Numerical methods for the Navier-Stokes equations

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Contents

The physical and mathematical problem

luid flow is one of the most common physical phenomena in nature and echnological devices. Examples include atmospheric flows ("weather"), global ocean currents, air flow around a car, breathing, and circulation of bl mention a few. The focus in the forthcoming text is on a subset of flows turbulence, where the flow can be considered as incompressible, and wl fluid's viscosity is constant. (Actually, the model to be discussed can be turbulence, in principle, but the computations are very heavy.)

1.1 The Navier-Stokes equations

For incompressible flow, the key unknowns are the pressure field $p(\boldsymbol{x},t)$ velocity field $\boldsymbol{u}(\boldsymbol{x},t)$. These quantities are governed by the a momentum equation,

$$\boldsymbol{u}_t + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} = -\frac{1}{\varrho}\nabla p + \nu\nabla^2\boldsymbol{u} + \boldsymbol{f},$$

and a mass balance equation

$$\nabla\cdot\boldsymbol{u} = 0\,.$$

Equations (1) and (2) are known as *Navier-Stokes equations* for incomp flow. The parameter ϱ is the fluid density, ν is the (kinematic) viscos $\boldsymbol{f}$ denotes body forces such as gravity. Geophysical applications often incorporate the Coriolis and centrifugal forces in $\boldsymbol{f}$. The Navier-Stokes ec are to be solved in a spatial domain Ω for $t\in(0,T]$.

1.2 Derivation

The derivation of the Navier-Stokes equations contains some equatic are useful for alternative formulations of numerical methods, so we shal recover the steps to arrive at (1) and (2). We start with the general mo balance equation for a continuum (arising from Newton's second law of

$$\varrho\frac{Du}{dt} = \nabla\cdot\boldsymbol{\sigma} + \varrho\boldsymbol{f},$$

where $\boldsymbol{\sigma}$ is the stress tensor and the operator D/dt is the material deri

$$\frac{D\boldsymbol{u}}{dt} = \boldsymbol{u}_t + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u},$$

here denoting acceleration. Therefore, $\varrho D\boldsymbol{u}/dt$ is density ("mass") tim eration, while the terms on the right-hand side are the forces that ind motion $\boldsymbol{u}$: the internal stresses $\boldsymbol{\sigma}$ and the external body forces $\boldsymbol{f}$.

The other fundamental equation for a fluid is that of mass conse called the continuity equation. It has the general form

$$\varrho_t + \nabla\cdot(\varrho\boldsymbol{u}) = 0,$$

which can be rewritten as

$$\nabla\cdot\boldsymbol{u} = \frac{1}{\varrho}\frac{D\varrho}{dt}\,.$$

n incompressible flow is defined as a flow where each fluid particle maintains s density. Since $\frac{D\varrho}{dt}$ is the rate of change of ϱ of a fluid particle, incompressible ow means $\frac{D\varrho}{dt} = 0$ and hence $\nabla\cdot\boldsymbol{u} = 0$. The latter is the most useful equation a PDE system for incompressible flow since it involves the unknown velocity .

Different types of fluids will have different relations between the motion $\boldsymbol{u}$ nd the internal stresses $\boldsymbol{\sigma}$. A *Newtonian* fluid has an isotropic, linear relation etween $\boldsymbol{u}$ and $\boldsymbol{\sigma}$:

$$\boldsymbol{\sigma} = -p\boldsymbol{I} + \mu(\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^T), \tag{6}$$

here $\boldsymbol{I}$ is the identity tensor, and μ is the dynamic viscosity ($\mu = \varrho\nu$). The elation (6) assumes incompressible flow. Inserting (6) in (3) gives (1) after ividing by ϱ, using $\nabla\cdot(p\boldsymbol{I}) = \nabla p$, and calculating $\nabla\cdot(\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^T)$ as $^2\boldsymbol{u} + \nabla(\nabla\cdot\boldsymbol{u}) = \nabla^2\boldsymbol{u}$. The vector operations involving the nabla operator are asiest performed by using index or dyadic notation, but the derivation of the articular terms is not important for the forthcoming text.

Some numerical methods apply the $\nabla\cdot\boldsymbol{\sigma} = -\nabla p + \nabla\cdot(\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^T)$ form (1):

$$\boldsymbol{u}_t + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} = \frac{1}{\varrho}\nabla\cdot\boldsymbol{\sigma} + \boldsymbol{f}, \tag{7}$$

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$$\boldsymbol{u}_t + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} = -\frac{1}{\varrho}\nabla p + \nu\nabla\cdot(\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^T) + \boldsymbol{f}\,. \tag{8}$$

Other formulations add a ϱ_t term to the continuity equation, usually by ssuming slight compressibility. Then $\varrho = \varrho(p)$ and we have

$$\varrho_t = \frac{\partial\varrho}{\partial p}p_t\,.$$

is common to evaluate $\partial\varrho/\partial p$ for some fixed reference value ϱ_0 so that $1/c^2 =$ $\varrho/\partial p$ can be treated as a constant. The parameter c is the speed of sound in e fluid. The equation of continuity is in such cases often written as

$$p_t + c^2\nabla\cdot\boldsymbol{u} = 0, \tag{9}$$

here we have used the simplification $\nabla(\varrho\boldsymbol{u}) = \varrho_0\nabla\boldsymbol{u}$ for a slightly incompressible uid and divided the original equation by ϱ_0.

