Study Guide: Vibration ODEs

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A simple vibration problem

$$u''t + \omega^2 u = 0, \quad u(0) = I,\ u'(0) = 0,\ t \in (0, T]\,. \tag{1}$$

Exact solution:

$$u(t) = I\cos(\omega t)\,. \tag{2}$$

$u(t)$ oscillates with constant amplitude I and (angular) frequency ω. Period: $P = 2\pi/\omega$.

A centered finite difference scheme; step 1 and 2

- Strategy: follow the four steps of the finite difference method.
- Step 1: Introduce a time mesh, here uniform on $[0, T]$: $t_n = n\Delta t$
- Step 2: Let the ODE be satisfied at each mesh point:

$$u''(t_n) + \omega^2 u(t_n) = 0, \quad n = 1, \ldots, N_t\,. \tag{3}$$

A centered finite difference scheme; step 3

Step 3: Approximate derivative(s) by finite difference approximation(s). Very common (standard!) formula for u'':

$$u''(t_n) \approx \frac{u^{n+1} - 2u^n + u^{n-1}}{\Delta t^2} \,. \tag{4}$$

Use this discrete initial condition together with the ODE at $t = 0$ to eliminate u^{-1} (insert (**??**) in (**??**)):

$$\frac{u^{n+1} - 2u^n + u^{n-1}}{\Delta t^2} = -\omega^2 u^n \,. \tag{5}$$

A centered finite difference scheme; step 4

Step 4: Formulate the computational algorithm. Assume u^{n-1} and u^n are known, solve for unknown u^{n+1}:

$$u^{n+1} = 2u^n - u^{n-1} - \omega^2 u^n \,. \tag{6}$$

Nick names for this scheme: Stormer's method or Verlet integration.

Computing the first step

- The formula breaks down for u^1 because u^{-1} is unknown and outside the mesh!
- And: we have not used the initial condition $u'(0) = 0$.

Discretize $u'(0)$ by a centered difference

$$\frac{u^1 - u^{-1}}{2\Delta t} = 0 \quad \Rightarrow \quad u^{-1} = u^1\,. \tag{7}$$

Inserted in (**??**) for $n = 0$ gives

$$u^1 = u^0 - \frac{1}{2}\Delta t^2\omega^2 u^0\,. \tag{8}$$

The computational algorithm

1. $u^0 = I$
2. compute u^1 from (**??**)
3. for $n = 1, 2, \ldots, N_t - 1$:
    1. compute u^{n+1} from (**??**)

More precisely expressed in Python:

```
t = linspace(0, T, Nt+1)  # mesh points in time
dt = t[1] - t[0]          # constant time step.
u = zeros(Nt+1)           # solution

u[0] = I
u[1] = u[0] - 0.5*dt**2*w**2*u[0]
for n in range(1, Nt):
    u[n+1] = 2*u[n] - u[n-1] - dt**2*w**2*u[n]
```

Note: `w` is consistently used for ω in code.

Operator notation; ODE

With $[D_tD_tu]^n$ as the finite difference approximation to $u''(t_n)$ we can write

$$[D_tD_tu + \omega^2 u = 0]^n\,. \tag{9}$$

$[D_tD_tu]^n$ means applying a central difference with step $\Delta t/2$ twice:

$$[D_t(D_tu)]^n = \frac{[D_tu]^{n+1/2} - [D_tu]^{n-1/2}}{\Delta t}$$

which is written out as

$$\frac{1}{\Delta t}\left(\frac{u^{n+1}-u^n}{\Delta t} - \frac{u^n - u^{n-1}}{\Delta t}\right) = \frac{u^{n+1} - 2u^n + u^{n-1}}{\Delta t^2}\,.$$

Operator notation; initial condition

$$[u = I]^0, \quad [D_{2t} u = 0]^0, \tag{10}$$

where $[D_{2t} u]^n$ is defined as

$$[D_{2t} u]^n = \frac{u^{n+1} - u^{n-1}}{2\Delta t} \,. \tag{11}$$

Computing u'

u is often displacement/position, u' is velocity and can be computed by

$$u'(t_n) \approx \frac{u^{n+1} - u^{n-1}}{2\Delta t} = [D_{2t} u]^n \,. \tag{12}$$

