

# INF5620 Lecture: Analysis of finite difference schemes for diffusion processes

Hans Petter Langtangen<sup>1,2</sup>

<sup>1</sup>Center for Biomedical Computing, Simula Research Laboratory

<sup>2</sup>Department of Informatics, University of Oslo

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## 1 Analysis of schemes for the diffusion equation

### 1.1 Properties of the solution

The PDE

$$u_t = \alpha u_{xx} \tag{1}$$

admits solutions

$$u(x, t) = Q e^{-\alpha k^2 t} \sin(kx) \tag{2}$$

Observations from this solution:

- The initial shape  $I(x) = Q \sin kx$  undergoes a damping  $\exp(-\alpha k^2 t)$
- The damping is very strong for short waves (large  $k$ )
- The damping is weak for long waves (small  $k$ )
- Consequence:  $u$  is smoothened with time

## 1.2 Example

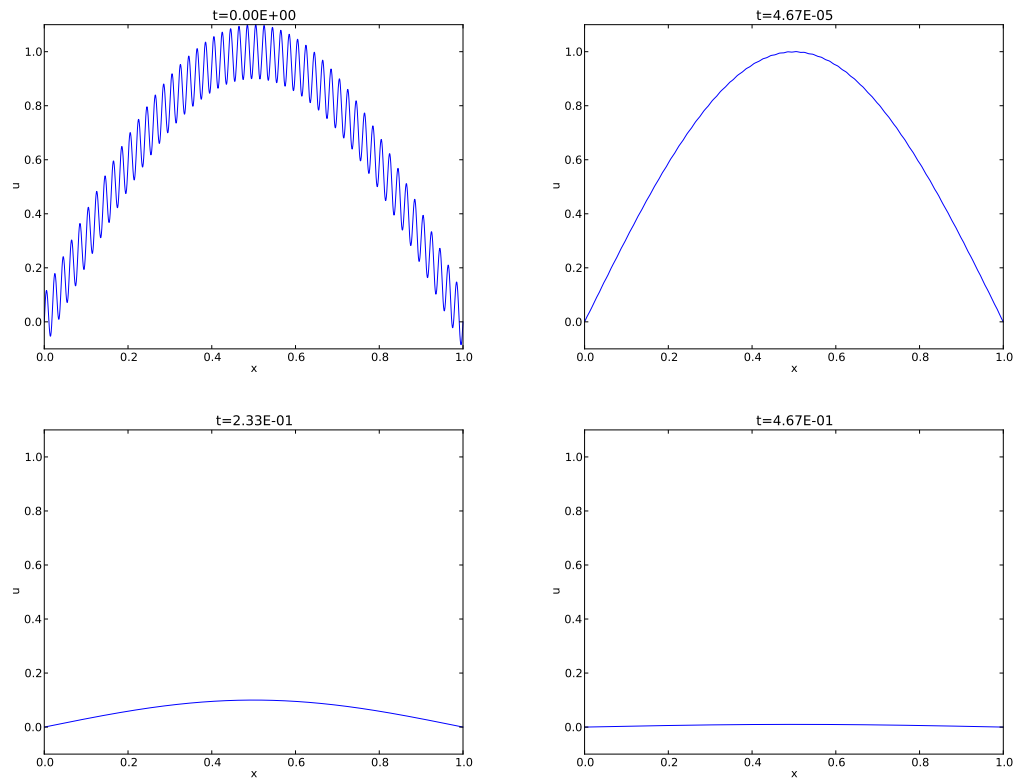
Test problem:

$$\begin{aligned} u_t &= u_{xx}, & x &\in (0, 1), \ t \in (0, T] \\ u(0, t) &= u(1, t) = 0, & t &\in (0, T] \\ u(x, 0) &= \sin(\pi x) + 0.1 \sin(100\pi x) \end{aligned}$$

Exact solution:

$$u(x, t) = e^{-\pi^2 t} \sin(\pi x) + 0.1 e^{-\pi^2 10^4 t} \sin(100\pi x) \tag{3}$$

### 1.3 Visualization of the damping in the diffusion equation



### 1.4 Damping of a discontinuity; problem and model

#### Problem.

Two pieces of a material, at different temperatures, are brought in contact at  $t = 0$ . Assume the end points of the pieces are kept at the initial temperature. How does the heat flow from the hot to the cold piece?

