Exercise 3: Identify the F vector of nonlinear equations and derive the Jacobian.

$$F = (F_0, F_1, \ldots, F_N)$$

$$F_i = \frac{1}{2\Delta x^2}\Big((\alpha(U_i) + \alpha(U_{i+1}))(U_{i+1} - U_i) - (\alpha(U_{i-1}) + \alpha(U_i))(U_i - U_{i-1}) \Big)$$

$$J_{i,j} = \frac{\partial F_i}{\partial U_j}$$

Observation: only $\frac{\partial F_i}{\partial U_{i-1}}$, $\frac{\partial F_i}{\partial U_i}$, and $\frac{\partial F_i}{\partial U_{i+1}}$ are nonzero (i.e. $j = i, i\pm 1$)

$$\frac{\partial F_i}{\partial U_{i-1}} = \frac{1}{2\Delta x^2}\Big(-\alpha'(U_{i-1})(U_i - U_{i-1}) - \alpha(U_{i-1})(-1)\Big)$$

$$\frac{\partial F_i}{\partial U_i} = \frac{1}{2\Delta x^2}\Big(\alpha'(U_i)(U_{i+1} - U_i) + \alpha(U_i)(-1) - \alpha'(U_i)(U_i - U_{i-1}) - \alpha(U_i)\cdot 1\Big)$$

$$\frac{\partial F_i}{\partial U_{i+1}} = \frac{1}{2\Delta x^2}\Big(\alpha'(U_{i+1})(U_{i+1} - U_i) + \alpha(U_{i+1})\cdot 1\Big)$$

In the finite element method we can also derive the Jacobian from the variational form, prior to integration over the reference element and assembly.

$$F_i = \int_0^1 \alpha(u)\, u' \varphi_i' \, dx, \quad u \doteq \sum_k \varphi_k U_k$$

$$= \sum_k \int_0^1 \alpha\Big(\sum_\ell \varphi_\ell U_\ell\Big) \varphi_k' \varphi_i' \, dx \cdot U_k$$

$$\frac{\partial F_i}{\partial U_j} = \sum_k \int_0^1 \Big(\underbrace{\frac{\partial}{\partial U_j}\Big(\alpha\big(\textstyle\sum_\ell \varphi_\ell U_\ell\big)\Big)}_{= \frac{d\alpha}{du}\frac{\partial u}{\partial U_j} = \alpha'(u)\cdot\varphi_j} \varphi_k' \varphi_i' U_k + \underbrace{\alpha(u)\varphi_k'\varphi_i' \underbrace{\frac{\partial U_k}{\partial U_j}}_{=1 \text{ when } k=j}}_{\varphi_j'\varphi_i'} \Big) dx$$

$$\frac{\partial F_i}{\partial U_j} = \int_0^1 \Big(\alpha'(u)\varphi_j \varphi_i' u' + \alpha(u)\varphi_j'\varphi_i' \Big) dx$$

Trapezoidal rule on the reference element:

$$\tilde{J}^{(i)}_{r,s} = \int_{-1}^1 \Big(\alpha'(u)\tilde{\varphi}_s \frac{2}{h}\frac{d\tilde{\varphi}_r}{d\bar{X}} u' + \alpha(u)\frac{2}{h}\frac{d\tilde{\varphi}_s}{d\bar{X}}\frac{2}{h}\frac{d\tilde{\varphi}_r}{d\bar{X}} \Big)\frac{h}{2}\, d\bar{X}$$

$$u(-1) = U_i, \quad u(1) = U_{i+1}, \quad u'(-1) = \sum_r \frac{2}{h}\frac{d\tilde{\varphi}_r}{d\bar{X}} \cdot U_{r+i} = \frac{2}{h}\Big(\frac{1}{2}U_i + \big(-\tfrac{1}{2}\big)U_{i+1}\Big)$$

The term $\alpha(u)\tilde{\varphi}_s'\tilde{\varphi}_r'$ assembles to the equations in Exercise 1 (and 2).
Concentrating on the new term:

$$\Big(\alpha'(u)\tilde{\varphi}_s \frac{2}{h}\big(-\tfrac{1}{2}\big)^r u'\Big)_{\bar{X}=-1} = \alpha'(U_i)\,\delta_{s0}\,\frac{2}{h}\big(\tfrac{1}{2}\big)^r \frac{1}{h}(U_i - U_{i+1})$$

δ_{s0}: 1 $s=0$, 0 $s\neq 0$

$$\Big(\alpha'(u)\tilde{\varphi}_s \frac{2}{h}\big(-\tfrac{1}{2}\big)^r u'\Big)_{\bar{X}=1} = \alpha'(U_{i+1})\,\delta_{s1}\,\frac{2}{h}\big(-\tfrac{1}{2}\big)^r \frac{1}{h}(U_i - U_{i+1})$$

$$(U_i - U_{i+1})\frac{1}{h}\begin{bmatrix} \alpha'(U_i) & +\alpha'(U_{i+1}) \\ -\alpha'(U_i) & -\alpha'(U_{i+1}) \end{bmatrix}$$

When this matrix is assembled it gives a contribution to the global Jacobian that is

$$\frac{1}{h}\alpha'(U_{i+1})(U_i - U_{i+1}) \quad \text{for element } J_{i,i+1}$$

$$-\frac{1}{h}\alpha'(U_{i-1})(U_{i-1} - U_i) \quad \text{for element } J_{i,i-1}$$

The end result should be the same whether we integrate and assemble first and then differentiate to obtain the Jacobian or whether we first differentiate and then integrate and assemble.

Exercise 4: Sparsity of the Jacobian.

Typically, eq. no. i involves the same unknowns as in a corresponding linear problem. In 1D (P1 elements):

$$F_i = F_i(U_{i-1}, U_i, U_{i+1})$$

Then $\frac{\partial F_i}{\partial U_j} \neq 0$ for $j = i-1, i, i+1$ and we get a tridiagonal matrix

In general:

$$F_i = F_i\big(\{U_k\}_{k\in K}\big)$$ K is a small set of unknowns entering eq. no. i

$\frac{\partial F_i}{\partial U_j} \neq 0$ only for $j \in K$.

Exercise 5: Newton's method converges in one iteration in linear problems.

$F = 0$: $Ax = b$, $F = b - Ax$. $J = A$

Given some x_0,

Solve $J\delta x = -F$: $A\delta x = Ax^0 - b \Rightarrow \delta x = x^0 - A^{-1}b$

$x^1 = x^0 - \delta x$: $x^1 = x^0 - x^0 + A^{-1}b = A^{-1}b = x$ (exact)