Implementation

Core algorithm

```
from numpy import *
from matplotlib.pyplot import *

def solver(I, w, dt, T):
    """
    Solve u'' + w**2*u = 0 for t in (0,T], u(0)=I and u'(0)=0,
    by a central finite difference method with time step dt.
    """
    dt = float(dt)
    Nt = int(round(T/dt))
    u = zeros(Nt+1)
    t = linspace(0, Nt*dt, Nt+1)

    u[0] = I
    u[1] = u[0] - 0.5*dt**2*w**2*u[0]
    for n in range(1, Nt):
        u[n+1] = 2*u[n] - u[n-1] - dt**2*w**2*u[n]
    return u, t
```

Plotting

```
def exact_solution(t, I, w):
    return I*cos(w*t)

def visualize(u, t, I, w):
    plot(t, u, 'r--o')
    t_fine = linspace(0, t[-1], 1001)  # very fine mesh for u_e
    u_e = exact_solution(t_fine, I, w)
    hold('on')
    plot(t_fine, u_e, 'b-')
    legend(['numerical', 'exact'], loc='upper left')
    xlabel('t')
    ylabel('u')
    dt = t[1] - t[0]
    title('dt=%g' % dt)
    umin = 1.2*u.min();  umax = -umin
    axis([t[0], t[-1], umin, umax])
    savefig('vib1.png')
    savefig('vib1.pdf')
    savefig('vib1.eps')
```

Main program

```
I = 1
w = 2*pi
dt = 0.05
num_periods = 5
P = 2*pi/w    #  one period
T = P*num_periods
u, t = solver(I, w, dt, T)
visualize(u, t, I, w, dt)
```

User interface: command line

```
import argparse
parser = argparse.ArgumentParser()
parser.add_argument('--I', type=float, default=1.0)
parser.add_argument('--w', type=float, default=2*pi)
parser.add_argument('--dt', type=float, default=0.05)
parser.add_argument('--num_periods', type=int, default=5)
a = parser.parse_args()
I, w, dt, num_periods = a.I, a.w, a.dt, a.num_periods
```

First steps for testing and debugging

- **Testing very simple solutions**: $u = \text{const}$ or $u = ct + d$ do not apply here (without a force term in the equation: $u'' + \omega^2 u = f$).
- **Hand calculations**: calculate u^1 and u^2 and compare with program.

Checking convergence rates

The function below

- performs m simulations with halved time steps: $2^{-k}\Delta t$, $k = 0, \ldots, m-1$,
- computes the L_2 norm of the error, $E = \sqrt{\Delta t_i \sum_{n=0}^{N_t-1}(u^n - u_{\rm e}(t_n))^2}$ in each case,
- estimates the rates r_i from two consecutive experiments $(\Delta t_{i-1}, E_{i-1})$ and $(\Delta t_i, E_i)$, assuming $E_i = C\Delta t_i^{r_i}$ and $E_{i-1} = C\Delta t_{i-1}^{r_i}$:

Implementational details

```python
def convergence_rates(m, num_periods=8):
    """
    Return m-1 empirical estimates of the convergence rate
    based on m simulations, where the time step is halved
    for each simulation.
    """
    w = 0.35; I = 0.3
    dt = 2*pi/w/30  # 30 time step per period 2*pi/w
    T = 2*pi/w*num_periods
    dt_values = []
    E_values = []
    for i in range(m):
        u, t = solver(I, w, dt, T)
        u_e = exact_solution(t, I, w)
        E = sqrt(dt*sum((u_e-u)**2))
        dt_values.append(dt)
        E_values.append(E)
        dt = dt/2

    r = [log(E_values[i-1]/E_values[i])/
         log(dt_values[i-1]/dt_values[i])
         for i in range(1, m, 1)]
    return r
```

Result: `r` contains values equal to 2.00 - as expected!

Nose test

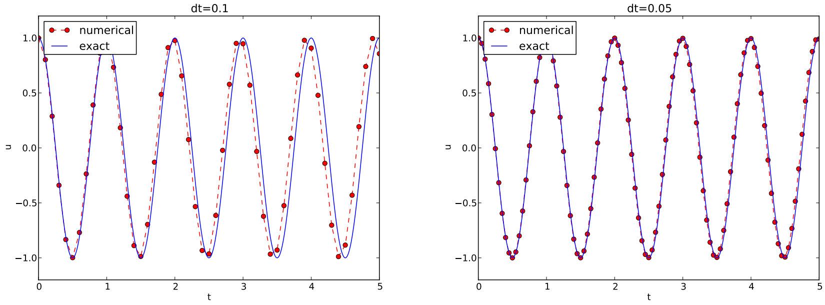
Use final `r[-1]` in a unit test:

```
def test_convergence_rates():
    r = convergence_rates(m=5, num_periods=8)
    # Accept rate to 1 decimal place
    nt.assert_almost_equal(r[-1], 2.0, places=1)
```

Complete code in `vib_undamped.py`.