.3 Boundary conditions

he incompressible Navier-Stokes equations need three scalar conditions on the elocity components or the stress vector at each point on the boundary. The oundary conditions can be classified as follows.

- Dirichlet conditions: components of $\boldsymbol{u}$ are known.

- Neumann conditions:
  - Stress condition: components of $\boldsymbol{\sigma}\cdot\boldsymbol{n}$ are prescribed.
  - Outflow or symmetry condition: $\partial\boldsymbol{u}/\partial n = 0$ (or components vector are zero).

We have here introduce the notion of Dirichlet and Neumann conditio similarities with Laplace and Poisson problems (i.e., whether the c regards the unknown itself or its derivative).

A combination of velocity and stress boundary conditions at a possible. For example, at a symmetry boundary we set the normal vel be zero.

2 The classical splitting method

The earliest and still the most widely applied numerical method for th pressible Navier-Stokes equations is based on splitting the PDE syst simpler components for which we can apply standard discretization n Such a strategy is known as *operator splitting*.

2.1 A simple, naive approach

The equation (1) looks similar to a convection-diffusion equation. The possible numerical method for such equations applies an explicit Forwa scheme in time. It is therefore tempting to advance (1) in time using a s Forward Euler discretization:

$$\frac{\boldsymbol{u}^{n+1} - \boldsymbol{u}^n}{\Delta t} + (\boldsymbol{u}^n\cdot\nabla)\boldsymbol{u}^n = -\frac{1}{\varrho}\nabla p^n + \nu\nabla^2\boldsymbol{u}^n + \boldsymbol{f}^n,$$

which yields an explicit formula for $\boldsymbol{u}^{n+1}$:

$$\boldsymbol{u}^{n+1} = \boldsymbol{u}^n - \Delta t(\boldsymbol{u}^n\cdot\nabla)\boldsymbol{u}^n - \frac{\Delta t}{\varrho}\nabla p^n + \Delta t\,\nu\nabla^2\boldsymbol{u}^n + \Delta t\boldsymbol{f}^n\,.$$

There are two fundamental problems with this method:

- the new $\boldsymbol{u}^{n+1}$ will in general not satisfy (2), i.e., $\nabla\cdot\boldsymbol{u}^{n+1} \neq 0$,
- there is no strategy for computing p^{n+1}.

We may say that the incompressible Navier-Stokes equations are diff solve numerically because of the incompressibility constraint $\nabla\cdot\boldsymbol{u} = 0$ pressure term ∇p.

.2 A working scheme

ıtuitively speaking, the fulfillment $\nabla\cdot\boldsymbol{u}^{n+1}$ requires us to have "more unknowns ɔ play with" when advancing (1). The idea is to basically use the Forward Euler :heme, but evaluate the pressure term at the new time level $n+1$:

$$\boldsymbol{u}^{n+1}=\boldsymbol{u}^{n}-\Delta t(\boldsymbol{u}^{n}\cdot\nabla)\boldsymbol{u}^{n}-\frac{\Delta t}{\varrho}\nabla p^{n+1}+\Delta t\,\nu\nabla^{2}\boldsymbol{u}^{n}+\Delta t\boldsymbol{f}^{n}\,. \tag{12}$$

/e must also require

$$\nabla\cdot\boldsymbol{u}^{n+1}=0\,. \tag{13}$$

he equations (12)-(13) constitute 3+1 coupled PDEs for the 3+1 unknowns $^{n+1}$ and p^{n+1}.

The method for solving (12)-(13) is based on a splitting idea where we rst propagate the velocity from old values to some intermediate velocity $\boldsymbol{u}^{*}$, sing (12). Then we enforce the incompressibility constraint (13) to compute a ɔrrection to $\boldsymbol{u}^{*}$ and also the new pressure p^{n+1}.

