Logic M384 Hw 3

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Problems from **?**

Problem: 1

Let Γ be any set of sentences in $\mathscr{S}$. Following the steps below, show that there is a model M with the following properties:

(a) $\mathcal{M} \models \Gamma_{all,no}$

Take $\mathcal{M} = (M, [\![\ ]\!])$ s.t

$M = \{A \subseteq P : (\forall v, w \in P, (v \in A) \wedge (\Gamma \vdash \text{All } v \text{ are } w) \implies w \in A) \wedge (\forall v, w \in A \implies \Gamma \nvdash \text{No } v \text{ are } w)\}$

$$[\![u]\!] = \{A \in M : \ u \in A\}$$

(first condition of A is that it is logically closed, like the State definition kind of).

1. Take $\Psi \in \Gamma_{all,no}$ s.t $\Psi = \text{All } x \text{ are } y$. W.t.s. $\mathcal{M} \models \Psi$.
   $\forall V \in [\![x]\!]$, by def. $x \in V$ and $\Gamma \vdash \text{All } x \text{ are } y$ is given by $\Psi \in \Gamma$, thus $y \in V$ by condition 1. Thus by def. since $\forall V \in [\![x]\!]$, $y \in V$ we have $V \in [\![y]\!]$, $\therefore [\![x]\!] \subseteq [\![y]\!]$, so $\mathcal{M} \models \Psi$.
2. Take $\Omega \in \Gamma_{all,no}$ s.t $\Omega = \text{No } x \text{ are } y$. W.t.s. $\mathcal{M} \models \Omega$.
   By def. $\forall A \in M,\ x \in A \implies y \notin A,\ y \in A \implies x \notin A$, else condition 2 will be violated. From this, $\forall G \in [\![x]\!]$, $y \notin G$ which is enough to show $[\![x]\!] \cap [\![y]\!] = \varnothing$, so $\mathcal{M} \models \Omega$.

(b) If φ is any sentence in $\mathscr{S}$(all,no), and $\mathcal{M} \models \varphi$, then $\Gamma \vdash \varphi$.

1. Take $\varphi_1 = \text{All } x \text{ are } y$. Assume $\mathcal{M} \models \varphi_1$. W.t.s $\Gamma \vdash \varphi_1$.
   Assume $\Gamma \nvdash \varphi_1$. Take $A = \{z : \Gamma \vdash \text{All } x \text{ are } z\}$, this set axiomatically contains x, and under our assumption $y \notin A$. To show $A \in M$, looking at the first condition: for $v \in A$ we have $\Gamma \vdash \text{All } x \text{ are } v$. If $\exists w \in P$ s.t $\Gamma \vdash \text{All } v \text{ are } w$ then we can apply BARBARA to get $\Gamma \vdash \text{All } x \text{ are } w$ and thus $w \in A$, satisfying condition 1. For condition 2: take $v, w \in A$, assume for contradiction that $\Gamma \vdash \text{No } v \text{ are } w$. Then applying CAMESTRES to this and $\Gamma \vdash \text{All } x \text{ are } w$, we get $\Gamma \vdash \text{No } w \text{ are } x$, but by def. of $w \in A$ we also have $\Gamma \vdash \text{All } x \text{ are } w$, an inconsistency thus $\Gamma \nvdash \text{No } v \text{ are } w$ for $v, w \in A$, satisfying condition 2. Therefore, if $\Gamma \nvdash \varphi_1$ there is some $A \in M$ s.t $x \in A,\ y \notin A$ which implies $\mathcal{M} \not\models \varphi_1$, a contradiction, thus $\forall x, y \in P,\ \mathcal{M} \models \text{All } x \text{ are } y \implies \Gamma \vdash \text{All } x \text{ are } y$
2. Take $\varphi_2 = \text{No } x \text{ are } y$. Assume $\mathcal{M} \models \varphi_2$. W.t.s. $\Gamma \vdash \varphi_2$.
   Assume $\Gamma \nvdash \varphi_2$. Take $A = \{z : \Gamma \vdash \text{All } x \text{ are } z \vee \Gamma \vdash \text{All } y \text{ are } z\}$, first this set axiomatically contains both x and y. Looking at condition 1: take $v \in A$, giving $\Gamma \vdash \text{All } x \text{ are } v$ or $\Gamma \vdash \text{All } y \text{ are } v$. If $\exists w \in A$ s.t $\Gamma \vdash \text{All } v \text{ are } w$ then by BARBARA we achieve $\Gamma \vdash \text{All } x \text{ are } w$ or $\Gamma \vdash \text{All } y \text{ are } w$ depending on which ever v satisfies by being in A, thus condition 1 holds. For condition 2, take $v, w \in A$ and assume for contradiction $\Gamma \vdash \text{No } v \text{ are } w$. We can look at 2 cases w.l.o.g.: $\Gamma \vdash \text{All } x \text{ are } v$, $\Gamma \vdash \text{All } x \text{ are } w$, or $\Gamma \vdash \text{All } x \text{ are } v$, $\Gamma \vdash \text{All } y \text{ are } v$. For case 1 we achieve inconsistency with CAMESTRES getting $\Gamma \vdash \text{No } v \text{ are } x$ which with $\Gamma \vdash \text{All } x \text{ are } v$ is an inconsistency. Case 2 using CAMESTRES and NO/ZERO gets $\Gamma \vdash \text{No } v \text{ are } y$ and $\Gamma \vdash$