#### Solution.

Assume a 1D model is sufficient (insulated rod):

$$u(x, 0) = \begin{cases} U_L, & x < L/2 \\ U_R, & x \geq L/2 \end{cases}$$

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2}, \quad u(0, t) = U_L, \quad u(L, t) = U_R$$

## 1.5 Damping of a discontinuity; Backward Euler simulation

Movie<sup>1</sup>

## 1.6 Damping of a discontinuity; Forward Euler simulation

Movie<sup>2</sup>

## 1.7 Damping of a discontinuity; Crank-Nicolson simulation

Movie<sup>3</sup>

## 1.8 Fourier representation

Represent  $I(x)$  as a Fourier series

$$I(x) \approx \sum_{k \in K} b_k e^{ikx} \quad (4)$$

The corresponding sum for  $u$  is

$$u(x, t) \approx \sum_{k \in K} b_k e^{-\alpha k^2 t} e^{ikx}. \quad (5)$$

Such solutions are also accepted by the numerical schemes, but with an amplification factor  $A$  different from  $\exp(-\alpha k^2 t)$ :

$$u_q^n = A^n e^{ikq\Delta x} = A^n e^{ikx} \quad (6)$$

## 1.9 Analysis of the finite difference schemes

Stability:

- $|A| < 1$ : decaying numerical solutions (as we want)
- $A < 0$ : *oscillating* numerical solutions (as we do not want)

Accuracy:

- Compare numerical and exact amplification factor:  $A$  vs  $A_e = \exp(-\alpha k^2 \Delta t)$

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<sup>1</sup>[http://tinyurl.com/k3sdbuv/pub/mov-diffu/BE\\_C0.5/index.html](http://tinyurl.com/k3sdbuv/pub/mov-diffu/BE_C0.5/index.html)

<sup>2</sup>[http://tinyurl.com/k3sdbuv/pub/mov-diffu/FE\\_C0.5/index.html](http://tinyurl.com/k3sdbuv/pub/mov-diffu/FE_C0.5/index.html)

<sup>3</sup>[http://tinyurl.com/k3sdbuv/pub/mov-diffu/CN\\_C5/index.html](http://tinyurl.com/k3sdbuv/pub/mov-diffu/CN_C5/index.html)

### 1.10 Analysis of the Forward Euler scheme

$$[D_t^+ u = \alpha D_x D_x u]_q^n$$

Inserting

$$u_q^n = A^n e^{ikq\Delta x}$$

leads to

$$A = 1 - 4C \sin^2 \left( \frac{k\Delta x}{2} \right), \quad C = \frac{\alpha \Delta t}{\Delta x^2} \quad (7)$$

The complete numerical solution is

$$u_q^n = (1 - 4C \sin^2 p)^n e^{ikq\Delta x}, \quad p = k\Delta x/2 \quad (8)$$

### 1.11 Results for stability

We always have  $A \leq 1$ . The condition  $A \geq -1$  implies

$$4C \sin^2 p \leq 2$$

The worst case is when  $\sin^2 p = 1$ , so a sufficient criterion for stability is

$$C \leq \frac{1}{2} \quad (9)$$

or:

$$\Delta t \leq \frac{\Delta x^2}{2\alpha} \quad (10)$$

#### Implications of the stability result.

Less favorable criterion than for  $u_{tt} = c^2 u_{xx}$ : halving  $\Delta x$  implies time step  $\frac{1}{4}\Delta t$  (not just  $\frac{1}{2}\Delta t$  as in a wave equation). Need very small time steps for fine spatial meshes!

