Running Time

Problem 1

Consider Euclidean algorithm. The input integers n and m are given in their decimal representations.

Part A

What is the size of the input for the algorithm?

Solution:

The initial size of the input of the algorithm would be the sum of the number of digits of m and n.

Or mathematically we could say Input $= (\lfloor \log(m) \rfloor + 1) + (\lfloor \log(n) \rfloor + 1)$

Part B

Explain why the running time of the algorithm is linear on the size of the input. For the analysis, assume that the division process takes a constant amount of time.

Solution:

Let's assume the worst case scenario: Where $n \neq m$ and $n > m$

After two iterations of the algorithm, the input changes as such:

$$\gcd(n, m) \rightarrow \gcd(m, n \bmod m) \rightarrow \gcd(n \bmod m, m \bmod (n \bmod m))$$

Also note that our first parameter changes from $n \rightarrow n \bmod m$ after 2 iterations.

Let's look at a single iteration of our algorithm:

Case 1: If $\left(m < \frac{n}{2}\right)$, then $n \bmod m < m$ which is less than $\frac{n}{2}$.

Case 2: If $\left(m > \frac{n}{2}\right)$, then $n \bmod m = n - m$ which is less than $\frac{n}{2}$.

In general, this means after any two consecutive iterations, both of the input arguments n and m must at least be halved in value.

This shows a logarithmic decrease in input size $(\lfloor \log(m) \rfloor + 1) + (\lfloor \log(n) \rfloor + 1)$ and the algorithm runs relative to the input size. Therefore this proves a linear upper bound of $\mathcal{O}(n)$, where n here represent the total input size.

Problem 2

Let p be a polynomial with positive coefficients. Show that p is $\mathcal{O}(n^{\log(n)})$.

Solution:

Suppose $p = a_0 + a_1 n + a_2 n^2 + ... + a_i n^i$ then consider $\frac{p}{n^{\log(n)}}$ if we can show that $\lim_{n\to\infty} \frac{p}{n^{\log(n)}} = 0$ then we are done so let's try to prove that.

$$
\begin{aligned}
L &= \lim_{n\to\infty} \frac{p}{n^{\log n}} \\
L &= \lim_{n\to\infty} \frac{a_0 + a_1 n + a_2 n^2 + ... + a_i n^i}{n^{\log n}} \\
L &= \lim_{n\to\infty} \frac{a_0}{n^{\log n}} + \frac{a_1}{n^{\log n - 1}} + \frac{a_2}{n^{\log n - 2}} + ... + \frac{a_i}{n^{\log n - i}}
\end{aligned}
$$

Now since all fractions have constants in the numerator and $n^{\text{log(n) - i}}$ in the denominator if we take the limit as $n \to \infty$ (for a sufficient $\log(n) > i$ or $n > 10^i$), we will always have the case that the denominator is larger than the numerator and this will grow much faster, so each fraction will tend to 0 and therefore $L = 0$. And by the definition of Big-Oh we have that: $p = \mathcal{O}(n^{\log(n)})$.

Problem 3

Show that $\sum_{i=1}^{n} i^2 = \Theta(n^3)$

Solution:

If we can prove that $\lim_{n\to\infty} \frac{\sum\limits_{i=1}^{n} i^2}{n^3} = C$ where $C > 0$ then we are done since that is the definition of Big-Theta.

Since it is a known fact that $\sum\limits_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} = \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}$ so now we try prove the limit:

$$
\begin{aligned}
L &= \lim_{n\to\infty} \frac{\sum\limits_{i=1}^{n} i^2}{n^3} \\
L &= \lim_{n\to\infty} \frac{\frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}}{n^3} \\
L &= \lim_{n\to\infty} \frac{1}{3} + \frac{1}{2n} + \frac{1}{6n^2} \\
L &= \frac{1}{3}
\end{aligned}
$$

Since $L = \frac{1}{3}$ which is a non-negative constant we are done. Therefore $\sum\limits_{i=1}^{n} i^2 = \Theta(n^3)$.

