Linear elasticity - Finte elements

INF5620 - Numerical methods for partial differential equations

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1 Mathematical problem

We consider the general time-dependent equations for linear elasticity,

$$
\begin{aligned}
\rho\mathbf{u}_{tt} &= \nabla\cdot\sigma + \rho\mathbf{b}, && (1)\\
\sigma &= \sigma(\mathbf{u}) = 2\mu\epsilon(\mathbf{u}) + \lambda tr(\epsilon(\mathbf{u}))I && (2)\\
\mathbf{u}(\mathbf{x},0) &= \mathbf{u_0} && (3)\\
\mathbf{u}_t(\mathbf{x},0) &= \mathbf{v_0} && (4)\\
\mathbf{u}(\mathbf{0},t) &= \mathbf{g} && (5)
\end{aligned}
$$

where $\epsilon(\mathbf{u}) = \frac{1}{2}\left(\nabla\mathbf{u} + \nabla\mathbf{u}^T\right)$ is the strain tensor, μ and λ are the Lamé parameters, and I is the identity tensor. We will consider Dirichlet boundary conditions, as shown in (5). The initial conditions are given by (3) and (4).

2 Discretization

2.1 Finite Difference in time

We begin by discretizing the time derivative, using a centered difference approximation:

$$
\frac{\partial^2\mathbf{u}}{\partial t^2} \approx \frac{\mathbf{u^{n+1}} - 2\mathbf{u^n} + \mathbf{u^{n-1}}}{\Delta t^2} \tag{6}
$$

We insert this approximation into (1) and evaluate at time $t = t_n$:

$$
\rho\frac{\mathbf{u^{n+1}} - 2\mathbf{u^n} + \mathbf{u^{n-1}}}{\Delta t^2} = \nabla\cdot\sigma(\mathbf{u}^n) + \rho\mathbf{b}^n \tag{7}
$$

For simplicity, we let $\sigma^n = \sigma(u^n)$. We solve for $\mathbf{u^{n+1}}$:

$$
\begin{aligned}
\rho(\mathbf{u^{n+1}} - 2\mathbf{u^n} + \mathbf{u^{n-1}}) &= \Delta t^2\nabla\cdot\sigma^n + \rho\mathbf{b}^n\\
\mathbf{u^{n+1}} &= 2\mathbf{u^n} - \mathbf{u^{n-1}} + \frac{\Delta t^2}{\rho}\nabla\cdot\sigma^n + \Delta t^2\mathbf{b}^n && (8)
\end{aligned}
$$

We now have a discretization in time.

2.2 Finite Elements in space

We use a Galerkin method to find the variational form of (1), by introducing a test function $\mathbf{v} \in V$:

$$\int\limits_{\Omega} \mathbf{u^{n+1}} \cdot \mathbf{v} dx = 2 \int\limits_{\Omega} \mathbf{u^{n}} \cdot \mathbf{v} dx - \int\limits_{\Omega} \mathbf{u^{n-1}} \cdot \mathbf{v} dx + \frac{\Delta t^2}{\rho} \int\limits_{\Omega} (\nabla \cdot \sigma^n) \cdot \mathbf{v} dx + \Delta t^2 \int\limits_{\Omega} \mathbf{b}^n \cdot \mathbf{v} dx \tag{9}$$

We need to integrate the $\int\limits_{\Omega} (\nabla \cdot \sigma^n) \cdot v dx$ -term by parts, as it contains second derivatives. We get

$$\int\limits_{\Omega} (\nabla \cdot \sigma^n) \cdot \mathbf{v} dx = - \int\limits_{\Omega} \sigma^n : \nabla \mathbf{v} dx + \int\limits_{\partial\Omega} (\sigma^n \cdot \mathbf{n}) \cdot \mathbf{v} ds \tag{10}$$