Long time simulations

Effect of the time step on long simulations



- The numerical solution seems to have right amplitude.
- There is a phase error which is reduced by reducing the time step.
- The total phase error seems to grow with time.

Using a moving plot window

- In long time simulations we need a plot window that follows the solution.
- Method 1: `scitools.MovingPlotWindow`.
- Method 2: `scitools.avplotter` (ASCII vertical plotter).

Example:

```
Terminal> python vib_undamped.py --dt 0.05 --num_periods 40
```

Movie of the moving plot window.

Analysis of the numerical scheme

Deriving an exact numerical solution; ideas

- Linear, homogeneous, difference equation for u^n.
- Has solutions $u^n \sim A^n$, where A is unknown (number).
- Here: $u_e(t) = I\cos(\omega t) \sim I\exp(i\omega t) = I(e^{i\omega\Delta t})^n$
- Trick for simplifying the algebra: $A = \exp(i\tilde{\omega}\Delta t)$ (ansatz)
- $\tilde{\omega}$: unknown *numerical frequency* (easier to calculate than A)
- $\omega - \tilde{\omega}$ is the *phase error*
- Use the real part as the physical relevant part of a complex expression

Deriving an exact numerical solution; calculations (1)

$$A^n = \exp(\tilde{\omega}\Delta t\, n) = \exp(\tilde{\omega} t) = \cos(\tilde{\omega} t) + i\sin(\tilde{\omega} t)\,.$$

$$\begin{aligned}
[D_tD_tu]^n &= \frac{\exp(i\tilde{\omega}(t+\Delta t)) - 2\exp(i\tilde{\omega} t) + \exp(i\tilde{\omega}(t-\Delta t))}{\Delta t^2}\\
&= \exp(i\tilde{\omega} t)\frac{1}{\Delta t^2}\left(\exp(i\tilde{\omega}(\Delta t)) + \exp(i\tilde{\omega}(-\Delta t)) - 2\right)\\
&= \exp(i\tilde{\omega} t)\frac{2}{\Delta t^2}\left(\cosh(i\tilde{\omega}\Delta t) - 1\right)\\
&= \exp(i\tilde{\omega} t)\frac{2}{\Delta t^2}\left(\cos(\tilde{\omega}\Delta t) - 1\right)\\
&= -\exp(i\tilde{\omega} t)\frac{4}{\Delta t^2}\sin^2(\frac{\tilde{\omega}\Delta t}{2})
\end{aligned}$$

Deriving an exact numerical; calculations (2)

The scheme (**??**) with $u^n = \exp\left(i\omega\tilde{\Delta} t\, n\right)$ inserted gives

$$-\exp\left(i\tilde{\omega}t\right)\frac{4}{\Delta t^2}\sin^2(\frac{\tilde{\omega}\Delta t}{2}) + \omega^2\exp\left(i\tilde{\omega}t\right) = 0, \qquad (13)$$

which after dividing by $\exp\left(i\tilde{\omega}t\right)$ results in

$$\frac{4}{\Delta t^2}\sin^2(\frac{\tilde{\omega}\Delta t}{2}) = \omega^2\,. \qquad (14)$$

Solve for $\tilde{\omega}$:

$$\tilde{\omega} = \pm\frac{2}{\Delta t}\sin^{-1}\left(\frac{\omega\Delta t}{2}\right)\,. \qquad (15)$$

- Phase error because $\tilde{\omega} \neq \omega$.
- But how good is the approximation $\tilde{\omega}$ to ω?