Problem 4

Show that $n \log(n) = \Omega(\log(n!))$.

Solution:

To prove $n \log(n) = \Omega(\log(n!))$ we can show that $n \log(n) \geq c \log(n!)$ or we can equivalently prove $n \log(n) \geq \log(n!)$.

However since $\log(x)$ is an increasing function we can equivalently show that $\log(n^n) \geq \log(n!)$ or $n^n \geq n!$.

Therefore we can prove this by induction (for $n > 0$):

Base Case: $n^n \geq n! \Leftrightarrow 1^1 \geq 1! \Leftrightarrow 1 \geq 1 \quad \checkmark$

Induction Case: Assume $P(k)$ is true: $k^k \geq k!$

Show $P(k+1)$ is true: $(k+1)^{k+1} \geq (k+1)!$

$$\begin{aligned}
(k+1)! &= (k+1) \cdot (k) \cdot (k-1) ... (3) \cdot (2) \cdot (1) \\
&= (k+1) \cdot (k!) \\
&\leq (k+1) \cdot k^k \quad \text{By hypothesis} \\
&\leq (k+1) \cdot (k+1)^k \quad \text{Since } k+1 > k \\
&= (k+1)^{k+1}
\end{aligned}$$

Therefore we can conclude by induction that: $n \log(n) \geq c \log(n!)$ is true for $n > 0$ and by definition of Big Omega we are done.

Problem 5

Show that $a^n = \mathcal{O}(n!)$ for any positive integer $a > 1$.

Solution

To prove that $a^n = \mathcal{O}(n!)$ we must have that $a^n \leq c \cdot n!$ so if we can prove that $n!$ grows faster than a^n then we are done.

We can first consider the series $S = \sum_{n=0}^{\infty} \frac{a^n}{n!}$ if this series converges to 0 we are done.

Now consider the ratio test for series: Consider $t_n = \frac{a^n}{n!}$ then

$$\begin{aligned}
\frac{t_{n+1}}{t_n} &= \frac{a^{n+1}}{(n+1)!} \cdot \frac{n!}{a^n} = \frac{a}{n+1} \\
\therefore \lim_{n \to \infty} \frac{a}{n+1} &= 0
\end{aligned}$$

Hence we can conclude $n!$ grows much faster than a^n and because of this we can conclude that after a certain value the function $n!$ will surpass a^n and therefore $a^n = \mathcal{O}(n!)$.

Problem 6

Show that $\sum\limits_{i=1}^{n} \log\left(\frac{n}{i}\right) = \Theta(n)$

If we can prove that $\lim_{n\to\infty} \frac{\sum\limits_{i=1}^{n} \log\left(\frac{n}{i}\right)}{n} = c$ where $c > 0$ then we are done since that is the definition of big-theta. So let's attempt to do that.

$$
\begin{aligned}
\sum_{i=1}^{n} \log\left(\frac{n}{i}\right) &= \sum_{i=1}^{n} \log(n) - \log(i) \\
\sum_{i=1}^{n} \log\left(\frac{n}{i}\right) &= (\log(n) - \log(1)) + (\log(n) - \log(2)) + (\log(n) - \log(3)) + ... + (\log(n) - \log(n)) \\
\sum_{i=1}^{n} \log\left(\frac{n}{i}\right) &= \underbrace{\log(n) + \log(n) + \log(n) + ... + \log(n)}_{n} - (\log(1) + \log(2) + \log(3) + ... + \log(n)) \\
\sum_{i=1}^{n} \log\left(\frac{n}{i}\right) &= n\log(n) - \log(n!) \\
\sum_{i=1}^{n} \log\left(\frac{n}{i}\right) &= \log\left(\frac{n^n}{n!}\right)
\end{aligned}
$$

