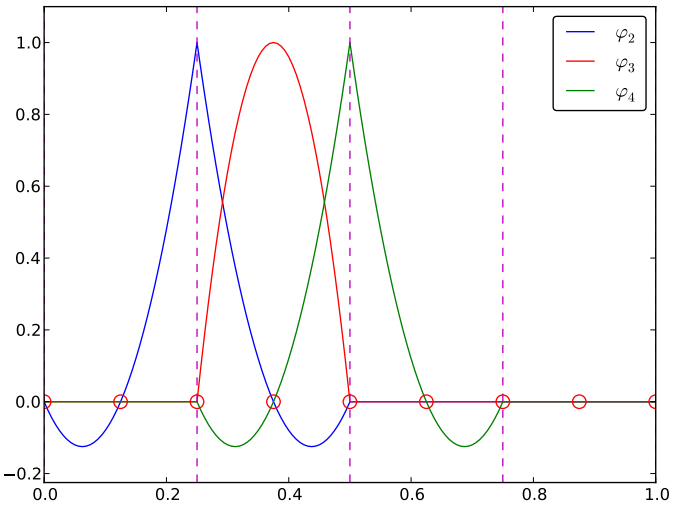



Figure 17: Illustration of the piecewise quadratic basis functions ass with nodes in element 1.

Because of this interpretation, the coefficient $c_i$ is by many named $u_i$ o

Second, $\varphi_i(x)$ is mostly zero throughout the domain:

- $\varphi_i(x) \neq 0$ only on those elements that contain global node $i$,
- $\varphi_i(x)\varphi_j(x) \neq 0$ if and only if $i$ and $j$ are global node numbers in t element.

Since $A_{i,j}$ is the integral of $\varphi_i\varphi_j$ it means that *most of the element coefficient matrix will be zero*. We will come back to these properties them actively in computations to save memory and CPU time.

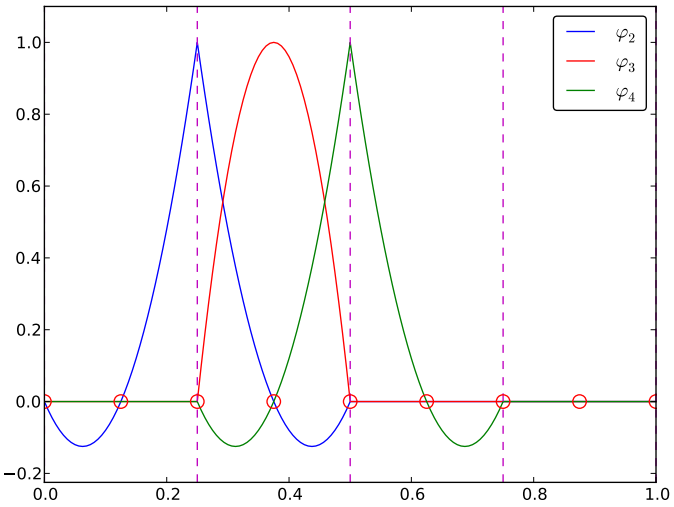
We let each element have $d+1$ nodes, resulting in local Lagrange poly of degree $d$. It is not a requirement to have the same $d$ value in each but for now we will assume so.

### 3.3 Example on piecewise quadratic finite element tions

Figure 18 illustrates how piecewise quadratic basis functions can look like We work with the domain $\Omega = [0, 1]$ divided into four equal-sized elemen having three nodes. The `nodes` and `elements` lists in this particular become

```
nodes = [0, 0.125, 0.25, 0.375, 0.5, 0.625, 0.75, 0.875, 1.0]
elements = [[0, 1, 2], [2, 3, 4], [4, 5, 6], [6, 7, 8]]
```

igure 19 sketches the mesh and the numbering. Nodes are marked with circles n the $x$ axis and element boundaries are marked with vertical dashed lines in igure 18.




igure 18: Illustration of the piecewise quadratic basis functions associated ith nodes in element 1.

Let us explain in detail how the basis functions are constructed according o the principles. Consider element number 1 in Figure 18, $\Omega^{(1)} = [0.25, 0.5]$, ith local nodes 0, 1, and 2 corresponding to global nodes 2, 3, and 4. The oordinates of these nodes are 0.25, 0.375, and 0.5, respectively. We define three agrange polynomials on this element:

1. The polynomial that is 1 at local node 1 ($x = 0.375$, global node 3) makes up the basis function $\varphi_3(x)$ over this element, with $\varphi_3(x) = 0$ outside the element.

2. The Lagrange polynomial that is 1 at local node 0 is the "right part" of the global basis function $\varphi_2(x)$. The "left part" of $\varphi_2(x)$ consists of a Lagrange polynomial associated with local node 2 in the neighboring element $\Omega^{(0)} = [0, 0.25]$.

3. Finally, the polynomial that is 1 at local node 2 (global node 4) is the "left part" of the global basis function $\varphi_4(x)$. The "right part" comes from the



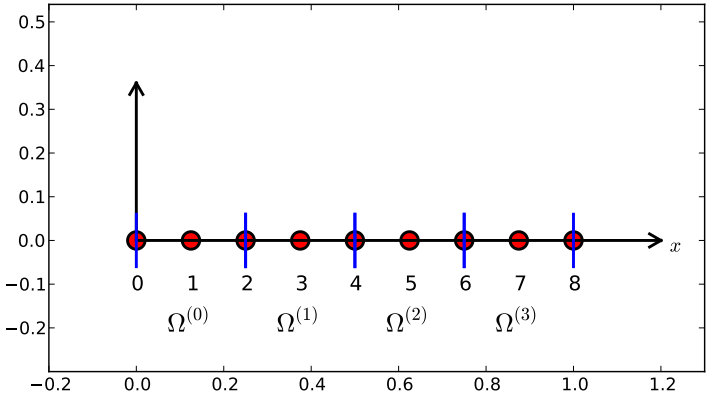



Figure 19: Sketch of mesh with 4 elements and 3 nodes per eleme

Lagrange polynomial that is 1 at local node 0 in the neighboring $\Omega^{(2)} = [0.5, 0.75]$.

As mentioned earlier, any global basis function $\varphi_i(x)$ is zero on eleme do not contain the node with global node number $i$.

The other global functions associated with internal nodes, $\varphi_1$, $\varphi_5$, and all of the same shape as the drawn $\varphi_3$, while the global basis functions as with shared nodes also have the same shape, provided the elements ar same length.

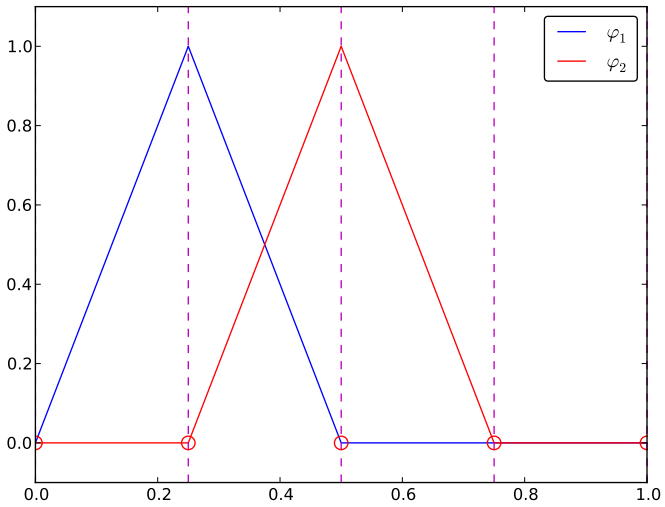
## 3.4 Example on piecewise linear finite element func

Figure 20 shows piecewise linear basis functions ($d = 1$). Also here four elements on $\Omega = [0, 1]$. Consider the element $\Omega^{(1)} = [0.25, 0.5]$. No are no internal nodes in the elements so that all basis functions are as with nodes at the element boundaries and hence made up of two L polynomials from neighboring elements. For example, $\varphi_1(x)$ results fi Lagrange polynomial in element 0 that is 1 at local node 1 and 0 at loc 0, combined with the Lagrange polynomial in element 1 that is 1 at loca and 0 at local node 1. The other basis functions are constructed simila

Explicit mathematical formulas are needed for $\varphi_i(x)$ in computations piecewise linear case, one can show that

$$\varphi_i(x) = \begin{cases} 0, & x < x_{i-1}, \\ (x - x_{i-1})/(x_i - x_{i-1}), & x_{i-1} \leq x < x_i, \\ 1 - (x - x_i)/(x_{i+1} - x_i), & x_i \leq x < x_{i+1}, \\ 0, & x \geq x_{i+1}\,. \end{cases}$$

Here, $x_j$, $j = i - 1, i, i + 1$, denotes the coordinate of node $j$. For elen equal length $h$ the formulas can be simplified to




igure 20: Illustration of the piecewise linear basis functions associated with odes in element 1.

$$\varphi_i(x) = \begin{cases} 0, & x < x_{i-1}, \\ (x - x_{i-1})/h, & x_{i-1} \leq x < x_i, \\ 1 - (x - x_i)/h, & x_i \leq x < x_{i+1}, \\ 0, & x \geq x_{i+1} \end{cases} \tag{54}$$

## .5 Example on piecewise cubic finite element basis functions

iecewise cubic basis functions can be defined by introducing four nodes per ement. Figure 21 shows examples on $\varphi_i(x)$, $i = 3, 4, 5, 6$, associated with ement number 1. Note that $\varphi_4$ and $\varphi_5$ are nonzero on element number 1, while 3 and $\varphi_6$ are made up of Lagrange polynomials on two neighboring elements.

We see that all the piecewise linear basis functions have the same "hat" shape. hey are naturally referred to as *hat functions*, also called *chapeau functions*. he piecewise quadratic functions in Figure 18 are seen to be of two types. Rounded hats" associated with internal nodes in the elements and some more sombrero" shaped hats associated with element boundary nodes. Higher-order asis functions also have hat-like shapes, but the functions have pronounced scillations in addition, as illustrated in Figure 21.

A common terminology is to speak about *linear elements* as elements with two cal nodes associated with piecewise linear basis functions. Similarly, *quadratic lements* and *cubic elements* refer to piecewise quadratic or cubic functions



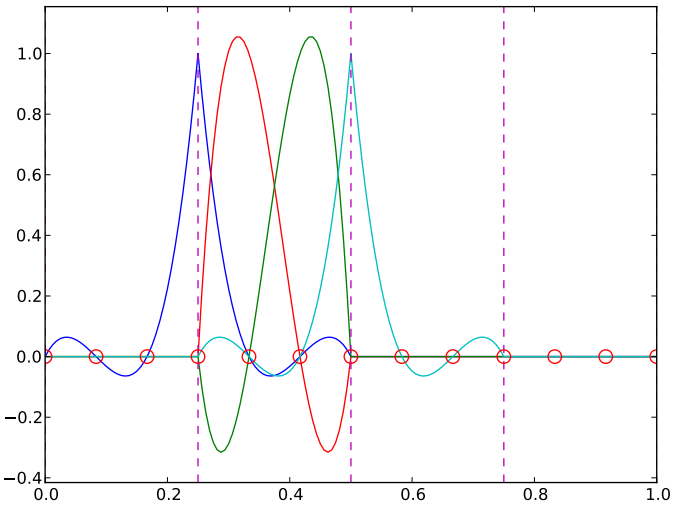



Figure 21: Illustration of the piecewise cubic basis functions associat nodes in element 1.

over elements with three or four local nodes, respectively. Alternative frequently used later, are P1 elements for linear elements, P2 for qu elements, and so forth: P$d$ signifies degree $d$ of the polynomial basis fu

## 3.6 Calculating the linear system

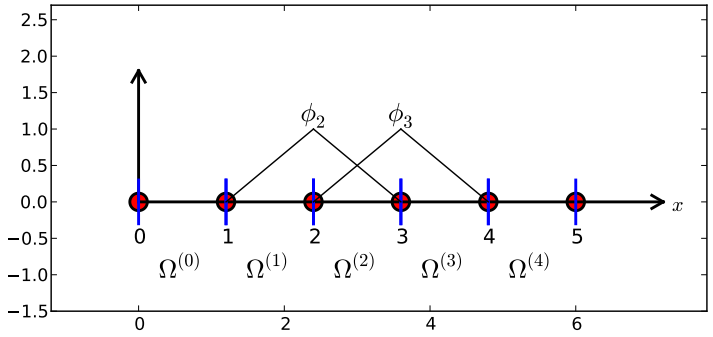
The elements in the coefficient matrix and right-hand side are given formulas (27) and (28), but now the choice of $\psi_i$ is $\varphi_i$. Consider P1 e where $\varphi_i(x)$ piecewise linear. Nodes and elements numbered consecutive left to right in a uniformly partitioned mesh imply the nodes

$$x_i = ih, \quad i = 0, \ldots, N,$$

and the elements

$$\Omega^{(i)} = [x_i, x_{i+1}] = [ih, (i+1)h], \quad i = 0, \ldots, N_e = N - 1\,.$$

We have in this case $N$ elements and $N + 1$ nodes, and $\Omega = [x_0, x_N$ formula for $\varphi_i(x)$ is given by (54) and a graphical illustration is prov Figures 20 and 23. First we clearly see from the figures the very im property $\varphi_i(x)\varphi_j(x) \neq 0$ if and only if $j = i - 1$, $j = i$, or $j = i$ alternatively expressed, if and only if $i$ and $j$ are nodes in the same Otherwise, $\varphi_i$ and $\varphi_j$ are too distant to have an overlap and consequent product vanishes.




igure 22: Illustration of the piecewise linear basis functions corresponding to lobal node 2 and 3.

