S.E. Theory: SOFTENG211 Assignment #2

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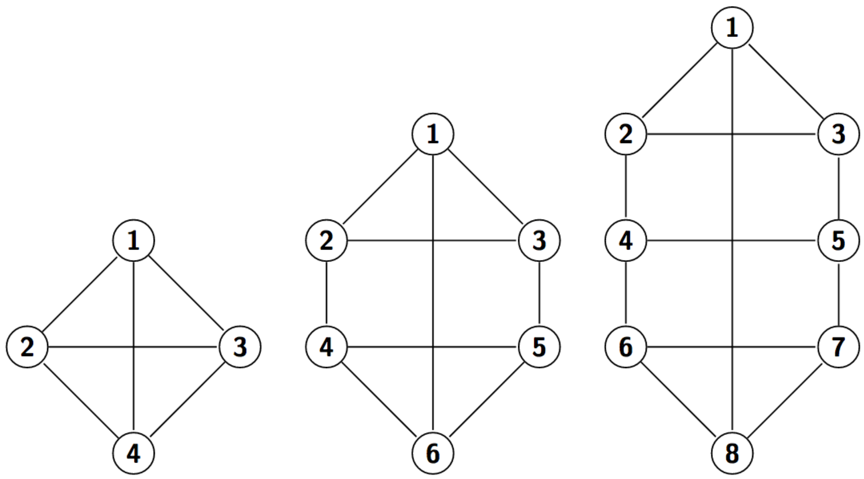
Problem 1

Question:

We say that a graph is k**-regular**, where k is a positive integer, if every vertex v in the graph has exactly k neighbors. For every positive *even* integer $n > 2$ construct a 3-regular graph with n vertices. (*Hint:* First, construct 3-regular graphs with 4, 6, and 8 vertices)

Answer:

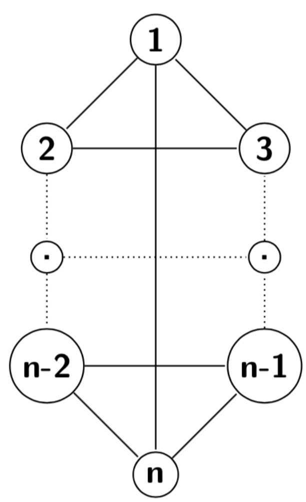
Consider 3-regular graphs for $4, 6, 8$ vertices:



From a recurring pattern, a 3 regular graph with n (where n is even and $n > 2$) vertices can be constructed as such:

Consider a graph $G = (V, E)$ where $V = \{1, 2, ...n\}$ and $E = \{(x, y) \in V \mid |x-y| = 2$ or $|x-y| = 1$ and $(1, n)\}$. Then G is 3 regular for any even $n > 2$.

This can be graphically represented below.



Problem 2

Question:

For each of the following statements, decide whether it is true and give a proof.

1. For any sets A and B, if $(A \cup B) \setminus (A \cap B) = \varnothing$ then $A = B$.
2. For any sets A and B, $A \times (B \cap C) = (A \times B) \cap (A \times C)$
3. For any sets A and B, $\mathcal{P}(A \cup B) = \mathcal{P}(A) \cup \mathcal{P}(B)$

Answer:

1. This statement is true.

   Proof. Suppose for the sake of contradiction that $A \neq B$, this means that these two sets are distinct...

   Consider the following 2 cases.

   - Case 1: Suppose that there exists an element $x \in A$ and $x \notin B$. If we consider the hypothesis: $(A \cup B) \setminus (A \cap B)$ then by this $x \in A \cup B$ but $x \notin A \cap B$ so $x \in (A \cup B) \setminus (A \cap B)$ which contradicts the fact that $(A \cup B) \setminus (A \cap B) = \varnothing$ since x is an element of $(A \cup B) \setminus (A \cap B)$.
   - Case 2: Suppose that there exists an element $x \notin A$ and $x \in B$. If we consider the hypothesis: $(A \cup B) \setminus (A \cap B)$ then by this $x \in A \cup B$ but $x \notin A \cap B$ so $x \in (A \cup B) \setminus (A \cap B)$ which contradicts the fact that $(A \cup B) \setminus (A \cap B) = \varnothing$ since x is an element of $(A \cup B) \setminus (A \cap B)$.