Polynomial approximation of the phase error

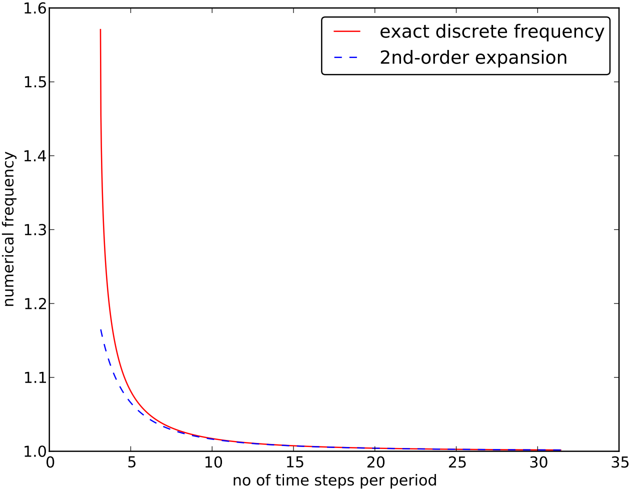
Taylor series expansion for small Δt gives a formula that is easier to understand:

```
>>> from sympy import *
>>> dt, w = symbols('dt w')
>>> w_tilde = asin(w*dt/2).series(dt, 0, 4)*2/dt
>>> print w_tilde
(dt*w + dt**3*w**3/24 + O(dt**4))/dt  # observe final /dt
```

$$\tilde{\omega} = \omega\left(1 + \frac{1}{24}\omega^2\Delta t^2\right) + \mathcal{O}(\Delta t^3)\,. \tag{16}$$

The numerical frequency is too large (to fast oscillations).

Plot of the phase error



Exact discrete solution

$$u^n = I\cos(\tilde{\omega} n\Delta t), \quad \tilde{\omega} = \frac{2}{\Delta t}\sin^{-1}\left(\frac{\omega\Delta t}{2}\right). \tag{17}$$

Ideal for verification and analysis!

Stability

- Numerical solution has constant amplitude (desired!), but phase error.
- Constant amplitude requires $\sin^{-1}(\omega\Delta t/2)$ to be real-valued $\Rightarrow |\omega\Delta t/2| \leq 1$.
- $\sin^{-1}(x)$ is complex if $|x| > 1$, and then $\tilde{\omega}$ becomes complex.

What is the consequence of complex $\tilde{\omega}$?

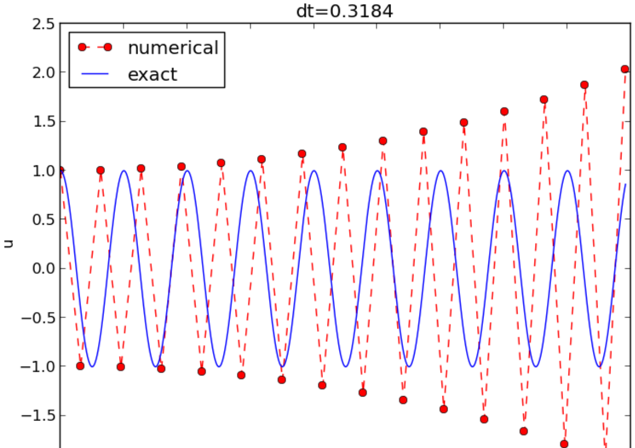
- Set $\tilde{\omega} = \tilde{\omega}_r + i\tilde{\omega}_i$.
- Since $\sin^{-1}(x)$ has a negative* imaginary part for $x > 1$, $\exp\left(i\omega\tilde{t}\right) = \exp\left(-\tilde{\omega}_i t\right)\exp\left(i\tilde{\omega}_r t\right)$ leads to exponential growth $e^{-\tilde{\omega}_i t}$ when $-\tilde{\omega}_i t > 0$.
- This instability because the qualitative behavior is wrong.

The stability criterion

Cannot tolerate growth and must therefore demand a *stability criterion*

$$\frac{\omega \Delta t}{2} \leq 1 \quad \Rightarrow \quad \Delta t \leq \frac{2}{\omega} \,. \tag{18}$$

Try $\Delta t = \frac{2}{\omega} + 9.01 \cdot 10^{-5}$ (π^{-1}:



Summary of the analysis

We can draw three important conclusions:

1. The key parameter in the formulas is $p = \omega\Delta t$.
   1. Period of oscillations: $P = 2\pi/\omega$
   2. Number of time steps per period: $N_P = P/\Delta t$
   3. $\Rightarrow\ p = \omega\Delta t = 2\pi N_P \sim N_P$
   4. The smallest possible N_P is $2 \Rightarrow p \in (0, \pi]$
2. For $p \leq 2$ the amplitude of u^n is constant (stable solution)
3. u^n has a relative phase error $\tilde{\omega}/\omega \approx 1 + \frac{1}{24}p^2$, making numerical peaks occur too early

Alternative schemes based on 1st-order equations

Rewriting 2nd-order ODE as system of two 1st-order ODEs

$$u'' + \omega^2 u = 0$$

Introduce auxiliary variable $v = u'$:

$$u' = v, \tag{19}$$

$$v' = -\omega^2 u\,. \tag{20}$$

Initial conditions: $u(0) = I$ and $v(0) = 0$.