A plain Forward Euler discretization of (1), but with a weight β on the ∇p^{n} ɜrm, reads

$$\boldsymbol{u}^{*}=\boldsymbol{u}^{n}-\Delta t(\boldsymbol{u}^{n}\cdot\nabla)\boldsymbol{u}^{n}-\beta\frac{\Delta t}{\varrho}\nabla p^{n}+\Delta t\,\nu\nabla^{2}\boldsymbol{u}^{n}+\Delta t\boldsymbol{f}^{n} \tag{14}$$

he intermediate velocity $\boldsymbol{u}^{*}$ does not fulfill the incompressibility constraint .3), but we seek a correction $\delta\boldsymbol{u}$,

$$\boldsymbol{u}^{n+1}=\boldsymbol{u}^{*}+\delta\boldsymbol{u}, \tag{15}$$

ıch that $\nabla\cdot\boldsymbol{u}^{n+1}=0$. Since $\delta\boldsymbol{u}=\boldsymbol{u}^{n+1}-\boldsymbol{u}^{*}$, we can subtract (14) from (12) ɔ find $\delta\boldsymbol{u}$.

$$\delta\boldsymbol{u}=\boldsymbol{u}^{n+1}-\boldsymbol{u}^{*}=-\frac{\Delta t}{\varrho}\nabla\Phi\,. \tag{16}$$

he quantity Φ is introduced as a kind of pressure change:

$$\Phi=p^{n+1}-\beta p^{n}\,. \tag{17}$$

Inserting $\delta\boldsymbol{u}$ in the incompressibility constraint,

$$\nabla\cdot(\boldsymbol{u}^{*}+\delta\boldsymbol{u})=0,$$

r equivalently,

$$\nabla\cdot\delta\boldsymbol{u}=-\nabla\cdot\boldsymbol{u}^{*},$$

ɜsults in

$$\nabla^{2}\Phi=\frac{\varrho}{\Delta t}\nabla\cdot\boldsymbol{u}^{*}, \tag{18}$$

nce $\nabla\cdot\nabla\Phi=\nabla^{2}\Phi$.

As soon as Φ is computed from the Poisson equation (18), we can c

$$\boldsymbol{u}^{n+1}=\boldsymbol{u}^{*}-\frac{\Delta t}{\varrho}\nabla\Phi,$$

and

$$p^{n+1}=\Phi+\beta p^{n}\,.$$

The solution algorithm at a time level then consists of the following

1. Compute the intermediate velocity $\boldsymbol{u}^{*}$ from (14).
2. Solve the Poisson equation (18) for Φ.
3. Update the velocity from (19).
4. Update the pressure from (20).

Remarks. The literature is full of papers and books with methods eq or almost equivalent to the scheme above. Many schemes apply β = replace Φ by p^{n+1}.

2.3 Boundary conditions

What boundary conditions should we assign to $\boldsymbol{u}^{*}$ when solving (14)? A s choice is to apply the same boundary conditions as those specified fc follows that $\delta\boldsymbol{u}=0$ on the boundary where $\boldsymbol{u}$ is subject to Dirichlet coı We let $\partial\Omega_{D,u}$ denote the part of the boundary $\partial\Omega$ with Dirichlet coı while $\partial\Omega_{N,u}$ denotes the boundary where Neumann conditions involving apply.

The boundary condition on the pressure in the original incompressible Stokes equations is simply to prescribe p at a single point, potentia function of time. However, when solving the Poisson equation (18) v Dirichlet or Neumann boundary conditions for Φ (the pressure change) whole boundary. Sometimes the pressure is prescribed at an inlet o boundary, which then yields a Dirichlet condition for $\Phi=p^{n+1}-\beta p^{n}$. boundaries where $\boldsymbol{u}$ is subject to Dirichlet conditions, $\boldsymbol{u}^{*}$ has the same coı and $\delta\boldsymbol{u}=0$, which implies $\nabla\Phi=0$. In particular, $\partial\Phi/\partial n=0$, and homo Neumann conditions are therefore used on such boundaries when solv Poisson equation for Φ. Also, at symmetry boundaries, $\partial\Phi/\partial n=0$. At boundary, a pressure gradient in the flow direction is often known, say implying that we can compute $\partial\Phi/\partial n=-(f(t^{n+1})-\beta f(t^{n}))$.

We let $\partial\Omega_{D,\Phi}$ be the part of the boundary where Φ is subject to let conditions, while $\partial\Omega_{N,p}$ is the remaining part where Neumann coı involving $\partial\Phi/\partial n$ are assigned.

.4 Spatial discretization by the finite element method

he equations to be solved, (14), (18), (19), and (20), are of two types: explicit pdates (approximations a la $u = f$) and the Poisson equation. We introduce a ector test function $\boldsymbol{v}^{(u)} \in V^{(u)}$ for the vector equations (14) and (19), and a alar test function $v^{(\Phi)} \in V^{(\Phi)}$ for the Poisson equation and the update (20). Iodulo nonzero Dirichlet conditions, we seek $\boldsymbol{u}^*, \boldsymbol{u}^{n+1} \in V^{(u)}$ and $p^{n+1} \in V^{(\Phi)}$.