   In both cases we reach a contradiction therefore $A = B$. □

2. This statement is true.

   Proof. Required to prove: $A \times (B \cap C) \subseteq (A \times B) \cap (A \times C)$ and $(A \times B) \cap (A \times C) \subseteq A \times (B \cap C)$

$$
\begin{aligned}
&\text{Suppose that } (x, y) \in A \times (B \cap C) \\
&\Leftrightarrow (x \in A) \text{ and } (y \in (B \cap C)) \\
&\Leftrightarrow (x \in A) \text{ and } (y \in B \text{ and } y \in C) \\
&\Leftrightarrow [(x \in A \text{ and } y \in B)] \text{ and } [(x \in A \text{ and } y \in C)] \quad \text{Because } y \in B,\ (x, y) \in A \times B \\
&\Leftrightarrow [(x, y) \in A \times B] \text{ and } [(x, y) \in A \times C] \quad \text{and because } y \in C,\ (x, y) \in A \times C \\
&\Leftrightarrow (x, y) \in (A \times B) \cap (A \times C)
\end{aligned}
$$

   Since we have proved the claim both ways we are done, thus $A \times (B \cap C) = (A \times B) \cap (A \times C)$ □

3. This statement is false

   Proof by Counterexample. Let $A = \{x\}$ and $B = \{y\}$

   Note that $A \cup B = \{x, y\}$ and thus $\mathcal{P}(A \cup B) = \{\varnothing, \{x\}, \{y\}, \{x, y\}\}$

   Also note that $\mathcal{P}(A) = \{\varnothing, x\}$ and $\mathcal{P}(B) = \{\varnothing, y\}$ and thus $\mathcal{P}(A) \cup \mathcal{P}(B) = \{\varnothing, x, y\}$

   Thus $\mathcal{P}(A \cup B) \neq \mathcal{P}(A) \cup \mathcal{P}(B)$ □

Problem 3

Question:

Define a binary relation $\sim$ on the set of integers $\mathbb{Z}$ by

$$x \sim y \text{ if and only if } xy \equiv 1 \pmod 3$$

Decide if the relation $\sim$ is an equivalence relation. Prove your claim.

Answer:

$\sim$ is not an equivalence relation. This is because an equivalence relation must satisfy the properties: Reflexive, Symmetric, Transitive.

However since $\sim$ is defined on the set of integers $\mathbb{Z}$, we have that $\sim$ is not *reflexive* because $(3, 3)$ is not in the relation.

$(3, 3)$ is not in the binary relation since $xy \equiv 1 \pmod 3$ then $3 \cdot 3 = 9 \equiv 0 \pmod 3$ thus 3 is not related to itself.

From this example we can further conclude that numbers $(3k, 3k)$ for any integer k cannot be in the binary relation since $3k \cdot 3k = 9k^2 \equiv 0 \pmod 3$ since $3|9$ and k^2 is an integer.

Since we have shown $\sim$ is not reflexive, this is enough to conclude that $\sim$ is not an equivalence relation. $\square$

Problem 4

Question:

Define a binary relation R on the set $\mathbb{Q}$ of rational numbers by

$$x\,R\,y \text{ if and only if } y = 2^k \cdot x \text{ for some integer } k.$$

Explain why R is an equivalence relation. Give one equivalence class for the relation R that has cardinality 1.

Answer:

We define xRy iff $y = 2^k \cdot x$ for $k \in \mathbb{Z}$ and $(x, y) \in \mathbb{Q}$

For R to be an equivalence relation it must satisfy 3 key properties:

- **Reflexive**
  Required to prove: For all $x \in \mathbb{Q}$, xRx

  Proof. For all $x \in \mathbb{Q}$ we have that $x = 2^k \cdot x$ which is true for $k = 0$ hence xRx will always be satisfied. □

- **Symmetric**
  Required to prove: For all $x, y \in \mathbb{Q}$, if xRy then yRx.