### 1.12 Analysis of the Backward Euler scheme

$$[D_t^- u = \alpha D_x D_x u]_q^n$$

$$u_q^n = A^n e^{ikq\Delta x}$$

$$A = (1 + 4C \sin^2 p)^{-1} \quad (11)$$

$$u_q^n = (1 + 4C \sin^2 p)^{-n} e^{ikq\Delta x} \quad (12)$$

### 1.13 Stability

We see from (11) that  $|A| < 1$  for all  $\Delta t > 0$  and that  $A > 0$  (no oscillations).

### 1.14 Analysis of the Crank-Nicolson scheme

The scheme

$$[D_t u = \alpha D_x D_x \bar{u}^x]_q^{n+\frac{1}{2}}$$

leads to

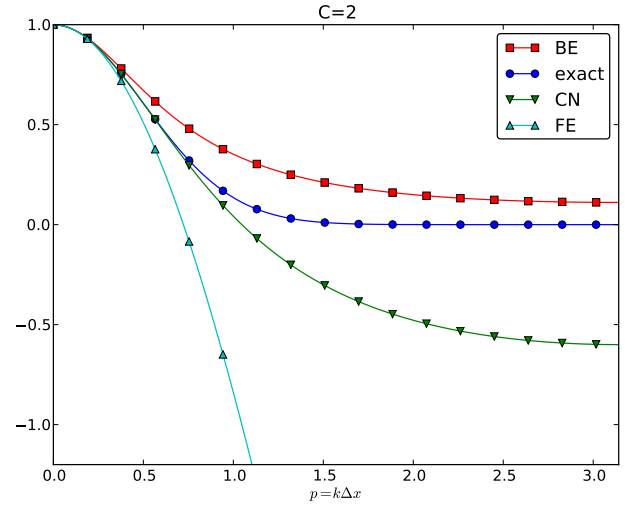
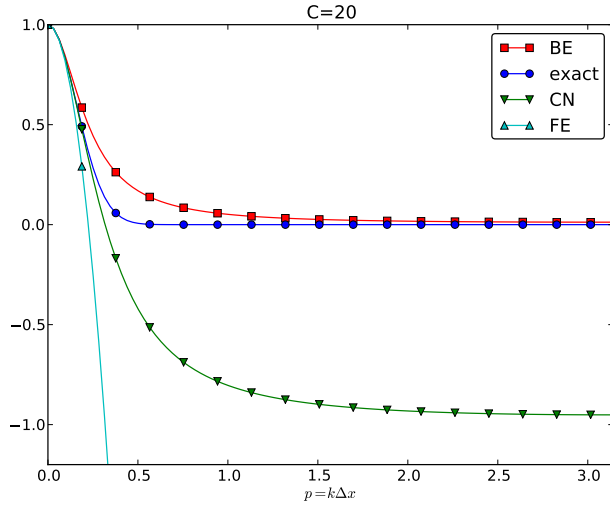
$$A = \frac{1 - 2C \sin^2 p}{1 + 2C \sin^2 p} \quad (13)$$

$$u_q^n = \left( \frac{1 - 2C \sin^2 p}{1 + 2C \sin^2 p} \right)^n e^{ikp\Delta x} \quad (14)$$