The Forward Euler scheme

$$
\begin{aligned}
[D_t^+ u &= v]^n, \\
[D_t^+ v &= -\omega^2 u]^n,
\end{aligned}
$$

or written out,

$$
u^{n+1} = u^n + \Delta t v^n, \tag{21}
$$

$$
v^{n+1} = v^n - \Delta t \omega^2 u^n \,. \tag{22}
$$

The Backward Euler scheme

$$
\begin{aligned}
[D_t^- u = v]^{n+1}, \qquad (23)\\
[D_t^- v = -\omega u]^{n+1}\,. \qquad (24)
\end{aligned}
$$

Written out:

$$
\begin{aligned}
u^{n+1} - \Delta t v^{n+1} = u^n, \qquad (25)\\
v^{n+1} + \Delta t \omega^2 u^{n+1} = v^n\,. \qquad (26)
\end{aligned}
$$

This is a *coupled* 2×2 system for the new values at $t = t_{n+1}$!

The Crank-Nicolson scheme

$$[D_t u = \overline{v}^t]^{n+\frac{1}{2}}, \tag{27}$$

$$[D_t v = -\omega \overline{u}^t]^{n+\frac{1}{2}} \,. \tag{28}$$

Also a coupled system:

$$u^{n+1} - \frac{1}{2}\Delta t v^{n+1} = u^n + \frac{1}{2}\Delta t v^n, \tag{29}$$

$$v^{n+1} + \frac{1}{2}\Delta t\omega^2 u^{n+1} = v^n - \frac{1}{2}\Delta t\omega^2 u^n \,. \tag{30}$$

Comparison of schemes via Odespy

Can use Odespy to compare many methods for first-order schemes:

```
import odespy
import numpy as np

def f(u, t, w=1):
    # u is array of length 2 holding our [u, v]
    u, v = u
    return [v, -w**2*u]

def run_solvers_and_plot(solvers, timesteps_per_period=20,
                         num_periods=1, I=1, w=2*np.pi):
    P = 2*np.pi/w  # one period
    dt = P/timesteps_per_period
    Nt = num_periods*timesteps_per_period
    T = Nt*dt
    t_mesh = np.linspace(0, T, Nt+1)

    legends = []
    for solver in solvers:
        solver.set(f_kwargs={'w': w})
        solver.set_initial_condition([I, 0])
        u, t = solver.solve(t_mesh)
```

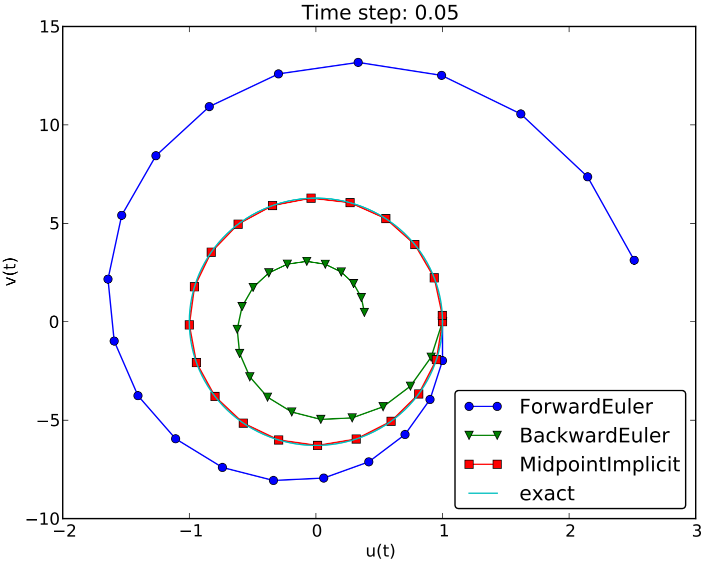
Forward and Backward Euler and Crank-Nicolson

```
solvers = [
    odespy.ForwardEuler(f),
    # Implicit methods must use Newton solver to converge
    odespy.BackwardEuler(f, nonlinear_solver='Newton'),
    odespy.CrankNicolson(f, nonlinear_solver='Newton'),
    ]
```