**alculating a specific matrix entry.** Let us calculate the specific matrix ntry $A_{2,3} = \int_\Omega \varphi_2\varphi_3\,\mathrm{d}x$. Figure 22 shows how $\varphi_2$ and $\varphi_3$ look like. We realize om this figure that the product $\varphi_2\varphi_3 \neq 0$ only over element 2, which contains ode 2 and 3. The particular formulas for $\varphi_2(x)$ and $\varphi_3(x)$ on $[x_2, x_3]$ are found om (54). The function $\varphi_3$ has positive slope over $[x_2, x_3]$ and corresponds to ne interval $[x_{i-1}, x_i]$ in (54). With $i = 3$ we get

$$\varphi_3(x) = (x - x_2)/h,$$

hile $\varphi_2(x)$ has negative slope over $[x_2, x_3]$ and corresponds to setting $i = 2$ in 4),

$$\varphi_2(x) = 1 - (x - x_2)/h\,.$$

/e can now easily integrate,

$$A_{2,3} = \int_\Omega \varphi_2\varphi_3\,\mathrm{d}x == \int_{x_2}^{x_3} \left(1 - \frac{x - x_2}{h}\right)\frac{x - x_2}{h}\,\mathrm{d}x = \frac{h}{6}\,.$$

The diagonal entry in the coefficient matrix becomes

$$A_{2,2} = \int_{x_1}^{x_2} \left(\frac{x - x_1}{h}\right)^2 \mathrm{d}x + \int_{x_2}^{x_3} \left(1 - \frac{x - x_2}{h}\right)^2 \mathrm{d}x = \frac{h}{3}\,.$$

he entry $A_{2,1}$ has an the integral that is geometrically similar to the situation ı Figure 22, so we get $A_{2,1} = h/6$.

**alculating a general row in the matrix.** We can now generalize the alculation of matrix entries to a general row number $i$. The entry $A_{i,i-1} = {}_2\,\varphi_i\varphi_{i-1}\,\mathrm{d}x$ involves hat functions as depicted in Figure 23. Since the integral geometrically identical to the situation with specific nodes 2 and 3, we realize iat $A_{i,i-1} = A_{i,i+1} = h/6$ and $A_{i,i} = h/3$. However, we can compute the ıtegral directly too:



$$A_{i,i-1} = \int_\Omega \varphi_i\varphi_{i-1}\,\mathrm{d}x = \underbrace{\int_{x_{i-2}}^{x_{i-1}} \varphi_i\varphi_{i-1}\,\mathrm{d}x}_{\varphi_i=0} + \int_{x_i}^{x_i} \varphi_i\varphi_{i-1}\,\mathrm{d}x + \underbrace{\int_{x_i}^{x_{i+1}} \varphi_i}_{\varphi_{i-1}}$$

$$= \int_{x_{i-1}}^{x_i} \underbrace{\frac{x - x_i}{h}}_{\varphi_i(x)} \underbrace{\left(1 - \frac{x - x_{i-1}}{h}\right)}_{\varphi_{i-1}(x)} \mathrm{d}x = \frac{h}{6}\,.$$

The particular formulas for $\varphi_{i-1}(x)$ and $\varphi_i(x)$ on $[x_{i-1}, x_i]$ are foun (54): $\varphi_i$ is the linear function with positive slope, corresponding to the $[x_{i-1}, x_i]$ in (54), while $\phi_{i-1}$ has a negative slope so the definition in $[x_i, x_{i+1}]$ in (54) must be used. (The appearance of $i$ in (54) and the might be confusing, as we speak about two different $i$ indices.)

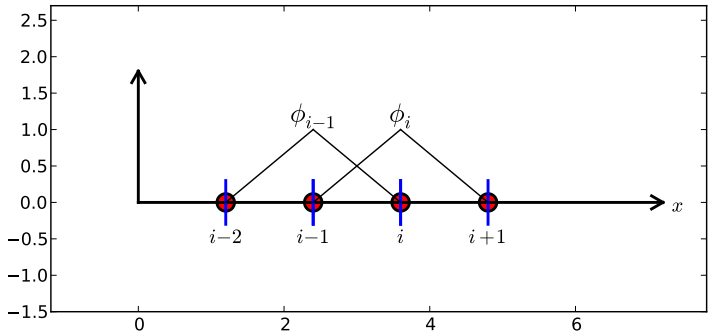



Figure 23: Illustration of two neighboring linear (hat) functions with node numbers.

The first and last row of the coefficient matrix lead to slightly c integrals:

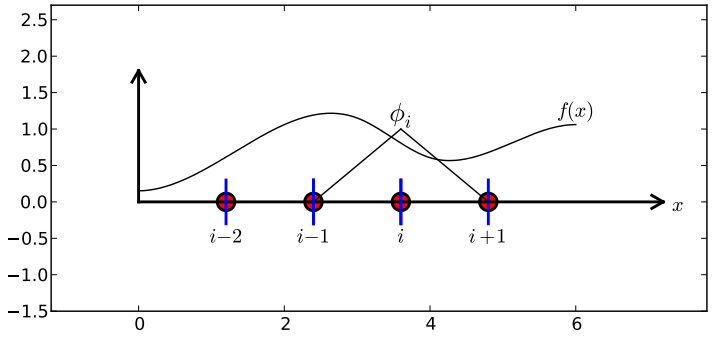
$$A_{0,0} = \int_\Omega \varphi_0^2\,\mathrm{d}x = \int_{x_0}^{x_1} \left(1 - \frac{x - x_0}{h}\right)^2 \mathrm{d}x = \frac{h}{3}\,.$$

Similarly, $A_{N,N}$ involves an integral over only one element and equals he

The general formula for $b_i$, see Figure 24, is now easy to set up

$$b_i = \int_\Omega \varphi_i(x)f(x)\,\mathrm{d}x = \int_{x_{i-1}}^{x_i} \frac{x - x_{i-1}}{h} f(x)\,\mathrm{d}x + \int_{x_i}^{x_{i+1}} \left(1 - \frac{x - x_i}{h}\right) f$$

We need a specific $f(x)$ function to compute these integrals. With two equ elements in $\Omega = [0, 1]$ and $f(x) = x(1 - x)$, one gets




igure 24: Right-hand side integral with the product of a basis function and ıe given function to approximate.

$$A = \frac{h}{6}\begin{pmatrix} 2 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 2 \end{pmatrix}, \quad b = \frac{h^2}{12}\begin{pmatrix} 2-3h \\ 12-14h \\ 10-17h \end{pmatrix}.$$

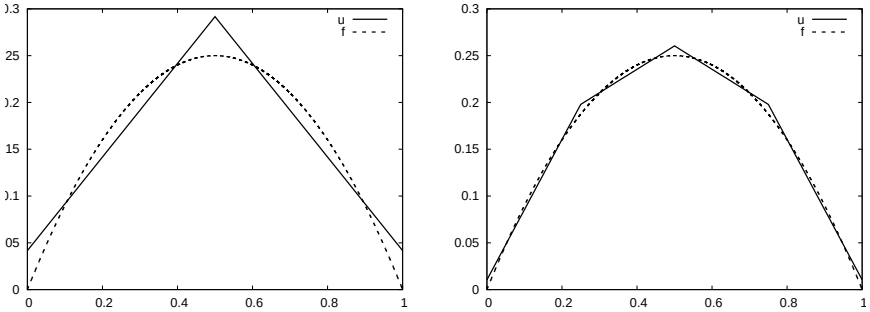
he solution becomes

$$c_0 = \frac{h^2}{6}, \quad c_1 = h - \frac{5}{6}h^2, \quad c_2 = 2h - \frac{23}{6}h^2\,.$$

he resulting function

$$u(x) = c_0\varphi_0(x) + c_1\varphi_1(x) + c_2\varphi_2(x)$$

displayed in Figure 25 (left). Doubling the number of elements to four leads ) the improved approximation in the right part of Figure 25.




igure 25: Least squares approximation using 2 (left) and 4 (right) P1 elements.



## 3.7 Assembly of elementwise computations

The integrals are naturally split into integrals over individual elemen the formulas change with the elements. This idea of splitting the int fundamental in all practical implementations of the finite element meth

Let us split the integral over $\Omega$ into a sum of contributions from each

$$A_{i,j} = \int_\Omega \varphi_i\varphi_j\,\mathrm{d}x = \sum_e A^{(e)}_{i,j}, \quad A^{(e)}_{i,j} = \int_{\Omega^{(e)}} \varphi_i\varphi_j\,\mathrm{d}x\,.$$

Now, $A^{(e)}_{i,j} \neq 0$ if and only if $i$ and $j$ are nodes in element $e$. Introduce $i$ as the mapping of local node number $r$ in element $e$ to the global node n This is just a short mathematical notation for the expression `i=elements` in a program. Let $r$ and $s$ be the local node numbers corresponding to th node numbers $i = q(e,r)$ and $j = q(e,s)$. With $d$ nodes per element, nonzero elements in $A^{(e)}_{i,j}$ arise from the integrals involving basis functic indices corresponding to the global node numbers in element number $e$

$$\int_{\Omega^{(e)}} \varphi_{q(e,r)}\varphi_{q(e,s)}\,\mathrm{d}x, \quad r,s = 0,\ldots,d\,.$$

These contributions can be collected in a $(d+1)\times(d+1)$ matrix know *element matrix*. We introduce the notation

$$\tilde{A}^{(e)} = \{\tilde{A}^{(e)}_{r,s}\}, \quad r,s \in I_d = \{0,1,\ldots,d\},$$

for the element matrix. For the case $d = 2$ we have

$$\tilde{A}^{(e)} = \begin{bmatrix} \tilde{A}^{(e)}_{0,0} & \tilde{A}^{(e)}_{0,1} & \tilde{A}^{(e)}_{0,2} \\ \tilde{A}^{(e)}_{1,0} & \tilde{A}^{(e)}_{1,1} & \tilde{A}^{(e)}_{1,2} \\ \tilde{A}^{(e)}_{2,0} & \tilde{A}^{(e)}_{2,1} & \tilde{A}^{(e)}_{2,2} \end{bmatrix}.$$

Given the numbers $\tilde{A}^{(e)}_{r,s}$, we should according to (57) add the contribu the global coefficient matrix by

$$A_{q(e,r),q(e,s)} := A_{q(e,r),q(e,s)} + \tilde{A}^{(e)}_{r,s}, \quad r,s \in I_d\,.$$

This process of adding in elementwise contributions to the global matrix *finite element assembly* or simply *assembly*. Figure 26 illustrates how matrices for elements with two nodes are added into the global matri specifically, the figure shows how the element matrix associated with ele and 3 assembled, assuming that global nodes are numbered from left to the domain.

The right-hand side of the linear system is also computed elementw

$$b_i = \int_\Omega \varphi_i\varphi_j\,\mathrm{d}x = \sum_e b^{(e)}_i, \quad b^{(e)}_i = \int_{\Omega^{(e)}} f(x)\varphi_i(x)\,\mathrm{d}x\,.$$

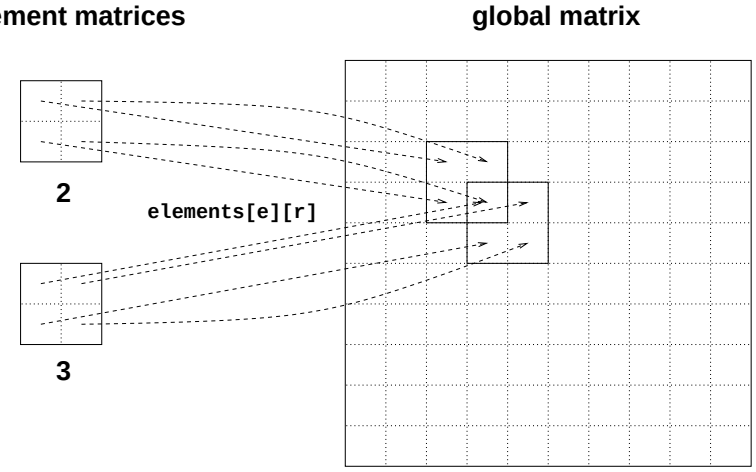



Figure 26: Illustration of matrix assembly.

We observe that $b_i^{(e)} \neq 0$ if and only if global node $i$ is a node in element $e$. With $d$ nodes per element we can collect the $d+1$ nonzero contributions $b_i^{(e)}$, for $i = q(e,r)$, $r \in I_d$, in an *element vector*

$$\tilde{b}_r^{(e)} = \{\tilde{b}_r^{(e)}\}, \quad r \in I_d\,.$$

These contributions are added to the global right-hand side by an assembly process similar to that for the element matrices:

$$b_{q(e,r)} := b_{q(e,r)} + \tilde{b}_r^{(e)}, \quad r, s \in I_d\,. \tag{60}$$

## .8 Mapping to a reference element

Instead of computing the integrals

$$\tilde{A}_{r,s}^{(e)} = \int_{\Omega^{(e)}} \varphi_{q(e,r)}(x)\varphi_{q(e,s)}(x)\,\mathrm{d}x$$

over some element $\Omega^{(e)} = [x_L, x_R]$, it is convenient to map the element domain $[x_L, x_R]$ to a standardized reference element domain $[-1, 1]$. (We have now introduced $x_L$ and $x_R$ as the left and right boundary points of an arbitrary element. With a natural numbering of nodes and elements from left to right through the domain, $x_L = x_e$ and $x_R = x_{e+1}$.) Let $X$ be the coordinate in the reference element. A linear or *affine mapping* from $X$ to $x$ reads

$$x = \frac{1}{2}(x_L + x_R) + \frac{1}{2}(x_R - x_L)X\,. \tag{61}$$

This relation can alternatively be expressed by

$$x = x_m + \frac{1}{2}hX, \tag{62}$$



where we have introduced the element midpoint $x_m = (x_L + x_R)/2$ and element length $h = x_R - x_L$.

Integrating on the reference element is a matter of just changing the integration variable from $x$ to $X$. Let

$$\tilde{\varphi}_r(X) = \varphi_{q(e,r)}(x(X))$$

be the basis function associated with local node number $r$ in the reference element. The integral transformation reads

$$\tilde{A}_{r,s}^{(e)} = \int_{\Omega^{(e)}} \varphi_{q(e,r)}(x)\varphi_{q(e,s)}(x)\,\mathrm{d}x = \int_{-1}^{1} \tilde{\varphi}_r(X)\tilde{\varphi}_s(X)\frac{dx}{dX}\,\mathrm{d}X\,.$$

The stretch factor $dx/dX$ between the $x$ and $X$ coordinates becomes the determinant of the Jacobian matrix of the mapping between the coordinates in 2D and 3D. To obtain a uniform notation for 1D, 2D, and 3D problems we therefore replace $dx/dX$ by $\det J$ already now. In 1D, $\det J = dx/dX$. The integration over the reference element is then written as

$$\tilde{A}_{r,s}^{(e)} = \int_{-1}^{1} \tilde{\varphi}_r(X)\tilde{\varphi}_s(X)\det J\,dX\,.$$

The corresponding formula for the element vector entries becomes

$$\tilde{b}_r^{(e)} = \int_{\Omega^{(e)}} f(x)\varphi_{q(e,r)}(x)dx = \int_{-1}^{1} f(x(X))\tilde{\varphi}_r(X)\det J\,dX\,.$$

Since we from now on will work in the reference element, we need mathematical formulas for the basis functions $\varphi_i(x)$ in the reference element only, i.e., we only need to specify formulas for $\tilde{\varphi}_r(X)$. This is a very considerable simplification compared to specifying piecewise polynomials in the physical domain.

