Remaining: impl. of $\left.\frac{\partial u}{\partial x}\right|_{x=L}$ (done as for the wave equation)

Alternative: Crank-Nicolson

$$\frac{u^{n+1}-u^n}{\Delta t} = \alpha \frac{1}{2}\left([D_x D_x u]_i^{n+1} + [D_x D_x u]_i^n \right)$$

$\Rightarrow$ Linear system for u_i^{n+1}, $i=1,\ldots,N$,

Forward Euler: explicit scheme with stability restriction on Δt

Backward Euler and Crank-Nicolson: implicit schemes with need for solving linear systems, but no restriction on Δt

(The same as for $u' = -au$)

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Finite elements:

- Finite differences in time
- Finite elements in space

$u_t = \alpha u_{xx}$: $\quad [D_t^+ u = \alpha u_{xx}]^n$

$u^{n+1} = u^n + \alpha \Delta t\, u_{xx}^n$ discrete in t, continuous in x

$u^n(x) \approx \sum_{j=0}^{N} c_j^n \varphi_j(x)$, $\quad u^{n+1} \approx \sum_{j=0}^{n} c_j^{n+1} \varphi_j(x)$

Galerkin method with $v(x)$ as test function:

$$\int_0^L u^{n+1} v\, dx = \int_0^L u^n v\, dx + \underbrace{\int_0^L \alpha \Delta t\, u_{xx}^n v\, dx}$$

$\downarrow$ integr. by parts

$$-\alpha \Delta t \int u_x^n v_x\, dx + \alpha \Delta t \left[u_x^n v\right]_0^L$$

$$\left[u_x^n v\right]_0^L = u_x^n(L)v(L) - u_x^n(0)v(0)$$

$u_x^n(L) \to D$, $v(L)$; $\quad u_x^n(0)v(0) \to 0$ Dirichlet cond. $x=0$, so $v(0)=0$

$= Dv(L)$

Variational form:

$$\int_0^L u^{n+1} v\, dx = \int_0^L u^n v\, dx - \alpha \Delta t \int_0^L u_x^n v_x\, dx + Dv(L) \quad \forall v \in V$$

To derive the linear system: $u^n = \sum_{j=0}^{N} c_j^n \varphi_j$, $\quad v = \varphi_i$, $i=0,\ldots,N$

$$\sum_{j=0}^{N} \underbrace{\left(\int_0^L \varphi_i \varphi_j\, dx\right)}_{M_{ij}} c_j^{n+1} = \sum_{j=0}^{N} \left(\int_0^L \varphi_i \varphi_j\right) c_j^n - \alpha \Delta t \sum_{j=0}^{N} \underbrace{\left(\int_0^L \varphi_i' \varphi_j'\, dx\right)}_{K_{ij}} c_j^n + D\varphi_i(L)$$

Let's write this on matrix form

$$M c^{n+1} = \underbrace{M c^n - \alpha \Delta t K c^n}_{\text{matrices times known vector } c^n} + b$$

M: coefficient matrix; c^{n+1}: unknown

$\Rightarrow$ We have to solve a linear system, despite having used Forward Euler on u_t !!!

Finite elements give an implicit scheme! (Still strong stability restrictions on Δt)