Nonlinear problems

Model problem: $u' = u(1-u)$, $u(0)=1$

$= u - u^2$ nonlinear

Simplest approach: explicit scheme

Forward Euler: $\frac{u^{n+1}-u^n}{\Delta t} = u^n(1-u^n) = u^n - (u^n)^2$ known u^n $\Rightarrow$ no problem

LeapFrog $\frac{u^{n+1}-u^{n-1}}{2\Delta t} = u^n(1-u^n)$ nonlinearity involves known quantities

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Implicit methods: nonlinear algebraic equations

Backward Euler: $\frac{u^n - u^{n-1}}{\Delta t} = u^n(1-u^n)$ u^{n-1}: known, u^n: unknown

$$\Downarrow$$

$$\underbrace{u^n - \Delta t u^n + \Delta t (u^n)^2}_{\text{nonlinear}} = u^{n-1}$$

General case: $u' = f(u,t)$

Backw. Euler: $\underbrace{u^n - \Delta t f(u^n, t_n) - u^{n-1}}_{} = 0$

$$g(x) = x - \Delta t f(x,t_n) - u^{n-1}$$

$g(x)=0$ can be solved by Newton's method, bisection, the secant method

Newton's method:

$$g'(x) = 1 - \Delta t \frac{\partial f}{\partial u}$$ need to compute this manually and feed to software

$f(u,t)$ $\qquad \frac{\partial f}{\partial x}$

$f(u,t) = u(1-u)$

$\frac{\partial f}{\partial u} = 1-2u$

Alternatives:

$$u^n - \Delta t u^n + \Delta t (u^n)^2 = u^{n-1}$$

$$\downarrow$$

$\Delta t u^{n-1} \cdot u^n$ (rough approx., ok for small Δt ??)

Improvement: iterate!

$u^{n,0} = u^{n-1}$, $\underbrace{u^{n,i} - \Delta t u^{n,i} + \Delta t u^{n,i-1} u^{n,i}}_{\text{linear in } u^{n,i}} = u^{n-1}$, $i=1,2,\dots$ until convergence

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See slides for more info on solving PDEs.