The $\tilde{\varphi}_r(x)$ functions are simply the Lagrange polynomials defined through the local nodes in the reference element. For $d = 1$ and two nodes per element we have the linear Lagrange polynomials

$$\tilde{\varphi}_0(X) = \frac{1}{2}(1 - X)$$
$$\tilde{\varphi}_1(X) = \frac{1}{2}(1 + X)$$

Quadratic polynomials, $d = 2$, have the formulas

$$\tilde{\varphi}_0(X) = \frac{1}{2}(X - 1)X$$
$$\tilde{\varphi}_1(X) = 1 - X^2$$
$$\tilde{\varphi}_2(X) = \frac{1}{2}(X + 1)X$$

ı general,

$$\tilde{\varphi}_r(x) = \prod_{s=0, s\neq r}^{d} \frac{X - X_{(s)}}{X_{(r)} - X_{(s)}}, \tag{72}$$

here $X_{(0)}, \ldots, X_{(d)}$ are the coordinates of the local nodes in the reference ement. These are normally uniformly spaced: $X_{(r)} = -1 + 2r/d$, $r \in I_d$.

## .9 Integration over a reference element

o illustrate the concepts from the previous section in a specific example, we ow consider calculation of the element matrix and vector for a specific choice f $d$ and $f(x)$. A simple choice is $d = 1$ and $f(x) = x(1-x)$ on $\Omega = [0, 1]$. We ave the general expressions (65) and (66) for $\tilde{A}^{(e)}_{r,s}$ and $\tilde{b}^{(e)}_r$. Writing these out or the choices (67) and (68), and using that $\det J = h/2$, we get

$$\begin{aligned}
\tilde{A}^{(e)}_{0,0} &= \int_{-1}^{1} \tilde{\varphi}_0(X)\tilde{\varphi}_0(X)\frac{h}{2}dX \\
&= \int_{-1}^{1} \frac{1}{2}(1-X)\frac{1}{2}(1-X)\frac{h}{2}dX = \frac{h}{8}\int_{-1}^{1}(1-X)^2dX = \frac{h}{3}, \qquad (73)\\
\tilde{A}^{(e)}_{1,0} &= \int_{-1}^{1} \tilde{\varphi}_1(X)\tilde{\varphi}_0(X)\frac{h}{2}dX \\
&= \int_{-1}^{1} \frac{1}{2}(1+X)\frac{1}{2}(1-X)\frac{h}{2}dX = \frac{h}{8}\int_{-1}^{1}(1-X^2)dX = \frac{h}{6}, \qquad (74)\\
\tilde{A}^{(e)}_{0,1} &= \tilde{A}^{(e)}_{1,0}, \qquad (75)\\
\tilde{A}^{(e)}_{1,1} &= \int_{-1}^{1} \tilde{\varphi}_1(X)\tilde{\varphi}_1(X)\frac{h}{2}dX \\
&= \int_{-1}^{1} \frac{1}{2}(1+X)\frac{1}{2}(1+X)\frac{h}{2}dX = \frac{h}{8}\int_{-1}^{1}(1+X)^2dX = \frac{h}{3}\,. \qquad (76)
\end{aligned}$$

$$\begin{aligned}
\tilde{b}^{(e)}_0 &= \int_{-1}^{1} f(x(X))\tilde{\varphi}_0(X)\frac{h}{2}dX \\
&= \int_{-1}^{1} (x_m + \frac{1}{2}hX)(1 - (x_m + \frac{1}{2}hX))\frac{1}{2}(1-X)\frac{h}{2}dX \\
&= -\frac{1}{24}h^3 + \frac{1}{6}h^2x_m - \frac{1}{12}h^2 - \frac{1}{2}hx_m^2 + \frac{1}{2}hx_m \qquad (77)\\
\tilde{b}^{(e)}_1 &= \int_{-1}^{1} f(x(X))\tilde{\varphi}_1(X)\frac{h}{2}dX \\
&= \int_{-1}^{1} (x_m + \frac{1}{2}hX)(1 - (x_m + \frac{1}{2}hX))\frac{1}{2}(1+X)\frac{h}{2}dX \\
&= -\frac{1}{24}h^3 - \frac{1}{6}h^2x_m + \frac{1}{12}h^2 - \frac{1}{2}hx_m^2 + \frac{1}{2}hx_m\,. \qquad (78)
\end{aligned}$$



In the last two expressions we have used the element midpoint $x_m$.

Integration of lower-degree polynomials above is tedious, and higher polynomials that very much more algebra, but `sympy` may help. For ex

```
>>> import sympy as sm
>>> x, x_m, h, X = sm.symbols('x x_m h X')
>>> sm.integrate(h/8*(1-X)**2, (X, -1, 1))
h/3
>>> sm.integrate(h/8*(1+X)*(1-X), (X, -1, 1))
h/6
>>> x = x_m + h/2*X
>>> b_0 = sm.integrate(h/4*x*(1-x)*(1-X), (X, -1, 1))
>>> print b_0
-h**3/24 + h**2*x_m/6 - h**2/12 - h*x_m**2/2 + h*x_m/2
```

For inclusion of formulas in documents (like the present one), `sympy` ca expressions in LaTeX format:

```
>>> print sm.latex(b_0, mode='plain')
- \frac{1}{24} h^{3} + \frac{1}{6} h^{2} x_{m}
- \frac{1}{12} h^{2} - \frac{1}{2} h x_{m}^{2}
+ \frac{1}{2} h x_{m}
```

# 4 Implementation

Based on the experience from the previous example, it makes sense t some code to automate the integration process for any choice of finite basis functions. In addition, we can automate the assembly process an system solution. Appropriate functions for this purpose document all d all steps in the finite element computations and can found in the mo `fe_approx1D.py`[6]. Some of the functions are explained below.

## 4.1 Integration

First we need a Python function for defining $\tilde{\varphi}_r(X)$ in terms of a L polynomial of degree `d`:

```
import sympy as sm
import numpy as np

def phi_r(r, X, d):
    if isinstance(X, sm.Symbol):
        h = sm.Rational(1, d)  # node spacing
        nodes = [2*i*h - 1 for i in range(d+1)]
    else:
        # assume X is numeric: use floats for nodes
        nodes = np.linspace(-1, 1, d+1)
    return Lagrange_polynomial(X, r, nodes)

def Lagrange_polynomial(x, i, points):
    p = 1
    for k in range(len(points)):
```

[6] `http://tinyurl.com/jvzzcfn/fem/fe_approx1D.py`

```
        if k != i:
            p *= (x - points[k])/(points[i] - points[k])
    return p
```

)bserve how we construct the `phi_r` function to be a symbolic expression for $_r(X)$ if `X` is a `Symbol` object from `sympy`. Otherwise, we assume that `X` is a `loat` object and compute the corresponding floating-point value of $\tilde{\varphi}_r(X)$. The `agrange_polynomial` function, copied here from Section 2.7, works with both ymbolic and numeric `x` and `points` variables.

The complete basis $\tilde{\varphi}_0(X), \ldots, \tilde{\varphi}_d(X)$ on the reference element is constructed y

```
def basis(d=1):
    X = sm.Symbol('X')
    phi = [phi_r(r, X, d) for r in range(d+1)]
    return phi
```

ow we are in a position to write the function for computing the element matrix:

```
def element_matrix(phi, Omega_e, symbolic=True):
    n = len(phi)
    A_e = sm.zeros((n, n))
    X = sm.Symbol('X')
    if symbolic:
        h = sm.Symbol('h')
    else:
        h = Omega_e[1] - Omega_e[0]
    detJ = h/2  # dx/dX
    for r in range(n):
        for s in range(r, n):
            A_e[r,s] = sm.integrate(phi[r]*phi[s]*detJ, (X, -1, 1))
            A_e[s,r] = A_e[r,s]
    return A_e
```

n the symbolic case (`symbolic` is `True`), we introduce the element length as symbol `h` in the computations. Otherwise, the real numerical value of the ement interval `Omega_e` is used and the final matrix elements are numbers, not ymbols. This functionality can be demonstrated:

```
>>> from fe_approx1D import *
>>> phi = basis(d=1)
>>> phi
[1/2 - X/2, 1/2 + X/2]
>>> element_matrix(phi, Omega_e=[0.1, 0.2], symbolic=True)
[h/3, h/6]
[h/6, h/3]
>>> element_matrix(phi, Omega_e=[0.1, 0.2], symbolic=False)
[0.0333333333333333, 0.0166666666666667]
[0.0166666666666667, 0.0333333333333333]
```

The computation of the element vector is done by a similar procedure:



```
def element_vector(f, phi, Omega_e, symbolic=True):
    n = len(phi)
    b_e = sm.zeros((n, 1))
    # Make f a function of X
    X = sm.Symbol('X')
    if symbolic:
        h = sm.Symbol('h')
    else:
        h = Omega_e[1] - Omega_e[0]
    x = (Omega_e[0] + Omega_e[1])/2 + h/2*X  # mapping
    f = f.subs('x', x)  # substitute mapping formula for x
    detJ = h/2  # dx/dX
    for r in range(n):
        b_e[r] = sm.integrate(f*phi[r]*detJ, (X, -1, 1))
    return b_e
```

Here we need to replace the symbol `x` in the expression for `f` by the n formula such that `f` contains the variable `X`.

The integration in the element matrix function involves only proc polynomials, which `sympy` can easily deal with, but for the right-ha `sympy` may face difficulties with certain types of expressions `f`. The r the integral is then an `Integral` object and not a number as when s integration is successful. It may therefore be wise to introduce a fall numerical integration. The symbolic integration can also take much tim an unsuccessful conclusion so we may introduce a parameter `symbolic` it to `False` to avoid symbolic integration:

```
def element_vector(f, phi, Omega_e, symbolic=True):
        ...
        if symbolic:
            I = sm.integrate(f*phi[r]*detJ, (X, -1, 1))
        if not symbolic or isinstance(I, sm.Integral):
            h = Omega_e[1] - Omega_e[0]  # Ensure h is numerical
            detJ = h/2
            integrand = sm.lambdify([X], f*phi[r]*detJ)
            I = sm.mpmath.quad(integrand, [-1, 1])
        b_e[r] = I
        ...
```

Successful numerical integration requires that the symbolic integrand is cc to a plain Python function (`integrand`) and that the element length `h` number.

## 4.2 Linear system assembly and solution

The complete algorithm for computing and assembling the elementwise c tions takes the following form

```
def assemble(nodes, elements, phi, f, symbolic=True):
    N_n, N_e = len(nodes), len(elements)
    if symbolic:
        A = sm.zeros((N_n, N_n))
        b = sm.zeros((N_n, 1))    # note: (N_n, 1) matrix
    else:
        A = np.zeros((N_n, N_n))
```

```
        b = np.zeros(N_n)
    for e in range(N_e):
        Omega_e = [nodes[elements[e][0]], nodes[elements[e][-1]]]

        A_e = element_matrix(phi, Omega_e, symbolic)
        b_e = element_vector(f, phi, Omega_e, symbolic)

        for r in range(len(elements[e])):
            for s in range(len(elements[e])):
                A[elements[e][r],elements[e][s]] += A_e[r,s]
            b[elements[e][r]] += b_e[r]
    return A, b
```

he `nodes` and `elements` variables represent the finite element mesh as explained arlier.

Given the coefficient matrix `A` and the right-hand side `b`, we can compute ıe coefficients $\{c_i\}_{i\in I}$ in the expansion $u(x) = \sum_j c_j\varphi_j$ as the solution vector `c` f the linear system:

```
if symbolic:
    c = A.LUsolve(b)
else:
    c = np.linalg.solve(A, b)
```

When `A` and `b` are `sympy` arrays, solution procedure implied by `A.LUsolve` is ymbolic, otherwise, when `A` and `b` are `numpy` arrays, a standard numerical solver called. The symbolic version is suited for small problems only (small $N$ values) nce the calculation time becomes prohibitively large otherwise. Normally, the ymbolic integration will be more time consuming in small problems than the ymbolic solution of the linear system.

## .3 Example on computing approximations

We can exemplify the use of `assemble` on the computational case from Section 3.6 ith two P1 elements (linear basis functions) on the domain $\Omega = [0, 1]$. Let us rst work with a symbolic element length:

```
>>> h, x = sm.symbols('h x')
>>> nodes = [0, h, 2*h]
>>> elements = [[0, 1], [1, 2]]
>>> phi = basis(d=1)
>>> f = x*(1-x)
>>> A, b = assemble(nodes, elements, phi, f, symbolic=True)
>>> A
[h/3,   h/6,   0]
[h/6, 2*h/3, h/6]
[  0,   h/6, h/3]
>>> b
[     h**2/6 - h**3/12]
[      h**2 - 7*h**3/6]
[5*h**2/6 - 17*h**3/12]
>>> c = A.LUsolve(b)
>>> c
[                           h**2/6]
[12*(7*h**2/12 - 35*h**3/72)/(7*h)]
[  7*(4*h**2/7 - 23*h**3/21)/(2*h)]
```



We may, for comparison, compute the `c` vector for an interpolation/col method, taking the nodes as collocation points. This is carried out by ev `f` numerically at the nodes:

```
>>> fn = sm.lambdify([x], f)
>>> c = [fn(xc) for xc in nodes]
>>> c
[0, h*(1 - h), 2*h*(1 - 2*h)]
```

The corresponding numerical computations, as done by `sympy` and st on symbolic integration, goes as follows:

```
>>> nodes = [0, 0.5, 1]
>>> elements = [[0, 1], [1, 2]]
>>> phi = basis(d=1)
>>> x = sm.Symbol('x')
>>> f = x*(1-x)
>>> A, b = assemble(nodes, elements, phi, f, symbolic=False)
>>> A
[ 0.166666666666667, 0.0833333333333333,                  0]
[0.0833333333333333,  0.333333333333333, 0.0833333333333333]
[                 0, 0.0833333333333333,  0.166666666666667]
>>> b
[          0.03125]
[0.104166666666667]
[          0.03125]
>>> c = A.LUsolve(b)
>>> c
[0.0416666666666666]
[ 0.291666666666667]
[0.0416666666666666]
```

The `fe_approx1D` module contains functions for generating the no `elements` lists for equal-sized elements with any number of nodes per The coordinates in `nodes` can be expressed either through the elemen symbol `h` or by real numbers. There is also a function

```
def approximate(f, symbolic=False, d=1, N_e=4, filename='tmp.pdf'
```

which computes a mesh with `N_e` elements, basis functions of degree approximates a given symbolic expression `f` by a finite element expansion $\sum_j c_j\varphi_j(x)$. When `symbolic` is `False`, $u(x)$ can be computed at a (large) of points and plotted together with $f(x)$. The construction of $u$ poi the solution vector `c` is done elementwise by evaluating $\sum_r c_r\tilde{\varphi}_r(X)$ at number of points in each element, and the discrete $(x, u)$ values on each e are stored in arrays that are finally concatenated to form global arrays $x$ and $u$ coordinates for plotting. The details are found in the `u_glob` in `fe_approx1D.py`.

