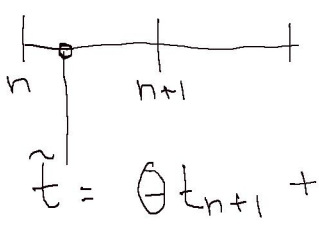
θ-rule:



$\tilde{t} = \theta t_{n+1} + (1-\theta) t_n$ weighted average

$\theta \in [0,1]$

$\hat{t} = t_{n+\theta}$ Sample the ODE at $t_{n+\theta}$

$$U'(t_{n+\theta}) \approx \frac{U^{n+1} - U^n}{\Delta t}$$

$$-aU(t_{n+\theta}) \approx -a\left(\theta U^{n+1} + (1-\theta) U^n\right)$$

weighted average

$$\Rightarrow \frac{U^{n+1} - U^n}{\Delta t} = -a\theta U^{n+1} - a(1-\theta) U^n$$

solve wrt U^{n+1}:

$$U^{n+1} = \frac{1 - (1-\theta) a \Delta t}{1 + \theta a \Delta t} U^n$$

Valid for any $\theta \in [0,1]$

$\theta = 0$: Forward Euler

$\theta = \frac{1}{2}$: Crank-Nicolson

$\theta = 1$: Backward Euler