The variational form of a vector equation like (14) is derived by taking the ner product of the equation and $\boldsymbol{v}^{(u)}$. The Laplace term is integrated by parts, s usual, but this time vectors are involved. The relevant rule takes the form

$$\int_\Omega (\nabla^2 \boldsymbol{u}) \cdot \boldsymbol{v} \, \mathrm{d}x = -\int_\Omega \nabla \boldsymbol{u} : \nabla v \, \mathrm{d}x + \int_{\partial\Omega} \frac{\partial \boldsymbol{u}}{\partial n} \cdot \boldsymbol{v} \, \mathrm{d}s,$$

here $\nabla \boldsymbol{u} : \nabla \boldsymbol{v}$ means the *inner tensor product*: $\boldsymbol{A} : \boldsymbol{B} = \sum_j \sum_j A_{ij} B_{ij}$ (when . has elements $A_{i,j}$ and $\boldsymbol{B}$ has elements $B_{i,j}$. Alternatively, we may say that . : $\boldsymbol{B}$ is simply the scalar product of the tensors $\boldsymbol{A}$ and $\boldsymbol{B}$ when these are viewed s vectors (of length 9 instead of tensors of dimension 3×3 in 3D problems). he normal derivative has the usual definition: $\partial \boldsymbol{u}/\partial n = \boldsymbol{n} \cdot \nabla \boldsymbol{u}$.

The $\int_\Omega \nabla p^n \cdot \boldsymbol{v}^{(u)} \, \mathrm{d}x$ integral can also be a candidate for integrated by parts, desired. The relevant rule reads

$$\int_\Omega \nabla p \cdot \boldsymbol{v} \, \mathrm{d}x = -\int_\Omega p \nabla \cdot \boldsymbol{v} \, \mathrm{d}x + \int_{\partial\Omega} p \boldsymbol{n} \cdot \boldsymbol{v} \, \mathrm{d}s \,.$$

/e use such an integration by parts below. The advantage is that we get boundary integral involving $p\boldsymbol{n}$, which is advantageous if we want to set a ondition on p, especially at an outflow boundary, but also on an inflow boundary.

For notational simplicity and close correspondence to computer code, we troduce the subscript 1 on quantities from the previous time level n and drop ny superscript $n+1$ for quantities to be computed at the new time level. The esulting variational form can be written as

$$\int_\Omega \Big(\boldsymbol{u}^* \cdot \boldsymbol{v}^{(u)} + \Delta t ((\boldsymbol{u}_1 \cdot \nabla) \nabla \boldsymbol{u}_1) \cdot \boldsymbol{v}^{(u)} - \frac{\Delta t}{\varrho} p \nabla \cdot \boldsymbol{v}^{(u)} + \\ \Delta t \, \nu \nabla \boldsymbol{u}_1 : \nabla \boldsymbol{v}^{(u)} - \Delta t f_1 \Big) \, \mathrm{d}x = \int_{\partial\Omega_{N,u}} \left(\nu \frac{\partial \boldsymbol{u}}{\partial n} - p\boldsymbol{n} \right) \cdot \boldsymbol{v}^{(u)} \, \mathrm{d}s, \qquad (21)$$

$\boldsymbol{v}^{(u)} \in V^{(u)}$. The variational form corresponding to the Poisson equation ecomes

$$\int_\Omega \nabla \Phi \cdot \nabla v^{(\Phi)} \, \mathrm{d}x = -\frac{\varrho}{\Delta t} \int_\Omega \nabla \cdot \boldsymbol{u}^* \, v^{(\Phi)} \, \mathrm{d}x + \int_{\partial\Omega_{N,p}} \frac{\partial \Phi}{\partial n} v^{(\Phi)} \, \mathrm{d}s, \quad \forall v^{(\Phi)} \in V^{(\Phi)} \,. \qquad (22)$$

The variational form for the velocity update (19) is based on taking the inner roduct of $\boldsymbol{v}^{(u)}$ and (19):

$$\int_\Omega \boldsymbol{u} \cdot \boldsymbol{v}^{(u)} \, \mathrm{d}x = \int_\Omega (\boldsymbol{u}^* - \frac{\Delta t}{\varrho} \nabla \Phi) \cdot \boldsymbol{v}^{(u)} \, \mathrm{d}x, \quad \forall \boldsymbol{v}^{(u)} \in V^{(u)} \,.$$

Note that this is the same form as in a vector approximation probl proximate a given vector field $\boldsymbol{f}$ by a $\boldsymbol{u}$, where the components of $\boldsymbol{u}$ ar finite element functions. Also note that $\boldsymbol{u}$ in (23) is actually $\boldsymbol{u}^{n+1}$, but superscript is dropped since we do not use that in an implementation.

The pressure update has the variational form

$$\int_\Omega p v^{(\Phi)} \, \mathrm{d}x = \int_\Omega (\Phi + \beta p_1) v^{(\Phi)} \, \mathrm{d}x, \quad \forall v^{(\Phi)} \in V^{(\Phi)} \,.$$

(Also here, p denotes p^{n+1} and p_1 is p^n.)

The splitting method presented above allows flexible choices of elem $\boldsymbol{u}$ and p. In the early days of the finite element method for incompressi fully implicit formulations were used and these require the $\boldsymbol{u}$ element one polynomial degree higher than the p element. This restriction does n to the splitting scheme, so one may, e.g., choose P1 elements for the components and the pressure.