## 4.4 The structure of the coefficient matrix

Let us first see how the global matrix looks like if we assemble symbolic matrices, expressed in terms of `h`, from several elements:

```
>>> d=1; N_e=8; Omega=[0,1]  # 8 linear elements on [0,1]
>>> phi = basis(d)
>>> f = x*(1-x)
>>> nodes, elements = mesh_symbolic(N_e, d, Omega)
>>> A, b = assemble(nodes, elements, phi, f, symbolic=True)
>>> A
[h/3,   h/6,     0,     0,     0,     0,     0,     0,   0]
[h/6, 2*h/3,   h/6,     0,     0,     0,     0,     0,   0]
[  0,   h/6, 2*h/3,   h/6,     0,     0,     0,     0,   0]
[  0,     0,   h/6, 2*h/3,   h/6,     0,     0,     0,   0]
[  0,     0,     0,   h/6, 2*h/3,   h/6,     0,     0,   0]
[  0,     0,     0,     0,   h/6, 2*h/3,   h/6,     0,   0]
[  0,     0,     0,     0,     0,   h/6, 2*h/3,   h/6,   0]
[  0,     0,     0,     0,     0,     0,   h/6, 2*h/3, h/6]
[  0,     0,     0,     0,     0,     0,     0,   h/6, h/3]
```

The reader is encouraged to assemble the element matrices by hand and verify his result, as this exercise will give a hands-on understanding of what the assembly is about.) In general we have a coefficient matrix that is tridiagonal:

$$A = \frac{h}{6}\begin{pmatrix} 2 & 1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ 1 & 4 & 1 & \ddots & & & & & \vdots \\ 0 & 1 & 4 & 1 & \ddots & & & & \vdots \\ \vdots & \ddots & & \ddots & \ddots & 0 & & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & & 0 & 1 & 4 & 1 & \ddots & \vdots \\ \vdots & & & & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & & & & \ddots & 1 & 4 & 1 \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 1 & 2 \end{pmatrix} \tag{79}$$

The structure of the right-hand side is more difficult to reveal since it involves an assembly of elementwise integrals of $f(x(X))\tilde{\varphi}_r(X)h/2$, which obviously depend on the particular choice of $f(x)$. It is easier to look at the integration in $X$ coordinates, which gives the general formula (56). For equal-sized elements of length $h$, we can apply the Trapezoidal rule at the global node points to arrive at a somewhat more specific expression than (56):

$$b_i = h\left(\frac{1}{2}\varphi_i(x_0)f(x_0) + \frac{1}{2}\varphi_i(x_N)f(x_N) + \sum_{j=1}^{N-1}\varphi_i(x_i)f(x_i)\right) \tag{80}$$

$$= \begin{cases} \frac{1}{2}hf(x_i), & i = 0 \text{ or } i = N, \\ hf(x_i), & 1 \leq i \leq N-1 \end{cases} \tag{81}$$

The reason for this simple formula is simply that $\varphi_i$ is either 0 or 1 at the nodes and 0 at all but one of them.



Going to P2 elements (`d=2`) leads to the element matrix

$$A^{(e)} = \frac{h}{30}\begin{pmatrix} 4 & 2 & -1 \\ 2 & 16 & 2 \\ -1 & 2 & 4 \end{pmatrix}$$

and the following global assembled matrix from four elements:

$$A = \frac{h}{30}\begin{pmatrix} 4 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 16 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & 8 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 16 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & 8 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 16 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & 8 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 16 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & 4 \end{pmatrix}$$

In general, for $i$ odd we have the nonzeroes

$$A_{i,i-2} = -1, \quad A_{i-1,i} = 2, \quad A_{i,i} = 8, \quad A_{i+1,i} = 2, \quad A_{i+2,i} = -1$$

multiplied by $h/30$, and for $i$ even we have the nonzeros

$$A_{i-1,i} = 2, \quad A_{i,i} = 16, \quad A_{i+1,i} = 2,$$

multiplied by $h/30$. The rows with odd numbers correspond to nodes element boundaries and get contributions from two neighboring element assembly process, while the even numbered rows correspond to internal n the elements where the only one element contributes to the values in th matrix.

## 4.5 Applications

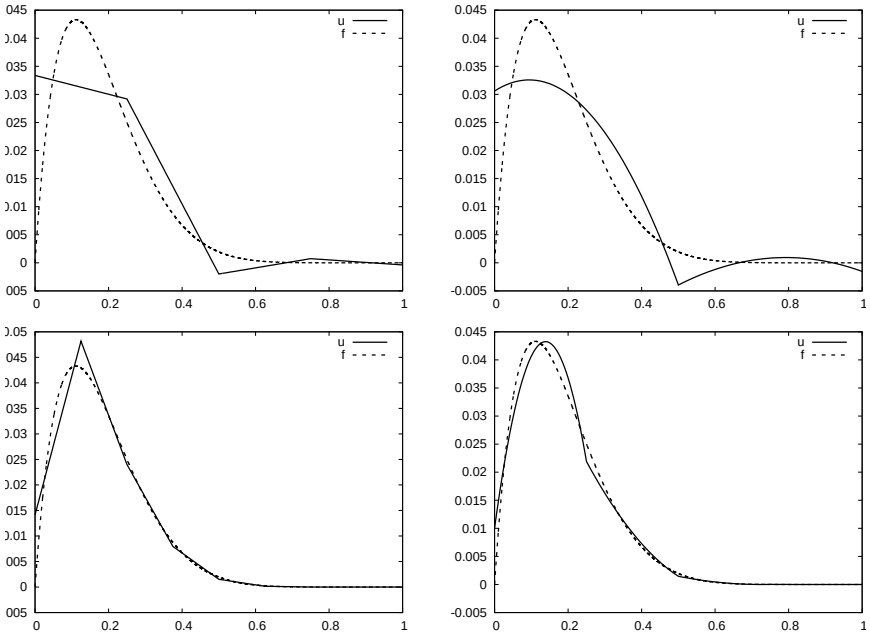
With the aid of the `approximate` function in the `fe_approx1D` module easily investigate the quality of various finite element approximations given functions. Figure 27 shows how linear and quadratic elements appro the polynomial $f(x) = x(1-x)^8$ on $\Omega = [0,1]$, using equal-sized elemer results arise from the program

```
import sympy as sm
from fe_approx1D import approximate
x = sm.Symbol('x')

approximate(f=x*(1-x)**8, symbolic=False, d=1, N_e=4)
approximate(f=x*(1-x)**8, symbolic=False, d=2, N_e=2)
approximate(f=x*(1-x)**8, symbolic=False, d=1, N_e=8)
approximate(f=x*(1-x)**8, symbolic=False, d=2, N_e=4)
```

The quadratic functions are seen to be better than the linear ones for t value of $N$, as we increase $N$. This observation has some generality:

egree is not necessarily better on a coarse mesh, but it is as we refined the ıesh.




igure 27: Comparison of the finite element approximations: 4 P1 elements with nodes (upper left), 2 P2 elements with 5 nodes (upper right), 8 P1 elements ith 9 nodes (lower left), and 4 P2 elements with 9 nodes (lower right).

## .6 Sparse matrix storage and solution

ome of the examples in the preceding section took several minutes to compute, ven on small meshes consisting of up to eight elements. The main explanation or slow computations is unsuccessful symbolic integration: `sympy` may use a ot of energy on integrals like $\int f(x(X))\tilde{\varphi}_r(X)h/2dx$ before giving up, and the rogram resorts to numerical integration. Codes that can deal with a large umber of basis functions and accept flexible choices of $f(x)$ should compute all ıtegrals numerically and replace the matrix objects from `sympy` by the far more fficient array objects from `numpy`.

A matrix whose majority of entries are zeros, are known as a *sparse* matrix. /e know beforehand that matrices from finite element approximations are sparse. he sparsity should be utilized in software as it dramatically decreases the storage emands and the CPU-time needed to compute the solution of the linear system. his optimization is not critical in 1D problems where modern computers can fford computing with all the zeros in the complete square matrix, but in 2D



and especially in 3D, sparse matrices are fundamental for feasible finite computations.

For one-dimensional finite element approximation problems, using bering of nodes and elements from left to right over the domain, the as coefficient matrix has only a few diagonals different from zero. More p $2d+1$ diagonals are different from zero. With a different numbering o nodes, say a random ordering, the diagonal structure is lost, but the nu nonzero elements is unaltered. Figures 28 and 29 exemplifies sparsity p

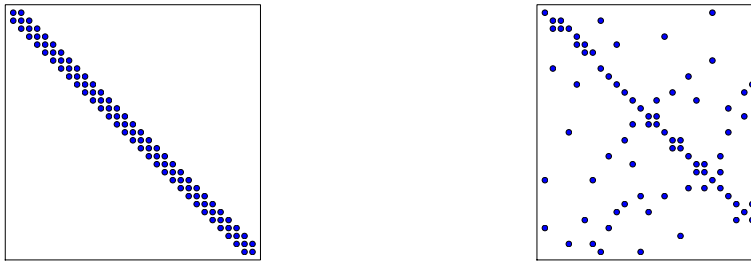


Figure 28: Matrix sparsity pattern for left-to-right numbering (left) and numbering (right) of nodes in P1 elements.

Figure 29: Matrix sparsity pattern for left-to-right numbering (left) and numbering (right) of nodes in P3 elements.

The `scipy.sparse` library supports creation of sparse matrices an system solution.

- `scipy.sparse.diags` for matrix defined via diagonals
- `scipy.sparse.lil_matrix` for creation via setting elements
- `scipy.sparse.dok_matrix` for creation via setting elements

Examples to come....

# Comparison of finite element and finite difference approximation

he previous sections on approximating $f$ by a finite element function $u$ utilize the rojection/Galerkin or least squares approaches to minimize the approximation :ror. We may, alternatively, use the collocation/interpolation method. Here we ıall compare these three approaches with what one does in the finite difference ıethod when representing a given function on a mesh.

## .1 Collocation or interpolation

et $x_i$, $i \in I$, be the nodes in the mesh. Collocation means

$$u(x_i) = f(x_i), \quad i \in I\,. \tag{84}$$

/ith $u(x) \approx \sum_{j\in I} c_j\psi_j(x)$ and $\psi_j$ chosen as $\varphi_j$, we get

$$\sum_{j\in I} c_j\varphi_j(x_i) = f(x_i),$$

ut $\varphi_j(x_i) = 0$ if $i \neq j$ so the sum collapses to one term $c_i\varphi_i(x_i) = c_i$, and we ave the result

$$c_i = f(x_i)\,. \tag{85}$$

hat is, $u$ *interpolates* $f$ at the node points (the values coincide at these points, ut the variation between the points is dictated by the type of polynomials used ı the expansion for $u$). The collocation/interpolation approach is obviously ıuch simpler and faster to use than the least squares or projection/Galerkin pproach.

**:emark.** When dealing with approximation of functions via finite elements, ll the three methods are in use, while the least squares and collocation methods re used to only a small extend when solving differential equations.

## .2 Finite difference approximation of given functions

pproximating a given function $f(x)$ on a mesh in a finite difference context ill typically just sample $f$ at the grid points. That is, the discrete version of $(x)$ is the set of point values $f(x_i)$, $i \in I$, where $x_i$ denotes a mesh point. The ollocation/interpolation method above gives exactly the same representation.

How does a finite element Galerkin or least squares approximation differ from ıis straightforward interpolation of $f$? This is the question to be addressed ext.

## .3 Finite difference interpretation of a finite element approximation

/e now limit the scope to P1 elements since this is the element type that gives ɔrmulas closest to what one gets from the finite difference method.



The linear system arising from a Galerkin or least squares approx reads in general

$$\sum_{j\in I} c_j(\psi_i, \psi_j) = (f, \psi_i), \quad i \in I\,.$$

In the finite element approximation we choose $\psi_i = \varphi_i$. With $\varphi_i$ corres to P1 elements and a uniform mesh of element length $h$ we have in Sec calculated the matrix with entries $(\varphi_i, \varphi_j)$. Equation number $i$ reads

$$\frac{h}{6}(u_{i-1} + 4u_i + u_{i+1}) = (f, \varphi_i)\,.$$

The finite difference counterpart of this equation is just $u_i = f_i$. (T and last equation, corresponding to $i = 0$ and $i = N$ are slightly differ Section 4.4.)

The left-hand side of (86) can be manipulated to equal

$$h(u_i - \frac{1}{6}(-u_{i-1} + 2u_i - u_{i+1}))\,.$$

Thinking in terms of finite differences, we can write this expression usir difference operator notation:

$$[h(u - \frac{h^2}{6}D_xD_xu)]_i,$$

which is nothing but the standard discretization of

$$h(u - \frac{h^2}{6}u'')\,.$$

Before interpreting the approximation procedure as solving a diff equation, we need to work out what the right-hand side is in the conte elements. Since $\varphi_i$ is the linear function that is 1 at $x_i$ and zero at a nodes, only the interval $[x_{i-1}, x_{i+1}]$ contributes to the integral on the rig side. This integral is naturally split into two parts according to (54):

$$(f, \varphi_i) = \int_{x_{i-1}}^{x_i} f(x)\frac{1}{h}(x - x_{i-1})dx + \int_{x_i}^{x_{i+1}} f(x)\frac{1}{h}(1 - (x - x_i))dx$$

However, if $f$ is not known we cannot do much else with this expressio clear that many values of $f$ around $x_i$ contributes to the right-hand s just the single point value $f(x_i)$ as in the finite difference method.