  Proof. If xRy then $y = 2^k \cdot x$ for $k \in \mathbb{Z}$.

$$\begin{aligned} y &= 2^k \cdot x \\ 2^{-k} \cdot y &= 2^{-k} \cdot 2^k \cdot x \\ 2^{-k} \cdot y &= x \\ x &= 2^{-k} \cdot y \\ x &= 2^m \cdot y \quad \text{Since } -k \text{ is just an integer, let } -k \text{ be } m \text{ for some } m \in \mathbb{Z} \end{aligned}$$

  This means that yRx so we are done. □

- **Transitive**
  Required to prove: For all $x, y, z \in \mathbb{Q}$, if xRy and yRz then xRz.

  Proof. If xRy then $y = 2^k \cdot x$ for some $k \in \mathbb{Z}$
  If yRz then $z = 2^m \cdot y$ for some $m \in \mathbb{Z}$

  Then by using the first relation we have that $z = 2^m \cdot y = 2^m \cdot (2^k \cdot x) = 2^{m+k} \cdot x = 2^n \cdot x$

  Since n is an integer we have that xRz so we are done. □

Consider the equivalence class of 0: $[0] = \{0\}$ which clearly has a cardinality of 1.

Problem 5

Question:

For each of the following binary relations on a set S, determine which properties the relation satisfies by ticking the correct boxes below.

The relations are:

(i) $S = \{a, b\}$, $R_1 = \{(a, a), (b, b)\}$ where a, b are distinct.

(ii) $S = \{a, b, c\}$, $R_2 = \{(a, b), (b, c), (a, c)\}$ where a, b, c are distinct

(iii) $S = \mathbb{N}$, $R_3 = \{(x, x^2) \mid x \in S\} \cup \{(x^2, x) \mid x \in S\}$.

Answer:

Relations	Reflexive	Symmetric	Anti-symmetric	Transitive
R_1	✓	✓	✓	✓
R_2			✓	✓
R_3		✓		

Problem 6

Question:

Prove that for any set S, the subset relation $\subseteq$ defined on the power set $P(S)$ of S is a partial order.

Answer:

Let S be a set and $\mathcal{P}(S)$ be the power set of S

We are required to prove the set $(\mathcal{P}(\mathcal{S}), \subseteq)$ is of partial order.

- **Reflexive**
  Required to prove: For all $X \in P(S)$, $X \subseteq X$

  Proof. Since we know that a set is always a subset of itself, For all $X \in P(S)$ we have that $X \subseteq X$.

  This implies the reflexive property for the subset relation. □

- **Antisymmetric**
  Required to prove: For all $X, Y \in P(S)$, if $X \subseteq Y$ and $Y \subseteq X$ then $X = Y$.

  Proof. By the definition of set equality (shown in class) we have that if $X \subseteq Y$ and $Y \subseteq X$ then it must be the case $X = Y$.

  This implies the antisymmetry property for the subset relation. □

- **Transitive**
  Required to prove: For all $X, Y, Z \in P(S)$, if $X \subseteq Y$ and $Y \subseteq Z$ then $X \subseteq Z$.

  Proof. Suppose that $X \subseteq Y$ and $Y \subseteq Z$.

  Suppose there exists an $a \in X$ then $a \in Y$ because $X \subseteq Y$ and since $a \in Y$ we have that $a \in Z$ because $Y \subseteq Z$.

  Since a is arbitrary, the implication $\forall a,\ a \in X \implies a \in Z$ is true for all a. Hence by the definition of subset we have $X \subseteq Z$

  This implies the transitive property for the subset relation. □

Problem 7

Question:

Consider the set $A = \{1, 2, 3\}$. Let $B = \{2^x \mid x \in A\}$. Draw the Hasse diagram of the partially ordered set $(P(A) \cup P(B), \subseteq)$.