Remark on boundary integrals. The boundary integral in (21) co play at element faces on the boundary if the nodes on a face are not su Dirichlet conditions. As for scalar PDEs, Dirichlet conditions either me $\boldsymbol{v} = 0$ on that part of the boundary, or the element matrix and vector global coefficient matrix and right-hand side) are manipulated to enforc values of the unknown such that any boundary integral is erased and r by the boundary value.

The boundary integral most often applies to inflow and outflow bou $x = \text{const}$ where we assume unidirectional flow, $\boldsymbol{u} = u\boldsymbol{i}$. Because of ∇ we have $\partial \boldsymbol{u}/\partial x = \partial \boldsymbol{u}/\partial n = 0$ and $p = \text{const}$. Very often, the boundary in (21) is zero, because we apply it to an outflow boundary where $\partial \boldsymbol{u}/\partial n = 0$ and then we fix the pressure at $p = 0$. Note that in the Navier-Stokes equations, p enters just through ∇p so a boundary cond one point is necessary to uniquely determine p (otherwise p is known free additive constant). At inflow boundaries, $\boldsymbol{u}$ is either known, which that the boundary integral does not apply, or we have $\partial \boldsymbol{u}/\partial n = 0$ and In this latter case, the boundary integral involves an integration of $p\boldsymbol{n}$

The boundary integral involving $\partial \Phi/\partial n$ is usually omitted since w the condition $\partial \Phi/\partial n = 0$, see Section 2.3.

2.5 Stress formulations

As mentioned in Section 1.2, we may exchange the $\nu \nabla^2 \boldsymbol{u}$ term in (1) stress term $\varrho^{-1} \nabla \cdot \boldsymbol{\sigma}$, where $\boldsymbol{\sigma}$ is given by (6). Occasionally, the $\nabla \cdot \boldsymbol{\sigma}$ advantageous, because integration by parts of $\int_\Omega \nabla \cdot \boldsymbol{\sigma} \cdot \boldsymbol{v}^{(u)} \, \mathrm{d}x$ gives a b integral with the stress vector $\boldsymbol{\sigma} \cdot \boldsymbol{n}$. This is convenient when boundary co are formulated in terms of stresses.

.6 Increasing the implicitness

he explicit scheme (14) resembles the same stability problems as when a orward Euler scheme is applied to the diffusion equation. However, there is also convection term $\boldsymbol{u}\cdot\nabla\boldsymbol{u}$ that reduces the time step restrictions. The stability iterion reads

$$\Delta t \leq \frac{h^2}{2\nu + Uh}, \tag{25}$$

here h is the minimum element size and U is a characteristic size of the elocity. The term 2ν stems from the viscous (Laplace) term while Uh arises om the convection term in the Navier-Stokes equations. Which of the term at dominates in the denominator therefore depends on whether viscous forces convection is important in the equation.

Treating the viscosity term $\nu\nabla^2\boldsymbol{u}$ implicitly helps greatly on the stability roperties of the scheme for $\boldsymbol{u}^*$. We may, for example, apply a Backward Euler heme. Instead of (12) we then have

$$\boldsymbol{u}^{n+1} = \boldsymbol{u}^n - \Delta t(\boldsymbol{u}^{n+1}\cdot\nabla)\boldsymbol{u}^{n+1} - \frac{\Delta t}{\varrho}\nabla p^{n+1} + \Delta t\,\nu\nabla^2\boldsymbol{u}^{n+1} + \Delta t\boldsymbol{f}^{n+1}, \tag{26}$$

$$\nabla\cdot\boldsymbol{u}^{n+1} = 0\,. \tag{27}$$

n intermediate velocity can be computed from the first equation if we replace $^{n+1}$ by βp^n as done earlier:

$$\boldsymbol{u}^* = \boldsymbol{u}^n - \Delta t(\boldsymbol{u}^*\cdot\nabla)\boldsymbol{u}^* - \beta\frac{\Delta t}{\varrho}p^{n+1} + \Delta t\,\nu\nabla^2\boldsymbol{u}^* + \Delta t\boldsymbol{f}^{n+1}$$

o simplify the nonlinearity in $(\boldsymbol{u}^*\cdot\nabla)\boldsymbol{u}^*$ we may use an old value in the onvective velocity:

$$(\boldsymbol{u}^*\cdot\nabla)\boldsymbol{u}^* \approx (\boldsymbol{u}^n\cdot\nabla)\boldsymbol{u}^*\,. \tag{28}$$

his approximation is essentially one Picard iteration using u^n as initial guess.