Now consider:

$$
\begin{aligned}
L &= \lim_{n\to\infty} \frac{\sum\limits_{i=1}^{n} \log\left(\frac{n}{i}\right)}{n} \\
L &= \lim_{n\to\infty} \frac{\log\left(\frac{n^n}{n!}\right)}{n} \\
L &= \lim_{n\to\infty} \frac{\log\left(\frac{(n+1)^{n+1}}{(n+1)!}\right) - \log\left(\frac{n^n}{n!}\right)}{(n+1)-(n)} \qquad \text{Using Stolz-Cesaro} \\
L &= \lim_{n\to\infty} \log\left(\frac{(n+1)^{n+1}}{(n+1)!}\,\frac{n!}{n^n}\right) \\
L &= \lim_{n\to\infty} \log\left(\frac{(n+1)^{n+1}}{n^n} \cdot \frac{1}{n+1}\right) \\
L &= \lim_{n\to\infty} \log\left(1 + \frac{1}{n}\right)^n \\
L &= \log(e) \quad \text{Assuming log is natural log} \\
L &= 1
\end{aligned}
$$

Since we have proved that $\lim_{n\to\infty} \frac{\sum\limits_{i=1}^{n} \log\left(\frac{n}{i}\right)}{n} = 1$ we are done due to the definition of Big-Theta.

$\therefore \sum\limits_{i=1}^{n} \log\left(\frac{n}{i}\right) = \Theta(n)$ $\square$

Problem 7

For the codes below analyse the running times in terms of Θ notation.

Algorithm 1

For $i = 1$ to n do
    $j = n - i$
    while $j \geq 0$ do
        $j = j - 3$

Analysis: Outer Loop (i) in each iteration: $1, 2, 3, ..., n-1$

Inner Loop (j) in each iteration: $\frac{n-1}{3}, \frac{n-2}{3}, \frac{n-3}{3}, ..., 1 = \left(\frac{n}{3} - \frac{1}{3}\right), \left(\frac{n}{3} - \frac{2}{3}\right), \left(\frac{n}{3} - \frac{3}{3}\right), ..., 1$

Updating j takes constant time.

Sum: $T(n) = (n)\left(\frac{n}{3} - k\right) = \frac{n^2}{3} - kn = \boxed{\Theta(n^2)}$

Algorithm 2

Set $s = 0$
    For $i = 1$ to n do
        For $j = 3 \cdot i$ to n do
            $s = s + 1$

For this algorithm we can determine the run time mathematically:

$$T(n) = \sum_{i=1}^{n} \sum_{j=3i}^{n} c = c \sum_{i=1}^{n} \sum_{j=3i}^{n} 1 = c \sum_{i=1}^{n} (n - 3i + 1) = c(n+1) \sum_{i=1}^{n} 1 - 3 \sum_{i=1}^{n} i$$
$$T(n) = c(n+1)(n) - 3\left(\frac{n(n+1)}{2}\right) = (n^2 + n) \cdot \left(c - \frac{3}{2}\right) = k(n^2 + n) = \boxed{\Theta(n^2)}$$

This is because we have 2 for loops and each i,j are incrementing by 1 and updating s takes a constant time hence the c inside the series.

Algorithm 3

For $i = 1$ to n do
    $j = i$
    while $j < n$ do
        $j = 2 \cdot j$

Analysis: Outer Loop (i) in each iteration: $1, 2, 3, ..., n$

Updating j takes constant time.

Inner Loop (j) in each iteration: $\log(n)$, $\log(n) - \log(2)$, $\log(n) - \log(3)$,..,0 which is equivalent to $\log(n)$, $\log\left(\frac{n}{2}\right)$, $\log\left(\frac{n}{3}\right), ..., 0$. So $\log(n) - \log(i)$ = start the loop from number i as when i gets larger, the number of multiplication before j reaches n will decrease logarithmically.

And since $\sum_{i=1}^{n} \log\left(\frac{n}{i}\right) = \Theta(n)$ (from problem 6) we have that $T(n) = \boxed{\Theta(n)}$.