where $\sigma^n : \nabla \mathbf{v}$ is a tensor inner product. The term $\int\limits_{\partial\Omega} (\sigma^n \cdot n) \cdot \mathbf{v} ds$ vanishes because of the Dirichlet boundary condition. We are now left with

$$\int\limits_{\Omega} \mathbf{u^{n+1}} \cdot \mathbf{v} dx = \int\limits_{\Omega} (2\mathbf{u^{n}} - \mathbf{u^{n-1}} + \Delta t^2 \mathbf{b}^n) \cdot \mathbf{v} dx - \frac{\Delta t^2}{\rho} \int\limits_{\Omega} \sigma^n : \nabla \mathbf{v} dx \tag{11}$$

This is the general scheme. We need to implement the initial conditions in order to get a special scheme for the first time step:

$$\begin{aligned}
&\mathbf{u}^0 = \mathbf{u_0} \\
&\mathbf{u}_t(\mathbf{x}, 0) = \mathbf{v_0} \Rightarrow \frac{\mathbf{u^{n+1}} - \mathbf{u^{n-1}}}{2\Delta t} \approx \mathbf{v_0} \\
&\Rightarrow \mathbf{u}^{-1} = \mathbf{u}^1 - 2\Delta t \mathbf{v_0}
\end{aligned}$$

We put this expression for $\mathbf{u}^{-1}$ into (11):

$$\begin{aligned}
2 \int\limits_{\Omega} \mathbf{u}^1 \cdot \mathbf{v} dx &= \int\limits_{\Omega} (2\mathbf{u}^0 - 2\Delta t \mathbf{v_0} + \Delta t^2 \mathbf{b}^0) \cdot \mathbf{v} dx - \frac{\Delta t^2}{\rho} \int\limits_{\Omega} \sigma^0 : \nabla \mathbf{v} dx \\
\int\limits_{\Omega} \mathbf{u}^1 \cdot \mathbf{v} dx &= \int\limits_{\Omega} (\mathbf{u_0} - \Delta t \mathbf{v_0} + \frac{\Delta t^2}{2} \mathbf{b}^0) \cdot \mathbf{v} dx - \frac{\Delta t^2}{2\rho} \int\limits_{\Omega} \sigma(\mathbf{u}_0) : \nabla \mathbf{v} dx
\end{aligned} \tag{12}$$

We are now ready to express (1) as a variational problem. We let $u = \mathbf{u^{n+1}}$ represent the unknown at the next time level:

$$a_0(\mathbf{u}, \mathbf{v}) = \int\limits_{\Omega} \mathbf{u} \cdot \mathbf{v} dx \tag{13}$$

$$L_0(\mathbf{v}) = \int\limits_{\Omega} \mathbf{u_0} \cdot \mathbf{v} dx \tag{14}$$

$$a_1(\mathbf{u}, \mathbf{v}) = \int\limits_{\Omega} \mathbf{u} \cdot \mathbf{v} dx \tag{15}$$

$$L_1(\mathbf{v}) = \int\limits_{\Omega} (\mathbf{u_0} - \Delta t \mathbf{v}_0 + \frac{\Delta t^2}{2} \mathbf{b}^0) \cdot \mathbf{v} dx - \frac{\Delta t^2}{2\rho} \int\limits_{\Omega} \sigma(\mathbf{u_0}) : \nabla \mathbf{v} dx \tag{16}$$

$$a(\mathbf{u}, \mathbf{v}) = \int\limits_{\Omega} \mathbf{u} \cdot \mathbf{v} dx \tag{17}$$

$$L(\mathbf{v}) = \int\limits_{\Omega} (2\mathbf{u^n} - \mathbf{u^{n-1}} + \Delta t^2 \mathbf{b}^n) \cdot \mathbf{v} dx - \frac{\Delta t^2}{\rho} \int\limits_{\Omega} \sigma^n : \nabla \mathbf{v} dx \tag{18}$$

$$L(\mathbf{v}) = \int\limits_{\Omega} (2\mathbf{u^n} - \mathbf{u^{n-1}} + \Delta t^2 \mathbf{b}^n) \cdot \mathbf{v} dx - \frac{\Delta t^2}{\rho} \int\limits_{\Omega} (2\mu\epsilon(\mathbf{u}) + \lambda tr(\epsilon(\mathbf{u}))I) : \nabla \mathbf{v} dx \tag{19}$$

With this formulation, we could implement the problem in FEniCS.

