

Experiments with Schemes for Exponential Decay

Hans Petter Langtangen^{1,2} (hpl@simula.no)

¹Center for Biomedical Computing, Simula Research Laboratory

²Department of Informatics, University of Oslo.

Aug 18, 2013

Abstract

This report investigates the accuracy of three finite difference schemes for the ordinary differential equation $u' = -au$ with the aid of numerical experiments. Numerical artifacts are in particular demonstrated.

Contents

1	Mathematical problem	2
2	Numerical solution method	2

3	Implementation	3
4	Numerical experiments	3
4.1	The Backward Euler method	4
4.2	The Crank-Nicolson method	5
4.3	The Forward Euler method	6
4.4	Error vs Δt	6

1 Mathematical problem

We address the initial-value problem

$$u'(t) = -au(t), \quad t \in (0, T], \quad (1)$$

$$u(0) = I, \quad (2)$$

where a , I , and T are prescribed parameters, and $u(t)$ is the unknown function to be estimated. This mathematical model is relevant for physical phenomena featuring exponential decay in time, e.g., vertical pressure variation in the atmosphere, cooling of an object, and radioactive decay.

2 Numerical solution method

We introduce a mesh in time with points $0 = t_0 < t_1 \cdots < t_{N_t} = T$. For simplicity, we assume constant spacing Δt between the mesh points: $\Delta t = t_n - t_{n-1}$, $n = 1, \dots, N_t$. Let u^n be the numerical approximation to the exact solution at t_n .

The θ -rule [1] is used to solve (1) numerically:

$$u^{n+1} = \frac{1 - (1 - \theta)a\Delta t}{1 + \theta a\Delta t} u^n,$$

for $n = 0, 1, \dots, N_t - 1$. This scheme corresponds to

- The **Forward Euler** scheme when $\theta = 0$
- The **Backward Euler** scheme when $\theta = 1$
- The **Crank-Nicolson** scheme when $\theta = 1/2$

3 Implementation

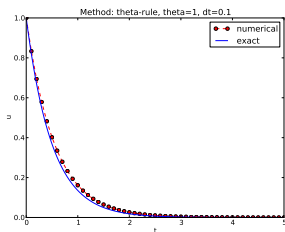
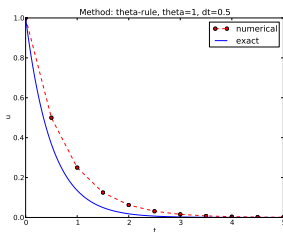
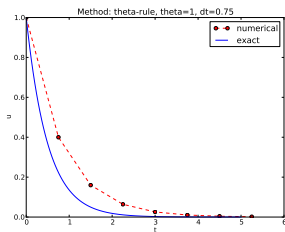
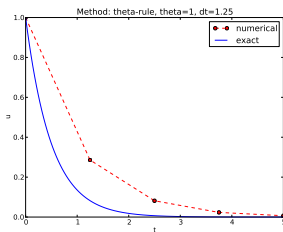
The numerical method is implemented in a Python function [2] `solver` (found in the `decay_mod` module):

```
def solver(I, a, T, dt, theta):  
    """Solve u'=-a*u, u(0)=I, for t in (0,T] with steps  
    dt = float(dt) # avoid integer division  
    Nt = int(round(T/dt)) # no of time intervals  
    T = Nt*dt # adjust T to fit time steps  
    u = zeros(Nt+1) # array of u[n] values  
    t = linspace(0, T, Nt+1) # time mesh  
  
    u[0] = I # assign initial condition  
    for n in range(0, Nt): # n=0,1,...,Nt-1  
        u[n+1] = (1 - (1-theta)*a*dt)/(1 + theta*dt*a)*u[n]  
    return u, t
```