No w are x. However, applying NO/ZERO and CAMESTRES with these new No statements on our case 2 assumptions gets $\Gamma \vdash$ No x are y and $\Gamma \vdash$ No y are x, which is an inconsistency with our assumption that $\Gamma \nvdash \varphi_2$, thus we must have $\forall v, w \in A$, $\Gamma \nvdash$ No v are w, and so condition 2 holds and $A \in M$. Therefore if $\Gamma \nvdash \varphi_2$ there is some $A \in M$ s.t $x, y \in A$ which implies $\mathcal{M} \nvDash \varphi_2$, which is a contradiction so we must have $\Gamma \vdash \varphi_2$. And from these cases of φ we see that $\forall \varphi \in \mathscr{S}$(all,no), $\mathcal{M} \vDash \varphi \implies \Gamma \vdash \varphi$.

Problem: 2

Finish the completeness proof of the logic for $\mathscr{S}$ given in the lecture (see Sep 9 notes). Here is an outline.

(a) Suppose that $\Gamma \vDash \varphi$, with φ of the form Some p are q. In this case, we use partial completeness result we did in the lecture (see Sep 11 notes).

From lecture: taking $\mathcal{M}_s = (\Gamma_{all,some}, [\![\ ]\!])$, $[\![u]\!] = \{\varphi \in \Gamma_{all,some} : \Gamma_{all,some} \vdash \text{All } v \text{ are } u \implies v \in \varphi\}$, our Lemma 1 showed $\mathcal{M}_s \vDash \Gamma_{all,some}$, our second lemma showed $\mathcal{M}_s \vDash$ Some p are $q \implies \Gamma_{all,some} \vdash$ Some . Our theorem showed for $\Gamma \subset \mathscr{S}$, $\Gamma \vDash$ Some p are $q \implies \Gamma \vdash$ Some p are q via two cases:
1. $\mathcal{M}_s \vDash \Gamma_{no}$. By lemma 2 and our assumption we have $\Gamma \vdash$ Some p are q.
2. $\mathcal{M}_s \nvDash \Gamma_{no}$. Then $\mathcal{M}_s \vDash$ Some m are n which by lemma 2 gives $\Gamma \vdash$ Some m are n. Thus by applying X rule we have $\Gamma \vdash$ No n are n thus $\Gamma \vdash$ Some p are q.

(b) Suppose that $\Gamma \vDash \varphi$, with φ of the form All p are q or No p are q. Let $\mathcal{M}$ be the model from previous exercise. We have two cases, depending on whether $M \vDash \Gamma_{some}$ or not. Argue that either way, $\Gamma \vdash \varphi$.

Case 1: $\mathcal{M} \vDash \Gamma_{some}$. This with $\mathcal{M} \vDash \Gamma_{all,no}$ from Problem 1 provides $\mathcal{M} \vDash \Gamma$. Since $\mathcal{M} \vDash \Gamma$ and $\Gamma \vDash \varphi$ by assumption, we know $\forall \psi \in \Gamma$, $\mathcal{M} \vDash \psi$, and from Problem 1 (b) we know $\mathcal{M} \vDash \varphi \implies \Gamma \vdash \varphi$ in this case.

Case 2: $\mathcal{M} \nvDash \Gamma_{some}$. Like lemma 2, this case implies there is some $x, y \in P$ s.t $\Gamma \vdash$ Some x are y and $\mathcal{M} \nvDash$ Some x are y. The later means $[\![x]\!] \cap [\![y]\!] = \varnothing$, but then $\mathcal{M} \vDash$ No x are y and again by 1 (b) this implies $\Gamma \vdash$ No x are y. Thus with both $\Gamma \vdash$ Some x are y and $\Gamma \vdash$ No x are y we use X rule to achieve $\Gamma \vdash \varphi$.

References