The intermediate velocity $\boldsymbol{u}^*$ is now governed by a linear problem

$$\boldsymbol{u}^* = \boldsymbol{u}^n - \Delta t(\boldsymbol{u}^n\cdot\nabla)\boldsymbol{u}^* - \beta\frac{\Delta t}{\varrho}\nabla p^n + \Delta t\,\nu\nabla^2\boldsymbol{u}^* + \Delta t\boldsymbol{f}^{n+1}$$

he correction $\delta\boldsymbol{u} = \boldsymbol{u}^{n+1} - \boldsymbol{u}^*$ becomes

$$\delta\boldsymbol{u} = \Delta t((\boldsymbol{u}^{n+1}\cdot\nabla)\boldsymbol{u}^{n+1} - (\boldsymbol{u}^n\cdot\nabla)\boldsymbol{u}^*) - \frac{\Delta t}{\varrho}\nabla\Phi + \Delta t\,\nu(\nabla^2(\boldsymbol{u}^{n+1} - \boldsymbol{u}^*)\,.$$

nder the assumption that $\boldsymbol{u}^*$ is close to $\boldsymbol{u}^{n+1}$, we may drop the terms involving $^{n+1} - \boldsymbol{u}^*$ and just keep the $\nabla\Phi$ term. Then

$$\delta\boldsymbol{u} = \boldsymbol{u}^{n+1} - \boldsymbol{u}^* = -\frac{\Delta t}{\varrho}\nabla\Phi,$$

as before, and the incompressibility constraint $\nabla\cdot\delta\boldsymbol{u} = -\nabla\cdot\boldsymbol{u}^*$ gives $\frac{\varrho}{\Delta t}\nabla\cdot\boldsymbol{u}^*$.

The algorithm becomes the same as for a Forward Euler discretization that (14) is replaced by (2.6).

3 Methods based on slight compressibility

By allowing a slight compressibility we can replace the problematic co $\nabla\cdot\boldsymbol{u}$ by an evolution equation (9) for p. Essentially, we then have two e equations for $\boldsymbol{u}$ and p:

$$\boldsymbol{u}_t = -(\boldsymbol{u}\cdot\nabla)\boldsymbol{u} - \frac{1}{\varrho}\nabla p + \nu\nabla^2\boldsymbol{u} + \boldsymbol{f},$$
$$p_t = -c^2\nabla\cdot\boldsymbol{u}\,.$$

The simplest method is a Forward Euler scheme:

$$\boldsymbol{u}^{n+1} = \boldsymbol{u}^n - \Delta t(\boldsymbol{u}^n\cdot\nabla)\boldsymbol{u}^n - \frac{\Delta t}{\varrho}\nabla p^n + \Delta t\,\nu\nabla^2\boldsymbol{u}^n + \Delta t\boldsymbol{f}^n,$$
$$p^{n+1} = p^n - \Delta t c^2\nabla\cdot\boldsymbol{u}^n\,.$$

The major problem with this scheme is the stability constraint, which is by the c parameter (velocity of sound): $\Delta t \sim 1/c$. Usually, c is taken as parameter and values much less than the speed of sound may give solutic acceptable compressibility.

Any other explicit scheme, say a 2nd- or 4th-order Runge-Kutta me easily applied. Implicit schemes are of course also possible, but then on solve linear systems, and the original formulation with a true incompre constraint $\nabla\cdot\boldsymbol{u} = 0$ is not more complicated and usually preferred. In the method based on slight compressibility and explicit time integration computationally very heavy and is not competitive unless one can use a much lower than the speed of sound.

4 Fully implicit formulation

Early attempts to use the finite element method to solve the Navier equations were based on fully implicit formulations. This is easily der applying a Backward Euler scheme to the system (1)-(2):

$$\frac{\boldsymbol{u}^n - \boldsymbol{u}^{n-1}}{\Delta t} + (\boldsymbol{u}^n\cdot\nabla)\boldsymbol{u}^n = -\frac{1}{\varrho}\nabla p^n + \nu\nabla^2\boldsymbol{u}^n + \boldsymbol{f}^n,$$
$$\nabla\cdot\boldsymbol{u}^n = 0$$

e introduce the test functions $\boldsymbol{v}^{(u)} \in V^{(u)}$ for the momentum balance equation nd seek $\boldsymbol{u}^{n+1} \in V^{(u)}$ (or more precisely, the part of $\boldsymbol{u}^{n+1}$ without nonzero irichlet conditions). We seek $p \in V^{(p)}$ and use $v^{(p} \in V^{(p}$ as test function for e continuity equation. We may write the system of PDEs as

$$\mathcal{L}_u(\boldsymbol{u}^n, p^n, \boldsymbol{u}^{n-1}) = 0,$$
$$\nabla\cdot\boldsymbol{u}^n = 0\,.$$

variational formulation can be based on treating the two equations separately,

$$\int_\Omega \mathcal{L}_u(\boldsymbol{u}^n, p^n, \boldsymbol{u}^{n-1})\cdot\boldsymbol{v}^{(u)}\,\mathrm{d}x = 0,$$
$$\int_\Omega \nabla\cdot\boldsymbol{u}^n v^{(p)}\,\mathrm{d}x = 0,$$

