Partial Differential Equations M441 Hw 4

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Problems from Levandosky et al. (2008)

Problem: 2.3.1

Consider the diffusion equation $u_t = u_{xx}$ in the interval $(0, 1)$ with $u(0, t) = u(1, t) = 0$ and $u(x, 0) = 1 - x^2$. Note that this initial function does not satisfy the boundary condition at the left end, but that the solution will satisfy it for all $t > 0$.

(a) Show that $u(x, t) > 0$ at all interior points $0 < x < 1, 0 < t < \infty$.

We see that the max value is taken at $u(0, 0) = 1$, and further, the minimum value can be verified to be 0 on the boundary, thus by the max-min principle we know $u(x, t) \in (0, 1)$ for all interior points.

(b) For each $t > 0$, let $\mu(t)$ = the maximum of $u(x, t)$ over $0 \leq x \leq 1$. Show that $\mu(t)$ is a decreasing (i.e., nonincreasing) function of t. (Hint: Let the maximum occur at the point $X(t)$, so that $\mu(t) = u(X(t), t)$. Differentiate $\mu(t)$, assuming that $X(t)$ is differentiable.)

Taking $u(X(t), t) = \mu(t)$, w.t.s $0 > \frac{\partial \mu}{\partial t} = X'(t)u_x(X(t), t) + u_t(X(t), t) = X'(t)u_x + u_x x$. Since μ is always a max of u, by the second derivative test we will have $u_x = 0, u_{xx} < 0$ thus the inequalities hold and $\frac{d\mu}{dt} < 0$ so μ is a decreasing function.

(c) Draw a rough sketch of what you think the solution looks like (u versus x) at a few times. (If you have appropriate software available, compute it.)

https://www.desmos.com/3d/kamla7y3ot give it a few seconds to graph.

Problem: 2.3.6

Prove the comparison principle for the diffusion equation: If u and v are two solutions, and if $u \leq v$ for $t = 0$, for $x = 0$, and for $x = l$, then $u \leq v$ for $0 \leq t < \infty,\ 0 \leq x \leq l$.

Taking $w = v - u$ as a solution, the max-min states that the minimum is obtained on the boundaries, and $w \geq 0$ on the boundaries, the the minium for the whole solution must be ≥ 0 thus $v - u \geq 0 \implies v \geq u$.

Problem: 2.3.8

Consider the diffusion equation on $(0,l)$ with the Robin boundary conditions $u_x(0,t) - a_0 u(0,t) = 0$ and $u_x(l,t) + a_l u(l,t) = 0$. If $a_0 > 0$ and $a_l > 0$, use the energy method to show that the endpoints contribute to the decrease of $\int_0^l u^2(x,t)\,dx$. (This is interpreted to mean that part of the "energy" is lost at the boundary, so we call the boundary conditions "radiating" or "dissipative.")

$$u_t = ku_{xx} \xrightarrow{\cdot x} uu_t = kuu_{xx} = \left(\frac{1}{2}u^2\right)_t = (kuu_x)_x - ku_x^2 \xrightarrow{\int} \frac{1}{2}\frac{d}{dt}\int u^2 dx = kuu_x|_0^l - k\int u_x^2 dx$$
$$= k\left[u(l,t)\left(-a_l u(l,t)\right) - u(0,t)\left(a_0 u(0,t)\right)\right] - k\int u_x^2 dx = -k(a_l u(l,t)^2 + a_0 u(0,t)^2) - k\int u_x^2 dx$$
$$\frac{d}{dt}\int u^2 dx = \frac{-k}{2}(\text{ positive stuff })$$

Thus for positive k's we have $\frac{d}{dt}\int_0^l u^2 dx \leq 0$, showing it to be a decreasing function with respect to time.

Problem: 2.4.3

Use (8) to solve the diffusion equation if $\phi(x) = e^{3x}$. (You may also use Exercises 6 and 7 below.)

$$\frac{1}{\sqrt{4\pi kt}}\int_{-\infty}^{\infty} e^{-(x-y)^2/4kt}(e^{3y})\,dy = \frac{1}{\sqrt{4\pi kt}}\int_{-\infty}^{\infty} e^{\frac{-y^2+2xy-x^2+12kty}{4kt}}dy$$
$$\frac{e^{-(x^2)/4kt}}{\sqrt{4\pi kt}}\int_{-\infty}^{\infty} e^{-(y^2-y(12kt+2x))/4kt}dy = \frac{e^{-x^2}}{\sqrt{4\pi kt}}\int_{-\infty}^{\infty} e^{-((y-(12kt+2x)/2)^2-(12kt+2x)^2/4)/4kt}dy$$
$$\frac{e^{(-x^2+(6kt+x)^2)/4kt}}{\sqrt{4\pi kt}}\int_{-\infty}^{\infty} e^{-(y-(12kt+2x)/2)^2/4kt}dy$$