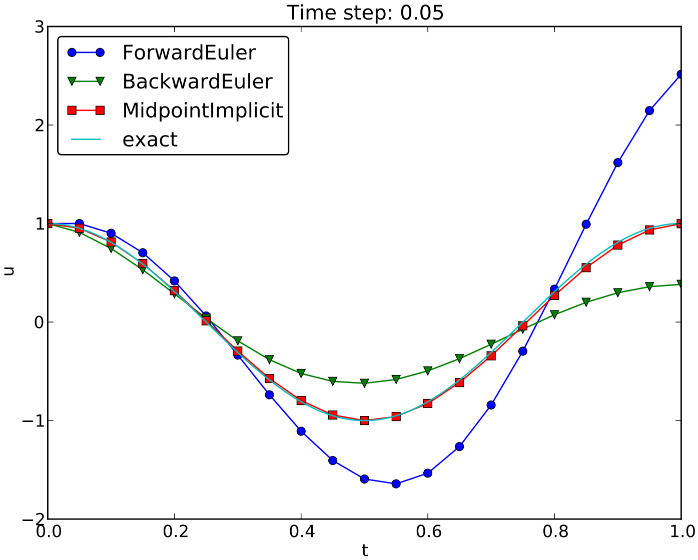
Two plot types:

- $u(t)$ vs t
- Parameterized curve $(u(t), v(t))$ in *phase space* (exact: $(I\cos\omega t, -\omega I\sin\omega t)$ = ellipse)
- Notice: The curve $(u(t), v(t))$ is closed and periodic ($P = 2\pi/\omega$)