r we may use an inner product of the two equations $(\mathcal{L}_u, \nabla\cdot\boldsymbol{u})$ and the test ector $(\boldsymbol{v}^{(u)}, v^{(p)})$:

$$\int_\Omega \left(\mathcal{L}_u(\boldsymbol{u}^n, p^n, \boldsymbol{u}^{n-1})\cdot\boldsymbol{v}^{(u)} + \nabla\cdot\boldsymbol{u}^n v^{(p)}\right)\mathrm{d}x = 0\,.$$

o minimize the distance between code and mathematics, we introduce new ymbols: $\boldsymbol{u}$ for $\boldsymbol{u}^n$, $\boldsymbol{u}_1$ for $\boldsymbol{u}^{n-1}$, and p for p^n. Integrating the pressure and iscous terms by parts yields

$$\int_\Omega \Big(\boldsymbol{u}\cdot\boldsymbol{v}^{(u)} + \Delta t((\boldsymbol{u}\cdot\nabla)\nabla\boldsymbol{u})\cdot\boldsymbol{v}^{(u)} - \frac{\Delta t}{\varrho}p\nabla\cdot\boldsymbol{v}^{(u)} +$$
$$\Delta t\,\nu\nabla\boldsymbol{u}\cdot\nabla\boldsymbol{v}^{(u)} - \Delta t f\Big)\,\mathrm{d}x + \int_{\partial\Omega_{N,u}}\left(\nu\frac{\partial\boldsymbol{u}}{\partial n} - p\boldsymbol{n}\right)\cdot\boldsymbol{v}^{(u)}\,\mathrm{d}s +$$
$$\int_\Omega \nabla\cdot\boldsymbol{u}\,v^{(p)}\,\mathrm{d}x = 0\,. \quad (33)$$

his is nothing but a coupled, nonlinear equation system for $\boldsymbol{u}$ and p. Inserting nite element expansions for $\boldsymbol{u}$ and p yields discrete equations that can be ritten in matrix form as

$$Mu + \Delta t C(u)u = -\frac{\Delta t}{\varrho}Lp + \nu Kp + f, \quad (34)$$
$$L^T u = 0, \quad (35)$$

where M is the usual mass matrix, but here for a vector function, u all coefficients for the $\boldsymbol{u}$ field, $C(u)$ is a matrix arising from the cor term $(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}$, L is a matrix arising from the $p\nabla\cdot\boldsymbol{v}^{(u)}$ term, K is the corresponding to the Laplace operator (acting on a vector), and f is a of the source terms arising from $\boldsymbol{f}$. The nonlinearity is typically handl Newton method.

The simplified system arising from dropping the time derivative a convection term $(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}$ can be analyzed. It turns out that only combinations of $V^{(u)}$ and $V^{(p)}$ can guarantee a stable solution. The poly in $V^{(u)}$ must be (at least) one degree higher than those in $V^{(p)}$. For exam may use P2 elements for $\boldsymbol{u}$ and P1 elements for p. This combination is k the famous Taylor-Hood element. Numerical experimentation indicates t same stability restriction on the combination of spaces is also important fully nonlinear Navier-Stokes equations when solved by a fully implicit The splitting into simpler systems, as shown in Section 2.2, introduces approximations that stabilize the problem such that the same type of can be used for velocity components and pressure.

The splitting method is much more widely used than the fully formulation. Although the latter is more robust and much better su stationary flow, it is also involves much heavier computations. In each iteration, a linear system involving all the coefficients in $\boldsymbol{u}$ and p must b and it is non-trivial to construct efficient iterative solution methods (es preconditioners).

5 Applications

Figure 1 exemplifies the boundary conditions for flow in a channel betw infinite plates. This flow configuration is assumed to be stationary, $\boldsymbol{u}_t$ = a simple analytical solution can be found in this particular case.

Note that the numerical solution method described above requires dependent problem. Stationary problems must be simulated by starti some initial condition and letting the flow develop toward the stationary as $t\to\infty$.

The velocity field in channel flow is symmetric with respect to the line. It is therefore sufficient to calculate the flow in half the channel. F displays the computational domain and the relevant boundary conditio

Figure 3 depicts a more complex flow geometry, leading to a more c velocity field. The boundary conditions are, however, similar to those for flow.