However, it is possible to go further, in order to make the implementation more efficient. We introduce the approximations

$$\mathbf{u^{n+1}} \approx \sum_{j}^{N} c_j^{n+1} \Phi_j \tag{20}$$

$$\mathbf{u^{n}} \approx \sum_{j}^{N} c_j^{n} \Phi_j \tag{21}$$

$$\mathbf{u^{n-1}} \approx \sum_{j}^{N} c_j^{n-1} \Phi_j \tag{22}$$

$$\mathbf{b}^{n} \approx \sum_{j}^{N} b_j^{n} \Phi_j \tag{23}$$

where $\Phi_j = (\phi_1, ..., \phi_d)^T$ are prescribed basis functions, and c_j^n and b_j^n are coefficients to be determined. Let $\mathbf{v} = \Phi_i$.

$$\int\limits_{\Omega} \mathbf{u^{n+1}} \cdot \mathbf{v} dx = \int\limits_{\Omega} \left(\sum_{j}^{N} c_j^{n+1} \Phi_j \right) \cdot \Phi_i dx = \sum_{j}^{N} \left(\int\limits_{\Omega} \Phi_i \cdot \Phi_j dx \right) c_j^{n+1} \tag{24}$$

$$\int_{\Omega}(2\mathbf{u^n} - \mathbf{u^{n-1}} + \Delta t^2 \mathbf{b}^n)\cdot \mathbf{v} dx = \int_{\Omega}\left(2\sum_j^N c_j^n \Phi_j - \sum_j^N c_j^{n-1}\Phi_j + \Delta t^2 \sum_j^N b_j^n \Phi_j\right)\cdot \Phi_i dx \tag{25}$$

$$= 2\sum_j^N \left(\int_{\Omega} \Phi_i \cdot \Phi_j dx\right) c_j^n - \sum_j^N \left(\int_{\Omega} \Phi_i \cdot \Phi_j dx\right) c_j^{n-1} + \Delta t^2 \sum_j^N \left(\int_{\Omega} \Phi_i \cdot \Phi_j dx\right) b_j^n \tag{26}$$

$$\begin{aligned}\int_{\Omega} \sigma^n : \nabla \mathbf{v} dx &= \int_{\Omega} \sigma(\sum_j^N c_j^n \Phi_j) : \nabla \Phi_i dx \\ &= \sum_j^N \left(\int_{\Omega} \sigma(\Phi_j) : \nabla \Phi_i dx\right) c_j^n\end{aligned}$$

We define the matrices M and K, with elements $M_{i,j} = \int_{\Omega} \Phi_i \cdot \Phi_j dx$ and $K_{i,j} = \int_{\Omega} \sigma(\Phi_j) : \nabla \Phi_i dx$. Since the coefficients c^n are the nodal values of $\mathbf{u}$ at time $t = t^n$, we get the linear system

$$M\mathbf{u}^{n+1} = 2M\mathbf{u}^n - M\mathbf{u}^{n-1} - \frac{\Delta t^2}{\rho} K\mathbf{u}^n + \Delta t^2 M\mathbf{b}^n \tag{27}$$

$$= \left(2M - \frac{\Delta t^2}{\rho}K\right)\mathbf{u}^n - M\mathbf{u}^{n-1} + \Delta t^2 M\mathbf{b}^n \tag{28}$$

where $\mathbf{u}^n$ and $\mathbf{u}^{n-1}$ are known.