Phase plane plot of the numerical solutions



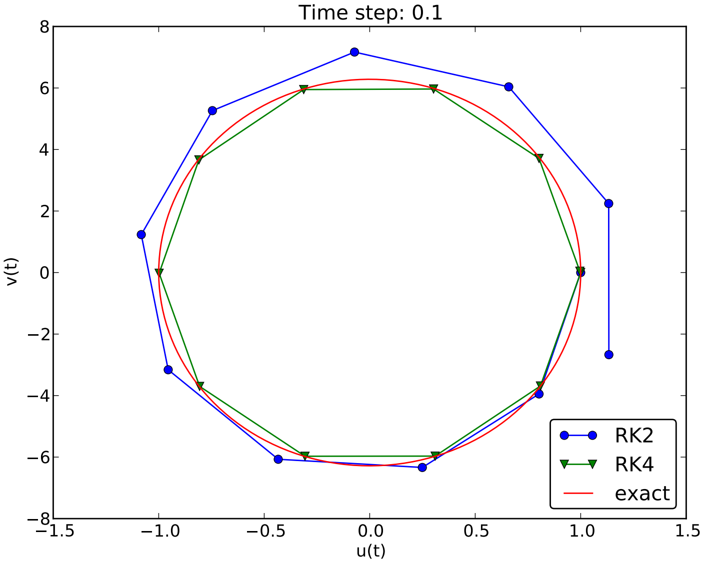
Plain solution curves



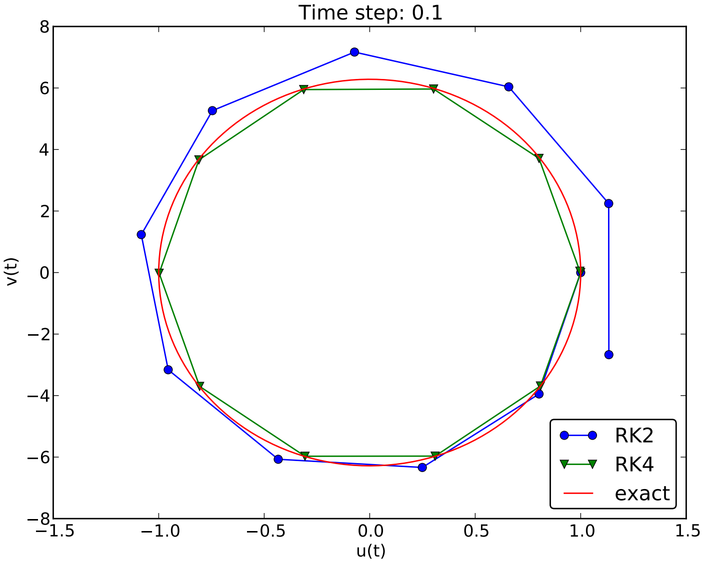
Observations from the figures

- Forward Euler has growing amplitude and outward (u, v) spiral - pumps energy into the system.
- Backward Euler is opposite: decreasing amplitude, inward sprial, extracts energy.
- **Forward and Backward Euler are useless for vibrations.**
- Crank-Nicolson (MidpointImplicit) looks much better.

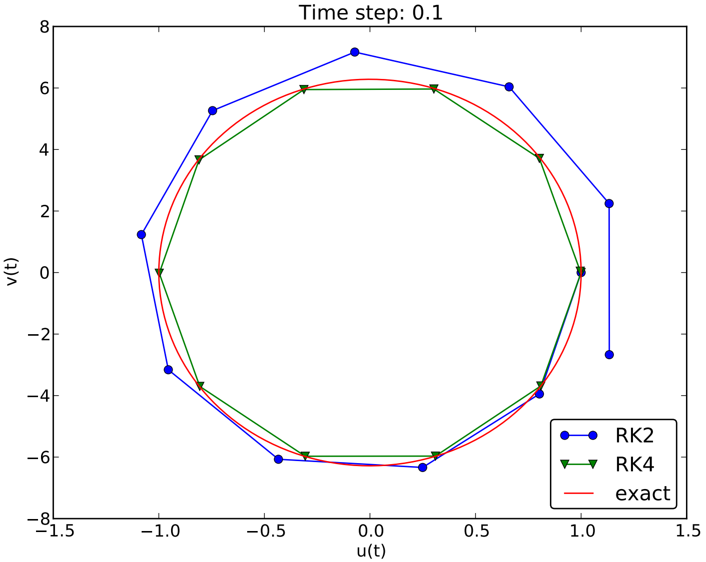
Runge-Kutta methods of order 2 and 4; short time series



Runge-Kutta methods of order 2 and 4; longer time series



Crank-Nicolson; longer time series



Observations of RK and CN methods

- 4th-order Runge-Kutta is very accurate, also for large Δt.
- 2th-order Runge-Kutta is almost as bad as Forward and Backward Euler.
- Crank-Nicolson is accurate, but the amplitude is not as accurate as the difference scheme for $u'' + \omega^2 u = 0$.

The Euler-Cromer method

Forward-backward discretization:

- Update u with Forward Euler
- Update v with Backward Euler, using latest u

$$[D_t^+ u = v]^n, \tag{31}$$

$$[D_t^- v = -\omega u]^{n+1}\,. \tag{32}$$

Written out:

$$u^{n+1} = u^n + \Delta t v^n, \tag{33}$$

$$v^{n+1} = v^n - \Delta t \omega^2 u^{n+1}\,. \tag{34}$$

Names: Forward-backward scheme, Semi-implicit Euler method, symplectic Euler, semi-explicit Euler, Newton-Stormer-Verlet, and Euler-Cromer.
Forward Euler and Backward Euler are both $\mathcal{O}(\Delta t))$ approximations. What about the overall scheme? Expect $\mathcal{O}(\Delta t)$...

Equivalence with the scheme for the second-order ODE

Eliminate v^n:

From (**??**):

$$v^n = v^{n-1} - \Delta t \omega^2 u^n,$$

which can be inserted in (**??**) to yield

$$u^{n+1} = u^n + \Delta t v^{n-1} - \Delta t^2 \omega^2 u^n. \quad (35)$$

Using (**??**),

$$v^{n-1} = \frac{u^n - u^{n-1}}{\Delta t},$$

and when this is inserted in (**??**) we get

$$u^{n+1} = 2u^n - u^{n-1} - \Delta t^2 \omega^2 u^n, \quad (36)$$

which is nothing but the centered scheme (??)! The previous analysis of this scheme then also applies to the Euler-Cromer method!

The initial condition $u' = 0$:

$$u' = v = 0 \quad \Rightarrow \quad v^0 = 0,$$

and (**??**) implies $u^1 = u^0$, while (**??**) says $v^1 = -\omega^2 u^0$