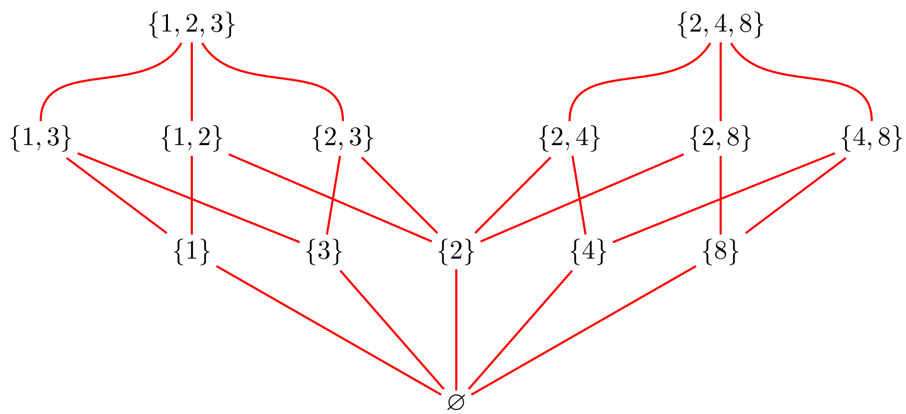
Answer:

Since $A = \{1, 2, 3\}$ we have that $\mathcal{P}(A) = \{\varnothing, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$

and for $B = \{2^x \mid x \in A\} = \{2, 4, 8\}$ we have that $\mathcal{P}(B) = \{\varnothing, \{2\}, \{4\}, \{8\}, \{2, 4\}, \{2, 8\}, \{4, 8\}, \{2, 4, 8\}\}$.

$\therefore \mathcal{P}(A) \cup \mathcal{P}(B) = \{\varnothing, \{1\}, \{2\}, \{3\}, \{4\}, \{8\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{2, 4\}, \{2, 8\}, \{4, 8\}, \{1, 2, 3\}, \{2, 4, 8\}\}$

Hasse Diagram

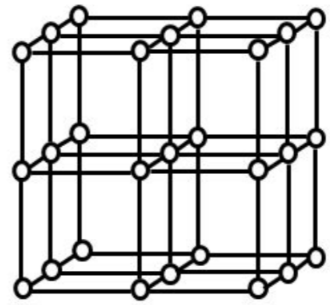


Problem 8

Question:

A block of cheese is made up of $3 \times 3 \times 3$ cubes as in the figure below. Is it possible for a mouse to tunnel its way through this block of cheese by (a) starting at a corner, (b) eating its way from cube to adjacent cube, (c) never passing through any cube twice, and, finally, (d) finishing at the center cube? Prove your answer.

The block of cheese



The corresponding graph

Hint. This question asks you to find a Hamiltonian path that starts from one of the corner vertices and ends at the center vertex (the only vertex with degree 6). Does such a Hamiltonian path exist?

Solution

There does not exist such a Hamiltonian path.

Proof. Since the block of cheese is a $3 \times 3 \times 3$ cube, we know that there must be 27 vertices (with coordinates (x, y, z)).

WLOG, suppose our corner vertex where we start is $(0, 0, 0)$, Let the coordinate of the center vertex be $(1, 1, 1)$. Consider the sum of coordinates (parity $= x + y + z$) for the corner vertex and center vertex: The corner vertex has a even parity and the center vertex has an odd parity.

Everytime we traverse along an edge to a vertex, only *one* of our coordinates gets shifted along by one (either x, y or z) which means the parity changes from even to odd to even to odd and so forth.

One can observe that a Hamiltonian path that travels along 27 vertices must have 26 edges. Therefore since 26 is even we will have that the parity of the final vertex will be the same as the starting vertex (even).

However we know that the parity of $(1, 1, 1)$ the center vertex is odd hence there is no Hamiltonian path that starts from one of the corner vertices and ends at a center vertex. □