$$u'' + f(u') + s(u) = F(t)$$

Discretization

$$[D_tD_t u'' + f(D_{2t}u) + s(u) = F]^n$$

$$\frac{u^{n+1} - 2u^n + u^{n-1}}{\Delta t^2} + f\left(\frac{u^{n+1} - u^{n-1}}{2\Delta t}\right) + s(u^n) = F^n$$

u^{n+1}: unknown, nonlinear in u^{n+1} if f is nonlinear

Linear damping: $f(u') = bu'$, $b = \text{const} \geq 0$

$f(u') \approx b\dfrac{u^{n+1} - u^{n-1}}{2\Delta t}$, can solve wrt u^{n+1}

$\Rightarrow u^{n+1} = \ldots u^n, u^{n-1}$ explicit formula

Quadratic f: $f(u') = b|u'|u'$

$$\frac{u^{n+1} - 2u^n + u^{n-1}}{\Delta t^2} + b\left|\frac{u^{n+1} - u^{n-1}}{2\Delta t}\right|\frac{u^{n+1} - u^{n-1}}{2\Delta t} + s(u^n) = F^n$$

nonlinear in u^{n+1}.

Smart trick: approx $|u'|u'$ by a geometric mean

$$[|u'|u']^n \approx |u'|^{n-\frac{1}{2}}[u']^{n+1/2} + \mathcal{O}(\Delta t^2)$$

$$\approx \left|[D_t u]^{n-\frac{1}{2}}\right| [D_t u]^{n+\frac{1}{2}}$$

$$= \left|\frac{u^n - u^{n-1}}{\Delta t}\right| \cdot \frac{u^{n+1} - u^n}{\Delta t} \quad \text{linear in } u^{n+1}\text{ !!}$$

$\Rightarrow u^{n+1} = \ldots\ldots u^n, u^{n-1} \ldots$ explicit scheme

Note: $s(u)$ is evaluated at t_n and is therefore known.

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$u^1 = ?$ Need to derive a special formula

$$\left.\begin{array}{l}\text{scheme at } n=0, t=0\\ \text{initial cond } [D_{2t}u]^0 = V\end{array}\right\} \Rightarrow u^1 = \ldots\ldots$$

—

First-order formulation and Staggered meshes:

First-order form:

$$\left\{\begin{array}{l} u' = v\\ v' = -f(v) - s(u) + F(t)\\ u(0) = I, \quad v(0) = V.\end{array}\right\}$$