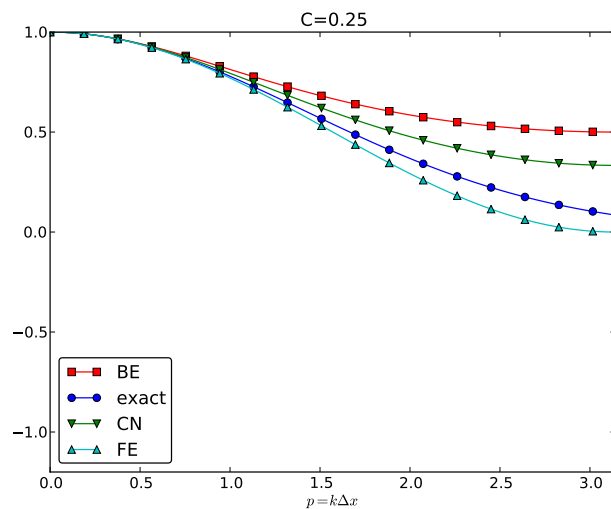
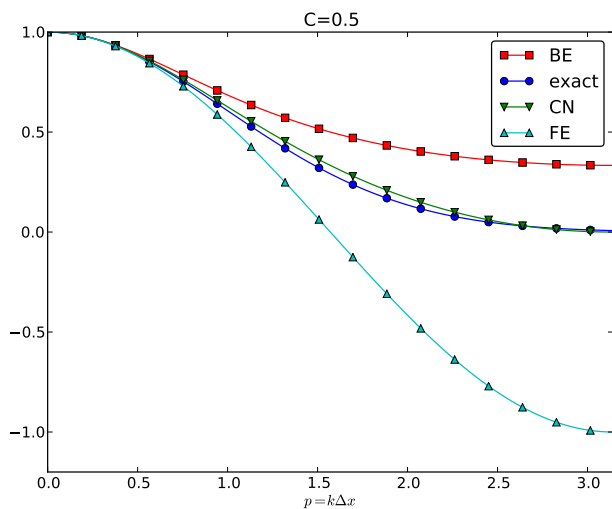
### 1.15 Stability

The criteria  $A > -1$  and  $A < 1$  are fulfilled for any  $\Delta t > 0$ .

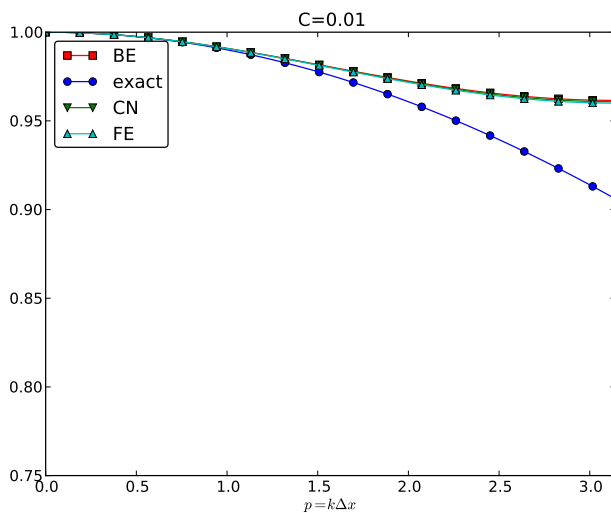
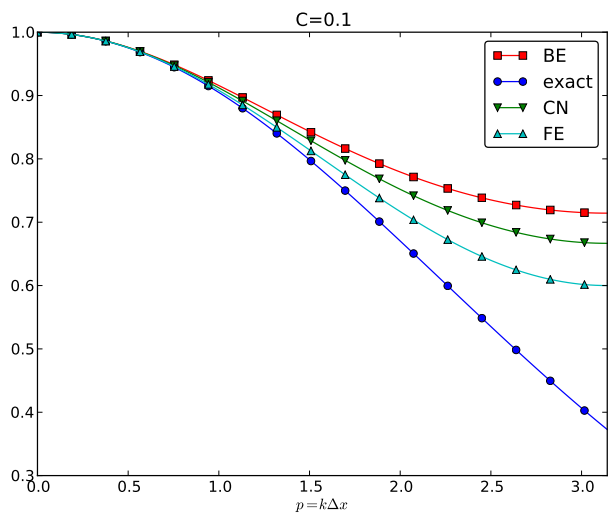
### 1.16 Summary of accuracy of amplification factors; large time steps



### 1.17 Summary of accuracy of amplification factors; time steps around the Forward Euler stability limit



### 1.18 Summary of accuracy of amplification factors; small time steps



### 1.19 Observations

- Crank-Nicolson gives oscillations and not much damping of short waves for increasing  $C$ .
- These waves will manifest themselves as high frequency oscillatory noise in the solution.
- All schemes fail to dampen short waves enough

The problems of correct damping for  $u_t = u_{xx}$  is partially manifested in the similar time discretization schemes for  $u'(t) = -\alpha u(t)$ .