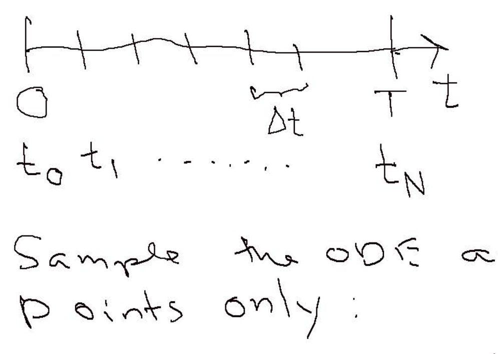
Decay: $u'(t) = -au$, $u(0) = I$

$t \in (0, T)$



mesh (grid)

Sample the ODE at the mesh points only:

$$u'(t_n) = -a u(t_n)$$

Replace u' by finite difference (FD)

$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$$u'(t_n) \approx \frac{u(t_{n+1}) - u(t_n)}{\Delta t}$$

FD → ODE:

$$\frac{u(t_{n+1}) - u(t_n)}{\Delta t} = -a u(t_n)$$

Notation: $u(t_n) = u^n$

Assumption/idea: u^n is known
u^{n+1} is unknown

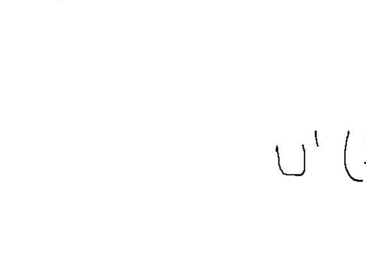
Solve wrt u^{n+1}:

$$\boxed{u^{n+1} = u^n - \Delta t\, a u^n}$$

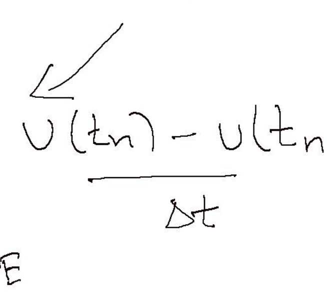
Start $u^0 = I$, $u^1 = u^0 - \Delta t\, a u^0$
$u^2 = u^1 - \Delta t\, a u^1$

Forward Euler scheme

Alternative FD approx:



Forward



Backward

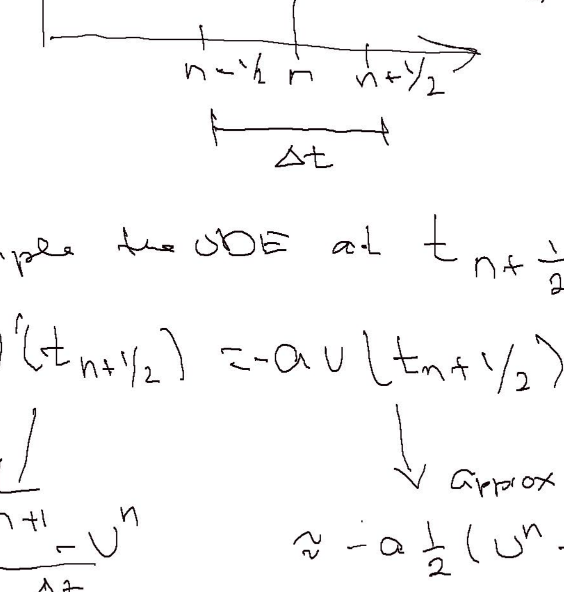
$$u'(t_n) = \frac{u(t_n) - u(t_{n-1})}{\Delta t} = \frac{u^n - u^{n-1}}{\Delta t}$$

↓ ODE

$$\frac{u^n - u^{n-1}}{\Delta t} = -a u^n,$$ u^{n-1} is known
u^n is unknown

$$\boxed{u^n = u^{n-1} \cdot \left(\frac{1}{1 + a\Delta t}\right)}$$

$u^0 = I$, $u^1 = \ldots$, $u^2 = \ldots$, ...

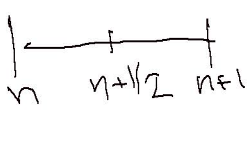


Sample the ODE at $t_{n+\frac{1}{2}}$

$$u'(t_{n+1/2}) = -a u(t_{n+1/2})$$

approx ↙ ↘ approx

$$\approx \frac{u^{n+1} - u^n}{\Delta t} \qquad \approx -a \frac{1}{2}(u^n + u^{n+1})$$



Assume: u^n is known
u^{n+1} is unknown

$$\boxed{u^{n+1} = \frac{1 - \frac{1}{2} a \Delta t}{1 + \frac{1}{2} a \Delta t} u^n}$$

Crank-Nicolson scheme