To proceed with the right-hand side, we turn to numerical integration Let us say we use the Trapezoidal method for $(f, \varphi_i)$, based on the nod $x_i = ih$:

$$(f, \varphi_i) = \int_\Omega f\varphi_i dx \approx h\frac{1}{2}(f(x_0)\varphi_i(x_0) + f(x_N)\varphi_i(x_N)) + h\sum_{j=1}^{N-1} f(x_j)\varphi$$

ince $\varphi_i$ is zero at all these points, except at $x_i$, the Trapezoidal rule collapses
› one term:

$$(f, \varphi_i) \approx hf(x_i), \tag{88}$$

›r $i = 1, \ldots, N-1$, which is the same result as with collocation/interpolation,
nd of course the same result as in the finite difference method. For $i = 0$ and
$= N$ we get contribution from only one element so

$$(f, \varphi_i) \approx \frac{1}{2} hf(x_i), \quad i = 0,\ i = N\,. \tag{89}$$

Turning to Simpson's rule with sample points also in the middle of the
.ements, $x_i = ih/2$, $i = 0, \ldots, 2N$, it reads in general

$$\int_\Omega f(x)dx \approx \frac{\tilde{h}}{3}\left(f(x_0) + 2\sum_{j=2,4,6,\ldots} f(x_j) + 4\sum_{j=1,3,5,\ldots} f(x_j) + f(x_{2N})\right),$$

here $\tilde{h} = x_i - x_{i-1} = h/2$ is the spacing between the sample points. We see that
ıe midpoints with odd numbers have the weight $2h/3$ while the node points
ith even numbers have the weight $h/3$. Since $\varphi_i = 0$ at the even numbers,
xcept for $x_{2i} = x_i$, and $\varphi_i = 0$ at all the midpoints, on the midpoints and $4h/3$
ı the node points. Since $\varphi_i$ vanishes at all the node points, except $\xi i$, and
xcept $x_{2i-1} = \xi i - h/2$ and $x_{2i+1} = \xi i + h/2$, where $\varphi_i = 1/2$, we get

$$(f, \varphi_i) \approx \frac{h}{3}(f(x_i - \frac{1}{2}h) + f(x_i) + f(x_i + \frac{1}{2}h)\,. \tag{90}$$

ı a finite difference context we would typically express this formula as

$$\frac{h}{3}(f_{i-\frac{1}{2}} + f_i + f_{i+\frac{1}{2}})\,.$$

his shows that, with Simpson's rule, the finite element method operates with
ıe average of $f$ over three points, while the finite difference method just applies
at one point. We may interpret this as a "smearing" or smoothing of $f$ by the
nite element method.

We can now summarize our findings. With the approximation of $(f, \varphi_i)$ by
ıe Trapezoidal rule, P1 elements give rise to equations that can be expressed
ş a finite difference discretization of

$$u + \frac{h^2}{6}u'' = f, \quad u'(0) = u'(L) = 0, \tag{91}$$

xpressed with operator notation as

$$[u + \frac{h^2}{6}D_xD_xu = f]_i\,. \tag{92}$$



As $h \to 0$, the extra term proportional to $u''$ goes to zero, and the two ı
are then equal.

With the Simpson's rule, we may say that we solve

$$[u + \frac{h^2}{6}D_xD_xu = \bar{f}]_i,$$

where $\bar{f}_i$ means the average $\frac{1}{3}(f_{i-1/2} + f_i + f_{i+1/2})$.

The extra term $\frac{h^2}{6}u''$ represents a smoothing: with just this term, w
find u by integrating $f$ twice and thereby smooth $f$ considerably. In a
the finite element representation of $f$ involves an average, or a smoothi
on the right-hand side of the equation system. If $f$ is a noisy functior
interpolation $u_i = f_i$ may result in a noisy $u$ too, but with a Galerkin
squares formulation and P1 elements, we should expect that $u$ is smootl
$f$ unless $h$ is very small.

The interpretation that finite elements tend to smooth the solution
in applications far beyond approximation of 1D function.

## 5.4 Making finite elements behave as finite differen

With a simple trick, using numerical integration, we can easily prod
same result $u_i = f_i$ with the Galerkin or least square formulation
elements. This is useful in many occasions when we deal with more
differential equations and want the finite element method to have pr
like the finite difference method (solving standard linear wave equation
primary example).

We have already seen that applying the Trapezoidal rule to the rig
side $(f, \varphi_i)$ simply gives $f$ sampled at $x_i$. Using the Trapezoidal rule
matrix entries $(\varphi_i, \varphi_j)$ involves a sum

$$\sum_k \varphi_i(x_k)\varphi_j(x_k),$$

but $\varphi_i(x_k) = 0$ for all $k$, except $k = i$, and $\varphi_j(x_k) = 0$ for all $k$, excep
The product $\varphi_i\varphi_j$ is then different from zero only when sampled at $x_i$ a
The approximation to the integral is then

$$(\varphi_i, \varphi_j) \approx h, \quad i = j,$$

and zero if $i \neq j$. This means that we have obtained a diagonal matrix!
and last diagonal elements, $(\varphi_0, \varphi_0)$ and $(\varphi_N, \varphi_N)$ get contribution on
the first and last element, respectively, resulting in the approximate
value $h/2$. The corresponding right-hand side also has a factor $1/2$ for $i$
$i = N$. Therefore, the least squares or Galerkin approach with P1 eleme
*Trapezoidal* integration results in

$$c_i = f_i\,.$$

Simpsons's rule can be used to achieve a similar result for P2 elements, i.e, a iagonal coefficient matrix, but with the previously derived average of $f$ on the ght-hand side.

**lementwise computations.** Identical results to those above will arise if e perform elementwise computations. The idea is to use the Trapezoidal rule n the reference element for computing the element matrix and vector. When ssembled, the same equations $c_i = f(x_i)$ arise. Exercise 16 encourages you to arry out the details.

**erminology.** The matrix with entries $(\varphi_i, \varphi_j)$ typically arises from terms roportional to $u$ in a differential equation where $u$ is the unknown function. his matrix is often called the *mass matrix*, because in the early days of the nite element method, the matrix arose from the mass times acceleration term n Newton's second law of motion. Making the mass matrix diagonal by, e.g., umerical integration, as demonstrated above, is a widely used technique and is alled *mass lumping*. In time-dependent problems it can enhance the numerical ccuracy and computational efficiency of the finite element method. However, here are also examples where mass lumping destroys accuracy.

# A generalized element concept

o far, finite element computing has employed the `nodes` and `element` lists ogether with the definition of the basis functions in the reference element. uppose we want to introduce a piecewise constant approximation with one basis nction $\tilde{\varphi}_0(x) = 1$ in the reference element. Although we could associate the nction value with a node in the middle of the elements, there are no nodes at ne ends, and the previous code snippets will not work because we cannot find ne element boundaries from the `nodes` list.

## .1 Cells, vertices, and degrees of freedom

/e now introduce *cells* as the subdomains $\Omega^{(e)}$ previously referred as elements. he cell boundaries are denoted as *vertices*. The reason for this name is that cells re recognized by their vertices in 2D and 3D. Then we define a set of *degrees of reedom*, which are the quantities we aim to compute. The most common type f degree of freedom is the value of the unknown function $u$ at some point. For xample, we can introduce nodes as before and say the degrees of freedom are the alues of $u$ at the nodes. The basis functions are constructed so that they equal nity for one particular degree of freedom and zero for the rest. This property nsures that when we evaluate $u = \sum_j c_j\varphi_j$ for degree of freedom number $i$, we et $u = c_i$. Integrals are performed over cells, usually by mapping the cell of nterest to a *reference cell*.

With the concepts of cells, vertices, and degrees of freedom we increase the ecoupling the geometry (cell, vertices) from the space of basis functions. We can ssociate different sets of basis functions with a cell. In 1D, all cells are intervals, hile in 2D we can have cells that are triangles with straight sides, or any olygon, or in fact any two-dimensional geometry. Triangles and quadrilaterals



are most common, though. The popular cell types in 3D are tetrahe hexahedra.

## 6.2 Extended finite element concept

The concept of a *finite element* is now

- a *reference cell* in a local reference coordinate system;
- a set of *basis functions* $\tilde{\varphi}_i$ defined on the cell;
- a set of *degrees of freedom* that uniquely determine the basis fu such that $\tilde{\varphi}_i = 1$ for degree of freedom number $i$ and $\tilde{\varphi}_i = 0$ for a degrees of freedom;
- a mapping between local and global degree of freedom numbers;
- a *mapping* of the reference cell onto to cell in the physical domai

There must be a geometric description of a cell. This is trivial in 1D s cell is an interval and is described by the interval limits, here called ver the cell is $\Omega^{(e)} = [x_L, x_R]$, vertex 0 is $x_L$ and vertex 1 is $x_R$. The refere in 1D is $[-1, 1]$ in the reference coordinate system $X$.

Our previous P1, P2, etc., elements are defined by introducing $d+1$ spaced nodes in the reference cell and saying that the degrees of freedom $d+1$ function values at these nodes. The basis functions must be 1 at o and 0 at the others, and the Lagrange polynomials have exactly this p The nodes can be numbered from left to right with associated degrees of that are numbered in the same way. The degree of freedom mapping b what was previously represented by the `elements` lists. The cell mappi same affine mapping (61) as before.

The expansion of $u$ over one cell is often used. In terms of reference coo we have

$$u(x) = \sum_r c_r\tilde{\varphi}_r(X),$$

where the sum is taken over the numbers of the degrees of freedom and value of $u$ for degree of freedom number $r$.

## 6.3 Implementation

Implementationwise,

- we replace `nodes` by `vertices`;
- we introduce `cells` such that `cell[e][r]` gives the mapping fro vertex `r` in cell `e` to the global vertex number in `vertices`;
- we replace `elements` by `dof_map` (the contents are the same).

onsider the example from Section 3.1 where $\Omega = [0, 1]$ is divided into two cells, $^{(0)} = [0, 0.4]$ and $\Omega^{(1)} = [0.4, 1]$. The vertices are $[0, 0.4, 1]$. Local vertex 0 and are 0 and 0.4 in cell 0 and 0.4 and 1 in cell 1. A P2 element means that the egrees of freedom are the value of $u$ at three equally spaced points (nodes) in ach cell. The data structures become

```
ertices = [0, 0.4, 1]
ells = [[0, 1], [1, 2]]
lof_map = [[0, 1, 2], [1, 2, 3]]
```

If we would approximate $f$ by piecewise constants, we simply introduce one oint or node in an element, preferably $X = 0$, and choose $\tilde{\varphi}_0(X) = 1$. Only the `of_map` is altered:

```
lof_map = [[0], [1], [2]]
```

We use the `cells` and `vertices` lists to retrieve information on the geometry f a cell, while `dof_map` is used in the assembly of element matrices and vectors. or example, the `Omega_e` variable (representing the cell interval) in previous ode snippets must now be computed as

```
Omega_e = [vertices[cells[e][0], vertices[cells[e][1]]
```

he assembly is done by

```
A[dof_map[e][r], dof_map[e][s]] += A_e[r,s]
b[dof_map[e][r]] += b_e[r]
```

We will hereafter work with `cells`, `vertices`, and `dof_map`.

## .4 Cubic Hermite polynomials

he finite elements considered so far represent $u$ as piecewise polynomials with iscontinuous derivatives at the cell boundaries. Sometimes it is desired to ave continuous derivatives. A primary examples is the solution of differential quations with fourth-order derivatives where standard finite element formula- ons lead to a need for basis functions with continuous first-order derivatives. he most common type of such basis functions in 1D is the cubic Hermite olynomials.

There are ready-made formulas for the cubic Hermite polynomials, but it instructive to apply the principles for constructing basis functions in detail. iven a reference cell $[-1, 1]$, we seek cubic polynomials with the values of the *unction* and its *first-order derivative* at $X = -1$ and $X = 1$ as the four degrees f freedom. Let us number the degrees of freedom as

- 0: value of function at $X = -1$
- 1: value of first derivative at $X = -1$



- 2: value of function at $X = 1$
- 3: value of first derivative at $X = 1$

By having the derivatives as unknowns, we ensure that the derivative of function in two neighboring elements is the same at the node points.

The four basis functions can be written in a general form

$$\tilde{\varphi}_i(X) = \sum_{j=0}^{3} C_{i,j} X^j,$$

with four coefficients $C_{i,j}$, $j = 0, 1, 2, 3$, to be determined for each constraints that basis function number $i$ must be 1 for degree of freedom $i$ and zero for the other three degrees of freedom, gives four equations to de $C_{i,j}$ for each $i$. In mathematical detail,

$$\begin{aligned}
\tilde{\varphi}_0(-1) &= 1, \quad \tilde{\varphi}_0(1) = \tilde{\varphi}_0'(-1) = \tilde{\varphi}_i'(1) = 0, \\
\tilde{\varphi}_1'(-1) &= 1, \quad \tilde{\varphi}_1(-1) = \tilde{\varphi}_1(1) = \tilde{\varphi}_1'(1) = 0, \\
\tilde{\varphi}_2(1) &= 1, \quad \tilde{\varphi}_2(-1) = \tilde{\varphi}_2'(-1) = \tilde{\varphi}_2'(1) = 0, \\
\tilde{\varphi}_3'(1) &= 1, \quad \tilde{\varphi}_3(-1) = \tilde{\varphi}_3'(-1) = \tilde{\varphi}_3(1) = 0\,.
\end{aligned}$$

These four $4 \times 4$ linear equations can be solved, yielding these formulas cubic basis functions:

$$\begin{aligned}
\tilde{\varphi}_0(X) &= 1 - \frac{3}{4}(X+1)^2 + \frac{1}{4}(X+1)^3 \\
\tilde{\varphi}_1(X) &= -(X+1)(1 - \frac{1}{2}(X+1))^2 \\
\tilde{\varphi}_2(X) &= \frac{3}{4}(X+1)^2 - \frac{1}{2}(X+1)^3 \\
\tilde{\varphi}_3(X) &= -\frac{1}{2}(X+1)(\frac{1}{2}(X+1)^2 - (X+1))
\end{aligned}$$

Remaining tasks:

- Global numbering of the dofs
- `dof_map`
- 4x4 element matrix

# Numerical integration

inite element codes usually apply numerical approximations to integrals. Since ıe integrands in the coefficient matrix often are (lower-order) polynomials, ıtegration rules that can integrate polynomials exactly are popular.