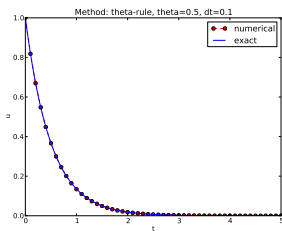
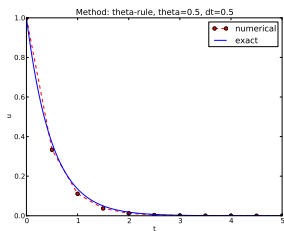
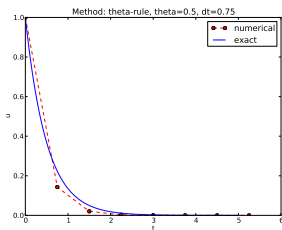
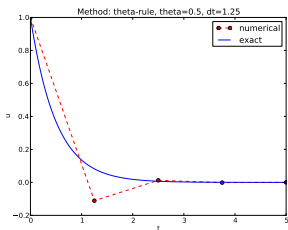
4 Numerical experiments

We define a set of numerical experiments where I , a , and T are fixed, while Δt and θ are varied. In particular, $I = 1$, $a = 2$, $\Delta t = 1.25, 0.75, 0.5, 0.1$.

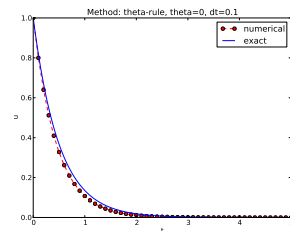
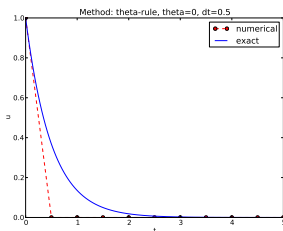
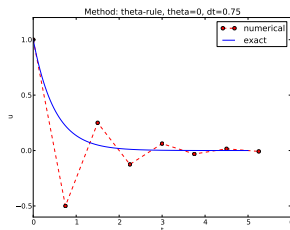
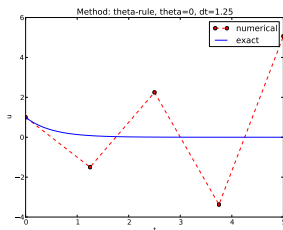
4.1 The Backward Euler method



4.2 The Crank-Nicolson method



4.3 The Forward Euler method

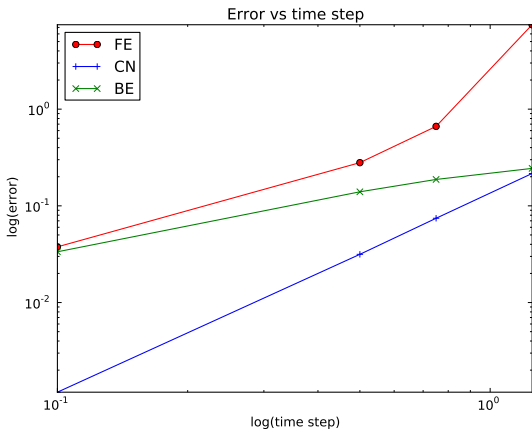


4.4 Error vs Δt

How E varies with Δt for $\theta = 0, 0.5, 1$ is shown below.

Observe:

The data points for the three largest Δt values in the Forward Euler method are not relevant as the solution behaves non-physically.



Summary.

1. $\theta = 1$: $E \sim \Delta t$ (first-order convergence).
2. $\theta = 0.5$: $E \sim \Delta t^2$ (second-order convergence).
3. $\theta = 1$ is always stable and gives qualitatively correct results.
4. $\theta = 0.5$ never blows up, but may give oscillating solutions if Δt is not sufficiently small.
5. $\theta = 0$ suffers from fast-growing solution if Δt is not small enough, but even below this limit one

can have oscillating solutions that disappear if Δt is sufficiently small.

References

- [1] A. Iserles. *A First Course in the Numerical Analysis of Differential Equations*. Cambridge Texts in Applied Mathematics. Cambridge University Press, second edition, 2009.
- [2] H. P. Langtangen. *A Primer on Scientific Programming With Python*. Texts in Computational Science and Engineering. Springer, third edition, 2012.