Let $p = f(y) = \dfrac{(y-(6kt+x)/2)}{\sqrt{4kt}}$, t, k ought be positive so should be safe. Jacobian $\dfrac{\partial y}{\partial p} = \sqrt{4kt}$ so we get :

$$\frac{e^{(-x^2+(6kt+x)^2)/4kt}}{\sqrt{4\pi kt}}\sqrt{4kt}\int_{-\infty}^{\infty} e^{-(p^2)}dp = \frac{e^{(-x^2+(6kt+x)^2)/4kt}}{\sqrt{\pi}}\left[\sqrt{\pi}\right]$$
$$e^{(-x^2+36(k^2t^2)+12ktx+x^2)/4kt} = e^{9kt+3x}$$

Problem: 2.4.9

Solve the diffusion equation $u_t = ku_{xx}$ with the initial condition $u(x,0) = x^2$ by the following special method. First show that u_{xxx} satisfies the diffusion equation with zero initial condition. Therefore, by uniqueness, $u_{xxx} \equiv 0$. Integrating this result thrice, obtain $u(x,t) = A(t)x^2 + B(t)x + C(t)$. Finally, it's easy to solve for A, B, and C by plugging into the original problem.
Differentiating the diffusion equation: $(u_t)_{xxx} = k(u_{xx})_{xxx}$, under our assumptions we can change the orders: $(u_{xxx})_t = k(u_{xxx})_{xx}$, showing u_{xxx} to satisfy diffusion eq. Plugging in after integrating thrice with respect to t:

$$A'(t)x^2 + B'(t)x + C'(t) = 2kA(t) \longrightarrow A'(t)x^2 + B'(t)x + C'(t) - 2kA(t) = 0$$

For this to always hold we take $A'(t), B'(t) = 0$, meaning $A(t), B(t)$ are just constants (say A, B) giving:

$$C'(t) - 2kA = 0 \xrightarrow{\int} C(t) - 2kAt = c_1 \longrightarrow C(t) = 2kAt + c_1$$
$$Ax^2 + Bx + (2kA \cdot 0 + c_1) = x^2 \longrightarrow A = 1,\ B = 0,\ c_1 = 0$$
$$u(x,t) = x^2 + 2kt$$

Problem: 2.4.10

(a) Solve Exercise 9 using the general formula discussed in the text. This expresses $u(x,t)$ as a certain integral. Substitute $p = (x-y)/\sqrt{4kt}$ in this integral.

First notice that the change of variables reverses the domain: $y \in [a,b] \implies p \in [a,b]$, then using (8):

$$\frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} y^2 e^{-(x-y)^2/4kt} dy \longrightarrow \frac{1}{\sqrt{4\pi kt}} \int_{\infty}^{-\infty} (x - p\sqrt{4kt})^2 e^{-p^2} (-\sqrt{4kt}) dp$$

Flip bounds switch signs and expand: $\frac{1}{\sqrt{\pi}} \left[x^2 \int_{-\infty}^{\infty} e^{-p^2} dp - 2x\sqrt{4kt} \int_{-\infty}^{\infty} pe^{-p^2} dp + 4kt \int_{-\infty}^{\infty} p^2 e^{-p^2} dp \right]$

From prev. results and integration by parts: $\frac{1}{\sqrt{\pi}} \left[x^2\sqrt{\pi} + 0 + 4kt \left(\frac{-1}{2} e^{-p^2} - -\frac{1}{2} \left(\sqrt{\pi}\right) \Big|_{-\infty}^{\infty} \right] \right.$

$$\frac{1}{\sqrt{\pi}} \left(x^2\sqrt{\pi} + 4kt\frac{1}{2}\sqrt{\pi} \right) = x^2 + 2kt$$

(b) Since the solution is unique, the resulting formula must agree with the answer to Exercise 9. Deduce the value of

$$\int_{-\infty}^{\infty} p^2 e^{-p^2} dp$$

By parts from (a): $\begin{matrix} pe^{-p^2} & \frac{-1}{2}e^{-p^2} \\ p & 1 \end{matrix} \longrightarrow \left[\frac{-pe^{-p^2}}{2} - \left(\frac{-1}{2} \int_{-\infty}^{\infty} e^{-p^2} dp \right) \right|_{-\infty}^{\infty} = 0 + \frac{1}{2}\sqrt{\pi}$

References

Levandosky, J., Levandosky, S., and Strauss, W. (2008). *Partial Differential Equations: An Introduction, 2e Student Solutions Manual.* Wiley.