The numerical integration rules can be expressed in a common form,

$$\int_{-1}^{1} g(X)dX \approx \sum_{j=0}^{M} w_j \bar{X}_j, \tag{100}$$

here $\bar{X}_j$ are *integration points* and $w_j$ are *integration weights*, $j = 0, \ldots, M$. ifferent rules correspond to different choices of points and weights.

The very simplest method is the *Midpoint rule*,

$$\int_{-1}^{1} g(X)dX \approx 2g(0), \quad \bar{X}_0 = 0,\ w_0 = 2, \tag{101}$$

hich integrates linear functions exactly.

## .1 Newton-Cotes rules

he Newton-Cotes[7] rules are based on a fixed uniform distribution of the points. he first two formulas in this family is the well-known *Trapezoidal rule*,

$$\int_{-1}^{1} g(X)dX \approx g(-1) + g(1), \quad \bar{X}_0 = -1,\ \bar{X}_1 = 1,\ w_0 = w_1 = 1, \tag{102}$$

nd *Simpson's rule*,

$$\int_{-1}^{1} g(X)dX \approx \frac{1}{3}\left(g(-1) + 4g(0) + g(1)\right), \tag{103}$$

here

$$\bar{X}_0 = -1,\ \bar{X}_1 = 0,\ \bar{X}_2 = 1,\ w_0 = w_2 = \frac{1}{3},\ w_1 = \frac{4}{3}. \tag{104}$$

Newton-Cotes rules up to five points is supported in the module file `numint.y`[8].

For higher accuracy one can divide the reference cell into a set of subintervals nd use the rules above on each subinterval. This approach results in *composite* ıles, well-known from basic introductions to numerical integration of $\int_a^b f(x)dx$.

[7] http://en.wikipedia.org/wiki/Newton%E2%80%93Cotes_formulas
[8] http://tinyurl.com/jvzzcfn/fem/numint.py



## 7.2 Gauss-Legendre rules with optimized points

All these rules apply equally spaced points. More accurate rules, for a g arise if the location of the points are optimized for polynomial integran Gauss-Legendre rules[9] (also known as Gauss-Legendre quadrature or G quadrature) constitute one such class of integration methods. Two widely Gauss-Legendre rules in this family have the choice

$$M = 1: \quad \bar{X}_0 = -\frac{1}{\sqrt{3}},\ \bar{X}_1 = \frac{1}{\sqrt{3}},\ w_0 = w_1 = 1$$
$$M = 2: \quad \bar{X}_0 = -\sqrt{\frac{3}{5}},\ \bar{X}_0 = 0,\ \bar{X}_2 = \sqrt{\frac{3}{5}},\ w_0 = w_2 = \frac{5}{9},\ w_1 = \frac{8}{9}.$$

These rules integrate 3rd and 5th degree polynomials exactly. In gen $M$-point Gauss-Legendre rule integrates a polynomial of degree $2M + 1$ The code `numint.py`[10] contains a large collection of Gauss-Legendre ru

# 8 Approximation of functions in 2D

All the concepts and algorithms developed for approximation of 1D fu $f(x)$ can readily be extended to 2D functions $f(x, y)$ and 3D functions $f$ Basically, the extensions consists of defining basis functions $\varphi_i(x, y)$ or $\varphi$ over some domain $\Omega$, and for the least squares and Galerkin metho integration is done over $\Omega$.

## 8.1 Global basis functions

An example will demonstrate the necessary extensions to use global ba tions and the least squares, Galerkin/projection, or interpolation/coll methods in 2D. The former two lead to linear systems

$$\sum_{j\in I} A_{i,j}c_j = b_i, \quad i \in I,$$
$$A_{i,j} = (\varphi_i, \varphi_j),$$
$$b_i = (f, \varphi_i),$$

where the inner product of two functions $f(x, y)$ and $g(x, y)$ is defined co analogously to the 1D case (24):

$$(f, g) = \int_\Omega f(x, y)g(x, y)dxdy$$

**Constructing 2D basis functions from 1D functions.** One str ward way to construct a basis in 2D is to combine 1D basis functions. have the 1D basis

[9] http://en.wikipedia.org/wiki/Gaussian_quadrature
[10] http://tinyurl.com/jvzzcfn/fem/numint.py

$$\{\hat{\varphi}_0(x), \ldots, \hat{\varphi}_{N_x}(x)\}\,.$$

/e can now form 2D basis functions as products of 1D basis functions: $\hat{\varphi}_p(x)\hat{\varphi}_q(y)$ or $p \in \mathcal{I}_x = \{0, \ldots, N_x\}$ and $q \in \mathcal{I}_y = \{0, \ldots, N_y\}$. We can either work with ouble indices, $\varphi_{p,q}(x,y) = \hat{\varphi}_p(x)\hat{\varphi}_q(y)$, and write

$$u = \sum_{p\in\mathcal{I}_x}\sum_{q\in\mathcal{I}_y} c_{p,q}\varphi_{p,q}(x,y),$$

r we may transform the double index $(p,q)$ to a single index $i$, using $i = pN_y + q$ r $i = qN_x + p$.

Suppose we choose $\hat{\varphi}_p(x) = x^p$, and try an approximation with $N_x = N_y = 1$:

$$\varphi_{0,0} = 1, \quad \varphi_{1,0} = x, \quad \varphi_{0,1} = y, \quad \varphi_{1,1} = xy\,.$$

sing a mapping to one index like $i = qN_x + p$, we get

$$\varphi_0 = 1, \quad \varphi_1 = x, \quad \varphi_2 = y, \quad \varphi_3 = xy\,.$$

**land calculations.** With the specific choice $f(x,y) = (1+x^2)(1+2y^2)$ on $= [0, L_x] \times [0, L_y]$, we can perform actual calculations:

$$\begin{aligned}
A_{0,0} &= (\varphi_0, \varphi_0) = \int_0^{L_y}\int_0^{L_x} \varphi_0(x,y)^2 dxdy = \int_0^{L_y}\int_0^{L_x} dxdy = L_x L_y,\\
A_{1,0} &= (\varphi_1, \varphi_0) = \int_0^{L_y}\int_0^{L_x} xdxdy = \frac{1}{2}L_x^2 L_y,\\
A_{0,1} &= (\varphi_0, \varphi_1) = \int_0^{L_y}\int_0^{L_x} ydxdy = \frac{1}{2}L_y^2 L_x,\\
A_{0,1} &= (\varphi_0, \varphi_1) = \int_0^{L_y}\int_0^{L_x} xydxdy = \int_0^{L_y} ydy\int_0^{L_x} xdx = \frac{1}{4}L_y^2 L_x^2\,.
\end{aligned}$$

he right-hand side vector has the entries



$$\begin{aligned}
b_0 &= (\varphi_0, f) = \int_0^{L_y}\int_0^{L_x} 1\cdot(1+x^2)(1+2y^2)dxdy\\
&= \int_0^{L_y}(1+2y^2)dy\int_0^{L_x}(1+x^2)dx = (L_y + \frac{2}{3}L_y^3)(L_x + \frac{1}{3}L_x^3)\\
b_1 &= (\varphi_1, f) = \int_0^{L_y}\int_0^{L_x} x(1+x^2)(1+2y^2)dxdy\\
&= \int_0^{L_y}(1+2y^2)dy\int_0^{L_x} x(1+x^2)dx = (L_y + \frac{2}{3}L_y^3)(\frac{1}{2}L_x^2 + \frac{1}{4}L_x^4)\\
b_2 &= (\varphi_2, f) = \int_0^{L_y}\int_0^{L_x} y(1+x^2)(1+2y^2)dxdy\\
&= \int_0^{L_y} y(1+2y^2)dy\int_0^{L_x}(1+x^2)dx = (\frac{1}{2}L_y + \frac{1}{2}L_y^4)(L_x + \frac{1}{3}L_x^3)\\
b_3 &= (\varphi_2, f) = \int_0^{L_y}\int_0^{L_x} xy(1+x^2)(1+2y^2)dxdy\\
&= \int_0^{L_y} y(1+2y^2)dy\int_0^{L_x} x(1+x^2)dx = (\frac{1}{2}L_y^2 + \frac{1}{2}L_y^4)(\frac{1}{2}L_x^2 + \frac{1}{4}I
\end{aligned}$$

There is a general pattern in these calculations that we can explo arbitrary matrix entry has the formula

$$\begin{aligned}
A_{i,j} &= (\varphi_i, \varphi_j) = \int_0^{L_y}\int_0^{L_x} \varphi_i\varphi_j dxdy\\
&= \int_0^{L_y}\int_0^{L_x} \varphi_{p,q}\varphi_{r,s}dxdy = \int_0^{L_y}\int_0^{L_x} \hat{\varphi}_p(x)\hat{\varphi}_q(y)\hat{\varphi}_r(x)\hat{\varphi}_s(y)dx\\
&= \int_0^{L_y} \hat{\varphi}_q(y)\hat{\varphi}_s(y)dy\int_0^{L_x} \hat{\varphi}_p(x)\hat{\varphi}_r(x)dx\\
&= \hat{A}^{(x)}_{p,r}\hat{A}^{(y)}_{q,s},
\end{aligned}$$

where

$$\hat{A}^{(x)}_{p,r} = \int_0^{L_x} \hat{\varphi}_p(x)\hat{\varphi}_r(x)dx, \quad \hat{A}^{(y)}_{q,s} = \int_0^{L_y} \hat{\varphi}_q(y)\hat{\varphi}_s(y)dy,$$

are matrix entries for one-dimensional approximations. Moreover, $i =$ and $j = sN_y + r$.

With $\hat{\varphi}_p(x) = x^p$ we have

$$\hat{A}^{(x)}_{p,r} = \frac{1}{p+r+1}L_x^{p+r+1}, \quad \hat{A}^{(y)}_{q,s} = \frac{1}{q+s+1}L_y^{q+s+1},$$

and

$$A_{i,j} = \hat{A}^{(x)}_{p,r}\hat{A}^{(y)}_{q,s} = \frac{1}{p+r+1}L_x^{p+r+1}\frac{1}{q+s+1}L_y^{q+s+1},$$

or $p, r \in \mathcal{I}_x$ and $q, s \in \mathcal{I}_y$.

Corresponding reasoning for the right-hand side leads to

$$\begin{aligned}
b_i = (\varphi_i, f) &= \int_0^{L_y}\int_0^{L_x} \varphi_i f\, dxdx \\
&= \int_0^{L_y}\int_0^{L_x} \hat{\varphi}_p(x)\hat{\varphi}_q(y) f\, dxdx \\
&= \int_0^{L_y} \hat{\varphi}_q(y)(1+2y^2)dy \int_0^{L_y} \hat{\varphi}_p(x)x^p(1+x^2)dx \\
&= \int_0^{L_y} y^q(1+2y^2)dy \int_0^{L_y} x^p(1+x^2)dx \\
&= (\frac{1}{q+1}L_y^{q+1} + \frac{2}{q+3}L_y^{q+3})(\frac{1}{p+1}L_x^{p+1} + \frac{2}{q+3}L_x^{p+3})
\end{aligned}$$

Choosing $L_x = L_y = 2$, we have

$$A = \begin{bmatrix} 4 & 4 & 4 & 4 \\ 4 & \frac{16}{3} & 4 & \frac{16}{3} \\ 4 & 4 & \frac{16}{3} & \frac{16}{3} \\ 4 & \frac{16}{3} & \frac{16}{3} & \frac{64}{9} \end{bmatrix}, \quad b = \begin{bmatrix} \frac{308}{9} \\ \frac{140}{3} \\ 44 \\ 60 \end{bmatrix}, \quad c = \begin{bmatrix} -\frac{1}{9}, \\ \frac{4}{3}, \\ -\frac{2}{3}, \\ 8 \end{bmatrix}.$$

Figure 30 illustrates the result.

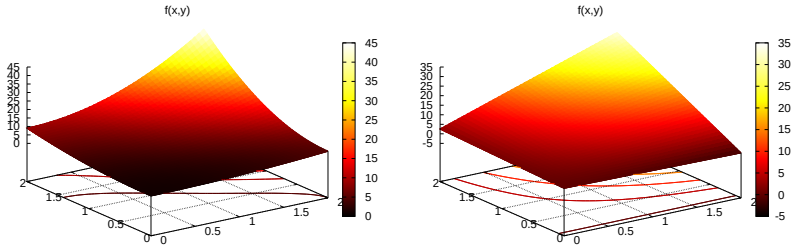



Figure 30: Approximation of a 2D quadratic function (left) by a 2D bilinear function (right) using the Galerkin or least squares method.

## .2 Implementation

The `least_squares` function from Section 2.8 and/or the file `approx1D.py`[11] can with very small modifications solve 2D approximation problems. First, let `Omega` now be a list of the intervals in $x$ and $y$ direction. For example, $\Omega = [0, L_x] \times [0, L_y]$ can be represented by `Omega = [[0, L_x], [0, L_y]]`.

Second, the symbolic integration must be extended to 2D:

[11] `http://tinyurl.com/jvzzcfn/fem/fe_approx1D.py`



```
import sympy as sm

integrand = phi[i]*phi[j]
I = sm.integrate(integrand,
                 (x, Omega[0][0], Omega[0][1]),
                 (y, Omega[1][0], Omega[1][1]))
```

provided `integrand` is an expression involving the `sympy` symbols `x` and 2D version of numerical integration becomes

```
if isinstance(I, sm.Integral):
    integrand = sm.lambdify([x,y], integrand)
    I = sm.mpmath.quad(integrand,
                       [Omega[0][0], Omega[0][1]],
                       [Omega[1][0], Omega[1][1]])
```

The right-hand side integrals are modified in a similar way.

Third, we must construct a list of 2D basis functions, e.g.,

```
def taylor(x, y, Nx, Ny):
    return [x**i*y**j for i in range(Nx+1) for j in range(Ny+1)]

def sines(x, y, Nx, Ny):
    return [sm.sin(sm.pi*(i+1)*x)*sm.sin(sm.pi*(j+1)*y)
            for i in range(Nx+1) for j in range(Ny+1)]
```

The complete code appears in `approx2D.py`[12].