The boundary conditions in Figure 4 are not listed in the figure there are multiple options. The inflow boundary must have a prescribed $\boldsymbol{u}$, and on the cylinder we must have $\boldsymbol{u} = 0$. On the remaining three bou we have some freedom in what to assign. At the outflow one typica $\partial\boldsymbol{u}/\partial n = 0$ and fix the pressure at one value. Alternatively, one ma a stress-free condition $\boldsymbol{\sigma}\cdot\boldsymbol{n} = 0$, which implicitly also sets the pressu

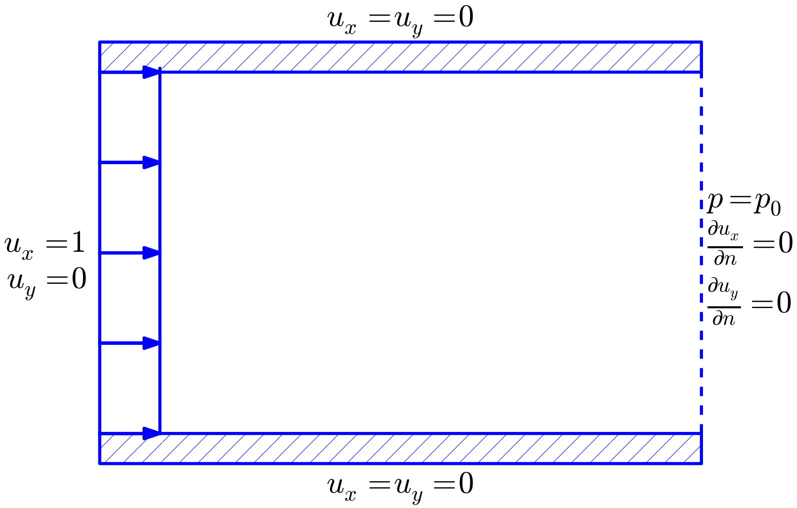

Figure 1: Flow in a channel.

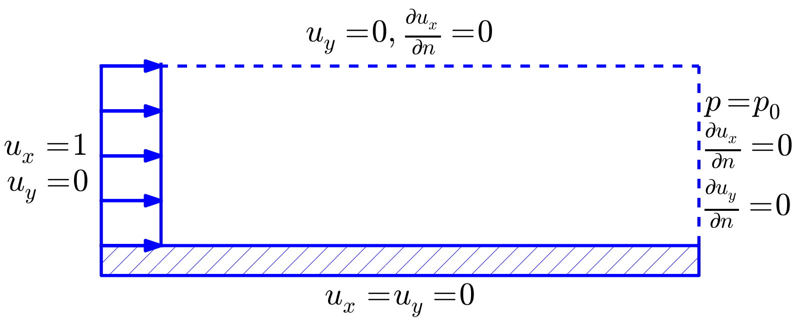

Figure 2: Flow in a half a channel with a symmetry line.

ıe boundaries AB and DC there is more freedom. The weakest condition is $\boldsymbol{u}/\partial n = 0$, assuming that the boundary is far enough away from the cylinder ıch that the flow field changes very little. Some prefer to set $\boldsymbol{\sigma} \cdot \boldsymbol{n} = 0$ here ıstead. A stronger condition is to require $u_y = 0$ and $\partial u_x/\partial n = 0$. However, $_y = 0$ requires the boundary to be far away from the cylinder.

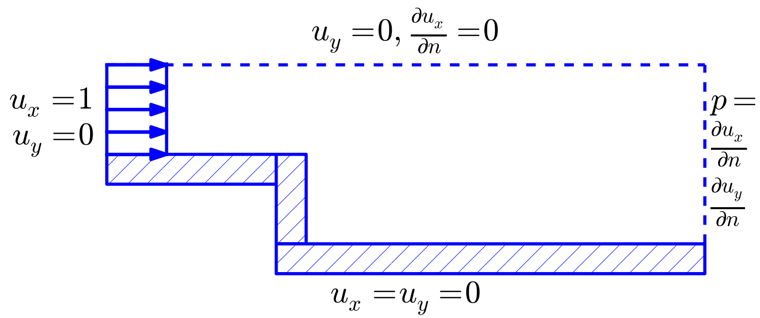

Figure 3: Flow over a backward facing step.

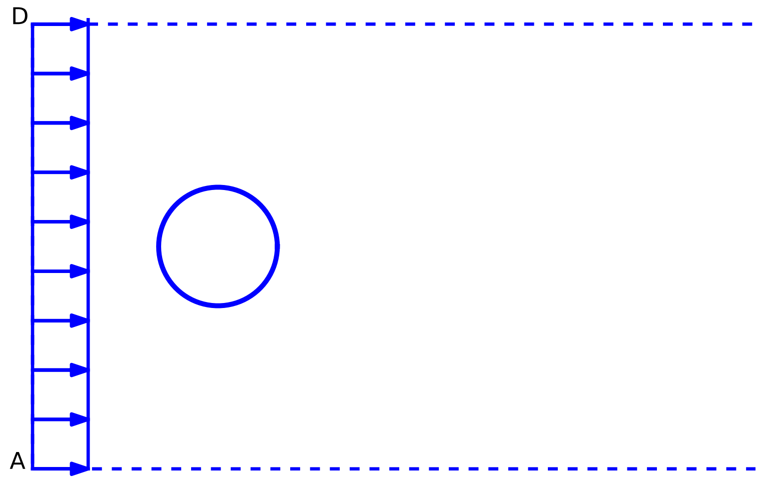

Figure 4: Flow around a cylinder.

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