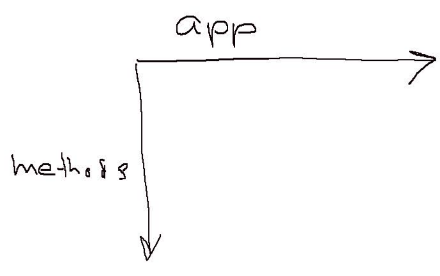
Sept. 6, 2012



$u' = -au, \quad u(0) = I$

Operator notation:

$$\left.\frac{du}{dt}\right|_{t_n} = u'(t_n) \approx \frac{u^n - u^{n-1}}{\Delta t} = [D_t^- u]^n$$

$$\text{—''—} \approx \frac{u^{n+1} - u^n}{\Delta t} = [D_t^+ u]^n$$

Differential eq.:

$$\frac{u^{n+1} - u^n}{\Delta t} = -a u^n$$

$$[D_t^+ u]^n = [-au]^n \longrightarrow [D_t^+ u = -au]^n$$

$$\frac{u^n - u^{n-1}}{\Delta t} = -a u^n \longrightarrow [D_t^- u = -au]^n$$

Centered difference:

$$u'(t_{n+1/2}) \approx \frac{u^{n+1} - u^n}{\Delta t} = [D_t u]^{n+\frac{1}{2}}$$

Crank-Nic. scheme:

$$[D_t u = -au]^n$$

Consider a wave PDE: $\quad [D_t D_t u]^n = \frac{u^{n+1} - 2u^n + u^{n-1}}{\Delta t} \approx \left.\frac{d^2 u}{dt^2}\right|_{t_n}$

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)$$

$$\downarrow$$

$$[D_t D_t u = c^2 (D_x D_x u + D_y D_y u)]^n_{i,j}$$

$$\downarrow$$

$$\frac{u^{n+1}_{i,j} - 2u^n_{i,j} + u^{n-1}_{i,j}}{\Delta t^2} = c^2 \left(\frac{u^n_{i+1,j} - 2u^n_{i,j} + u^n_{i-1,j}}{\Delta x^2} + \frac{u^n_{i,j+1} - 2u^n_{i,j} + u^n_{i,j-1}}{\Delta y} \right)$$

$\downarrow$ solve for $u^{n+1}_{i,j}$

$$u^{n+1}_{i,j} = \ldots\ldots$$

—

Generalize the ODE problem: $u'(t) = -a(t)u(t) + b(t)$

Forward Euler:

$$[D_t^+ u = -au + b]^n$$

$$\downarrow$$

$$\frac{u^{n+1} - u^n}{\Delta t} = -a(t_n)u^n + b(t_n) = -a^n u^n + b^n$$

$\downarrow$ algorithm: solve wrt. unknown (u^{n+1})

$$u^{n+1} = u^n - \Delta t\, a^n u^n + \Delta t\, b^n$$

Crank-Nicolson:

$$[D_t u = -au + b]^{n+1/2}$$

$$[au + b]^{n+1/2} \text{ ??}$$

$$-a(t_{n+1/2})\, u^{n+1/2} + b(t_{n+1/2})$$

$$u^{n+1/2} \approx \tfrac{1}{2}(u^n + u^{n+1}) = [\overline{u}^t]^{n+1/2}$$

$$[D_t u = -a\overline{u}^t + b]^{n+1/2}$$

Alt: $\quad [D_t u = \overline{-au + b}^t]^{n+1/2}$

$$\downarrow$$

$$\frac{u^{n+1} - u^n}{\Delta t} = \frac{1}{2}\left(-a^n u^n + b^n - a^{n+1} u^{n+1} + b^{n+1} \right)$$

θ-rule: Operator notation also possible (see notes)

—

How to verify an implementation?

1. Linear solutions, $u_e(t) = ct + d$, are often exactly reproduced by most numerical schemes

   Insert in ODE: $\quad c = -a(ct + d) + b, \quad u(0) = d = I$

   Choose an $a(t)$, choose a c, then $b = c + a(t)(ct + I)$.

2. Method of manufactured solutions (MMS):

   Choose some $u_e(t)$, say $u_e(t) = \sin \pi t$

   Fit a and b so that u_e fulfills the ODE:

   $$\pi \cos \pi t = -a(t) \sin \pi t + b(t)$$

   Pick $a = 1$, $b = \pi \cos \pi t + \sin \pi t$.

   $$u' = -u + \pi \cos \pi t + \sin \pi t, \quad u(0) = 0.$$

   Problem: We don't know $e(t_n) = u_e(t_n) - u^n$

   Introduce a scalar (one number) measure of $e(t)$:

   $$\tilde{E} = \int_0^T e(t)\, dt \approx \text{num. integration}$$

   $$E = \sqrt{\int_0^T e(t)^2\, dt} \approx \sqrt{\Delta t \sum_{i=0}^{N} \left(u_e(t_i) - u^i\right)^2}$$

   Experiments:

   $$(\Delta t_1, E_1), (\Delta t_2, E_2), \ldots, (\Delta t_m, E_m)$$

   Assumption (good!):

   $$E = C \Delta t^r \qquad \text{(valid when } \Delta t \to 0\text{)}$$