The previous hand calculation where a quadratic $f$ was approximat bilinear function can be computed symbolically by

```
>>> f = (1+x**2)*(1+2*y**2)
>>> phi = taylor(x, y, 1, 1)
>>> Omega = [[0, 2], [0, 2]]
>>> u = least_squares(f, phi, Omega)
>>> print u
8*x*y - 2*x/3 + 4*y/3 - 1/9
>>> print sm.expand(f)
2*x**2*y**2 + x**2 + 2*y**2 + 1
```

We may continue with adding higher powers to the basis and check th $N_x \geq 2$ and $N_y \geq 2$ we recover the exact function $f$:

```
>>> phi = taylor(x, y, 2, 2)
>>> u = least_squares(f, phi, Omega)
>>> print u
2*x**2*y**2 + x**2 + 2*y**2 + 1
>>> print u-f
0
```

[12] `http://tinyurl.com/jvzzcfn/fem/fe_approx2D.py`

# Finite elements in 2D and 3D

inite element approximation is particularly powerful in 2D and 3D because ıe method can handle a geometrically complex domain $\Omega$ with ease. The rincipal idea is, as in 1D, to divide the domain into cells use polynomials for pproximating a function over a cell. Two popular cell shapes are triangles and ıe quadrilaterals. Figures 31, 32, and 33 provide examples. P1 elements means near functions $(a_0 + a_1x + a_2y)$ over triangles, while Q1 elements have bilinear unctions $(a_0 + a_1x + a_2y + a_3xy)$ over rectangular cells. Higher-order elements an easily be defined.

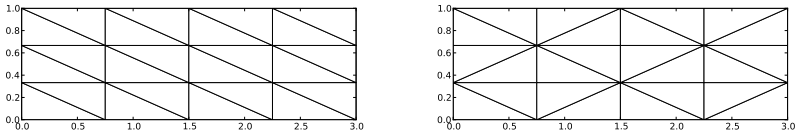


Figure 31: Examples on 2D P1 elements.

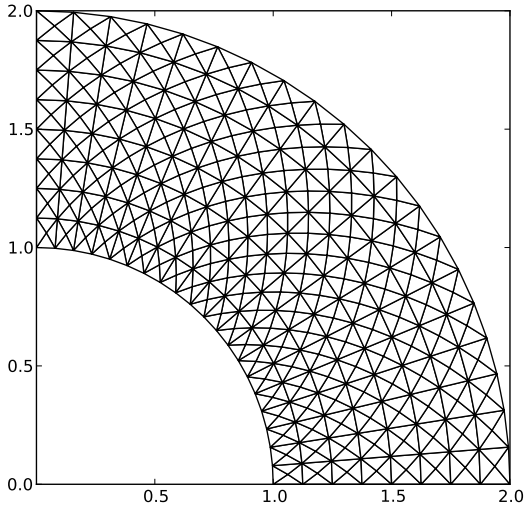


Figure 32: Examples on 2D P1 elements in a deformed geometry.

## .1 Basis functions over triangles in the physical domain

ells with triangular shape will be in main focus here. With the P1 triangular ement, $u$ is a linear function over each cell, with discontinuous derivatives at ıe cell boundaries, as depicted in Figure 34.



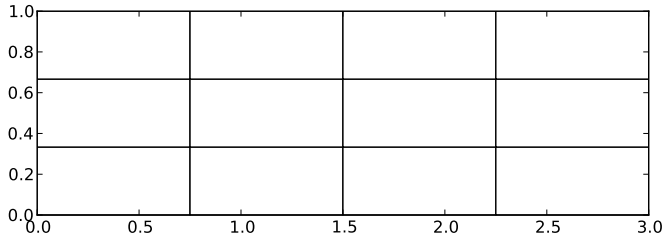


Figure 33: Examples on 2D Q1 elements.

We give the vertices of the cells global and local numbers as in 1 degrees of freedom in the P1 element are the function values at a set o which are the three vertices. The basis function $\varphi_i(x, y)$ is then 1 at th with global vertex number $i$ and zero at all other vertices. On an elem three degrees of freedom uniquely determine the linear basis functions element, as usual. The global $\varphi_i(x, y)$ function is then a combination of tl functions (planar surfaces) over all the neighboring cells that have vertex $i$ in common. Figure 35 tries to illustrate the shape of such a "pyram function.

**Element matrices and vectors.** As in 1D, we split the integral ove a sum of integrals over cells. Also as in 1D, $\varphi_i$ overlaps $\varphi_j$ (i.e., $\varphi_i\varphi_j$ and only if $i$ and $j$ are vertices in the same cell. Therefore, the integral over an element is nonzero only when $i$ and $j$ run over the vertex nun the element. These nonzero contributions to the coefficient matrix are, a collected in an element matrix. The size of the element matrix becom since there are three degrees of freedom that $i$ and $j$ run over. Again, a we number the local vertices in a cell, starting at 0, and add the entrie element matrix into the global system matrix, exactly as in 1D. All det code appear below.

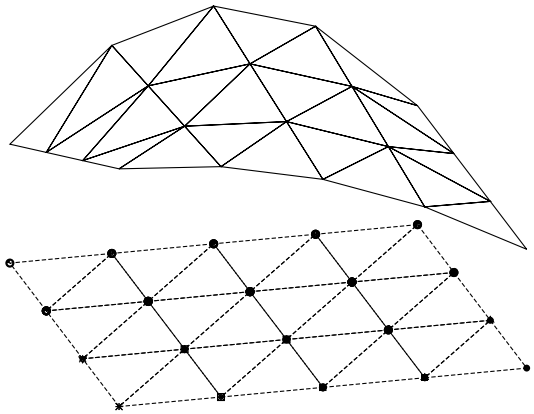


Figure 34: Example on piecewise linear 2D functions defined on triangles.

## .2 Basis functions over triangles in the reference cell

s in 1D, we can define the basis functions and the degrees of freedom in a eference cell and then use a mapping from the reference coordinate system to ne physical coordinate system. We also have a mapping of local degrees of eedom numbers to global degrees of freedom numbers.

The reference cell in an $(X, Y)$ coordinate system has vertices $(0, 0)$, $(1, 0)$, nd $(0, 1)$, corresponding to local vertex numbers 0, 1, and 2, respectively. The 1 element has linear functions $\tilde{\varphi}_r(X, Y)$ as basis functions, $r = 0, 1, 2$. Since a near function $\tilde{\varphi}_r(X, Y)$ in 2D is on the form $C_{r,0} + C_{r,1}X + C_{r,2}Y$, and hence as three parameters $C_{r,0}$, $C_{r,1}$, and $C_{r,2}$, we need three degrees of freedom. hese are in general taken as the function values at a set of nodes. For the P1 ement the set of nodes is the three vertices. Figure 36 displays the geometry f the element and the location of the nodes.

Requiring $\tilde{\varphi}_r = 1$ at node number $r$ and $\tilde{\varphi}_r = 0$ at the two other nodes, gives ree linear equations to determine $C_{r,0}$, $C_{r,1}$, and $C_{r,2}$. The result is

$$\tilde{\varphi}_0(X, Y) = 1 - X - Y, \tag{108}$$
$$\tilde{\varphi}_1(X, Y) = X, \tag{109}$$
$$\tilde{\varphi}_2(X, Y) = Y \tag{110}$$

Higher-order approximations are obtained by increasing the polynomial order, dding additional nodes, and letting the degrees of freedom be function values t the nodes. Figure 37 shows the location of the six nodes in the P2 element.



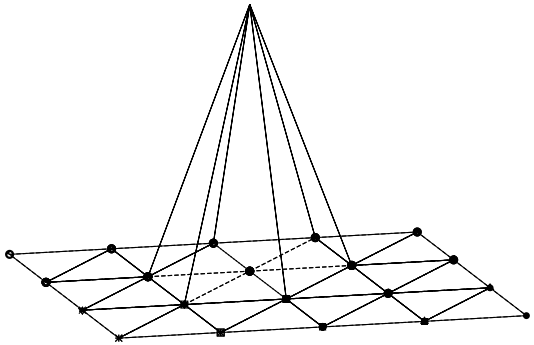


Figure 35: Example on a piecewise linear 2D basis function over a p triangles.

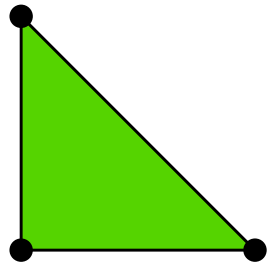


Figure 36: 2D P1 element.

A polynomial of degree $p$ in $X$ and $Y$ has $n_p = (p + 1)(p + 2)/2$ te hence needs $n_p$ nodes. The values at the nodes constitute $n_p$ degrees of The location of the nodes for $\tilde{\varphi}_r$ up to degree 6 is displayed in Figure 3

The generalization to 3D is straightforward: the reference eleme tetrahedron[13] with vertices $(0, 0, 0)$, $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$ in a reference coordinate system. The P1 element has its degrees of freedom nodes, which are the four vertices, see Figure 39. The P2 element adds ad nodes along the edges of the cell, yielding a total of 10 nodes and de freedom, see Figure 40.

[13] `http://en.wikipedia.org/wiki/Tetrahedron`

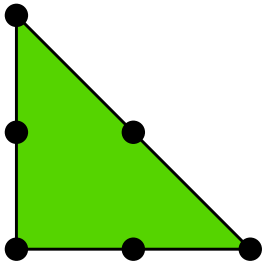


Figure 37: 2D P2 element.

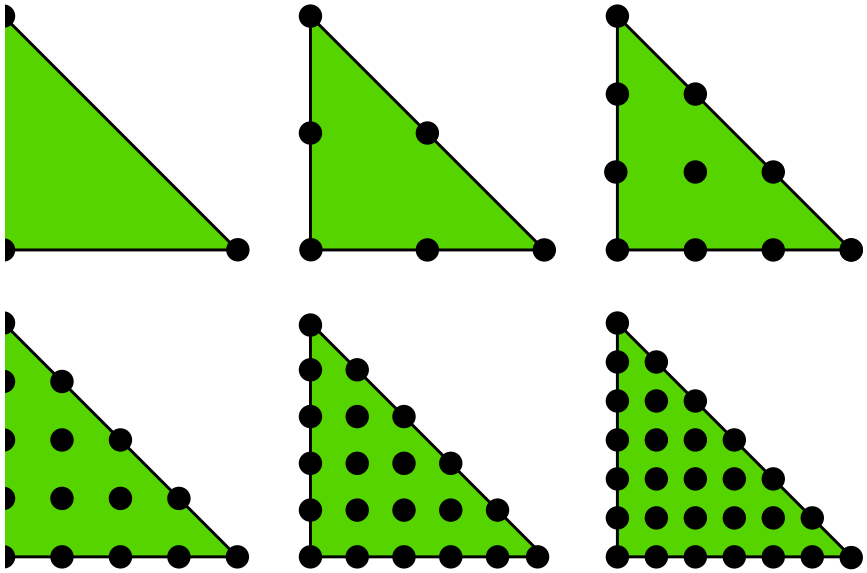


Figure 38: 2D P1, P2, P3, P4, P5, and P6 elements.

The interval in 1D, the triangle in 2D, the tetrahedron in 3D, and its eneralizations to higher space dimensions are known as *simplex* cells (the eometry) or *simplex* elements (the geometry, basis functions, degrees of freedom, c.). The plural forms simplices[14] and simplexes are also a much used shorter erms when referring to this type of cells or elements. The side of a simplex is alled a *face*, while the tetrahedron also has *edges*.

**cknowledgment.** Figures 36 to 40 are created by Anders Logg and taken om the FEniCS book[15]: *Automated Solution of Differential Equations by the*

[14] http://en.wikipedia.org/wiki/Simplex
[15] https://launchpad.net/fenics-book



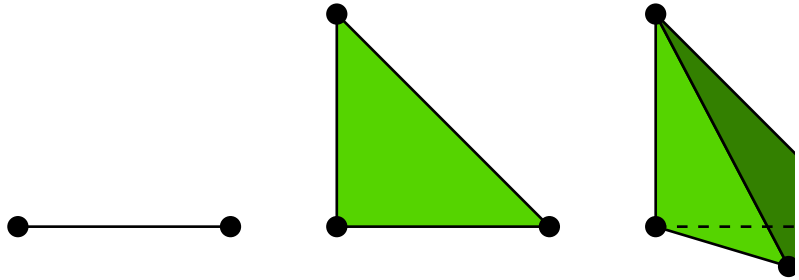


Figure 39: P1 elements in 1D, 2D, and 3D.

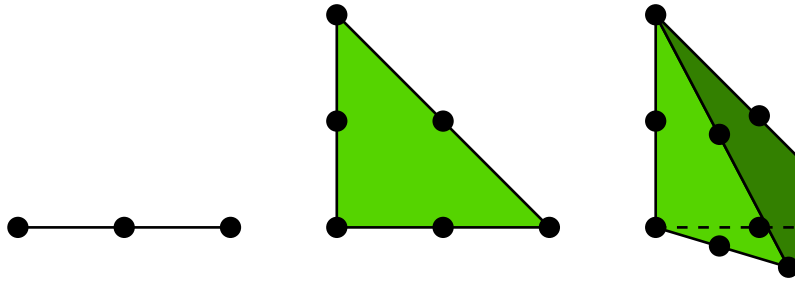


Figure 40: P2 elements in 1D, 2D, and 3D.

*Finite Element Method*, edited by A. Logg, K.-A. Mardal, and G. N published by Springer[16], 2012.

## 9.3 Affine mapping of the reference cell

Let $\tilde{\varphi}_r^{(1)}$ denote the basis functions associated with the P1 element in 1D 3D, and let $\boldsymbol{x}_{q(e,r)}$ be the physical coordinates of local vertex number $r$ Furthermore, let $\boldsymbol{X}$ be a point in the reference coordinate system corres to the point $\boldsymbol{x}$ in the physical coordinate system. The affine mapping o onto $\boldsymbol{x}$ is then defined by

$$\boldsymbol{x} = \sum_r \tilde{\varphi}_r^{(1)}(\boldsymbol{X})\boldsymbol{x}_{q(e,r)},$$

where $r$ runs over the local vertex numbers in the cell. The affine mappi the straight or planar faces of the reference cell onto straight or plan in the physical coordinate system. The mapping can be used for both higher-order elements.