Graphs

Problem 8

Let G be any graph. Explain why the sum of all degrees of the vertices of any graph G equals twice the number of edges of G.

Solution:

Let $e \in E(G)$ by any edge, then we can say e connects 2 vertices together. By adding another edge to the graph G, 1 degree must be added to each vertex the edge connects to (i.e. if vertex A is connected to vertex B then vertex B must be connected to vertex A) and thus the sum increases by 2. Therefore $\sum\limits_{V \in V(G)} deg(V) = 2|E(G)|$.

Problem 9

How many edges does a complete bipartite graph $K_{n,m}$ have?

Solution: A complete bipartite graph has $m \cdot n$ edges.

Problem 10

Prove that if there is a path in a graph from vertex x to vertex y and $x \neq y$ then there is a simple path from x to y. Recall that simple path is a path in which every vertex appears at most once.

Solution: If a vertex appears more than once on the path from x to y, that means that there is a cycle in that path and if there is a cycle we can remove all the nodes in that cycle.

For example consider the path: $\{x \to c_1 \to c_2 \to c_3 \to c_4 \to c_1 \to y\}$, there is a cycle in this path: $C = \{c_1 \to c_2 \to c_3 \to c_4 \to c_1\}$ since we can simply remove this cycle from our path we can thus reduce our path to a simple path which is just $\{x \to c_1 \to y\}$

If we have more cycles we can repeat this process. Since there can only be a finite amount of nodes at a certain time after we have completed removing all the cycles, we will have a path between x and y that which does not contain a cycle and hence a simple path.

Problem 11

Show that every finite connected graph G with more than 1 vertex has two vertices of the same degree.

Solution:

As the graph is connected, no vertex can have a degree of 0 and at most a vertex can have is a degree of $n-1$ which means it is connected to every other vertex.

Let's use a proof by contradiction to prove this:

If vertex 1 has a degree of 1 and vertex 2 has a degree of 2... At vertex $n-1$, it will have a degree of $(n-1)$. However, at the last vertex n, it cannot have a degree of n as vertices cannot connect to itself in an undirected graph which means vertex n has at most $n-1$ degrees. This proves that it will have the same degree as one of the previous vertices.

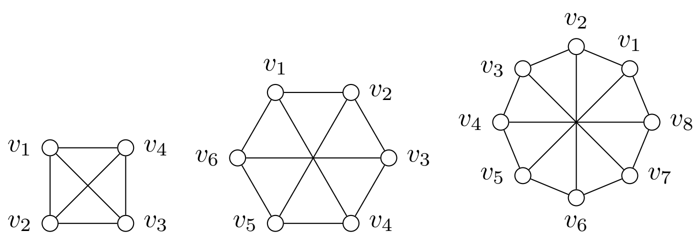
A contradiction to our initial assumption is found, therefore in graph G, at least two vertices must have the same degree.

Problem 12

We call a graph 3-regular if every vertex of the graph has degree 3. Draw 3-regular connected graphs consisting of 4 vertices, 6 vertices, and 8 vertices.

Solution:

I have drawn them below respectively:



Problem 13

Let G be a connected graph. For any two vertices u, v let $d(u, v)$ be the distance from u to v. Recall that the distance from u to v is the length of the shortest path from u to v.

Prove the following triangle inequality. For all vertices x, y and z of the graph we have $d(x, z) \leq d(x, y) + d(y, z)$.

Solution: Let's show this using a direct proof:

If you simply connect the paths from x to y to the path connecting y to z you will have a valid path of length $d(x, y) + d(y, z)$.

However $d(x, y) + d(y, z)$ might not be the *shortest* path from $x \to z$, it could have a lot of cycles and detours and since by definition $d(x, z)$ will be a shorter or equidistant path from x to z, this path will be shorter or equal than $d(x, y) + d(y, z)$ above and thus the triangle inequality will be satisfied.

$\therefore d(x, z) \leq d(x, y) + d(y, z) \quad \square$