[16] http://goo.gl/1byVMH

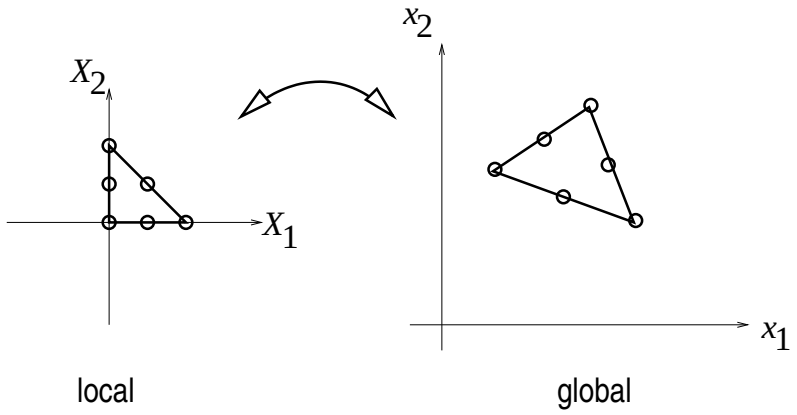



Figure 41: Affine mapping of a P1 element.

## .4 Isoparametric mapping of the reference cell

ıstead of using the P1 basis functions in the mapping (111), we may use the asis functions of the actual element:

$$\boldsymbol{x} = \sum_r \tilde{\varphi}_r(\boldsymbol{X})\boldsymbol{x}_{q(e,r)}, \tag{112}$$

here $r$ runs over all nodes, i.e., all points associated with the degrees of freedom. his is called an *isoparametric mapping*. For P1 elements it is identical to ıe affine mapping (111), but for higher-order elements the mapping of the :raight or planar faces of the reference cell will result in a *curved* face in the hysical coordinate system. For example, when we use the basis functions of the iangular P2 element in 2D in (112), the straight faces of the reference triangle re mapped onto curved faces of parabolic shape in the physical coordinate ystem, see Figure 42.

From (111) or (112) it is easy to realize that the vertices are correctly mapped. onsider a vertex with local number $s$. Then $\tilde{\varphi}_s = 1$ at this vertex and zero at ıe others. This means that only one term in the sum is nonzero and $\boldsymbol{x} = \boldsymbol{x}_{q(e,s)}$, hich is the coordinate of this vertex in the global coordinate system.

## .5 Computing integrals

et $\tilde{\Omega}^r$ denote the reference cell and $\Omega^{(e)}$ the cell in the physical coordinate ystem. The transformation of the integral from the physical to the reference oordinate system reads

$$\int_{\Omega^{(e)}} \varphi_i(\boldsymbol{x})\varphi_j(\boldsymbol{x})d\boldsymbol{x} = \int_{\tilde{\Omega}^r} \tilde{\varphi}_i(\boldsymbol{X})\tilde{\varphi}_j(\boldsymbol{X}) \det J \, d\boldsymbol{X}, \tag{113}$$

$$\int_{\Omega^{(e)}} \varphi_i(\boldsymbol{x})f(\boldsymbol{x})d\boldsymbol{x} = \int_{\tilde{\Omega}^r} \tilde{\varphi}_i(\boldsymbol{X})f(\boldsymbol{x}(\boldsymbol{X})) \det J \, d\boldsymbol{X}, \tag{114}$$



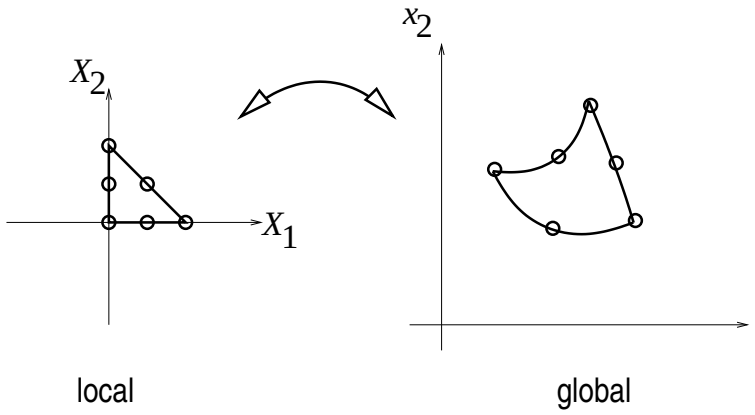



Figure 42: Isoparametric mapping of a P2 element.

where $d\boldsymbol{x} = dxdy$ in 2D and $d\boldsymbol{x} = dxdydz$ in 3D, with a similar definitio The quantity $\det J$ is the determinant of the Jacobian of the mapping $\boldsymbol{x}$ 2D,

$$J = \begin{bmatrix} \frac{\partial x}{\partial X} & \frac{\partial x}{\partial Y} \\ \frac{\partial y}{\partial X} & \frac{\partial y}{\partial Y} \end{bmatrix}, \quad \det J = \frac{\partial x}{\partial X}\frac{\partial y}{\partial Y} - \frac{\partial x}{\partial Y}\frac{\partial y}{\partial X}.$$

With the affine mapping (111), $\det J = 2\Delta$, where $\Delta$ is the area or vo the cell in the physical coordinate system.

**Remark.** Observe that finite elements in 2D and 3D builds on the sar and *concepts* as in 1D, but there is simply more to compute because the mathematical formulas in 2D and 3D are more complicated.

# 10 Exercises

## Exercise 1: Linear algebra refresher I

Look up the topic of *vector space* in your favorite linear algebra book o for the term at Wikipedia. Prove that vectors in the plane $(a, b)$ form space by showing that all the axioms of a vector space are satisfied. S prove that all linear functions of the form $ax + b$ constitute a vecto $a, b \in \mathbb{R}$.

On the contrary, show that all quadratic functions of the form $1 + a$ *do not* constitute a vector space. Filename: `linalg1`.

## Exercise 2: Linear algebra refresher II

As an extension of Exercise 1, check out the topic of *inner product spaces.* a possible inner product for the space of all linear functions of the form $a, b \in \mathbb{R}$. Show that this inner product satisfies the general requiremen inner product in a vector space. Filename: `linalg2`.

## Exercise 3: Approximate a three-dimensional vector in a plane

Given $\boldsymbol{f} = (1,1,1)$ in $\mathbb{R}^3$, find the best approximation vector $\boldsymbol{u}$ in the plane spanned by the unit vectors $(1,0)$ and $(0,1)$. Repeat the calculations using the vectors $(2,1)$ and $(1,2)$. Filename: `vec111_approx`.

## Exercise 4: Approximate the sine function by power functions

Let $V$ be a function space with basis functions $x^{2i+1}$, $i = 0,1,\ldots,N$. Find the best approximation to $f(x) = \sin(x)$ among all functions in $V$, using $N = 8$ for a domain that includes more and more half-periods of the sine function: $\Omega = [0, k\pi/2]$, $k = 2,3,\ldots,12$. How does a Taylor series of $\sin(x)$ around $x$ up to degree 9 behave for the largest domain?

**Hint.** One can make a loop over $k$ and call the `least_squares` and `comparison_plot` from the `approx1D` module.

Filename: `sin_powers.py`.

## Exercise 5: Approximate a steep function by sines

Find the best approximation of $f(x) = \tanh(s(x-\pi))$ on $[0, 2\pi]$ in the space $V$ with basis $\varphi_i(x) = sin((2i+1)x)$, $i \in I$. Make a movie showing how $u = \sum_{j\in I} c_j\varphi_j(x)$ approximates $f(x)$ as $N$ grows. Choose $s$ such that $f$ is steep ($s = 20$ may be appropriate).

**Hint.** One may naively call the `least_squares_orth` and `comparison_plot` from the `approx1D` module in a loop and extend the basis with one new element in each pass. This approach implies a lot of recomputations. A more efficient strategy is to let `least_squares_orth` compute with only one basis function at a time and accumulate the corresponding `u` in the total solution.

Filename: `tanh_sines_approx1.py`.

## Exercise 6: Fourier series as a least squares approximation

Given a function $f(x)$ on an interval $[0, L]$, find the formula for the coefficients of the Fourier series of $f$:

$$f(x) = a_0 + \sum_{j=1}^{\infty} a_j \cos\left(j\frac{\pi x}{L}\right) + \sum_{j=1}^{\infty} b_j \sin\left(j\frac{\pi x}{L}\right).$$

Let an infinite-dimensional vector space $V$ have the basis functions $\cos j\frac{\pi x}{L}$ for $j = 0,1,\ldots,\infty$ and $\sin j\frac{\pi x}{L}$ for $j = 1,\ldots,\infty$. Show that the least squares approximation method from Section 2 leads to a linear system whose solution coincides with the standard formulas for the coefficients in a Fourier series of $f(x)$ (see also Section 2.7). You may choose

$$\varphi_{2i} = \cos\left(i\frac{\pi}{L}x\right), \quad \varphi_{2i+1} = \sin\left(i\frac{\pi}{L}x\right),$$

for $i = 0,1,\ldots,N \to \infty$.



Choose $f(x) = \tanh(s(x-\frac{1}{2}))$ on $\Omega = [0,1]$, which is a smooth funct with considerable steepness around $x = 1/2$ as $s$ grows in size. Calcu coefficients in the Fourier expansion by solving the linear system, arisi the least squares or Galerkin methods, by hand. Plot some truncated of the series together with $f(x)$ to show how the series expansion conve $s = 10$ and $s = 100$. Filename: `Fourier_approx.py`.

## Exercise 7: Approximate a steep function by Lagrange nomials

Use interpolation/collocation with uniformly distributed points and Ch nodes to approximate

$$f(x) = -\tanh(s(x-\frac{1}{2}))$$

by Lagrange polynomials for $s = 10, 100$ and $N = 3,6,9,11$. Make s plots of the approximation for each combination of $s$, point type (Cheby uniform), and $N$. Filename: `tanh_Lagrange.py`.

## Exercise 8: Define finite element meshes

Consider a domain $\Omega = [0,2]$ divided into the three elements $[0,1]$, and $[1.2,2]$, with two nodes in each element (P1 elements). Set up th coordinates and nodes (`nodes`) and the numbers of the nodes that be each element (`elements`) in two cases: 1) nodes and elements number left to right, and 2) nodes and elements numbered from right to left.

Thereafter, subdivide the element $[1.2,2]$ into two new equal-sized e Add the new node and the two new elements to the data structures above, and try to minimize the modifications. Filename: `fe_numbering`

## Exercise 9: Construct matrix sparsity patterns

Exercise 8 describes a element mesh with a total of five elements, but w different element and node orderings. For each of the two orderings, $5 \times 5$ matrix and fill in the entries that will be nonzero.

**Hint.** A matrix entry $(i,j)$ is nonzero if $i$ and $j$ are nodes in the same

Filename: `fe_sparsity_pattern.pdf`.

## Exercise 10: Perform symbolic finite element comput

Find formulas for the coefficient matrix and right-hand side when approx $f(x) = sin(x)$ on $\Omega = [0,\pi]$ by two P1 elements of size $\pi/2$. Solve the and compare $u(\pi/2$ with the exact value 1.

Filename: `sin_approx_P1.py`.

## Exercise 11: Approximate a steep function by P1 a elements

Given

$$f(x) = \tanh(s(x - \frac{1}{2}))$$

se the Galerkin or least squares method with finite elements to find an pproximate function $u(x)$. Choose $s = 40$ and try $N_e = 4, 8, 16$ P1 ele- ıents and $N_e = 2, 4, 8$ P2 elements. Integrate $f\varphi_i$ numerically. Filename: `anh_fe_P1P2_approx.py`.

### Exercise 12: Approximate a tanh function by P3 and P4 lements

olve Exercise 11 using $N_e = 1, 2, 4$ P3 and P4 elements. How will a colloca- on/interpolation method work in this case with the same number of nodes? ilename: `tanh_fe_P3P4_approx.py`.

### Exercise 13: Investigate the approximation errors in finite lements

fundamental question is how accurate the finite element approximation is ı terms of the cell length $h$ and the degree $d$ of the basis functions. We can ıvestigate this empirically by choosing an $f$ function, say $f(x) = A\sin(\omega x)$ on $= [0, 2\pi/\omega]$, and compute the approximation error for a series of $h$ and $d$ alues. The theory predicts that the error should behave as $h^{d+1}$. Use experi- ıents to verify this asymptotic behavior (i.e., for small enough $h$). Filename: `sinwt_interpolation_error.py`.

### Exercise 14: Approximate a step function by finite elements

pproximate the step function

$$f(x) = \begin{cases} 1 & x < 1/2, \\ 2 & x \geq 1/2 \end{cases}$$

y 2, 4, and 8 P1 and P2 elements. Compare approximations visually.

**lint.** This $f$ can also be expressed in terms of the Heaviside function $H(x)$: $(x) = H(x - 1/2)$. Therefore, $f$ can be defined by

```
 = sm.Heaviside(x -  sm.Rational(1,2))
```

aking the `approximate` function in the `fe_approx1D.py` module an obvious andidate to solve the problem. However, `sympy` does not handle symbolic ıtegration with this particular integrand, and the `approximate` function faces a roblem when converting `f` to a Python function (for plotting) since `Heaviside` not an available function in `numpy`. It is better to make special-purpose code or this case or perform all calculations by hand.

Filename: `Heaviside_approx_P1P2.py`..



### Exercise 15: 2D approximation with orthogonal func

Assume we have basis functions $\varphi_i(x, y)$ in 2D that are orthogonal su $(\varphi_i, \varphi_j) = 0$ when $i \neq j$. The function `least_squares` in the file approx will then spend much time on computing off-diagonal terms in the co matrix that we know are zero. To speed up the computations, make a `least_squares_orth` that utilizes the orthogonality among the basis fu Apply the function to approximate

$$f(x, y) = x(1 - x)y(1 - y)e^{-x-y}$$

on $\Omega = [0, 1] \times [0, 1]$ via basis functions

$$\varphi_i(x, y) = \sin(p\pi x)\sin(q\pi y), \quad i = qN_x + p\,.$$

**Hint.** Get ideas from the function `least_squares_orth` in Section file `approx1D.py`[18].

Filename: `approx2D_lsorth_sin.py`.

### Exercise 16: Use the Trapezoidal rule and P1 elemer

Consider approximation of some $f(x)$ on an interval $\Omega$ using the least sq Galerkin methods with P1 elements. Derive the element matrix and vect the Trapezoidal rule (102) for calculating integrals on the reference Assemble the contributions, assuming a uniform cell partitioning, ar that the resulting linear system has the form $c_i = f(x_i)$ for $i \in I$. Fi `fe_trapez.pdf`.

[17] `http://tinyurl.com/jvzzcfn/fem/fe_approx2D.py`
[18] `http://tinyurl.com/jvzzcfn/fem/fe_approx1D.py`

# ndex